Formal Groups of Height One

The title is defined

Canonical Example: The multiplicative group

$$\mathbb{G}_m(X,Y) = X + Y + XY \quad \text{over } A$$

$$[n]_{\mathbb{G}_m}(X) = (1+X)^n - 1 \qquad \text{by } \underline{\text{induction}}:$$

$$\left([n]_{\mathbb{G}_m}(X) = \mathbb{G}_m([n-1]_{\mathbb{G}_m}(X), X) \quad , \quad [0]_{\mathbb{G}_m}(X) = 0. \right)$$

$\vdash$: A of char $p \Rightarrow [p]_{\mathbb{G}_m}(X) = (1+X)^p - 1 = X^p$.

Definition 1: Let A = ring of char p
F groupلaw over A, such that
$[p]_F(X) = u X^p + \cdots$, $u \in A^\times$.
Then F is of height 1.

Explanation / Lemma: A of char p, F grouplaw over $A \Rightarrow$
$[p]_F(T) = \varphi(T^p)$, $\varphi \in A[[T]]$.

~~Proof~~: Suppose $f : F \to G$ is a homomorphism such that $f(T) = 0 \cdot T$ + higher order terms, i.e. $f'(0) = 0$. (Example $[p]_F$). Then $f(T) = f_0(T^p)$

Suff. to prove $f'(T) = 0$ (since A of char p).
(i.e. $f(T) = \sum_{i \geq 1} f_i T^{pi}$).

But $f F(X,Y) = G(fX, fY)$ differentiate wrt Y

$$f' F(X,Y)\, F_2(X,Y) = G_2(fX, fY)\, f'Y \qquad \text{(chain rule)}$$

let $Y = 0$: $f'(X)\, F_2(X,0) = G_2(fX, 0) \cdot 0$ $\quad (f'(0) = 0)$

But $F_2(X,0) = 1 + \cdots$ is a unit, so $f'(X) = 0$.

Endomorphism ring is computed

Prop 1. F of height one over k = field of char p. Then the natural map $\mathbb{Z} \to \mathrm{End}_k(F)$ extends to an isomorphism $\hat{\mathbb{Z}}_p \to \mathrm{End}_k(F)$.

Canonical Example $\quad [n]_{\mathbb{G}_m}(T) = \sum_{m>0}^{n} \binom{n}{m} T^m$.

This is defined for n any p-adic integer; for the residue (mod p) of $\binom{n}{m}$ is defined for $m \in \mathbb{Z}$, $n \in \hat{\mathbb{Z}}_p$.

Thus $[\cdot]_{\mathbb{G}_m} : \mathbb{Z} \to \mathrm{End}_k \mathbb{G}_m$ certainly

extends to an injection $[\cdot]_{\mathbb{G}_m} : \hat{\mathbb{Z}}_p \to \mathrm{End}_k \mathbb{G}_m$

Sketch proof:

1) $\mathrm{End}_k(F)$ is an integral domain, since k is.
2) $\mathbb{Z} \to \mathrm{End}_k(F)$ is injective. (otherwise $[n]_F = 0$ for some n; WMA $n = p$, but $[p]_F = uT^p + \cdots$ by height 1 assumption.
3) $\mathrm{End}_k F$ is complete in the power-series topology; $\therefore$ in p-adic top by 2)
4) $\therefore$ $\mathrm{End}_k F$ is free $\hat{\mathbb{Z}}_p$-module.

Hence it suffices to prove $\mathrm{End}_k F / p\,\mathrm{End}_k F$ is

one-dimensional over $\mathbb{F}_p$.

simplifying assumption. $\exists v \in k$, $v^{p-1} = u$ (removable.

Change variables $T \mapsto v^{-1}T$. Get a fgp law
$\bar{F}(X,Y) = vF(v^{-1}X, v^{-1}Y)$, with
$$[p]_{\bar{F}}(T) = v[p]_F(v^{-1}T) = v^{1-p}u\, T^p + \cdots$$
$$= T^p + \cdots$$

Now an element of $\mathrm{End}_k F / p\,\mathrm{End}_k F$ is represented by a finite truncation of its power-series:

$$f : \in \mathrm{End}_k F \longmapsto \underbrace{\sum_{i=1}^{p-1} f_i T^i}_{(\text{mod deg } p)} \in E/pE.$$

Step 6): $f \in \mathrm{End}_k F \Rightarrow f[p]_F = [p]_F f$

$$\Rightarrow \sum_{i=1}^{p-1} f_i T^{pi} = \left(\sum_{i=1}^{p-1} f_i T^i\right)^p \text{ mod deg } p$$

$\Rightarrow f_i = f_i^p \Rightarrow f_i \in \mathbb{F}_p$ (a priori, only $f_i \in k$).

Step 6) Hence suff. to show: $f \in \mathrm{End}_k F$, $f'(0) = 0 \Rightarrow f = 0 \bmod p\,\mathrm{End}_k F$.

(For then $\mathrm{End}/p\mathrm{End} \xrightarrow{\text{injects}} \mathbb{F}_p \subset k$)
$$f \longmapsto f'(0)$$

But we know (see Explanation/ Lemma)
$f'(0) = 0 \Rightarrow f(X) = f_0(X^p)$.

The invariant is defined. Let F be a formal group law over $\mathbb{F}_p$.

Prop 2: The map $\mathcal{F}(X) = X^p \in \mathrm{End}_{\mathbb{F}_p} F$.

Proof: Suff. to show

$$\mathcal{F} F(X,Y) = F(\mathcal{F}X, \mathcal{F}Y)$$

but

$$\left(\sum a_{ij} X^i Y^j\right)^p \qquad \sum a_{ij} (X^p)^i (Y^p)^j$$

and

$$\sum a_{ij}^p X^{pi} Y^{pj} = \sum a_{ij} X^{pi} Y^{pj}$$

since $(a_{ij}^p = a_{ij}$ since $a_{ij} \in \mathbb{F}_p)$. //.

DEFINITION:

The element $\mathcal{F} \in \mathrm{End}_{\mathbb{F}_p} F$ is the Frobenius invariant of F.

Suppose F is of height 1. Then we have a canonical isomorphism $\hat{\mathbb{Z}}_p \xrightarrow{[\cdot]_F} \mathrm{End}_k F$ by prop 1. Consequently

(F; ht. 1) $\mathcal{F}_F \in \hat{\mathbb{Z}}_p$ (The Frobenius invariant of F is defined by $[\mathcal{F}_F]_F = \mathcal{F}$

(Since $[p]_F(T) = \varphi(\mathcal{F}(T))$, with $\varphi(T) = uT + \cdots$ we know that $\mathcal{F}_F = u_F^{-1} \cdot p$, with $u_F \in \hat{\mathbb{Z}}_p^\times$, a p-adic unit

Proposition 3.

Let $u \in \hat{\mathbb{Z}}_p^\times$ be given. There is then a formal group law of height one over $\mathbb{F}_p$, with Frobenius invariant $= u^{-1}p$.

Proof:

§4: Examples are constructed

LEMMA (LUBIN-TATE):

Let $L(X_1, \ldots, X_m) \equiv L(\underline{X})$ be a linear form $\sum_{i=1}^{m} l_i X_i$, $l_i \in \hat{\mathbb{Z}}_p$

Let $\pi \in \hat{\mathbb{Z}}_p$, $\pi = u^{-1}p$ with $u \in \hat{\mathbb{Z}}_p^\times$,
$[\pi](X) = \pi X + X^p$.

Then $\exists$ unique $\phi(\underline{X}) \in \hat{\mathbb{Z}}_p[[X_1, \ldots, X_m]]$, such that

1) $\phi(\underline{X}) = L(\underline{X})$ mod deg 2
2) $[\pi]\phi(\underline{X}) = \phi(\underline{[\pi](X)})$.

Proof of lemma: We construct ϕ by successive approximations. Suppose ϕ_n is defined, such that 1), 2) hold mod deg $n+1$. Write $\phi_{n+1} = \phi_n + \phi_{(n+1)}$, where $\phi_{(n+1)}$ is a homogeneous fn. of $X_1, \ldots, X_m$ of deg $n+1$.

Write $E_{n+1}(\underline{X}) = [\pi] \circ \phi_{n+1} - \phi_{n+1} \circ [\pi]$ mod deg $n+2$

Since $[\pi] \circ (\phi_n + \phi_{(n+1)}) = [\pi] \circ \phi_n + \pi \cdot \phi_{(n+1)}$ mod deg $n+2$

$$\phi_{n+1}\circ[\pi] = \phi_n\circ[\pi] + \phi_{(n+1)}\circ[\pi]$$

$$= \phi_n\circ[\pi] + \pi^{n+1}\cdot\phi_{(n+1)} \mod \deg\ n+2$$

$$\Rightarrow\quad E_{n+1} = \pi(\pi^n - 1)\,\phi_{(n+1)}.$$

Hence we <u>define</u> $\phi_{(n+1)}(-) = (1-\pi^n)^{-1}(\pi^{-1}E_{n+1})$;

This will lie in $\hat{\mathbb{Z}}_p[[\underline{X}]]$ provided $E_{n+1} \equiv 0 \mod$

But reducing over $\mathbb{F}_p$, we have $[\pi](X) = X^p$,

and since $\bar{\phi}_n(X)^p = \bar{\phi}_n(X^p)$ $\quad [\phi_n \in \hat{\mathbb{Z}}_p[[\underline{X}]] \Rightarrow \bar{\phi}_n \in \mathbb{F}_p[[\underline{X}]]$

we know $\bar{E}_{n+1} = 0 \pmod p$.

Now $\phi = \lim_{n\to\infty} \phi_n$ is defined — QED —

<u>Construction of The Examples</u>

Given $\pi = u^{-1}p$,

let $F_\pi(X,Y) \in \hat{\mathbb{Z}}_p[[X,Y]]$ be the <u>unique</u> extension of the linear form $X+Y$ such that $F_\pi([\pi](X), [\pi](Y)) = [\pi]\, F_\pi(X,Y)$

and

let $[a]_\pi(X) \in \hat{\mathbb{Z}}_p[[X]]$ be the <u>unique</u> extension of the linear form aX $(a \in \hat{\mathbb{Z}}_p)$ such

That $[a]_\pi \circ [\pi] = [\pi] \circ [a]_\pi$.

(Note: by uniqueness, $[\pi]_\pi = [\pi]$.)

<u>Prop 4</u>: F_π is a formal group (over $\hat{\mathbb{Z}}_p$!), and

$$[\cdot]_\pi : \hat{\mathbb{Z}}_p \to \mathrm{End}_{\hat{\mathbb{Z}}_p}(F_\pi) \text{ is an injection}$$

<u>Proof</u>:

1) $F_\pi(X, F_\pi(Y,Z))$ and $F_\pi(F_\pi(X,Y),Z)$ are extensions $\phi_0(X,Y,Z)$ and $\phi_1(X,Y,Z)$ of $X+Y+Z$ over $\hat{\mathbb{Z}}_p[[X,Y,Z]]$ satisfying (by def. of F_π) $\phi([\pi](X), [\pi](Y), [\pi](Z))$
$= [\pi](\phi(X,Y,Z))$.

Hence they are equal by the uniqueness.

2) Similarly for
$[a]_\pi F_\pi(X,Y)$ and $F_\pi([a]_\pi(X), [a]_\pi(Y))$ which extend $aX + aY$.

<u>Corollary</u>, Let $\overline{F}_\pi$ be the reduction of F_π over $\mathbb{F}_p$

Then $\overline{F}_\pi$ is of height one, with Frobenius invariant $= \pi$.

<u>Proof</u>: Evidently $[\pi](X) = [\pi]_\pi \in \mathrm{End}_{\hat{\mathbb{Z}}_p} \overline{F}_\pi$ maps
$= \pi X + X^p$

to $\underset{(\mathrm{mod}\, p)}{[\pi](X)} = X^p = \mathcal{F} \in \mathrm{End}_{\mathbb{F}_p} \bar{F}_\pi$. Hence if we know $[p]_\pi(X) = uX^p$ mod deg $p+1$, with $u \in \mathbb{F}_p^\times$, we are through.

But $[p]_\pi = [u]_\pi [\pi]_\pi$ since $u^{-1}p = \pi$, and $[\cdot]_\pi$ is a ring homomorphism (by uniqueness part of Lubin-Tate lemma again). Hence

$$[p]_\pi(X) = \bar{u} X^p + \cdots,$$

with $\bar{u}$ the mod p reduction of $u \in \hat{\mathbb{Z}}_p^\times$, since $[u]_\pi(X) = uX + \cdots$ by definition.

—//—

Corollary.

5 The moral is drawn

All possible values of the Frobenius invariant are assumed.

For a proof that (iso classes of) formal group laws over $\mathbb{F}_p$ (or $\hat{\mathbb{Z}}_p$) are uniquely defined by their Frobenius invariant, see W.L. Hill, Inventiones Math. 12 (1971) p321–336, in particular Theorem E, p330.

A more explicit description of the formal group laws just defined is possible. Let $l_\pi(X) = \sum_{n \geq 0} \pi^{-n} X^{p^n}$. Then

$$F_\pi(X,Y) = \ell_\pi^{-1}(\ell_\pi(X) + \ell_\pi(Y)),$$
$$[a]_\pi(X) = \ell_\pi^{-1}(a\,\ell_\pi(X)).$$

There is a fancier way to describe F_π. Let $\alpha = u + pu^{-1}$. We define the

ζ- function of F_π, $\zeta_\pi(s) = (1 - \alpha p^{-s} + p^{1-2s})^{-1}$.

Its Mellin transform ($\sum a_n n^{-s} \mapsto \sum' \frac{a_n}{n} X^n$)

The author goes off the deep end

is ~~as~~ the logarithm for a formal group (canonically) isomorphic to F_π. ζ_π is the ζ- fn. of a certain elliptic curve over $\mathbb{F}_p$ (one whose Weil # $= \alpha$), cf. W.C. Waterhouse, Ann. Sci. of ENS 1969(2) 521–560 particularly §4.

1)

Very Rough Notes

About L's Thm.

H machine for producing ring spectra is recalled.

Suppose $\rho: U^*(pt) \longrightarrow Z_{(p)}$ ringhom $\ni$ $\rho(\mathbb{C}P(p-1)) \in$ units.

(Then ρ is a hom. of height 1)

$\left\{ \text{General defn: } \begin{array}{ll} \rho(V_{p^i-1}) \equiv 0 \mod p, & 0<i<n \\ \not\equiv 0 \mod p & i=n \end{array} \right\} \Leftrightarrow \rho$ is of height $\underline{\underline{n}}$.

Application of L's thm:

$X \mapsto U^*(X) \underset{\rho}{\otimes} Z_{(p)}$ is an (ungraded) coh. theory

$K_\rho(X; Z_{(p)}) \ni$

1) $\exists$ canonical iso $K_\rho(S^{2(p-1)} \wedge X; Z_{(p)}) \xrightarrow{\approx} K_\rho(X; Z_{(p)})$

2) K_ρ is multiplicative.

Classification: Associated to K_ρ is the formalgrouplaw $F_\rho(X,Y)$ over $Z_{(p)}$ induced by ρ; $= \sum \rho(c_{ij}) X^i Y^j$,

$$\Delta_\rho(T) = F_\rho(T\otimes 1, 1\otimes T) \in K_\rho(\mathbb{C}P(\infty)).$$

Proposition 1: Two homomorphisms ρ, ρ' of ht. 1 give iso coh. theories (mult.) iff their formal group laws (over $Z_{(p)}$) are iso.

2)

i.e. let $f : F_\rho \longrightarrow F_{\rho'}$ be an iso of fgplaws,

$$f \in \mathbb{Z}_{(p)}[[T]], \quad f(T) = \sum f_i T^{i+1} \quad f_0 = 1$$

$$\in \Gamma_0(\mathbb{Z}_{(p)})$$

$$f : F_\rho \Rightarrow F_{\rho'} : f\, F_\rho(X,Y) = F_{\rho'}(fX, fY).$$

auxilliary information about LN. op's.

Last Tues (Lect IV) I showed that the Landweber-Novikov operation

$$s_f = \sum s_\alpha f^\alpha \in S^* \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)}, \quad f^\alpha = f_1^{\alpha_1} \cdots f_n^{\alpha_n} \cdots$$

has the properties

1) $s_f : U^*(X)_{(p)} \longrightarrow U^*(X)_{(p)}$ is a ringhom

2) $s_f(T) = f(T)$ if $T \in U^2(\mathbb{C}P(\infty))$

<u>Lemma 1</u> The diagram

$$\begin{tikzcd} U^*_{(p)}(pt) \arrow[dr, "\rho'"] \arrow[d, "s_f"] & \\ U^*_{(p)}(pt) \arrow[r, "\rho"] & \mathbb{Z}_{(p)} \end{tikzcd}$$

commutes.

<u>Proof</u>: Suff. to show $\rho s_f(c_{ij}) = \rho'(c_{ij})$, since the coefficients c_{ij} of the univ. gplaw $F_c(X,Y) = \sum c_{ij} X^i Y^j$ generate $U^*(pt)$.

3)

But I showed last time

$$\sum s_f(c_{ij})\, X^i Y^j = \bar{f}^{\,1} F_c(fX, fY). \quad \text{Consequently}$$

$$\sum \rho s_f(c_{ij}) X^i Y^j = \bar{f}^{\,1} (\rho F_c)(fX, fY)$$ [$\rho : U^*(pt)_{(p)} \to Z_{(p)}$ is $Z_{(p)}$ - linear]

$$= \bar{f}^{\,1} F_\rho(fX, fY)$$

$$= F_{\rho'}(X, Y) \quad \text{by assumption.}$$

Hence $\rho s_f(c_{ij}) = \rho'(c_{ij})$ //

Corollary 1:

s_f induces a map of $U^*(X) \underset{\rho'}{\otimes} Z_{(p)} \to U^*(X) \underset{\rho}{\otimes} Z_{(p)}$

of coh. Theories.

[Proof: Recall

$0 \to I_{\rho'} \to U^*(X) \underset{Z}{\otimes} Z_{(p)} \to U^*(X) \underset{\rho'}{\otimes} Z_{(p)}$ is exact by defn,

where I is the submodule generated (over $Z_{(p)}$) by

$x u \otimes 1 - x \otimes \rho'(u)$ $\qquad x \in U^*(X),\ u \in U^*(pt).$

$\therefore$ Suff. to prove $s_f : I_{\rho'} \to I_\rho$: but

$$s_f(xu \otimes 1 - x \otimes \rho'(u)) = \underset{\text{(by mult.)}}{s_f(x)\cdot s_f(u) \otimes 1} - s_f(x) \otimes \rho'(u)$$

and $\rho'(x) = \quad \rho S_f(u)$ by preceding lemma

$$\therefore \quad S_f(xu\otimes 1 - x\otimes \rho'(u)) = S_f(x)\cdot S_f(u)\otimes 1 - f(x)\otimes \rho S_f(u) \in I_p.$$

//

Now $S_f : K_p \longrightarrow K_{p'}$ is iso on groundring. QED.

Fgplaws over $\hat{Z}_p$ are classified by $\alpha \in \hat{Z}_p^\times$, α^p = Frobenius inv., cf. "C. i fgplaws"

[To every ht. 1 fgplaw F over $Z_{(p)}$, $\exists$ corresponding ringspectrum X_F

$$X_F \xrightarrow{\sim} \Omega^{2(p-1)} X_F \qquad \pi_i(X_F) = Z_{(p)} \quad i \equiv 0 \bmod 2(p-1)$$
$$= 0 \quad \text{otherwise}$$

//

So it is asserted that the mult. structure is not necessary.

