18.917 Tues 2/4

Compact Lie Group Examples 1) G = finite

2) $G = S^1 \subseteq \mathbb{C}^*$ 3) $S^1 \times S^1$ = Torus

Need $G \times G \to G$, $G \xrightarrow{(\,)^{-1}} G$ } conts get a (~~topological~~ differentiable) group

$G = U(n), SO(n), Sp(n) \quad n \geq 0$

Any compact lie group w/ finite

→ Can you do this completely from homotopy theory

Classification - not quite, structure

Techniques all homotopy

Main Techniques

Construct a "Dictionary"

G(geometry) Homotopy Theory

2) Compact Lie Group ⇝ (~~Finite Loop~~ Space) Study p-adic Finite loop spaces

1) Topological Group G ⇝ Loop Space X

Def'n $f: X \to Y$ is a weak equivalence if f induces

$$\pi_i(X, x) \xrightarrow{\cong} \pi_i(Y, f(x))$$

for all $x \in X$ and $i \geq 0$.

Prop: For a map $f: X \to Y$, the following are equivalent:

① f is a weak equivalence.

② For any CW complex K, f induces a bijection $[K, X] \xrightarrow{f} [K, Y]$

③ for CW K, f induces a weak equivalence $\mathrm{Map}(K, X) \longrightarrow \mathrm{Map}(K, Y)$ (on mapping spaces)

Sketch of proof 1 ⟺ 2

Lemma If X is any space then there exists a CW complex $C(X)$ and a weak equivalence $C(X) \to X$

(Inductive proof: put on pts, circles, 2-cells to kill off etc. . . .)

Pf 1 ⟺ 2: ① ⇒ ② Obstruction theory argument (induction on # cells

K = one cell = sphere get in terms of homotopy . . .

② ⇒ ① interesting:

base pt. problem need ptd homotopy

Sublemma: If $f: K \to L$ is a weak equivalence between CW complexes, then f is a homotopy eq.

Pf: by ① ⇒ ②, f induces bijection from $[L, K] \xrightarrow{f_*} [L, L]$

$g \longmapsto id$

So get a homotopy inverse g: So then $fg = id$ + $gf = id$

Back to ② ⇒ ① have bijection $[K, X] \to [K, Y]$

$$\begin{tikzcd} C(X) \arrow[r, dashed, "\Phi"] \arrow[d, "w.eq."'] & C(Y) \arrow[d, "w.eq."] \\ X \arrow[r] & Y \end{tikzcd}$$

need $C(X)$ & on π_0 on all components ⟵ homotopy commutes

w.hm. $C(Y) \to Y$

$[C(X), C(Y)] \xleftrightarrow{bij.} [C(X), Y]$

& by sublemma $(\overset{\Phi}{- - \to})$ is a hm. eq.

Thus a weak eq.

Thus have ≅ on π_i's all around.

Question from this: w. hm. eq ptd homotopy for 1 ⟺ 2 just test w/ CW w/ free homotopy classes

→ Suppose that $f: X \to Y$ induces a bijection $[K, X] \to [K, Y]$ for all finite CW K is f a weak equivalence?

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Cont Pf. of Prop: ③ w.eq. $Map(K,X) \to Map(K,Y)$

③ ⇒ ① Take $K = *$ so $Map(*,X) = X$ get $X \to Y$ w.eq.

② ⇒ ③ Take any L cw,

Then

$$\begin{tikzcd}
{[L, Map(K,X)]} \arrow[r, "\simeq", "\text{get from other 3}"'] \arrow[d, "\simeq"', no head] & {[L, Map(K,Y)]} \arrow[d, "\simeq\ \text{Ran eq}", no head] \\
{[L\times K, X]} \arrow[r, "\simeq", "\text{by ②}"'] & {[L\times K, Y]}
\end{tikzcd}$$

Thus by 1⟺2 have f w.eq. on mapping spaces

Dictionary

Defn: A ~~map~~ homomorphism $f: G_1 \to G_2$ of top. groups is a weak iso. if f is a weak eq. of the underlying spaces

Thm (Milnor/Kan)

There Exist functors

$$\begin{tikzcd}
\{\text{Top. Groups}\} \arrow[r, shift left, "B"] & \{\text{Pointed connected Spaces}\} \arrow[l, shift left, "L"]
\end{tikzcd}$$

Notes ✪
X, Y are w. equiv. if they are in same w.eq. c[lass]
Top. grps w. ≃ if related by eq. c[lass] of w ≃
So have zig-za[g] of w.eq.'s (some going wrong way) finite ch[ain]

S.T. ① for any top group G, $L(BG)$ is w. iso. to G.

② for any ptd space X, $B(L(X))$ is w. eq. to X.

So bijection w. iso. classes ⟺ w. eq. classes

For compact Lie groups G, $BG \simeq EG/G$

$$\begin{tikzcd}
G \arrow[r] & EG \arrow[d] \\
 & BG
\end{tikzcd}$$

EG contractible space w/ a free G-acti[on]

This is the classifying space of G.

The other functor L: For any X

$L(X)$ is w. eq. to $\Omega X = \{ w: [0,1] \to X \mid w(0) = w(1) = *$

modify ΩX carefully to get connect (Milnor Bundles II)

$$\begin{tikzcd} G \arrow[r] & EG \sim * \arrow[d, "\text{fibre bundle}"] \\ & BG \end{tikzcd}$$

$$\begin{tikzcd} \Omega X \arrow[r] & PX = \left\{ \begin{matrix} w : [0,1] \to X \\ w(0) = * \end{matrix} \right\} \sim * \arrow[d, "\text{fibration}"] \\ & X \end{tikzcd}$$

Remark:

Homotopy groups shift up Ⓑ $\pi_{i-1}(G) \approx \pi_i BG$
or down Ⓛ $\pi_{i-1}(LX) \approx \pi_i(X)$

Defn A loop space X is a triple
(X, BX, e)
where ① X is a space
② BX is a ptd. space
③ $e : X \to \Omega BX$ or $L(BX)$
is a weak equivalence
So X is the loop space that is homotopy analogue of group

So Dictionary Top Group $G \longrightarrow$ Loop Space X
Compact Lie Groups:
Get finite CW $H_*\mathbb{Z}$ = f.g. ab. groups, + 0 above finite

Finite Loop Space ≡ X is a f.l.s. if it is a loop space s.t. the integral Homology groups $H_*(X, \mathbb{Z})$ are f.g. + almost all 0.

Bill's comments: (Hom eq. to finite CW complex is not assumed!) ?$K(\pi,1)$ loop space abelian??

Main Goal of Class:

Thm: If X is a finite loop space and p is a prime number then $H^*(BX; \mathbb{F}_p)$ is finitely gen. as algebra graded commutative ring.

Remarks: Com. Alg. ⇒ Noetherian ring (a.c.c. ⇔ f.g. ideals) From here on just say)

True for X compact Lie group (Pf uses geometric, representation theory) techniques more interesting than theorem itself.

Turns out finite loop space hard to work with.
Actually want to specialize to a prime p.
Let p be a prime.

Defn: A map $f: X \to Y$ is an $\mathbb{F}_p$-equivalence if f induces an isomorphism
$$H_*(X, \mathbb{F}_p) \to H_*(Y, \mathbb{F}_p)$$
(equiv. $H^*(X, \mathbb{F}_p) \leftarrow H^*(Y, \mathbb{F}_p)$ (use fact of UCT for fields)

Paper Topology 70's On localization of

Defn (Bousfield) A space X is $\mathbb{F}_p$-local if the map
$$[A, X] \xleftarrow{f^\#} [B, X]$$
is a bijection whenever $f: A \to B$ is an $\mathbb{F}_p$-equivalence of CW complexes

So X only "sees" $\mathbb{F}_p$ information.

Prop (Bousfield)

① If $f: X_1 \to X_2$ is an $\mathbb{F}_p$ equivalence between $\mathbb{F}_p$-local spaces, then f is a weak equivalence

② For any space X, ∃ a functorially associated $\mathbb{F}_p$-local space $L_{\mathbb{F}_p}X$ together w/ a natural $\mathbb{F}_p$-equivalence $X \to L_{\mathbb{F}_p}(X)$

Sketch of proof ① (Assume X_1, X_2 are CW by arg's of before.)
Then get bijectn mapping into $\mathbb{F}_p$-local spaces

$$[X_1, X_1] \xleftarrow{f^\#} [X_2, X_1] \quad \text{is a bijectn}$$

$$id \longleftarrow\!\shortmid g \quad \text{thus } g \text{ is a hom. inverse to } f$$

② a type of CW equivalence, uses a trick see paper

Remark: suppose that $X \to Y$ is some other $\mathbb{F}_p$-equivalence from X to a $\mathbb{F}_p$-local space Y

construct the diagram

$$\begin{tikzcd} X \arrow[r, "\mathbb{F}_p", "="'] \arrow[d] & Y \arrow[d, "\mathbb{F}_p \text{ equiv.} \Rightarrow \text{w. eq. by } ①"] \\ L_{\mathbb{F}_p} X \arrow[r, "\simeq"', "\text{by } ①"'] & L_{\mathbb{F}_p} Y \end{tikzcd}$$

$\leftarrow$ local-$\mathbb{F}_p$

so Y is w.eq. to $L_{\mathbb{F}_p} X$

so Y is w. eq. to $L_{\mathbb{F}_p} X$.

So for any space X, get unique up to homotopy $L_{\mathbb{F}_p} X$
& there is a functorial construction if you want.

Remark: To get to $L_{\mathbb{F}_p}$:

Let $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n$ p-adic integers

① domain

② P.I.D. max ideal $= (p)$ powers $\mathfrak{a} = (p^n)$

$\mathbb{Z}_p$ module = free + f.g. p-group

Then if X is a 1-connected space with each $H_i(X; \mathbb{Z})$ f.g., then

$$\pi_i(L_{\mathbb{F}_p}(X)) \simeq \mathbb{Z}_p \otimes \pi_i X$$

($\mathbb{Z}_p$ is uncountable

Def: A space X is a p-adic finite loop space if it sats (TFAE) 2 conds

① $H_*(X; \mathbb{F}_p)$ finite and BX is $\mathbb{F}_p$-local

$\Updownarrow$

② $H_*(X; \mathbb{F}_p)$ finite and X is $\mathbb{F}_p$-local and $\pi_0 X$ is a (finite) p-group

Loop Space & Top group so $\pi_0 X = \pi_1 BX$

(equivalence on thms. so Finite loop Space $\rightarrow$

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So Finite loop space → p-adic finite loop space

Goal becomes:

Thm' If X is an p-adic finite loop space, then $H^*(BX, \mathbb{F}_p)$ is noetherian.

Thus: show Thm' ⇒ thm of goal first
& why this is better — ① too many finite
② too few

① Too many
S^3 finite loop structures
$S^3 \xrightarrow{\simeq} \Omega Y$ uncountable # of them
p-adic loop ⇒ only 1 for $L_{\mathbb{F}_p} S^3$

② Too few $2n-1 > 3$
S^{2n-1} no finite loop structure (pure 2 phenomenon)
p-completions are finite loop space
if $n \mid p-1$
& for certain p's get wider picture
& examples are more rigid & algebraic.

Bill 6 Feb.

Th. X finite loop space $\Rightarrow H^*BX$ is Noetherian.

Th'. X a p-adic finite Loop space $\Rightarrow H^*BX$ Noetherian

Prop. X a loop sp. with H_*X finite. Then TFAE

(1) BX is $\mathbb{F}_p$-local

(2) X is $\mathbb{F}_p$-local & $\pi_0 X$ is a finite p-gp.

Pf of prop will use a lemma of Bousfield:

Prop. Suppose Y is connected & $\pi_1 Y$ a finite p-gp. Then $L_p \Omega Y \xrightarrow{\simeq} \Omega L_p Y$ (weak equiv.).

$L_p =$ Bousfield $H\mathbb{F}_p$-loc.

Notice that this says that in this case L_p doesn't alter π_1.

Lemma. If Y is $\mathbb{F}_p$-local & 0-connected ~~& $\pi_1 Y$ is finite,~~ then $\pi_1 Y$ ~~is a finite~~ ~~p-group~~ has no q-tors. for $(q,p)=1$

pf. Let $x \in \pi_1 Y$ have order q prime to p. The map $f: S^1 \longrightarrow Y$ representing x. extends over $S^1 \cup_q e^2$. $\tilde{H}_*(S^1 \cup_q e^2; \mathbb{F}_p) = 0$. So

$$\begin{tikzcd}
S^1 \cup_q e^2 \arrow[r, "q"] \arrow[d] & Y \arrow[d, "\simeq"] \\
* = L_p(S^1 \cup_q e^2) \arrow[r] & L_q Y
\end{tikzcd}$$

.

//—

Pf of Prop. (1) ⇒ (2): $\overset{\text{finite}}{\pi_0(X)} \Rightarrow \pi_1(BX)$ finite

so BX p-local ⇒ $\pi_1 BX$ is a finite p-gp.

By Bousfield's lemma,

$$X \simeq \Omega BX \simeq \Omega L_p BX \simeq L_p \Omega BX \cong L_p X.$$

(Bousfield's map?

$$\begin{tikzcd} \Omega Y \arrow[r] \arrow[d] & PY \arrow[d] & & L_p\Omega Y \arrow[r] \arrow[d] & L_pPY \arrow[d] \\ * \arrow[r] & Y & \arrow[r] & & L_p * \arrow[r] & L_pY \end{tikzcd}$$

$$\begin{tikzcd} \Omega L_pY \arrow[d] & \Omega \arrow[d] & \\ PL_pY \arrow[d] & L_pPY \arrow[d] & * \arrow[l, "\simeq"'] \\ L_pY & L_pY & \end{tikzcd}$$

made a fibration

$\xrightarrow{?}$)

---

(2) ⇒ (1). $\pi_1 BX$ is a finite p- gp., so again by Bousfield's lemma,

$$\Omega L_p BX \simeq L_p \Omega BX \cong L_p X \simeq \underset{\substack{\wr\wr \\ \Omega BX.}}{X}$$

Looking, this is $\Omega(BX \to L_p BX)$.

Since BX is 0-connected, this implies $BX \xrightarrow{\simeq} L_p BX$.

Δ.

Thm' ⇒ Thm. First case: $\pi_0 X$ is a finite p-group

By Bousfield's lemma:

$$L_pX \cong L_p\Omega BX \simeq \Omega L_p BX$$

$H_*(X;\mathbb{F}_p)$ is finite, so $H_*(L_pX;\mathbb{F}_p)$ is too:

(L_pX, L_pBX) is a p-adic finite loop space.
So $H^*(L_pBX)$ is Noetherian by Thm'.
So $H^*(BX)$ is too. △.

General case. $\pi_0(X)$ is a finite group; let H be a p-Sylow subgp. & let

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ H \arrow[r] \arrow[ur, phantom, "\Nwarrow"] & \pi_0 X \end{tikzcd}$$

ie $X' = \Omega BX'$ where BX' is the covering space of BX corresp. to $H \subset \pi_1(BX)$.
So (X', BX') is a finite loop space with $\pi_0 X'$ a p-gp. Hence $H^*(BX')$ is Noetherian, by the above. Transfer $\Rightarrow$ $H^*BX \hookrightarrow H^*BX'$. △

Transfer: $\pi : E \downarrow B$ finite cover.
$t : S_*B \longrightarrow S_*E$ singular chains
by sending σ to the sum of its lifts (as a map)

Dualize to get $t : S^*E \longrightarrow S^*B$.
S^*E is an S^*B-module, and t is an S^*B-map
(via the Alexander-Whitney map!)

So $t : H^*E \longrightarrow H^*B$ is an H^*B-module map.

Algebraic setup: $q : A \to B$ gr. com. rings.
$r : B \to A$ an A-module map

st. $rq|A = id$. Suppose B Noetherian.
Then A is Noetherian. **Exercise.**

△

Sketch of a "geometric" pf that H^*BG is Noetherian, G cpt conn Lie.

① Find a maximal torus in G: ie $T \subset G$ st $Z_G(T) = T$.

② $N_G T / T$ is finite & $\chi(G/NT) = 1$.

③ prove directly that H^*BNT is Noetherian.

④ Use above to get result.

For a start on ①, we'll show any con. nontriv. cpt Lie gp has a central elt of order p.

11 Feb

Th. If G is a nontrivial connected cpt. Lie gp., then G has an elt of order p.

– by a "transformation group method": here's a finite gp analogue:

Suppose G finite of even order n. G contains an elt of order 2: because $\Gamma = \mathbb{Z}/2$ acts on ~~[illegible]~~ G by $\gamma g = g^{-1}$. Then $G = G^\Gamma$ ~~[illegible]~~ $\sqcup$ (free orbits).

$G^\Gamma = \{g : g^2 = 1\}$. must have even order. $1 \in G^\Gamma$, so there's another.

Say $p \mid \#G$. $\Gamma = \mathbb{Z}/p$ acts on $X : G^p$ ~~by~~ $/\Delta(G)$ ~~by~~

~~$\gamma(g_1, \dots g_p) =$~~ $= \{(g_1, \dots g_p) : g_1 \cdots g_p = 1\}$

by $(g_1, \dots g_p) \longmapsto (g_1 g_2^{-1}, g_2 g_3^{-1}, \dots g_p g_1^{-1})$.

Γ acts on both by cyclic permutation, and $X^\Gamma = \{g \in G : g^p = 1\}$. $p \mid \#X$, so $\#X^\Gamma \equiv 0\,(p)$.

Lefschetz & Euler.

M^* graded finite type finitely nonzero vs / k.

$\chi(M) = \Sigma (-1)^i \dim M^i$.

$\chi(M) = \chi(M') + \chi(M'')$ if $0 \to M' \to M \to M'' \to 0$

Prop. If $H_*(X; \mathbb{Z})$ is finitely gen & finitely nonzero, then $\chi(H_*(X;\mathbb{Q})) = \chi(H_*(X; \mathbb{F}_p))$ $\forall p$.

pf. Write $H_*(X;\mathbb{Z})$ as sum of cyclic groups, and trace the effect of each on X. $\triangle$

Write $\chi(X)$ for this common value.

Let $f: M^* \to M^*$ be an endomorphism. Assume k has char 0.
Then $\Lambda(f) = \sum (-1)^i \operatorname{tr}(f|M^i)$.
So $\Lambda(1) = \underline{\chi(M^*)}$ (in $\mathbb{Z} \subset k$).

Prop. Let Γ be a finite cyclic gp. ~~of order~~ w/ gen γ, acting smoothly on cpt manifold M. Then

$$\Lambda(\gamma_* | H_*(M;\mathbb{Q})) = \chi(M^\Gamma).$$

pf. (sketch). M^Γ is a manifold. Pick a Γ-invariant metric on M.
Let T be a tubular nd of M^Γ in M, equivariant.
Let $M' = M - \operatorname{int} T$, so

$$\begin{tikzcd} & \partial T \arrow[r] \arrow[d] \arrow[dr, Rightarrow] & M' \arrow[d] \\ M^\Gamma \arrow[r, phantom, "\simeq"] & T \arrow[r] & M \end{tikzcd}$$

& mvs., with Γ-action, in $\underline{H_*(-;\mathbb{Q})}$.
The Λ's add:

$$\chi(M^\Gamma) + 0 =$$
$$\Lambda(\gamma | H_*(M^\Gamma;\mathbb{Q})) + \Lambda(\gamma_* | H_*(M',\mathbb{Q})) =$$
$$\Lambda(\gamma_* | H_*(\partial T;\mathbb{Q})) + \Lambda(\gamma_* | H_*(M;\mathbb{Q})) =$$
$$0 + \Lambda(\gamma_* | H_*(M;\mathbb{Q}))$$ $\triangle$

Counterexample: let $\mathbb{Z}$ act on $\mathbb{R}$ by $\gamma : x \mapsto x+1$, & ~~let~~ extend to an action of $\mathbb{Z}$ on $\mathbb{R}_+$. Then there's no invariant tubular nd. of the fixed pt set, $\{\infty\}$.

Example. S^3 has an nontrivial elt. of order 2.

- Construct an action of $\Gamma = \mathbb{Z}/2$ on $G = S^3$: $\gamma g = g^{-1}$. $G^\Gamma = \{g : g^2 = 1\}$ is a submanifold, and $\chi(G^\Gamma) = \Lambda(\gamma_* | H_*(G;\mathbb{Q})) = 1-(-1) = 2$ since $\gamma_* | H_3 = -1$:

$$* : S^3 \xrightarrow{\Delta} S^3 \times S^3 \xrightarrow{1\times\gamma} S^3 \times S^3 \xrightarrow{\mu} S^3$$

Chase this in H_3.

