THE ADAMS-NOVIKOV SPECTRAL SEQUENCES FOR PROJECTIVE SPACES

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Introduction

Recently, there has been a great deal of activity in understanding the Adams-Novikov spectral sequence for the sphere spectrum, especially making use of the Brown-Peterson spectra. In this paper, we make some observations on the analogues for $\mathbb{F}P^k$ where $\mathbb{F} = \mathbb{C}$ (the complex numbers) or $\mathbb{H}$ (the quaternions). Some of our results will be used in a joint paper with N. Ray and F. Clarke on $\pi_*^S BU$ and thus can be viewed as the first steps in a larger programme of computations in $\pi_*^S BU$.

All notation will hopefully be well known to workers in this field and corresponds to that of [Ad] and [Sw].

I wish to thank Francis Clarke (to whom my understanding of Γ_n^K and introduction to the type of calculations in §1 is due) and Nigel Ray for much help and encouragement; also, Memorial University of Newfoundland for a Post-Doctoral Fellowship in the period 1980-81, especially Renzo Piccinini for many sympathetic discussions. Finally, many thanks to Sandra Crane for her nearly flawless typing of two totally different versions of this paper.

In §1, we describe the coaction primitives in $MU_*\mathbb{F}P^k$. Rather than simply use the method of [Se1], we project into $K_*\mathbb{F}P^k$ and use some properties of Stirling numbers. We also make use of a consequence of the Hattori-Stong Theorem that $PMU_*\mathbb{F}P^k = PK_*\mathbb{F}P^k$.

In §2, we consider the exact sequences of Ext-groups induced by the cellular decomposition $\ldots \subset \mathbb{F}P^{k-1} \subset \mathbb{F}P^k \subset \ldots \subset \mathbb{F}P^\infty$. Unfortunately, we are unable to prove algebraically a result of [Se1] on $\mu_{8k+1}x \in \pi^s_{8k+3}\mathbb{C}P^{4k+2}$ - this would then avoid the use of high degree differentials in the classical Adams spectral sequence. We also give a new proof of a result of [Sn] on the torsion in $\pi^S_*\mathbb{C}P^\infty$.

In §3, we use Novikov's calculation of π_*MSU to show that half of the generators of $PMU_*\mathbb{H}P^\infty$ are not infinite cycles; also, we calculate $PMSp_*\mathbb{H}P^k$.

In §4, we use the generators of $PK_*\mathbb{F}P^\infty$ to determine the e-invariants of the elements in the image of the transfer maps

$$tr_{G*} : \pi^S_*(BG(1)_+) \to \pi^S_*.$$

§1 We begin by establishing certain notations.

$H_*MU = \mathbb{Z}[b_1, b_2, \ldots]$, where $b_n \in H_{2n}MU$ is the canonical generator as in [Ad].

$MU_*\mathbb{C}P^k$ is the free MU_* module on generators $\beta_n \in MU_{2n}\mathbb{C}P^k$ $(n \leq k \leq \infty)$, dual to $cf_1^{\,n}$, again as in [Ad]; we also use this to denote the image under $\underline{h} : MU_*(-) \to (H \wedge MU)_*(-)$.

Similarly, $q_n \in MU_{4n}\mathbb{H}P^k$ will denote the canonical generator, dual to $pf_1^{\,n}$, together with $\underline{h}(q_n)$. Actually, q_n comes from $MSp_{4n}\mathbb{H}P^k$.

$$b(t) = \sum_{0 \leq k} b_k t^k$$

More standard to say $\exp(T) = \sum b_k t^{k+1}$

$$\bar{b}(t) = \sum_{0 \leq k} (-1)^k b_k t^k. \;= b(-t)$$

We will use the symbol $[f(t)]_n$ to denote the coefficient of t^n in the power series $f(t)$.

1.1 Definition. $\Gamma_n = n! \sum_{1 \leq j \leq n} [b(t)^j]_{n-j}\, \beta_j \in (H \wedge MU)_{2n}\mathbb{C}P^k$

these are then nicer

$$\Xi_n = (2n)! \sum_{1 \leq j \leq n} [b(t)^j \bar{b}(t)^j]_{2n-2j}\, \frac{q_j}{2^j} \in (H \wedge MU)_{4n}\mathbb{H}P^k.$$

More natural choice would be as shown.

(Here $1 \leq n \leq k \leq \infty$).

1.2 Proposition.

(a) Γ_n, Ξ_n are primitive under the MU-coaction

$$\psi : (H \wedge MU)_*X \to (H \wedge MU)_*MU \otimes (H \wedge MU)_*X$$

$\times$ choice

(b) $\Gamma_n \in MU_{2n}\mathbb{C}P^k \subset (H \wedge MU)_{2n}\mathbb{C}P^k$

$\Xi_n \in MU_{4n}\mathbb{H}P^k \subset (H \wedge MU)_{4n}\mathbb{H}P^k$ and are coaction primitive therein.

(c) Γ_n and Ξ_n are indivisible in $MU_*\mathbb{C}P^k$ and $MU_*\mathbb{H}P^k$, respectively.

Proof. (a) is proved as in [Se2]. (b) and (c) are consequences of the following which we present as an alternative to the proof in [Se2].

1.3 Lemma. The K-theory orientation map $\tau : MU \to K$ induces an isomorphism $\underline{\tau} : PMU_*X \to PK_*X$ whenever X has torsion free homology. (Here, PE_*X denotes the primitive subgroup of E_*X under the E-theory coaction.)

1.4 Proposition. Let $\Gamma_n^K = \underline{\tau}\,\Gamma_n \in (H \wedge K)_{2n}\mathbb{C}P^k$

$$\Xi_n^K = \underline{\tau}\,\Xi_n \in (H \wedge K)_{4n}\mathbb{H}P^k.$$

(a) Γ_n^K and Ξ_n^K are primitive under the K-coaction.

(b) $\Gamma_n^K \in K_{2n}\mathbb{C}P^k \subset (H \wedge K)_{2n}\mathbb{C}P^k$

$\Xi_n^K \in K_{4n}\,\mathbb{H}P^k \subset (H \wedge K)_{4n}\mathbb{H}P^k$.