Claim. This classification really doesn't depend on the ringstructure.

Theorem 1: Let X be a space with a homotopy-eq $X \longrightarrow \Omega^{2(p-1)}X$,

$$\pi_i(X) = \mathbb{F}_p \qquad i \equiv 0 \bmod 2(p-1)$$
$$= 0 \qquad \text{otherwise.}$$

Then $\exists$ canonically associated isomorphism class of fgplaws over $\mathbb{F}_p$.

[If this iso. class is of height one, then X can be explicitly constructed via Landweber. But this is not nec. for obvious reasons: $X = \prod_{n \in \mathbb{Z}} K(\mathbb{F}_p, 2(p-1)n)$.

has the *additive* group as associated gplaw

problem is avoided + postponed

(For periodic spectra $\exists$ elementary analogue of Wilson's Thm: (at least over $\mathbb{F}_p$)
$\forall$ thing splits, perhaps after extending $\mathbb{F}_p$ to $\overline{\mathbb{F}}_p$, to a product of things with primitive period $2(p^i-1)$ (at least if p is odd.)
Cf. forthcoming Appendix on Small Coh. Theories

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Sketch Proof of Thm 1.

An ignorant ringstructure is constructed.

Lemma 2. The Atiyah-Hirzebruch spectral sequence
$$H^*(MU; h_X^*(pt)) \Longrightarrow h_X^*(\underline{MU}) \quad \text{collapses.}$$

[Here $h_X^i(Y) = [Y, \Omega^i X]$, $i \leq 0$, (extended by periodicity)].

Proof: $h_X^*(pt)$ is concentrated in even dimensions. //

Pick $\mu \in h_X^0(\underline{MU})$ which lifts 1, i.e.
$$1 = gr\,\mu \in gr\, h_X^0(\underline{MU}) = H^*(\underline{MU}; h^*(pt))^{\dim 0}.$$

Corollary 2: $\mu : U^*(pt) \longrightarrow h_X^*(pt)$ (extended to a transf of Theories by periodicity) is onto.

Proof: It is so in dimension zero, by assumption, and it extends by periodicity. //

Corollary 3. $h_X^*(\mathbb{C}P(\infty))$ is a free $\mathbb{F}_p$-module with basis $1 \underset{\text{def}}{\equiv} \mu(1), \mu(T), \mu(T^2), \ldots$, where $T^i \in U^{2i}(\mathbb{C}P(\infty))$.

Proof: The AH sseq. $h_X^*(\mathbb{C}P(\infty)) \Leftarrow H^*(\mathbb{C}P(\infty); h_X^*(pt))$ also collapses. /

Def $h_X^*(\mathbb{C}P(\infty))$ is an $\mathbb{F}_p$-algebra, with product defined by

$$\mu(T^i) \cdot \mu(T^j) = \mu(T^{i+j}).$$

(Since these form a basis, we can define it as we please). //

Similarly for $h_X^*(\mathbb{C}P(\infty) \times \mathbb{C}P(\infty))$: it has a natural basis as $\mathbb{F}_p$-module given by $\{\mu(T^i \otimes T^j) : i, j \geq 0\}$. We define a product by $\mu(T^i \otimes T^j) \cdot \mu(T^a \otimes T^b) = \mu(T^{i+a} \otimes T^{j+b})$.

Consequently

$$\begin{array}{ccc} U^*(\mathbb{C}P(\infty)) & \xrightarrow{\Delta} & U^*(\mathbb{C}P(\infty) \times \mathbb{C}P(\infty)) \\ \downarrow \mu & & \downarrow \mu \\ h_X^*(\mathbb{C}P(\infty)) & \longrightarrow & h_X^*(\mathbb{C}P(\infty) \times \mathbb{C}P(\infty)) \end{array}$$

commutes, and $h_X^*(\mathbb{C}P(\infty))$ carries a hopf alg. (formal gp) structure $F_\mu(\mu(T \otimes 1), \mu(1 \otimes T)) = \Delta\mu(T)$.

7)

(Evidently F_μ is the formal gp. structure induced by $\mu : U^*(pt) \longrightarrow h_X^*(pt) \longrightarrow \mathbb{F}_p$ (by periodicity).)

It suffices to prove

__Lemma__ 3. Let $\mu, \mu' \in h_X^0(\underline{MU})$ with $\operatorname{gr}\mu = \operatorname{gr}\mu' = 1$.

Then $\exists$ formal series $f \in \Gamma_0(\mathbb{F}_p)$ $\ni$

$$\begin{tikzcd} U^*(\mathbb{C}P(\infty)) \arrow[r, "\mu'"] \arrow[d, "s_f"] & h_X^*(\mathbb{C}P(\infty)) \\ U^*(\mathbb{C}P(\infty)) \arrow[ur, "\mu"'] & \end{tikzcd}$$

__commutes__.

— Notice its consequence: —

__Corollary__ 4: The formal group F_μ is isomorphic to the formal group $F_{\mu'}$; i.e. the iso. type of F_μ is dependent only on h_X.

__Proof__: Notice that the analogous diagram

$$\begin{tikzcd} U^*(\mathbb{C}P(\infty) \times \mathbb{C}P(\infty)) \arrow[r, "\mu'"] \arrow[d, "f"] & h_X^*(\mathbb{C}P(\infty) \times \mathbb{C}P(\infty)) \\ U^*(\mathbb{C}P(\infty) \times \mathbb{C}P(\infty)) \arrow[ur, "\mu"'] & \end{tikzcd}$$

__also__ commutes:

$$\begin{aligned} \mu s_f(T^i \otimes T^j) &= \mu\left[f(T)^i \otimes f(T)^j\right] \\ &= f(\mu T)^i \otimes f(\mu T)^j \\ &= (\mu' T)^i \otimes (\mu' T)^j = \mu'(T^i \otimes T^j). \end{aligned}$$

Hence the Hopf alg's induced by μ, μ' are iso. : QED.

Proof of Lemma 3:

$\mu'(T)$, $\mu(T) \in h_X^*(\mathbb{C}P(\infty))$, and $\mu'(T) \equiv \mu(T)$ (mod stuff of higher filtration in nat'l grading on $h^*(\mathbb{C}P(\infty))$.

Consequently $\mu'(T) = \sum_{i \geq 0} f_i \mu(T^{i+1})$, $g_0 = 1.$,

since $\mu(T^i)$ form a basis by Cor 3.

Consider

$$\begin{array}{ccc} U^*(\mathbb{C}P(\infty)) & \xrightarrow{\mu'} & h_X^*(\mathbb{C}P(\infty)) \\ \downarrow s_f & \nearrow \mu & \\ U^*(\mathbb{C}P(\infty)) & & \end{array}$$

(by def. in case μ, μ').

Since s_f, μ, μ' are ringhom's, it suffices to check commutativity on generators. But

$$\begin{array}{ccc} T \in U^2(\mathbb{C}P(\infty)) & \longrightarrow & \mu'(T) \\ \downarrow & & \| \text{ by defn of } f. \\ \sum f_i T^{i+1} & \xrightarrow{\mu} & \sum_{i \geq 0} f_i \mu(T^{i+1}) \end{array}$$

//

Thus let X be a space as in Theorem 1; suppose that its associated formal group law (over $\mathbb{F}_p$) has height one.

A Theorem of Hill/Cartier asserts that such an iso. class has a <u>canonical</u> lifting to a fgplaw over $\hat{\mathbb{Z}}_p$.

Hence we can find

1) a fgplaw F of height 1 over $\hat{\mathbb{Z}}_p$, classified by $\rho : U^*(pt) \longrightarrow \hat{\mathbb{Z}}_p$

2) $\mu \in h_X^*(\underline{MU})$ with $\mathrm{gr}\,\mu = 1$,

such that the diagram

$$\begin{tikzcd} U^*(pt) \arrow[r, "\rho"] \arrow[d, "\mu_{(pt)}"] & \hat{\mathbb{Z}}_p \arrow[d] \\ h_X^*(pt) \arrow[r, "\simeq"] & \mathbb{F}_p \end{tikzcd}$$

<u>commutes</u>.

It follows that $\exists$ <u>factorization</u> of μ

$$\begin{tikzcd} U^*(-) \arrow[r] \arrow[dr, "\mu"'] & K_\rho^*(-) \arrow[d] \\ & h_X^*(-) \end{tikzcd}$$

[since $\mu(ux) = \rho(u)\mu(x)$, $\forall x \in U^*(X)$, $u \in U^*(pt)$.]

<u>Consequently</u> $K_\rho^*(-) \longrightarrow h_X^*(-)$ is the natural mod p reduction, i.e. K_ρ "lifts h_X^* to a theory over $\hat{\mathbb{Z}}_p$, with a <u>ringstructure</u>; consequently h_X^* also has a ringstructure.

Lect. VI : Localization, continued.

more or less all philosophy.

<u>Part I</u>: The sheaves get reformulated

§1. U^* as a sheaf over the cat. of f.gps.

§2. Cartier's conj:

a) Kraines - Segal coh. theory, for example

b) Speculations about alg. K-Theory

<u>Part II</u>: Structure of U (page 7 ..)

§1. Prelim's about orbits

§2. Canonical examples. What happens when the field is alg. closed

§3. The invariant ideal Theorem

for fanatics { §4. The stabilizers

§5 The number theoretic structure.

§6 Example: $U^*(BZ/p)$ [This isn't done very well, apologies].

Lecture V: Localization (part 1)

Recollections: We have now proved the "fundamental" results about complex cobordism; it's convenient to recall them here:

0) $\exists$ a cohomology functor. $X \longmapsto U^*(X)$, endowed with Gysin homomorphisms for proper, complex-oriented maps, and a multiplicative structure.

1) $U^*(\mathbb{C}P(\infty)) = U^*(pt)[[T]]$ is a Hopf algebra with diagonal $\Delta T = F_C(T\otimes 1, 1\otimes T)$, where $F_C(X,Y) = \Sigma c_{ij} X^i Y^j$ is a formal group law. This group law is <u>universal</u> (i.e. <u>any</u> formal group law over a commutative ring is a unique image of F_C) and the ring $U^*(pt)$, (which is generated by the $c_{i,j}$) is a polynomial ring on countably many variables $\mathbb{Z}[x_1, \ldots, x_n, \ldots]$.

2) $\exists$ Landweber-Novikov cohomology operations $s_{\underline{t}} : U^*(X) \longrightarrow U^*(X)[t_1, \ldots, t_n \ldots] = U^*(X)[\underline{t}]$ (where the t_i are indeterminates). The operations $s_{\underline{t}}$ are <u>multiplicative</u>:

$$s_{\underline{t}}(x \cdot y) = s_{\underline{t}}(x) \cdot s_{\underline{t}}(y) \quad \begin{bmatrix} x \in U^*(X) \\ y \in U^*(Y) \end{bmatrix}$$

as well as being natural ($s_{\underline{t}} f^* = f^* s_{\underline{t}}$) and stable ($s_{\underline{t}} \sigma = \sigma s_{\underline{t}}$, with $\sigma : U^*(X) \to U^*(X \times \mathbb{R})$ the suspension).

If $T \in U^2(\mathbb{C}P(\infty))$ is the canonical element, then $s_{\underline{t}}(T) = \sum_{i \geq 0} t_i T^{i+1}$.

We use the following formalism with the s_f:

Let $\Gamma_0(\mathbb{Z}) = \{f \in \mathbb{Z}[[T]] \mid f(T) = T + \cdots\}$ be the *group* (under composition) of invertib[le] formal power series in T, with integer coefficients. If $f \in \Gamma_0(\mathbb{Z})$, we write $f(T) = \sum_{i \geq 0} f_i T$[$^{i+1}$] with $f_i \in \mathbb{Z}$. ($f_0 = 1$)

N.B. s_f is an infinite sum; but if X is a finite cx, so $s_\alpha = 0$ on $U^*(X)$ if $|\alpha| \gg 0$. Hence s_f is a well-defined coh. operation. (a big space X ($\mathbb{C}P(\infty)$) there is a natural topology on $U^*(X)$, in which the sum converges.

Let $s_f = \sum s_\alpha f^\alpha$ be the sum over all multiindices α, where $f^\alpha = f_1^{\alpha_1} \cdots f_n^{\alpha_n} \cdots \in \mathbb{Z}$ and s_α is the α^{th} Landweber-Novikov operat[ion] (Cf. Lect IV, Cor. 2).

Evidently $s_f(T) = \sum_{i \geq 0} f_i T^{i+1} = f(T)$. Consequently

$$s_f s_g(T) = s_f(g(T)) = s_f\left(\sum g_i T^{i+1}\right)$$

$$= \sum g_i \left(s_f(T)\right)^{i+1} \quad \text{(since T[he] } s_f \text{ are } \underline{\text{mul[tiplicative]}}\text{)}$$

$$= g(s_f(T)) = g(f(T)).$$

Consequently $s_f \cdot s_g = s_{g \cdot f}$.

Thus $\Gamma_0(\mathbb{Z})$ acts naturally on $U^*(X)$, preserving the multiplicative structure.

3)

<u>Remark</u>. That $f \mapsto s_f$ is an <u>anti</u>automorphism is related to the traditional composition of functions fuckup. If $f, g \in \Gamma_0(R)$ define formal automorphisms $\underline{f}, \underline{g}$ of R, preserving the origin, then $\underline{f} \circ \underline{g} = \underline{g \circ f}$. I prefer to think of Γ_0 as "the group of formal coordinate transformations of the line" and things untwist themselves.

§2 Some definitions. The action of $\Gamma_0(\mathbb{Z})$ on $U^*(pt)$ is of great <u>arithmetic</u> complexity, and although it can now be described, the best results are available only over fields. Thus we will try to avoid the arithmetic as much as possible by working mod p.

We first establish a useful general result.

<u>Prop 1</u>. Let $U^*(pt) \xrightarrow{x} A$ be a ringhomomorphism. Let $F_x(X,Y) = \Sigma\, x(a_{ij}) X^i Y^j \in A[[X,Y]]$ be the induced groupplaw. Let $f \in \Gamma_0(A)$, $s_f = \Sigma\, s_\alpha f^\alpha \in S^* \underset{\mathbb{Z}}{\otimes} A$, $f^\alpha = f_1^{\alpha_1} \cdots f_n^{\alpha_n} \cdots \in A$.

Then $U^*(pt) \xrightarrow{s_f} U^*(pt) \underset{\mathbb{Z}}{\otimes} A \xrightarrow{x \otimes 1_A} A \otimes A \xrightarrow{\text{mult.}} A$ is a ringhomomorphism, inducing the formal group law

$$f F_x(\bar{f}^1(X), \bar{f}^1(Y)) \underset{\text{def}}{\equiv} F_x^f(X,Y).$$

<u>Proof</u>: $x \circ s_f\, F_C(X,Y) = x(f F_C(\bar{f}^1 X, \bar{f}^1 Y)) = f\,(x F_C)(\text{etc}$

—//—

<u>Def. 1</u> If A is a commutative ring, let

$U_A = \mathrm{Hom}_{(\text{Rings})}(U^*(pt); A)$ be the set of ringhomomorphisms from $U^*(pt)$ to A.

(Since $U^*(pt) = \mathbb{Z}[x_1, \ldots, x_n, \ldots]$, the set U_A is equivalent to the set of infinite sequences $(a_1, \ldots, a_n, \ldots)$, $a_i \in A$.)

<u>Def. 2</u>. If $x \in U_A$, $f \in \Gamma_0(A)$, then $f(x) : U^*(pt) \longrightarrow A$ is defined by

$$U^*(pt) \xrightarrow{s_f} U^*(pt) \otimes_{\mathbb{Z}} A \xrightarrow{x} A.$$

Hence there is a natural group action

$$\Gamma_0(A) \times U_A \longrightarrow U_A.$$

<u>Def. 3</u>, Let M be a complex-oriented closed manifold. Define a function $\check{M} : U_A \longrightarrow A$ as follows:

if $x \in U_A$, then $\check{M}(x) = x(M) \in A$: here we interpret x as a <u>genus</u> of closed complex-oriented manifolds.

We will topologize U_A, so that $\check{M}$ is a continuous function

<u>1st mystical principle</u> : A manifold is a <u>function</u> on U_A.

<u>Exercise</u>: Let $A = \mathbb{Q}$. Show that M bounds $\Longleftrightarrow$ $\check{M} = 0$.

2nd mystical principle: U_A is the set of all formal group laws over A; the $\Gamma_0(A)$-action changes the coordinates of the group law.

[U_A is canonically iso. to $\{F \in A[[X,Y]] \mid F(X,F(Y,Z)) = F(F(X,Y),Z),\ F(X,Y) = F(Y,X)$ etc.

In this natural identification, we have

$$(f, F) \longmapsto F^f(X,Y) = fF(f^{-1}(X), f^{-1}(Y)).$$
$$\Gamma_0(A) \times U_A \longrightarrow U_A \ ]$$

Notation. I'll write $F \in U_A$ to mean "F is a f.gp law over A; $\chi_F : U^*(pt) \to A$ is its associated ringhomomorphism.

Example. $\Gamma_0(Q)$ acts transitively, and without fixed points, on U_Q.

Proof: Recall that $F \in U_Q$ is canonically represented as $F(X,Y) = \ell_F^{-1}(\ell_F(X) + \ell_F(Y))$, where $\ell_F \in Q[[X]]$ is an invertible series, the logarithm of F. Clearly the logarithm of F^f is $\ell_F \circ f^{-1}$; since F is defined by its logarithm, we're through.

Remark 1: The algebra $S^* \otimes_{\mathbb{Z}} Q$ has an interesting Lie-algebraic description. The Lie algebra of $\Gamma_0(Q)$ is the Lie algebra of "formal vector fields near 0", with elements $f(T)\frac{\partial}{\partial T}$, $f \in Q[[T]]$. The universal enveloping algebra of this

Lie algebra is $S^* \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

<u>Remark 2</u>. Our eventual aim will be to describe the action of $\Gamma_0(k)$ on U_k, when k is any field. The situation when characteristic$(k)>0$ is quite unlike the char $k=0$ case.

§3. The Zariski topology

For simplicity, we assume from now on that $A=k$ is a field. The set U_k has a natural topology, in which the functions $\check{M}: U_k \to k$ are continuous.

The description of the topology is completely general, so we work with a field k, and a commutative k-algebra B. (In our case, $B = U^*(pt) \underset{\mathbb{Z}}{\otimes} k$).

Write $B(k)$ for the set $\mathrm{Hom}_{(k\text{-algs})}(B,k)$.

($= \mathrm{Hom}_{\text{Rings}}(U^*(pt); k) = U_k$ in our case).

A subbasic <u>closed</u> set in the Zariski topology on $B(k)$ is a set of the form $\{x \in B(k) \mid x(a)=0\} = V_a$, where $a \in B$.

In other words, V_a is the set of zeros of the function $\check{a}: B(k) \to k$ defined by $\check{a}(x) = x(a)$, so we have given $B(k)$ the smallest topology which makes all the functions $\check{a}$ continuous.

Conversely, $B(k) - V_a = \{x \in B(k) \mid x(a) \neq 0\}$ is a subbasic open set.

§4: Sheaves Now let M be a B-module. We can define a sheaf $\tilde{M}$ over $B(k)$ in a natural way: it suffices to define the set of sections of $\tilde{M}$ over any open set, and verify that they fit properly together. In fact it suffices to work with the basic open sets $D(a) = \{x \in B(k) \mid x(a) \neq 0\}$ and their generalizations $D(a_1, \ldots, a_n) = D(a_1) \cap \cdots \cap D(a_n)$.

