To. Haynes
with best regards,
Pete

Homogeneous Functors and their Derived Functors

by

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§1. Introduction

Dold and Puppe have generalized [5] the classical theory of derived functors [2] to the case of non-additive functors between abelian categories. Such derived functors have various applications ([1] and [5]) in algebraic topology, but are typically quite difficult to compute. However, most of the common non-additive functors on abelian categories may be decomposed as sums of homogeneous (3.1) functors, whose derived functors are more accessible and have special properties (§5 and §9). Under mild restrictions, this approach permits a simple determination (§9) of derived functors for functors from the category of abelian groups to vector spaces over the rationals. This reinforces the principle that most problems concerning derived functors are easily solved modulo torsion.

We give special attention to three functors on abelian groups:

(i) the symmetric algebra functor (3.5)

$$SP = \sum_{r=0}^{\infty} SP^r$$

(ii) the exterior algebra functor (3.6)

$$\bigwedge = \sum_{r=0}^{\infty} \bigwedge^r$$

(iii) the gamma functor (2.1)

$$\Gamma = \sum_{r=0}^{\infty} \Gamma^r$$

We prove (§7) that the homogeneous functors SP^r, $\bigwedge^r$, and Γ^r all have the same derived functors except for shifts in degree.

In a future article, we will define an algebra of operators on derived functors and will use the theory of homogeneous functors to determine the action of these operators in certain examples.

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§2. The Gamma Functor

The gamma functor of Eilenberg-MacLane [6, §18] is needed for the definition of a homogeneous functor.

2.1. Definition of the gamma functor. Let M be an abelian group. Then ΓM is the commutative ring with identity which has generators $\gamma_r(x)$ for each $x \in M$ and integer $r \geq 0$, and has relations:

(i) $\gamma_0(x) = 1$

(ii) $\gamma_r(x)\ \gamma_s(x) = \binom{r+s}{r}\ \gamma_{r+s}(x)$

(iii) $\gamma_t(x+y) = \sum_{r+s=t} \gamma_r(x)\ \gamma_s(y)$

(iv) $\gamma_r(nx) = n^r\ \gamma_r(x)$ for each integer n.

If $\gamma_r(x)$ is assigned degree r, then

$$\Gamma M = \sum_{r=0}^{\infty} \Gamma^r M$$

where $\Gamma^r M$ is additively generated by products of total degree r. Thus $\Gamma^0(M) = Z$ and $\Gamma^1(M) = M$. Eilenberg-MacLane have shown [6, §18]:

(i) If $r \geq 1$ then $\Gamma^r(Z) = Z$ with generator $\gamma_r(1)$.

(ii) If $r \geq 1$ and $n \geq 2$ then $\Gamma^r(Z_n)$ is cyclic of order $n(r,n^\infty)$ with generator $\gamma_r(1)$, where (r,n^∞) denotes the common value of the greatest common divisor (r,n^e) for large e.

2.2. A natural pairing. Let M and N be abelian groups and $r \geq 1$. Define a pairing

$$\mu\colon \Gamma^r M \otimes \Gamma^r N \to \Gamma^r(M \otimes N)$$

as follows. If $u = \gamma_{s_1}(m_1)\ldots\gamma_{s_h}(m_h) \in \Gamma^r(M)$ and $v = \gamma_{t_1}(n_1)\ldots\gamma_{t_k}(n_k) \in \Gamma^r N$, then let

$$\mu(u \otimes v) = \sum_f \prod_{i,j} \gamma_{f(i,j)}(m_i \otimes n_j)$$

where f ranges over the h by k matrices in non-negative integers such that

$$s_i = \sum_{j=1}^{k} f(i,j)$$

$$t_j = \sum_{i=1}^{h} f(i,j)$$

The pairing μ is associative and commutative in the obvious sense.

2.3. The category. $\Gamma^r \mathcal{U}$. Let $\mathcal{U}$ be a preadditive category, i.e., $\mathcal{U}$ satisfies the axioms for an additive category except for the existence of finite direct sums. For $r \geq 1$ define another preadditive category $\Gamma^r \mathcal{U}$ by the conditions:

(i) $\Gamma^r \mathcal{U}$ has the same objects as $\mathcal{U}$

(ii) For objects X and Y

$$\mathrm{Hom}_{\Gamma^r \mathcal{U}}(X,Y) = \Gamma^r \mathrm{Hom}_{\mathcal{U}}(X,Y)$$

(iii) For objects X, Y, and Z, the composition in $\Gamma^r \mathcal{U}$ is given by

$$\Gamma^r \mathrm{Hom}_{\mathcal{U}}(X,Y) \otimes \Gamma^r \mathrm{Hom}_{\mathcal{U}}(Y,Z) \xrightarrow{\mu} \Gamma^r(\mathrm{Hom}_{\mathcal{U}}(X,Y) \otimes \mathrm{Hom}_{\mathcal{U}}(Y,Z))$$

$$\to \Gamma^r \mathrm{Hom}_{\mathcal{U}}(X,Z)$$

where the second map is induced by composition in $\mathcal{U}$.

(iv) The identity map for an object X in $\Gamma^r \mathcal{U}$ is $\gamma_r(1_X)$ where 1_X is the identity for X in $\mathcal{U}$.

2.4. If $\mathcal{U}$ is a preadditive category and $r \geq 1$, define a functor

$$\gamma^r : \mathcal{U} \longrightarrow \Gamma^r \mathcal{U},$$

where γ^r sends each object to itself and sends each map $f \in \mathrm{Hom}_{\mathcal{U}}(X,Y)$ to $\gamma_r(f) \in \Gamma^r \mathrm{Hom}_{\mathcal{U}}(X,Y)$.

§3. Homogeneous Functors

All functors will be assumed covariant unless otherwise specified. Let $\mathcal{U}$ and $\mathcal{V}$ be preadditive categories and $r \geq 1$.

<u>Definition 3.1.</u> A functor $T: \mathcal{U} \to \mathcal{V}$ is r-<u>homogeneous</u> if there exists an additive functor $\phi: \Gamma^r \mathcal{U} \to \mathcal{V}$ such that $T = \phi \circ \gamma^r$. Such a functor ϕ is an r-<u>homogeneous structure</u> for T.

<u>Example 3.2.</u> If $T: \mathcal{U} \to \mathcal{V}$ is additive, then T is 1-homogeneous.

A functor $T: \mathcal{U} \to \mathcal{V}$ is <u>quadratic</u> [6,§9] if

$$0 = T(f_1+f_2+f_3)-T(f_1+f_2)-T(f_2+f_3)-T(f_1+f_3)+T(f_1)+T(f_2)+T(f_3)$$

for any maps $f_1, f_2, f_3 \in \mathrm{Hom}_{\mathcal{U}}(X,Y)$.

<u>Example 3.3.</u> If $T: \mathcal{U} \to \mathcal{V}$ is quadratic and $T(f) = T(-f)$ for each map f in $\mathcal{U}$, then T is 2-homogeneous.

Let $\mathcal{A}$ denote the category of abelian groups.

<u>Example 3.4.</u> The r-fold <u>tensor power</u> functor

$$\otimes^r : \mathcal{A} \to \mathcal{A}$$

is r-homogeneous (see 3.9).