(c) Γ_n^K and Ξ_n^k are indivisible in $K_*\mathbb{C}P^k$ and $K_*\mathbb{H}P^k$, respectively.

Proof of (1.4). (a) is obvious since τ is a map of ring spectra, hence, preserves coactions. To see (b) and (c), we need to explicitly describe these elements.

Recall that $H_*K = \mathbb{Q}[u, u^{-1}]$

$\pi_*K = \mathbb{Z}[u, u^{-1}]$ where $\underline{h} : \pi_*K \hookrightarrow H_*K$ is the hurewicz homomorphism.

We need to describe explicitly the homomorphism of rings $\tau_* : H_*MU \to H_*K$. This is a result given in [Sw]:

$$\tau_* b_n = \frac{u^n}{(n+1)!}$$

Hence, $\tau_* b(t) = u^{-1}t^{-1}(e^{ut} - 1)$

$\tau_* \overline{b}(t) = -u^{-1}t^{-1}(e^{-ut} - 1)$

Let

$$\tau_* \beta_n = u^n \beta_n^K$$

$$\tau_* q_n = u^{2n} q_n^K$$

$$\Gamma_n^K = n! \sum_{1 \leq j \leq n} [(\frac{e^{ut} - 1}{ut})^j]_{n-j}\; u^j \beta_j^K$$

$$\Xi_n^K = \frac{1}{2}(2n)! \sum_{1 \leq j \leq n} [e^{-jut}(\frac{e^{ut} - 1}{ut})^{2j}]_{2n-2j}\; u^{2j} q_j^K$$

Now, recall the <u>Stirling Numbers</u> <u>(of the second kind)</u> [Ri]:

$$S(p, q) = \frac{1}{q!} \sum_{1 \leq j \leq q} (-1)^{q-j} \binom{q}{j} j^p$$

We will use the notation

$$A(p, q) = q! S(p, q).$$

It is well known that:

1.6 $$(e^t - 1)^q = \sum_{q \leq j} \frac{A(j, q)}{j!} t^j .$$

Hence, $\Gamma_n^K = u^n \sum_{1 \leq j \leq n} A(n, j) q_j^K$

$$\Xi_n^K = u^{2n} \sum_{1 \leq j \leq n} \sum_{0 \leq r \leq 2n-2j} \tfrac{1}{2}(-1)^r j^r \binom{2u}{r} A(2n - r, 2j) q_j^K .$$

Note that

1.7 $S(p, q)$ is always an integer, and $A(p, 2s) = (2s)! S(p, 2s)$ - so $\frac{1}{2}A(p, 2s)$ is an integer.

Hence, since the β_j^K and q_j^K form bases over K_*, we can see that (b) holds.

1.8 $A(n, 1) = 1$

$A(r, 2) = 2^r - 2$ (by an easy induction). Therefore, the coefficient of $u^n \beta_1^K$ in Γ_n^K is 1, and that of $u^{2n} q_1^K$ in Ξ_n^K is

$$\sum_{2 \leq r \leq 2n} (-1)^r \binom{2n}{r} (2^{r-1} - 1) = 1.$$

So (c) also holds true.

1.9 Corollary. $PK_{2n}\mathbb{C}P^k$ has generator

$$\Gamma_n^K = u^n \sum_{1 \leq j \leq n} A(n, j) \beta_j^K .$$

$PK_{4n}\mathbb{H}P^k$ has generator

$$\Xi_n^K = u^{2n} \sum_{1 \leq j \leq n} \sum_{0 \leq r \leq 2n-2j} \tfrac{1}{2}(-1)^r j^r \binom{2n}{r} A(2n - r, 2j) q_j^K .$$

Proof of (1.3). Recall that if $E = MU, K$, then

$$PE_*X = \mathrm{Ext}^{0*}_{E_*E}(E_*, E_*X).$$

The unit map $mu : S^0 \to MU$ gives rise to a natural homomorphism

$$mu_* : \mathrm{Ext}^{0*}_{K_*K}(K_*, K_*X) \to \mathrm{Ext}^{0*}_{K_*}(K_*, K_*(X \wedge MU))$$

and, hence, a homomorphism

$$\Phi : \mathrm{Ext}^{0*}_{K_*K}(K_*, K_*X) \to \mathrm{Ext}^{0*}_{MU_*MU}(MU_*, PK_*(X \wedge MU))$$

where Ext_{MU_*MU} is taken in the category of right MU_*MU comodules, as in [Cl].

A reformulation of the Hattori-Stong theorem is provided by [Sm]:

1.10 The unit $ku : S^0 \to K$ induces an isomorphism

$$\underline{ku} : \pi_*(X \wedge MU) \to PK_*(X \wedge MU)$$

if X has torsion free homology.

Hence, we obtain an isomorphism

$$SH : \mathrm{Ext}^{0*}_{MU_*MU}(MU_*, \pi_*(X \wedge MU)) \to \mathrm{Ext}^{0*}_{MU_*MU}(MU_*, PK_*(X \wedge MU)).$$

I claim the following is a commutative diagram:

1.11

$$\begin{array}{ccc} \mathrm{Ext}^{0*}_{MU_*MU}(MU_*, \pi_*(X \wedge MU)) & \xrightarrow{I} & \mathrm{Ext}^{0*}_{K_*K}(K_*, K_*X) \\ & \searrow^{SH} & \downarrow \Phi \\ & & \mathrm{Ext}^{0*}_{MU_*MU}(MU_*, PK_*(X \wedge MU)) \end{array}$$

Here, $\underline{\tau}$ is the "change of theories" homomorphism induced by τ.