Def. $\tilde{M}(D(a)) = M[a^{-1}]$ is the set of sections of $\tilde{M}$ over the open set $D(a)$; here $M[a^{-1}]$ is the module of fractions (m/a^n), $m \in M$, n a positive integer, under the standard equivalence relation.

More generally if $S \subset B$ is any finitely generated multiplicatively closed subset of B, we write $D(S)$ for the open set $\{x \in B(k) \mid x(s) \neq 0, \forall s \in S\}$. The set of sections of $\tilde{M}$ over $D(S)$ is just $S^{-1}M$.

Example. Let $M = B$. We will write $\mathcal{O}_B$ for the resulting sheaf on $B(k)$; it is evidently a sheaf of k-algebras.

<u>Proposition</u>. If $M \in B\text{-Mod}$, then $\tilde{M}$ is a sheaf of $\mathcal{O}_B$- modules.

<u>Proof by Explanation and omission</u>.

1) There is a natural map

$$\mathcal{O}_B \otimes_k \tilde{M} \longrightarrow \tilde{M}$$ of sheaves. [If D is an open set, then the group of sections of the left-hand side of the arrow is just $\mathcal{O}_B(D) \otimes_k \tilde{M}(D)$, that of the right-hand side being $\tilde{M}(D)$. If $D = D(a)$, we have to construct an arrow

$$B[\bar{a}'] \otimes_k M[\bar{a}'] \longrightarrow M[\bar{a}'].$$ But this is just a $B[\bar{a}']$-module structure on $M[\bar{a}']$, which is clear.

<u>Theorem</u> [Serre] .

The map $M \in (B\text{-Mod}) \longmapsto \tilde{M} \in (\mathcal{O}_B\text{- Mod})$ is a full, faithful, and exact functor.

<u>Proof</u>: Exactness is a corollary of the well-known fact that localization is exact. To see the rest of the conclusion, we use the global section functor: if $\underline{E}$ is a sheaf of $\mathcal{O}_B$- modules, then ΓE is a B-module.

There are obvious maps $M \to \Gamma\tilde{M}$, $(\Gamma \underset{\sim}{E})^{\sim} \to \underset{\sim}{E}$, and clearly if M is free over B, or $\underline{E}$ is free over $\underline{O}_B$, these maps are isomorphisms.

A sheaf $\underline{E}$ of $\underline{O}_B$-modules is said to be quasicoherent if (locally on $B(k)$) there is an exact sequence

$$(\underline{O}_B|U)^J \to (\underline{O}_B|U)^I \to \underline{E}|U \to 0$$

$U^{\text{open}} \subset B(k)$, I, J arbitrary index sets. It follows from the exactness of $\sim$ and Γ that the category of B-modules is equivalent to the category of quasi-coherent $\underline{O}_B$-modules. //

Some terminology.

Let $x : B \to k$ be a homomorphism of k-algebras; let $B \supset S_x = \{b \in B / x(b) \neq 0\}$. Evidently S_x is a multiplicative set, and the map x factors

$$B \to S_x^{-1}B \xrightarrow{\bar{x}} k$$

. Here $S_x^{-1}B$ is a local k-algebra, called the stalk of B at x (it is iso. to the stalk $\underline{O}_{B,x}$).

If M is any B-module, $S_x^{-1}M = \tilde{M}_x$ is the stalk of M at x. Evidently there is a homomorphism

$$M \to S_x^{-1}M \to M \underset{x}{\otimes} k.$$

It is convenient to call $M \underset{x}{\otimes} k$ the fiber over x.

§5 Back to cobordism

Let $M(p) = S^1 \cup_p E^2$, p a prime. We define

$$U^*(X;\mathbb{F}_p) = U^{*+2}(X \wedge M(p)).$$

THEOREM OF ARAKI–TODA:

$X \mapsto U^*(X;\mathbb{F}_p)$ is a cohomology theory, taking values in the category of $U^*(pt;\mathbb{F}_p) = U^*(pt) \otimes_{\mathbb{Z}} \mathbb{F}_p$-modules.

Remark. There is a natural multiplicative structure on $U^*(X;\mathbb{F}_p)$, but when $p=2$, that structure is in general not multiplicative. (e.g. $X = \mathbb{R}P(2)$) (Araki–Toda study $K^*(-;\mathbb{F}_2)$ in detail, where the same thing happens.)

Note that there is a natural action of $\Gamma_0(\mathbb{Z})$ on $U^*(X;\mathbb{F}_p)$, since this group is $U^*(X \wedge M(p))$ in disguise, but evidently the group action factors through $\Gamma_0(\mathbb{F}_p)$.

3rd mystical principle

$X \mapsto \widetilde{U^*(X;\mathbb{F}_p)}$ is a cohomology theory taking values in the category of sheaves of $\mathcal{O}_U$-modules over $U_{\mathbb{F}_p}$ (= the space of all formal group laws over $\mathbb{F}_p$).

Lecture VI : Localization (Part II)

We have shown

THEOREM 4. There is a cohomology functor

$X \longmapsto \widehat{U^*}(X;k) = \widehat{U^*(X;k)}$, taking values in the category of sheaves of modules over U_k; these sheaves are equivariant under the action of $\Gamma_0(k)$ given by the Landweber–Novikov operations.

Here k is any field, and $U^*(X;k)$ is defined to be

$$\left.\begin{array}{l} U^*(X)\underset{\mathbb{Z}}{\otimes} k \\ \\ U^*(X;\mathbb{F}_p)\underset{\mathbb{Z}}{\otimes} k \end{array}\right\} \text{if} \begin{array}{l} k \text{ is of char. } 0 \\ \\ k \text{ is of char. } p>0. \end{array}$$

(Since any field of char p is flat over $\mathbb{F}_p$ (indeed over any subfield) the second group gives a coh. theory.

(In fact the introduction of fields is not essential; one could work over $\mathbb{Z}$.

In the 1st part of this talk I will try to reformulate Thm 4 more conceptually. In the 2nd part, I will summarize some detailed results from the theory of formal groups, and show by examples (lens spaces) that this structure shows up pretty quickly.

§1 What U^* means to me.

U_k is the set of all formal groups over the field k, so Thm 4 says 1st that $U(X;k)$ is a sheaf over the formal groups over k.

However it is really more reasonable to think of formal groups over k as a category.

Def 1. The cat. of fgplaws over a field k consists of
objects: $F \in k[[X,Y]]$ such that $F(X,F(Y,Z)) = F(F(X,Y),$ ...
morphisms $f: F \to G$ series $f \in k[[T]]$, $f = T + \cdots$ such that

$$fF(X,Y) = G(fX, fY)$$

[invariant sheaves are dispensed with.]

Remark 1 This is a very small cat; its objects are contained in $k[[X,Y]]$, its morphisms are contained in $k[[T]]$.

Remark 2: The category FG/k just defined is a groupoid; all its morphisms are isomorphisms. [I didn't allow noninvertible morphisms, for this reason.]

Remark 3. Let G be a group acting on a set X. The associated groupoid X_G is the category whose objects are points $x \in X$; if $x, y \in X$, then $\mathrm{Hom}(x,y) = \{g \in G \mid gx = y\}$.
The set of isomorphism classes of objects in $X_G \overset{1-1}{\longleftrightarrow} X/G$ = orbits.

Similarly, a G-action on a top. space X defines a <u>topological</u> groupoid.

<u>Remark 4</u>. Consider a sheaf $\mathcal{F} \xrightarrow{\pi} X$; here π is an honest topological space. A G-structure on $\mathcal{F}$ is an action of G on X, and on $\mathcal{F}$, $\ni \pi$ becomes a G-map.

Consequently we get a map $\pi_G : \mathcal{F}_G \to X_G$ of groupoids, and

<u>a G-equivariant sheaf on X is really just a sheaf on the category X_G.</u>

<u>Theorem 4*</u> : The cobordism functor $X \to U^*(X;k)$ is really a sheaf on the <u>category</u> of fgplaw over k.

This has the advantage of putting the cohomology operations into your viewpoint.

The general philosophy is that if something happens in $U^*(X;k$ you should be able to look at it "locally", i.e. in a neighborhood (on the space U_k) of the relevant formal group. The Theorem above says that cobordism "localizes" well.

§2 The behavior of cobordism could be "explained" if one could prove the following

Cartier Conjecture.

Let A be a commutative, cocommutative Hopf algebra over the field k.

field restriction might be superfluous; A should probably be noetherian

Then there should be a cohomology theory $H(-;A)$ (N.B. not necessarily graded),

$$X \longmapsto H(X;A) \in k\text{-Modules}$$ which

acts like cohomology "with coefficients in A". In particular, the ring of Hopf algebra endomorphisms on A should act naturally on $H(-;A)$.

... A is the additive Hopf alg., get the alg. of reduced p^{th} powers (Bocksteins not included). Mult. Hopf alg $\Rightarrow$ Adams operations.

$\Delta T = T \otimes T$ $\qquad$ $\Delta T = T \otimes 1 + 1 \otimes T$.

Remark. The examples I construct fit this conjecture: $A = k[[T]]$. The conj. fits ordinary K-theory and ordinary cohomology. Moreover, it fits the following

$\Delta T = F(T\otimes 1, 1\otimes T)$, Fancy fgp law work

Kraines-Segal Cohomology Theory

Consider the functor $X \longmapsto$ (Units of $H^*(X;\mathbb{F}_p)[[t]]$) where by "units of $H^*[[t]]$" I mean sequences

5)

$1 + x_1 t + \cdots + x_n t^n + \cdots$ with $x_i \in H^{2i}$, i.e. $\deg t = -2$.

The composition in the group on the right is multiplication of power series.

Evidently this functor is represented by $\prod_{j \geq 0} K(\mathbb{F}_p, 2j)$, and the composition law gives it a funny H-space structure.

<u>Example</u>. if $X = \mathbb{C}P(\infty)$, units $H^*(\mathbb{C}P(\infty); \mathbb{F}_p)[t] =$ [illegible] strict units of $\mathbb{F}_p[[t]]^* \cong \hat{\mathbb{Z}}_p$ (p-adic integers, by the Artin-Hasse exponential).

<u>Theorem</u>. The functor $X \longmapsto$ units $H^*(X; \mathbb{F}_p)[t]$ is represented by an ∞ loopspace.

Proof uses Peter May's <u>operads</u>: one defines the things on the

This would fit Cartier's conjecture: A is the Hopf algebra corresponding to the <u>Witt group</u> over $\mathbb{F}_p$.

<u>THE</u> basic <u>problem</u> of this whole business is probably to "prove" this conjecture: i.e. to construct a functor from

$$\left(\begin{array}{l}\text{comm.}\\\text{cocomm. Hopf alg's}/k\end{array}\right) \longmapsto (\infty\text{-loop spaces})$$

which makes the conj. true.

6)

Remark. There is now a powerful machine for constructing ∞-loopspaces: e.g. if $\mathcal{C}$ is a small cat. with coherently comm. + assoc. composition law, then $\exists$ classifying ∞-loopspectrum: May / Beck / Segal / Vogt - Boardman.

So all you have to do is find a "natural" construction

comm + cocomm Hopf alg's/k $\longmapsto$ Categories with $\oplus$

For ex., mult. gp. $\longmapsto$ Cat. of $\mathbb{C}$-Vector spaces $\longmapsto$ Classical K-theory.

A major objection. But I can't figure out a reasonable way to construct ordinary coh. from the additive gp!

Remark. In "Forms of K-theory" there's a lot of stuff about elliptic curves. The pt. is that the machine also works for elliptic curves, (i.e. compact alg. gp's over $\mathbb{C}$) which means that there might be an analytic way to build loopspaces. I can't stand to think about it though.

7)

part II: Structure of U_k.

1. preliminaries

<u>Defn's</u>. U_k is the set of formal group laws over k,

$\Gamma_0(k)$ .... auto by

$$\Gamma_0 \times U \ni (f, F) \mapsto fF(f^{-1}(x), f^{-1}(y)) = F^f(x,y)$$

i.e. $\{gF \mid g \in \Gamma_0(k)\} = \mathcal{O}_F$

<u>⊢1</u>: An orbit of $\Gamma_0(k)$ on U_k is an iso. class of fgplaws over k.

<u>⊢2</u>: If $\mathcal{F}$ is an eq. sheaf on U_k, and $F, G \in U_k$ be on the same orbit, then the stalk $\mathcal{F}_F$ of $\mathcal{F}$ over F, and the stalk $\mathcal{F}_G$ over G, are isomorphic.

<u>Pf</u>: By def., $\exists g \in \Gamma_0 \ni gF = G$. Hence by def. of eq. structure, if V is any open set containing G, $\exists$ map $S_g(V): \mathcal{F}(V) \longrightarrow \mathcal{F}(gV)$.

Take lim. over open nbds of F: get map

$$\varinjlim_{F \in V} \mathcal{F}(V) = \mathcal{F}_F$$

$$\Big\downarrow \varinjlim S_g(V)$$

$$\varinjlim_{G = gF \in gV} \mathcal{F}(gV) = \mathcal{F}_{gF} = \mathcal{F}_G.$$

But g has an inverse, $g^{-1}: \mathcal{F}_G \longrightarrow \mathcal{F}_F$, since $g \in \Gamma_0$ - a group. //

<u>Ideological</u> <u>Cor.</u> To understand a sheaf over U_k, it suffices to

understand its behavior at a single point on each orbit.

⟨ The orbits of $\Gamma_0(k)$ on U_k are thus good to know. ⟩

Def'n. If $F \in U_k$, $S_F = \{ \gamma \in \Gamma_0(k) \mid \gamma(F) = F \}$ is the <u>stabilizer</u> (or isotropy group) of F.

$= \{ \gamma \in \Gamma_0(k) \mid \gamma F(X,Y) = F(\gamma X, \gamma Y) \}$ is the automorphism group of the fglaw F.

<u>L 3</u>: If $\mathcal{F}$ is an eq. sheaf, then the stabilizer S_F acts naturally on $\mathcal{F}_F$.

<u>Pf</u>: Since $g \in S_F \Rightarrow gF = g$, $\exists$ map $S_g : \mathcal{F}_F \to \mathcal{F}_{gF} = \mathcal{F}_F$. //

⟨ The automorphism gp. of F is thus the "Galois group" of the stalk $\mathcal{F}_F$, and good to know. ⟩

<u>Case of an alg. closed field of char p</u>

2. Examples. The alg. closed case.

<u>THM of Lazard</u>. $\exists$ a single pos. integer (or $+\infty$) invariant which classifies fglaws over such a field: the <u>height</u>.

Examples: ht. of $\oplus$'ive gp $= +\infty$, ht. of $\otimes$ gp. $= 1$

<u>Def</u>: $[n]_F(T) = F([n-1]_F(T), T)$, $[0]_F(T) = 0$.

Let q be the smallest integer $\ni T^q$ has a nonvanishing coeff. in $[p]_F(T) = 0 + \cdots + 0 + uT^q + \cdots$. One

Exercise: $\begin{cases} [p]_{F} f(X) = \quad [p]_{F'}(X) \Rightarrow \\ \text{ht. is an iso. invariant.} \end{cases}$

Can show $q = p^{h_F}$ for $h = 1, 2, \ldots$: h_F is the <u>height</u> of F.

<u>Ex:</u> $F = \mathbb{G}_m$, $\mathbb{G}_m(X,Y) = X + Y - XY$ the mult. gp.

$$[p]_{\mathbb{G}_m}(X) = (1+X)^p - 1 \bmod p = X^p \quad \text{so} \quad h_{\mathbb{G}_m} = 1.$$

$$F = \mathbb{G}_a, \quad [p]_{\mathbb{G}_a}(X) = 0 \bmod p = X^{p^\infty} \quad \text{so} \quad h_{\mathbb{G}_a} = +\infty.$$

<u>The canonical gp. law of height $n = 1, 2, \ldots$</u>

Let $\ell_n(X) = \sum_{r \geq 0} p^{-r} X^{p^{nr}}$; let $F_n(X,Y) = \ell_n^{-1}(\ell_n(X) + \ell_n(Y))$

<u>THM</u> (Honda 1968) : $F_n(X,Y) \in \mathbb{Z}_{(p)}[[X,Y]]$, i.e. the series has rational coeffs with no p's in the denominator.

So $\overline{F}_n(X,Y)$, the reduction of F_n mod p, defines a 1gplaw $\in \mathbb{F}_p[[X,Y]]$.

moreover $[p]_{F_n} \in \mathbb{Z}_{(p)}[[X]]$ is given by

$$[p]_{F_n}(X) = pX + X^{p^n}.$$

Consequently $\overline{F}_n$ is of ht. n over $\mathbb{F}_p$.

Of course we can now extend the field to <u>any</u> field of char p, & we get a gplaw of ht n again.

3. comp. of U into orbits.

Suppose k is a field of char p. I'll supress it from the notation, & write

$$U_n = \{ F \in U \mid ht. F = n \}.$$

The invariant ideal Theorem

THEOREM 5 $\exists\, w_1, \dots, w_n, \dots \in U^*(pt)$, which are (not all of the) generators for $U^*(pt) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, with $\deg w_i = 2(p^i - 1)$, such that

$$U_n = \{ x \in U \mid x(w_i) = 0, \quad i = 1, \dots, n-1;\ x(w_n) \neq 0 \}.$$

Remark: These generators w are Milnor generators in the appropriate dimensions, i.e. manifolds whose char. #'s satisfy a certain congruence mod p. Such manifolds are usually denoted v_i; but Hazewinkel has very nice explicit manifolds which he calls v_i, which are different from the w_i's (obs. of Ravenel), & I don't want to confuse them.

Proof. We need a lemma which was implicit in the defn of height:

Lemma. F gplaw over $\forall$ ring of char p. Then $[p]_F(X) = \varphi(X^{p^n})$ for some integer n, where $\varphi(T) = aT + \dots$, $a \neq 0$ in R.

Proof: By calculus. Cf. Iwasawa.

Let $[p]_U(T) \in U^*(pt)[[T]]$ be the p-series for the *univ.* fgplaw, Write

$$[p]_U(X) = pX + \text{junk} + w_1 X^p + \text{junk} + w_2 X^{p^2} + \text{more junk} + w_n X^{p^n} + \cdots$$

Cor. $[p]_U(X) = \quad w_n X^{p^n}$ + higher order terms
modulo $p, w_1, \ldots, w_{n-1}$.

Pf: apply the lemma.

Cor. Let $x : U^*(pt) \to k$ induce the groupplaw F_x over the field k. Then F_x is of height $n \iff$
$x(w_i) = 0 \quad , \; i = 1, \ldots, n-1$
$\qquad \neq 0 \quad , \; i = n.$

It remains to show that the w_i can be chosen as generators for $U^*(pt) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ (of course they're not a *complete* set of generators).

The proof is purely fgpleaetic; I'll only sketch it, since it's best done when one proves Lazard's Thm: That $U^*(pt)$ is torsion-free. All known proofs of the theorem induct on the # of generators; as you do the induction, use:

cf. Fröhlich or Iwasawa.

