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EQUIVARIANT K-THEORY

These notes are based on lectures given at a seminar on K-theory and representation theory held at Oxford University in Summer 1965. It is assumed that the reader is already familiar with vector bundles and K-theory : these notes are concerned with the more general theory in which a compact Lie group G acts on the spaces and vector bundles involved. Results on differentiable manifolds and on representations which were needed in the lectures are collected and proved in the appendix [which will be available shortly, and can be obtained by sending a postcard to the undersigned].

We wish to thank Michael Atiyah and Graeme Segal for their co-operation throughout the writing and revising of these notes; they are nevertheless in no way to be held responsible for any errors which remain.

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Lecture 1 G-vector-bundles

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In this lecture we introduce the concept of G-vector-bundle and define the ring $K_G(X)$. Throughout we let G denote a fixed topological group; from Lecture 2 onwards we assume that G is a compact Lie group.

1.1 Definitions

A G-space is a topological space X together with a continuous map $G \times X \longrightarrow X$ satisfying the associativity condition $g_1(g_2x) = (g_1g_2)x$ for all $g_1, g_2 \in X$ and $x \in X$. We say " X is a G-space".

A G-map between two G-spaces is a continuous map which commutes with the action of G. More generally, if X is a G-space and Y is an H-space and $\theta: H \longrightarrow G$ is a continuous homomorphism, we say that $f: Y \longrightarrow X$ is a θ - equivariant map if it is continuous and if $f(hy) = \theta(h)\, f(y)$ for all $h \in H$, $y \in Y$.

A G-vector-bundle over the G-space X consists of a vector bundle $p: E \longrightarrow X$, together with a G-space structure on E, such that

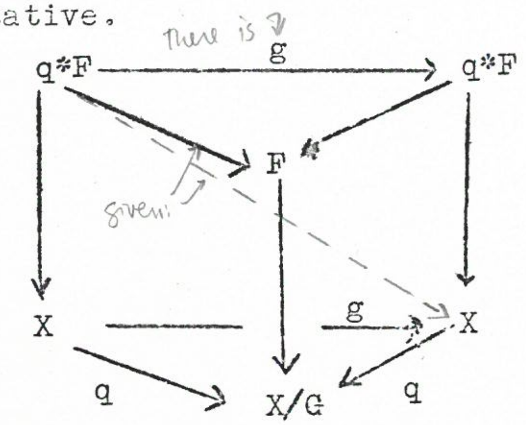
(i) p is a G-map

(ii) if $g \in G$ then $g: p^{-1}(x) \longrightarrow p^{-1}(gx)$ is a linear map.

A G-homomorphism $E \longrightarrow F$ between G-vector-bundles E,F is a map which is both a vector bundle homomorphism and a G-map.

If X is a point then a G-vector-bundle over X is a representation of G. At the other extreme, if G = 1 then a G-vector-bundle is an ordinary vector bundle.

If X is a G-space, we have the quotient map $q:X \longrightarrow X/G$. A vector bundle F over X/G gives rise to an induced bundle q^*F over X. By the universal property of the induced bundle there is a unique G-vector-bundle structure on q^*F which makes the following diagram commutative.



It follows similarly that q^* is a functor from the category of vector bundles and homomorphisms over X/G to the category of G-vector-bundles and G-homomorphisms over X.

Proposition 1.1.1. If X is a principal G-bundle then q^* is an equivalence.

Proof We have to find a functor r such that both q^*r and rq^* are naturally isomorphic to the identity functor. Given a G-vector-bundle E over X we define r(E) to be the map $E/G \longrightarrow X/G$. We first prove that this is a vector bundle over X/G; the only problem is to show that $E/G \longrightarrow X/G$ is locally trivial. Let K be the field of scalars and n the fibre dimension of E. Let U be a neighbourhood of a given point in X/G such that $X \longrightarrow X/G$ is trivial over U.

we need only show that the restriction of $E/G \longrightarrow X/G$ to U is locally trivial. This reduces the problem to the case where $X = G \times X^1$ and $X \longrightarrow X/G = X^1$ is the product projection. Let $x^1 \in X^1$. There is a neighbourhood V of the identity $e \in G$, and a neighbourhood W of x^1 such that $p: E \longrightarrow G \times X^1$ is trivial over $V \times W$. Then we have an isomorphism of vector bundles over $e \times W$

$$e \times W \times K^n \longrightarrow p^{-1}(e \times W)$$

which can be extended uniquely to a G-isomorphism

$$G \times W \times K^n \longrightarrow p^{-1}(G \times X)$$

of G-vector-bundles over $G \times W$. This induces an isomorphism of vector bundles between $W \times K^n \longrightarrow W$ and the restriction to W of $E/G \longrightarrow X^1$. Thus $E/G \longrightarrow X/G$ is a vector bundle $r(E)$ over X/G.

The commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & E/G \\ \downarrow & & \downarrow \\ X & \longrightarrow & X/G \end{array}$$

shows that $E \longrightarrow X$ is the induced bundle, and hence that q^*r is naturally isomorphic to the identity. (We remarked above that there is a unique G-structure on the induced bundle).

To show that rq^* is naturally isomorphic to the identity we write $q^*F = X \times_{X/G} F \subset X \times F$. G acts on the first factor. The map $q^*F \longrightarrow F$ induces an isomorphism $q^*F/G \longrightarrow F$ of vector bundles over X/G,

<u>Corollary</u> 1.1.2. The category of G-vector-bundles over G is equivalent to the category of vector spaces.

In the sequel we shall make use of the following generalisation of Corollary 1.1.2. Let H be a subgroup of G, and Y an H-space. Let $G \times_H Y$ denote the identification space obtained from $G \times Y$ by the equivalence relation:

$$(g_1, y_1) \sim (g_2, y_2) \text{ if and only if } g_2 = g_1 h^{-1},\ y_2 = h y_1 \text{ for some } h \in H.$$

Then $G \times_H Y$ admits a G-space structure : we define

$$g(g_1, y) = (gg_1, y)$$

and note that $g(g_1 h^{-1}, hy) = (gg_1 h^{-1}, hy) \sim (gg_1, y) = g(g_1, y)$.

<u>Proposition</u> 1.1.3. The category of H-vector-bundles over Y is equivalent to the category of G-vector-bundles over $G \times_H Y$.

<u>Proof</u> Consider the subspace $H \times_H Y$ of $G \times_H Y$. Clearly this space is homeomorphic to Y and is stable under the action of the subgroup H of G. Therefore any G-vector-bundle over $G \times_H Y$ defines by restriction an H-vector-bundle over $H \times_H Y = Y$.

Now let F be a H-vector bundle over Y and consider the map $G \times_H F \longrightarrow G \times_H Y$. Here $G \times_H F$ is a G-space, with action of $g \in G$ given by $g(g_1, f) = (gg_1, f)$, and also a vector bundle. The restriction of $G \times_H F$ to $H \times_H Y$ is clearly $H \times_H F = F$. On the other hand if E is a G-vector-bundle over $G \times_H Y$ then there is a map

$$G \times_H (E | H \times_H Y) \longrightarrow E$$

defined by $(g, e) \longmapsto ge$ which is a G-isomorphism.

<u>Corollary</u> 1.1.4. The category of G-vector-bundles over G/H is equivalent to the category of H-vector-spaces.

1.2. Trivial G-spaces

We say that X is a trivial G-space if $gx = x$ for all $g \in G$, $x \in X$. If X is a trivial G-space and V is a G-vector-space then we let $\underline{\underline{V}}$ denote the G-vector-bundle $X \times V \longrightarrow X$.

Now let G be a compact Hausdorff group and let E be a G-vector-space. In other words E is a representation of G. We shall restrict ourselves to complex representations. Using the Haar measure on G, it is easy to construct a G-invariant Hermitian inner product on E. Hence E is completely reducible and we can write

$$E = \oplus_i \quad n_i E_i$$

where E_i is an irreducible G-vector-space and $n_i E_i$ denotes the direct sum of n_i copies of E_i. The integers n_i are determined uniqely by E.

By Schur's lemma we have, for an irreducible G-module E, that the G-endomorphisms of E are just the scalar multiples of the identity. In symbols $\mathrm{End}_G E = \mathbb{C}$.

Let $\{E_\pi\}_{\pi \in S}$ be a complete set of irreducible inequivalent representations of G. If E is any G-vector-spce we write $\pi E = \mathrm{Hom}_G(E_\pi, E)$. There is a G-map

$$\oplus_\pi \quad (E_\pi \otimes \pi E) \longrightarrow E$$

which by the above results is a G-isomorphism. This discussion can be regarded as a discussion of G-vector-bundles over a point. We generalize to arbitrary G-trivial spaces in the following proposition.