In general case we want to compute Λ without knowing H_*:

~~Lemma~~. $\chi(M)(t) = \sum (\dim M^i) t^i$.

$\Lambda(f)(t) = \sum \operatorname{tr}(f | M^i) t^i$.

Lemma Let X have $H_*(X;\mathbb{Q})$ finitely nonzero & fi type. Let $\gamma \in \Gamma = \mathbb{Z}/p$ act on X^p cyclicly. Then

$$\Lambda(\gamma_* | H_*(X^p;\mathbb{Q}))(t) = \chi(H_*(X;\mathbb{Q}))((-1)^{p-1}t^p).$$

Pf seems messy. Here's the key idea: The lemma actually holds for $\underline{\underline{M^*}}$.

Key comp: $(p=2)$

$$(\underline{M\oplus\mathbb{Q}}) \otimes (M\oplus\mathbb{Q}) = M\otimes M \oplus \mathbb{Q}\otimes\mathbb{Q}$$

$$\begin{tikzcd} \oplus\; M\otimes\mathbb{Q} \arrow[r, leftrightarrow, bend right, "\text{permuted by } \Gamma"'] & \oplus\; \mathbb{Q}\otimes M \end{tikzcd}$$

$\square$

Note ① if X, Y as in lem, then
$\underline{\Lambda(\gamma_*|H_*(X\times Y))(t)} = \underline{\Lambda(\gamma_*|H_*X)(t)\,\Lambda(\gamma_*|H_*Y)(t)}.$

(since $\operatorname{tr}(A\otimes B) = (\operatorname{tr}A)(\operatorname{tr}B)$

② if G is a nontrivial con. cpt. Lie gp, then $H^*(G;\mathbb{Q})$ is exterior on r generators

$1\neq g\in G$ has no fixedpts, so $\chi(G)=0$: so there's <u>some</u> higher rational homology: so $r>0$.

Bill 13 Feb 92.

G cpt Lie gp.

${}_pG$ elts of order p: $p^G \cong (G^p/\Delta)^G \cong \{(g_1, \dots g_p) : g_1 \cdots g_p = 1\}^G$

& $\chi({}_pG) = \Lambda(\gamma | H_*(G^p/\Delta, \mathbb{Q}))$.

To compute this we showed

$$\Lambda(\gamma | H_*(X^p); t) = \chi(H_*X ; (-1)^{p-1} t^p)$$

and by Hopf, $H^*(G;\mathbb{Q})$ is exterior, on $r > 0$ gen's (if $G \neq 1$), r = rational rank.
Say dim's of gen's $m_1, \dots m_r$: odd.

Prop. $\Lambda(\gamma | H_*(G^p/\Delta)) = p^r$.

pf. $G \longrightarrow G^p \longrightarrow G^p/\Delta$, pr. fib; trivial since there's a section. T acts. Look at Serre s'seq; γ acts. So there's an equiv a filt. on $H_*(G^p)$ s.t.

$$E^0 \cong H_*(G^p/\Delta) \otimes H_*(G)$$

By additivity of traces, $\chi(H_*G, (-1)^{p-1}t^p) =$

$$\Lambda(\gamma | H_*(G^p), t) = \Lambda(\gamma | E^0, t)$$

$$= \Lambda(\gamma | H_*(G^p/\Delta), t) \; \chi(H_*G, t).$$

$\chi(t) = \prod_i (1 + t^{m_i})$, so

$$\Lambda(G^p/\Delta;\ t) = \prod_i \frac{1 + ((-1)^{p-1} t^p)^{m_i}}{1 + t^{m_i}}$$

m_i is odd. $t = -1$ gives Λ: do this, to get p^r.

$$\boxed{\cancel{\Lambda = \prod 1 + ((-1)^{2p-1})^{m_i}}}$$

$${}_pG = \coprod \text{ conjugacy classes of elts of order } p.$$

The isotropy of g is the centralizer of g, which is a closed subgroup: so fibr seq.

(*)
$$ {}_pG \longrightarrow \coprod_{\langle g \rangle} BZ_G(g) \longrightarrow BG$$

Motivating facts: 1) For a cpt Lie gp. G,

$$\mathrm{Rep}(\mathbb{Z}/p,\ G) \xrightarrow{\cong} [B\mathbb{Z}/p,\ BG]$$

2) For any hom $\rho: \mathbb{Z}/p \to G$, $\quad BZ_G(g)^\wedge_p \xrightarrow{\simeq} \mathrm{map}(B\mathbb{Z}/p, BG^\wedge_p)_{B\rho}$

So the homotopy analogue of (*) is

$$\mathrm{map}(B\mathbb{Z}/p,\ BG) \xrightarrow{\text{eval}} BG$$

whose fiber is $\mathrm{map}_*(B\mathbb{Z}/p,\ BG)$

So now let X be a $p$adic finite loopspace, & consider

$$map_*(B\mathbb{Z}/p, BX) \to map(B\mathbb{Z}/p, BX) \longrightarrow BX .$$

Let's try to show that there's a nontrivial map $B\mathbb{Z}/p \to X$ (if X is nontrivial) ~~[illegible]~~ and indeed that each component of $map(B\mathbb{Z}/p, X)$ is the classifying space of a mod p finite loop space. and that $H_*(map_*(B\mathbb{Z}/p, BX); \mathbb{F}_p)$ is finite with Euler characteristic p^r, $r > 0$; where r is a "rational rank" of X.

Our computation of $\chi({}_pG)$ used a group action & Lefschetz numbers. Let's interpret $map_*(B\mathbb{Z}/p, X)$ as a (homotopy) fixed pt. set.

Def $\quad X^{h\Gamma} = map^{\Gamma}(E\Gamma, X)$.

Then (1) $\quad X^{\Gamma} \to X^{h\Gamma}$

(2) $\quad X^{h\Gamma} \cong$ sections of $\left(\begin{array}{c} E\Gamma \times_\Gamma X \\ \downarrow \\ B\Gamma \end{array}\right)$

by $\quad (E\Gamma \to X) \mapsto (E\Gamma \xrightarrow{(1,f)} E\Gamma \times X)/\Gamma$

Conversely, $s : B\Gamma \to E\Gamma \times_\Gamma X$ lifts to Γ-equivariant map $E\Gamma \to (E\Gamma \times_\Gamma X)^\sim \cong E\Gamma \times X \longrightarrow X$.

(3) So if $f: X \longrightarrow Y$ is a Γ· map & a weak equivalence then $f^{h\Gamma}: X^{h\Gamma} \longrightarrow Y^{h\Gamma}$ is a w. eq.:

$$\begin{tikzcd}
X^{h\Gamma} \arrow[r] \arrow[d] & Y^{h\Gamma} \arrow[d] \\
\mathrm{map}(B\Gamma, E\Gamma \times_\Gamma X) \arrow[r, "\simeq"] \arrow[d] & \mathrm{map}(B\Gamma, E\Gamma \times_\Gamma Y) \arrow[d] \\
\mathrm{map}(B\Gamma, B\Gamma) \arrow[r, "="'] & \mathrm{map}(B\Gamma, B\Gamma)
\end{tikzcd}$$

since

$$\begin{tikzcd}
X \arrow[r, "\simeq"] \arrow[d] & Y \arrow[d] \\
E\Gamma \times_\Gamma X \arrow[r] \arrow[d] & E\Gamma \times_\Gamma Y \arrow[d] \\
B\Gamma \arrow[r, "="] & B\Gamma
\end{tikzcd}$$

Eg, $E\Gamma \times X \longrightarrow X$ induces w. eq. of $(\ )^{h\Gamma}$, but on $(\ )^{\Gamma}$, $\emptyset \longrightarrow X^{\Gamma}$.

(4) If $E \xrightarrow{\pi} B\Gamma$ is a fibration, there is a space F' h. eq. to the fiber of F, with a Γ action, s.t. $(F')^{h\Gamma} \simeq$ sections(π). Also
$$\begin{tikzcd}
E\Gamma \times_\Gamma F' \arrow[rr, "\simeq"] \arrow[dr] & & E \arrow[dl] \\
& B\Gamma &
\end{tikzcd}$$
.

Indeed,

$$\begin{tikzcd}
F' \arrow[r] \arrow[d] \arrow[dr, phantom, "\Leftarrow"] & E\Gamma \arrow[d] \\
E \arrow[r] & B\Gamma
\end{tikzcd}$$

18 Feb.

G top. gp. ${}_pG = (G^p/\Delta)^{\Gamma}$ $\Gamma = \mathbb{Z}/p$.

and ${}_pG \underset{p}{\simeq} \mathrm{map}_*(B\Gamma, BG)$.

Prop. X a loop space. Then $\mathrm{map}_*(B\Gamma, X) \simeq (X^p/\Delta)^{h\Gamma}$.

Here X^p/Δ is the homotopy fiber of $BX \xrightarrow{\Delta} BX^p$.

Lemma. Let $E \to B$ be a fibration with Γ-action, and $* \in B^{\Gamma}$. Let F = fiber over $*$. Then

$$F^{h\Gamma} \longrightarrow E^{h\Gamma} \longrightarrow B^{h\Gamma}$$

is a fibration sequence.

(What's a ctrex. to same w/o the h?)

Sketch. Check Serre condition: K cw cx.

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & E^{h\Gamma} \arrow[d] & & K \times E\Gamma \arrow[r] \arrow[d] & E \arrow[d] \\
K \times I \arrow[r] \arrow[ur, dashed] & B^{h\Gamma} \arrow[r, phantom, "\Longleftrightarrow"] & {} & K \times E\Gamma \times I \arrow[r] \arrow[ur, dashed, "\Gamma\text{-eq.}"'] & B
\end{tikzcd}$$

Now produce the lift by induction on relative cells. Use freeness of Γ action.

Lem. 2. Y space. $\boxed{\cancel{Y^h = \mathrm{map}(E\Gamma, Y)}}$

$$\mathrm{map}(\Gamma, Y)^{h\Gamma} \xrightarrow{\cong} \cancel{\mathrm{map}}\; Y^h = \mathrm{map}(E\Gamma, Y)$$

pf $$\mathrm{map}(\Gamma, Y)^{h\Gamma} = \mathrm{map}^{\Gamma}(E\Gamma, \mathrm{map}(\Gamma, Y))$$

$$= \mathrm{map}^{\Gamma}(\Gamma \times E\Gamma, Y) \cong \mathrm{map}(E\Gamma, Y) \quad \triangle.$$

pf of Prop. $$X^p/\Delta \longrightarrow BX \longrightarrow \mathrm{map}(\Gamma, BX)$$

$$\begin{tikzcd}
\Rightarrow \quad (X^p/\Delta)^{h\Gamma} \arrow[r] \arrow[d, phantom, sloped, "\cong"] & (BX)^{h\Gamma} \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \cancel{B}\,\mathrm{map}(\Gamma, BX)^{h\Gamma} \arrow[d, phantom, sloped, "\cong"] \\
\cancel{\mathrm{map}(\Gamma, BX)}F \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \mathrm{map}(B\Gamma, BX) \arrow[r] \arrow[d, phantom, "\parallel"] & \mathrm{map}(E\Gamma, BX) \arrow[d, phantom, sloped, "\cong"] \\
\mathrm{map}_*(B\Gamma, BX) \arrow[r] & \mathrm{map}(B\Gamma, BX) \arrow[r] & BX
\end{tikzcd} \quad \triangle.$$

Goal. Make sense of and prove: if X is a nontrivial p-adic finite loop space, then

$$\chi((X^p/\Delta)^{h\Gamma}) = \Lambda(\gamma | H^*(X^p/\Delta; \mathbb{Q})) = p^r, \; r > 0.$$

And we'll have to see that the component of null-maps contributes $\chi = 1$.

Example. $X = S^1_p = \Omega\mathbb{C}P^\infty_p = K(\mathbb{Z}_p, 1)$

Study rational cohomology:

$$H_*(X;\mathbb{F}_p) = \begin{cases} \mathbb{F}_p & * = 0,1 \\ 0 & \text{else.} \end{cases}$$

We used $X_p = X_0$ before. How about here:

From rational homotopy theory, $H_*(K(\mathbb{Z}_p,1);\mathbb{Q}) = H_*(K(\mathbb{Q}_p,1);\mathbb{Q})$ and $\mathbb{Q}_p$ is an uncountable $\mathbb{Q}$-vectorspace.
So H_* is uncountable for all $* > 0$. H^* is worse!

As a replacement, consider $\mathbb{F}_p$-local.

$$H^{\cancel{*}\,n}(X_p;\mathbb{Z}_p) = [X_p, K(\mathbb{Z}_p,n)] = [X, K(\mathbb{Z}_p,n)] = H^*(X;\mathbb{Z}_p)$$

so we'll use $H^*(X;\mathbb{Z}_p)\otimes\mathbb{Q}$ for $\mathbb{F}_p$-local X.
~~write~~ write $H^*(X;\mathbb{Q}_p)$ for this!

Def. X is $\mathbb{F}_p$-finite iff $H_*(X;\mathbb{F}_p)$ is f.type&finitely nonzero

Prop. Let X be $\mathbb{F}_p$ finite. Then $H^*(X;\mathbb{Q}_p)$ is f.type $/\mathbb{Q}_p$ & finitely nonzero, and

$$\chi_{\mathbb{F}_p}(H^*(X;\mathbb{F}_p)) = \chi_{\mathbb{Q}_p}(H^*(X;\mathbb{Q}_p)).$$

Lemma. If X is $\mathbb{F}_p$-finite, then $H^*(X;\mathbb{Z}_p)$ is f.gen $/\mathbb{Z}_p$ in each deg. and finitely nonzero.

Lem. ⇒ Prop. $H^*(X;\mathbb{Z}_p)$ is a finite $\oplus$ of $\mathbb{Z}_p$'s & finite abelian p-groups

$$0 \to H^*(X;\mathbb{Z}_p)_p \to H^*(X;\mathbb{F}_p) \to {}_pH^*(X;\mathbb{Z}_p) \to 0.$$

Now count. △

Tools for lemma:

① Nakayama's lemma. Let M be a module over $\mathbb{Z}_p$ is s.t. (1) M_p is finite & (2) $M = \varprojlim M/p^nM$. Then M is finitely generated.

② A tower of ab. gps. is $M_\bullet : \cdots \xrightarrow{q} M_2 \xrightarrow{q} M_1 \xrightarrow{q} M_0$.

$\Pi(M_\cdot) = \Pi M_s$; $\quad s : \Pi M_\cdot \to \Pi M_\cdot$

$$\begin{tikzcd} \Pi M_\cdot \arrow[r, "s"] \arrow[d, "pr_{s+1}"'] & \Pi M_\cdot \arrow[d, "pr_s"] \\ M_{s+1} \arrow[r, "q"] & M_s \end{tikzcd}$$

Then

$$0 \to \varprojlim M_\cdot \to \Pi M_\cdot \xrightarrow{1-s} \Pi M_\cdot \to \varprojlim{}^1 M_\cdot \to 0.$$

Then (a) $\quad 0 \to M_\cdot' \to M_\cdot \to M_\cdot'' \to 0$

$$\Rightarrow \begin{tikzcd} 0 \arrow[r] & \varprojlim M' \arrow[r] & \varprojlim M \arrow[r] & \varprojlim M'' \arrow[dll] \\ & \varprojlim{}^1 M' \arrow[r] & \varprojlim{}^1 M \arrow[r] & \varprojlim{}^1 M'' \arrow[r] & 0. \end{tikzcd}$$

(b) if $M_\cdot$ is pro-trivial — ie, $\forall n \geq 0$, $\exists k \geq n$ st. $M_k \to M_n$ is zero. — then $\varprojlim M_\cdot = 0 = \varprojlim{}^1 M_\cdot$.

(pf: $(1-s)^{-1} = 1 + s + s^2 + \cdots$; ~~on any M_s~~ the rhs is finite)

(d) $\varprojlim^1 M_\cdot = 0$ if each M_s is finite.

pf Let $M_s(\infty) = \bigcap_{k>s} \operatorname{im}(M_k \longrightarrow M_s)$

Ses. of towers: $0 \longrightarrow M_\cdot(\infty) \longrightarrow M_\cdot \longrightarrow M_\cdot/M_\cdot(\infty) \to 0.$

By Artinianness, $M_\cdot/M_\cdot(\infty)$ is protrivial. So $M(\infty) \longrightarrow M$ is iso in $\lim^*$. But $M_\cdot(\infty)$ is a tower of epis, & we have:

(c) $\lim^1 M_\cdot = 0$ if M is a tower of epis.

So lim is <u>exact</u> on towers of finite ab. gps.

Feb 20

Th. X $\mathbb{F}_p$-finite type $\Leftrightarrow$ $\mathbb{Z}_p$-finite type. ($\Leftarrow$ easy)

& then: $\chi_{\mathbb{F}_p}(H^*(X;\mathbb{F}_p)) = \chi_{\mathbb{Q}_p}(H^*(X;\mathbb{Q}_p))$ as before.

(recall $H^*(X;\mathbb{Q}_p) = \mathbb{Q} \otimes H^*(X;\mathbb{Z}_p)$.)

Example of $\lim^1$:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d, dashed] & & \vdots \arrow[d, dashed] & \\
\mathbb{Z} \arrow[r, "p^3"] \arrow[d, "p"] & \mathbb{Z} \arrow[r] \arrow[d] & & \mathbb{Z}/p^3 \arrow[r] \arrow[d] & 0 \\
\mathbb{Z} \arrow[r, "p^2"] \arrow[d, "p"] & \mathbb{Z} \arrow[r] \arrow[d] & & \mathbb{Z}/p^2 \arrow[r] \arrow[d] & 0 \\
\mathbb{Z} \arrow[r, "p"] & \mathbb{Z} \arrow[r] & & \mathbb{Z}/p \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow$ LES: $0 \to 0 \to \mathbb{Z} \to \mathbb{Z}_p \to \lim^1 \mathbb{Z} \to 0$.

- in fact $\lim^1 \mathbb{Z}$ is an uncountable $\mathbb{Q}$-$\overset{p}{\mathrm{vs}}$.

Pf of Th: if X is $\mathbb{F}_p$-finite type, then it's also $\mathbb{Z}/p^n$-finite type $\forall n$.

Now

$$C^n(X;\mathbb{Z}_p) = \mathrm{map}(\Delta_n X, \varprojlim \mathbb{Z}_p) = \varprojlim \mathrm{map}(\Delta_n X, \mathbb{Z}/p^n)$$
$$= \varprojlim C^n(X;\mathbb{Z}/p^n).$$

$\{C^*(X;\mathbb{Z}/p^n)\}$ is epimorphic, so its $\lim^1 = 0$: so exact:

$$0 \to C^*(X;\mathbb{Z}_p) \longrightarrow \prod_n C^*(X;\mathbb{Z}/p^n) \xrightarrow{1-s} \prod_n C^*(X;\mathbb{Z}/p^n) \to 0$$

so get LES:

$$\begin{tikzcd}
H^*(X;\mathbb{Z}_p) \arrow[r] & \prod H^*(X;\mathbb{Z}/p^n) \arrow[r, "1-s"] & \prod H^*(X;\mathbb{Z}/p^n) \arrow[ll, bend left]
\end{tikzcd}$$

ie $0 \longrightarrow \varprojlim^1 H^{i-1}(X;\mathbb{Z}/p^n) \longrightarrow H^i(X;\mathbb{Z}_p) \longrightarrow \varprojlim H^i(X;\mathbb{Z}/p^n) \longrightarrow 0.$

Since $H^*(X;\mathbb{F}_p)$ is finite type, $\lim^1 = 0$.
(We also see that if $H^*(X;\mathbb{F}_p)$ quits, then so does $H^*(X;\mathbb{Z}_p)$.)

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[r] & H^i(X;\mathbb{Z}_p)\otimes \mathbb{Z}/p^{n+1} \arrow[r] \arrow[d] & H^i(X;\mathbb{Z}/p^{n+1}) \arrow[d] \\
0 \arrow[r] & H^i(X;\mathbb{Z}_p)\otimes \mathbb{Z}/p^{n} \arrow[r] \arrow[d] & H^*(X;\mathbb{Z}/p^{n}) \arrow[d] \\
 & \perp & \perp
\end{tikzcd}$$

and the map from the constant $H^*(X;\mathbb{Z}_p)$ tower factors. Thus

$$\begin{tikzcd}
H^*(X;\mathbb{Z}_p) \arrow[rr, "\cong"] \arrow[dr] & & \varprojlim H^*(X;\mathbb{Z}/p^n) \\
 & \varprojlim (H^*(X;\mathbb{Z}_p)\otimes \mathbb{Z}/p^n) \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

Thus $H^*(X;\mathbb{Z}_p) \xrightarrow{\cong} \varprojlim (H^*(X;\mathbb{Z}_p)\otimes \mathbb{Z}/p^n)$.