Lemma. Suppose F, G fgplaws, $F \equiv G$ mod deg n. Then $F \equiv G + \lambda C_n$ mod deg $n+1$, and $[p]_F = [p]_G + u_n \lambda X^n$ mod deg $n+1$, where $u_n \in \mathbb{Z}_{(p)}$ and
$$u_n \equiv 0 \text{ mod } p \quad \text{if } n \neq p^r - 1$$
$$\not\equiv 0 \text{ mod } p \quad \text{if } n = p^r - 1 \text{ for some } r.$$

[Suppose $U_n(X,Y)$ is the univ. gp law up to deg n; then U_{n+1} is defined as $U_n + \alpha_n C_n$, where α_n is the new generator creeping in. From the lemma it follows that $u_n \alpha_n = W_n \bmod p, W_1, \dots, W_{n-1}$, and u_n is a unit mod p (whenever the dimensions are right]

—//—

<u>Corollary</u>. The closure of U_n (in the Zariski topology) is $\{x \in U \mid x(W_1) = \cdots = x(W_{n-1}) = 0\}$,

$$\overline{U_n} = \bigcup_{m \geq n} U_m .$$

<u>Remark</u>. The orbit structure on the "affine space" U has the "hyperplane stratification":

$\mathbb{C}P(n) = e^{2n} \cup e^{2n-1} \cup \cdots \cup pt.$

Top orbit : U − hyperplane $\overline{U}_1$

next orbit : $\overline{U}_1$ − hyperplane $\overline{U}_2$

...

n^{th} orbit : $\overline{U}_{n-1}$ − hyperplane $\overline{U}_n$

...

in fact it's better to pass to the associated <u>projective</u> space. Then the top orbit = top cell | in the standard cell decomp.

next orbit = next cell

etc.-

† The stabilizers.

Those not terribly excited by the stabilizers should read only the †'ed parts (if those).

$l_n(X) = \sum_{r\geq 0} p^{-r} X^{p^{nr}}$

If two gplaws are isomorphic, they have isomorphic stabilizers. Hence it suffice (over an alg. closed field) to describe the stabilizer of $F_n(X,Y) = l_n^{-1}(l_n(X)+l_n(Y))$

__Lemma 1.__ F fgplaw $\Rightarrow$ $\mathrm{End}\, F = \{f \in k[[T]] \mid fF(X,Y) = F(fX, fY)\}$ is a ring.

N.B. f need not lie in $\Gamma_0(k)$.

__Lemma 2.__ k is a field $\Rightarrow$ $\mathrm{End}\, F$ is an integral domain

__Lemma 3.__ $\exists$ canonical injection $\hat{\mathbb{Z}}_p \to \mathrm{End}(\overline{F}_n)$

($\alpha \in \hat{\mathbb{Z}}_p \mapsto [\alpha]_{F_n}(X) = l_n^{-1}(\alpha l_n(X)) \in \hat{\mathbb{Z}}_p[[X]]$; Reduce mod p.)

__Lemma 4.__ If F is a gplaw over $\mathbb{F}_p$, then the Frobenius $\mathcal{F}(X) = X^p$ lies in $\mathrm{End}_{\mathbb{F}_p}(F)$.

__Pf:__ $F(X,Y) = \sum a_{ij} X^i Y^j$, $a_{ij} \in \mathbb{F}_p \Rightarrow (a_{ij})^p = a_{ij}$. Hence

$F(X^p, Y^p) = F(X,Y)^p$ //

Let ω be a (p^n-1)th root of 1. Adjoin it to $\mathbb{Q}_p$.

__Lemma 5.__ $\omega l_n(X) = l_n(\omega X)$; hence

$F_n(\omega X, \omega Y) = \omega F_n(X,Y)$.

Pf: $\ln(\omega X) = \sum p^{-r} (\omega X)^{p^{nr}}$; but $\omega^{p^n} = \omega$
$= \sum p^{-r} \omega X^{p^{nr}}$
$= \omega \ln(X)$ //.

★ **Lemma 6** Let E_n be the algebra over $\hat{\mathbb{Z}}_p$ generated by indeterminates ω, $\mathcal{F}$ subject to the relations

1) ω is a primitive (p^n-1) th root of 1
2) $\mathcal{F}\omega = \omega^p \mathcal{F}$
3) $\mathcal{F}^n = p$.

Then $\exists$ ringhom $E_n \longrightarrow \mathrm{End}(\bar{F}_n)$.

Pf: Here ω maps to the endo. of $\bar{F}_n$ induced by $T \mapsto \bar{\omega} T$; $\bar{\omega} \in k$ is a (p^n-1) th root of 1, the mod p reduction of ω; $\mathcal{F} \mapsto X^p$. It suffices to check the relations 2), 3); but 2) is obvious, and 3) is the assertion $[p]_{F_n}(X) = X^{p^n} (= \mathcal{F}^n(X))$, cf. the canonical example, page 9

★ **Proposition.** The map above is an iso.

cf. Fröhlich, or my Tata notes, such as they are.

★ **Cor.** S_{F_n} is the group of units of E_n

§5. The delicate structure.

I should say something about the classification of formal groups over $\mathbb{F}_p$. The subject is very number-theoretic, so I will not try to give proofs.

I'll omit the p from now on.

Definition. A (p-)Eisenstein number is a number which satisfies a (p-) Eisenstein equation

Definition. An Eisenstein polynomial $E \in \mathbb{Z}[X]$ is an irred. poly. of degree n, $\ni$

Examples: $X^2 + pX + p$, $X^n - p$

1) $E(X) \equiv aX^n \mod p$, $a \not\equiv 0 \mod p$

2) $p^{-1} E(0) \not\equiv 0 \mod p$.

(An Eisenstein eq'n. is $E(X) = 0$, so $\pi \in \mathbb{C}$ is an Eisenstein # if $E(\pi) = 0$)

Similarly, π is a p-adic Eisenstein # iff it satisfies an Eisenstein eq'n with p-adic coeffs.

Honda-Hill-Cartier **Theorem.** (isomorphism classes of Formal group laws over $\mathbb{F}_p$) correspond 1-1 with (p-adic Eisenstein numbers).

<u>Explanation</u>. If F is a gp law over $\mathbb{F}_p$, then $\exists$ canonical Frobenius $\mathcal{F} \in \mathrm{End}_{\mathbb{F}_p}(F)$. Now $\mathrm{End}_{\mathbb{F}_p}(F)$ is a finite dimensional integral domain over $\hat{\mathbb{Z}}_p$ hence $\mathcal{F}$ satisfies a char. polynomial over $\hat{\mathbb{Z}}_p$.

cf. <u>Hill</u>

[It can be shown that this polynomial must be Eisenstein; hence $\mathcal{F}$ is a p-adic Eisenstein number. For example

(the canonical law of ht n)

$\mathcal{F} \in \mathrm{End}(\bar{F}_n)$ satisfies $\mathcal{F}^n - p = 0$, so its associated Eisenstein # is $\sqrt[n]{p}$.

In the ht. 1 case, the Eisenstein poly. is of the form $X - up$, where u is the invariant introduced earlier.

<u>Explicit Formulas</u>.

A good thing to know:

<u>Thm of Cartier-Hill</u>: Two fgps over $\hat{\mathbb{Z}}_p$ are <u>iso</u> $\Leftrightarrow$ their reductions over $\mathbb{F}_p$ are iso.

(N.B. The <u>cat</u> of fgps/$\hat{\mathbb{Z}}_p$ is different from the <u>cat</u> of fgps/$\mathbb{F}_p$, because they have very different morphisms. But iso. classes of obj's are the same).

So we can ask for a description of each iso. class over $\hat{\mathbb{Z}}_p$; we give the <u>classifying map</u>

(a ringhomomorphism)

$\underline{\Pi} : U^*(pt) \longrightarrow \hat{\mathbb{Z}}_p$ which associates to each Eisenstein #, a fgplaw over $\hat{\mathbb{Z}}_p$.

Since $\hat{\mathbb{Z}}_p$ has no torsion, the map $\underline{\Pi}$ is specified by its value on projective spaces; hence by the generating series

$$\boxed{\sum_{n\geq 1} \underline{\Pi}\,(\mathbb{C}P(n-1))\, n^{-s} \;=\; \frac{E(0)}{E(p^{1-s})}}$$

where E is the Eisenstein poly. of π, normalized so that $E(X) = X^n + \cdots$.

The resulting gplaw is of ht $\underline{\underline{n}}$.

<u>Example</u>. Take $E(X) = X^n - p$. Then

$$\sum \underline{\Pi}\,(\mathbb{C}P(n-1))\, n^{-s} = \left(1 - p^{n(1-s)-1}\right)^{-1}$$

(In particular, $n=1$ gives the local factor of the Riemann ζ-fn. (up to iso)).

There is a similar description of all fgplaws <u>over $\mathbb{Z}$</u>; (due to Honda 1970).

Example.

It is a priori possible that none of the more complicated phenomena occur — e.g. that nothing nontrivial happens on the higher strata. The best counterexample to this seems to be $U^*(B\mathbb{Z}/p)$; for this group is closely related to the study of periodic maps and is thus of geometric significance.

Theorem (Peter L.) $U^*(B\mathbb{Z}/p) = U^*(pt)[[T]] \big/ \left([p]_U(T)\right)$.

Pf: Look at the Gysin sequence of the circle bundle $S^1 \underset{\mathbb{Z}/p}{\times} S^{2n-1} \longrightarrow pt \underset{\mathbb{Z}/p}{\times} S^{2n-1}$ as $n \to +\infty$. //

Consider $U^*(B\mathbb{Z}/p) \underset{\mathbb{Z}}{\otimes} \mathbb{F}_p$ as a sheaf over $U_{\mathbb{F}_p}$.

[It is actually more natural to look at $U^*(B\mathbb{Z}/p\,; \mathbb{F}_p)$, which is a bigger group, but the above problem is a little simpler].

Proposition. $U^*(B\mathbb{Z}/p) \underset{\mathbb{Z}}{\otimes} \mathbb{F}_p$ induces a sheaf of algebras over $U_{\mathbb{F}_p}$; if $F \in U_{\mathbb{F}_p}$ is a fglaw of height n, then the fiber of this sheaf of algebras over F is just $\mathbb{F}_p[T] \big/ (T^{p^n})$.

This is really the picture over the alg. closure since we've collapsed the orbits to points

Cor $U^*(BZ/p)\otimes \mathbb{F}_p$ looks like

$\mathbb{F}_p[T]/(T^p)$	$\mathbb{F}_p[T]/(T^{p^2})$	$\cdots$	$\mathbb{F}_p[T]/(T^{p^n})$	$\mathbb{F}_p[[$
pt. of ht = 1	ht = 2	$\cdots$	ht = n	ht =
looks like $K(BZ/p)$				looks like $H(BZ/$

__Proof.__ The fiber of $U^*(BZ/p) = U^*(pt)[[T]]/([p](T))$ over F

is just $U^*(pt)[[T]]/([p]_U(T)) \otimes_F \mathbb{F}_p =$

$\mathbb{F}_p[[T]]/([p]_F(T))$. Now $[p]_F(T) = T^{p^n} +$ higher order stuff; we can change coordinates over $\overline{\mathbb{F}}_p$ so that F is iso. to the __standard__ group F_n, with $[p]_{F_n}(T) = T^{p^n}$. Then we're through.

1

Appendix to Lect. VI

[this is intended to explain the ravings of p. 10-11 (Part II §3)]

Invariant prime ideal in $U^*(pt)$ is defined.

Def. 1. Let $U^*(pt)$ be the cobordism ring, $\hat{S}^*$ the completed algebra of operations ($\hat{S}^* \ni \sum s_\alpha c_\alpha$, $c_\alpha \in \mathbb{Z}$, s_α the preferred L-N basis elements, the series being *formal*)

A sub-abelian group $M \subset U^*(pt)$ is *invariant* (under $\hat{S}^*$) iff $\hat{S}^* \cdot M \subset M$, i.e. $m \in M \Rightarrow s_\alpha(m) \in M$, $\forall$ multiindex α.

Variant def: Let $s_{\underline{t}} : U^*(pt) \longrightarrow U^*(pt)[\underline{t}]$ be the giant L-N operation of Boardman. Then M is invariant iff $s_{\underline{t}}(M) \subset M[\underline{t}]$.

Def. 2 A (finitely generated) prime invariant ideal in $U^*(pt)$ is a (finitely generated) prime ideal which is also invariant (under $\hat{S}^*$).

A form of the main Thm. is stated.

Prime Ideal Theorem. Let $I \subset U^*(pt)$ be a f.g. prime invariant ideal. Then $I = (p, \alpha_{p-1}, \alpha_{p^2-1}, \ldots, \alpha_{p^n-1})$ for some prime p, and integer n, where the α's are polynomial generators of $U^*(pt)$ in the indicated dimension.

Setting up to state the main lemma.

Notation: We use Iwasawa (+ Lazard's) notation, cf. Iwasawa's lecture of 16/10/73 (on the inductive construction of the universal groplaw).

In part. $U(X,Y) \in A[[X,Y]]$ is the universal gplaw

$A = \mathbb{Z}[x_1, \ldots] = \mathbb{Z}[\underline{x}]$ being the ground ring.

$[n](X) \in A[[X]]$ where $[n](X) = U(X, [n-1](X))$.

Definition: $w_{p^i-1} \in A$ is the coefficient of X^{p^i} in $[p](X)$, i.e.

Main technical lemma:

$$(-1)^i \alpha_{p^i-1} = w_{p^i-1} \text{ modulo all } w_{p^j-1} \text{ with } j<i.$$

[Corollary. $(-1)^i w_{p^i-1} = \alpha_{p^i-1}$ modulo all α_{p^j-1}, with $j<i$, (provided we set $\alpha_0 = p$).]

I will only prove a weaker result here:

The author chickens out

$$(-1)^i w_{p^i-1} = \alpha_{p^i-1} \text{ modulo all } \alpha_n \text{, with } n<p^i-1.$$

The proof can be strengthened to the full Theorem by either using Adam's proof of Lazard's Thm (cf. Adams notes) or by using BP.

Proof of Main Technical Lemma

It suffices to prove $[p](X) = \alpha_{p^i-1} X^{p^i} +$ higher order terms modulo $p, w_{p-1}, \ldots, w_{p^i-1}$.

Step 1 $\vdash$: $[p](X) \equiv 0$ modulo deg p^i and $p, w_{p-1}, \ldots, w_{p^j-1}, j<$

Proof. We require

First Height Lemma; cf. Iwasawa This week

1st lemma from fgl theory

(If F is a fgplaw over a ring R, $pR=0$ i.e. R is of char p, then $[p]_F(X) = \varphi(X^{p^n})$ for some $n>0$, where $\varphi(X) = aX + \cdots$ with $0 \neq a \in R$.

Proof by calculus: use X^p has 0 derivative in char p.)

Proof of Step 1: Consider the image
$W(X,Y) \in A/(p, w_{p-1}, \ldots, w_{p^j-1}; j<i)\ [[X,Y]]$ of the
universal law under reduction modulo the ideal.

Then $[p]_W(X) \in A/(p, \ldots)\ [[X]]$ satisfies

$$[p]_W(X) = \varphi(X^{p^n}) = aX^{p^n} + \cdots \qquad 0 \neq a \in R$$

for some n. Clearly $n > i$, so
$\varphi(X^{p^n}) \equiv 0 \mod \deg p^i$ //

Step 2 $\vdash$: $[p]_W(X) = \pm\, \alpha_{p^i-1} X^{p^i} +$ higher order terms
in $A/(p, \ldots)$.

<u>Proof</u>: We require

<u>Second Height Lemma</u>:

2nd lemma from fgptheory

If F, G are fgplaws over a ring R of char p, with $F \equiv G + a C_m$ mod deg $m+1$
(i.e. $F \equiv G$ mod deg m) then (mod deg $m+1$)
$[p]_F(X) \equiv [p]_G(X) - aX^m$ if $m = p^r$ for some r;
$\equiv [p]_G(X)$ otherwise

<u>Proof</u>: <u>deferred</u>. (see page)

Now recall Iwasawa-Lazard's inductive construction of the universal grouplaw $U(X,Y) \in A[[X,Y]]$:

details of the proof of L's thm.

inductive hypothesis:
$\exists\, U_n \in A_n[X,Y]$ such that $(A_n = \mathbb{Z}[\alpha_1, \ldots, \alpha_n])$.
1) $U_n = U_{n+1}$ mod deg $n+2$ (U_n is a poly of deg $\leq n+1$)
2) $U_n - \alpha_n C_{n+1} \in A_{n-1}[X,Y]$ if $n \geq 1$.

<u>Proof of Step 2</u> :

1) $(U_{n-1} \equiv U_n \bmod \deg n+1) \Rightarrow \exists \lambda_n \in A_n \ni U_n = U_{n-1} + \lambda_n C_{n+1} \bmod \deg n+2$

2) $U_n - \alpha_n C_{n+1} \in A_{n-1}[X,Y]$

and $U_n - U_{n-1} = \lambda_n C_{n+1} \bmod \deg n+2 \Rightarrow \lambda_n - \alpha_n \in A_{n-1}$

$\Rightarrow \lambda_n = \alpha_n$ modulo lower dimensional α's

The point that needs strengthening for the proof of the strongest theorem.

<u>Define</u> $[p]_n(X) \in A_n[X]$ is defined by $[m]_n(X) = U_n(X, [m-1]_n(X))$

(it's only interesting modulo deg $n+1$)

From the 2nd Height Lemma, and 1) above,

$$[p]_n(X) = [p]_{n-1}(X) - \lambda_n X^{n+1} \quad \text{if } n+1 = p^i$$
$$= [p]_{n-1}(X) \quad \text{otherwise.}$$

$n = p^i - 1$

Now consider $[p]_W(X)$ mod deg $n+1$ over $A/(p, \dots)$:

we know $[p]_W(X) \equiv 0 \bmod p^i = n+1$, so

$$[p]_W(X) = -\lambda_{p^i-1} X^{p^i} \bmod \deg n+2 \text{. Hence}$$

$$-\lambda_{p^i-1} = w_{p^i-1} \text{ modulo } (p, w_{p-1}, \dots, w_{p^i-1}).$$

But by 2) above, $\lambda_{p^i-1} = \alpha_{p^i-1}$ modulo lower dimensional α's.

5)

Proof of

<u>2^{nd} Height Lemma</u>:

If F, G fgplaws over a ring R of char p, such that $F \equiv G + aC_m \mod \deg m+1$, $a \in R$, then

$$[p]_F(X) = [p]_G(X) - aX^m \quad \text{if } m \text{ is a power of } p$$
$$= [p]_G(X) \quad \text{otherwise.}$$

Iwasawa will prove this, possibly this Thurs.

Evidently the lemma is a corollary of a more general

<u>LEMMA</u>: F, G as above, R <u>any</u> comm. ring. Then for $\forall r \in \mathbb{Z}$,
$$[r]_F(X) = [r]_G(X) + a \cdot \{\epsilon_m(r^m - r)\} X^m \mod \deg m$$
where
$$\epsilon_m = 1 \text{ if } m \text{ is } \underline{\text{not}} \text{ a prime power,}$$
$$= p^{-1} \text{ if } m = p^n, \text{ some } n \geq 0$$
$$(\text{N.B. } C_m = \epsilon_m B_m)$$

Proof of <u>more general lemma</u>: by induction on r.

Let $l_r \in R[X]$ be the poly. of degree $\leq m$ such that
$$l_r(X) \equiv [r]_F(X) - [r]_G(X) \mod \deg m+1.$$

I claim $l_{r+1}(X) = l_r(X) + a\{\epsilon_m((r+1)^m - r^m - 1)\}X^m$, which establishes the result inductively: for

(mod deg $m+1$)
$$[r+1]_F(X) = F(X, [r]_F(X)) \equiv G(X, [r]_F(X)) + aC_m(X, [r]_F(X$$
$$\equiv G(X, [r]_G(X)) + l_r(X) + aC_m(X, rX)$$
$$\equiv [r+1]_G(X) + l_r(X) + aC_m(X, rX)$$
and $C_m(X, rX) = \epsilon_m((r+1)^m - r^m - 1)X^m$. //

To deduce the prime ideal Theorem from Thm 5 of Lect. VI part II, we need two lemmas connecting our geometric language to commutative algebra

Suppose $A = k[x_1, \ldots, x_n, \ldots]$ is a polynomial ring over a field, and suppose $\mathfrak{p}, \mathfrak{q} \subset A$ are prime ideals. Write

$$V(\mathfrak{p}) = \{ \chi \in A(k) = \mathrm{Hom}_{k\text{-}Alg}(A; k) \mid \chi(x) = 0, \forall x \in \mathfrak{p} \}$$; similarly,

$$V(\mathfrak{q}) = \{ \chi \in A(k) \mid \chi(\mathfrak{q}) = 0 \}.$$

Lemma 1: If $V(\mathfrak{p}) = V(\mathfrak{q})$ then $\mathfrak{p} = \mathfrak{q}$.