Proposition 1.2.1. Let G be a compact Hausdorff group and X a trivial G-space. Let E and F be complex G-vector-bundles over X.

Then

(i) E^G, the subspace of E pointwise invariant under G, is a vector sub-bundle of E,

(ii) $\mathrm{Hom}_G(E,F)$ is a vector sub-bundle of $\mathrm{Hom}(E,F)$

(iii) Let $\pi E = \mathrm{Hom}_G(\underline{\underline{E}}_\pi, E)$. By (ii) this is a vector bundle. The natural map $\oplus_\pi (\underline{\underline{E}}_\pi \otimes \pi E) \longrightarrow E$ is an isomorphism of G-vector-bundles.

Proof (ii) follows trivially from (i). (iii) makes sense because of (ii) and is proved by checking at each point of X. In order to prove (i) we use the Haar measure on G. If V is a G-vector-space we define the *averaging map* $\mu: V \longrightarrow V$ by

$$\mu(v) = \int_G gv$$

Obviously μ is linear. It is also a projection operator - that is $\mu^2 = \mu$ - and its image is V^G. Similarly there is a continuous projection operator $\mu: E \longrightarrow E$ with image E^G. But the image of a projection operator is always a vector sub-bundle (Atiyah-Bott [2], Lemma 1.4).

Propositions 1.1.1. and 1.2.1. describe two extreme situations: when X is a principal G-bundle and when X is a trivial G-space. In general the description of the G-vector-bundle is much more complicated .

1.3. The Grothendieck ring

We now come to the definition of the ring $K_G(X)$. The definition makes sense for arbitrary G-spaces X, but turns out to be inconvenient when X is not compact. We therefore assume that X is a compact G-space. The definition of $K_G(X)$ when X is non-compact is then given in 2.9.

Let X be a compact G-space. The equivalence classes of G-vector-bundles form a commutative semigroup $E_G(X)$ under direct sum (logical difficulties can be avoided by one of the usual devices - see Steenrod [5] for example). We define $K_G(X)$ to be the group which solves the following universal problem.

There is a semigroup homomorphism $\theta : E_G(X) \longrightarrow K_G(X)$ such that, given any semigroup homomorphism $\varphi : E_G(X) \longrightarrow H$ from $E_G(X)$ to a group H, there is a unique homomorphism $\psi : K_G(X) \longrightarrow H$ such that $\psi\theta = \varphi$.

$K_G(X)$ is the quotient of the free abelion group on $E_G(X)$ by the subgroup generated by elements of the form $E \oplus F - E - F$.

The tensor product of two G-vector-bundles becomes a G-vector-bundle if we allow G to act by the diagonal action. This introduces a commutative bilinear product into the free abelian group on $E_G(X)$, and induces a commutative bilinear product in $K_G(X)$. Thus $K_G(X)$ is a commutative ring with unit represented by the G-vector-Bundle $X \times K \longrightarrow X$ where G acts trivially on the field of scalars K.

If G = 1 we obtain the familiar ring K(X). If X is a point we obtain the ring R(G) of virtual representations. We recall that if G is compact then R(G) is the free abelian group on the irreducible representations.

The G-map from X to a point induces a ring homomorphism $R(G) \longrightarrow K_G(X)$, which maps a G-vector space V to the G-vector-bundle $X \times V \longrightarrow X$.

here the map is the product projection and the action of g is given by

$$g(x,v) = (gx,gv).$$

Note that this homomorphism makes $K_G(X)$ an $R(G)$ - module.

We have a category whose objects are pairs (G,X) where X is a G-space, and whose maps $(H,Y) \longrightarrow (G,X)$ are pairs (θ,f) where θ: $H \longrightarrow G$ is a homomorphism and $Y \longrightarrow X$ is θ-equivariant. Given a G-vector-bundle on X, (θ,f) induces an H-vector-bundle over Y. In this way $K_G(X)$ defines a contravariant functor from the above category to the category of commutative rings with a unit.

Thus the map $(G,X) \longrightarrow (G,\text{point})$ induces the ring homomorphism $R(G) \longrightarrow K_G(X)$ mentioned above. The obvious map $(G,X) \longrightarrow (1,X/G)$ induces a ring of homomorphism $K(X/G) \longrightarrow K_G(X)$ which, when X is a trivial G-space, becomes a ring homorphism $K(X) \longrightarrow K_G(X)$. We use these homomorphisms without comment in Propositions 1.3.1. and 1.3.2.