To see that this claim is true, we observe that we are actually investigating the commutativity of the diagram below on the subgroup

$$PMU_*X \subset \pi_*(X \wedge MU).$$

$$\begin{array}{ccccc}
\pi_*(X \wedge S^0 \wedge MU) & & \xrightarrow{\quad c_* \quad} & & \pi_*(X \wedge MU \wedge S^0) \\
& \searrow^{(1\wedge mu \wedge 1)_*} & & {}^{(1\wedge 1 \wedge mu)_*}\swarrow & \\
\Big\downarrow (1\wedge ku \wedge 1)_* & & \pi_*(X \wedge MU \wedge MU) & & \Big\downarrow (1\wedge 1 \wedge ku)_* \\
& & \Big\downarrow (1\wedge \tau \wedge 1)_* & & \\
\longrightarrow & & \pi_*(X \wedge K \wedge MU) & & \longleftarrow
\end{array}$$

(Here, c denotes the switch map.)

The definition of primitivity ensures that the top triangle commutes on PMU_*X, and the fact that $\tau \cdot mu \simeq ku$ completes the verification.

We can now use (1.11) and the facts that SH is an isomorphism and $\underline{\tau}$ an injection, to deduce the result (1.3) for X with torsion free homology.

1.12 <u>Note</u>: In fact, the homomorphism Φ is the edge homomorphism of a spectral sequence with E_2-term of form

$$E_2^{pq*} = \mathrm{Ext}^{p*}_{MU_*MU}(MU_*, \mathrm{Ext}^{q*}_{K_*K}(K_*, K_*(X \wedge MU)))$$

converging to $\mathrm{Ext}^{p+q*}_{K_*K}(K_*, K_*X)$. The details generalize those in [C1], and also show that Φ is monic for <u>all</u> X.

As corollaries of our result (1.9), we can relatively easily show that

1.13 <u>Proposition</u>. Under the tensor product map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$, we have $(\Gamma_1)^n = \Gamma_n$ and $\Gamma_m\Gamma_n = \Gamma_{m+n}$. Similarly for Γ_n.

The proof is achieved in $K_*\mathbb{C}P^\infty$, and uses the well-known result that as a Pontrjagin ring $K_0\mathbb{C}P^\infty \approx \{f(x) \in \mathbb{Q}[x] \mid f(\mathbb{Z}) \subset \mathbb{Z}\}$ with $\beta^K_n \leftrightarrow \dfrac{x(x-1)\ \ldots\ (x-n+1)}{n!}$. In fact, $\Gamma^K_n \leftrightarrow u^n \cdot x^n$.

1.14 <u>Corollary</u>.
$$PMU_*(\mathbb{C}P^\infty_+) = \mathbb{Z}[\Gamma_1]$$
$$PK_*(\mathbb{C}P^\infty_+) = \mathbb{Z}[\Gamma_1{}^K].$$

We will use this result in a later section.

§2. In this section, we make some observations on the Adams-Novikov spectral sequences for $\pi_*^s\mathbb{C}P^k$ and $\pi_*^s\mathbb{H}P^k$ $(1 \leq k \leq \infty)$. Recall that there is a natural spectral sequence $\{E_r^{**}(X), d_r\}_{1\leq r}$ converging to $\pi_*{}^sX$, and with

$$E_2^{pq}(X) = \mathrm{Ext}^{pq}_{MU_*MU}(MU_*, MU_*X) \qquad \text{[Ad]}$$

(Here, X is a connective spectrum or a space.)

Our elements Γ_n and Ξ_n are, of course, in the E_2-terms for $X = \mathbb{C}P^k$ or $X = \mathbb{H}P^k$. In fact, it is easily seen that Γ_n is an infinite cycle - for if $x \in \pi_2{}^s\mathbb{C}P^k$ is represented by the inclusion of the bottom cell, then $\underline{mu}x = \Gamma_1$, and so $\underline{mu}(x^n) = \Gamma_n$, by (1.13).

2.1 <u>Proposition.</u> [Mo]. $\pi_*^s(\mathbb{C}P_+^\infty)/\mathrm{Torsion} = \mathbb{Z}[x]$ and in $E_*^{**}(\mathbb{C}P^k)$, each $\Gamma_n (n \leq k)$ is an infinite cycle.

The analogous technique fails for $\mathbb{H}P^\infty$, since there is no product on $\mathbb{H}P^\infty$. However, there is a map

$$h : \mathbb{C}P^r \to \mathbb{H}P^k$$

for $r = 2k, 2k + 1$, which classifies the canonical $Sp(1)$ structure on $y \to \mathbb{C}P^r$ (the Hopf bundle). It is well known that $h_* : H_{4t}\mathbb{C}P^r \to H_{4t}\mathbb{H}P^k$ is then an isomorphism, and since the relevant Atiyah-Hirzebruch spectral sequences collapse,

$$h_*\beta_{2n} = q_n + \text{(higher filtration terms)}.$$

Thus, $h_*\Gamma_{2n} = 2\Xi_n$.

2.2 Proposition. For each $n \leq k$, $\pi^S_{4n}\mathbb{H}P^k$ contains an element which is represented by $2\Xi_n$ in $E_*^{**}(\mathbb{H}P^k)$ - so this is an infinite cycle.

Note that $h_*\Gamma_{2n+1} = 0$; in fact, $h_*(x^{2n+1})$ must be an element of filtration at least 2.

Now recall that if $Y = X \vee e^d$ is such that

$$0 \to MU_*X \to MU_*Y \to MU_*S^d \to 0$$

is short exact, then there is a long exact sequence for each r:

$$0 \to E_2^{0r}(X) \to E_2^{0r}(Y) \to E_2^{0r}(S^d) \xrightarrow[\delta]{} E_2^{1r}(X) \longrightarrow \ldots \qquad (2.3)$$

Take $X = \mathbb{C}P^{k-1}$, $Y = \mathbb{C}P^k$. Let $i : \mathbb{C}P^{k-1} \longrightarrow \mathbb{C}P^k$ be the inclusion and $p : \mathbb{C}P^k \to S^{2k}$ the projection.

2.4 Proposition. In the sequence (2.3), we have

$$\delta(\sigma_{2k}) \in E_2^{1\ 2k}(\mathbb{C}P^{k-1})$$

is an element of order $k!$, which detects $f \in \pi^S_{2k-1}\mathbb{C}P^{k-1}$, the attaching map of the top cell of $\mathbb{C}P^k$; in fact, no non-zero multiple of f is of filtration greater than 1. ambiguous.