Proof: If A is a comm. ring, its nilradical $\mathfrak{N}(A)$ is the set of nilpotents $= \bigcap$ of all prime ideals. The jacobson radical $= \bigcap$ of all max. ideals $= \{ x \in A \mid 1 - xy$ is a unit in A for $\forall y \in A \}$.

cf Atiyah-MacDonald p 5

If A is a polynomial ring, it is easy to see that these two radicals coincide. In fact a polynomial ring over a field is a Jacobson ring; the nilradical + jacobson radical agree for any quotient ring $A/\mathfrak{r}$ of A

Atiyah-Mac.D Ex.'s 23-24 p71

In particular: if $\mathfrak{q}$ is prime, then the intersection of the maximal ideals of $A/\mathfrak{q}$ is zero (since it is the nilradical, which is zero since $A/\mathfrak{q}$ is a domain

Thus if $x \in A/\mathfrak{q}$ is such that $\chi(x) = 0$, $\forall$ ring-homomorphism $\chi : A/\mathfrak{q} \to k$, it follows that $x = 0$ (since $x \in \bigcap$ max ideals).

<u>Pf. of Lemma 1</u>: Suppose $V(\mathfrak{p}) = V(\mathfrak{q})$, $\mathfrak{p} \neq \mathfrak{q}$.

Then $\exists\, z \in \mathfrak{p}$, $z \notin \mathfrak{q}$ (say); so the image $\underline{z} \in A/\mathfrak{q}$ is <u>nonzero</u>.

But every ring-homomorphism $x : A \to k$ which annihilates $\mathfrak{q}$ (i.e. factors through $A/\mathfrak{q}$) annihilates $\mathfrak{p}$, since $V(\mathfrak{p}) = V(\mathfrak{q})$. Thus $x(\underline{z}) = 0$ for $\forall\, x : A/\mathfrak{q} \to k$, and by the above, $\underline{z} \in A/\mathfrak{q}$ is <u>zero</u>, a contradiction //

This means we can work over an alg. closed field if we want

<u>Lemma 2</u>. If $\mathfrak{p}, \mathfrak{q}$ are distinct ideals in a k-algebra A, then $\mathfrak{p} \otimes_k k'$, $\mathfrak{q} \otimes_k k' \subset A \otimes_k k'$ are distinct, where k'/k is a field extension.

<u>Proof</u>: Let $V_k(\mathfrak{p}) = \{x \in A(k) \mid x(\mathfrak{p}) = 0\}$,
$V_{k'}(\mathfrak{p}) = \{x \in A(k') \mid x(\mathfrak{p}) = 0\}$.

Then we have a diagram

$$\begin{array}{ccc} V_k(\mathfrak{p}) & \neq & V_k(\mathfrak{q}) \\ \cap & & \cap \\ V_{k'}(\mathfrak{p}) & = & V_{k'}(\mathfrak{q}) \end{array}$$

$V_k \subset V_{k'}$ by $(x : A \to k) \mapsto (x : A \to k \subset k')$.

if the lemma is false; but this is absurd. //

§1: The Baas-Sullivan Theorem

Let $\Xi = (\Xi_1, \dots, \Xi_n)$ denote a sequence of (closed compact) cx-oriented manifolds; let $\xi_1, \dots, \xi_n \in U^*(pt)$ denote their cobordism classes. [*Notation*: $\dim \xi_i = 2|\xi_i|$].

cf. N.A. Baas, 'On the Bordism Theory of Manifolds with Singularities', *Math. Scand.* to appear soon and D. Sullivan, *Liverpool Singularities II*, "Singularities in Spaces"

Then there exists a *cobordism category* $(\text{cx-Oriented Manifolds})_\Xi$

whose associated cobordism functor $X \longmapsto U^*_\Xi(X)$

is a cohomology theory with the following properties:

0) $U^*_\Xi(X)$ is a $U^*(pt)$-module.

1) If $\hat{\Xi}_i = (\Xi_1, \dots, \Xi_{i-1}, \Xi_{i+1}, \dots, \Xi_n)$ omits the i^{th} manifold, then there is a natural exact Δ of $U^*(pt)$-modules & homomorphisms

$$\begin{tikzcd} U_{\hat{\Xi}_i}^*(X) \arrow[rr, "\cdot \xi_i"] & & U_{\hat{\Xi}_i}^*(X) \arrow[dl, "\mu_i"] \\ & U_\Xi^*(X) \arrow[ul, "\bar{\beta}_i"] & \end{tikzcd}$$

(Here $\cdot \xi_i$ denotes right-mult. by $\xi_i \in U^*(pt)$, and thus changes dimension by $-2|\xi_i|$, μ_i "forgets" that a $\hat{\Xi}_i$-manifold has no Ξ_i-singularity, and β_i is a Bockstein operation which changes dimension by $+2|\xi_i|+1$.)

2)

Corollary 1: Suppose $\xi_1 \neq 0$ in $U^*(pt)$, and that in general ξ_{i+1} is not a 0-divisor in $U^*(pt)/(\xi_1, \dots, \xi_i)$. Then

$$U^*_{\Xi}(pt) = U^*(pt)/(\xi_1, \dots, \xi_n) = U^*(pt)/\underline{\xi},$$

Proof: Use the Δ's and induct //

Definition. A sequence $\underline{\xi} \in U^*(pt)$ having the property of the corollary is a regular sequence of manifolds

Remark. Let $\xi_1, \dots, \xi_n \in U^*(pt)$ be any sequence of manifolds.
Define the Koszul complex of ξ_i to be $K_U(\xi_i) = 0 \to U^*(pt) \xrightarrow{\xi_i} U^*(pt) \to 0$
Define the Koszul complex of $\underline{\xi}$ to be $K_U(\underline{\xi}) = K(\xi_1) \otimes_{U^*(pt)} K(\xi_2) \otimes \cdots \otimes K$

$$\left(\underline{\text{Ex}}: K_U(\xi_1, \xi_2) = 0 \to U \xrightarrow{x \mapsto (\xi_1 x, \xi_2 x)} U \oplus U \xrightarrow{(y,z) \mapsto \xi_2 y - \xi_1 x} U \to 0\right).$$

Then we can restate the corollary as follows:

Cor.* : $\exists$ a spectral sequence converging to $U^*_{\Xi}(pt)$, whose E_1-term is $K_U(\xi_1, \dots, \xi_n)$.

(Then we use)

[Koszul theorem: if $(r_1, \dots, r_n) = \underline{r}$ is a regular sequence in a commutative ring R. Then $K_R(\underline{r})$ is a resolution of $R/(r_1, \dots, r_n)$ by free R-modules, (and conversely)].

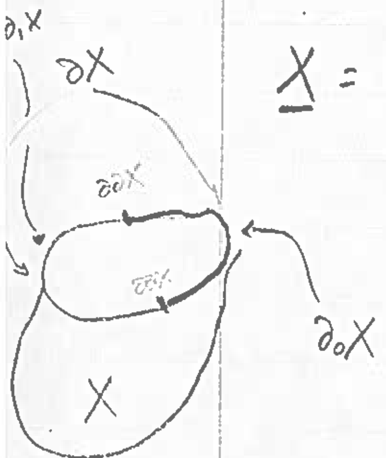
(The advantage of this formulation of Cor. is that it gives some indication of what happens when Ξ is not regular i.e. if $\Xi_i = 0$)

§2:

is one I understand

<u>Example</u>. Suppose $n=1$, $\Xi = (\Xi_1)$.

Then a Ξ-singular (cx-oriented) manifold consists of a <u>quintuple</u>



$\underline{X} = (X, \partial_0 X, \partial_1 X, \sigma X, \alpha_X)$ where

1) $\partial_0 X$, $\partial_1 X$ are submanifolds of ∂X, such that $\partial\partial_0 X = \partial\partial_1 X = \partial_0 X \cap \partial_1 X$,
$\partial X = \partial_0 X \cup \partial_1 X$.

2) $\alpha_X : \partial_1 X \longrightarrow \Xi \times \sigma X$ is a diffeo. of cx-oriented manifolds

The <u>boundary</u> of $\underline{X}$ is

$$\partial \underline{X} = (\partial_0 X, \phi, \partial\partial_0 X, \alpha_X | \partial\partial_1 X)$$

(Clearly $\partial\partial\underline{X} = \phi$)

A <u>morphism</u> $f : \underline{X} \longrightarrow \underline{Y}$ is a pair $(f, \sigma f)$:

$f : (X, \partial X) \longrightarrow (Y, \partial Y)$ taking $\partial_0 X$ to $\partial_0 Y$
$\partial_1 X$ to $\partial_1 Y$

$\sigma f : \sigma X \longrightarrow \sigma Y$ such that

$$\begin{array}{ccc} \Xi \times \sigma X & \xrightarrow{1_\Xi \times \sigma f} & \Xi \times \sigma Y \\ \downarrow & & \downarrow \\ \partial_1 X & \xrightarrow{f} & \partial_1 Y \end{array}$$

commutes.

The underlying space of $\underline{X}$ is $X \underset{\alpha_X}{\cup} C\Xi \times \sigma X$ where $C\Xi$ is the cone on Ξ.

Def: The Bockstein $\beta_i : U_{\Xi}^*(X) \longrightarrow U_{\Xi}^{*+2|\xi_i|+1}(X)$ is

defined as the composition $\beta_i = \mu_i \bar{\beta}_i$;

we can construct it geometrically in the example:

$\underline{X}$ closed, i.e. $\partial \underline{X} = \emptyset$

$\beta_{\Xi}(\underline{X}) = (\sigma X, \emptyset, \emptyset, \emptyset, 1 : \emptyset \to \emptyset)$. Evidently $\beta_{\Xi} \circ \beta_{\Xi} = ($

Ex. if $\dim \Xi = 0$, i.e. Ξ is an integer, then we get the usual Bockstein.

Remark. From Baas's results, it follows that the diagram

$$\begin{array}{ccc} U^*_{\hat{\Xi}_{i,j}} & \xrightarrow{\mu_j} & U_{\hat{\Xi}_i} \\ \downarrow \mu_i & & \downarrow \mu_i \\ U^*_{\Xi_j} & \xrightarrow{\mu_j} & U^*_{\Xi} \end{array}$$

commutes, and

$$\begin{array}{ccc} U^*_{\Xi} & \xrightarrow{\bar{\beta}_i} & U_{\Xi_i} \\ \downarrow \bar{\beta}_j & & \downarrow \bar{\beta}_j \\ U_{\Xi_j} & \xrightarrow{\bar{\beta}_i} & U_{\hat{\Xi}_{i,j}} \end{array}$$

anticommute (at least if all the Ξ_i are even-dim

Consequently $\beta_i \beta_j = -\beta_j \beta_i$ as maps of $U_{\Xi}^*(-)$ to itself; in other words

Corollary 2. There is a natural action of the exterior algebra $\Lambda^*(\beta_1, \dots, \beta_n)$ on $U_{\Xi}^*(-)$.

5)

§3 Coh. Op's on U^*_{Ξ}.

It is tempting to hope that the coalgebra S_* coacts on U_{Ξ}, at least if $\underline{\xi} \subset U^*(pt)$ is <u>invariant</u>, i.e. an S_*-comodule.

Indeed if $s_{\underline{t}} : \underline{\xi} \longrightarrow \underline{\xi}[\underline{t}] = \underline{\xi} \otimes_{\mathbb{Z}} S_*$ is the coaction map there is an induced map

$$s_{\underline{t}} : \underline{\xi}/\underline{\xi}^2 \longrightarrow \underline{\xi}/\underline{\xi}^2 \otimes_{\mathbb{Z}} S_* \quad \text{making}$$

$\underline{\xi}/\underline{\xi}^2$ into an S_*-comodule.

β_{ξ} has degree +1 if $\underline{\xi}/\underline{\xi}^2$ gets its natural grading.

<u>Def</u>: $\beta_{\underline{\xi}} : U_{\Xi}^*(X) \longrightarrow U_{\Xi}^* \otimes_{U^*(pt)} \underline{\xi}/\underline{\xi}^2$ is the <u>Total</u> <u>Bockstein</u>

$$\beta_{\underline{\xi}}(x) = \sum \beta_i x \otimes [\xi_i], \quad [\xi_i] \in \underline{\xi}/\underline{\xi}^2 \text{ being a generator.}$$

<u>Conjecture</u>. If $\underline{\xi}$ is regular and invariant, then the operations $s_{\underline{t}}$ act on U^*_{Ξ}, and are related to the Bocksteins by

$$\underline{s_{\underline{t}}\, \beta_{\underline{\xi}}(x)} = \beta_{s_{\underline{t}}(\underline{\xi})}(s_{\underline{t}}\, x).$$

<u>TRUE</u> using results of <u>Johnson-Wilson</u>; cf p 11, if $\underline{\xi}$ is e.g. prime, or even $(p, w_{p-1}, \ldots, w_{p^{n-1}-1}, \widehat{w_{p^n-1}})$.

6)

The proof uses

<u>Prop 1</u>. $U^*_\Xi(-)$ is a $U^*(pt)/(\xi)$ - module; i.e. mult. by ξ_i on $U^*_\Xi(-)$ is zero.

—— Proof deferred ——

Correct statement of Prop 1 for $\forall$ cobordism theory Ω: associate to Ξ_i, the cobordism class $\xi_i \in \Omega^*(pt)$ of the manifold

$\Xi_i \times [0,1] \times \Xi_i$ modulo the equivalence relation $(x,0,y) \sim (y,1,x)$.

x. Ξ_i = two points. ξ_i = 3 circles, one with a funny frame.

If $\xi_i = 0$ in $M^*(pt)$, then ξ_i - mult. on M_Ξ is zero.

(True in general for $U^*(pt)$, since there's nothing in the odd dimensions

In the next 4 pages I will sketch my attempts at a proof, which have been superceded by J & W; an ill-favored thing, gentlemen, but mine own.

§4

An Auxilliary Construction

We write 2^Ξ for the set of subsets of the set $(\Xi_1, \dots, \Xi_n)$.

I will construct a contravariant functor

$$MU_\Xi : 2^\Xi \longrightarrow (\text{Suspension Spectra})$$ [with honest maps, not homotopy classes, as morphisms]

(morphisms in 2^Ξ are inclusions) together with a homotopy equivalence $\epsilon_S : MU_\Xi(S) \longrightarrow MU$ = the usual Thom spectrum.

The diagram

$$\begin{tikzcd}[column sep=small]
U^*(pt) = \pi_*(MU(\phi)) \arrow[r, "\epsilon_S^{-1}"] \arrow[drrr, "\xi^S_T = \prod_{i\in S-T} \xi_i\text{-mult.}"'] & \pi_*(MU(S)) \arrow[r] & \pi_*(MU(T)) \arrow[r, "\epsilon_T"] & \pi_*(MU(\phi)) \arrow[d, phantom, "\shortparallel"] \\
& & & U^*(pt)
\end{tikzcd}$$

$T \subset S$

commutes.

Note that the $MU(S)$ are homotopy-equivalent, but not necessarily the same; while the diagram

$$\begin{tikzcd}
MU(S) \arrow[r] & MU(S\cap T) \\
MU(S\cup T) \arrow[u] \arrow[r] & MU(T) \arrow[u]
\end{tikzcd}$$

is to commute <u>on the nose</u>.

Cf. for two singularities we consider diagrams of the form

$$\begin{tikzcd}[column sep=tiny]
& X & \\
X_1 \arrow[ur] \arrow[d] & X_{12} \arrow[l] \arrow[r] \arrow[u] \arrow[dl] \arrow[d] \arrow[dr] & X_2 \arrow[ul] \arrow[d] \\
\Xi_1 & \Xi_{12} \arrow[l] \arrow[r] & \Xi_2
\end{tikzcd}$$

in which every straight seq. $A \leftarrow B \rightarrow C$ expresses B as the product of A and C, cf. G. Segal, classifying spaces & cohomology theories.

We can assume that the maps $MU(S) \to MU(T)$ are cofibrations. I'll do my construction some other time: essentially, $MU(S)$ is the Quinn classifying spectrum of the cobordism category of manifolds "that look like $\prod_{i\in S} \Xi_i \times X$" for some manifold X. The map from $MU(S)$ to $MU(T)$, $T\subset S$, is induced by the forgetful functor.

<u>Def</u>. Consider the spaces $MU_{\Xi}[n]$ defined as follows:

$$MU_{\Xi}[n] = \bigvee_{\text{card } S = n} \underline{MU}(S).$$

n.b. any $C = 2^{\Xi}$ inherits a nat'l order from Ξ.

There is a natural map $\partial_i : MU_{\Xi}[n] \to MU_{\Xi}[n-1]$, $i = 1, \dots, n$ defined on $\underline{MU}(S)$ to be the map $\underline{MU}(S) \to \underline{MU}(\partial_i S)$ induced by $\partial_i S \subset S$, where $\partial_i S$ is the subset ommitting the i^{th} element in S.

§5. Spectral Sequences

This is then a functor from the cat. Δ (no degeneracies) to (Spectra), so it has a geometric realization

$$MU_{\Xi}. = \coprod_{\partial's} \Delta^n \times MU_{\Xi}[n].$$

I'll try to do all my Hopf algebra structure in homology

<u>Proposition 2</u>: There is a spectral sequence converging to $\pi_*(MU_{\Xi} \wedge X)$, with E^1-term $= K_U(\xi_1, \dots, \xi_n) \otimes_{U^*(pt)} U_*($

The Koszul cx. is actually a DGA-algebra, with underlying algebra $\Lambda^*_U(\xi/\xi^2)$

<u>Proof</u>: Use the spectral sequence of a filtered space, applied to the skeleton filtration of MU_{Ξ}.

(More generally suppose h_* is any homology theory, and $X : 2^{\Xi} \to (\text{Spaces})_*$ is a functor as above. Then we define $|X|$ for the associated simplicial realization, and there is a s.s. converging to $h_*(|X|)$ whose E^1_*-term is the complex

$$h_* X : \cdots \to \bigoplus_{\text{card } S = q} h_*(X(S)) \xrightarrow{d} \bigoplus_{\text{card } S = q-1} h_*(X(S)) \to$$

where $d = \sum_{i=1}^{q} (-1)^i h_*(\partial_i)$.

Now there is a canonical iso.

$$\epsilon_S : \pi_*(X \wedge MU(S)) \to \pi_*(X \wedge MU(\emptyset)) = U_*(X),$$

so applying the above with the homology theory $h = \pi_*(X \wedge -)$, we get

$\bigoplus_{\mathrm{card}\,S=q} \pi_*(X \wedge MU(S)) \cong \bigoplus_{\binom{n}{q}\text{ times}} U_*(X)$. This complex is

then just $(\text{Koszul}) \otimes_{U_*(pt)} U_*(X)$.

—//—

<u>Prop. 3</u>. There is a <u>trigraded</u> spectral sequence

converging to $\pi_*(MU_{\Xi} \wedge MU_{\Xi})$, with E^1_{***} term $=$

$$K_U^*(\underline{\xi}) \otimes_U \pi_*(MU \wedge MU) \otimes_U K_U^*(\underline{\xi})$$

<u>Proof</u>: This is just the s.seq. of a <u>bi</u>filtered space, assuming such a thing exists. —//—

If $\underline{\xi}$ is a <u>regular</u> sequence, and generates an <u>invariant</u> ideal, then we can compute something.

