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ELLIPTICAL GENERA OF THE N-th LEVEL AND UMBRAL ANALYSIS

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1. Elliptical genera $T_N(M^n)$ for quasi-complex manifolds were introduced in [6], [14]. Following Hirzebruch [8], the elliptical genus of order N is a modular form of the weight n with respect to the group

$$\Gamma_1(N)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(\mathbb{Z}),\ c\equiv 0,\ a\equiv b\equiv 1 \mod N\right\}$$

At points cusp-α of the group $\Gamma_1(N)$ ($\alpha=2\pi i(\frac{K}{N}\tau+\frac{\ell}{N})$, $L=2\pi i(\mathbb{Z}\tau+\mathbb{Z})$ is a lattice, $\alpha\in \mathbb{C}/L$, $N\alpha=0$ in $\mathbb{C}/L$, $\mathbb{C}$- are complex numb .s) this genus degenerates and takes the form:

if $K=0$, then

$T_N(M^n)=\chi_y(M^n)/(1+y)^n$, where $-y=\xi^\ell$, $\chi_y(M^n)$ is the genus from [4].

If $K>0$, then

$T_N(M^n)=\chi(M^n, K^{k/N})$, which corresponds to the power -ies

$Q(x)=\frac{x}{1-e^{-x}}e^{-k/N}$.

Similar genera were studied in [11], [9], [3], [4].

2. Generalization of Hirzebruch's genus.

Consider a formal group

$f(u,v)=\dfrac{u+v+\gamma uv}{1-\omega uv}$, where γ and ω are arbitrary numbers.

The logarithm of this formal group is

$$g(u)=\int_0^u \frac{dz}{1+\gamma z+\omega z^2} \qquad (*)$$

This formal group can be obtained from Euler's formal group [5], whose logarithm is equal to the elliptical integral

$$g(u)=\int_0^u \frac{dz}{\sqrt{1-2\delta z^2+\varepsilon z^4}}$$

applying the linear-fractional transform $u \to \frac{au+b}{cu+d}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(R)$.

When

$$\gamma = y-1,$$
$$\varepsilon = -y,$$

we obtain that χ_y is Hirzebruch's genus [4], [10].

<u>Lemma 1</u>.

$$\Theta_1(x,y)=\frac{2x+2y+(8\omega-\gamma^2)xy+(2\omega^2+\gamma^2)(x^2y+xy^2)+\omega^2\gamma^2x^2y^2}{1-(2\omega^2+\omega\gamma^2)xy+\omega^3\gamma^2(x^2y+xy^2)+(\omega^4-\omega^3\gamma^2)x^2y^2}$$

$$\Theta_2(x,y)=\frac{(x-y)^2}{1-(2\omega^2+\omega\gamma^2)xy+\omega^3\gamma^2(x^2y+xy^2)+(\omega^4-\omega^3\gamma^2)x^2y^2},$$

$$H_1(x,y)=\frac{1}{1-(2\omega^2+\omega\gamma^2)xy+\omega^3\gamma^2(x^2y+xy^2)+(\omega^4-\omega^3\gamma^2)x^2y^2}!$$

where θ_1, θ_2, H are coefficients of the two-valued formal group [2].

Using the results from [10], [4] we obtain

<u>Theorem 1</u>. $\mathrm{Im}\left(H : \Omega^{*}_{SC} \to Q[\omega, \gamma] \right) =$

$$= \mathbb{Z}[h_4, h_8, h_{12}, h_{16}];$$

H – is the homeomorphism that corresponds to the formal group (*);

$$h_4 = 8\omega - \gamma^2,$$
$$h_8 = 2\omega^2 + \gamma^2\omega,$$
$$h_{12} = \omega^2\gamma^2,$$
$$h_{16} = \omega^4 - \omega^3\gamma^2.$$

3. Connection with Umbral Analysis

From [10] $\quad g^{-1}(x) = \dfrac{x}{Q(x)} = e^{(1/2+1)x}(e^x - 1).$

Consider the general case

$$g^{-1}(x) = \frac{1}{\beta} e^{\alpha x}(e^{\beta x} - 1).$$

Applying the results of umbral analysis [13], [12], we have

$$g(x) = \sum \frac{G'_k(0, \alpha, \beta)}{k!} \beta^k x^k,$$

where

$$G_k(x, a, \beta) = \frac{x}{x - ak}\left(\frac{x - ak}{\beta}\right)_k \quad \text{are Gould's polynomials,}$$

$$(c)_n = c(c-1)\cdots c(c-n+1).$$

Theorem 2. $f(u, v) = g^{-1}(g(u) + g(v)) = uR(v) + vR(u)$, where

$$R(u) = e^{ag(u)} + \frac{\beta}{2}u.$$

<u>Remark</u>. From [12] we have:

1. $$e^{xg(u)} = \sum_{k=0}^{\infty} \frac{G_k(x, a, \beta)}{k!} u^k.$$

2. If $2a + \beta$, then $g^{-1}(x) = \dfrac{\sin\frac{\beta x}{2}}{\frac{\beta}{2}}$.

The corresponding formal group has the form

$$f(u,v) = uR(v) + vR(u),$$

where $R(u) = \sqrt{1 - \frac{b^2}{4}u^2}$, which is the particular case of Euler's formal group [11], [9], [4].

3. If $b \neq 0$, then $g^{-1}(x) = xe^{ax}$ and according to [13]

$$g(x) = \sum \frac{(-ak)^{k-1}}{k!} t^k.$$

The corresponding formal group has the form

$$f(u,v) = uR(v) + vR(u),$$

where $R(u) = \sum \frac{a^k (1-k)^{k-1}}{k!} u^k$ is Abel's polynomial.

Theorem 2. The function $R(u)$ satisfies the equation

$$R'(u)[ku + R(u)] = kR(u) + cu$$

where $k = R'(0) = a + \frac{b}{2}$, $c = \frac{b^2}{2}$.

Corollary 1. Coefficients of the series $f(u,v)$ lie in the ring

$$\mathbb{Q}[2a+b, b^2].$$

Let $G : \Omega_U^* \to \mathbb{Q}[2a+b, b^2]$ be a homomorphism that corresponds to the formal group.

Corollary 2. $\operatorname{Im} G$ is generated by Gould's polynomials.

Corollary 3. $G(CP(1)) = -2a - b,$
$G(CP(2)) = \frac{1}{2}(9a^2 - 9ab + 2b^2).$

Theorem 3. 1. $\Theta_1(x,y) = 2x + 2y - b^2 xy,$

2. $\Theta_2(x,y) = (x-y)^2,$

3. $H_1(x,y) = 1.$

Corollary 4. $\operatorname{Im}(G : \Omega_{SC}^* \to \mathbb{Q}[2a+b, b^2]) =$
$= \mathbb{Z}[b^2].$

Литература

I. В.М.Бухштабер, С.П.Новиков. Матем.сб.,84, I26, I97I,II6-I53.

2. В.М.Бухштабер. Современные проблемы математики. Т.I0, I978, 5-I78.

3. Р.Г.Надирадзе. У.М.Н. Т.XXXУ, вып.3, I980, 206-206-209.

4. Р.Г.Надирадзе. Эллиптические роды разных теорий кобордизмов. (в печати). Сообщения АН ГССР, I988.

5. L.Euler. De integratione aequationis differentialis $m\,dx/\sqrt{1-x^4} = n\,dy/\sqrt{1-y^4}$. Opera omnia, XX,58-79, Teubner-Füssli, 1911-1979.

6. A.Hatori. Spin - structures and s'-actions. Inventiones math., 48, 1978, 7-31.

7. F.Hirzebruch. Topological methods in algebraic geometry, Springer-Verlag, 1966.

8. F.Hirzebruch. Elliptic genera of level N for complex manifolds. Preprint MPI/88-24.

9. P.S.Landweber. Elliptic cohomology and modular forms. To appear. Springer Lecture Notes in Mathematics, 1326 (1988).

10. R.G.Nadiradze. Characteristic classes in SC-theory and their applications. To appear. Springer Lecture Notes in Mathematics.

11. S.Ochanine. Sur les genres multiplicatifs définis par des intégrales elliptiques. Topology 26, 1987, 143-151.

12. N.Ray. Extensions of umbral calculus: penumbral coalgebras and generalized Bernoulli numbers. Adv. in Math.,61,1986, 49-106.

13. S.Roman. The umbral calculus. Academic Press, 1984.

14. E. Witten. The index of the Dirac operator in loop space. To appear, Springer Lecture Notes in Mathematics, 1326 (1988).