Nakayama: M/p finite & $M \xrightarrow{\cong} \varprojlim M_{p^n} \Rightarrow M$ f gen.
so done. ↯

=

$$X \longrightarrow X^p \longrightarrow X^p/X \longrightarrow BX \longrightarrow BX^p \quad \circlearrowright \Gamma$$

Lemma: $X^p/X \simeq X^{p-1}$

Well, $BX \to BX^p$ is split monic. Since $X^p/X \to BX^p$ is null, this implies that $X^p/X \to BX$ is null. Thus $X \to X^p \to X^p/X$ is a trivial fibr; $X^p \simeq X \times X^p/X$.

Need another argument for the lemma, but we know now anyway that ~~$H^*($~~ X^p/X is $\mathbb{F}_p$-finite.

Def If X is $\mathbb{F}_p$-finite & $\gamma : X \to X$, then

$$\Lambda(\gamma) = \sum_i (-1)^i \operatorname{tr} H^i(X;\mathbb{Q}_p) \in \mathbb{Z}_p.$$

Prop. Let X be a nontrivial connected p-adic finite loop space. Then

$$\Lambda(\gamma \mid X^p/X) = p^r, \qquad r > 0.$$

Lemma. $H^*(X;\mathbb{Q}_p)$ is an exterior algebra on odd classes. If it's $\mathbb{Q}_p$, then $X \simeq *$.

pf. $H^*(X;\mathbb{Q}_p)$ is a Hopf algebra. $(\mathbb{Q} \otimes \mathrm{Tor}_1^{\mathbb{Z}_p}(M,N) = 0)$

In Lie case, $\chi(G) \neq 1$ for various reasons: $g\cdot \simeq 1$ has no fixed pts if g is near 1; or G is parallelizable.
But here: say p odd.

$$H^*(X;\mathbb{F}_p) = (\text{Exterior on odd}) \otimes (\text{T.P. on even}).$$

The $\mathbb{Q}_p$ case shows $\chi_{\mathbb{Q}} = \begin{cases} 1 & r = 0 \\ 0 & r > 0. \end{cases}$

The $\mathbb{F}_p$ case shows $\chi = \begin{cases} 0 & \text{~~$r>0$~~ some exterior gens} \\ p^k,\ k>0 & \text{~~$r=0$~~, ~~Some~~ no expl gens} \end{cases}$

by nontriviality

Conclusion: <u>must be 0</u>. ↯

Pf of Prop Same as in Lie case: study the collapsed SS for $X \to X^p \to X^p/X$, with its Γ action.

Next: relate $\Lambda(\gamma \mid X^p/X)$ and $\chi((X^p/X)^{h\Gamma})$:

Rabbit out of a hat: Y a space with Γ-action.

eval: $E\Gamma \times \mathrm{Map}^\Gamma(E\Gamma, Y) \longrightarrow Y$

is equivariant and induces maps

$$\begin{tikzcd}
{} & E\Gamma \times Y^{h\Gamma} \arrow[r] & E\Gamma \times Y \\
\&\ B\Gamma \times Y^{h\Gamma} \arrow[r, phantom, "="] & E\Gamma \times_\Gamma Y^{h\Gamma} \arrow[r] & E\Gamma \times_\Gamma Y \\
B\Gamma \times Y^\Gamma \arrow[u] \arrow[urr]
\end{tikzcd}$$

Theorem. (Lannes; D-W). Let Y be $\mathbb{F}_p$-finite Γ-space. ($|\Gamma| = p$) Assume that

(1) Y is $\mathbb{F}_p$-complete

(2) $Y^{h\Gamma}$ is $\mathbb{F}_p$-complete.

Then $Y^{h\Gamma}$ is $\mathbb{F}_p$-finite, and $(E\Gamma \times_\Gamma Y, B\Gamma \times Y^{h\Gamma})$ is $\mathbb{F}_p$-finite (ie the rel. cohomology is finite).

Plausibility: say M cpt mnfld with smooth Γ action.

T = tub. nd.

$M' = M - \mathrm{int}\, T$.

Then $H^*(M^\Gamma)$ is finite since it's a cpt mnfld.

Also $H^*(E\Gamma\times_\Gamma M^{\xi}, B\Gamma\times M^\Gamma) \cong H^*(E\Gamma\times_\Gamma M^{\bullet}, B\Gamma\times_\Gamma T)$

$$\overset{\text{exc}}{\cong} H^*(E\Gamma\times_\Gamma M', B\Gamma\times\partial T).$$

$$\cong H^*(M'/\Gamma, \partial T/\Gamma)$$

<u>Cor.</u> Y $\mathbb{F}_p$-finite Γ space, $\mathbb{F}_p$-complete and st $Y^{h\Gamma}$ is $\mathbb{F}_p$-complete. Then

$$\chi(Y^{h\Gamma}) = \Lambda(\gamma\,|\,Y).$$ (using $H^*(; \mathbb{Q}_p)$.)

<u>Proof.</u> Γ has only two distinct irreps $/\mathbb{Q}_p$:

ρ_0 = trivial rep. $\quad tr(\gamma\,|\,\rho_0) = 1.$

ρ_1 = reduced reg. rep.: $\quad tr(\gamma\,|\,\rho_1) = -1$

(ie $\mathbb{Q}_p[\Gamma] = \mathbb{Q}_p\times\mathbb{Q}_p(\omega)$, ω prim $(p-1)^{\text{st}}$ root of 1).

$H^*(Y;\mathbb{Q}_p) = H_0\oplus H_1$. $\qquad \chi$'s $= \chi_0, \chi_1$

$$\Lambda(\gamma\,|\,Y) = \chi_0 - \frac{\chi_1}{p-1}$$

$$\chi(Y) = \chi_0 + \chi_1$$

<u>Notice:</u> ② $\chi(Y, E\Gamma\times Y^{h\Gamma}) = p\,\chi(E\Gamma\times_\Gamma Y, B\Gamma\times Y^{h\Gamma})$

① $\qquad$ .. $\qquad = \chi(E\Gamma\times_\Gamma Y, B\Gamma\times Y^{h\Gamma})$

25 Feb. Similarly,

$$H^*(Y, E\Gamma \times Y^{h\Gamma}) = H_0 \oplus H_1$$ so

$$\chi(\qquad) = \chi_0 + \chi_1$$

$$\Lambda(\gamma|\qquad) = \chi_0 - \frac{\chi_1}{p-1}.$$

(know the pair is $\mathbb{F}_p$-finite, by Lannes' thm). Two claims:

1) $\chi(E\Gamma \times_\Gamma Y, B\Gamma \times Y^{h\Gamma}) = \chi_0(Y, E\Gamma \times Y^{h\Gamma})$

2) $p\chi(E\Gamma \times_\Gamma Y, B\Gamma \times Y^{h\Gamma}) = \chi(Y, E\Gamma \times Y^{h\Gamma})$.

Ad 1): Use serre sseq to compute $H^*(\ ; \mathbb{Q}_p)$ of the ~~total~~ base space of the covering space pair.
$E_2 = H^0(\Gamma; H^*(Y, E\Gamma \times Y^{h\Gamma}))$
i.e. the invariants. △

Ad 2) In a principal Γ-bundle pair,

$$H^*(E\Gamma \times Y; E\Gamma \times Y^{h\Gamma}; \mathbb{F}_p) = H^*(E\Gamma \times_\Gamma Y; B\Gamma \times Y^{h\Gamma}; \mathbb{F}_p[\Gamma])$$

twisted coefficients. (use Serre spectral sequence of the bundle projection.)

Let $I = \ker(\mathbb{F}_p\Gamma \xrightarrow{\epsilon} \mathbb{F}_p)$.

Let $t = \gamma - 1$, so $\mathbb{F}_p\Gamma = \mathbb{F}_p[t]/t^p$.

So $E_0\mathbb{F}_p\Gamma \cong \mathbb{F}_p[\bar{t}]/\bar{t}^p$, $t \in E_0^1$

Γ acts on this and in the quotient it acts trivially.

(2) follows by additivity of χ under s.e.s. of coeff systems. $\triangle$

$p\times(1) = (2)$: $\quad p\chi_0 = \chi_0 + \chi_1$

ie $0 = \chi_0 - \frac{1}{p-1}\chi_1 \underset{\uparrow}{=} \Lambda(\gamma|(Y, E\Gamma\times Y^{h\Gamma}))$

<u>Now we notice</u>!

so by additivity,

$$\Lambda(\gamma|Y) = \Lambda(\gamma|E\Gamma\times Y^{h\Gamma}) = \chi(Y^{h\Gamma}) \triangle.$$

We used the conclusion of Lannes' thm only; but now let's investigate the $\mathbb{F}_p$-complete condition.

X is $\mathbb{F}_p$-local if $\forall$ $H\mathbb{F}_p$-equiv. $A \to B$,
$\text{map}(B,X) \longrightarrow \text{map}(A,X)$ is a weq.

Bousfield $\mathbb{F}_p$-localization L_pX:

B-K <u>$\mathbb{F}_p$-completion</u> C_pX: a $\mathbb{F}_p$-local space; &

$$\begin{tikzcd} X \arrow[r, "\simeq_p"] \arrow[d] & L_pX \arrow[d, "w. eq."] \\ C_pX \arrow[r] & C_pL_pX \end{tikzcd}$$

Good things:

1) $C_pX = \text{tot}\ \mathbb{F}_p^{\bullet}X$, so CASS $\Rightarrow \pi_* C_pX$. & assoc. obst thy..

2) $C_pX \longrightarrow C_pL_pX$ is often an equivalence.

Bad news: $C_pX \longrightarrow C_pL_pX$ is not always an equivalence.

3) $X \xrightarrow{f} Y$ $\mathbb{F}_p$-equiv. $\Rightarrow C_pX \xrightarrow{\simeq} C_pY$.

X is $\mathbb{F}_p$-good if $C_pX \xrightarrow{\simeq} C_pL_pX$:
ie $X \longrightarrow C_pX$ is $H\mathbb{F}_p$-iso.

Examples. 1) Any discrete space / in fact
$C_p(X \amalg Y) \simeq C_pX \amalg C_pY$.

2) Any abelian $K(\pi, n)$ is $\mathbb{F}_p$-good
(see BK for a formula for $\pi_* C_pK(\pi,n)$.)

3) $F \to E \to B$ fibration, B connected. Assume $\pi_1 B$ acts nilpotently on $H_n(F; \mathbb{F}_p)$. $\forall n$.
["unipotently" might be better terminology].
If F, B are $\mathbb{F}_p$-good, so is E.

4) (Since $C_pX = \lim\limits_{\overleftarrow{k}} C_p P_k X$, P_kX = Postnikov stage)

if P_kX is $\mathbb{F}_p$-good $\forall k$, then X is $\mathbb{F}_p$-good.

So any simply connected space is $\mathbb{F}_p$-good.
In fact by B-K's formula you get a formula for the homotopy of C_pX, up to some short exact sequences.

Ctrex. $S^1 \vee S^1$ is not $\mathbb{F}_p$-good (Bousfield)
It's a $K(\pi,1)$, but the mod p homology of π is still not known.

5) Any loop space is $\mathbb{F}_p$-good.
~~(for $X \longrightarrow X \longrightarrow$~~
(Reduce to connected case; remember that in $\tilde{X} \longrightarrow X$ univ. cover, ~~each~~ deck-transformations are given by left translation by elts in kernel.
Since $\tilde{X}$ is connected, these are homotopic to the id., so the action is trivial.}

Def. X is $\mathbb{F}_p$ complete iff X is p-local & p-good.

ie: $X \longrightarrow C_{\mathbb{F}_p}X$ is a weak equivalence.

Th. $\forall X$, TFAE:

1) X is p-good.

2) C_pX is p-complete.

X p-good $\Rightarrow$ $X \longrightarrow C_pX$ $H\mathbb{F}_p$-equiv.

$\Rightarrow$ $C_pX \longrightarrow C_p^2X$ weak equivalence.

ie C_pX is p-complete

~~(Since.~~

The other direction is harder.

X finite p-adic loop space, $f: \mathbb{Z}/p \to X$ a hom.

ie $Bf: B\mathbb{Z}/p \longrightarrow BX$, ptd map.

Prop. $C_X(f)$ ~~$(B\mathbb{Z}/p)$~~ is a p-adic finite loop space.

Recall: $C_X(f) = \Omega(\mathrm{Map}(B\mathbb{Z}/p, BX), Bf)$

~~& we're claiming $BC_X(f) = \mathrm{Map}(B\mathbb{Z}/p, B$~~

so we must just show $H_*(C_X(f); \mathbb{F}_p)$ finite.

$$\Omega\, \mathrm{map}(B\mathbb{Z}/p, BX)_{Bf} = \left\{ \begin{tikzcd} & X \arrow[d] \\ & \mathrm{map}(S^1, BX) \arrow[d, "\mathrm{eval}"] \\ B\mathbb{Z}/p \arrow[r, "Bf"] \arrow[ur, dashed] & BX \end{tikzcd} \right\}$$

$$= \mathrm{Sections}\left\{ \begin{tikzcd} X \arrow[d] \\ E \arrow[d] \\ B\mathbb{Z}/p \end{tikzcd} \right\} \simeq X^{h\Gamma},$$

E the pull-back.
for a suitable action of Γ on $X' \simeq X$.

27 Feb 91.

Apply the Thm to $Y = X^p/\Delta$:
Need to check Y $\mathbb{F}_p$-complete, $Y^{h\Gamma}$ $\mathbb{F}_p$-complete
We had started :

Prop. If $B\mathbb{Z}/p \longrightarrow BX$, X finite p-adic loopspace, then $\mathrm{Cent}_X(\cancel{\mathbb{Z}/p}\, f)$ is a p-adic finite loop space.

pf. By def, $Z_X f = \Omega\, \mathrm{map}(B\mathbb{Z}/p, BX)_{Bf} = X^{h\mathbb{Z}/p}$.
(The action is the "conjugation" action.
Must show
1) $Z_X f$ is $\mathbb{F}_p$-finite
2) $Z_X f$ is $\mathbb{F}_p$-local
3) $\pi_0 Z_X f$ is a finite p-group.

(1): To apply the prop, need
a) X is $\mathbb{F}_p$-finite
b) X is $\mathbb{F}_p$-complete $= \begin{cases} \text{local} & \text{by asm.} \\ \text{good} & \text{since } \text{~~loop~~ space.} \end{cases}$
c) $X^{h\mathbb{Z}/p}$ is $\mathbb{F}_p$-complete $\begin{cases} \text{local. } * \\ \text{good since it's a loop space too!} \end{cases}$

(*): Lemma: Let Γ be any group acting on an $\mathbb{F}_p$-local space X. Then $X^{h\Gamma}$ is $\mathbb{F}_p$ local.
(Generally any homotopy limit preserves local)

pf. $A \to B$ an $\mathbb{F}_p$-equiv. Must check

$$\begin{tikzcd}
\mathrm{map}(A, X^{h\Gamma}) \arrow[d, equal] & \mathrm{map}(B, X^{h\Gamma}) \arrow[l, "\simeq"'] \arrow[d, equal] \\
\mathrm{map}^\Gamma(E\Gamma \times A, X) \arrow[d, equal] & \mathrm{map}^\Gamma(E\Gamma \times B, X) \arrow[l] \arrow[d, equal] \\
\mathrm{map}^\Gamma(E\Gamma, \mathrm{map}(A,X)) & \mathrm{map}^\Gamma(E\Gamma, \mathrm{map}(B,X)) \arrow[l, "\simeq"]
\end{tikzcd}$$

Since $\mathrm{map}^\Gamma(E\Gamma, -)$ carries w.eq's to w.eq's. ∎

In the course of this we showed $Z_X f$ is local (it's a loop space). This leaves 3):

$$\pi_0 Z_X f = \pi_1(\mathrm{map}(B\mathbb{Z}/p, BX), Bf)$$

$$= \pi_1(BX^{h\mathbb{Z}/p}_{(\text{trivial action})}, Bf)$$

BX is $\mathbb{F}_p$-local; so $BX^{h\mathbb{Z}/p}$ is $\mathbb{F}_p$ local; so each component is. We showed early on that all the torsion in π_1 of an $\mathbb{F}_p$-local space is of order p. But we know that $Z_X f$ is $\mathbb{F}_p$-finite: so $\underline{\underline{\pi_0}}$ is finite! ∎

(cf: the centralizer of an elt. of order p in a cpt connected Lie group has centralizer with π_0 a finite p-group.)

X conn. p-adic finite loop space.

On to $X^p/\Delta = Y$. Must check:

1) Y is $\mathbb{F}_p$-finite : it's $\simeq X^{p-1}$.

2) Y is $\mathbb{F}_p$-complete { local: since X is local
good: since X is a loop space.

(actually a product of good spaces is good - BK).

3) $Y^{h\Gamma}$ is complete: local as before.
~~&~~ good? Y isn't a loop space anymore.
Well,

$$\begin{tikzcd}
Y^{h\Gamma} = \mathrm{map}_*(B\mathbb{Z}/p, BX) \arrow[d] \\
\mathrm{map}(B\mathbb{Z}/p, BX) \arrow[d, "\mathrm{eval}"] \\
BX
\end{tikzcd}$$

— that's ~~the~~ why we care!

so $X \to Y^{h\Gamma} \to \mathrm{map}(B\mathbb{Z}/p, BX)$.

Remember the lemma: F, B good, & nilpotent action, $\Rightarrow$ good total space.
X is a loop space. (note that all fibers are X since this is a principal fibration).
Trivial action: ~~the~~ since this fibr. is pulled back from the path-space fibration

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y^{h\Gamma} \arrow[r] \arrow[d] & \mathrm{map}(B\mathbb{Z}/p, BX) \arrow[d] \\ X \arrow[r] & PBX \arrow[r] & BX \end{tikzcd}$$

so the action of $\pi_1(\mathrm{map}(B\mathbb{Z}/p, BX), c)$ on H_*X factors through the action of $\pi_1 BX = 0$ on H_*X.

~~Another~~ Why is $\mathrm{map}(B\mathbb{Z}/p, BX)$ good?

We just saw that π_1 of each component is a finite p-group!

Lemma: if Y is a connected space with $\pi_1 Y$ a finite p-group, then Y is $\mathbb{F}_p$-good.

pf First, if π is a finite $p$group then $B\pi$ is good. Use ind. on order, using fibration + nilpotent (or even trivial) action. Then fiber $Y \to B\pi_1(Y)$. & use the fact that any $\pi_1 Y$-action on $\mathbb{F}_p$ vs is nilpotent.

(First assume V is f dim: $\pi_1 \longrightarrow GL_n\mathbb{F}_p$; p-Sylow subgp = upper triangular with 1's on diag. So the action is nilpotent. (or count to find fixed ~~vector~~ line) So, esp, π acts nilpotently on $\mathbb{F}_p\pi$? so $I^n = 0$ (I acts trivially on a trivial mod)

Know:

X conn. & p-adic finite loop space. Then

$(X^p/\Delta)^{h\Gamma} = \mathrm{map}_*(B\mathbb{Z}/p, BX)$ is $\mathbb{F}_p$-finite,

and $\chi = p^r$, $r > 0$. To finish, we must make sure that $\mathrm{map}_*(B\mathbb{Z}/p, BX)_0$ doesn't contribute the whole lot! In classical case, the 0-component corresponds to the trivial map: one pt.

Th. $\mathrm{map}_*(B\mathbb{Z}/p, BX)_0 \simeq *$.

proof. Enough to show that

$$\mathrm{map}(B\mathbb{Z}/p, BX)_0 \longrightarrow BX$$

is an equivalence: look at the loops.

$$\Omega\mathrm{map} = \left\{ \begin{tikzcd} & (BX)^{S^1} \arrow[d] \\ B\mathbb{Z}/p \arrow[ur, dashed] \arrow[r, "*"'] & BX \end{tikzcd} \right\} = \mathrm{map}(B\mathbb{Z}/p, X)$$

$\simeq X$ by Sullivan conjecture. //.