Proposition 1.3.1. If X is a principal G-bundle $K(X/G) \longrightarrow K_G(X)$ is an isomorphism.

Proof By Proposition 1.1.1.

Proposition 1.3.2. Let K be the field of complex numbers. If X is a trivial G-space and G is compact and Hausdorff then the composite

$$R(G) \otimes K(X) \longrightarrow K_G(X) \otimes K_G(X) \longrightarrow K_G(X)$$

is an isomorphism.

Proof We have a map $E_G(X) \longrightarrow R(G) \otimes K(X)$ which, in the notation of Proposition 1.2.1. sends a G-vector-bundle E over X to $\Sigma_\pi (\underline{E}_\pi \otimes \pi E)$. This obviously factors to give an additive homomorphism

$$K_G(X) \longrightarrow R(G) \otimes K(X)$$

The composition

$$E_G(X) \longrightarrow K_G(X) \longrightarrow R(G) \otimes K(X) \longrightarrow K_G(X)$$

is the canonical projection $E_G(X) \longrightarrow K_G(X)$ as we see from 1.2.1 (iii). Therefore $K_G(X) \longrightarrow R(G) \otimes K(X) \longrightarrow K_G(X)$ is the identity. The composition

$$R(G) \otimes K(X) \longrightarrow K_G(X) \longrightarrow R(G) \otimes K(X)$$

is seen to be the identity by checking on elements of the form $\underline{E}_\pi \otimes F$ where F is a vector bundle over X.

Propositions 1.3.1 and 1.3.2 describe $K_G(X)$ in the two extreme situations: X a principal G-bundle and X a trivial G-space. In general the description of $K_G(X)$ is much more complicated. Note however that from Proposition 1.1.3 we obtain

<u>Proposition</u> 1.3.3: Let H be a subgroup of G. The map $(H,Y) \longrightarrow (G, G \times_H Y)$ defines an isomorphism $K_G(G \times_H Y) \longrightarrow K_H(Y)$ for every H-space Y.

<u>Corollary</u> 1.3.4. $K_G(G/H) \cong R(H)$.

Consider the case in which X is a G-space and H is a subgroup of G. Then X can also be regarded as an H-space. There is a G-map

$$G \times_H X \longrightarrow X$$

defined by $(g,x) \longmapsto gx$. If H is of finite index n in G then this is a covering map of degree n. The direct image construction (Atiyah [1], §1) associates to each G-vector-bundle over $G \times_H X$ with fibre dimension q, a G-vector-bundle over X with fibre dimension nq. In fact, in the present case $G \times_H X$ is homeomorphic to $(G/H) \times X$, that is, to the disjoint union of n copies of X; therefore the direct image can be defined as a direct sum.

This defines a homomorphism (Atiyah [1], §2)

$$K_G(G \times_H X) \longrightarrow K_G(X)$$

and hence a homomorphism $K_H(X) \longrightarrow K_G(X)$. This is the <u>transfer homomorphism</u>.

When X is a point this is the "induced representation familiar to students of group theory.

REFERENCES

[1] M. F. Atiyah. Characters and cohomology of finite groups. Publ. Math. Inst. Hautes Etudes Scientifiques 9(1961).

[2] M. F. Atiyah and R. Bott, On the periodicity theorem for complex vector-bundles. Acta Math. 112(1964), 229-247.

[3] N. Steenrod. Topology of Fibre Bundles. Princeton University Press.

Lecture 2. Elementary properties of K_G

G. SEGAL

In this lecture we summarise those properties of K_G - theory which can be obtained by slight modifications of the arguments for ordinary K - theory to be found in Atiyah - Bott [1]

We assume that G is a compact Lie group and consider only complex vector bundles. For convenience all G-spaces will also be assumed compact.

2.1. The space of sections of a G-vector-bundle

Let $p: E \longrightarrow X$ be a vector bundle over a compact space X. A section of E is a map $s : X \longrightarrow E$ such that $ps = 1_X$. The sections of E form a vector space $\Gamma(E)$. Since X is compact, $\Gamma(E)$ is a Banach space and a subspace (in the compact open topology) of E^X.