(Here, $\sigma_d \in MU_dS^d$ is the canonical generator.)

Proof. Firstly, note that $p_*\Gamma_k = n!\sigma_{2k}$, and then recall the "Geometric Boundary Theorem" of [J - M - W - Z].

Need ref. for upper bd for order of f.

A corollary of this is that

$$\ker[i_* : E_2^{1*}(\mathbb{C}P^{k-1}) \to E_2^{1*}(\mathbb{C}P^k)] = \mathbb{Z}/(k!)\{\delta(\sigma_{2k})\}.$$

Theorem (1.2) of [Sel] can be interpreted in our context as saying:

2.5 Theorem. Let $\alpha_{4n+1} \in E_2^{1\ n+2}(S^0)$ denote the 2-torsion element detecting Adams' μ_{8n+1} (see [Rav]). Then $\alpha_{4n+1}\Gamma_1 = \frac{1}{2}(4n+2)!\delta(\sigma_{8n+4})$ in the sequence (2.3) with $k = 4n + 2$.

Equivalently, $\mu_{8n+1}x \in \pi^S_{8n+3}\mathbb{C}P^{4n+1}$ is non-zero, but $\mu_{8n+1}x = \frac{1}{2}(4n+2)!f = 0$ as an element of $\pi^S_{8n+3}\mathbb{C}P^{4n+2}$. — why not just higher filt?

Unfortunately, we have not been able to give a direct proof of this result in our present setting. Such a proof would avoid the need to use differentials of high order as in [Sel]. However, we can prove Theorem (1.1) of [Sel] as a corollary of (2.5).

Repeating all of the above for $X = \mathbb{H}P^{k-1}$, $Y = \mathbb{H}P^k$, with $j : \mathbb{H}P^{k-1} \to \mathbb{H}P^k$ the inclusion, $q : \mathbb{H}P^k \to S^{4k}$ the projection, we have

2.7 Proposition. In the sequence of (2.3),

$$\delta(\sigma_{4k}) \in E_2^{1\ 4k}(\mathbb{H}P^{k-1})$$

is an element of order $\frac{1}{2}(2k)!$, detecting $g \in \pi^S_{8k-1}\mathbb{H}P^{k-1}$, the attaching map of the top cell of $\mathbb{H}P^k$; in fact, g has order a divisor of $(2k)!$

The proof uses details already mentioned and is analogous to that of (2.4).

Now consider the following commutative diagram

2.9

$$\begin{array}{ccccc} \mathbb{C}P^{4n+1} & \xrightarrow{i} & \mathbb{C}P^{4n+2} & \xrightarrow{p} & S^{8n+4} \\ \downarrow h & & \downarrow h & & \downarrow Id \\ \mathbb{H}P^{2n} & \xrightarrow{j} & \mathbb{H}P^{2n+1} & \xrightarrow{q} & S^{8n+4} \end{array}$$

This induces a commutative diagram with exact rows:

$$\begin{array}{ccccccc} \pi^s_{8n+4}\mathbb{C}P^{4n+1} & \xrightarrow{i_*} & \pi^s_{8n+4}\mathbb{C}P^{4n+2} & \xrightarrow{p_*} & \pi^s_{8n+4}S^{8n+4} & \xrightarrow{f_*} & \pi^s_{8n+3}\mathbb{C}P^{4n+1} \\ \downarrow h_* & & \downarrow h_* & & \downarrow Id & & \downarrow h_* \\ \pi^s_{8n+4}\mathbb{H}P^{2n} & \xrightarrow{j_*} & \pi^s_{8n+4}\mathbb{H}P^{2n+1} & \longrightarrow & \pi^s_{8n+4}S^{8n+4} & \xrightarrow{g_*} & \pi^s_{8n+3}\mathbb{H}P^{2n} \end{array}$$

Now notice that $h_*(\mu_{8n+1}x) = 0$, since $h_*x = 0$. Hence, by (2.5), $\frac{1}{2}(4n+2)!h_*f = 0$ and so $\frac{1}{2}(4n+2)!h_*g = 0$

We, therefore, have some element of $\pi^s_{8n+4}\mathbb{H}P^{2n+1}$ hitting $\frac{1}{2}(4n+2)!\sigma_{8n+4} \in \pi^s_{8n+4}$ ~~$\mathbb{H}P^{2n+1}$~~ S^{8n+4} - but it is easy to see that this is represented by $\Xi_{2n+1} \in E_2^{0\ 8n+4}(\mathbb{H}P^{2n+1})$.

2.11 Theorem. In $E_*^{**}(\mathbb{H}P^k)$, each Ξ_{2n+1} is an infinite [permanent.] cycle $(1 \le 2n+1 \le$ ~~∞~~ $k)$ - hence, the attaching map of the top cell of $\mathbb{H}P^{2n+1}$ is $\frac{1}{2}(4n+2)!$, and no non-zero multiple has filtration greater than 1.

Similarly, each $2\Xi_{2n}$ is an infinite cycle, but Ξ_{2n} supports a non-zero differential - hence, the top cell of $\mathbb{H}P^{2n}$ has attaching map g of order $(4n)!$ and $\frac{1}{2}(4n)!g$ has filtration greater than 1.

(We leave the proof of the last part until the next section - see (3.2).)

2.12 Note: Explicit formulae for $\delta(\sigma_{2k})$ and $\delta(\sigma_{4k})$ in the above can be found from the definition of δ as

$$\delta(\sigma_{2k}) = \psi\beta_k - 1 \otimes \beta_k$$

$$\delta(\sigma_{4k}) = \psi q_k - 1 \otimes q_k$$

We end this section with the following result, which is obtained in [Sn] by very different means.

2.13 Theorem. Let $y \in \pi_n^S \mathbb{C}P^\infty$ be a torsion element. Then for some k, $yx^k = 0$.

Equivalently, $\pi_*^S(\mathbb{C}P_+)[x^{-1}] = \mathbb{Z}[x, x^{-1}]$.