3 March 1992

Recall: if $\Gamma = \mathbb{Z}/p$ acts on X,

$$\left.\begin{array}{l} X \ \mathbb{F}_p\text{-finite} \\ X \ \mathbb{F}_p\text{-complete} \\ X^{h\Gamma} \ \mathbb{F}_p\text{-complete} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} X^{h\Gamma} \quad \mathbb{F}_p\text{-finite} \\ (B\Gamma(X), B\Gamma(X^{h\Gamma})) \ \mathbb{F}_p\text{-finite} \\ \chi(X^{h\Gamma}) = \Lambda(H^*_{\mathbb{Q}_p}(X), \gamma_*). \end{array}\right.$$

We have shown that for X a p-adic loop space,

$$\begin{tikzcd} \mathrm{Map}_{\ast}(B\mathbb{Z}/p, BX) \arrow[r] \arrow[d, phantom, "\wr"] & \mathrm{Map}(B\mathbb{Z}/p, BX) \arrow[r] & BX \ ; \\ (X^p/\mathrm{diag}(X))^{h\mathbb{Z}/p} \end{tikzcd}$$

$\left.\begin{array}{l}\mathbb{F}_p\text{-finite} \\ \mathbb{F}_p\text{-complete} \\ \text{with } \chi = p^r\end{array}\right\{$

another interpretation of these spaces:

$$\coprod_{f:\mathbb{Z}/p\to X} X/\mathrm{Cent}_X f(\mathbb{Z}/p) \longrightarrow \coprod_{f:\mathbb{Z}/p\to X} \overbrace{B\mathrm{Cent}_X f(\mathbb{Z}/p)}^{B(p\text{-adic finite loop space})} \longrightarrow BX.$$

<u>in our dictionary</u>

$f: X \to Y$	$Bf: BX \to BY$
$Y/f(X)$	homotopy fibre of $Bf: BX \to BY$
f is a monomorphism	$Y/f(X)$ is $\mathbb{F}_p$-finite, and fibre of $Bf: BX \to BY$ is $\mathbb{F}_p$-finite.
f is an epimorphism	$Y/f(X)$ = class. space of a p-adic finite loop space

Theorem If X is a p-adic finite loop space and $f: \mathbb{Z}/p \to X$, then $\mathrm{Cent}_X f(\mathbb{Z}/p)$ is a p-adic finite loop space, and the natural homom. $\mathrm{Cent}_X f(\mathbb{Z}/p) \to X$ is a monomorphism.

Theorem′ If X is as above, G a finite p-group, and $f: G \to X$, then $\mathrm{Cent}_X f(G)$ is a p-adic finite loop space, and $\mathrm{Cent}_X f(G)$ is a monomorphism.

[$f: G \to X$ corresp. to $Bf: BG \to BX$;
$\mathrm{Cent}_X f(G) = \Omega\,\mathrm{Map}(BG, BX)_{Bf}$.]

prop. Let G be a finite p-group acting on a space X.

Assume that
- X is $\mathbb{F}_p$-finite
- X^{hK} is $\mathbb{F}_p$-complete for every subgp. $K \subseteq G$.

Then X^{hG} is $\mathbb{F}_p$-finite and $\chi(X^{hG}) \underset{\text{mod } p}{\equiv} \chi(X)$.

proof of prop. (induction on $|G|$)

Assume true for all proper subgps. K of G; assume $G \neq \{1\}$.

Choose $f : G \twoheadrightarrow \mathbb{Z}/p$ with kernel K.

By inductive assumption, X^{hK} is $\mathbb{F}_p$-finite and $\chi(X^{hK}) \underset{p}{\equiv} \chi(X)$.

By the first prop., $X^{hG} \simeq (X^{hK})^{h\underset{\cong \mathbb{Z}/p}{\underline{G/K}}}$.

So applying the theorem we recalled at the beginning,

$$X^{hG} \text{ is } \mathbb{F}_p\text{-finite and } \chi(X^{hG}) = \Lambda(H^*_{\mathbb{Q}_p}(X^{hK}), \gamma_*). \qquad (\gamma \text{ gen. of } \mathbb{Z}/p.)$$

Now we need only to prove the following lemma:

lemma If X is an $\mathbb{F}_p$-finite space with an action of $\mathbb{Z}/p$, then for γ gen. of $\mathbb{Z}/p$,

$$\Lambda(H^*_{\mathbb{Q}_p}(X), \gamma_*) \underset{p}{\equiv} \chi(X).$$

pf. of lemma

Write $H^*_{\mathbb{Q}_p}(X) = (H^*_{\mathbb{Q}_p}(X))^{(0)} \oplus (H^*_{\mathbb{Q}_p}(X))^{(1)}$ corresp. to the two representations (trivial and non-trivial).

Then

$$\left.\begin{aligned} \chi(X) &= \chi^0 + \chi^1 \\ \Lambda(H^*_{\mathbb{Q}_p}(X), \gamma_*) &= \chi^0 - \frac{\chi^1}{p-1} \end{aligned}\right\} \text{ and these are the same mod } p.$$

☒

☒

5 March 1992

Some facts about $f : G \to X$, G a finite p-group, X a p-adic finite loop space:

① $\mathrm{Cent}_X f(G)$ is a p-adic finite loop space. (proved last time)

② If K is a subgp. of G,

$$\mathrm{Cent}_X f(G) \to \mathrm{Cent}_X f(K) \text{ is a monomorphism.}$$

③ If X is a connected p-adic finite loop space, then any homom. $\mathbb{Z}/p^n \to X$ extends to a homom. $\mathbb{Z}/p^{n+1} \to X$.

__proof of ②__

Consider the restriction map

$$\begin{tikzcd} \mathrm{Map}(BG, BX) \arrow[r] \arrow[d, phantom, "\wr\|"] & \mathrm{Map}(BK, BX) \arrow[d, phantom, "\wr\|"] \\ \coprod_{f : G \to X} B\mathrm{Cent}_X f(G) \arrow[r] & \coprod_{f : G \to X} B\mathrm{Cent}_X f(K) \end{tikzcd}$$

(Are the homotopy fibres F finite?)

By lemma, $\mathrm{Map}(BG, BX) \to \mathrm{Map}(BK, BX)$ can be identified with the map

$$(BX)^{hG} \longrightarrow (\mathrm{Map}(G/K, BX))^{hG} \qquad (*)$$

induced by the diagonal map $BX \to \mathrm{Map}(G/K, BX)$:

The homotopy fibre of this diagonal map is

$$X^{|G/K|}/\mathrm{diag}(X) \longrightarrow BX \longrightarrow \mathrm{Map}(G/K, BX).$$

Now, we would like to see that the homotopy fibre of the induced map (*) is $\text{"}\left(\frac{X^{|G/K|}}{\mathrm{diag}(X)}\right)\text{"}$.

__prop.__ Let $f : X \to Y$ be a map of G-spaces for G a discrete group, with homotopy fibre F (assume Y is connected). Then each $y \in Y^{hG}$ gives an action of G up to homotopy on F (call the action F_y) and the homotopy fibre of $X^{hG} \to Y^{hG}$ over y is $(F_y)^{hG}$.

__pf.__

Y^{hG} and X^{hG} are spaces of lifts, as indicated:

$$\begin{tikzcd} & F \arrow[d] \\ & \mathrm{Borel}(G, X) \arrow[d] \\ & \mathrm{Borel}(G, Y) \arrow[d] \\ BG \arrow[r] \arrow[uur, dashed, "X^{hG}"] \arrow[ur, dashed, "Y^{hG}"'] & BG \end{tikzcd}$$

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proof of ③ (cont'd)

The fibration on the left gives the required action of $\mathbb{Z}/p^{n+1}$ on $X^p/\mathrm{diag}(X)$.

$$\begin{tikzcd}
F = X^p/\mathrm{diag}(X) \arrow[d] & X^p/\mathrm{diag}(X) \arrow[d] & X^p/\mathrm{diag}(X) \arrow[d] \\
E_\alpha \arrow[r] \arrow[d] & E\mathbb{Z}/p^{n+1} \times_{\mathbb{Z}/p^{n+1}} BX \arrow[r] \arrow[d] & E\mathbb{Z}/p \times_{\mathbb{Z}/p} BX \arrow[d] \\
B\mathbb{Z}/p^{n+1} \arrow[r, "\alpha"] & E\mathbb{Z}/p^{n+1} \times_{\mathbb{Z}/p^{n+1}} BX^p \arrow[r] & E\mathbb{Z}/p \times_{\mathbb{Z}/p} BX^p
\end{tikzcd}$$

conclude:

Any generator of $\mathbb{Z}/p^{n+1}$ acts on $H^*_{\mathbb{Q}_p}(F_\alpha)$ with a Lefschetz number p^n, for $n>0$. $\boxtimes$

Theorem. Let X be an $\mathbb{F}_p$-finite space with an action of the group $\mathbb{Z}/p^n$. Assume that X^{hK} is $\mathbb{F}_p$-complete for all $K \subseteq \mathbb{Z}/p^n$. Then

$$\chi(X^{h\mathbb{Z}/p^n}) = \Lambda(H^*_{\mathbb{Q}_p} X, \gamma_*)$$

where $\gamma \in \mathbb{Z}/p^n$ is a generator.

10 Mar.

Th. i: $H \leq K$ finite p-gps., X p-adic finite loop space.
$f: K \to X$ a homomorphism.
Then $Z_X(f) \longrightarrow Z_X(f \circ i)$ is an embedding of p-adic finite loop spaces.

Th. X conn. p-adic finite loop space, $f: \mathbb{Z}/p^n \longrightarrow X$
$\Rightarrow \exists$ extension to $\mathbb{Z}/p^{n+1} \longrightarrow X$.

We're in the midst of proving the second thm, by Lefschetz argument. Here's a step:

Prop. X a space with $\mathbb{Z}/p^n$-action (γ a gen). Assume that X is $\mathbb{F}_p$-finite & X^{hK} $\mathbb{F}_p$-complete $\forall$ $K \leq \mathbb{Z}/p^n$. Then $X^{h\mathbb{Z}/p^n}$ is $\mathbb{F}_p$-finite, and

$$\chi(X^{h\mathbb{Z}/p^n}) = \Lambda(\gamma \mid H^*(X; \mathbb{Q}_p))$$

pf. By induction on n: True for $n=1$, by earlier wk. Let $\mathbb{Z}/p^n > K = p(\mathbb{Z}/p^n)$. Then (using $n=1$) X^{hK} is $\mathbb{F}_p$-finite. By previous lemma,

$$X^{h\mathbb{Z}/p^n} = (X^{hK})^{h\mathbb{Z}/p^{n-1}}$$

so by ind. we know $X^{h\mathbb{Z}/p^n}$ is $\mathbb{F}_p$-finite and

$$\chi(X^{h\mathbb{Z}/p^n}) = \Lambda(\gamma \mid H^*(X^{hK}; \mathbb{Q}_p))$$

So we need to show

$$\Lambda(\gamma \mid H^*(X; \mathbb{Q}_p)) = \Lambda(\gamma \mid H^*(X^{hK}; \mathbb{Q}_p)).$$

Know that (X, X^{hK}) & $(EK \times_K X, EK \times_K X^{hK})$ are $\mathbb{F}_p$-finite.

<u>Lemma</u>. Let (X, Y) be an $\mathbb{F}_p$-finite pair, with an action of $\mathbb{Z}/p^n$. Again $K = {}_p(\mathbb{Z}/p^n)$. Assume $(EK \times_K X, EK, \times_K Y)$ is $\mathbb{F}_p$-finite. (This is a relative homological version of saying the action is free.) Then $\Lambda(\gamma \mid H^*(X, Y; \mathbb{Q}_p)) = 0$.

<u>pf</u> (Not completely satisfactory). Using rep. thy. of $\mathbb{Z}/p^n$ over $\mathbb{Q}_p$: there are $n+1$ irreps., $\rho_0, \dots \rho_n$. (also over $\mathbb{Q}$):

$\rho_0 = \text{triv}$.

$\rho_i = \mathbb{Q}_p(\zeta_i)$ by $\gamma \mapsto$ mult. by ζ_i

<u>(ζ_i = prim. p^{i}th root of 1)</u>.

dimensions: $\dim \rho_0 = 1$ Gal gp: 1

$\dim \rho_i = p^{i-1}(p-1)$ $(\mathbb{Z}/p^i)^\times$.

acting in obvious ways.

trace of γ: on ρ_0: 1

on ρ_1: -1

on ρ_i: 0 $i > 1$.

Write $H^* = H^*(X, Y; \mathbb{Q}_p)$. Then $H^* = \bigoplus_{i=0}^{n} H^*(i)$ where $H^*_\xi(i) = \oplus \rho_i$. Then

$$\chi_\xi(X, Y) = \sum \chi_{\xi i} \qquad \chi_i = \chi(H^*(i))$$

& $$\Lambda(\gamma \mid H^*\xi) = \chi_0 - \frac{1}{p-1}\chi_1 .$$

Let $K_i = p^i(\mathbb{Z}/p^n)$: $0 \subset K_1 \subset \cdots \subset K_n = \mathbb{Z}/p^n$.

	Rational χ	$\mathbb{F}_p \cdot \chi$
(X, Y)	$\chi_0 + \cdots + \chi_n$	χ
$\xi EK_1 \times_{K_1} (X, Y)$	$\chi_0 + \cdots + \chi_{n-1}$	$p^{-1}\chi$
$\vdots$	$\vdots$	$\vdots$
$EK_{n-1} \times_{K_{n-1}} (X, Y)$	$\chi_0 + \chi_1$	$p^{-(n-1)}\chi$
$EK_n \times_{K_n} (X, Y)$	χ_0.	$p^{-n}\chi$.

$EK_1 \times_{K_1} (X, Y)$ is a p-fold cover. Study the way K_1 acts in each rep: trivially except on ρ_n. So in $H^*(-; \mathbb{Q}_p)$, get $\chi_0 + \cdots + \chi_{n-1}$. Etc.

<u>Mod p</u>: $E \downarrow B$. p^n-fold cover $\Rightarrow \chi(H^*(B; \mathbb{F}_p)) \cdot [E|B] = \chi(H^*(E; \mathbb{F}_p))$.

by filtering $\mathbb{F}_p$[covering group].

What's the gap here? See ⊛ below

Look at last two rows: key equality, get

$$pX_0 = X_0 + X_1$$

ie $$X_0 - \frac{1}{p-1}X_1 = 0 \qquad ! \; \triangle$$

*. Gap: ~~why~~ ~~How do~~ We know that $EK_1 \times_{K_1} (X, Y)$ is $\mathbb{F}_p$-finite. But why should $EK_i \times_{K_i} (X, Y)$ be?

Lemma. Let (X, Y) be a pair, with an action of $\mathbb{Z}/p^n$. Assume that $EK \times_K (X, Y)$ is $\mathbb{F}_p$-finite, $K = {}_p(\mathbb{Z}/p^n)$. Then $E\mathbb{Z}/p^n \times_{\mathbb{Z}/p^n} (X, Y)$ is $\mathbb{F}_p$-finite as well.

Proof. $0 \to K \to \mathbb{Z}/p^n \to \mathbb{Z}/p^{n-1} \to 0$.

$\Rightarrow$ fib. seq.

$$EK \times_K (X, Y) \longrightarrow E\mathbb{Z}/p^n \times_{\mathbb{Z}/p^n} (X, Y) \to B\mathbb{Z}/p^{n-1}$$

& so serre s'seq; $E_2 = H^*(B\mathbb{Z}/p^{n-1}; H^*(EK \times_K (X, Y)))$

$H^*(B\mathbb{Z}/p^{n-1})$ is Noetherian; E_2 is f gen. as module (by same filt argument, if action is nontrivial.)

So E_∞ is fg.; so $H^*(E)$ is fg. over $H^*(B)$ (NB: the filtration is by submodules of the base.)

The whole s'seq. ~~is a~~ maps to $BK \to B\mathbb{Z}/p^r \to B\mathbb{Z}/p^{r-1}$. And in that sseq, $H^*(B\mathbb{Z}/p^{r-1}) \longrightarrow H^*(B\mathbb{Z}/p^r)$

factors through the exterior algebra quotient ~~is trivial~~ : so the $H^*(B\mathbb{Z}/p^{n-1})$ - action ~~is~~ trivial ! factors through the action by a finite algebra. //.

Now we can ~~try to~~ form successive extensions

$$\begin{tikzcd}
B\mathbb{Z}/p \arrow[rr] \arrow[d] & & X \\
B\mathbb{Z}/p^2 \arrow[urr] \arrow[d] & & \\
\vdots & & \\
B\mathbb{Z}/p^\infty \arrow[uuurr, "f_\infty"] & &
\end{tikzcd}$$

We study next the centralizer of f_∞.

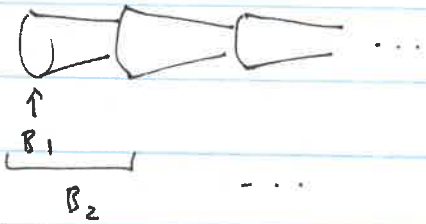
Theorem. Let X be a p-adic finite loop space, $f: \mathbb{Z}/p^\infty \to X$ a homomorphism, Then $Z_X(f) \to X$ is an embedding of p-adic finite loop spaces.

pf. Must study $\mathrm{map}(B\mathbb{Z}/p^\infty, X)_f \longrightarrow BX$.

$B\mathbb{Z}/p^\infty = \underrightarrow{\mathrm{holim}}\ B\mathbb{Z}/p^n$.

12 March

Claim : $Z_X f$ is a p-adic finite loop space, where $f : \mathbb{Z}_{p^\infty} \longrightarrow X$ is the map we constructed.



$B_n \xrightarrow{\simeq}$ right end ;

Each $B_n \longrightarrow B_{n+1}$ is cofibr.

eg. $B\mathbb{Z}/p \longrightarrow B\mathbb{Z}/p^2 \longrightarrow B\mathbb{Z}/p^3 \longrightarrow \cdots$

$\mathrm{map}(B\mathbb{Z}_{p^\infty}, BX) \simeq$ ~~ho~~ $\underleftarrow{\lim}$ $\mathrm{map}(B_n, BX)$.

Regard $f : B\mathbb{Z}_{p^\infty} \longrightarrow BX$ as $f : B_\infty \longrightarrow BX$, & let $f_n = f \mid B_n$. Then

$$\mathrm{map}(B\mathbb{Z}_{p^\infty}, BX)_f \simeq \underleftarrow{\lim} \ \mathrm{map}(B_n, BX)_{f_n}.$$

This is a tower of fibrations, with htpy types

$$\begin{tikzcd}
\mathrm{map}(B\mathbb{Z}/p, BX)_{f_1} \arrow[d, equal] & \mathrm{map}(B\mathbb{Z}/p^2, BX)_{f_2} \arrow[l] \arrow[d, equal] & \cdots \arrow[l] \\
B Z_X(f_1) & B Z_X(f_2) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

These should stabilize : there is an artinian property for subgps of cpt Lie groups

This will prove what we want.

Claim: $\exists n$ st $\forall i > n$, $Z_X(f_i) \xrightarrow{\simeq} Z_X(f_n)$.

Steps. 1: consider dimensions: show they decrease; so they stabilize

2. Then show that beyond that point you're just taking subgroups of π_0 : & this stabilizes too.

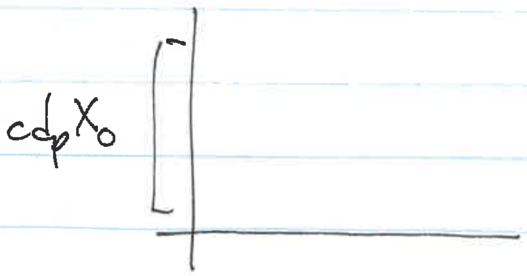
For any ptd Y & $\pi_1(Y)$-module M, define

$$cd(Y;M) = \sup\{ i : H^i(Y;M) \neq 0 \}$$

& $cd_p(Y) = cd(Y;\mathbb{F}_p)$. $\quad cd \in \{-\infty, 0, 1, \ldots \infty\}$.

Lemma. If $X_0 \longrightarrow X_1$ is an inclusion of p-adic finite loop spaces, then $cd_p X_1 = cd_p X_0 + cd_p(X_1/X_0)$. Moreover if $cd_p X_1 = cd_p X_0$, then X_0 is (up to homotopy) a union of components in X_1.