<u>Example 3.5.</u> The <u>symmetric algebra</u> functor

$$SP = \sum_{r=0}^{\infty} SP^r : \mathcal{A} \to \mathcal{A}$$

is formed by defining SP(X) as the quotient of the tensor algebra $\otimes(X) = \sum_{r=0}^{\infty} \otimes^r(X)$ by the two-sided ideal with generators $x \otimes y - y \otimes x$ for $x, y \in X$. For $r \geq 1$ the functor

$$SP^r : \mathcal{A} \to \mathcal{A}$$

is r-homogeneous (see 4.6).

<u>Example 3.6.</u> The <u>exterior algebra</u> functor

$$\Lambda = \sum_{r=0}^{\infty} \Lambda^r : \mathcal{A} \to \mathcal{A}$$

is formed by defining $\Lambda(X)$ as the quotient of the tensor algebra $\otimes(X)$ by the two-sided ideal with generators $x \otimes x$ for $x \in X$. For $r \geq 1$ the functor

$$\Lambda^r : \mathcal{A} \to \mathcal{A}$$

is r-homogeneous (see 4.6).

<u>Example 3.7.</u> The <u>gamma</u> functor

$$\Gamma^r : \mathcal{A} \to \mathcal{A}$$

is r-homogeneous. Its homogeneous structure is determined by the map

$$\Gamma^r \mathrm{Hom}(X,Y) \to \mathrm{Hom}(\Gamma^r X, \Gamma^r Y)$$

adjoint to the composition

$$\Gamma^r \mathrm{Hom}(X,Y) \otimes \Gamma^r X \xrightarrow{\mu} \Gamma^r(\mathrm{Hom}(X,Y) \otimes X) \xrightarrow{\Gamma^r(e)} \Gamma^r Y$$

where e is the evaluation map.

Further examples of homogeneous functors may be constructed using the following propositions.

Let $\mathscr{U}$, $\mathscr{V}$, and $\mathscr{W}$ be preadditive categories.

Proposition 3.8. *If* $V: \mathscr{U} \to \mathscr{V}$ is r-*homogeneous and* $W: \mathscr{V} \to \mathscr{W}$ *is* s-*homogeneous, then the composed functor* $W \circ V: \mathscr{U} \to \mathscr{W}$ *is* rs-*homogeneous*.

Proof. For $M \in \mathscr{A}$ let $h: \Gamma^t M \to \otimes^t M$ be the homomorphism such that

$$h(\gamma_{r_1}(m_1)\cdots\gamma_{r_k}(m_k)) = \sum_{(i_1,\dots,i_t)} m_{i_1}\otimes\cdots\otimes m_{i_t}$$

where $(i_1,\dots,i_t)$ ranges over those integral t-tuples in which n appears r_n times for $1 \le n \le k$. Now let $\eta: \Gamma^{rs}M \to \Gamma^s\Gamma^r M$ be the unique natural homomorphism such that the diagram

$$\begin{array}{ccc} \Gamma^{rs}M & \xrightarrow{\eta} & \Gamma^s\Gamma^r M \\ \downarrow & & \downarrow \\ \otimes^{rs}M & \to & \otimes^s\otimes^r M \end{array}$$

commutes, where the vertical maps are induced by h and the lower map "inserts parentheses". An rs-homogeneous structure for $W \circ V$ is induced by

$$\Gamma^{rs}\mathrm{Hom}_{\mathcal{U}}(X,Y) \xrightarrow{\eta} \Gamma^{s}\Gamma^{r}\mathrm{Hom}_{\mathcal{U}}(X,Y) \xrightarrow{\Gamma^{s}(\phi)} \Gamma^{s}\mathrm{Hom}_{\mathcal{V}}(VX,VY)$$

$$\xrightarrow{\phi'} \mathrm{Hom}_{\mathcal{W}}(WVX,WVY)$$

where ϕ and ϕ' are homogeneous structures for V and W.

<u>Proposition 3.9</u>. <u>Let</u> $V_i: \mathcal{U} \to \mathcal{V}_i$ <u>be an</u> r_i<u>-homogeneous functor for</u> $1 \leq i \leq n$; <u>and let</u> $W(\cdot,\ldots,\cdot): \mathcal{V} \to \mathcal{W}$ <u>be an additive functor in</u> n <u>variables. Then the functor</u> $T: \mathcal{U} \to \mathcal{W}$ <u>with</u> $T(X) = W(V_1(X),\ldots,V_n(X))$ <u>is</u> $r_1+\ldots+r_n$ <u>homogeneous.</u>

<u>Proof</u>. For $M \in \mathcal{A}$ let

$$d: \Gamma^{r_1+\ldots+r_n}M \to \Gamma^{r_1}M \otimes \ldots \otimes \Gamma^{r_n}M$$

be the unique natural homomorphism such that the diagram

$$\begin{array}{ccc} \Gamma^{r_1+\ldots+r_n}M & \xrightarrow{d} & \Gamma^{r_1}M\otimes\ldots\otimes\Gamma^{r_n}M \\ \downarrow & & \downarrow \\ \otimes^{r_1+\ldots+r_n}M & \to & (\otimes^{r_1}M)\otimes\ldots\otimes(\otimes^{r_n}M) \end{array}$$

commutes, where the vertical maps are induced by h (see 3.8) and the lower map "inserts parentheses". An $r_1+\ldots+r_n$-homogeneous structure for T is induced by the composition

$$\Gamma^{r_1+\dots+r_n}\mathrm{Hom}_{\mathcal{U}}(X,Y) \xrightarrow{d} \Gamma^{r_1}\mathrm{Hom}_{\mathcal{U}}(X,Y)\otimes\dots\otimes\Gamma^{r_n}\mathrm{Hom}_{\mathcal{U}}(X,Y)$$

$$\xrightarrow{\phi_1\otimes\dots\otimes\phi_n} \mathrm{Hom}_{\mathcal{V}_1}(V_1X,V_1Y)\otimes\dots\otimes\mathrm{Hom}_{\mathcal{V}_n}(V_nX,V_nY)$$

$$\xrightarrow{W} \mathrm{Hom}_{\mathcal{W}}(W(V_1X,\dots,V_nX),W(V_1Y,\dots,V_nY)) = \mathrm{Hom}_{\mathcal{W}}(TX,TY)$$

where ϕ_i is an r_i-homogeneous structure for V_i.

Proposition 3.10. *If $T,T' : \mathcal{U} \to \mathcal{V}$ are r-homogeneous where $\mathcal{V}$ is an additive category, then $T+T' : \mathcal{U} \to \mathcal{V}$ is r-homogeneous.*

The proof is trivial.

§4. Cross-effects for Homogeneous Functors

Let $\mathcal{U}$ and $\mathcal{V}$ be abelian categories and $T: \mathcal{U} \to \mathcal{V}$ be a functor with $T(0) = 0$. Eilenberg-MacLane have defined [6, §9] <u>cross-effect</u> functors for T, which are functors $T_k(X_1, \ldots, X_k)$ from $\mathcal{U}$ to $\mathcal{V}$ for $k \geq 1$. If $X_1, \ldots, X_n \in \mathcal{U}$, then

$$\Sigma\, T_k(X_{i_1}, \ldots, X_{i_k}) \approx T(X_1 + \ldots + X_n)$$

where the sum ranges over all $(i_1, \ldots, i_k)$ with $1 \leq k \leq n$ and $1 \leq i_1 < \ldots < i_k \leq n$.