Now suppose that E is a G-vector-bundle. Then G acts on $\Gamma(E)$ by

$$(g.s)\ (x) = g.s(g^{-1}.x) \quad g\in G, s\in\Gamma(E)$$

and defines

a homomorphism $G \longrightarrow \mathrm{Aut}\ \Gamma(E)$, corresponding to
a G-action $G \times \Gamma(E) \longrightarrow \Gamma(E)$.

To prove that this G-action is continuous, consider the maps

$$\begin{array}{ccccccc} G \times \Gamma(E) \times X & \longrightarrow & G \times \Gamma(E) \times_1 X & \longrightarrow & G \times E & \longrightarrow & E \\ g \times s \times x & \longmapsto & g \times s \times g^{-1}.x & \longmapsto & g \times s(g^{-1}.x) & \longmapsto & (g.s)(x) \end{array}$$

Since each map is continuous, the composite is continuous and defines a continuous map

$$\begin{array}{ccc} G \times \Gamma(E) & \longrightarrow & \Gamma(E) \subset E^X, \\ g \times s & \longmapsto & g.s \end{array}$$

We conclude that there is a continous action of G on the Banach space $\Gamma(E)$. Since G is compact, the averaging map $s \longmapsto \int_G g.s$ defines a projection

$$\mu: \Gamma(E) \longrightarrow \Gamma^G(E)$$

from $\Gamma(E)$ to the subspace $\Gamma^G(E)$ of G-invariant sections of E. The projection μ can be used to extend results on sections of vector bundles to results on G-invariant sections of G-vector-bundles.

Lemma 2.1.1. Let s' be a G - invariant section of E over a closed stable subspace A of X. Then s' can be extended to a G - invariant section over X.

Proof Choose a cover $\{U_i\}$ of X such that $E|U_i$ is trivial. Extend $s_i' : A \cap U_i \longrightarrow C^n$ to $s_i: U_i \longrightarrow C^n$ and define $s = \Sigma\, \varphi_i s_i$ where $\{\varphi_i\}$ is a partition of unity associated to $\{U_i\}$. Then s is a section of E over X. Consider the G - invariant section μs. Since A is stable, μs coincides with s and s' on A.

2.2 Extension of homomorphisms

Consider two G-vector bundles E,F over X. There is a vector bundle Hom (E,F) with fibre

$$\text{Hom }(E,F)_x = \text{Hom }(E_x,F_x)$$

Moreover Hom (E,F) is a G-vector-bundle with action of G defined by

$$(g.\varphi_x)\ (e) = g.\varphi_x(g^{-1}e)$$

for $g \in G$, $e \in Eg.x$, $\varphi_x \in$ Hom (E_x, F_x). The subspace Γ^G (Hom (E,F)) of G - invariant sections of Hom (E,F) consists of all vector bundle homomorphisms φ: E ⟶ F such that $g.\varphi_x(e) = \varphi_{g.x}(g.e)$ for all $x \in E$, $e \in E_x$, $g \in G$. That is, the space of G-vector bundle homomorphisms φ: E ⟶ F can be identified with Γ^G (Hom (E,F)).

Lemma 2.2.1. Let φ' : E|A ⟶ F|A be a G-vector-bundle homomorphism over a closed G-stable subspace A of X. Then φ' can be extended to a G-vector-bundle homomorphism φ: E ⟶ F over X. If moreover φ' is an isomorphism then there is a G-stable open neighbourhood U of A such that $\varphi|U$ is an isomorphism.

Proof Apply lemma 2.1.1. to the G-invariant section φ' of Hom (E,F)|A to obtain a G-invariant section φ of Hom (E,F). By continuity there is an open neighbourhood V of A such that $\varphi|V$ is an ismorphism. The result follows by defining $U = \bigcap_{g\in G} gV$.

Remark Let φ_0, φ_1 be two extensions of an isomophism $\varphi' : E|A \longrightarrow F|A$ Then $\varphi_t = (1-t)\varphi_0 + t\varphi_1$ is also a homomorphism $E \longrightarrow F$ which when restricted to A gives φ'. Therefore there is an open neighbourhood V of A such that $\varphi_t|V$ is an isomorphism for all $0 \leqslant t \leqslant 1$. Therefore φ_0, φ_1 are homotopic through isomorphisms in some neighbourhood of A.

2.3 Homotopy properties of bundles

Let Y be a G-space and let I be the unit interval $0 \leqslant t \leqslant 1$. Then $Y \times I$ is a G-space with group action $g(y,t) = gy \times t$.