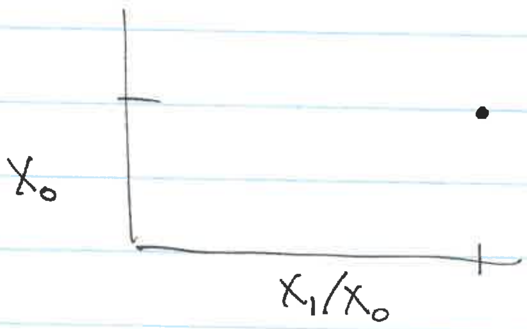
Pf. Consider Serre s'seq for $X_0 \to X_1 \to X_1/X_0$:



The action of $\pi_1(X_1/X_0)$ on $H_*(X_0)$ extends to the action of $\pi_1(BX_0)$ on $H_*(X_0)$. $\pi_1(BX_0)$ is a finite p-group, so this action is nilpotent

Thus $cd_p(X_1/X_0; H_i(X_0)) \leq cd_p(X_1/X_0)$, so the sseq lies in a rectangle. ~~Actually,~~ ~~equality~~, ~~since $H_d X_0 \neq 0$ if $d = cd_p X_0$~~.

In fact the corner group is nonzero, since $H_d X_0 \neq 0$ for $d = cd_p X_0$:



(in fact $\pi_1(X_1/X_0)$ acts trivially on $H_d X_0 = \mathbb{F}_p$.)
The corner must persist △

(The components of the htpy fiber of a map between connected spaces are h. eq; π_1(base) acts transitively on components. Thus all components of X_1/X_0 are the same.)

Now suppose $cd_p X_0 = cd_p X_1$. Then $cd_p X_1/X_0 = 0$. X_1/X_0 is $\mathbb{F}_p$-local (as we've seen; see the sheet, for ex: it's the htpy fiber of a map between $\mathbb{F}_p$-local spaces. Thus the projection to π_0, being $H\mathbb{F}_{p*}$-iso, is a w. eq.

Thus $BX_0 \to BX_1$ is homotopically a finite cover. △

Why $B\mathbb{Z}/p^\infty$ is a circle -

$$K(\mathbb{Z}_{p^\infty}, 1) \longrightarrow K(\mathbb{Z}, 2) \longrightarrow K(\mathbb{Z}_p, 2).$$

are $H\mathbb{F}_p$ - iso's.

(Problem: compute $H^*(B\mathbb{Z}_{p^\infty}; \mathbb{Z})$ as ring)

To see 1st, what is the htpy. fiber? First, how about $\mathrm{Fib}(K(\mathbb{Z}/p^n, 1) \longrightarrow K(\mathbb{Z}, 2))$? It's S^1,

Equally,

$$\mathbb{Z} \longrightarrow \mathbb{Z}[\tfrac{1}{p}] \longrightarrow \mathbb{Z}_{p^\infty} \Rightarrow \mathrm{fib}.$$

$$S^1 \to B\mathbb{Z}[\tfrac{1}{p}] \longrightarrow B\mathbb{Z}_{p^\infty} \longrightarrow \mathbb{CP}^\infty.$$

So $K(\mathbb{Z}_{p^\infty}, 1) \longrightarrow K(\mathbb{Z}_p, 2)$ is $H\mathbb{F}_p$ - localization and $H\mathbb{F}_p$ - completion.

12 March

GOODNESS

We will describe below a general result which implies that almost all of the homotopy fixed point sets of interest in the theory of p-adic finite loop spaces are $\mathbf{F}_p$-complete. Recall that a space X is $\mathbf{F}_p$-complete iff X is both $\mathbf{F}_p$-local and $\mathbf{F}_p$-good.

The following lemmas were proved or discussed in class.

1.1 Lemma. *Any loop space is $\mathbf{F}_p$-good. Any 0-connected space X with $\pi_1 X$ a finite p-group is $\mathbf{F}_p$-good. A space is $\mathbf{F}_p$-good iff each of its path components is $\mathbf{F}_p$-good. If $F \longrightarrow E \longrightarrow B$ is a homotopy fibre sequence such that B is 0-connected. the spaces F and B are $\mathbf{F}_p$-good, and the action of $\pi_1 B$ on each $H_i F$ is nilpotent (e.g., $\pi_1 B$ is a finite p-group), then E is $\mathbf{F}_p$-good.*

1.2 Lemma. *Any product of $\mathbf{F}_p$-local spaces is $\mathbf{F}_p$-local. Any homotopy fibre of a map between $\mathbf{F}_p$-local spaces is $\mathbf{F}_p$-local. If X is $\mathbf{F}_p$-local and Y is any CW complex, then* $\mathrm{Map}(Y, X)$ *is $\mathbf{F}_p$-local. If X is an $\mathbf{F}_p$-local space and G is a discrete group acting on X, then $X^{\mathrm{h}G}$ is $\mathbf{F}_p$-local. A space X is $\mathbf{F}_p$-local iff each of its path components is $\mathbf{F}_p$-local.*

Remark. Lemma 1.2 could be generalized by saying that the class of $\mathbf{F}_p$-local spaces contains all discrete spaces and is closed under arbitrary *homotopy inverse limits.*

1.3 Lemma. *If X is a 0-conected $\mathbf{F}_p$-local space and $\pi_1 X$ is a finite group, then $\pi_1 X$ is a finite p-group.*

1.4 Lemma. *Let X be an $\mathbf{F}_p$-finite space and G a finite p-group acting on X. Suppose that $X^{\mathrm{h}K}$ is $\mathbf{F}_p$-complete for each subgroup K of G. The $X^{\mathrm{h}G}$ is $\mathbf{F}_p$-finite.*

We will begin with a slight generalization of a result proved in class.

1.5 Proposition. *Let X be a p-adic finite loop space and G a finite p-group acting on BX. Then each component of $(BX)^{\mathrm{h}G}$ is the classifying space of a p-adic finite loop space.*

Proof. Choose a point $a : BG \to \mathrm{Borel}(BX, G)$ of $BX^{\mathrm{h}G}$ and construct a (homotopy) fibre square

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "q"'] & \mathrm{Borel}(\mathrm{Map}(S^1, BX), G) \arrow[d] \\
BG \arrow[r, "a"] & \mathrm{Borel}(BX, G)
\end{tikzcd}$$

in which the right hand vertical map is derived from the free loop space fibration over BX. The space S of sections of the fibration q is the loop space of the component of $BX^{\mathrm{h}G}$ containing a, so it is necessary to prove that S is $\mathbf{F}_p$-finite, $\mathbf{F}_p$-local and has a p-group as its group of components. Let Y be the pullback over q of the universal cover of BG. The group G acts in a natural way on Y, and the space S is weakly equivalent to the homotopy fixed point set $Y^{\mathrm{h}G}$. If K is any subgroup of G, the space $Y^{\mathrm{h}K}$ is weakly equivalent to

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Have $\mathbb{Z}_{p^\infty} \xrightarrow{f} X$. what is $\mathbb{Z}_{p^\infty}$? -

How to get a max. torus: -

Let $X' = Z_X f$. "$\mathbb{Z}_{p^\infty}$ is in the center." (by completion).
So you can form the quotient. The quotient will have smaller dimension than X'. Pull back the max torus in $X'/\mathbb{Z}_{p^\infty}$; this is a max. torus in X', & in X.

$B\mathbb{Z}_{p^\infty} \longrightarrow BS^1 \longrightarrow BS^{1\wedge}$ all H_* iso's.
This a homotopical analogue of the fact that the closure of $\mathbb{Z}_{p^\infty}$ is S^1.

If X is a p-adic finite loop space, then any hom. $\mathbb{Z}_{p^\infty} \to X$ extends to ~~B~~$S^{1\wedge} \longrightarrow X$: since BX is p-local.

Def. A p-toral group is an extension of a finite p-gp by a torus: $1 \to T \to P \to \Gamma \to 1$

A discrete p toral gp. is an extension of finite p-gp. by a $(\mathbb{Z}_{p^\infty})^r$.

A complete p-Toral gp. is an extension of a finite p-group by $(S^{1\wedge})^r$.

Eg.
$$\begin{tikzcd} 1 \arrow[r] & S^1 \arrow[r] & O(2) \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[l, bend right] & 1 \end{tikzcd} ,$$
$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}_{2^\infty} \arrow[r] & D_{2^\infty} \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[l, bend right] & 1 \end{tikzcd}$$

& fibr: $BS^{1\wedge} \to BX \to \mathbb{R}P^\infty$.

So more precisely, a completed p-toral group is a space in ~~an~~ ~~extension~~ a fibration.

$$\widehat{BT} \longrightarrow BX \longrightarrow B\Gamma .$$

BX here is automatically p-local: Γ acts nilpotently on $H_*(\widehat{BT};\mathbb{F}_p)$. ~~And on looping,~~ Its then complete, since fiber & base do. On looping, ΩBX is a finite disjoint union of $\hat{T}$'s. So BX is automatically a p-adic loopspace.

Def. Let G be a complete p-toral ~~finite~~ ~~p-adic~~ group. A *discrete approximation* to G is a discrete p-toral group H together with a homomorphism $H \to G$ inducing an iso $H_*(BH;\mathbb{F}_p) \to H_*(BG;\mathbb{F}_p)$.

Def. Let H be a discrete p-toral group. A *closure* of H is a complete p-toral group G together with a homomorphism $H \to G$ inducing $H_*(BH) \longrightarrow H_*(BG)$.

Th. 1 Every complete p-Toral gp has a discrete approx.

Th. 2 Every discrete p-toral gp has a closure, which is unique up to equivalence.

pf of 2. $B\mathbb{Z}_{\infty}^r \to BP \to B\Gamma$. Complete:

$\widehat{BT}^r \longrightarrow \widehat{BP} \to B\Gamma$.

By fiber lemma, this is a fibration.

pf of 1. To construct $B\mathbb{Z}_{p^\infty}^{\wedge}$, first rationally localize:

$$F \longrightarrow K(\mathbb{Z}_p^{\wedge r}, 2) \longrightarrow K(\mathbb{Q}_p^{\wedge r}, 2)$$

$\bar{H}_*(\text{base}; \mathbb{F}_p) = 0$, so $\bar{H}_* F \xrightarrow{\cong} H_*(K(\mathbb{Z}_p^{\wedge}, 2))$.
And $F = K((\mathbb{Q}_p/\mathbb{Z}_p)^r, 1) = K(\mathbb{Z}_{p^\infty}^{\wedge r}, 1)$.

Now look at $\widehat{BT}^r \longrightarrow BP \to B\Gamma$.
We can't just rationalize; just think of what happens when $r = 0$. Instead, complete fiberwise: get new fibr.

$$K(\mathbb{Q}_p^r, 2) ~~\widehat{BT}^r~~ \longrightarrow X \longrightarrow B\Gamma.$$

Next, this fibration has a section: by obstruction theory.

Pull back

$$\begin{tikzcd} BG \arrow[d] & BH \arrow[l] \arrow[d] \\ X & B\Gamma \arrow[l, "s"'] \end{tikzcd}$$

Fiber is $K((\mathbb{Z}/p^\infty)^r, 1)$.

s is unique up to homotopy, but the space of such sections is not contractible (unless $\mathbb{Q}_p^r$ has no nontrivial ~~summan~~ summands as a Γ-rep.). So BH here is abstractly unique, but not by a unique~~ly~~ iso.

Th. If G is a discrete p-toral group, then $H^*(BG)$ is Noetherian.

(Recall Noetherian $\iff$ FGen for graded connected R-algebras, R Noetherian.)

Lemma. Let $f: R_1 \to R_2$ be graded conn. ring hom. If R_1 is Noetherian, & R_2 is finitely gen R_1-mod, then R_2 is Noetherian. $\triangle$

Lemma. Let $F \to E \to B$ be a fibration, in which F & B are connected & $\pi_1 B$ acts trivially on H_*F. If $x \in H^*F = E_2^{0*}$ is transgressive, and θ is a Steenrod op., then θx is again transgressive.

Sketch: Eliminate bottom row in sseq:

$$H^*(B; \tilde{H}^*(F \to *)) \Rightarrow H^*(E \to B).$$

ie mapping cones.

~~$H^iF \to H$~~

$H^*(B; \tilde{H}^1F)$

$H^*(B; \tilde{H}^0F) = 0.$

$0 \longrightarrow 0 \longrightarrow$

The sseq coincides with abs. sseq except for shift in dimension & absense of bottom row.

So x is transgressive $\Leftrightarrow$ it survives $\in$ in the rel. sseq. : ie ~~$\Leftrightarrow$~~

$$\Sigma x \in \mathrm{Im}\left(H^*(E \longrightarrow B) \longrightarrow H^*(F \longrightarrow *)\right).$$

This ~~set~~ is clearly a submodule of H^*F. $\triangle$.

Case of a p-group. (Quillen, Venkov).

pf by induction on $|G|$. Pick ses., central

$$\mathbb{Z}/p \longrightarrow G \longrightarrow K.$$

By induction, $H^*(BK)$ is Noetherian.

$$H^*(BK) \otimes H^*(B\mathbb{Z}_{(p)}) \Rightarrow H^*(BG).$$

~~For~~ $H^* B\mathbb{Z}/p = \langle 1, x, y, xy, y^2, \dots \rangle$

x transgresses, so $y = \beta x$ transgresses, so

$P^1 y = y^p$ transgresses, etc. So some y^{p^n} survives.

~~This~~ The s'seq is one of $H^*K \otimes \mathbb{F}_p[y^{p^n}]$-modules, & fin. gen. at E_2; so it quits, & is fin gen at E_∞. Hence H^*BG is fgen over $H^*BK \otimes \mathbb{F}_p[z]$, z a lift of y^{p^n}: so it's Noetherian $\triangle$

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we saw: Th: if G is a finite p-group, then H^*BG is Noetherian

and Cor. if G is any finite group, then H^*BG is Noetherian.

Alt. pf of a this: $G \hookrightarrow U(n) \Rightarrow U(n)/G \longrightarrow BG \longrightarrow BU(n)$.
$\pi_1 BU(n) = 0$; $H^*BU(n)$ is Noetherian, so E_2 & E_∞ are f gen, so H^*BG is finite over $H^*BU(n)$. △

Cor 1. If G is a finite p-group and ~~M~~ M is a fd $\mathbb{F}_p$-vs with G-action, then $H^*(BG; M)$ is ~~a~~ f. gen over H^*BG.

Cor. if G is a finite p-gp. & $K \leq G$ is a subgp, then $H^*BG \longrightarrow H^*BK$ is finite.

pf. Shapiro's lemma: $H^*(BK) = H^*(BG; \mathbb{F}_p[G/K])$ (as H^*BG-modules) △

[The converse is true too: let $1 \to K \to H \to G$. There is a subgp. π of K of order p which is central in H. Consider $1 \to \pi \longrightarrow H \longrightarrow H/\pi \to 1$. The argument from 17 Mar shows that $H^*(BH)$ is not finite over $H^*(B(H/\pi))$; but $H^*BG \to H^*BH$ factors through this. △]

or complete

Th. If G is discrete p-Toral, then H^*BG is Noetherian.

pf. $\exists$ ses $1 \to D \to G \to P \to 1$
with $D = (\mathbb{Z}_{p^\infty})^\ell$, P a p-gp.
This extension is det'd by a cocycle
$c \in C^2(P; D)$ (whose coh. class
is the k-invariant $k \in H^2(BP; D)$.).

Since P is finite, the cocycle ~~[illegible]~~ has image in
a finite subgroup of D. Let $D_n = {}_{p^n}D$;
then $c \in \operatorname{Im} \; C^2(P; D_n) \to C^2(P; D)$
for some n: so for $m \geq n$ we have:

$$\begin{tikzcd}
1 \arrow[r] & D_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\searrow"] & G_m \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & D \arrow[r] & G \arrow[r] & P \arrow[r] & 1
\end{tikzcd}$$

(if the extension splits, the cocycle can be taken to be 0, and $G_m = D_m \tilde{\times} P$; $n = 0$).

Let $\bar{G} = \Omega(BG)^\wedge$ be the "closure" of G.
Consider the homotopy fiber F_n of the composite
$$BG_n \to BG \to B\bar{G} :$$

$$B\mathbb{Z}_{p^\infty}^\ell \to BG \to BP$$
$$\Rightarrow \quad B\hat{T}^\ell \to B\bar{G} \to BP \qquad \text{fibr.}$$

So

$$\begin{tikzcd}
(T^\ell/\mu_{p^m})^\wedge \arrow[r] \arrow[d] & B(\mathbb{Z}/p^m)^\ell \arrow[r] \arrow[d] & B\hat{T}^\ell \arrow[d] \\
F_m \arrow[r] \arrow[d] & BG_m \arrow[r] \arrow[d] & B\bar{G} \arrow[d] \\
* \arrow[r] & BP \arrow[r, phantom, "="] & BP
\end{tikzcd}$$

And under the equivalence

$$\begin{tikzcd}
\hat{T}^\ell \arrow[r, "\simeq"] \arrow[d, "p^{\text{th}}\text{ power}"'] & F_m \arrow[d] \\
\hat{T}^\ell \arrow[r, "\simeq"] & F_{m+1}
\end{tikzcd}$$

,

so induces the 0 map in $\bar{H}_*(; \mathbb{F}_p)$. Comparing the Serre s'seq's for $F_m \to BG_m \to B\bar{G}$, ~~they all collapse~~ ~~because~~ $E^2_{**t} \xrightarrow{0} E^2_{*t}$, $t>0$.

Any elt. in $H^*B\bar{G}$ eventually survives, ~~ie~~ for m large enough, since these s'seq's are all conc. in a band of height ℓ.

That is, $H_*BG_m \twoheadrightarrow H_*B\bar{G}$ for $m \geq m_0$.

Moreover, since $E^\infty \to E^\infty$ is 0 above bottom row :—

Regard the filtration of $H_k BG_m$ in a nonstandard way:

$F^0 = H_k BG_m = F_k$

$\cup|$

$F^1 = F_{k-1} H_k BG_m$ ie by rows at E^∞.

$\cup|$

$\vdots$

so $F^1(m) \longrightarrow F^2(m+1) \longrightarrow \cdots$

& so $F^1(m)$ is eventually killed: ie a factorization

$$\begin{tikzcd} H_* BG_m \arrow[rr] \arrow[dr] & & H_* BG_{m+\ell} \\ & H_* B\bar{G} \arrow[ur] & \end{tikzcd}$$

exists.

In H^*,

$$\begin{tikzcd} H^* BG_m & & H^* BG_{m+\ell} \arrow[ll, "\text{ring hom}"'] \arrow[dl, "f"] \\ & H^* B\bar{G} \arrow[ul, hook', "\text{ring hom.}"] & \end{tikzcd}$$

& we conclude that the factoring map f (which is unique) is a ring-hom. The top map is finite, and $H^* BG_{m+\ell}$ is Noetherian: so $H^* B\bar{G}$ is f. gen over $H^* BG_{m+\ell}$: so it's Noetherian.

Now remember $H^* B\bar{G} \xrightarrow{\cong} H^* BG.$ //

Remark.

Notice also that each $H^* BG_m$ is finite over $H^* B\bar{G}$. Moreover, any finite subgp $H < G$ lies in some G_m: so ~~it too has finite~~ $H^* BH$ is also finite over $H^* B\bar{G}$.

Better pf of converse on p. 1:

Say
$$\begin{tikzcd} \mathbb{Z}/p \arrow[r, hook] \arrow[rr, bend left, "*"] & K \arrow[r, hook] & G \end{tikzcd}.$$

$H^*B\mathbb{Z}/p$ is finite over H^*BK & the action factors ~~but~~ through $H^*BK \otimes_{H^*BG} \mathbb{F}_p$. So $H^*BK \otimes_{H^*BG} \mathbb{F}_p$ is infinite $\triangle$.

Dwyer. 31 Mar 92

X p-adic finite loop space. We constructed $\mathbb{Z}/p \hookrightarrow X$, & then $\mathbb{Z}/p^n \hookrightarrow X$ $\forall n$. and so by completing, $BS^{1\wedge} \longrightarrow X$.