Now suppose $T: \mathcal{U} \to \mathcal{V}$ is an r-homogeneous functor with homogeneous structure ϕ. <u>The cross-effect functors of T decompose into direct sums</u>, and these finer cross-effects will be used repeatedly in the study of homogeneous functors.

For $X_1, \ldots, X_k \in \mathcal{U}$ let

$$p_i: X_1 + \ldots + X_k \to X_1 + \ldots + X_k$$

be the projection onto the summand $X_i \subset X_1 + \ldots + X_k$. For $r_1 + \ldots + r_k = r$, $r_i > 0$, and $k \geq 1$, let

$$T^{r_1, \ldots, r_k}(X_1, \ldots, X_k)$$

denote the image of

$$\phi(\gamma_{r_1}(p_1)\ldots\gamma_{r_k}(p_k)) : T(X_1+\ldots+X_k) \to T(X_1+\ldots+X_k)$$

__Definition 4.1.__ The __homogeneous cross-effect functors__ of (T,ϕ) are the functors $T^{r_1,\ldots,r_k}(X_1,\ldots,X_k)$.

__Proposition 4.2.__ __For__ $X_1,\ldots,X_n \in \mathcal{U}$ __there is a natural isomorphism__

$$\Sigma\, T^{r_1,\ldots,r_k}(X_{i_1},\ldots,X_{i_k}) \cong T(X_1+\ldots+X_n)$$

__where the sum ranges over all__ $(r_1,\ldots,r_k,\ i_1,\ldots,i_k)$ __with__ $r_1+\ldots+r_k = r$, $r_i > 0$, $k \geq 1$, __and__ $1 \leq i_1<\ldots<i_k \leq n$. __The isomorphism is induced by inclusion maps__

$$T^{r_1,\ldots,r_k}(X_{i_1},\ldots,X_{i_k}) \subset T(X_{i_1}+\ldots+X_{i_k}) \subset T(X_1+\ldots+X_n).$$

__Proof.__ The image of the map

$$\phi(\gamma_{r_1}(p_{i_1})\ldots\gamma_{r_k}(p_{i_k})) : T(X_1+\ldots+X_n) \to T(X_1+\ldots+X_n)$$

equals $T^{r_1,\ldots,r_k}(X_{i_1},\ldots,X_{i_k}) \subset T(X_1+\ldots+X_n)$. Let F be the set

$$\{\phi(\gamma_{r_1}(p_{i_1})\cdots\gamma_{r_k}(p_{i_k})) \mid r_1+\cdots+r_k = r,\ 1 \le i_1<\cdots<i_k\le n\}$$

of maps $T(X_1+\cdots+X_n) \to T(X_1+\cdots+X_n)$. Now 4.2 follows from:

(i) $\sum_{f\in F} f = 1$

(ii) $f\circ f = f$ for $f\in F$

(iii) $f\circ g = 0$ for $f,g\in F$ distinct.

<u>Remark 4.3.</u> The cross-effect functors of T decompose as

$$\Sigma\, T^{r_1,\ldots,r_k}(X_1,\ldots,X_k) \approx T_k(X_1,\ldots,X_k)$$

where the sum ranges over all $(r_1,\ldots,r_k)$ with $r_1+\cdots+r_k = r$ and $r_i > 0$. In particular, T is of <u>degree</u> $\le r$ [6, p.86], i.e., $T_k(X_1,\ldots,X_k) = 0$ for $k > r$.

<u>Example 4.4.</u> For the tensor power functor (3.4)

$$T = \otimes^r : \mathcal{A} \to \mathcal{A}$$

the homogeneous cross-effects are

$$T^{r_1,\ldots,r_k}(X_1,\ldots,X_k) \approx \underbrace{(\otimes^{r_1}X_1)\otimes\cdots\otimes(\otimes^{r_k}X_k)}_{\text{sum of } r!/r_1!\cdots r_k! \text{ copies of this}}.$$

As an application of homogeneous cross-effects, we prove a lemma which will be used to impose homogeneous structures on SP^r and $\wedge^r$. Let $T: \mathcal{U} \to \mathcal{V}$ have r-homogeneous structure ϕ; and let $U \subset T$ be a subfunctor. Suppose for any $X_1, \ldots, X_n \in \mathcal{U}$ that the inclusion

$$U(X_1 + \ldots + X_n) \subset T(X_1 + \ldots + X_n)$$

is compatible with the decomposition 4.2 of $T(X_1 + \ldots + X_n)$, i.e., that $U(X_1 + \ldots + X_n)$ is generated by its intersections with the summands. Then:

Lemma 4.5. The r-homogeneous structure of T induces r-homogeneous structures for

$$U,\ T/U: \mathcal{U} \longrightarrow \mathcal{V}$$

Proof. It suffices to show for any $\gamma_{r_1}(f_1)\ldots\gamma_{r_k}(f_k) \in \Gamma^r \mathrm{Hom}_{\mathcal{U}}(X, Y)$ that

$$\phi(\gamma_{r_1}(f_1)\ldots\gamma_{r_k}(f_k)): TX \to TY$$

restricts to a map $UX \to UY$. This is easily proved from the fact that $\phi(\gamma_{r_1}(f_1)\ldots\gamma_{r_k}(f_k))$ equals the composition

$$TX \xrightarrow{\Delta} T^{r_1,\ldots,r_k}(X,\ldots,X) \xrightarrow{T^{r_1,\ldots,r_k}(f_1,\ldots,f_k)} T^{r_1,\ldots,r_k}(Y,\ldots,Y)$$

$$\xrightarrow{\nabla} TY$$

where Δ is the composition

$$TX \xrightarrow{T(\text{diag.})} T(\underbrace{X+\ldots+X}_{k\text{-times}}) \xrightarrow{\text{proj.}} T^{r_1,\ldots,r_k}(X,\ldots,X)$$

and where ∇ is the composition

$$T^{r_1,\ldots,r_k}(X,\ldots,X) \xrightarrow{\text{inj.}} T(\underbrace{X+\ldots+X}_{k\text{-times}}) \xrightarrow{T(\text{codiag.})} TX$$

<u>Example 4.6.</u> By 3.5 and 3.6 the functors SP^r and $\wedge^r$ are quotients of the r-homogeneous functor $\boxtimes^r$. Using 4.4 and 4.5 one obtains r-homogeneous structures for SP^r and $\wedge^r$. Then the homogeneous cross-effects for $T = SP^r$ are

$$T^{r_1,\ldots,r_k}(X_1,\ldots,X_k) \approx (SP^{r_1}X_1)\boxtimes\ldots\boxtimes(SP^{r_k}X_k)$$

and for $T = \wedge^r$ are

$$T^{r_1,\ldots,r_k}(X_1,\ldots,X_k) \approx (\wedge^{r_1}X_1)\boxtimes\ldots\boxtimes(\wedge^{r_k}X_k).$$

§5. Derived Functors of Homogeneous Functors

5.1. Dold-Puppe derived functors. The Dold-Puppe theory [5] of derived functors uses semi-simplicial (abbr. "s.s.") objects where the classical theory [2] uses chain complexes. If X is an s.s. object over an abelian category $\mathcal{U}$, let π_*X denote the homotopy groups of X, defined by $\pi_*X = H_*(NX)$ where NX is the normalized chain complex [7, p. 236] of X.