<u>Lemma</u>. If $\underline{\xi} \subset U^*(pt)$ is an <u>invariant</u> ideal, then $\underline{\xi} \cdot \pi_*(MU \wedge MU) \supset \pi_*(MU \wedge MU)\underline{\xi}$.

<u>Proof</u>: The <u>left map</u> $\pi_*(MU) \longrightarrow \pi_*(MU \wedge MU)$ (cf. Adams) sends $x \longmapsto x \otimes 1$ if we identify $\pi_*(MU \wedge MU)$ with $U \otimes S_*$

<u>right map</u> $\pi_*(MU) \longrightarrow \pi_*(MU \wedge MU)$ sends $x \longmapsto S_{\underline{t}}(x) \in U^*[\underline{t}]$ if we identify

$Z[t]$ with S_*

But $(\underline{\xi} \otimes 1)(U \otimes_{\mathbb{Z}} S_*) = \underline{\xi} \otimes S_* \supset (U \otimes S_*)(s_t \underline{\xi})$.

//

<u>Corollary</u>. Right-mult. of $U^*(pt)/(\underline{\xi}) \otimes_U \pi_*(MU \wedge MU)$ by $\sigma \in \underline{\xi}$ is zero. //

Now the bicomplex $K_U(\underline{\xi}) \otimes_U \pi_*(MU \wedge MU) \otimes_U K_U(\xi)$ is a $U^*(pt)$-free resolution of the complex $U^*(pt)/(\xi) \otimes_U \pi_*(MU \wedge MU) \otimes_U K_U(\xi)$; and by the corollary, the differentials in this latter complex (which are given by right-multiplication with various $\xi_i \in \xi$) are <u>zero</u>.

Consequently the homology of the total complex is $U^*(pt)/(\xi) \otimes_U \pi_*(MU \wedge MU) \otimes_U \Lambda^*_U(\underline{\xi}/\underline{\xi}^2)$.

[Thm. of Johnson-Wilson, if $\xi = (p, w_p, \ldots, w_{p^{n-1}})$.]

<u>Conjecture</u>. If $\underline{\xi}$ is regular and invariant, the trigrade spectral sequence collapses (for dimensional reasons?) to an isomorphism

$$\pi_*(MU_{\Xi} \wedge MU_{\Xi}) \cong U^*(pt)/(\underline{\xi}) \otimes S_* \otimes \Lambda^*_U(\underline{\xi}/\underline{\xi}^2)$$

<u>JW Thm</u>:

$$\{MU_{\boxminus}, MU_{\boxminus}\}_* \cong \Lambda^*_U(\xi/\xi^{\overset{\vee}{2}}) \underset{\mathbb{Z}}{\otimes} S^*, \text{ if}$$

[all the subsequences $(\xi_1, \ldots, \xi_i) \subset (\xi_1, \ldots, \xi_n)$ generate invariant ideals.]

<u>Proof</u>: directly from Baas's spaces, by induction.

Lect VII : Localization, for the last time.

Coherent sheaves are "generically" like vector bundles.

Let k be a field, $U_k = \mathrm{Hom}_{\mathrm{Rings}}(U^*(pt); k)$ is identified with the (ringed) space of all formal group laws over k

If X is a finite CW-cx.,

$X \longmapsto \tilde{U}^*(X;k)$ is a cohomology theory taking values in the cat. of
(Sheaves of $\mathcal{O}_U$-modules over U_k)
(with $\mathcal{O}_U$ the natural sheaf of rings)

We need a

Finiteness Thm: (Conner-Smith, Adams)

X a finite cx. $\Rightarrow \tilde{U}^*(X;k)$ is a coherent sheaf

(The cat. of coherent sheaves over U_k = the cat. of finitely-presented $U^*(pt;k)$-modules. Since $U^*(pt;k)$ is a poly. ring, these cat.'s are abelian and $U^*(X)$ gets a finite presentation by inducting up the cell structure of X). //

We need this finiteness because we want to apply a

Generic Flatness Theorem

Let $\mathcal{F}$ be a coherent sheaf of $\underline{O}_U$ - modules over U_k
Then $\exists$ a dense open $D \subset U$, such that

$\mathcal{F} | D$ (restricted to D) is locally free $\left\{ \begin{array}{l} \text{i.e. } \forall x \in D \text{ possesses an op} \\ \text{nbd } x \in D_x \subset D \text{, such th} \\ \mathcal{F} | D_x \cong \underline{O}_U^r | D_x \text{ for som} \\ \text{integer } r \geq 0. \end{array} \right.$

Proof: (This is a very special case of what's usually called the generic flatness thm, cf. below)

Here's the argument in the "of finite type" more familiar module-language.
Given $M \in$ A-mod, A commutative + e.g. noetherian.
Consider $\mathrm{Ass}\, M$ = set of associated prime ideals of M.

Atiyah-MacDonald p.52
Bourbaki Alg. Comm. IV §1 Def. 1

$\vdash$: Noetherian $\Rightarrow$ $\mathrm{Ass}\, M$ is a finite set of primes

[This is also true for finitely presented modules over $U^K(\mu)$]

$\mathfrak{p}$ prime in A is associated to M if it lies in some annihilator ideal $\mathrm{Ann}\, x \subset A$ for some $x \in M$.

Let S be the multiplicative set generated by $\{x \in \mathfrak{p} \in \mathrm{Ass} M\}$.

$\vdash$: $S^{-1}M$ is free over $S^{-1}A$.

Bourbaki IV §1, 4 Thm 1

Proof: Use the filtration thm: M has a filtration $M_0 \supset M_1 \supset \dots \supset M_n$ such that $M_i/M_{i+1} \cong A/\mathfrak{p}_i$ where $\mathfrak{p}_i \in \mathrm{Ass}\, M$. If we invert all the primes in $\mathrm{Ass}\, M$, $S^{-1}M$ has a filtration with quotients =

3

$S^{-1}A/\{0\}$, so $S^{-1}M$ is free over $S^{-1}A$. /

(assume for simplicity A is a k-algebra)

Now note that localizing with respect to S is the same (in the geo. language) as restricting the module $\tilde{M}$ over $\mathrm{Hom}_{k\text{-alg}}(A, k) = A(k)$ to the

set $D_M = \{ X : A \to k \mid X(s) \neq 0,\ s \in \wp \subset \mathrm{Ass}\, M \}$

$= \{ X : A \to k \mid X(a) \neq 0$ if $ax = 0$ for some $x \in M$ $(x \neq 0)$

In view of the finiteness of Ass M, D_M is open + dense: it's the complement, roughly speaking, of the support of the torsion of M.

e.g. a scheme

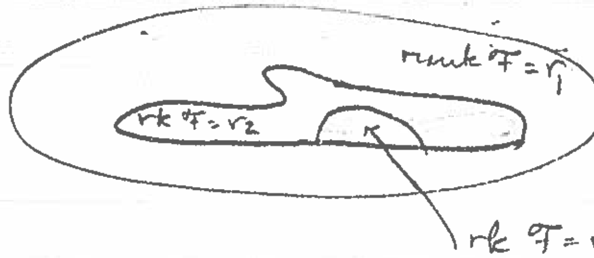
Mumford, Curves on an Alg. Surface, Lect 8 p 55.

* There's a more general theorem which appears to be deep: namely If $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules over some nice ringed space X, then $\exists$ stratification $X = X_0 \cup X_1 \cup \cdots$ of X into locally closed subsets, $\ni$ $\mathcal{F}|X_i$ is a free sheaf of $\mathcal{O}_{X_i}$-modules.

A is locally closed $\Leftrightarrow$ A = (Open Set) $\cap$ (Closed Set)

i.e. $\forall$ coh. sheaf looks like

(Notice that this stratification depends on $\mathcal{F}$)



etc

MAIN APPLICATION | Let $\mathcal{F}$ be a coherent sheaf on a space X, equivariant with respect to a group G. If G acts transitively, then $\mathcal{F}$ is locally free.

we dont need the fact that D is dense.

Proof: By the generic flatness theorem, $\exists\, D^{\text{open}} \subset X$ such that $\mathcal{F}|D$ is free. But by transitivity of G, the translates of D cover X; so each point has a neighborhood in which $\mathcal{F}$ is free. //

§2: Taking fibers at a point on an open orbit is an exact functor.

Lemma 1: Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of free A-modules. Let $x : A \to k$ be a ringhom. Then

$$0 \to M' \underset{x}{\otimes} k \to M \underset{x}{\otimes} k \to M'' \underset{x}{\otimes} k \to 0$$

is exact.

Proof: Free things split.

Locally free module is synonymous with projective.

Corollary 2: Let $0 \to \widetilde{M}' \to \widetilde{M} \to \widetilde{M}'' \to 0$ be an exact sequence of locally free sheaves of $\mathcal{O}_U$-modules on U_k; let $x \in U_k$. Then

$$0 \to M'_x \to M_x \to M''_x \to 0$$

is an exact sequence of k-vector-spaces.

[Here M_x is the fiber of $\widetilde{M}$ at x,

M is a $U^*(pt;k)$ module, and k is a $U^*(pt;k)$-module via x.

$$M_x = M \underset{x}{\otimes} k\Big] \Big(= \widetilde{M}_x \underset{\mathcal{O}_{U,x}}{\otimes} k$$

; but this is too elaborate.

$\widetilde{M}_x$ is the stalk of $\widetilde{M}$ at x, $= S_x^{-1} M$, where $S_x = \{u \in U^*(pt) \mid x(u) \neq 0\}$; similarly $\mathcal{O}_{U,x} =$

$S_x^{-1} U^*(pt;k)$ is a local k-algebra, with residue field $S_x^{-1} U^*(pt;k) \xrightarrow{x} k$.)

<u>Proof</u>: There exists an open D in U_k, containing x, such that

$\widetilde{M}'|D$, $\widetilde{M}|D$, and $\widetilde{M}''|D$ are all <u>free</u>. Take sections over D: we get an exact sequence

$$0 \to \Gamma_D \widetilde{M}' = S^{-1}M' \to \Gamma_D \widetilde{M} = S^{-1}M \to \Gamma_D \widetilde{M}'' = S^{-1}M'' \to 0$$

(where $S \subset U^*(pt;k)$ is the mult. set defining the open set D) of <u>free</u> modules over $S^{-1}U^*(pt;k)$. Since $x \in D$, there is a factorization

$$\begin{tikzcd} & S^{-1}U^*(pt;k) \arrow[dr, dashed, "\chi_D"] & \\ U^*(pt;k) \arrow[ur] \arrow[rr, "\chi"'] & & k \end{tikzcd}$$

and consequently

$$\begin{tikzcd} 0 \arrow[r] & S^{-1}M' \otimes_{\chi_D} k \arrow[r] \arrow[d] & S^{-1}M \otimes_{\chi_D} k \arrow[r] \arrow[d] & S^{-1}M'' \otimes_{\chi_D} k \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M'_x \arrow[r] & M_x \arrow[r] & M''_x \arrow[r] & 0 \end{tikzcd}$$

is exact by the lemma. //

MAIN APPLICATION REVISITED

Suppose h^*: (Finite Cxes) $\longrightarrow$ (G-equivariant Coherent Sheaves on X) is a cohomology theory, and suppose that G acts <u>transitively</u> on X. Then

$$A \longmapsto h^*(A)_x \quad (= \text{fiber at } x \in X)$$

is a cohomology theory.

<u>Proof</u>: Apply the <u>corollary</u> to the MAIN APPLICATION; all we need to check is exactness.

Faithfully Flat Descent

(my candidate for the funniest name in all of mathematics)

Suppose now k is a field, k' a Galois extension, with galois group Π. If E is a k-vectorspace, Then $E \otimes_k k'$ is a k'-vectorspace, with a natural Π-action: if $g \in \Pi$, $e \otimes r \in E \otimes_k k'$, Then

$$g(e\otimes r) = e \otimes gr.$$

Evidently E is canonically isomorphic to $(E \otimes_k k')^{\Pi} = \{v \in E \otimes_k k' \mid gv = v, \forall g \in \Pi\}$ = The fixpt. set of Π acting on $E \otimes_k k'$.

<u>Proposition 3</u>: Suppose $E \xrightarrow{\alpha} F \xrightarrow{\beta} G$ is a sequence of k-vectorspaces, such that $\ker \beta' = \operatorname{im} \alpha'$ in The diagram $E \otimes_k k' \xrightarrow{\alpha'} F \otimes_k k' \xrightarrow{\beta'} G \otimes_k k'$. Then

image α = kernel β.

<u>Proof</u>: Evidently

$$\text{image } \alpha = (\text{Image } \alpha')^{\Pi}$$
$$\text{kernel } \beta = (\text{kernel } \beta')^{\Pi}. \quad //$$

<u>Cor 4</u>: If $0 \to E \to F \to G \to 0$ ($\in k$-Vect) becomes exact after tensoring with k', Then it is exact in The first place. //

<u>Cor 5</u>. Suppose A is a commutative k-algebra, $M \in A$-mod, and $M \otimes_k k'$ is flat (projective) over $A \otimes_k k'$. Then M is flat (projective) over A.

<u>Proof</u>: M is flat $\Leftrightarrow$ $\otimes_A M$ preserves exactness. (If M is flat and f.g., then M is projective.)

§4 MAIN APPLICATION gets applied.

Consider the coherent sheaf $\widetilde{U}^*(X;\mathbb{F}_p)$ over the space $U_{\mathbb{F}_p}$ of all formal groups over $\mathbb{F}_p$.

we didn't prove this, but we're only using it for motivation

By the strong generic flatness theorem (cf. page 3), the sheaf $\widetilde{U}^*(X;\mathbb{F}_p)$ determines a stratification of $U_{\mathbb{F}_p} = \bigcup_{n\geq 1} U_{\mathbb{F}_p}(n)$

(which depends a priori on X) such that

$\widetilde{U}^*(X;\mathbb{F}_p)$ pulls back to a locally free sheaf over each stratum

Prop 5: The height stratification gives a universal stratification, independent of X, with the above property.

[i.e. the strata determined by $\widetilde{U}^*(X;\mathbb{F}_p)$, for $\forall X$, are unions of universal strata]

Proof: We set $U_{\mathbb{F}_p}(n) =$ space of formal groups of ht n over $\mathbb{F}_p$

$$= \{\chi : U^*(pt) \to k \mid \chi(w_{p^i-1}) = 0,\ i = 1,\dots,n-1;\ \chi(w_{p^n-1}) \neq 0\}$$

We have to prove that $\widetilde{U}^*(X;\mathbb{F}_p)$ pulls back to a locally free sheaf over $U_{\mathbb{F}_p}(n)$. It suffices to show that $\widetilde{U}^*(X;\overline{\mathbb{F}}_p)$ is locally free over $U_{\overline{\mathbb{F}}_p}(n)$, where $\overline{\mathbb{F}}_p$ = alg. closure of $\mathbb{F}_p$;
(by Cor. 5 above, projective (= locally free) over $\overline{\mathbb{F}}_p$ implies projective (= locally free) over $\mathbb{F}_p$.

But $\Gamma(\overline{\mathbb{F}}_p)$ acts transitively on $U_{\overline{\mathbb{F}}_p}(n)$, since any two formal group laws of the same height over an algebraically closed field are isomorphic, and $\Gamma(\overline{\mathbb{F}}_p)$ acts on $\widetilde{U}^*(X;\overline{\mathbb{F}}_p)$. Hence we apply Main Application (p. 3), $X = U_{\overline{\mathbb{F}}_p}(n)$

$$G = \Gamma(\overline{\mathbb{F}}_p) \qquad \mathcal{F} = \widetilde{U}^*(X;\overline{\mathbb{F}}_p). \quad //$$

. . p v.

Landweber's exact functor thm. gives a very quick proof that there are coh. theories, but this argument allows us to classify them.

<u>Prop 6.</u> Let $\chi : U^*(pt) \to k$ (= field of char $p>0$) have the property that $\chi(\mathbb{CP}(p-1)) \neq 0$. Then

$$X \longmapsto K_\chi^*(X;k) = U^*(X;\mathbb{F}_p) \underset{\chi}{\otimes} k \quad \text{is a cohomology}$$

theory; if k is alg. closed, then all such K_χ are equivalent.

<u>Proof:</u> Here $K_\chi^*(X;k)$ is the fiber of $\tilde{U}^*(X;k)$ over $\chi \in ($

By hypothesis, χ lies on the height one stratum

$U_k(1) = \{ \chi : U^*(pt) \to k \mid \chi(\mathbb{CP}(p-1)) \neq 0 \}$. Since

$U_k(1)$ is an orbit of Γ, it follows that $\tilde{U}^*(X;k) \mid U_k(1)$ is locally free, so taking fibers is exact, and we get a cohomology theory. //

<u>N.B.</u> If X is a ringed space, $D \subset X$ an <u>open</u> subspace, then the functor

<u>Restrict</u>: (Sheaves of Modules on X) $\longmapsto$ (Sheaves of Modules on D) is <u>exact</u>.

<u>For example</u>:

(Sheaves of modules on U_k) $\longmapsto$ (Sheaves of modules on $U_k(1)$).

is exact: this restriction is really localization: if $M \in (\tilde{U}^*(pt;k)-\text{Mod})$; it goes to $M[\mathbb{CP}(p-1)^{-1}]$.

But if D is <u>not</u> open, e.g. $D = U_k(2)$ (2^{nd} stratum)

<u>Restrict</u>: (Sheaves of Modules on X) $\longrightarrow$ (Sheaves of Mod's on D) is <u>not</u> in general exact; so we can't get a coh. theory by <u>algebraically</u> restricting to $U_k(2)$.

<u>Prop 1</u>. $U^*_\Xi(X)$ is a $U^*(pt)/(\Xi)$ – module.

<u>Proof</u>: This amounts to proving the following:

If $\underline{M} \xrightarrow{f} X$ is a Ξ-singular bordism class, [i.e. $\underline{M}$ is a <u>closed</u> Ξ-singular manifold $(M, \partial M, \sigma M, \alpha_M)$ with $\alpha_M : \partial M \xrightarrow{\cong} \Xi \times \sigma M$, and $f: M \to X$, $f_\sigma : \sigma M \to X$ are maps such that

$$\begin{tikzcd}
\partial M \arrow[r, "\alpha_M"] \arrow[ddr, "f|\partial M"'] & \Xi \times \sigma M \arrow[d, "pr_2"] \\
 & \sigma M \arrow[d, "f_\sigma"] \\
 & X
\end{tikzcd}$$

commutes]

Then $\exists\ \underline{W} \xrightarrow{F} X$ such that $\partial \underline{W} = \Xi \times \underline{M}$
$\underline{F}|\partial\underline{W} = \underline{f} \circ pr_2$.

i.e. if $[\underline{M}, f] \in U^*_\Xi(X)$, then $\Xi \times [\underline{M}, f] \underset{\Xi}{\simeq} 0$.

(In general if T is <u>any</u> closed cx-oriented manifold, $T \times \underline{M} = (T\times M, T\times\partial M, T\times\sigma M, \alpha_{T\times M})$ where

here's where the twist comes in

$$\alpha_{T\times M} : T\times\partial M \xrightarrow{1_T\times\alpha_M} T\times S\times\sigma M \xrightarrow{\text{twist}} S\times T\times\sigma M$$

(You can multiply by T on the right, without introducing the twist yet; but that just slips it under the rug.))