Th. If A is a toral p-adic ~~tora~~ loop space and $f: A \longrightarrow X$ is a homomorphism, then $C_X(f)$ is a p-adic finite loop space and $C_X(f) \longrightarrow X$ is a monomorphism

(Recall that a p-adic loop space A is toral if there's a fib seq. $BT^\wedge \longrightarrow A \longrightarrow B\pi$ where π is a finite p-group.)

pf. Let $A' \longrightarrow A$ be an approximating discrete p-toral grp & let $f': A' \longrightarrow A \longrightarrow X$.
For any finite subgroup $\Gamma \subset A'$, $C_X(\Gamma \to X)$ is a p-adic finite loop space and $C_X(\Gamma \to X) \to X$ is a monomorphism.
Since subgroups satisfy DCC, $C_X f' = C_X(\Gamma \to X)$ for suitably large Γ.
Now since $BA' \longrightarrow BA$ is an $\mathbb{F}_p$-equiv.,

$$\mathrm{map}(BA', BX)_{f'} \xleftarrow{\simeq} \mathrm{map}(BA, BX)_f$$

△

Def. A map $f : X \to Y$ of p-adic finite loop spaces is *central* if $C_Y(f) \xrightarrow{\simeq} Y$.

ie if $Y / C_Y(f) \simeq *$

eg. if A is toral p-adic finite loop space, and X is a finite loop space, then the trivial map $A \longrightarrow X$ is central.

pf. Let $A' \to A$ be a discrete approx. Show

$$\mathrm{map}(BA', BX)_{Bf \simeq *} \xrightarrow{\simeq} BX.$$

Now
$$\Omega\, \mathrm{map}(BA', BX)_* \simeq \mathrm{map}(BA', \Omega BX)$$
$$\simeq \mathrm{map}(BA', X).$$
$$\simeq \operatorname{holim}_{\alpha} \mathrm{map}(B\Gamma_\alpha, X).$$

by Sullivan conjecture.

$$\simeq \operatorname{holim}_{\alpha} X \simeq X. \quad \triangle.$$

(In fact, after the fact, we'll discover that for any p-adic finite loop space A, $C_X(* : A \to X) \xrightarrow{\simeq} X$.
Also, it is likely that JMO methods would show that any p-adic finite loop space has a discrete approximation).

Prop. Assume $*: X \to Y$ is central. Then $f : X \to Y$ is central iff $\exists$ (hom)

$$\begin{tikzcd} BX \times BY \arrow[r, "h"] & BY \\ BX \vee BY \arrow[u] \arrow[ur, "{(f,1)}"'] & \end{tikzcd}$$

st.

pf. Suppose f is central: $map(BX, BY)_{Bf} \xrightarrow{\simeq} BY$

$$\begin{tikzcd} BX \times BY \arrow[rr, dashed, "h"] & & BY \\ BX \times map(BX, BY)_{Bf} \arrow[u, "\simeq"] \arrow[rr, "eval"] & & BY \end{tikzcd}$$

Check axes.

Suppose you have h. We find an equivalence between $map(BX, BY)_{Bf}$ & $map(BX, BY)_*$:

$$\begin{tikzcd} BX \arrow[r, "in_1 = Bu"] \arrow[rr, bend left, "Bv"] \arrow[dr, "="'] & BX \times BY \arrow[r, "{(pr_1, h)}"] \arrow[d] & BX \times BY \arrow[d] \\ & BX \arrow[r, "="'] & BX \end{tikzcd}$$

ie $\boxed{Bv = (1, f)}$

(pr_1, h) is an equivalence, since $BY \xrightarrow{in_2} BX \times BY \xrightarrow{h} BY$.

$$\begin{tikzcd} BY \arrow[r, "in_2"] \arrow[rr, bend left, "1"] & BX \times BY \arrow[r, "h"] & BY \end{tikzcd}$$

So

$$\begin{tikzcd} map(BX, BX \times BY)_{Bu} \arrow[r, "\simeq"] \arrow[d] & map(BX, BX \times BY)_{Bv} \arrow[d] \\ map(BX, BX)_1 \arrow[r, "="] & map(BX, BX)_1 \end{tikzcd}$$

Top row is $map(BX, BX)_1 \times map(BX, BY)_* \longrightarrow map(BX, BX)_1 \times map(BX, BY)_f$

So on fibers you get the desired equivalence $/\!/\triangle$

(The same arg. backwards, shows that if there's any central hom ~~map~~ $X \to Y$, then $* : X \to Y$ is central.)

<u>Remark</u>: $\mathrm{map}(BX, BX)_1 = C_X(1) =$ "Bcenter of X"

<u>Defn</u>. A p-adic finite loop space X is <u>abelian</u> if $\mathrm{map}(BX, BX)_1 \xrightarrow{\simeq} BX$: ie iff $1 : X \to X$ is central.

If $* : X \to X$ is central, this is equivalent to: BX is an H-space. (And, if X is abelian, then $* : X \to X$ is central.)

<u>Warning</u>: $\mathrm{map}(BX, BX)_1 \simeq X$ ~~is not equivalent to~~ ~~does not imply that~~

$$\mathrm{map}_*(BX, BX)_1 \simeq * .$$

Let G be a center-free group. Then

$$\mathrm{map}_*(BG, BG)_1 \simeq * .$$

but $\mathrm{map}(BG, BG)_1 = *.$ as well!

<u>Q</u>: say BX is st $\mathrm{map}(BX, BX)_1 \xrightarrow{\simeq} BX$. Is such a space a product of E-M spaces? (Don't know.)

2 Apr

Prop. Let $f: A \to X$ be a hom. of p-adic finite loop spaces, with A abelian. Then f lifts to a central hom. $\tilde{f}: A \to C_X(f)$. We assume $*: A \to A$ is central.

[if the fiber of $X \to Y$ is finite, is it a 'mono' in cat of p-adic finite loop spaces? – Sullivan conj.

pf.

$$\begin{tikzcd}
BC(1) \arrow[d, equal] & BC(f) \arrow[d, equal] \\
\mathrm{map}(BA, BA)_1 \arrow[r, "Bf\circ -"] \arrow[d, "\simeq"'] & \mathrm{map}(BA, BX)_{Bf} \arrow[d, "eval"] \\
BA \arrow[r, "f"] & BX
\end{tikzcd}$$

commutes.

Why central? We use the cond'n that there be a hom. $A \times C(f) \to C(f)$: namely,

$$\begin{tikzcd}
BA \times BC(f) \arrow[d, "\cong"] & BC(f) \arrow[d, "\cong"] \\
\mathrm{map}(BA, BA)_1 \times \mathrm{map}(BA, BX)_{Bf} \arrow[r, "Compose"] & \mathrm{map}(BA, BX)_f
\end{tikzcd}$$

Forming the quotient. Grouplike topological monoids are for practical purposes groups, and we can form Borel constructions:

[Is any abelian p-adic finite loop space p-toral?]

* Thm. Let $f : A \to X$ be a central mono. of p-adic finite loop spaces (with A abelian — is this redundant?). Then f extends to a "s.e.s."

$$1 \to A \to X \to X/A \to 1$$

of p-adic finite loop spaces.

That is, there's a fib seq. $BA \to BX \to B(X/A)$.

[* : yes?

$$\begin{tikzcd}
\mathrm{map}(BA, X/A) \arrow[r, "\boxed{\simeq}"] \arrow[d] & X/A \arrow[d] & \underline{\underline{\text{finite}}} \arrow[l] \\
\mathrm{map}(BA, BA)_1 \arrow[r] \arrow[d] & BA \arrow[d] & \\
\mathrm{map}(BA, BX)_f \arrow[r, "\simeq"] & BX &
\end{tikzcd}$$

claim top is equivalence by Sullivan conj.].
– at least if A has a discrete approximation

[Facts about grouplike monoids.

Given one, M, ~~acting on Y~~, form simplicial space:

$$* \Leftarrow M \Lleftarrow M^2 \quad \cdots \quad ; \quad |M^{\cdot}| = BM.$$

Is s.t. $\Omega BM \simeq M$. If M acts on Y, form

$$Y \Leftarrow M \times Y \Lleftarrow M^2 \times Y \quad \cdots \quad , \quad |\ | = \text{"}EM \times_M Y\text{"}$$

in the sense that there is a "bundle $EM \times_M Y \downarrow BM$, with Y as fiber.

(Eg. ΩX; ~~we claimed earlier that~~ Moore loops do form a top. monoid.

pf. ~~From~~ $BA \simeq map(BA, BA)_1$ is a gplike top. monoid, so form B^2A, and the bundle

$$EBA \times_{BA} BX$$

over it, using $BX \simeq map(BA, BX)_f$. By Barratt-Puppe,

$$\Omega(EBA \times_{BA} BX) \simeq X/A,$$

Must still check X/A is p-local, & $\pi_0(X/A)$ is finite p-gp. It's p-local, being fiber of map between local spaces. And, $\pi_0(X/A) = \pi_0(X)/\pi_0(A)$. ◁

Towards the maximal torus.

Def. Let $f: A \to X$ be a homomorphism of p-adic finite loop spaces, where A is an abelian toral p-adic finite loop space. The map f is:

(1) self-centralizing if $A \xrightarrow{\simeq} C_X(f)$

(2) almost self-centralizing if $C_X(f)/A$ is htpically discrete.

Def. Let X be a p-adic finite loop space. A maximal torus in X is an almost self-centralizing hom. $T_p^\wedge \longrightarrow X$.

Thm. Every p-adic finite loop space has a max. torus.

Proof. By ind. on $\dim X$. If $\dim X = 0$, then $BX \simeq B\pi_0 X$ and we can take $T_p^\wedge = *$.

Let X_1 = id. component of X; $B(X_1) = \widetilde{BX}$.

Find a nontrivial hom $\mathbb{Z}/p \xrightarrow{f} X_1$.

(easy check if X is Lie: check degrees of p^{th} power map.)

Then (Lannes) $0 \neq Bf^* : \bar{H}^* BX_1 \longrightarrow \bar{H}^* B\mathbb{Z}/p$.

Extend f to hom. $\mathbb{Z}/p^\infty \longrightarrow X_1$, as above.

$$\begin{tikzcd}
\mathbb{Z}/p^\infty \arrow[rr] \arrow[dr] & & X_1 \\
& BS^{1\wedge} \arrow[ur] &
\end{tikzcd}$$

In order to show $H_*(X_1/\hat{S}^1)$ is finite, consider first $X_1/\mathbb{Z}_{p^\infty}$, in fibr.

$$\begin{tikzcd}
X_1 \arrow[r] & X_1/\mathbb{Z}_{p^\infty} \arrow[r] & B\mathbb{Z}_{p^\infty} \arrow[d] \\
& & BX_1
\end{tikzcd}$$

$\Rightarrow$ action of $\pi_1 B\mathbb{Z}_{p^\infty}$ is trivial.

We know $H^*X_1 \longrightarrow H^*B\mathbb{Z}_{p^\infty}$ is nonzero infinitely often; so some power of gen. of $H^*B\mathbb{Z}/p^\infty$ maps to 0 in $H^*(X/\mathbb{Z}_{p^\infty})$, so dies in s'seq; so the action of $H^*B\mathbb{Z}_{p^\infty}$ on s'seq factors through finite ring; so by Noetherianness, $H^*(X_1/\mathbb{Z}_{p^\infty})$ is finite.

– an old argument. We should have used $S^{1\wedge}$ instead of $\mathbb{Z}_{p^\infty}$: $X_1/S^{1\wedge}_p$ is $\mathbb{F}_p$-finite:

we have $S^{1\wedge} \xrightarrow{f} X_1$, mono: ~~The fact that the inclusion of both~~

$$\begin{tikzcd} S^{1\wedge} \arrow[r, "f"] \arrow[d, hook] & X_1 \\ C_{X_1}f \arrow[ur] & \end{tikzcd}$$

$C_{X_1}f \to X_1$ is mono since $C_{X_1}f = C_{X_1}(f_n)$,

$f_n : \mathbb{Z}_{p^n} \to X$, for some big n; & these

are monic by htpy fixed pt argument we did.

And the arg just above shows also that $S^1 \longrightarrow C_{X_1}f$ is monic.

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we have mono $\hat{S}^1 \xrightarrow{f} X$. and $X/C_X(f)$

$$\begin{tikzcd}
\hat{S}^1 \arrow[r, "f"] \arrow[dr] & X \\
& C_X f = Y \arrow[u]
\end{tikzcd}$$

Suppose we have a torus $T \xrightarrow{g}$ ~~$X/C_X f$~~ $Y/\hat{S}^1$
so Y/T is a finite set. From the pullback:

$$\begin{tikzcd}
B\tilde{T} \arrow[r, "B\tilde{g}"] \arrow[d] \arrow[dr, "B\tilde{g}'"] & BY \arrow[d] \\
BT \arrow[r, "Bg"] & B(Y/S^1)
\end{tikzcd}$$

Notice that $B\tilde{T} \longrightarrow BT \longrightarrow B^2\hat{S}^1$ is a fiber seq.;
but $H^3(BT; \hat{\mathbb{Z}}_p) = 0$, so $B\tilde{T} \simeq BT \times B\hat{S}^1$.

||| We claim that $\tilde{T} \xrightarrow{g} Y$ is a max. torus in Y.
Compute $\mathrm{map}(B\tilde{T}, BY)_{Bg}$: use

$$\begin{tikzcd}
\widetilde{BC_{Y/S^1}(g)} \arrow[r] \arrow[d] & BY \arrow[d] \\
BC_{Y/S^1}(g) \arrow[r] & B(Y/S^1) \arrow[ul, phantom, "\nwarrow"]
\end{tikzcd}$$

$\Rightarrow$

$$\begin{tikzcd}
\mathrm{map}(B\tilde{T}, \widetilde{BC(g)}) \arrow[r] \arrow[d] & \mathrm{map}(B\tilde{T}, BY) \arrow[d] \\
\mathrm{map}(B\tilde{T}, BC(g)) \arrow[r] & \mathrm{map}(B\tilde{T}, B(Y/S^1))
\end{tikzcd}$$

Claim: $\mathrm{map}(B\tilde{T}, B(Y/S^1))_{B\tilde{g}'} \xleftarrow{\simeq} \mathrm{map}(BT, B(Y/S^1))_{Bg}$

$$\mathrm{map}(B\tilde{T}, B(Y/S^1))_{Bg} = \mathrm{map}(BT \times BS^1, B(Y/S^1))_{Bg}$$

$$= \mathrm{map}(BT, \mathrm{map}(BS^1, B(Y/S^1))_*)_h$$

(since $BS^1 \longrightarrow B\tilde{T} \longrightarrow BT \longrightarrow B(Y/S^1)$ is null)

Now (by Sullivan conj, as above) $\Omega\,\mathrm{map}(B\mathbb{Z}/p^n, B(Y/S^1))_* \simeq Y/S^1$ so by stabilization of centralizers,

$$\mathrm{map}(BS^1, B(Y/S^1))_* \simeq B(Y/S^1).$$

— and under this id, $h = Bg$. so

$$BC_{Y/S^1}(g') = \mathrm{map}(B\tilde{T}, B(Y/S^1))_{Bg} \simeq BC_{Y/S^1}(g):$$

so our pullback square has a component:

$$\begin{tikzcd}
\mathrm{map}(B\tilde{T}, B\widetilde{C(g)})_{\text{suitable comp.}} \arrow[r] \arrow[d] & \mathrm{map}(B\tilde{T}, BY)_{B\tilde{g}} \arrow[d] \\
BC_{Y/S^1}(g') \arrow[r, "\simeq"] & BC_{Y/S^1}(g)
\end{tikzcd}$$

so ~~the~~ top map is h. eq.

Now $BC_{Y/S^1}(g)$ is an abelian p-toral finite loop space, since $T < Y/S^1$ was max torus. So ~~the centralizer of~~ $C_{Y/S^1}(g)$ is an extension of $\tilde{T}$ by a finite p-group. So the centralizer of $\tilde{T}$ in it is an extension of $\tilde{T}$ by a finite abelian p-group: so $\tilde{T}$ is almost self-centralizing : qed.

So we've got a max torus in $Y := C_X(f)$. Rename it $T \xrightarrow{f} Y$.

Want to show next that

$$\begin{tikzcd} & S^1 \arrow[d, hook] & \\ T \arrow[r] & Y \arrow[r] & X \end{tikzcd}$$

is ~~a~~ max torus. Idea:

1) Show T contains S^1.

2) Then $C_X(T) \subset C_X(S^1) = Y$; so $C_X(T) = C_Y(T)$.

and T will be almost self-centralizing in X.

Ad 1): Prop. Let Y be a p-adic finite loop space & $f: T \to Y$ a max. torus, & $i: S^1 \to Y$ a homomorphism. Then i lifts to a hom. to T.

proof.

$$\left\{ \begin{tikzcd} & BT \arrow[d] \\ BS^1 \arrow[r, "i"'] \arrow[ur, dashed] & BY \end{tikzcd} \right\} = \Gamma\left(\begin{tikzcd} Y/T \arrow[d] \\ E \arrow[d] \\ BS^1 \end{tikzcd} \right).$$

In cpt Lie case, one sees $\chi(Y/T) > 0$, so any S^1-action has a fixed point.

Here we want to see there's a homotopy fixed pt.

Try the technique of $\overset{\text{discrete}}{\wedge}$approximation

① ~~Keto~~

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ B\mathbb{Z}_{p^\infty} \arrow[r] \arrow[ur, phantom, "\Leftarrow"] & BS^1 \end{tikzcd}$$

claim $\Gamma(\pi') \xleftarrow{\simeq} \Gamma(\pi)$:

$$\begin{tikzcd} \Gamma(\pi) \arrow[r] & \mathrm{map}(BS^1, E) \arrow[r] \arrow[d, "\textcircled{\sim}"] & \mathrm{map}(BS^1, BS^1) \arrow[d, equal] \\ \Gamma(\pi') \arrow[r] & \mathrm{map}(B\mathbb{Z}_{p^\infty}, E) \arrow[r] & \mathrm{map}(B\mathbb{Z}_{p^\infty}, BS^1) \end{tikzcd}$$

$\textcircled{\sim}$ since $B\mathbb{Z}_{p^\infty} \longrightarrow BS^1$ is mod p equivalence and E is p-complete.

② $\Gamma\left(\begin{array}{c} E' \\ \downarrow \\ B\mathbb{Z}_{p^\infty} \end{array}\right) = (Y/T)^{h\mathbb{Z}_{p^\infty}} = \underset{\overleftarrow{n}}{\mathrm{holim}}\ (Y/T)^{h\mathbb{Z}/p^n}$.

We know each $(Y/T)^{h\mathbb{Z}/p^n}$ is $\mathbb{F}_p$-finite, & that its euler characteristic = the Lefschetz number.

But the $\mathbb{Z}/p^n$ action extends to an S^1-action, so the homology action is trivial: action of $\pi_1(B\mathbb{Z}/p^n)$ factors through ~~that of~~ $\pi_1(B\hat{S}^1) = 0$:

so $\chi((Y/T)^{h\mathbb{Z}/p^n}) = \chi(Y/T)$.

Then $\pi_0 \underleftarrow{\text{holim}}_n \left((Y/T)^{h\mathbb{Z}/p^n}\right) \twoheadrightarrow \underleftarrow{\lim}_n \pi_0 \left((Y/T)^{h\mathbb{Z}/p^n}\right)$

and $\underleftarrow{\lim}$ of finite nonempty sets. is nonempty.

What's needed is: $\boxed{\chi(Y/T) \neq 0}$.

Lemma. if $T \to Y$ is a max torus, then $\chi(Y/T) > 0$.

proof. In connected Lie case, $(Y/T)^T = NT/T = W$

W acts faithfully on T since $C_Y T = T$:

so $NT/T \hookrightarrow GL(m, \mathbb{Z})$: it's discrete

And cpt, so finite. And $\neq \emptyset$: fixes T/T.

So $0 < |W| = \chi((Y/T)^T) = \chi(Y/T)$.

For us: $Y/T \to BT \to BY$.

Choose a disc. approx $(\mathbb{Z}/p^\infty)^r \to T$, and a finite subgp. $i: A < (\mathbb{Z}/p^\infty)^r$ st

$$C_Y(T) \xrightarrow{\simeq} C_Y((\mathbb{Z}/p^\infty)^r) \xrightarrow{\simeq} C_Y(A).$$

Consider the A-action on Y/T;

$$(Y/T)^{hA} = \left\{ \begin{tikzcd} & BT \arrow[d] \\ BA \arrow[r] \arrow[ur, dashed] & BY \end{tikzcd} \right\} = \text{fiber}\left(\begin{tikzcd} \text{map}(BA, BT)_{\text{some comps.}} \arrow[d] \\ \text{map}(BA, BY)_{Bi} \end{tikzcd} \right)$$

$T \to Y$ was maximal, so $\text{map}(BA, BY)_{Bi} = BC_Y(A)$

which is the classifying space a finite extension of a torus: so has BT as its univ. cover.

The components of $\mathrm{map}(BA, BT)$ are BT. ($H^1(BA; \mathbb{Z}) = 0$).