Proof. Let $j_0 : \mathbb{C}P_+^\infty \to K$ be the stable map obtained by including $\mathbb{C}P^\infty$ in $BU \times \{1\} \subset BU \times \mathbb{Z} = K_0$. This is actually a map of ring spectra, if $\mathbb{C}P_+^\infty$ is interpreted as a suspension spectrum.

Arguments similar to those in [Ad] show that $j_{0*} : MU_*(\mathbb{C}P_+^\infty) \to MU_*K$ is injective and

$$j_{0*}\Gamma_n = j_{0*}(\Gamma_1{}^n) = V^n$$

where $V \in PMU_2K$ is the element mu u, for $u \in \pi_2 K$ the usual generator. There is a commutative diagram

2.14
$$\begin{array}{ccc} MU_*(\mathbb{C}P^\infty_+) & \xrightarrow{j_{0*}} & MU_*K \\ & \searrow \quad \nearrow J_0 & \\ & MU_*(\mathbb{C}P^\infty_+)[\Gamma_1^{-1}] & \end{array}$$

where J_0 is the unique algebra extension of j_{0*}. Notice that J_0 is actually an <u>isomorphism</u>, by arguments as in [Ad].

We can now obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^{**}_{MU_*MU}(MU_*, MU_*(\mathbb{C}P^\infty_+)) & \xrightarrow{j_{0*}} & \mathrm{Ext}^{**}_{MU_*MU}(MU_*, MU_*K) \\ & \searrow \quad \nearrow J_{0*} & \\ & \mathrm{Ext}^{**}_{MU_*MU}(MU_*, MU_*(\mathbb{C}P^\infty_+)[\Gamma_1^{-1}]) & \end{array}$$

in which again J_{0*} is an isomorphism. Also, the Conner-Floyd Theorem tells us that

$$MU_*MU \otimes_{\pi_*MU} \pi_*K \simeq MU_*K$$

and so MU_*K is an <u>extended</u> MU_*MU-comodule - therefore,

$$\begin{aligned} \mathrm{Ext}^{**}_{MU_*MU}(MU_*, MU_*K) &= \mathrm{Ext}^{0*}_{MU_*MU}(MU_*, MU_*K) \\ &= \mathbb{Z}[V, V^{-1}] \\ &\simeq \pi_*K. \end{aligned}$$

Finally, note that since Γ_1^{-1} is primitive,

$$\mathrm{Ext}^{**}_{MU_*MU}(MU_*, MU_*(\mathbb{C}P^\infty_+)[\Gamma_1^{-1}]) = \mathrm{Ext}^{**}_{MU_*MU}(MU_*, MU_*(\mathbb{C}P^\infty_+))[\Gamma_1^{-1}]$$

$$= \mathbb{Z}[\Gamma_1, \Gamma_1^{-1}].$$

So, for any $\gamma \in \mathrm{Ext}^{r*}_{MU_*MU}(MU_*, MU_*(\mathbb{C}P^\infty_+))$ which is torsion, for some power k of Γ_1,

$$\gamma\Gamma_1^{\ k} = 0.$$

So, if γ is an infinite cycle representing a homotopy element y, then yx^k is of filtration greater than r. But all elements of $\pi_*^s(\mathbb{C}P^\infty_+)$ have finite filtration.

It might be possible to prove Snaith's result that a spectrum $P(\mathbb{C}P^\infty)$ obtained by using the Hopf construction on $S^2 \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ is equivalent to K in this fashion if the convergence problems can be overcome (neither $P(\mathbb{C}P^\infty)$ nor K is connective). See [Sn].

§3. In this section, we will use the Adams-Novikov spectral sequences for $\pi_*^S \mathbb{H}P^\infty$, $\pi_* MSp$ and $\pi_* MSU$ to show that:

3.1 <u>Proposition.</u> $\Xi_{2n} \in MU_{4n} \mathbb{H}P^\infty$ is killed by d_3 in $E_*^{**}(\mathbb{H}P^\infty) \Rightarrow \pi_*^S \mathbb{H}P^\infty$; more generally, if $Q_{2n-1} \in MU_{8n-4} MSp$ is given by $Q_{2n-1} = j_{1*}\Xi_{2n}$, where $j_1 : \mathbb{H}P^\infty = MSp(1) \to \Sigma^4 MSp$ is the usual inclusion, then Q_{2n-1} is killed by d_3 in $E_*^{**}(MSp) \Rightarrow \pi_* MSp$.

If $Q_{2n} = j_{1*}\Xi_{2n+1}$, then Q_{2n} is an infinite cycle representing an indivisible, indecomposable element in $\pi_* MSp$.

3.2 <u>Corollary.</u> $\Xi_{2n} \in MU_{4n} \mathbb{H}P^r$ $(2n \leq r)$ is killed by d_3 in $E_*^{**}(\mathbb{H}P^r)$ (thus, the remainder of the proof of (2.11) follows).

3.3 <u>Corollary.</u> The coaction primitives in $MSp_{4n} \mathbb{H}P^k$ $(n \leq k \leq \infty)$ contain an infinite cyclic summand generated by an element Θ_n, where under the orientation $\phi : MSp \to MU$, we have

$$\phi\Theta_n = \Xi_n, \text{ if } n \text{ odd}$$
$$= 2\Xi_n, \text{ if } n \text{ even.}$$

In fact, under the hurewicz homomorphism

$$\underline{h} : MSp_* \mathbb{H}P^k \to (H \wedge MSp)_* \mathbb{H}P^k,$$

we have

$$\underline{h}\Theta_n = \frac{1}{2}(2n)! \sum_{1\le j\le n} [B(t)^j]_{n-j}\, q'_j, \quad \text{if } n \text{ odd;}$$

$$= (2n)! \sum_{1\le j\le n} [B(t)^j]_{n-j}\, q'_j, \quad \text{if } n \text{ even.}$$

Here, $B(t) = \sum_{0\le r} B_r t^r$, where $B_r \in H_{4r}MSp$ is the canonical generator, and $q'_r \in MSp_{4r}\,\mathbb{H}P^k$ (or $(H \wedge MSp)_{4r}\,\mathbb{H}P^k$) is the canonical generator.