<u>Step 1</u> : Compare the Ξ - singular manifolds

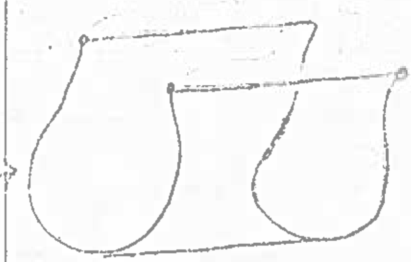
$\Xi\times\underline{M} = (\Xi\times M, \Xi\times\partial M, \Xi\times\sigma M, \alpha_{\Xi\times M} : \Xi\times\partial M \xrightarrow{1_\Xi\times\alpha_M} \Xi\times\Xi\times\sigma M \xrightarrow{\text{twist}\times 1_{\sigma M}} \Xi\times\Xi\times\sigma M)$

$\underline{N} = (\Xi\times M, \Xi\times\partial M, \Xi\times\sigma M, \beta_{\Xi\times M} : \Xi\times\partial M \xrightarrow{1_\Xi\times\alpha_M} \Xi\times\Xi\times\sigma M)$ (no twist)

Now N is obviously Ξ-cobordant to 0:

$I = [0,1]$ Let $\underline{W} = (\overset{W}{\Xi\times I\times M}, \overset{\partial_0 W}{\Xi\times 0\times M}, \overset{\partial_1 W}{\Xi\times 1\times M \cup_{1\times M} \Xi\times I\times \partial M}, \overset{\sigma W}{1\times M\cup I\times\partial M}, \text{identity})$

$\Xi\times 0\times M = \partial_0 W$



$\Xi\times I\times M$

Remaining $\partial = \partial_1 W$

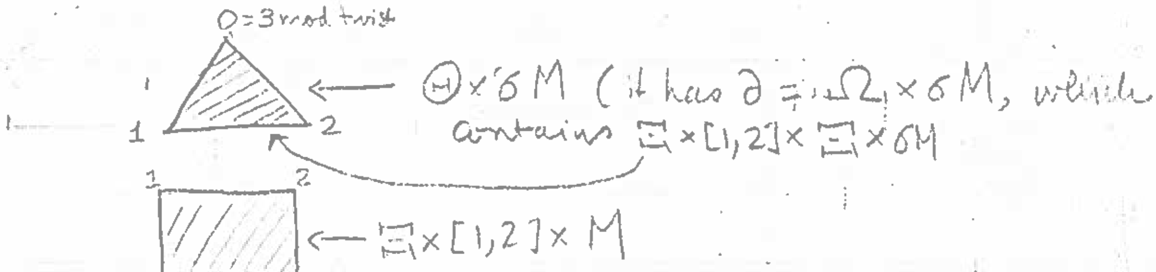
Hence to prove the prop., it suffices to construct

$\underline{V} \ni \partial\underline{V} = \Xi\times\underline{M} \cup \underline{N}$, together with a canonical extension of f to $F: \underline{W} \longrightarrow X$.

<u>Step 2</u>: Let $\Omega_1 = \Xi\times[0,3]\times\Xi$ modulo $(x,0,y)\sim(y,3,x)$. Ω_1 is a closed, odd-dim'l ϵ-oriented manifold, hence a boundary (in the usual sense). <u>Pick</u> $\Theta \ni \partial\Theta = \Omega_1$ (in the usual sense)

<u>Step 3</u>. Construct $\underline{V}$: The underlying manifold is

$\Xi\times[1,2]\times M \cup \Theta\times\sigma M$ like this:



3

Glue together:

this is $\partial_1 \underline{V}$ = image of $\Xi \times [2,3] \times \Xi \times \sigma M$

the rest of the boundary (i.e. ∂_0) decompo…

This part = $\Xi \times 1 \times M \cup \Xi \times [3,1] \times \Xi \times \sigma M$

This part = $\Xi \times 2 \times M$

Finally, define $\sigma V = [2,3] \times \Xi \times \sigma M$; let

$\alpha_V : \partial_1 V \longrightarrow \Xi \times \sigma V$ be the obvious map.

Step 4: α_V, restricted to $\partial\partial_1 V$, is the identity map

$\Xi \times 2 \times \Xi \times \sigma M \longrightarrow \Xi \times 2 \times \Xi \times \sigma M$ on one end

$\Xi \times 3 \times \Xi \times \sigma M \xrightarrow{\text{twist}} \Xi \times 3 \times \Xi \times \sigma M$ on the other.

Proof by geometric intuition.

Hence $\partial_0 \underline{V}$ is diffeo. to $\underline{N}$ (left side) $\amalg$ $\Xi \times \underline{M}$ (right)

Finally, we have to extend $f \circ pr_2 : \Xi \times \underline{M} \longrightarrow X$ canonically to $\underline{V}$. But we extend it in the obvious way to $\Xi \times [1,2] \times \underline{M}$, so we have to extend it compatibly on $\Theta \times \sigma M$: But, if we define it there by $\Theta \times \sigma M \xrightarrow{pr_2} \sigma M \xrightarrow{f_\sigma} \underline{X}$, then this is compatible, so we get a map of $\underline{V} = \Xi \times [1,2] \times M \cup \Theta \times \sigma M$.

1

Seattle notes: Category Theory + Homotopy III Springer?

§1: Cooperations, after Adams.

Theorem: (Adams)

[$\epsilon : S \to E$ is the 2-sided unit. $\mu : E\wedge E \to E$ is the product map.]

Suppose E is a ringspectrum. Then $\pi_*(E\wedge E)$ is a ~~coaided $\pi_*(E)$~~ algebra. If it is left (or eq. right) flat over $\pi_*(E)$, then it has a natural Hopf algebra structure, and there is a natural coaction map

...(X) = ...(E∧X).

$$\psi_X : \pi_*(E\wedge X) \longrightarrow \pi_*(E\wedge E) \underset{\pi_*(E)}{\otimes} \pi_*(E\wedge X)$$, making

$\pi_*(E\wedge X)$ a comodule over the Hopf algebra $\pi_*(E\wedge E)$.

(1) is the ...inal example, ... BU worked pretty well too.

Proof: The Hopf algebra structure comes out of the coaction map ψ_X when $X = E$, so the coaction map is the thing to define:

Definition: $\alpha_X : \pi_*(E\wedge E) \underset{\pi_*(E)}{\otimes} \pi_*(E\wedge X) \longrightarrow \pi_*(E\wedge E\wedge X)$ is defined by the composition

$$\begin{tikzcd}
I \arrow[d] & & \\
\pi_*(E\wedge E) \otimes_{\mathbb{Z}} \pi_*(E\wedge X) \arrow[r] \arrow[d] & \pi_*(E\wedge E\wedge E\wedge X) \arrow[r, "1_E\wedge\mu\wedge 1_X"] & \pi_*(E\wedge E\wedge X) \\
\pi_*(E\wedge E) \underset{\pi_*(E)}{\otimes} \pi_*(E\wedge X) \arrow[urr, dashed, "\alpha_X"'] \arrow[d] & & \\
0 & &
\end{tikzcd}$$

Evidently when $X = pt$, $\alpha_{pt.}$ is an isomorphism.

<u>Lemma</u>. α_X is an isomorphism.

<u>Pf</u>:

Both $X \longmapsto \pi_*(E\wedge E\wedge X)$

$X \longmapsto \pi_*(E\wedge E) \otimes_{\pi_*(E)} \pi_*(E\wedge X)$ are homology theories,

the latter since $\pi_*(E\wedge E)$ is <u>flat</u> and hence injective.

But α_X is a natural transf. of homology theories, and an iso. if $X = pt$. //

<u>Definition</u>. The structure map ψ_X is the composition

$$\pi_*(E\wedge X) \xrightarrow{\sim} \pi_*(E\wedge S\wedge X) \xrightarrow{1_E\wedge\epsilon\wedge 1_X} \pi_*(E\wedge E\wedge X) \xrightarrow[\cong]{\alpha_X^{-1}} \pi_*(E\wedge E) \otimes_{\pi_*(E)} \pi_*(E\wedge X)$$

I want to show that <u>if</u> $(\Xi_1, \ldots, \Xi_n)$ is a regular sequence of manifolds, generating an ideal $\xi \subset U^*(pt)$ which is <u>invariant</u> (as well as <u>regular</u>), and MU_Ξ is a spectrum representing $U_\Xi^*(X)$, <u>then</u>

there is a natural coaction map

$$\psi_X : \pi_*(MU_\Xi \wedge X) \longrightarrow \pi_*(MU_\Xi \wedge MU_\Xi) \otimes_{\pi_*(MU)} \pi_*(MU_\Xi \wedge X)$$

<u>even though</u> MU_Ξ is not (<u>known</u> to be) a ring spectrum.

[I use the fact that it's an <u>MU</u> - module spectrum, tho.]

Note that by the results of lecture VIII, $\pi_*(MU_\Xi \wedge MU_\Xi)$ has a natural Hopf algebra structure:

$$\pi_*(MU_\Xi \wedge MU_\Xi) \cong \pi_*(MU_\Xi) \underset{\pi_*(MU)}{\otimes} \pi_*(MU \wedge MU) \underset{\pi_*(MU)}{\otimes} \Lambda^*_U(\xi/\xi^2).$$

A useful thing not written down anywhere in Adams' lectures.

Here $\pi_*(MU \wedge MU) \cong U^*(pt) \otimes S_*$ if we want; the left $U^*(pt)$-module structure map sends $U^*(pt) \ni u \mapsto u \otimes 1$, the right $U^*(pt)$-module map sends $u \longmapsto s_{\underline{t}}(u)$, where $S_* = \mathbb{Z}[\underline{t}]$.

$$s_{\underline{t}}(u) = \sum b^E \otimes s_E(u)$$

(Recall that ξ invariant means that $s_{\underline{t}}(\xi) \subset \xi[\underline{t}]$, so ξ/ξ^2 has a natural S_*-comodule structure: $s_{\underline{t}} : \xi/\xi^2 \longrightarrow \xi/\xi^2[\underline{t}] = \xi/\xi^2 \underset{\mathbb{Z}}{\otimes} S_*$.

ξ/ξ^2 is a free $U^*(pt)/(\xi)$-module on $[\xi_1], \ldots, [\xi_n]$ and $\Lambda^*_U(\xi/\xi^2)$ is the exterior algebra over $U^*(pt)/(\xi)$.)

Construction of such a ψ_X requires:

<u>Lemma 1</u>. If Ξ is invariant and regular,

$$U_*(\underline{MU}_\Xi) \; (= \pi_*(MU \wedge MU_\Xi)) \cong \pi_*(MU_\Xi) \underset{\pi_*(MU)}{\otimes} \pi_*(MU \wedge MU)$$

as a <u>left</u> $\pi_*(MU)$-module.

<u>Remark</u>. It's easy to see that

$$U_*(\underline{MU}_\Xi) \cong \pi_*(MU \wedge MU) \underset{\pi_*(MU)}{\otimes} \pi_*(MU_\Xi)$$

as a <u>right</u> $\pi_*(MU)$-module.

Proof of <u>lemma 1</u>:

A

Use the s. seq. of a filtered space, on the filtration of $MU_{\boxminus}$ constructed last time.

$\exists$ s. seq. converging to $U_*(\underline{MU}_{\boxminus})$, whose E_1-term

is $\pi_*(MU \wedge MU) \underset{\pi_*(MU)}{\otimes} \underline{K}/(\xi)$.

[The Koszul cx. $K(\xi)$ has, as its underlying algebra, the exterior algebra over $U^*(pt)$ on symbols ξ_i ; the differential is a sum of multiplications by various ξ_i.]

Now if we identify $\pi_*(MU \wedge MU)$ with $U^*(pt) \otimes S_*$, then right-multiplication by ξ_i is the same as <u>left</u>-mult. in $U^*(pt)[\underline{t}] = U^*(pt) \otimes S_*$ by $S_{\underline{t}}(\xi_i)$.

cf. useful remark, p. 3

Thus $\pi_*(MU \wedge MU) \underset{\pi_*(MU)}{\otimes} K_{U^*(pt)}(\xi) = K_{U^*(pt)[\underline{t}]}(S_{\underline{t}}\xi_1, \ldots, S_{\underline{t}}\xi_n)$

If (ξ) is generated by a regular sequence, we can always pick a generating sequence ξ_i with $|\xi_i| > \dim \xi_{i-1}$

I assert that $(S_{\underline{t}}\xi_1, \ldots, S_{\underline{t}}\xi_n) \subset \xi[\underline{t}]$ is a regular sequence, if (ξ) is regular and invariant

[for $S_{\underline{t}}\xi_i = \xi_i +$ (lower-dimensional stuff in ξ)(nontrivial $\underline{t}^\alpha$)

so $(S_{\underline{t}}\xi_1, \ldots, S_{\underline{t}}\xi_n) = \xi[\underline{t}]$ by induction; so if $(S_{\underline{t}}\xi_i)$ isn't regular, it follows that (ξ_i) isn't regular either, Consequently $K(s_{\underline{t}}\xi_1, \ldots, s_{\underline{t}}\xi_n)$ is a resolution of

$$U^*(pt)[\underline{t}] \big/ (s_{\underline{t}}\xi_1, \ldots, s_{\underline{t}}\xi_n) = U^*(pt)[\underline{t}] / \xi[\underline{t}]$$

$$= \left(U^*(pt)/(\xi) \right)[\underline{t}].$$

$$\left[= U^*(pt)/(\xi) \otimes_{\mathbb{Z}} S_* \text{ by defn} \right].$$

———— END of Lemma 1 proof ————

<u>Corollary</u>: $\pi_*(MU \wedge MU_{\Xi})$ is a left flat module over $U^*(pt)/(\xi)$, and $\pi_*(MU_{\Xi} \wedge MU)$ is a right flat module over $U^*(pt)/(\xi)$.

<u>Corollary</u>: $\pi_*(MU_{\Xi} \wedge MU) \otimes_{\pi_*(MU_{\Xi})} \pi_*(MU_{\Xi} \wedge X)$ is a homology Theory

Note that this homology theory carries a natural transformation

$$\delta_X : \pi_*(MU_{\Xi} \wedge MU) \otimes_{\pi_*(MU_{\Xi})} \pi_*(MU_{\Xi} \wedge X) \to \pi_*(MU_{\Xi} \wedge MU_{\Xi} \wedge X)$$

which is <u>not</u> an isomorphism when $X = pt.$; indeed when $X = pt.$ we get

$$\pi_*(MU_{\Xi} \wedge MU) \xrightarrow{\simeq} \pi_*(MU_{\Xi}) \otimes_{\pi_*(MU)} \pi_*(MU \wedge MU) \longrightarrow \pi_*(MU_{\Xi}) \otimes_{\pi_*(MU)} \pi_*(MU \wedge MU) \otimes_{\pi_*(MU)} \Lambda^*(\xi)$$

$$\hookrightarrow \pi_*(MU_{\Xi} \wedge MU_{\Xi})$$

In other symbols, we get the inclusion

$$U^*(pt)/\xi \underset{\mathbb{Z}}{\otimes} S_* \longrightarrow U^*(pt)/\xi \underset{\mathbb{Z}}{\otimes} S_* \underset{\mathbb{Z}}{\otimes} \Lambda^*_{\mathbb{Z}}(B)$$

<u>Lemma 2</u>. $\pi_*(MU_\Xi \wedge MU_\Xi) = \pi_*(MU_\Xi \wedge MU) \underset{\pi_*(MU)}{\otimes} \Lambda^*_U(\xi/\xi^2)$

as a module over the Bockstein operations.

(Since $\pi_*(MU_\Xi \wedge MU_\Xi) = U_{\Xi *}(\underline{MU}_\Xi)$, the geometrically defined Bockstein operations act on it. If we identify $[\xi_i] \in \xi/\xi^2$ with $\beta_i \underline{1}$, we are first claiming that $U_{\Xi *}(MU_\Xi)$ is a free module over the Bocksteins, with $U_{\Xi *}(\underline{MU})$ as a basis.)

<u>Assuming</u> Lemma 2, we construct the coaction:

cf. page 8

Def: $\alpha_{\xi, X}$ is the composition

$$\pi_*(MU_\Xi \wedge MU_\Xi) \underset{\pi_*(MU_\Xi)}{\otimes} \pi_*(MU_\Xi \wedge X) \longrightarrow \text{[illegible]}$$

$$\downarrow$$

$$\Lambda^*(\xi/\xi^2) \underset{\pi_*(MU)}{\otimes} \pi_*(MU_\Xi \wedge MU) \underset{\pi_*(MU_\Xi)}{\otimes} \pi_*(MU_\Xi \wedge X)$$

$$\downarrow S_X$$

$$\Lambda^*(\xi/\xi^2) \underset{\pi_*(MU)}{\otimes} \pi_*(MU_\Xi \wedge MU_\Xi \wedge X) \longrightarrow \pi_*(MU_\Xi \wedge MU_\Xi \wedge X)$$

via the standard action of $\Lambda^*(\xi/\xi^2)$.

Note that when $X = pt$, we get

$$\begin{array}{l} \pi_*(MU_\Xi \wedge MU_\Xi) \\ \quad\downarrow\wr \\ \Lambda^*(\xi/\xi^2) \otimes \pi_*(MU_\Xi \wedge MU) \\ \quad\downarrow \delta_X \\ \Lambda^*(\xi/\xi^2) \otimes \pi_*(MU_\Xi \wedge MU_\Xi) \longrightarrow \pi_*(MU_\Xi \wedge MU_\Xi) \end{array}$$

or in other words

$$\begin{array}{l} U/\xi \otimes S \otimes \Lambda \\ \quad\downarrow \text{twist} \\ \Lambda \otimes U/\xi \otimes S \\ \quad\downarrow \delta \\ \Lambda \otimes U/\xi \otimes S \otimes \Lambda \xrightarrow{\theta} U/\xi \otimes S \otimes \Lambda \end{array}$$

$\delta(\lambda \otimes u \otimes s) = \lambda \otimes u \otimes s \otimes 1$

$\theta(\lambda \otimes u \otimes s \otimes \tau) = u \otimes s \otimes \lambda\tau$ by <u>lemma 2</u>

So $\theta\delta \cdot \text{twist}\,(u \otimes s \otimes \lambda) = u \otimes s \otimes \lambda$ is an <u>isomorphism</u>

<u>Corollary</u> $\alpha_{\xi, X} : \pi_*(MU_\Xi \wedge MU_\Xi) \otimes_{\pi_*(MU_\Xi)} \pi_*(MU_\Xi \wedge X)$

$$\downarrow\wr$$

$\pi_*(MU_\Xi \wedge MU_\Xi \wedge X)$ is an isomorphism

For it's an equivalence of homology theories.

Def.

$$\begin{tikzcd}
\pi_*(MU_\Xi \wedge X) \arrow[r] \arrow[dr] & \pi_*(MU_\Xi \wedge MU_\Xi \wedge X) \arrow[d, "\alpha_{\xi,X}^{-1}"] \\
& \pi_*(MU_\Xi \wedge MU_\Xi) \otimes_{\pi_*(MU_\Xi)} \pi_*(MU_\Xi \wedge X).
\end{tikzcd}$$

is the desired structure map.

Proof of Lemma 2:

Consider the spectral sequence converging to $U_{\Xi *}(MU_\Xi)$ obtained by applying the homology theory $U_{\Xi *}$ to the filtration on MU_Ξ constructed in the prev. lect.

The s.s. degenerates at the E_2-level; but this is a spectral sequence in the cat. of modules over the Bocksteins, and the E_2-term $=$ $\pi_*(MU_\Xi \wedge MU) \otimes \Lambda^*(\xi/\xi^2)$.