And the maps $BT \longrightarrow BC_Y(T)$ is ~~[illegible]~~ inclusion of the T: so this is ~~htpy dis~~ up (✱) to homotopy a finite cover: so $(Y/T)^{hA}$ is discrete

Again, A acts on $H_*(Y/T)$ trivially, so just as before, so $H(Y/T)^{hA}$ is ~~fp~~ finite and $\chi((Y/T)^{hA}) = \chi(Y/T)$.

and we just discovered the LHS is ≥ 0. It's > 0 since we do have the lift given by including $A \hookrightarrow T$.

See next lecture.

9 Apr.

Recall:

Prop. $G = (\mathbb{Z}/p^\infty)^n$ acts on X. Assume:

X^{hK} is $\mathbb{F}_p$-complete $\forall\ K \leq G$

X is $\mathbb{F}_p$-finite.

Then $\forall$ finite $K < G$, X^{hK} is $\mathbb{F}_p$-finite and $\chi(X^{hK}) = \chi(X)$.

Recall the proof: Induct on $|K|$; Let $H \triangleleft K$ of index p, so $X^{hK} = (X^{hH})^{h(K/H)}$

Thus for general reasons $\chi(X^{hK}) = L(T \mid X^{hH})$ where T generates K/H.

Now the K/H-action extends to a G/H-action on X^{hH}. In homology, this action ~~is~~ must thus be trivial; so $L(T \mid X^{hH}) = \chi(X^{hH})$. $\triangle$

PS: we mean the action on $H^*(X^{hH}; \mathbb{Q}_p)$, here; but $GL(n, \mathbb{Q}_p)$ has no infinitely p-divisible elements either. This is the algebraic analogue of the homotopy property of L: if action extends to an S^1-action, the homology action is trivial: Alg. analogue of connectedness.

($\mathbb{Z}_{p^\infty}$ has no finite quotient groups.)

pf - wma $f : (\mathbb{Z}_{p^\infty}) \longrightarrow GL(n, \mathbb{Q}_p)$ is injective. $f_k = f \mid (\mathbb{Z}_{p^k})$. is faithful

But the smallest faithful rep. of $\mathbb{Z}_{p^k}$ has dim. which increases with k. $\lhd$

Lem: The only hom. $\mathbb{Z}_{p^\infty} \longrightarrow GL_n \mathbb{Q}_p$ is trivial.

Prop. $A \to B \to C$ ses of p-adic finite loop spaces.

$f\ X \xrightarrow[\xi]{} Y$

Let ~~two~~ a hom. of p-adic finite loop spaces,

$g : B \longrightarrow X$.

Assume $C_X(A) \xrightarrow{\cong} C_Y(A)$.

Then $C_X(B) \xrightarrow{\cong} C_Y(B)$ also.

Idea: Given fibr. $F \to E \to B$, & $E \to W$.

There's a fibration $\mathrm{Map}(F, W) \to E' \to B$

st. map$(E, W) = \Gamma(E' \downarrow B)$. So

$$BC_X(B) = \mathrm{map}(BB, BX)_{Bf}$$

$$= \Gamma\begin{pmatrix} \mathrm{map}(BA, BX) \\ \downarrow \\ E' \\ \downarrow \\ BC \end{pmatrix}_{s}.$$

Just so,

$$BC_Y(B) = \mathrm{map}(BB, BY)_{Bgf} = \Gamma\left(\begin{array}{c} \mathrm{map}(BA, BY) \\ \downarrow \\ E'' \\ \downarrow \\ BC \end{array}\right)_{Bs'}$$

By assumption, $\mathrm{map}(BA, BX)_{Bf} \simeq \mathrm{map}(BA, BY)_{Bgf}$, which is the comp. of the fibers that Bs & Bs' land in. Actually we only have a right to ~~tan~~ specify an orbit of π_0(fiber) under $\pi_1(BC)$. ~~Tta~~ The components of such an orbit are all h. eq., so map by h. eq's under g. So its ok. $\triangle$

Wrong. ~~do~~ Rather: we have a section of each fibr: so π_1 ~~acts trivially on the fibers~~ & fixes this component of the fibers.

Recall: we had

$$\begin{tikzcd} & S' \arrow[dl] \arrow[d, hook] \arrow[r] & X \\ T \arrow[r] & C_X(S') \arrow[ur] & \end{tikzcd}$$

max torus

$$\begin{tikzcd} S' \arrow[d] & & \\ T \arrow[r] \arrow[d] & C_X(S') \arrow[r] & X \\ T/S' & & \end{tikzcd}$$

~~Gp~~

~~So now~~ $\mathcal{E}$ and $C_{C_X(S')}(S') \xrightarrow{\simeq} C_X(S')$

so we conclude $C_X(T) \xrightarrow{\simeq} C_{C_X(S')}(T)$

and this is a finite extension of T (by ind.) $\triangle$

This finishes the construction of a max. torus.

Towards NT:

Def. Let X be a p-adic finite loop space and $T \to X$ its max. torus. (Assume $BT \to BX$ a fibration & BT a CW cx.) The Weyl space is

$$\mathcal{W}(T \to X) = \operatorname{map}_{BX}(BT, BT).$$

Prop. $\mathcal{W}(T \to X)$ is homotopically discrete & finite and $|\pi_0| = \chi(X/T)$.

Proof. We already know this! –

$$\mathcal{W}(T \to X) = \left\{ \begin{tikzcd} & BT \arrow[d] \\ BT \arrow[r] \arrow[ur, dashed] & BX \end{tikzcd} \right\} = \Gamma\left(\begin{tikzcd} X/T \arrow[d] \\ BT \times_{BX} BT \arrow[d] \\ BT \end{tikzcd} \right)$$

$$= \Gamma\left(\begin{tikzcd} X/T \arrow[d] \\ B\check{T} \times_{BX} BT \arrow[d] \\ B\check{T} \end{tikzcd} \right) = (X/T)^{h\check{T}}$$

where $\check{T} \to T$ is a discrete approximation.

~~And~~ ~~So~~ by last time ⊛, this is htpically discrete finite, with order $= \chi(X/T)$. △

~~Prop~~. So $\mathcal{W}(T \to X) \xrightarrow{\simeq} \pi_0(\mathcal{W}(T \to X)) = W(T \to X)$, a finite ~~group~~ monoid.

Prop. $W(T \to X)$ is in fact a group.

ie Prop. Any

$$\begin{tikzcd} BT \arrow[rr] \arrow[dr] & & BT \arrow[dl] \\ & BX & \end{tikzcd}$$

is a h. eq.

The trouble with the "obvious" pf of this is that we don't yet know that H^*BT is finite over H^*BX — only that $H^*(X/T)$ is finite.

But we can argue: $f : BT \to BT$ is h. eq. $\iff$ f^* iso in mod p coh.

Say f isn't equiv. Then $\exists$ ~~BZ~~

$$\begin{tikzcd} B\mathbb{Z}/p \arrow[r, "x \neq 0"] \arrow[rr, bend left, "0"] & BT \arrow[r, "f"] \arrow[d, "i"'] & BT \arrow[dl] \\ & BX & \end{tikzcd}$$

So $ix \simeq *$: so $B\mathbb{Z}/p \to X/T$, non trivial in cohomology. ※ (don't need Sullivan conj.).

This doesn't use that $T \to X$ is maximal.

Prop. If $T \xrightarrow{i} X$ & $T' \xrightarrow{i'} X$ are both maximal tori, then $\exists$ $f : T \xrightarrow{\simeq} T'$ st. $if = i'$: ie $(Bi)(Bf) = Bi'$.

Pf. We saw that any toral subgp lifts: (the χ of X/T' [illegible] is > 0.).

So we get
$$\begin{tikzcd}
BT \arrow[r] \arrow[dr] & BT' \arrow[r] \arrow[d] & BT \arrow[dl] \\
& BX &
\end{tikzcd}$$

$\triangle$.

14 April

<u>Th</u>. $T \to X$ max torus. Then
(1) $W(T\to X)$ is htpically discrete
(2) $\pi_0 W(T\to X)$ is a finite group
(3) $\# \pi_0 W(T\to X) = \chi(X/T)$.

<u>pf</u>.

$$W(T\to X) = \left\{ \begin{tikzcd} & BT \arrow[d] \\ BT \arrow[r] \arrow[ur, dashed] & BX \end{tikzcd} \right\} = (X/T)^{hT}$$

Let $\check{T} \to T$ be a discrete approximation; Then

$$W(T\to X) = (X/T)^{h\check{T}}$$

We saw last week that <u>this</u> is htpically discrete and $\# \pi_0 ((X/T)^{h\check{T}}) = \chi(X/T)$.

We saw last time that it is a group. $\triangleleft$.

<u>Exercise</u>: Check that if

$$\begin{tikzcd} BX \arrow[rr, "f"] \arrow[dr] & & BX \arrow[dl] \\ & BT & \end{tikzcd}$$

and f is a htpy equivalence, then it has a htpy <u>inverse</u> <u>over</u> BT.

<u>Note</u>: ~~$W(T\to X)$ is~~ $T \to X$ is unique up to equivalence over X, by the same token.

Thus $W(T \to X)$ is isom. to $W(T' \to X)$; but it's not functorial in X.

We write $W(T \to X)$ for $\pi_0(\mathcal{W}(T \to X))$.

Now towards NT: Just try dividing out BT by W-action; this should divide $\chi(X/T)$ by $|W|$ to give $\chi(X/NT) = 1$.

Def. Let $T \xrightarrow{i} X$ ~~as~~ be a max. torus. The normalizer NT of i is the loop-space with classifying space $EW \times_W BT$.

~~This will never be a p-adic finite loop space; and even after p-completion. To fix this, let W_p be a p-Sylow~~

~~One~~ This isn't a p-adic finite loop space. We could complete this; but it's not well-understood how p-completion behaves on π_1. So instead:

Let $W_p \subset W$ be a p-Sylow subgroup, & $\mathcal{W}_p$ the corresp. union of components.

Then let N_pT be ~~a~~ the loop space with classifying space $BN_pT = EW_p \times_{W_p} BT$. Then N_pT is a toral p-adic finite loop space.

Since $BT \to BX$ is W-equivariant, it extends to

$$\begin{tikzcd} BN_pT \arrow[rr] \arrow[dr] & & BX \\ & BNT \arrow[ur] & \end{tikzcd}$$

<u>Thm.</u> H^*BX is Noetherian.

<u>Proof.</u> $X/N_pT \to BN_pT \xrightarrow{\pi} BX$

$X/T \to X/N_pT$ is a W_p-sheeted cover, so
$\chi(X/T) = \chi(X/N_pT) \#(W_p)$

so $\chi(X/N_pT) = [W : W_p]$ is prime to p.

Hence there is a transfer $H^*BN_pT \xrightarrow{t^*} H^*BX$
s.t ① $t^*\pi^* = [W : W_p]$
② t^* is a H^*BX-module hom.

We know (from a s'seq arg.) that H^*BN_pT is Noetherian, so the little algebraic lemma shows H^*BX is Noetherian. ◿

<u>2 Problems</u>: ① why is $N_pT \to X$ a subgroup?
② What transfer?

Ad ①: Kernels.

Def. Let G be a p-discrete ~~toral~~ group. & $f : G \to X$ a homomorphism. For $g \in G$, the characteristic map is $\kappa_g : \mathbb{Z}/|g| \to G$ is the hom. sending $1 \mapsto g$. The kernel of f is

$$\{g : f \circ \kappa_g = 0\}.$$

Of course $1 \in \ker f$ always.

Prop. $\ker f$ is a normal subgroup of G.

Prop Suppose that G is a toral p-adic finite loop space, and $f : G \to X$ a hom. Let $i : \check{G} \to G$ be a discrete approx. Then f is a mono iff $\ker(f \circ i) = \{1\}$.

Lemma. Let G a p-discrete toral gp., $f : G \to X$ a homomorphism. Let $i : K \subseteq G$ be a normal subgroup, s.t. $f \circ i = 0$. Then f factors uniquely through a hom. $h : G/K \to X$. Furthermore,

$$C_X(h) \simeq C_X(f).$$

pf. Let
$$U = \left\{ \begin{tikzcd} BK \arrow[d] \arrow[rd, "*"] & \\ BG \arrow[r, dashed] \arrow[ur, phantom, "\sim"] & BX \end{tikzcd} \right\}.$$

$$V = \left\{ B(G/K) \xrightarrow{\ \text{-----}\ } BX \right\}.$$

Of course, $V \longrightarrow U$; we claim this is an equivalence. This is equivalent to the conclusion of the lemma. Now the fibrseq. $BK \longrightarrow BG \longrightarrow B(G/K)$ gives

~~IS~~=

$$\begin{tikzcd} & \mathrm{map}(BG,BX) & = \Gamma\left(\begin{array}{c} \mathrm{map}(BK,BX) \\ \downarrow \\ E \\ \downarrow \\ B(G/K) \end{array}\right) \\ \text{and} & U \arrow[u, hook] & = \Gamma\left(\begin{array}{c} \mathrm{map}(BK,BX)_* \\ \downarrow \\ E' \\ \downarrow \\ B(G/K) \end{array}\right) \end{tikzcd}$$

To get $U \simeq V$, hope that E' is a product fibration, and $\mathrm{map}(BK,BX)_* \simeq BX$. The latter holds, ~~since ΛBK~~ as we saw:

$$\Omega \ \mathrm{map}(BK,BX)_* = \mathrm{map}(BK,X) \simeq X$$

by Sullivan conjecture. ~~This shows~~ ~~S~~ Now

$$\begin{tikzcd} BK \arrow[r, dash] \arrow[d] & * \arrow[d] \\ BG \arrow[r, dash] \arrow[d] & B(G/K) \arrow[d] \\ B(G/K) \arrow[r, dash] & B(G/K) \end{tikzcd}$$

So

$$\begin{tikzcd}
\mathrm{map}(BK, BX) \arrow[d] & \mathrm{map}(*, BX) \arrow[l, "\simeq"'] \arrow[d] \\
E'' \arrow[d] & E' \arrow[l] \arrow[d] \\
B(G/K) & B(G/K) \arrow[l, Leftarrow]
\end{tikzcd}$$

so $E'' \xLeftarrow{\simeq} E'$, and E' $\underline{\underline{\text{is}}}$ a product.

16 Apr.

For G p-discrete toral gp & X p-adic finite loopspace, & $f : G \longrightarrow X$ hom, ~~$g \in G$~~ & $g \in G$, let $\kappa_g : \mathbb{Z}/p^n \longrightarrow G \longrightarrow X$ where $|g| = p^n$. Define

$$\ker f = \{ g \in G : f \circ \kappa_g = 0 \}$$

Clearly $\ker f$ is closed under conjugation.

Prop. 1. $\ker(f)$ is a (normal!) subgroup, and f factors uniquely through a hom. $G/\ker f$.

Let G be a toral p-adic finite loop space, with discrete approx. $i : \breve{G} \to G$.

Prop 2. f is monic $\Longleftrightarrow$ $\ker(f \circ i) = 0$.

Prop. 0 If G is a p-discrete toral gp, $K \lhd G$, $f : G \to X$ hom is trivial on K, then f factors uniquely through a hom. $G/K \longrightarrow X$.

done (a)

Prop $\frac{1}{2}$ If G is p-discrete toral, & $f : G \to X$ hom, & $\ker f = G$, then $f = 0$.

pf. Wma G finite (For $G = \bigcup G_m$, G_m finite;

$$\mathrm{map}(BG, BX) = \varprojlim \mathrm{map}(BG_m, BX)$$ so

$$\mathrm{map}(BG, BX)_{\substack{\text{null on} \\ \text{all } BG_m}} = \varprojlim \mathrm{map}(BG_m, BX)_* = \varprojlim BX = BX.$$

which is connected.)

Wlog by ind. on $|G|$. $\exists$ central subgp. ~~elt~~ of order p: $K < G$ central. Then $BK \longrightarrow BG \longrightarrow BX$ is null so by Prop 0 $BG \to BX$ factors through $B(G/K) \xrightarrow{\bar f} BX$. We claim $\ker \bar f = G/K$. Let $x \in G/K$, $\bar x$ a pull-back to G:

⊛

$$\begin{tikzcd}
\mathbb{Z}/p^{j-i} \arrow[r, "\ell"] & \mathbb{Z}/p^j \arrow[r, "h"] \arrow[d, "\kappa_{\bar x}"'] & \mathbb{Z}/p^i \arrow[r, "g"] \arrow[d, "\kappa_x"'] & X \arrow[d, equal] \\
& G \arrow[r] & G/K \arrow[r, "\bar f"] & X
\end{tikzcd}$$

So $g \circ h \circ \ell = *$, so $B(g \circ h)$ factors uniquely through $B\mathbb{Z}/p^i$; but $B(g \circ h) = *$ too ~~(since $Bg = *$!)~~, so the null map works: So $Bg = *$ & $x \in \ker \bar f$. $\triangle$

Pf of Prop 1. By Prop 0, it's enough to show that $\ker f$ is a subgroup. wma G finite, since $K \leq G$ is a subgp iff $K \cap G'$ is a subgp $\lhd G'$ for all finite G'. (since G is locally finite). Work by ind on $|G|$. Assume $|G| > 1$.

Case 1. $\exists$ proper $K \lhd G$ with $K \cap \ker f \neq \{1\}$.
Case 2. $\nexists$ . . .

Case 2: If $\ker f = 1$ the conclusion is trivially true. If not, $\forall\ x \neq 1,\ x \in \ker f$, the normal closure of x in G is G. Thus x mod $[G,G]$ generates $G/[G,G]$, which is thus cyclic. Any such p-group is cyclic itself. (if $f : G \longrightarrow H$ is epi on $G_{ab} \longrightarrow H_{ab}$, then f is epi: a collection of elts generating H_{ab} generate H).

Thus $\ker f = G$ (Notice that $\ker f$ is clearly closed under powers: for these can be accomplished by precomposing K_x by a homomorphism) — and the thm is true $\measuredangle$

Case 1. Let $L = K \cap \ker f$. $L = \ker\{f|_K\}$ so by induction $L \lhd K$. L is a subgp of K — so of G; & it's closed under conjugation, so it's normal in G. By Prop $\frac{1}{2}$, $f|_L = 0$, so by Prop 0, f factors uniquely to $G \xrightarrow{q} G/L \xrightarrow{f} X$.

By $\circledast$, $\ker f = \ker(\bar{f} \circ q) = q^{-1} \ker \bar{f}$

By induction $\ker \bar{f}$ is a subgp, so $\ker f$ is too $\triangle$.

Pf of prop. 2. uses:

Prop. Let X be p-complete & p-finite. Let G be a finite p-group acting on X. Then $H^*(X_{hG})$ is finite $\Longleftrightarrow$ $X^{hK} = \emptyset$ for each subgp. of K of order p.

$\Longleftarrow$:

pf. [Induct on $|G|$: lets do the other way first.

$\Longrightarrow$: Say $X^{hK} \neq \emptyset$, $K < G$, $|K| = p$.

Let $s : BK \longrightarrow EK \times_K X$. ~~split~~ be a section; then

$$\begin{tikzcd}
 & EG \times_G X \arrow[d] \\
BK \arrow[ur] \arrow[r, hook, "i"] & BG
\end{tikzcd}$$

But i^* is nonzero in infinitely many dimensions.

$\Longleftarrow$ (cont'd) Say $X^{hK} = \emptyset$ $\forall$ $K < G$, $|K| = p$.

Let $L \triangleleft G$ have index p. By ind., X_{hL} is p-finite. We have the fibr.

$$X_{hL} \longrightarrow X_{hG} \longrightarrow B(G/L) = B\mathbb{Z}/p.$$

X_{hL} is p-complete.

Recall: if Y is p-complete, p-finite, ~~& Y^h~~ with $\mathbb{Z}/p$ -action st. $Y^{h\mathbb{Z}/p}$ is p-complete, then $Y^{h\mathbb{Z}/p}$ & $(Y_{h\mathbb{Z}/p}, B\mathbb{Z}/p \times Y^{h\mathbb{Z}/p})$

|| are p-finite. In particular, if $Y^{h\mathbb{Z}/p} = \emptyset$, then $H^*(Y_{h\mathbb{Z}/p})$ is finite.

So we'll show $(X_{hL})^{h\mathbb{Z}/p} = \emptyset$,

since $X_{hG} = (X_{hL})_{h\mathbb{Z}/p}$.