Let $\mathcal{U}$ and $\mathcal{V}$ be abelian categories such that $\mathcal{U}$ has enough projectives. A (non-additive) functor

$$T: \mathcal{U} \to \mathcal{V}$$

with $T(0) = 0$ has derived functors [5, §4]

$$L_qT(\cdot, n): \mathcal{U} \to \mathcal{V}$$

for $q, n \geq 0$. Thus for $G \in \mathcal{U}$

$$L_qT(G,n) \approx \pi_qTK(G,n)$$

where K(G,n) is a dimensionwise projective s.s. object over $\mathcal{U}$ with

$$\pi_i K(G,n) = \begin{cases} G & \text{for } i = n \\ 0 & \text{otherwise} \end{cases}$$

and where $TK(G,n)$ is the s.s. object formed from $K(G,n)$ by dimensionwise application of T.

Dold-Puppe have defined [5, 5.9] the <u>suspension homomorphism</u>

$$\sigma\colon L_q T(\cdot,n) \to L_{q+1} T(\cdot,n+1)$$

which is an isomorphism for $q < 2n$ and an epimorphism for $q = 2n$. The <u>stable derived functors</u> of T are defined as the limits under σ

$$L^S{}_k T = \lim_{n\to\infty} L_{k+n} T(\cdot,n)$$

The functors $L^S{}_k T$ are additive and $\{L^S{}_k T\}$ is an exact connected sequence of functors [4]. Indeed if T is additive then σ is always an isomorphism and $\{L^S{}_k T\}$ are the classical left derived functors of T.

We will be chiefly concerned with derived functors of homogeneous functors.

Proposition 5.2. *If* $T: \mathcal{U} \to \mathcal{V}$ *is r-homogeneous, then so is* $L_qT(\cdot,n): \mathcal{U} \to \mathcal{V}$ *for* $q,n \geq 0$.

Proof. Let ϕ be an r-homogeneous structure for T and define

$$\bar{\phi}: \Gamma^r \mathrm{Hom}_{\mathcal{U}}(G,G') \to \mathrm{Hom}_{\mathcal{V}}(L_qT(G,n), L_qT(G',n))$$

as follows. Let $\gamma_{r_1}(f_1)\ldots\gamma_{r_k}(f_k) \in \Gamma^r \mathrm{Hom}_{\mathcal{U}}(G,G')$ and choose s.s. maps $\bar{f}_1,\ldots,\bar{f}_k: K(G,n) \to K(G',n)$ which induce $f_1,\ldots,f_k$ in homotopy. Then $\bar{\phi}$ sends $\gamma_{r_1}(f_1)\ldots\gamma_{r_k}(f_k)$ to

$$\phi(\gamma_{r_1}(\bar{f}_1)\ldots\gamma_{r_k}(\bar{f}_k))_*: \pi_q TK(G,n) \to \pi_q TK(G',n)$$

Clearly $\bar{\phi}$ is well defined and gives an r-homogeneous structure for $L_qT(\cdot,n)$.

Corollary 5.3. *If* $T: \mathcal{U} \to \mathcal{V}$ *is r-homogeneous and* $G \in \mathcal{U}$ *is annihilated by the positive integer* m, *then* $L_qT(G,n)$ *is annihilated by* $m(r,m^\infty)$.

Proof. By 2.1 the element $\gamma_r(1) \in \Gamma^r \mathrm{Hom}(G,G)$ is of order $m(r,m^\infty)$. Hence by 5.2 the identity map on $L_qT(G,n)$ is of order $m(r,m^\infty)$.

Example 5.4. For $\Gamma^r, SP^r : \mathcal{A} \to \mathcal{A}$ a straightforward computation shows that

$$L_0\Gamma^r(G,0) \approx \Gamma^r(G)$$

which implies (see 7.2) that

$$L_{2r}SP^r(G,2) \approx \Gamma^r(G)$$

These show that 5.3 is best possible.

Proposition 5.5. *Let* $T: \mathcal{A} \to \mathcal{A}$ *be an r-homogeneous functor which preserves direct limits. If* $G \in \mathcal{A}$ *and* $n \geq 0$, *then* $L_qT(G,n)$ *is a torsion group for* $q \neq rn$.

Proof. We need:

(i) Let $T: \mathcal{A} \to \mathcal{A}$ be any functor with $T(0) = 0$ and let $s \geq 1$. Then $L_qT(\cdot,n)$ is of degree $\leq s-1$ for $q < sn$ by [5, 6.10].

(ii) If $T: \mathcal{A} \to \mathcal{A}$ is r-homogeneous and of degree $\leq r-1$, then $T(G)$ is annihilated by $r!$ for $G \in \mathcal{A}$. This holds since the composition of diagonal and codiagonal maps

$$T(G) \xrightarrow{\Delta} T_r(G, \ldots, G) \xrightarrow{\nabla} T(G)$$

is multiplication by $r!$.

(iii) If $T: \mathcal{A} \to \mathcal{A}$ is of degree $\leq r$ and G is free abelian, then $L_q T(G,n) = 0$ for $q > rn$ by [5, 4.23].

(iv) If $T: \mathcal{A} \to \mathcal{A}$ is r-homogeneous and $T(M) = 0$ for each free abelian M, then $T(G)$ is a torsion group for G finitely generated. This holds since $mG_t = 0$ for some m, where G_t is the torsion subgroup of G. Thus there are homomorphisms $G \xrightarrow{i} G/G_t \xrightarrow{j} G$ with $j \circ i = m$, and hence $T(G)$ is annihilated by m^r.

(v) If $T: \mathcal{A} \to \mathcal{A}$ preserves direct limits and $T(0) = 0$, then each $L_q T(\cdot, n)$ preserves direct limits.

Now 5.5 follows for $q < rn$ by 5.2, (i), and (ii), and follows for $q > rn$ by (iii), (iv), and (v).

Remark 5.6. If T did not preserve direct limits then 5.5 would hold for $q < rn$ but might fail for $q > rn$. For example, if we define

$$T(G) = \operatorname{Hom}(Q, G \otimes (Q/Z))$$

where Q is the group of rationals, then for $i \geq 0$

$$L_{i+1}T(G,i) \approx \operatorname{Hom}(Q, G_t)$$

where G_t is the torsion subgroup of G.

If $T: \mathcal{A} \to \mathcal{A}$ is r-homogeneous, one can prove various restrictions on the order of elements in the torsion groups $L_qT(G,n)$ for $q < rn$. Roughly speaking, as q increases elements of higher order are permitted. A result in this direction is:

Proposition 5.7. Let $T: \mathcal{A} \to \mathcal{A}$ be r-homogeneous with $r = m_0 + m_1 p + \ldots + m_k p^k$ where p is prime and $0 \leq m_i < p$. Then for $G \in \mathcal{A}$, $L_qT(G,n)$ has trivial p-primary component for $q < mn$ where $m = m_0 + \ldots + m_k$.