Lemma 2.3.1. Let Y be compact, $f_t: Y \longrightarrow X$ a homotopy of G-maps $(0 \leqslant t \leqslant 1)$, and E a G-vector-bundle over X. Then $f_0^* E \cong f_1^* E$

Proof (Atiyah-Bott [1], Prop. 1.3.) Let $f: Y \times I \longrightarrow X$ be the homotopy, so that $f(y,t) = f_t(y)$ and let $\pi: Y \times I \longrightarrow Y$ be the projection. Consider the closed subspace $Y = Y \times \{t\}$ of $Y \times I$. The G-vector bundles f^*E, $\pi^* f_t^* E$ are isomorphic when restricted to Y and so, by 2.3, they are isomorphic in some strip $Y \times \delta T$ where δT denotes some neighbourhood of $\{t\}$ in I. Therefore the isomorphism class of $f_t^* E$ is a locally constant and hence constant function of t. We conclude that $f_0^* E = f_1^* E$.

As in the usual theory, G-vector-bundles are often defined by clutching functions. Let $X = X_1 \cup X_2$, $A = X_1 \cap X_2$, where X_i is a G-stable closed subspace of X. Let E_i be a G-vector-bundle over X_i $(i = 1,2)$ and $\alpha: E_1|A \longrightarrow E_2|A$ an isomorphism. Then there is a G-vector-bundle $E = E_1 \cup_\alpha E_2$ over X such that $E|X_1 \cong E_1$, $E|X_2 \cong E_2$.

Lemma 2.3.2. The isomorphism class of $E_1 \cup_\alpha E_2$ depends only on the homotopy class of the isomorphism α.

Proof A homotopy of isomorphism $E_1|A \longrightarrow E_2|A$ means an isomorphism

$$\Phi : \pi^* E_1|A \times I \longrightarrow \pi^* E_2|A \times I$$

where $\pi : X \times I \longrightarrow X$ is the projection. Let $f_t: X \longrightarrow X \times I$ be defined by $f_t(x) = (x,t)$ and let $\varphi_t : E_1|A \longrightarrow E_2|A$ be the isomorphism induced from Φ by f_t. Then

$$E_1 \cup_{\varphi_t} E_2 \cong f_t^* (\pi^* E_1 \cup_\Phi \pi^* E_2)$$

Since f_0 and f_1 are homotopic it follows that

$$E_1 \cup_{\varphi_0} E_2 \cong E_1 \cup_{\varphi_1} E_2$$

2.4 Existence of complementary bundles

We shall make use of the following definition and lemma due to Mostow.

Definition Consider an action of G on a Banach space Γ. A vector $s \in \Gamma$ is periodic if G.s is contained in a finite dimensional subspace of Γ.

Lemma 2.4.1. The periodic vectors in Γ form a dense set. For the proof see § 2.16 of Mostow [2].

Let E be a G-vector bundle over X. The F is a <u>complementary</u> vector bundle if there exists a G-module M such that $E \oplus F = \underline{\underline{M}} = X \times M$. We wish to prove that such vector bundles exist with M finite dimensional.

Consider the G-invariant map $X \times \Gamma(E) \longrightarrow E$ defined by $x \times s \longmapsto s(x)$. Given a point $x \in X$ there exists a neighbourhood U_x of x and sections $s_1^x, \ldots, s_n^x \in \Gamma(E)$ such that, for all $y \in U_x$, the vectors $s_1^x(y), \ldots, s_n^x(y)$ span the fibre E_y over y. By the above lemma it is moreover possible to choose $s_1^x, \ldots s_n^x$ periodic. Since X is compact, the covering $\{U_n\}$ has a finite subcover $\{U_{x(i)}\}$. The corresponding finite set of sections $s_j^{x(i)}$ determine a finite dimensional G-stable subspace M of $\Gamma(E)$ such that the induced map

$$\underline{\underline{M}} \longrightarrow E$$

is a G-invariant epimorphism. Let F be the kernel of this map, so that there is an exact sequence of G-vector bundles

$$0 \longrightarrow F \longrightarrow \underline{\underline{M}} \longrightarrow E \longrightarrow 0$$

Finally use the averaging process μ defined in 2.1.to obtain a G-invariant metric on M, and hence a G-invariant splitting of the exact sequence. We conclude that $\underline{\underline{M}} = E \oplus F$.