Proof of (3.1). Recall that under $\tau : MU \to K$, we have $\tau\underline{\Xi}_n = u^{2n} q_1^K + \dots$. Hence, we see that $\tau\underline{Q}_n = u^{2n} + \dots$ in $K_{4n}MSp$, where we write $u^r \in K_{2r}MSp$ for the image of $u^r \in K_{2r}(S^0)$ under the inclusion induced by the unit $S^0 \longrightarrow MSp$. So if Q_n represents an element of π_*MSp, a representing Sp-manifold has "Todd genus" equal to 1. But it is a well-known result that every $(8k+4)$-dimensional SU-manifold has even Todd genus - essentially because its K-theory orientation class comes from KO - see [St].

Now let $\rho : MSp \to MSU$ denote the forgetful map, and consider the map

$$\rho_* : E_2^{0,*}(MSp) \to E_2^{0,*}(MSU).$$

In [Novikov], the structure of $E_*^{**}(MSU)$ is well documented, and it is shown that the spectral sequence is determined at the E_3-level. We have

$$d_3(\rho_* Q_{2k+1}) = \alpha_1^3 \rho_* Q_{2k} \neq 0$$

by the above remark and Novikov's calculations.

So Q_{2k+1} must also support a non-trivial d_3, and

$$d_3 Q_{2k+1} = \alpha_1^3 Q_{2k} + \dots$$

([Nov], Lemma (7.2)).

Similarly, we must also kill Ξ_{2k+2} in $E_3^{0,*}(\mathbb{H}P^{\infty})$ by a differential of form

$$d_3 \Xi_{2k+2} = \alpha_1^3 \Xi_{2k+1} + \dots .$$

The rest follows from (2.2).

Proof of (3.3). [Se2] actually defines primitives $\Phi_n = \sum_{1 \leq j \leq n} [\mathcal{B}(t)^j]_{2n-2j}\, q_j' \in (H \wedge MSp)_{4n}\mathbb{H}P^k$ which, of course, generate a cyclic subgroup. The problem is to decide when a multiple is in the image of $MSp_{4n}\mathbb{H}P^k$ in $(H \wedge MSp)_{4n}\mathbb{H}P^k$. We take

$$\Theta_n = \tfrac{1}{2}(2n)!\,\Phi_n, \quad \text{if } n \text{ odd};$$
$$= (2n)!\,\Phi_n, \quad \text{if } n \text{ even}.$$

Clearly, by Segal's definition,

$$\Xi_n = \underline{\phi}\Theta_n, \quad \text{if } n \text{ odd};$$
$$2\Xi_n = \underline{\phi}\Theta_n, \quad \text{if } n \text{ even}.$$

For any n, we have that Θ_n is in the image of $\underline{msp} : \pi_*^S \mathbb{H}P^k \to MSp_{4n}\mathbb{H}P^k$, by (2.2). So, if n is odd, we have

$$PMSp_{4n}\mathbb{H}P^k = \mathbb{Z}\{\Theta_n\}.$$

If n is even, then the only way that $\frac{1}{2}\Theta_n$ can be in $MSp_{4n}\mathbb{H}P^k$ is by supporting a non-zero differential in the MSp-Adams-Novikov spectral sequence

$$'E_*^{**}(\mathbb{H}P^k) \Rightarrow \pi_*^S\mathbb{H}P^k.$$

However, under j_{1*}, we would then get that $j_{1*}(\frac{1}{2}\Theta_n)$ would be an infinite cycle in $'E_*^{**}(MSp)$, because it is well known that

$$'E_2^{k*}(MSp) = Ext^{k*}_{MSp_*MSp}(MSp, MSp_*MSp)$$

is zero if $1 \leq k$ - hence, the spectral sequence collapses. But by (3.1), we cannot have this, since $\underline{\phi} j_{1*}(\frac{1}{2}\Theta_n) = Q_{n-1}$, which is <u>not</u> an infinite cycle in $E_*^{**}(MSp)$.

§4. In this section, we derive formulae for the e-invariants of elements in the image of the transfer maps

$$tr_{G*} : \pi_*^S(BG(1)_+) \to \pi_{*+d-1}^S$$

where $G = U$, $d = 2$, or $G = Sp$, $d = 4$. This makes good use of our formulae for Γ_n^K, Ξ_n^K of §1.

First, recall that a stable normal G-manifold $(M^n; \tilde{\nu})$ together with G(1)? a U(1) vector bundle $\lambda \to M^n$ determines a singular G-manifold $(M^n; \tilde{\nu}; \lambda)$ in $BG(1)$. We thus have a well defined bordism class

$$[M^n; \tilde{\nu}; \lambda]_G \in MG_n(BG(1)_+).$$

Denote by Tr_G the following composite

4.1 $$MG_n(BG(1)_+) \xrightarrow[\Phi]{} MG_{n+d}MG(1) \xrightarrow[i_*]{} MG_{n+d}BG \xrightarrow[\chi_*]{} MG_{n+d}BG \xrightarrow[\Psi]{} MG_{n+d}\overline{MG}$$

Here, we use the following notations:

$$\Phi : MG_n(BG(1)_+) \to MG_{n+d}(DG(1); SG(1)) = MG_{n+d}MG(1)$$

is the Thom isomorphism with

4.2 $$\Phi[M^n; \tilde{\nu}; \lambda]_G = [D\lambda; S\lambda; q^*\tilde{\nu}; \Lambda]_G$$

where $q : D\lambda \to M^n$ is the projection of the disc bundle, and Λ: $(D\lambda; S\lambda; q^*\tilde{\nu}) \to (DG(1); SG(1))$ is the unique Thom complexification of the classifying map of $\lambda \to M^n$.

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$i : MG(1) = BG(1) \longrightarrow BG$ denotes the standard inclusion. $\chi : BG \to BG$ is the Whitney sum inverse map. If we write

$$MG_*BG(1) = MG_*\{z_1, z_2, \ldots\} \quad \text{(i.e. free on } z_i\text{'s)}$$
$$MG_*(BG_+) = MG_*[Z_1, Z_2, \ldots]$$

then $i_*z_n = Z_n$. Also, if $Z(t) = \sum_{0\leq r} Z_r t^r$, then $\chi_*Z(t) = Z(t)^{-1}$.