Proof. By 5.2 and 5.5 (i), it suffices to show that if $U: \mathcal{A} \to \mathcal{A}$ is r-homogeneous and of degree $\leq m-1$, then U(G) has trivial p-primary component for $G \in \mathcal{A}$. Consider a decomposition $r = r_1 + \ldots + r_m$ in which each p^i appears as a summand m_i times. The composition

$$U(G) \xrightarrow{\Delta} U^{r_1,\ldots,r_m}(G,\ldots,G) \xrightarrow{\nabla} U(G)$$

is both zero and multiplication by $r!/r_1!\ldots r_m!$. Since $r!/r_1!\ldots r_m!$ is not divisible by p, U(G) has trivial p-primary component.

<u>Example 5.8.</u> For the functor

$$\Gamma^{p^k} : \mathcal{A} \to \mathcal{A}$$

with $k \geq 1$ and p prime, a computation using 6.1 shows

$$L_n \Gamma^{p^k}(G,n) \approx G \otimes Z_p$$

for $n \geq 1$. This easily implies that 5.7 is best possible.

We now show that certain of the Dold-Puppe results [5] on the suspension homomorphism can be sharpened when the functors are homogeneous.

Theorem 5.9. Let $T: \mathcal{A} \to \mathcal{A}$ be r-homogeneous with $r > 1$, and consider

$$\sigma: L_qT(G,n) \to L_{q+1}T(G,n+1)$$

If r is not a prime power, then $\sigma = 0$. If $r = p^j$ with p prime then:

(i) $p\sigma = 0$

(ii) For $q < pn$, σ restricts to an isomorphism of the p-primary components.

(iii) For $q = pn$ the image of σ is the p-primary component of $L_{pn+1}T(G,n+1)$.

Proof. The image functor of

$$\sigma: L_qT(\cdot,n) \to L_{q+1}T(\cdot,n+1)$$

is additive [5, 5.25] and r-homogeneous by 5.2 and 4.5. If $U: \mathcal{A} \to \mathcal{A}$ is additive and r-homogeneous for $r > 1$, then:

(i) $U = 0$ if r is not a prime power.

(ii) $pU = 0$ if $r = p^j$ with p prime.

This follows since the composition

$$U(G) \xrightarrow{\Delta} U^{i,r-i}(G,G) \xrightarrow{\nabla} U(G)$$

is both zero and multiplication by $\binom{r}{i}$ for $0 < i < r$.

It remains to prove 5.9 (ii) and (iii), so suppose $r = p^j$. Let X be an s.s. abelian group which is trivial below n [5, 6.8]. For $1 < s < r$ consider $T^{s,r-s}(X,X)$. Suppose $s = \ell_0 + \ell_1 p + \ldots + \ell_h p^h$ with $0 \leq \ell_i < p$ and $r-s = m_0 + m_1 p + \ldots + m_k p^k$ with $0 \leq m_j < p$. If $q < (\ell_0 + \ldots + \ell_h + m_0 + \ldots + m_k)n$ then $\pi_q T^{s,r-s}(X,X)$ is a torsion group with trivial p-primary component, as is shown by modifying proof 5.7 and using [5, 6.10]. Thus for $q < pn$, $\pi_q T_2(X,X)$ is a torsion group with trivial p-primary component. The argument generalizes to show that $\pi_q T_i(X,\ldots,X)$ is such a torsion group for $q < pn$ and $i \geq 2$. Using the Dold-Puppe spectral sequence [5, 6.7] one now shows that the suspension homomorphism

$$\sigma\colon \pi_q TX \to \pi_{q+1} TSX$$

is such that both (kernel σ) for $q < pn$ and (cokernel σ) for $q \leq pn$ are torsion groups with trivial p-primary component. Taking $X = K(G,n)$ this completes the proof.

Example 5.10. Consider the suspension map

$$\sigma: L_2\Gamma^3(G,1) \to L_3\Gamma^3(G,2)$$

One shows that

$$L_2\Gamma^3(G,1) \approx G \otimes G \otimes Z_2 + Tor(G,Z_3)$$

$$L_3\Gamma^3(G,2) \approx Tor(G,Z_3)$$

and σ is the projection map onto $Tor(G,Z_3)$.

A *stable* consequence of 5.9 is:

Corollary 5.11. Let $T: \mathcal{A} \to \mathcal{A}$ *be* r-*homogeneous with* $r > 1$. *Then*

(i) $L^s{}_*T(G) = 0$ *if* r *is not a prime power.*

(ii) $L^s{}_*T(G)$ *is a* Z_p *module if* $r = p^j$ *with* p *prime.*

§6. Stable Derived Functors

We here supplement 5.11 with some observations on more general stable derived functors. A crucial result is Dold's universal coefficient theorem [4]. Using a different proof, we give a version of this theorem which eliminates certain restrictions on splittability.

If $U: \mathcal{A} \to \mathcal{A}$ is an additive functor and $G \in \mathcal{A}$ let

$$\psi: G \otimes U(Z) \to U(G)$$

be the homomorphism such that

$$\psi(g \otimes x) = (U(i_g))(x)$$

for $g \in G$ and $x \in U(Z)$, where $i_g: Z \to G$ is the homomorphism with $i_g(1) = g$.

Theorem 6.1. (Dold) *Let $T: \mathcal{A} \to \mathcal{A}$ be a functor with $T(0) = 0$ which preserves direct limits. There is a splittable short exact sequence*

$$0 \to G \otimes L^s{}_i T(Z) \xrightarrow{\psi} L^s{}_i T(G) \to \mathrm{Tor}(G, L^s{}_{i-1} T(Z)) \to 0$$

Proof. If K is a set with basepoint $*$, let $AK \in \mathcal{A}$ denote the free abelian group generated by the elements of K with the relation $1[*] = 0$. For $G \in \mathcal{A}$ let

$$E: AK \otimes TG \to T(AK \otimes G)$$

be the homomorphism such that $E(1[u] \otimes v) = (T(f_u))(v)$ for $u \in K$ and $v \in TG$, where $f_u: G \to AK \otimes G$ is the homomorphism with $f_u(g) = 1[u] \otimes g$ for $g \in G$. Now prolong E to a map

$$E: AK \otimes TK(Z,n) \to T(AK \otimes K(Z,n))$$

where K is any s.s. set with basepoint, and $K(Z,n)$ is as in 5.1. We claim that

$$E_*: \pi_q(AK \otimes TK(Z,n)) \to \pi_q T(AK \otimes K(Z,n))$$

is an isomorphism for $q < 2n$ One first verifies the case where K is a set rather than an s.s. set. The general case then follows from the theory [5] of s.s. double objects, using the map

$$\hat{E}: AK \hat{\otimes} TK(Z,n) \to T(AK \hat{\otimes} K(Z,n))$$

of s.s. double objects which E induces, and applying a simple spectral sequence argument.