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "="'] & X_{hL} \arrow[r] \arrow[d] & BL \arrow[d] \\
X \arrow[r] \arrow[d] & X_{hG} \arrow[r] \arrow[d] & BG \arrow[d] \\
* \arrow[r] & B\mathbb{Z}/p \arrow[r] & B\mathbb{Z}/p
\end{tikzcd}$$

($\mathbb{Z}/p$ acts on L, but the resulting action on BL only agrees with the covering action of $\mathbb{Z}/p$ on $BL \downarrow BG$ in case $G \to \mathbb{Z}/p$ is split.
No splitting $\Longrightarrow$ $(BL)^{h\mathbb{Z}/p} = \emptyset$.
Since $(X_{hL})^{h\mathbb{Z}/p} \xrightarrow[q]{} (BL)^{h\mathbb{Z}/p}$, this is a good case.)

What's the fiber of q? — over $x \in (BL)^{h\mathbb{Z}/p}$ — it's

$$\left\{ \begin{tikzcd}
 & X \arrow[d] \\
 & X_{hG} \arrow[d] \\
B\mathbb{Z}/p \arrow[ur, dashed] \arrow[r, "x"'] & BG
\end{tikzcd} \right\} = X^{h\mathbb{Z}/p}.$$

using the $\mathbb{Z}/p$-action coming from $\mathbb{Z}/p \xrightarrow{x} G$.
By hypothesis, $X^{h\mathbb{Z}/p} = \emptyset$ so $(X_{hL})^{h\mathbb{Z}/p} = \emptyset$ △

23 Apr.

Prop. G p-complete toral gp, $f: G \to X$ hom. $i: \check{G} \to G$ a discrete approx. Then f is mono (ie X/G is ~~p-c~~ p-finite) $\iff$ $\ker(f \circ i) = \{1\}$.

pf. Aside on:

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d] & F_2 \arrow[d] \\ E_1 \arrow[r] \arrow[d] & G_2 \arrow[d] \\ B \arrow[r, equal] & B \end{tikzcd}$$

Say $E_1 \to E_2$ a H_* iso. Is $F_1 \to F_2$?

Convert to inclusions; look at Serre sseq (~~say B conn~~)

$$E^2 = H_*(B; H_*(F_2, F_1)) \Rightarrow H_*(E_2, E_1) = 0.$$

So $0 = H_0(E_2, E_1) = H_0(B; H_*(F_2, F_1))$

Say B s.c.; then $H_0(F_2, F_1) = 0$; continue by ind to see $H_*(F_2, F_1) = 0$.

Application:

$$\begin{tikzcd} X/\check{G} \arrow[r] \arrow[d] & X/G \arrow[d] \\ B\check{G} \arrow[r] \arrow[d] & BG \arrow[d] \\ BX \arrow[r, equal] & BX \end{tikzcd}$$

BX isn't generally 1-conn, but π_1 is a p-gp.

Back in the gen. case, if $\pi_1 B$ is a finite p-group, then

$$~~H_{or}~~ \quad H_0(F_2, F_1) \otimes_{\pi_1 B} \mathbb{F}_p = 0$$

which implies by Nakayama that $H_0(F_2, F_1)$; ~~etc~~.

~~But now why should $H_*(B; H)$~~

So $H_*(X/\breve{G}; \mathbb{F}_p) \xrightarrow{\cong} H_*(X/G; \mathbb{F}_p)$.

Assume $\ker(f \circ i) \neq \{1\}$; find a nontrivial elt of order p in ker, say x.

$$\begin{tikzcd}
& X/\breve{G} \arrow[d] \\
B\mathbb{Z}/p \arrow[ur, dashed] \arrow[r, "B\kappa_x"] \arrow[dr, "*"'] & B\breve{G} \arrow[d] \\
& BX
\end{tikzcd}$$

~~But~~ Then (as before) $X/\breve{G}$ isn't p-finite, so X/G isn't either, and so f isn't a mono.

Conversely, suppose $\ker(f \circ i) = \{1\}$. Let $K < \breve{G}$ be a finite subgroup, & consider X/K. Show first that X/K is p-finite.

From the lemma of last lecture, it suffices to show that $X^{hL} = \emptyset$ for each $L \leq K$ of order p.

~~Let~~ Fix such an L.

$$\begin{tikzcd} X/L \arrow[r] \arrow[d] & X/K \arrow[r] \arrow[d] & X/\breve{G} \arrow[d] \\ BL \arrow[r] & BK \arrow[r] & B\breve{G} \end{tikzcd}$$

$X/L = EL \times_L X$

So a htpy fixed pt in X $\Leftrightarrow$ section, ie
$$\begin{tikzcd} & X/\breve{G} \arrow[d] \\ BL \arrow[ur] \arrow[r] & B\breve{G} \end{tikzcd}.$$

This implies
$$\begin{tikzcd} BL \arrow[r, "i"] \arrow[dr, "*"'] & B\breve{G} \arrow[d, "f"] \\ & BX \end{tikzcd}$$
ie $L \subseteq \ker(f \circ i)$.

This contradiction forces $X^{hL} = \emptyset$.

Thus X/K is p-finite for any K.
Write $\breve{G} = \bigcup K_n$; then

$$X/\breve{G} = \underrightarrow{\text{holim}}\, X/K_n$$

we claim $\exists N$ st. $\forall n$, $H_k(X/K_n) = 0$ for $k > N$.

Now $H^*(X) = H^*(X/K_n ; \mathbb{F}_p[K_n])$

and π_1 acts nilpotently on $\mathbb{F}_p[K_n]$, so the top nonzero deg is same in $H^*(X)$ as in $H^*(X/K_n)$.

In fact, we saw in general that $\operatorname{cd} X = \operatorname{cd} Y + \operatorname{cd}(X/Y)$.

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & G_2 \arrow[d] \\
G \arrow[r] \arrow[d] & G_1 \arrow[d] \\
B \arrow[r, equal] & B
\end{tikzcd}$$

π perfect.

eg

π acyclic group

$$\boxed{\begin{tikzcd}
\pi \arrow[r] \arrow[d] & * \arrow[d] \\
E\pi \arrow[r] \arrow[d] & B\pi \arrow[d] \\
B\pi \arrow[r] & B\pi
\end{tikzcd}}$$

$$\begin{tikzcd}
X \arrow[d] \\
X/G \arrow[d] \\
BG
\end{tikzcd}$$

Elmendorf

Now by Serre s'seq of $X \longrightarrow X/\breve{G} \longrightarrow B\breve{G}$, since X & $B\breve{G}$ are f.type (& $\pi_1(G)$ is a finite p-group), you see $H_*(X/G)$ is too. $\triangle$

Classically, a max. torus in a connected cpt Lie gp is self-centralizing. Our defn. has almost self-centralizing; but

Th. X conn p-adic finite loop space, $T \hookrightarrow X$ max torus. Then $C_X T = T$.

pf. Suppose not. Let $G = C_X T$; this is a p-~~[illegible]~~ complete toral gp with con. cpt T. Let $\breve{G} \longrightarrow G$ be a discrete approx. It's an extension $\breve{T} \longrightarrow \breve{G} \longrightarrow \pi_0 G$ (*) where $\breve{T} \to T$ is a discrete approximation of T.

(*) is central, since ~~T~~ $T \to G$ was central.

Pick $x \in P$ with $x^p = 1$. Let $y \in \breve{G}$ project to x. $y^p \in \breve{T}$; let $t \in \breve{T}$ have $t^p = y^p$. Then $(yt^{-1})^p = y^p t^{-p} = 1$, and $yt^{-1} \longmapsto x \in P$. So wma $y^p = 1$.

~~So~~ Let $L = \breve{T} \times \langle y \rangle \subset \breve{G}$.

¶ Next lift L back to T: :

$$\begin{tikzcd}
 & & & BT \arrow[d] \\
BL \arrow[r] \arrow[urrr, dashed] & B\breve{G} \arrow[r] & BG \arrow[r] & BX
\end{tikzcd}$$

ie an elt of $(X/T)^{hL} = ((X/T)^{h\langle y\rangle})^{h\breve{T}}$.

$\chi(X/T) \neq 0$. The action of L ~~ex~~ restricts an action of X, which is connected. So L acts trivially in homology. This implies

$$\chi((X/T)^{h\langle y\rangle}) = L(y\,|X/T) = \chi(X/T) \neq 0$$

$(X/T)^{h\langle y\rangle}$ is p-complete & $\neq \emptyset$. We saw that $(\ )^{h\breve{T}} \neq \emptyset$ as well: so you can lift.

¶ By obst. thy you can lift further to $B\breve{T}$ ($T/\breve{T}$ is a rational Eilenberg MacLane space)

¶ We know that $BG \to BX$ is monic; so $BL \to BX$ is monic

So $BL \to B\breve{T}$ is monic (This uses the kernel stuff) But it can't be.

28 April

Remarks.

G conn. cpt Lie group, T^5 max torus, $W = NT/T$.

(1). The action of W on T induces

$$W \hookrightarrow \mathrm{Aut}(\pi_1 T) = \mathrm{Aut}(H_2 BT) = GL_5 \mathbb{Z}.$$

which presents W as a finite subgroup gen'd by reflections.

(A reflection is a matrix conjugate to $\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} =: r$

This is equivalent to

1) r has finite order

& 2) $\mathrm{rank}(r - 1) = 1$.

((2) $\Leftrightarrow$ (over $\mathbb{Q}$) $r \sim \begin{bmatrix} x & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$; $x = \pm 1$ by (1)).

(2). $H^*(BG; \mathbb{Q}) \longrightarrow H^*(BT; \mathbb{Q})^W$ is iso.

Thm. X conn. p-adic finite loop space, $T^5 \to X$ max torus, W_X the Weyl group. Then

1) The action of W_X on BT gives an injection

$$W_X \longrightarrow \mathrm{Aut}(H^2(BT; \mathbb{Z}_p)) = GL_5(\mathbb{Z}_p).$$

2) The natural map

$$H_q^*(BG;\mathbb{Q}_p) \longrightarrow H^*(BT;\mathbb{Q}_p)^W \quad \text{is iso.}$$

3) $W_X \subseteq GL_s(\mathbb{Q}_p)$ is a finite (pseudo) reflection subgroup.

Remark. (1) $\mathbb{Z}_p \longrightarrow \mathbb{F}_p$ induces iso from gp. of roots of 1 in $\mathbb{Q}_p$ to $\mathbb{F}_p^\times$. $(p \neq 2)$.

(2) All roots of 1 in $\mathbb{Q}_p$ are integral.

Topology in the proof:

Prop. 1. Let $F \longrightarrow E \xrightarrow{f} B$ be a fibration seq. st. F is p-finite. There exists "transfers":

$$f_*: H^*E \longrightarrow H^*B \; ; \; f_*: H^*(E;\mathbb{Q}_p) \longrightarrow H^*(B;\mathbb{Q}_p).$$

~~A~~ which are H^*B resp $H^*(B;\mathbb{Q}_p)$ - module hom's, s.t.

$$f_* f^* = \text{mult. by } \chi(F).$$

~~Prop~~ Algebra in the proof:

(Auslander-Buchsbaum)

Prop 2. $P_1 \hookrightarrow P_2$ com. graded f.gen. polynomial algebras over a field F. If P_2 is finite over P_1, then rank P_1 = rank P_2 & P_2 is free over P_1.

Prop 3. F a field of char. 0, W a finite subgp. of $GL_n(F) = \mathrm{Aut}_F(V)$ $(V = F^n)$.
Then $S(V^*)^W$ is poly $\Leftrightarrow$ W is gen'd by reflections.

Pf of Thm.

$$W(T \to X) = \mathrm{map}_{BX}(BT, BT) = \text{fiber}\left(\begin{matrix}\mathrm{map}(BT,BT)\\ \downarrow \\ \mathrm{map}(BT,BX)\end{matrix}\right) \text{ over } Bi.$$

Now $T \to X$ is self-centralizing (since X is connected)

$\Downarrow$

$$\begin{tikzcd} \mathrm{map}(BT,BT) \arrow[r, "~~\text{comps}~~"', "\text{comps over } Bi"'] \arrow[d, phantom, "\cong"] & \mathrm{map}(BT,BX)_{Bi} \cong BT. \\ \coprod BT. & \end{tikzcd}$$

and the map is an equiv on each comp.

so $W =$ ~~$\pi_0(BT, BT)$~~ $\pi_0\left(\mathrm{map}(BT,BT)_{\mathrm{comps}/Bi}\right)$

$\subseteq \pi_0(\mathrm{HAut}\ BT) \cong GL_r(\mathbb{Z}_p)$.

Recall that $H^*(BX; \mathbb{Q}_p)$ is polynomial on $r \geq 0$ gens (& $BX \simeq *$ if $r = 0$).

($H^*(X; \mathbb{Q}_p)$ is prim gen. exterior; so by EMSS...)

Consider $X/T \longrightarrow BT \longrightarrow BX$

$\chi(X/T) > 0$, so by transfer, $H^*(BX; \mathbb{Q}_p) \hookrightarrow H^*(BT; \mathbb{Q}_p)$.

$H^*(BX; \mathbb{Q}_p)$ is Noetherian, so by the Serre s'seq $H^*(BT; \mathbb{Q}_p)$ is finite over it.

So by Auslander-Buchsbaum $r = s$ ranks are the same, and $H^*(BT; \mathbb{Q}_p)$ is free over $H^*(BX; \mathbb{Q}_p)$.

(*) Thus (by EMSS) $\mathbb{Q}_p \otimes_{H^*(BX; \mathbb{Q}_p)} H^*(BT; \mathbb{Q}_p) \cong H^*(X/T)$,

ie $H^*(BT; \mathbb{Q}_p)$ takes ~~#W~~ rank $H^*(X/T)$. gen's as $H^*(BX; \mathbb{Q}_p)$-module.

We know X; but why are the $\mathbb{Q}_p$ gens of X/T in even degrees? By (*)! — so #gens = #W.

Let $R_T = H^*(BT; \mathbb{Q}_p)$ etc.
& let F_T = fraction field of R_T etc.

Then $\dim_{F_X} F_T = \# W$.

W acts faithfully, and $F_X \subseteq (F_T)^W$.

So by Galois theory $F_X = (F_T)^W$

Now

$$\begin{tikzcd} R_X \arrow[r, hook] \arrow[d, phantom, "\cap"] & (R_T)^W \arrow[d, phantom, "\cap"] \\ F_X \arrow[r, phantom, "="] & (F_T)^W \end{tikzcd}$$

By Hilbert basis thm, $(R_T)^W$ is fgen $/R_X$: ie integral

But a poly. ring is int. closed in its fraction field:

$$R_X = (R_T)^W$$

Finally, W is gen'd by reflections, since R_X is polynomial. $\triangle$.

Prop. If X is a p-adic finite loopspace & $T \to X$ a max torus, then $i' : N_pT \longrightarrow X$ is mono.

pf. Assume first X is connected. Then

$$BT \longrightarrow BN_pT \longrightarrow BW_p.$$

Pick a discrete approximation $j: \check{G} \to N_pT$, so

$$B\check{T} \longrightarrow B\check{G} \longrightarrow BW_p.$$

i' is monic $\Longleftrightarrow$ $\ker(i' \circ j) = \{1\}$. $\ker \triangleleft \check{G}$.

$i: T \to G$ is monic, so $\ker \cap \check{T} = \{1\}$.

W_p acts faithfully on $\pi_1 \check{T}$.

Say $x \in \ker$ projects nontrivially to W_p. Then it doesn't centralize $\check{T}$. Pick $y \in \check{T}$

st $xyx^{-1} \neq y$. Since ker is normal,

$$y^{-1}xy \in \ker$$

so also $y^{-1}xyx^{-1} \in \ker$.

But $y^{-1} \in \check{T}$ & $xyx^{-1} \in \check{T}$, so

$$1 \neq y^{-1}xyx^{-1} \in \check{T} \cap \ker. \qquad \# \qquad \triangle.$$

$$\begin{tikzcd}
X/N_pT \arrow[r] & BN_pT \arrow[r] & BX \\
X/T \arrow[u] \arrow[r] & BT \arrow[u] \arrow[r] & BX \arrow[u, no head]
\end{tikzcd}$$

So $\chi(X/N_pT) = \dfrac{\chi(N/T)}{|W_p|}$ is prime to p.

Hence by transfer, $H^*(BX)$ is Noetherian

(since H^*BN_pT is Noeth by calc.)

Apr 30.

Th. $F \to E \to B$ fibr, B conn, H_*F finite.
$\exists f_* : H^*E \longrightarrow H^*B$ st $f_*f^* =$ mult. by χ
f_* an H^*B-module map.

Preliminary remarks.

1) wma $B = K(\pi, 1)$: by Kan - Thurston.

($X \to Y$ iso in $H^*(- ; M)$ for any $\pi_1 Y$-module M
$\Leftrightarrow$ fiber is acyclic. So given two

$$\begin{tikzcd}
K(G'',1) \arrow[dr] & & \\
q & K' \times K'' \arrow[r] \arrow[d, "\times" marking] & K(G'',1) \arrow[d] \\
& K(G',1) \arrow[r] & X
\end{tikzcd}$$

get comparison: so f_* won't depend upon the K-T map.)

2) wma $E = EG \times_G X$:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & E \arrow[d] \\
\widetilde{BG} \arrow[r] \arrow[ur, phantom, "\Nwarrow"] & BG
\end{tikzcd}$$

Prop. $f : C_1 \to C_2$ chain map of nonneg. graded free R-chain cxs. If its iso in H_*, then it has a htpy inverse.

pf of Thm.

$C_\cdot$ a nonneg. ch. cx over $\mathbb{F}_p$.
$D = \mathrm{Hom}(C, \mathbb{F}_p)$, negatively graded.

$\mathrm{Hom}(C,C)_n = \prod_k \mathrm{Hom}(C_k, C_{k+n})$, $df = [d,f]$.

so $H_0 \, \mathrm{Hom}(C,C)_* = [C,C]$.
$Z_0 \, \mathrm{Hom}(C,C) =$ chain-maps $C \longrightarrow C$.

$$h : D \otimes C \longrightarrow \mathrm{Hom}(C,C).$$

– Iso if C is finite: so a w. eq. if H_*C is finite.
~~(since C is w. eq. to H_*C, and h is a~~ [illegible]
~~weak equiv~~
(since $C = H_*C \oplus$ (acyclic), and both sides are unchanged under addition of an acyclic complex.) – so a h. eq. in that case

We have $\alpha : \mathbb{F}_p \longrightarrow \mathrm{Hom}(C,C) \; : \; 1 \longmapsto id.$

$$\beta : D \otimes C \longrightarrow \mathbb{F}_p \; : \; f, x \longmapsto (-1)^{|f|} f(x)$$

and

$$\begin{tikzcd} \mathbb{F}_p \arrow[r] \arrow[drr, bend right] & \mathrm{Hom}(C,C) & & \underline{\underline{\chi(H_*C)}} \arrow[dl] \\ & D \otimes C \arrow[u, "h"] \arrow[r] & \mathbb{F}_p \end{tikzcd}$$

It's enough to check this f~~ae~~ in case C has 0 differential.

Case. $C = C_*(X; \mathbb{F}_p)$ X p-finite.

$$\begin{tikzcd}
\mathbb{F}_p \arrow[r, "\alpha"] & \mathrm{Hom}(C,C) \arrow[r, "h^{-1}"] & D\otimes C \arrow[r, "\beta"] \arrow[d, "1\otimes\Delta"] & \mathbb{F}_p . \\
 & & D\otimes C\otimes C \arrow[r, "\beta\otimes 1"] & C \arrow[u, "\epsilon"]
\end{tikzcd}$$

Still the composite is mult. by $\chi(H_* X)$.

Now suppose π acts on X: so on C. h^{-1} is not Equivariant. But let $W = C_* E\pi$;

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & W\otimes \mathrm{Hom}(C,C) \arrow[d] & W\otimes D\otimes C \arrow[l, "1\otimes h"'] \arrow[d] \\
\mathbb{F}_p \arrow[r, "\alpha"] & \mathrm{Hom}(C,C) & D\otimes C \arrow[l, "h"']
\end{tikzcd}$$

Now by lemma, we can find $(1\otimes h)^{-1}$, since $1\otimes h$ is a map between free chain cxs.

$\mathbb{H}_*(\pi, -)$ converts weak equivalences to iso's. We get

$$\begin{tikzcd}
H_*(\pi) \arrow[r] \arrow[rr, bend right, "f_*"'] & \cdots \arrow[r] & \mathbb{H}_*(\pi; C) \arrow[r] & H_*(\pi)
\end{tikzcd}$$

Ext acts by cap-product