We note two important corollaries of the existence of complementary bundles.

(1) Let G_n denote the Grassmannian of all n-dimensional subspaces of M. The action of G on M defines a universal G-vector bundle

$$K = \{g \times m; \ g \in G_n, m \in M, m \in g\} \subset G_n \times M$$

over G_n. The map $f: X \longrightarrow G_n$ defined by $x \longmapsto E_x$ gives $E = f^*K$. Thus every G-vector bundle is induced from a G-vector bundle over some suitable Grassmannian,

(2) The group $K_G(X)$ is the abelian group associated to the semi-group of G-vector bundles over X. Thus every element of $K_G(X)$ is represented by a pair $(E_0,E_1) = [E_0] - [E_1]$, where $[E_i]$ is the element represented by a G-vector bundle E_i over X. Then $K_G(X)$ is defined by the equivalence relation

$(E_0,E_1) \sim (F_0,F_1) \Longleftrightarrow$ there exists a G-vector bundle E and an isomorphism.

$$E_0 \oplus F_1 \oplus E \cong E_1 \oplus F_0 \oplus E.$$

Since there is a G-vector bundle F such that $E \oplus F$ is a trivial bundle $\underline{\underline{M}}$ we deduce

$(E_0,E_1) \sim (F_0,F_1) \Longleftrightarrow$ there exists a G-module M such that

$$E_0 \oplus F_1 \oplus \underline{\underline{M}} \cong E_1 \oplus F_0 \oplus \underline{\underline{M}}$$

In particular every element of $K_G(X)$ can be represented by a pair $(E,\underline{\underline{M}})$ where E is a G-vector bundle and M is a G-module.

2.5 Extension of bundles

Let E be a G-vector-bundle and consider the G-vector-bundle Hom (E,E) with fibre Hom (E_x,E_x) at $x \in X$. An element $p_x \in$ Hom (E_x,E_x) is a projection operator if $p_x^2 = p_x$. The set of all projection operators p_x, $x \in X$, is a closed subspace

$$\text{Proj}(E) \subset \text{Hom}(E,E).$$

Now consider the open subspace $Q(E) \subset \mathrm{Hom}(E,E)$ consisting of all $T \in \mathrm{Hom}(E,E)$ such that, for all eigenvalues z of T, $|z-1| \neq \frac{1}{2}$. Clearly $\mathrm{Proj}(E) \subset Q(E)$.

Lemma 2.5.1. There is a retraction $\alpha: Q(E) \longrightarrow \mathrm{Proj}(E)$.

Proof Given $T \in Q(E)$ define $\alpha T = \frac{1}{2\pi i}\int_{|z-1| = \frac{1}{2}} (x-T)^{-1}\, dz$

We have to show

(i) that α is well defined

(ii) that $\alpha T \in \mathrm{Proj}(E)$, and

(iii) that if $T \in \mathrm{Proj}(E)$ then $\alpha T = T$

(i) Consider the homomorphism $zI - T \in \mathrm{Hom}(E_x, E_x)$ where I denotes the identity. This admits an inverse provided $\det(zI-T) \neq 0$ i.e. provided z is not an eigenvalue of T. Since $T \in Q$ this is always the case on the circle $|z-1| = \frac{1}{2}$.

(ii) Let C be the circle $|z-1| = \frac{1}{2}$ and C' a smaller circle $|z-1| = \rho < \frac{1}{2}$ such that no eigenvalues z of T lie in the annulus $\rho \leqslant |z-1| \leqslant \frac{1}{2}$. Then

$$(2\pi i)^2 (\alpha T)^2 = \left(\int_C (z-T)^{-1}\, dz\right)\left(\int_{C'} (z'-T)^{-1}\, dz'\right)$$

$$= \int_C \int_{C'} \left(\frac{1}{z-T} - \frac{1}{z'-T}\right) \frac{1}{z'-z}\, dz\, dz'$$

$$= \int_C \frac{1}{z-T}\left(\int_{C'} \frac{dz'}{z'-z}\right) dz + \int_{C'} \frac{1}{z'-T}\left(\int_C \frac{dz}{z-z'}\right) dz'$$

$$= 0 + (2\pi i)^2\, \alpha T$$

Therefore $(\alpha T)^2 = \alpha T$ and $\alpha T \in \mathrm{Proj}\, E$.