4.2 $$\chi_*Z_k = [Z(t)^{-1}]_k.$$

Finally, $\overline{MG} = MG/S^0$, the cofibre of the unit $S^0 \longrightarrow MG$, and Ψ denotes the reduced Thom isomorphism.

Suppose now that $(M^n;\ fr;\ \lambda)$ is a <u>framed</u> singular manifold in $BG(1)$; we can consider the framing as a (trivial) G-structure $\tilde{fr}$. Hence, the framed bordism class

$$[M^n;\ fr;\ \lambda]_{fr} \in \pi_n^S\, BG(1)_+)$$

has $\underline{mg}[M^n;\ fr;\ \lambda]_{fr} = [M^n;\ \tilde{fr};\ \lambda]_G \in MG_n(BG(1)_+)$ where $\underline{mg} : \pi_*^S(-) \to MG_*(-)$ denotes the hurewicz homomorphism.

Recall next that the transfer tr_G associated to the principal universal $G(1)$ bundle $S^\infty \to BG(1)$ is a stable map $tr_G : \Sigma^{d-1}(BG(1)_+) \longrightarrow S^0$ [Be - G~~e~~o]. This has the well-known property that:

4.4 $$tr_{G*}[M^n;\ fr;\ \lambda]_{fr} = [S\lambda;\ q^*fr - q^*\lambda]_{fr} \in \pi_{n+d-1}^S.$$

Here, $q^*fr - q^*\lambda$ is a framing since $q^*\lambda$ is manifestly trivial. An analogous formula applies for G-bordism.

There is a canonical G-bordism Adams resolution for S^0, which ends in

4.5 $$\dots \to \Sigma^{-1}MG \to \Sigma^{-1}\overline{MG} \to S^0$$

It is easily seen that under the usual relative bordism interpretation of $\pi_{n+d-1}\Sigma^{-1}\overline{MG} = \pi_{n+d}\overline{MG}$,

$$[D\lambda;\ S\lambda;\ q^*\tilde{fr} - q^*\lambda]_{(G;fr)} \in \pi_{n+d}\overline{MG}$$

has boundary $[S\lambda;\ q^*fr - q^*\Lambda]_{fr} \in \pi_{n+d-1}S^0 = \pi^S_{n+d-1}$. Hence, an E_1-representative for $[S\lambda;\ q^*fr - q^*\lambda]_{fr}$ is provided by $\underline{mg}[D\lambda;\ S\lambda;\ q^*\tilde{fr} - q^*\lambda]_{(G;fr)}$ in the G-bordism Adams spectral sequence for the homotopy of S^0 - but this is precisely

$$Tr_G[M^n;\ \tilde{fr};\ \lambda]_G \in MG_{n+d}\overline{MG}.$$

So a representative in the E_2-term is

$$\{Tr_G[M^n;\ \tilde{fr};\ \lambda]_G\} \in Ext^{1\ n+d}_{MG_*MG}(MG_*,\ MG_*).$$

4.6 Proposition. The G-bordism e-invariant of

$$tr_{G*}[M^n;\ fr;\ \lambda]_{fr} = [S\lambda;\ q^*fr - q^*\lambda]_{fr} \in \pi^S_{n+d}$$

is

$$\{Tr_G\underline{mg}[M^n;\ fr;\ \lambda]_{fr}\} = \{Tr_G[M^n;\ \tilde{fr};\ \lambda]_G\} \in Ext^{1\ n+d}_{MG_*MG}(MG_*,\ MG_*).$$

We could now take any orientation $\sigma : MG \to E$ and deduce

4.7 Proposition. The E-theory e-invariant of $tr_{G*}[M^n; fr; \lambda]_{fr}$ is

$$\{\underline{\sigma}\, Tr_G\, \underline{mg}[M^n; fr; \lambda]_{fr}\} = \{Tr_E\, \underline{\sigma}[M^n; fr; \lambda]_{fr}\} \in Ext^{1\ n+d}_{E_*E}(E_*, E_*)$$

where $\underline{\sigma} : MG_*\overline{MG} \to E_*\overline{E}$ is the map induced from σ, and $\underline{\sigma} : \pi^S_*(-) \to E_*(-)$ is the hurewicz map; Tr_E denotes the composite

4.8
$$E_n(BG(1)_+) \xrightarrow{\Phi} E_{n+d}\, MG(1) \xrightarrow{i_*} E_{n+d}\, BG$$
$$\xrightarrow{\chi_*} E_{n+d}\, BG \xrightarrow{\Psi} E_{n+d}\, \overline{MG} \xrightarrow{\sigma_*} E_{n+d}\, \overline{E}$$

where all symbols are the same as in (4.1), except σ_*, which is the homomorphism induced by $\sigma : \overline{MG} \to \overline{E}$.

We will now perform these calculations with $G = U, Sp$, and $E = K$. We can, in fact, reduce the calculation to that of $Tr_K(\Gamma_n{}^K)$ and $Tr_K(\Xi_n{}^K)$ and then apply the results to the homotopy elements of (2.1) and (2.11).