—//—

§1

Three categories of cobordism comodules get defined

Cotruncated Cobordism Comodules

Let $[p](T) = pT + \cdots + w_{p-1} T^p + \cdots + w_{p^2-1} T^{p^2} + \cdots \in U^*(pt)[[T]]$ define the elements w_{p^i-1}. The ideal
$I_{p,n} = (p, w_{p-1}, \ldots, w_{p^{n-1}-1}) \subset U^*(pt)$ is invariant;

Hence there is a natural map

$$s_{\underline{t}} : U^*(pt)/I_{p,n} \longrightarrow \left(U^*(pt)/I_{p,n}\right)[\underline{t}] \quad , \text{ because}$$

$$s_{\underline{t}}(u + I_{p,n}) \subset s_{\underline{t}} u + I_{p,n} .$$

<u>Def. 1.</u> A cobordism comodule M^* is an (ungraded) finitely presented $U^*(pt)$ – module, together with a homomorphism $s_{\underline{t}} : M^* \longrightarrow M^*[\underline{t}]$, satisfying

$s_{\underline{t}}(x) = \sum_\alpha (s_\alpha x) t^\alpha$, 1) $s_{\underline{t}}(u \cdot m) = s_{\underline{t}}(u) \cdot s_{\underline{t}}(m)$ $\left[\begin{array}{l} u \in U^*(pt) \\ m \in M^* \end{array}\right.$

$t^\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n} \cdots$, 2) $s_{\underline{t}}(s_{\underline{t}'}(m)) = s_{\underline{t}''}(m)$.

where $\underline{t} = (t_0, t_1, \ldots, t_n, \ldots)$ $t(T) = \sum_{i \ge 0} t_i T^{i+1}$,

$\underline{t}' = (t_0, t_1, \ldots, t_n, \ldots)$ $t'(T) = \sum_{i \ge 0} t_i' T^{i+1}$

and $\underline{t}''$ is defined by

$$\sum_{i \ge 0} t_i'' T^{i+1} = t'(t(T)).$$

<u>Definition 2</u>.

A cobordism comodule M^*, cotruncated to height $\leq n$, is a finitely presented left $U^*(pt)/I_{p,n}$ – module with a homomorphism $s_{\underline{t}} : M^* \longrightarrow M^*[\underline{t}]$, satisfy 2) and the analogue of 1).

<u>Lemma 1</u>: $s_{\underline{t}} w_{p^n-1} = w_{p^n-1}$ modulo $I_{p,n}[\underline{t}]$.

[For $s_{\underline{t}} w_{p^n-1} = w_{p^n-1} + \sum_{|\alpha|>0} z_\alpha t^\alpha$, where

$$z_\alpha \in U^{2(p^n-1)-2|\alpha|}(pt) \cap I_{p,n+1}.$$

But $I_{p,n+1}^{2(p^n-1)-2|\alpha|} \subset I_{p,n}^{-2(p^n-1)-2|\alpha|}$ because

$p, w_{p-1}, \ldots, w_{p^{n-1}-1}$ generate both ideals up to deg. $2(p^n-1)-2$, and $|\alpha|>0$.]

<u>Corollary 1</u>: There is a natural map

$$s_{\underline{t}} : \left(U^*(pt)/I_{p,n}\right)[w_{p^n-1}^{-1}] \longrightarrow \left(U^*(pt)/I_{p,n}\right)[w_{p^n-1}^{-1}][\underline{t}]$$

Def 3. A cobordism comodule M^*, cotruncated to ht $\underline{= n}$ ($M^* \in$ CobComCotr $= n$) is a finitely presented left module over $_nU^*(pt; \mathbb{F}_p) \underset{def}{=} (U^*(pt)/I_{p,n})[w_{p^n-1}^{-1}]$. together with a map

$$S_{\underline{t}} : M^* \to M^*[\underline{t}]$$ satisfying 1), and the analogue of 2

Remark. Quillen's idempotents take $M \in$ CobComCotr $\leq n$ to a module over $\mathbb{F}_p[v_n, v_{n+1}, \ldots] =$ BP cotruncated. Similarly, $M \in$ CobCom Cotr $= n \mapsto$ module over $\mathbb{F}_p[v_n^{-1}, v_n, v_{n+1}, \ldots]$.

§2. The Structure of (CobComCotr $= n$).

Note that CobComCotr $= n$ is equivalent to a category of modules over a ring (e.g. $_nU^*(pt; \mathbb{F}_p) \underset{\mathbb{Z}}{\otimes} S^*$) and is thus abelian.

Our first aim is to define an exact functor

$$(\text{CobComCotr} \leq n) \ni M^* \longmapsto \overline{M} \in (\mathbb{F}_p\text{-Vector Spaces})$$
finite-dimensional

by adding extra structure, we will make this functor into an equivalence of the category (CobComCotr $= n$) to a cat. of group representations

Def 4: $\rho_n : U^*(pt) \to \mathbb{Q}$ is defined by $\rho(\mathbb{CP}(p^{nr}-1)) = p^{(n-1)r}$ if $r \geq 0$

$\rho(\mathbb{CP}(m-1)) = 0$ otherwise

<u>Lemma 2</u>. ρ_n factors through $\mathbb{Z}_{(p)}$, i.e. for any M, $\rho_n(M)$ has denominator prime to p.

<u>Proof</u>: The logarithm $\sum_{m\geq 1} \rho_n(\mathbb{C}P(m-1))\dfrac{X^m}{m} = \sum_{r\geq 0} p^{-r} X^{p^{nr}}$

defines a formal group law over $\mathbb{Z}_{(p)}$, by Honda / Iwasawa. Hence $\rho_n(c_{i,j}) \in \mathbb{Z}_{(p)}$, where the $c_{i,j}$ generate $U^*(pt)$, $F_c(X,Y) = \sum c_{i,j} X^i Y^j$. //

It can be shown that if $l_n(X) = \sum_{r\geq 0} p^{-r} X^{p^{nr}}$, then

$$
\begin{aligned}
[p]_n(X) = l_n^{-1}(p\, l_n(X)) &= pX + X^{p^n}. \quad \text{Hence} \\
&= \rho_n([p](X))
\end{aligned}
$$

$$
\begin{cases} \rho_n(W_{p^i-1}) \equiv 0 \mod p & \text{if } i<n \\ \qquad\qquad \equiv 1 \mod p & \text{if } i=n. \end{cases}
$$

<u>Corollary 2</u>. $\exists$ a factorization

$$
\begin{tikzcd}
U^*(pt) \arrow[r, "\rho_n"] \arrow[d] & \mathbb{Z}_{(p)} \arrow[d] \\
{}_nU^*(pt;\mathbb{F}_p) \arrow[r, "\overline{\rho_n}"] & \mathbb{F}_p
\end{tikzcd}
$$

<u>Def. 5</u>. If M^* is a cobordism comodule, cotruncated to height $\leq n$, then $\overline{M} = M^* \otimes_{\rho_n} \mathbb{F}_p$

Prop. 1. The functor

$$(\text{Comb Conn Cotr} \leq n) \ni M^* \longrightarrow \overline{M} \in (\mathbb{F}_p\text{-Vector spaces})$$

is exact.

Proof: Use Peter's exact functor Theorem, or think of $\overline{M}$ as the fiber of the sheaf $\widetilde{M}$ on the space of formal groups of height $\leq n$, over the point defined by $\overline{P_n}$. //

Def. 6. Let E_n = the $\hat{\mathbb{Z}}_p$-algebra on indeterminates F, ω where

$$\begin{cases} \omega \text{ is a primitive } (p^n-1)^{\text{th}} \text{ root of } 1 \\ F^n = p \\ F\omega = \omega^p F \end{cases}$$

I'll write S_n = group of units of $E_n = \mathbb{Z}_p[F, \omega]$.

It is a compact Hausdorff topological group.

Remark. E_n is an integral domain, & its quotient ring is a skew field. See Milnor, Lectures on Alg K-Theory, Chapter 8.

Proposition / Definition 2: If $M^* \in (\text{Comb Conn Cotr} \leq n)$ then there is a continuous representation of S_n defined on $\overline{M} \underset{\mathbb{F}_p}{\otimes} \mathbb{F}_{p^n}$.

Proof: The point is that S_n is the stabilizer of $\overline{P_n} \in U_{\mathbb{F}_{p^n}}$

(= space of formal groups over $\mathbb{F}_{p^n}$), but we can be very explicit.

We define a map (N.B. not a homomorphism)

$[\cdot] : E_n \longrightarrow \mathbb{F}_{p^n}[[T]]$ as follows:

1) if $\alpha \in \hat{\mathbb{Z}}_p$, then $\ell_n^{-1}(\alpha \ell_n(T)) \in \hat{\mathbb{Z}}_p[[T]]$; let $[\alpha](T) \in \mathbb{F}_p[[T]]$ be its mod p reduction.

$\ell_n = \sum_{r \geq 0} p^{-r} X^{p^{nr}}$

2) If ω is a primitive $(p^n-1)^{\text{th}}$ root of 1, then $\ell_n(\omega T) = \omega \ell_n(T)$, so if we set $[\omega](T) = \bar{\omega} T$ in $\mathbb{F}_{p^n}[[T]]$, where $\bar{\omega} \in \mathbb{F}_{p^n}$ is a nontrivial element, then $[\alpha] \circ [\omega] = [\omega] \circ [\alpha]$

3) Define $[F](T) = T^p$. Evidently $[\alpha] \circ [F] = [F] \circ [\alpha]$, $[\omega] \circ [F] = [F] \circ [\omega]$.

Now $[\cdot] : E_n \to \mathbb{F}_{p^n}[[T]]$ is the unique map satisfying

$a, b \in E_n$ $\qquad [a]([b](T)) = [ab](T)$

$$\overline{F_n}([a](T), [b](T)) = [a+b](T).$$

[Check the distributive law].

If $a \in E_n$, define $a_i \in \mathbb{F}_{p^n}$ by $[a](T) = \sum_{i \geq 0} a_i T^i$.

<u>Lemma 3</u> if $a \in E_n$ with $a_0 = 1$, then the diagram

$$\begin{tikzcd} {}_n\mathcal{U}^*(pt; \mathbb{F}_{p^n}) \arrow[rr, "S_a"] \arrow[dr, "\bar{\rho}_n"'] & & {}_n\mathcal{U}^*(pt; \mathbb{F}_{p^n}) \arrow[dl, "\bar{\rho}_n"] \\ & \mathbb{F}_{p^n} & \end{tikzcd}$$

<u>commutes.</u>

where $S_a = \sum S_\alpha \cdot a^\alpha$, $a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \cdots$.

<u>Proof:</u> E_n is mapped by $[\cdot]$ to the Endomorphism ring of F_n. Apply the formula

$$\sum s_a(a_{ij}) X^i Y^j = [a] F_c([a]^{-1}(X), [a]^{-1}(Y)). \quad //$$

<u>Corollary</u>. $S_a : \bar{M} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n} = M^* \otimes_{\bar{\rho}_n} \mathbb{F}_{p^n} \rightarrow \bar{M} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n}$

makes $\bar{M} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n}$ into a representation of S_n.

[$\bar{M} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n}$ is a continuous representation, for if

$a \in 1 + p^r E_n$ for suff. high r, S_a = identity on $\bar{M}$.

<u>THEOREM 1</u>: The category of cobordism comodules, concentrated to height $= n$, is <u>isomorphic over $\mathbb{F}_{p^n}$</u> to

Not over $\mathbb{F}_p$; there's some Galois twist!

the cat. of (continuous) S_n-representations on $\mathbb{F}_{p^n}$.

Main Corollary:

$$\mathrm{Ext}^*_{\left(\substack{\text{cob. com. Cotr.} = n \\ \text{over } \mathbb{F}_{p^n}}\right)}\left({}_nU^*(pt;\mathbb{F}_{p^n}),\ {}_nU^*(pt;\mathbb{F}_{p^n})\right)$$

$$\cong H^*(S_n;\mathbb{F}_{p^n}).$$

Remark: S_n is a compact Hausdorff top. group, and the cohomology is group cohomology with continuous cochains.

To get the result over $\mathbb{F}_p$, we take invariants under $Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$, but the Galois action on $H^*(S_n;\mathbb{F}_{p^n})$ is the obvious one.

Example $n=1$, $S_n = \hat{\mathbb{Z}}_p^*$

$$H^*(\hat{\mathbb{Z}}_p^*, \mathbb{F}_p) = \mathbb{F}_p \quad \text{in dim } 0$$

$$= \mathbb{F}_p \quad \text{in dim } 1$$

the 1-dim'l generator is the e-invariant of Adams.

General philosophy: An element of $H^q(S_n;\mathbb{F}_p)$ is a q-ary cobordism operation, defined when $w_{p-1} = \cdots = w_{p^{n-1}-1} = 0$, $w_{p^n-1} \neq 0$.

[Proof of Main Corollary: If G is a finite group, k a field (with trivial G-action) then $H^*(G;k) = \mathrm{Ext}^*_G(k,k)$. cf. e.g. Maclane, Cartan-Eilenberg.

There is a certain amount of general information about these groups.

Theorem 2.

a) If $n < p-1$, then

$H^*(S_n;\mathbb{F}_p)$ satisfies n^2-dimensional Poincaré duality. [in part, $H^{n^2}(-) \neq 0$].

b) if $p-1 \nmid n$ then $H^*(S_n;\mathbb{F}_p)$ is a nil-ring, hence finite-dimensional [Assuming the Coh. ring is finitely generated as a ring, which must follow from compactness.]

c) If $n = k(p-1)$, then

$\sum \operatorname{rank} H^q(S_n;\mathbb{F}_p)\, t^q$ is a rational function of t, with a pole of order $\underline{1}$ on the unit circle, i.e. $H^*(S_n;\mathbb{F}_p)$ contains $\mathbb{F}_p[x]$ for some x, with $x^n \neq 0$ $\forall n$.

NB this corresponds to periodic phenomena of period approx $= 2(p^{p-1}-1)$

Example. $H^*(S_2;\mathbb{F}_p)$ is 4-dimensional if $p \geq 5$, but ∞-dim'l if $p = 3$.

§3: Proof of the Main Theorem

It uses one deep theorem from algebraic geometry.

Let k be an algebraically closed field; let G be a smooth algebraic group over k, acting on a smooth algebraic variety X, $G \times X \to X$,

such that the action of G is transitive on the points of X.

Then $\bar{p}: G/S \xrightarrow{\sim} X$ is an isomorphism, where S is the stabilizer of a point $x \in X$, $p(g) = gx$ defines a map $G \to G/S \to X$, and the quotient is taken in the cat. of algebraic varieties.

N.B. in the cat. of varieties, the quotient by a general group action need not exist.

Note that S is a subgroup of an algebraic group, hence itself an algebraic group.

If X = Stratum of formal group laws of height $= n$, then X is the affine variety with coordinate ring ${}_nU^{*}(pt; k)$, $k = \bigcup_{n \geq 1} \mathbb{F}_{p^n} = \overline{\mathbb{F}_p}$.

$G = \Gamma(k)$ = group of formal series under composition

Then S = automorphism group of a gp law of ht n, e.g. $\overline{F}_n$ defined by $\overline{e}_n$.

<u>Proposition</u>. S_n is a (pro) <u>discrete</u> algebraic group.

<u>Proof</u>: Its Lie algebra is zero. Cf. my Tata notes on f gps. for the computation.

In general the stabilizers won't be discrete, so we could only reduce the problem down to that of computing the coh. of the <u>alg. group</u> S.

To prove the main theorem, we construct an inverse functor

(Continuous S_n - Representations over k) $\mapsto$ Sheaves over the space of ht. n formal groups.

If $V \in S_n\text{-Reps}_k$, consider the sheaf

[Since S_n is (pro) discrete, $\Gamma \underset{S_n}{\times} V$ is easily seen to exist]

$$\begin{array}{ccc} \Gamma \underset{S_n}{\times} V & = & \mathcal{V} \\ \downarrow & & \downarrow \\ \Gamma \underset{S_n}{\times} pt & = \Gamma/S_n = & (\text{ht. } n \text{ fgps}) \end{array}$$

Since $\mathcal{V} = \Gamma \underset{S_n}{\times} V$, it has a natural Γ-equivariant structure, hence determines a cobordism comodule cotruncated, etc. If, given a sheaf $\mathcal{M}$ over the space of ht. n fgps, we pass to the associated stalk $\overline{M} \in S_n$ - representations, then $\overline{\mathcal{M}} = \Gamma \times_{S_n} \overline{M}$ has a canonical map $\mathcal{M} \to \overline{\mathcal{M}}$ of $\Gamma(k)$ - equivariant sheaves which is an iso. at the point $\overline{P}_n$. Hence it is an iso. everywhere, since Γ acts transitively.

That $\overline{M} \to \dot{\overline{\mathcal{M}}} \to \overline{\overline{M}}$ is an isomorphism amounts to checking that the fiber of $\overline{\mathcal{M}}$ at $\overline{P}_n$ is $\overline{M}$, and requires nothing from Γ.

—/—

§5 Some Remarks on the Computation of $H^*(S_2; \mathbb{F}_p)$, $p \geq 5$.

Let $W = \hat{\mathbb{Z}}_p[\omega]$ be the extension of $\hat{\mathbb{Z}}_p$ by a primitive (p^2-1)-th root of 1. Write

$$\hat{\mathbb{Z}}_p[F, \omega] = W\langle F \rangle \quad \text{where } F^2 = p, \; F\omega = \omega^p F.$$

Evidently S_2 = units of $W\langle F\rangle = \{\alpha + \beta F \mid \alpha \in \text{units of } W\}$

$$= \mu_{p^2-1} \times_{sd} (1 + F\,W\langle F\rangle)^\times, \text{ where}$$

$(1 + FW\langle F\rangle)^\times$ is the group of <u>strict</u> units S_2^0 of E_2

[The action of μ_{p^2-1} = cyclic group generated by ω merely gives the grading; the cohomology of S_2 gives <u>graded</u> higher-order ops, that of S_2^0 = <u>ungraded</u> higher order ops.]

B

There is a <u>determinant</u>

$$\det : 1 + F\,W\langle F\rangle \overset{\cdot}{\longrightarrow} \hat{\mathbb{Z}}_p^*$$
$$= \{\alpha + \beta F \mid \alpha = 1 + p? \in W\}. \to \hat{\mathbb{Z}}_p^* \quad \text{via}$$

$\alpha + \beta F \longmapsto |\alpha|^2 - |\beta|^2 p$, where $|\alpha|^2 = \alpha\bar{\alpha}$, and
$\alpha \mapsto \bar{\alpha}$ is the generator of
$Gal(\mathbb{Q}_p[\omega]/\mathbb{Q}_p)$

If $p \geq 5$, $S_2^0 = (1 + p\hat{\mathbb{Z}}_p)^\times \times S_2^1$ (direct product)

where S_2^1 is a prop-group, satisfying 3-dimensional Poincare duality

$$= \{\alpha + \beta F \mid \alpha = 1 + p?, \ |\alpha|^2 - p|\beta|^2 = 1\}$$

<u>Prop</u>: $H^*(S_2^0; \mathbb{F}_p) \cong \Lambda(e_1) \otimes H^*(S_2^1; \mathbb{F}_p)$.

where $\Lambda(e_1)$ = 1-dim'l exterior algebra generated by Adams' e-invariant

Since $H^0(S_2^1) \cong H^3(S_2^1) \cong \mathbb{F}_p$,

$H^2(S_2^1) \cong (S_2^1)_{ab}$, dual to $H^1(S_2^1)$,

We know that $H^2(S_2^1) \neq 0$ [S_2^1 is a pro-p group, hence has nontrivial abelianiza