For $G \in \mathcal{A}$ let $M(G,2)$ be a Moore space of type $(G,2)$; and let the s.s. set $\bar{M}(G,2)$ be the singular complex of $M(G,2)$. For $q < 2n$ we obtain an isomorphism

$$E_*: \pi_q(A\bar{M}(G,2) \otimes TK(Z,n)) \overset{\approx}{\to} \pi_q T(A\bar{M}(G,2) \otimes K(Z,n))$$

Applying the Eilenberg-Zilber theorem and the Kunneth formula to the left side of this isomorphism, we obtain a splittable exact sequence

$$0 \to G \otimes L_{q-2}T(Z,n) \to L_q T(G,n+2) \to \mathrm{Tor}(G, L_{q-3}T(Z,n)) \to 0$$

for $q < 2n$. Since n can be arbitrarily large, it follows that for $i \geq 0$ there is a splittable exact sequence

$$0 \to G \otimes L^s{}_i T(Z) \overset{\psi'}{\to} L^s{}_i T(G) \to \mathrm{Tor}(G, L^s{}_{i-1} T(Z)) \to 0$$

which is natural in G. If $G = Z$ then ψ' gives an isomorphism

$$e: L^s{}_i T(Z) \to L^s{}_i T(Z)$$

and the diagram

$$\begin{array}{ccc} G \otimes L^s{}_i T(Z) & \overset{\psi'}{\to} & L^s{}_i T(G) \\ \downarrow 1 \otimes e & & \downarrow 1 \\ G \otimes L^s{}_i T(Z) & \overset{\psi}{\to} & L^s{}_i T(G) \end{array}$$

commutes by a naturality argument. Thus we can replace ψ' by ψ in the above exact sequence.

Remark 6.2. In practice there is a strong tendency for the groups $L^S{}_i T(Z)$ with $i > 0$ to decompose as sums of Z_p-modules for various primes p. This is true for homogeneous functors (5.11), and we now investigate this phenomenon for other functors.

Proposition 6.3. *For* $r > 1$ *let* $T: \mathcal{A} \longrightarrow \mathcal{A}$ *be such that* $T(nf) = n^r T(f)$ *for all maps* $f \in \mathcal{A}$ *and all* $n \in Z$. *Then* $L^S{}_* T(Z)$ *is a sum of* Z_p*-modules for primes* p *such that* p-1 *divides* r-1.

Proof. The map

$$L^S{}_* T(n): L^S{}_* T(Z) \to L^S{}_* T(Z)$$

equals multiplication by n and also by n^r. Let

$$\theta(r) = \text{G.C.D.} \{n^r - n \mid n \in Z\}$$

Elementary number theory shows that $\theta(r)$ is the product of those primes p such that p-1 divides r-1. Clearly $\theta(r)$ annihilates $L^S{}_* T(Z)$.

Remark 6.4. The hypotheses of 6.3 are satisfied by subfunctors and quotient functors of homogeneous functors, which need not themselves be homogeneous. For example, if $G \in \mathcal{A}$ and $r \geq 2$, let $T(G)$ be the subgroup of $\otimes^r(G)$ generated by $g \otimes \ldots \otimes g$ for $g \in G$. Since $L^s_0 T(Z) \approx Z_{\theta(r)}$, where $\theta(r)$ is as in proof 6.3, it follows using 5.11 that T is non-homogeneous for $r \neq 2^j$.

Proposition 6.5. *If $T: \mathcal{A} \longrightarrow \mathcal{A}$ is a functor of finite degree, then $L^s_i T(Z)$ is a torsion group for $i > 0$.*

This will follow from 9.2.

Remark 6.6. If T were of infinite degree, then 6.5 might fail. For $G \in \mathcal{A}$ let $T(G)$ be the cokernel of the monomorphism

$$d: A(G) \to A(G) \otimes A(G)$$

with $d(1[g]) = 1[g] \otimes 1[g]$, where $A(G)$ is as in proof 6.1. Then $L^s_1 T(Z) \approx Z$.

Remark 6.7. Under the hypotheses of 6.5, $L^s{}_i T(Z)$ is usually a sum of various Z_p-modules for $i > 0$. However, for $r \geq 2$ we shall define $T^r: \mathscr{A} \to \mathscr{A}$ of degree $\leq r$ such that $L^s{}_1 T^r(Z) \approx Z_{\phi(r)}$, where $\phi(r)$ is the least common multiple of $\{1,2,\ldots,r\}$. If $G \in \mathscr{A}$ then an associative multiplication on $A(G)$

$$\nu: A(G) \otimes A(G) \to A(G)$$

is defined by

$$\nu(1[g] \otimes 1[h]) = 1[g+h] - 1[g] - 1[h]$$

For $r \geq 1$ let $A^r(G)$ denote the cokernel of the $r+1$-fold multiplication map

$$\otimes^{r+1}(A(G)) \to A(G)$$

induced by ν. For $r \geq 2$ let $T^r(G)$ be the cokernel of the homomorphism

$$e: A^r(G) \to A^{r-1}(G) \otimes A^1(G)$$

induced by the map d in 6.6.

§7. Derived Functors of SP^r, $\wedge^r$, and Γ^r

The r-homogeneous functors

$$SP^r,\ \wedge^r, \Gamma^r : \mathcal{A} \to \mathcal{A}$$

have essentially the same derived functors.

<u>Theorem 7.1.</u> <u>There is a natural isomorphism</u>

$$L_q \wedge^r (G,n) \approx L_{q+r} SP^r (G,n+1)$$

<u>for</u> $q \geq 0$, $r \geq 1$, $n \geq 0$.

<u>Theorem 7.2.</u> <u>There is a natural isomorphism</u>

$$L_q \Gamma^r (G,n) \approx L_{q+2r} SP^r (G,n+2)$$

<u>for</u> $q \geq 0$, $r \geq 1$, $n \geq 0$.

<u>Remark 7.3.</u> Much is known about the functors $L_q SP^r (G,n)$ because of their connection [5, p. 231] with the homology of Eilenberg-MacLane spaces. Furthermore, the functors $L_q \wedge^2 (G,n)$ have been completely determined [3].

7.4. Natural homomorphisms. To prove 7.1 and 7.2 we need the following.

(i) Define $f\colon SP^r M \to \otimes^r M$ by

$$f(m_1 \ldots m_r) = \sum_{\sigma \in S(r)} m_{\sigma(1)} \otimes \ldots \otimes m_{\sigma(r)}$$

for $m_1, \ldots, m_r \in M = SP^1 M$, where $S(r)$ is the symmetric group on r elements.

(ii) Define $g\colon \wedge^r M \to \otimes^r M$ by

$$g(m_1 \ldots m_r) = \sum_{\sigma \in S(r)} \operatorname{sign}(\sigma) m_{\sigma(1)} \otimes \ldots \otimes m_{\sigma(r)}$$

for $m_1, \ldots, m_r \in M = \wedge^1 M$.

(iii) Let $h\colon \Gamma^r M \to \otimes^r M$ be as in 3.8.

(iv) For $r_1 + \ldots + r_k = r$, $r_i > 0$, let

$$f^{r_1, \ldots, r_k}\colon SP^r M \to SP^{r_1} M \otimes \ldots \otimes SP^{r_k} M$$

be the unique natural homomorphism such that

$$\begin{array}{ccc}
SP^r M & \xrightarrow{f^{r_1,\ldots,r_k}} & SP^{r_1}M \otimes \ldots \otimes SP^{r_k}M \\
\downarrow f & & \downarrow f\otimes\ldots\otimes f \\
\otimes^r M & \longrightarrow & (\otimes^{r_1}M)\otimes\ldots\otimes(\otimes^{r_k}M)
\end{array}$$

commutes, where the lower map "inserts parentheses".