First, take $G = U$, $E = K$. Then the orientation $\tau : MU \to K$ gives

$$Tr_K(\Gamma_n{}^K) = \underline{\tau}\Psi\chi_*i_*\Phi(u^n \sum_{1\leq j\leq n} A(n, j)\, \beta^K_j)$$
$$= \underline{\tau}(u^{n+1} \sum_{1\leq j\leq n} A(n; j)[b^K(t)^{-1}]_{j+1}).$$

Here, $b^K(t) = \sum_{0\leq r} b^K_r t^r$, with b^K_r the usual generator for $K_0BU[Ad]$. Recall from [Sw] that $\underline{\tau} : MU_*MU \to K_*K$ gives

$$\underline{\tau}(b^K(t)) = \sum_{0\leq r} \frac{(\omega - 1)(\omega - 2) \ldots (\omega - r)}{(r + 1)!} t^r$$

$$= \frac{(1 + t)^\omega - 1}{\omega t}$$

as formal power series where $(1 + t)^\omega = 1 + \omega t + \frac{\omega(\omega - 1)}{2!} t^2 + \ldots$. Here, $\omega = vu^{-1} \in K_0K$ ([Ad], [Sw]). Thus,

4.9 $$Tr_K(\Gamma_n^K) = u^{n+1} \sum_{1\leq j\leq n} A(n, j)\left[\frac{\omega t}{(1 + t)^\omega - 1}\right]_{j+1}$$

Put

$$\frac{\omega t}{(1 + t)^\omega - 1} = 1 + \sum_{0\leq k} \alpha_{k+1}(\omega) t^{k+1}$$

and use the change of variable $t = e^z - 1$. Then

$$\frac{\omega z}{e^{\omega z} - 1} = \frac{z}{e^z - 1} + z \sum_{0\leq k} \alpha_{k+1}(\omega)(e^z - 1)^k.$$

Comparing coefficients of powers of z gives

4.10 $$\frac{B_{n+1}}{(n + 1)!}(\omega^{n+1} - 1) = \sum_{0\leq k} \frac{A(n, k)}{n!} \alpha_{k+1}(\omega)$$

where B_r is the r^{th} Bernoulli number.

Together with (1.6), we obtain from this

4.11 $$Tr_K(\Gamma_n^K) = \frac{B_{n+1}}{(n + 1)}(v^{n+1} - u^{n+1}).$$

4.12 <u>Proposition</u>. Let $x^n \in \pi_{2n}^S(BU(1)_+)$ be as in (2.1). Then

$$e_{\mathfrak{C}}(tr_{u*}(x^n)) = \{\frac{B_{n+1}}{n+1}(v^{n+1} - u^{n+1})\} \in Ext^{1\ 2n+2}_{K_*K}(K_*, K_*).$$

Next, take $G = Sp$. From §1,

$$Tr_K(\Xi_n^K) = \frac{1}{2}(2n)!\ u^{2n+2} \sum_{1\leq j\leq n} [e^{-jt}(e^t - 1)^{2j}]_{2n}\ \underline{\sigma}\ [q^k(t)^{-1}]_{2j+2}$$

where $q^K(t) = \sum_{0\leq r} q_r^K t^2$, for $q_r^K \in K_0MSp$, the canonical generator.

Here, we take $\sigma : MSp \to K$ to be the obvious composite of orientations

$$MSp \to MU \xrightarrow[\tau]{} K.$$

According to [Sw],

$$\underline{\sigma}(q^K(t)) = 2 + \sum_{0\leq r} \frac{2}{(2r+2)!}(\omega^2 - 1^2) \ldots (\omega^2 - r^2)t^{2r}.$$

Put $\underline{\sigma}(q^K(t)^{-1}) = 1 + \sum_{0\leq k} \gamma_{k+1}(\omega)t^{2k+2}$. From [Ri], page 215 (28), we learn that

4.13 $$e^{\omega z} + e^{-\omega z} = 2 + \sum_{0\leq r} \frac{2}{(2r+z)!}\omega^2(\omega^2 - 1^2) \ldots (\omega^2 - r^2)t^{2r+2}.$$

Therefore, by changing variable by $t = e^{z/2} - e^{-z/2}$, we obtain:

4.14 $$\underline{\sigma}(q^K(e^{z/2} - e^{-z/2})^{-1}) = \frac{\omega^2(e^{z/2} - e^{-z/2})^2}{(e^{\omega z} + e^{-\omega z} - 2)}.$$

Algebra gives:

4.15 $$\frac{e^{\omega z}\,\omega^2 z^2}{(e^{\omega z} - 1)^2} = \frac{e^{z} z^2}{(e^{2} - 1)^2} + z^2 \sum_{0 \leq k} \gamma_{k+1}(\omega)\, e^{-kz}(e^{z} - 1)^{2k}.$$

The coefficient of z^{2n+2} in $\frac{e^{z} z^2}{(e^{2} - 1)^2}$ is

$$\frac{1}{2\pi i} \oint \frac{e^{z}\,dz}{z^{2n+1}(e^{z} - 1)^2} = \frac{-1}{2\pi i} \oint \frac{(2n+1)\,z\,dz}{z^{2n+3}(e^{z} - 1)}$$

by the Residue Theorem and Integration by Parts. This is equal to

$$-(2n+1)\frac{B_{2n+2}}{(2n+2)!}.$$

Hence,

4.16 $$-(2n+1)\frac{B_{2n+2}}{(2n+2)!}(\omega^{2n+2} - 1) = \sum_{0 \leq k} [e^{-kt}(e^{t} - 1)^{2k}]_{2n}\, \gamma_{k+1}(\omega).$$

4.17 $$Tr_K(\Xi_n^K) = \frac{-B_{2n+2}}{2(2n+2)}(v^{2n+2} - u^{2n+2}).$$

4.18 Proposition. Let $y_n \in \pi_{4n}^S(\mathbb{H}P_+^\infty)$ be such that

$$\underline{k}\underline{u}(y_n) = \Xi_n^K \text{ if } n \text{ odd;}$$

$$= 2\Xi_n^K \text{ if } n \text{ even} \quad \text{(see (2.4)).}$$

Then

$$e_{\mathbb{C}}(tr_{Sp*}(y_n)) = \left\{\frac{-B_{2n+2}}{2(2n+2)}(v^{2n+2} - u^{2n+2})\right\} \text{ if } n \text{ odd;}$$

$$= \left\{\frac{-B_{2n+2}}{(2n+2)}(v^{2n+2} - u^{2n+2})\right\} \text{ if } n \text{ even.}$$

Note that for n odd, $\mathbb{H}P^\infty$ gives a factor of 2 more of $(\text{im } J)_{8n-1} \subset \pi_{8n-1}^S$ than $\mathbb{C}P^\infty$. ? Not what you mean.

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