(v) Let $\eta\colon \wedge^r M \otimes \wedge^r N \longrightarrow SP^r(M \otimes N)$ be the unique natural homomorphism such that

$$\begin{array}{ccc}
\wedge^r M \otimes \wedge^r N & \xrightarrow{\eta} & SP^r(M \otimes N) \\
\downarrow g\otimes g & & \downarrow f \\
(\otimes^r M) \otimes (\otimes^r N) & \xrightarrow{\lambda} & \otimes^r(M \otimes N)
\end{array}$$

commutes, where

$$\lambda((m_1\otimes\ldots\otimes m_r) \otimes (n_1\otimes\ldots\otimes n_r)) = (m_1\otimes n_1)\otimes\ldots\otimes(m_r\otimes n_r)$$

(vi) Let $\nu\colon \Gamma^r M \otimes SP^r N \longrightarrow SP^r(M \otimes N)$ be the unique natural homomorphism such that

$$\begin{array}{ccc}
\Gamma^r M \otimes SP^r N & \xrightarrow{\nu} & SP^r(M \otimes N) \\
\downarrow h\otimes f & & \downarrow f \\
(\otimes^r M) \otimes (\otimes^r N) & \xrightarrow{\lambda} & \otimes^r(M \otimes N)
\end{array}$$

commutes.

The following lemma is proved by the method of [5].

Lemma 7.5.

(i) $L_* SP^r(Z,1) = 0$ *for* $r > 1$.

(ii) *For* $r \geq 1$,

$$f_*: L_* SP^r(Z,2) \to L_* \otimes^r(Z,2)$$

is an isomorphism.

(iii) *For* $r_1 + \ldots + r_k = r$ *with* $r_i > 0$,

$$f_*^{r_1,\ldots,r_k} : L_* SP^r(Z,2) \to L_*(SP^{r_1} \otimes \ldots \otimes SP^{r_k})(Z,2)$$

is an isomorphism.

(iv) *For* $r \geq 1$

$$g_*: L_* \Lambda^r(Z,1) \to L_* \otimes^r(Z,1)$$

is an isomorphism.

7.6. Proof of 7.1. It clearly suffices to show that for any s.s. free abelian group X

$$\eta_*: \pi_*(\wedge^r X \otimes \wedge^r K(Z,1)) \to \pi_* SP^r(X \otimes K(Z,1))$$

is an isomorphism. For this it suffices by the theory of s.s. double objects [5] to suppose X is merely a free abelian group. Choose an ordered basis $(b_1, b_2, \ldots)$ for X. Then by 4.6 and 7.5 (i), $SP^r(X \otimes K(Z,1))$ is the direct sum of a contractible complex and the complexes $(b_{i_1} \otimes K(Z,1)) \otimes \ldots \otimes (b_{i_r} \otimes K(Z,1))$ for $i_1 < \ldots < i_r$. Also $\wedge^r X \otimes \wedge^r K(Z,1)$ is the direct sum of complexes $(b_{i_1} \ldots b_{i_r}) \otimes \wedge^r K(Z,1)$ for $i_1 < \ldots < i_r$. The map η then restricts to maps

$$(b_{i_1} \ldots b_{i_r}) \otimes \wedge^r K(Z,1) \to (b_{i_1} \otimes K(Z,1)) \otimes \ldots \otimes (b_{i_r} \otimes K(Z,1))$$

which are equivalent to

$$g: \wedge^r K(Z,1) \to \otimes^r K(Z,1)$$

Thus by 7.5 (iv), η_* is an isomorphism.

7.7. Proof of 7.2. It suffices to show that for any s.s. free abelian group X

$$\nu_*: \pi_*(\Gamma^r X \boxtimes SP^r K(Z,2)) \to \pi_* SP^r(X \boxtimes K(Z,2))$$

is an isomorphism. For this it again suffices to assume X is a free abelian group. Choose an ordered basis $(b_1, b_2, \ldots)$ for X. Then by 4.6 $SP^r(X \boxtimes K(Z,2))$ is the direct sum of complexes $SP^{r_1}(b_{i_1} \boxtimes K(Z,2)) \boxtimes \ldots \boxtimes SP^{r_k}(b_{i_k} \boxtimes K(Z,2))$ for $i_1 < \ldots < i_k$ and $r_1 + \ldots + r_k = r$ with each $r_i > 0$. Also $\Gamma^r X \boxtimes SP^r K(Z,2)$ is the direct sum of corresponding complexes $(\gamma_{r_1}(b_{i_1}) \ldots \gamma_{r_k}(b_{i_k})) \boxtimes SP^r K(Z,2)$. The map ν then restricts to maps

$$(\gamma_{r_1}(b_{i_1}) \ldots \gamma_{r_k}(b_{i_k})) \boxtimes SP^r K(Z,2) \to$$

$$SP^{r_1}(b_{i_1} \boxtimes K(Z,2)) \boxtimes \ldots \boxtimes SP^{r_k}(b_{i_k} \boxtimes K(Z,2))$$

which are equivalent to $f^{r_1, \ldots, r_k}$. Thus by 7.5 (iii), ν_* is an isomorphism.

§8. Rational Functors

Let $T: \mathcal{A} \to \mathcal{Q}$ be a functor of finite degree $\leq r$, where $\mathcal{Q}$ is the category of vector spaces over the rational numbers Q. As a step toward determining derived functors of T, we prove the following.

Proposition 8.1. *There exists a unique decomposition* $T = \sum_{i=1}^{r} T^{(i)}$ *such that* $T^{(i)}: \mathcal{A} \to \mathcal{Q}$ *is i-homogeneous. Furthermore, suppose* $U = \sum_{i=1}^{r} U^{(i)}: \mathcal{A} \to \mathcal{Q}$ *where* $U^{(i)}$ *is i-homogeneous. If* $f: T \to U$ *is a natural transformation, then* $f = \sum_{i=1}^{r} f^{(i)}$ *where* $f^{(i)}: T^{(i)} \to U^{(i)}$.

Proof. The symmetric group $S(r)$ operates on the cross-effect $T_r(X,\ldots,X) \subset T(X+\ldots+X)$ for $X \in \mathcal{A}$ by permuting the copies of X. Let $T^{(r)}(X)$ be the quotient of $T_r(X,\ldots,X)$ by the relations $t \sim \sigma(t)$ for $t \in T_r(X,\ldots,X)$ and $\sigma \in S(r)$. The functor $T^{(r)}: \mathcal{A} \to \mathcal{Q}$ is r-homogeneous. Define $c: T^{(r)} \to T$ by the condition that the composition

$$T_r(X,\ldots,X) \xrightarrow{j} T^{(r)}X \xrightarrow{c} TX$$

equals the codiagonal map, where j is the quotient map. Let $d: T \to T^{(r)}$ be the composition

$$TX \xrightarrow{\Delta} T_r(X,\ldots,X) \xrightarrow{j} T^{(r)}X$$

where Δ is the diagonal map. Then the composition

$$T^{(r)} \xrightarrow{c} T \xrightarrow{d} T^{(r)}$$

is multiplication by $r!$. Hence

$$T \approx (\text{Image } c) + (\text{Kernel } d)$$

Clearly $(\text{Image } c) \approx T^{(r)}$ is r-homogeneous and $(\text{Kernel } d)$ is of degree $\leq r-1$. The desired decomposition $T \approx \sum_{i=1}^{r} T^{(i)}$ can thus be constructed inductively.

To complete the proof of 8.1 it suffices to show that if $U,V: \mathcal{A} \to \mathcal{Q}$ are respectively i,j-homogeneous with $i \neq j$, then any natural transformation $f: U \to V$ is zero. Consider the doubling map $2: X \to X$ in $\mathcal{A}$. Since

$$f \circ V(2) = V(2) \circ f: U(X) \to V(X)$$

it follows that $2^i f = 2^j f$ so $f = 0$.

The following notion will be useful in §9.

<u>8.2. Antifunctors.</u> If $T: \mathcal{A} \to \mathcal{Q}$ is r-homogeneous, we shall define its <u>antifunctor</u> $T^d: \mathcal{A} \to \mathcal{Q}$ which is r-homogeneous and such that $T \approx T^{dd}$. Consider $T_r(X,\ldots,X)$ as an $S(r)$-module; and let $T^d(X)$ be the quotient of $T_r(X,\ldots,X)$ by the relation $\sigma(t) \sim (\text{sign } \sigma)t$ for $\sigma \in S(r)$ and $t \in T_r(X,\ldots,X)$. Then $T^d: \mathcal{A} \to \mathcal{Q}$ is the antifunctor of T.

<u>Example 8.3.</u> A pair of antifunctors is

$$Q \boxtimes SP^r: \mathcal{A} \to \mathcal{Q}$$

$$Q \boxtimes \wedge^r: \mathcal{A} \to \mathcal{Q}$$

§9. Derived Functors of Rational Functors

If $T: \mathcal{A} \to \mathcal{Q}$ is a functor with $T(0) = 0$, then an associated additive functor $T^{ad}: \mathcal{A} \to \mathcal{Q}$ is defined by $T^{ad}(G) = T(G)/(\text{Image } \nabla)$, where $\nabla: T_2(G,G) \to T(G)$ is the codiagonal.

Proposition 9.1. *Let $T: \mathcal{A} \to \mathcal{Q}$ be of finite degree and $j: T \to T^{ad}$ be the quotient map. Then $j_*: L^s{}_*T \to L^s{}_*T^{ad}$ is an isomorphism.*

Proof. By 8.1 there is a decomposition $T = \sum_{i=1}^{r} T^{(i)}$ where $T^{(i)}$ is i-homogeneous, and the map $j: T \to T^{ad}$ is equivalent to the projection $T \to T^{(1)}$. By 5.5 and 5.6, $L^{(s)}{}_*T^{(i)} = 0$ for $i \geq 2$.

Remark 9.2. By 9.1 the stable derived functors of $T: \mathcal{A} \to \mathcal{Q}$ equal the classical left derived functors of T^{ad}. However, this does not hold for functors of infinite degree (see 6.6). If $U: \mathcal{A} \to \mathcal{A}$ is a functor with $U(0) = 0$, then

$$Q \otimes (L^s{}_i U) \approx L^s{}_i (Q \otimes U) : \mathcal{A} \to \mathcal{Q}$$

so 6.5 follows from 9.1.

To determine *unstable* derived functors, one combines 8.1 with the following.

Theorem 9.3. If $T: \mathcal{A} \to \mathcal{Q}$ *is an r-homogeneous functor which preserves direct limits, then:*

(i) $L_q T(\cdot, n) = 0$ *for* $q \neq rn$

(ii) $L_{rn} T(\cdot, n) \approx T$ *for* n *even*

(iii) $L_{rn} T(\cdot, n) \approx T^d$ *for* n *odd, where* T^d *is the anti-functor* (8.2) *of* T.

Proof. Part (i) follows from 5.5.

We construct a natural equivalence $\alpha: L_0 T(\cdot, 0) \to T$. Let $K(G,0)$ be as in 5.1, let $k(G)$ be the s.s. abelian group consisting of G in each dimension with s.s. operators equal to the identity, and let $h: K(G,0) \to k(G)$ be an s.s. homomorphism inducing the identity map on homotopy groups. Then

$$T(h)_*: \pi_0 TK(G,0) \to \pi_0 Tk(G)$$

gives

$$\alpha: L_0 T(G,0) \to T(G)$$

which is clearly an isomorphism for G free abelian. Using the quotient map $G \to G/G_t$, where G_t is the torsion subgroup of G, it is easily shown that α is an isomorphism for G finitely generated. By a direct limit argument this implies that α is a natural equivalence.

It now suffices to show that $L_{r(n+1)}T(\cdot, n+1)$ is the antifunctor of $L_{rn}T(\cdot, n)$ for $n \geq 0$. Let

$$\sigma_r: \pi_* T_r(K(G,n),\ldots,K(G,n)) \to \pi_{*+r} T_r(K(G,n+1),\ldots,K(G,n+1))$$

be the composition of suspension homomorphisms in successive variables from right to left. Then σ_r is an isomorphism since T_r is additive in each variable. Furthermore for $\tau \in S(r)$ the diagram

$$\begin{array}{ccc}
\pi_* T_r(K(G,n),\ldots,K(G,n)) & \xrightarrow{\sigma_r} & \pi_{*+r} T_r(K(G,n+1),\ldots,K(G,n+1)) \\
\downarrow \tau_* & & \downarrow \tau_* \\
\pi_* T_r(K(G,n),\ldots,K(G,n)) & \xrightarrow{\sigma_r} & \pi_{*+r} T_r(K(G,n+1),\ldots,K(G,n+1))
\end{array}$$

commutes for τ even and anticommutes for τ odd. Clearly

$$(L_* T(\cdot,n))_r(G,\ldots,G) \approx \pi_* T_r(K(G,n),\ldots,K(G,n))$$

For any r-homogeneous functor $U: \mathcal{A} \longrightarrow \mathcal{Q}$, $U(G)$ is the quotient of $U_r(G,\ldots,G)$ by the action of $S(r)$. These facts easily imply that $L_{r(n+1)}T(\cdot,n+1)$ is the antifunctor of $L_{rn}T(\cdot,n)$.

<u>Example 9.4.</u> By 8.3 and 9.3

$$Q \otimes (L_{rn} SP^r(G,n)) \begin{cases} Q \otimes SP^r(G) & \text{for } n \text{ even} \\ Q \otimes \Lambda^r(G) & \text{for } n \text{ odd} \end{cases}$$

$$Q \otimes (L_{rn} \Lambda^r(G,n)) \begin{cases} Q \otimes \Lambda^r(G) & \text{for } n \text{ even} \\ Q \otimes SP^r(G) & \text{for } n \text{ odd} \end{cases}$$

since for any functor $U: \mathcal{A} \longrightarrow \mathcal{A}$ with $U(0) = 0$

$$Q \otimes (L_* U(G,n)) \approx L_*(Q \otimes U)(G,n).$$

<u>Corollary 9.5.</u> <u>If</u> $T: \mathcal{A} \to \mathcal{A}$ <u>is of finite degree and preserves direct limits, then</u> $L_q T(G,n)$ <u>is a torsion group unless</u> q <u>is divisible by</u> n.

<u>Proof.</u> Apply 8.1 and 9.3 to the functor $Q \otimes T: \mathcal{A} \longrightarrow \mathcal{Q}$.

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