(iii) Consider a projection operator $p_x: E_x \longrightarrow E_x$ and choose a coordinate system for which p_x has diagonal form. If p_x has diagonal $(1,\ldots,1,0,\ldots,0)$ then $(z-p_x)^{-1}$ has diagonal $\left((z-1)^{-1},\ldots,(z-1)^{-1}, z^{-1},\ldots, z^{-1}\right)$ and

$\int_C (z-p_x)^{-1}\,dz$ has diagonal $(1,\ldots 1,0,\ldots,0)$.

Now let E' be a vector bundle over a G-stable closed subspace A of X. Then $E' \oplus F' = \underline{\underline{M}}$ for some G-module M. The map p': $\underline{\underline{M}} \longleftrightarrow \underline{\underline{M}}$ given by $e \oplus f \longmapsto e \oplus 0$ is a projection operator and $E' = \operatorname{im} p'$. We can extend p' to a section $p \in \Gamma(\operatorname{Hom}(E,E))$ over X.

There is a neighbourhood U of A such that $p|U \in Q(E)$. Therefore the section $\alpha(p) \in \Gamma(\operatorname{Hom}(E,E))$ is a projection operator over U which extends p'.

We conclude that there exists a bundle $E = \operatorname{im}\ \alpha(p)$ over U such that $E|A = E'$.

<u>Corollary</u> 2.5.2. $K_G(A) \cong \varinjlim_{U \supset A} K_G(U)$

<u>Proof</u> There is a homomorphism $\varinjlim_{U \supset A} K_G(U) \longrightarrow K_G(A)$ defined by restriction. It is onto by the above argument. It is one-one because (2.2) any two extensions of a bundle E' over A are isomorphic over some neighbourhood V of A.

2.6 Relative K_G-theory

Let $A \subset X$ be a G-stable subspace. Consider the semigroup of triples (E_0, E_1, α) where E_0, E_1 are G-vector-bundles over X and $\alpha: E_0|A \longrightarrow E_1|A$ is an isomorphism of G-vector-bundles over A. The equivalence relation

$(E_0, E_1, \alpha) \sim (F_0, F_1, \beta) \Longleftrightarrow$ There exists a G-vector bundle E and a G-isomorphism $E_0 \oplus F_1 \oplus E \cong E_1 \oplus F_0 \oplus E$ which over A restricts to $\alpha \oplus \beta^{-1} \oplus 1$.

The equivalence classes form the associated group $K_G(X,A)$.
Now assume that A is closed in X. There is a sequence

$$K_G(X,A) \longrightarrow K_G(X) \longrightarrow K_G(A) \qquad (1)$$

in which the first homomorphism is defined by ignoring A, the second by restricting to A.

Lemma 2.6.1. Sequence (1) is exact.

Proof An element of $K_G(X,A)$ is represented by a triple (E_0, E_1, α). Under (1) we have

$$(E_0, E_1, \alpha) \longmapsto (E_0, E_1) \longmapsto (E_0|A, E_1|A) \sim (0,0)$$

since α is an isomorphism. Now consider an element of $K_G(X)$ represented by a pair (E_0, E_1) and suppose that $(E_0|A, E_1|A) \sim (0,0)$. Then there exists a G-module M and an isomophism

$$\alpha : E_0|A \oplus \underline{\underline{M}} \cong E_1|A \oplus \underline{\underline{M}}$$

The triple $(E_0 \oplus \underline{\underline{M}}, E_1 \oplus \underline{\underline{M}}, \alpha)$ determines the required element of $K_G(X,A)$.

Let $X|A$ denote the space obtained by collapsing the G-stable subspace A to a point $* \in X|A$. Then $X|A$ is a G-space and $r: X \longrightarrow X|A$ is a G-map.

<u>Lemma</u> 2.6.2. $K_G(X,A) \cong K_G(X|A, *)$.

<u>Proof</u> By 2.4 we may represent every element of $K_G(X|A,*)$ by a triple $(E', \underline{\underline{M}}, \alpha')$ where E' is a G-vector-bundle over $X|A$, M is a G-module and α' is an isomorphism $E'_* \cong M$. The triple $(r^*E', \underline{\underline{M}}, r^*\alpha')$ represents an element of $K_G(X,A)$ and therefore r defines a homomorphism $K_G(X|A,*) \longrightarrow K_G(X,A)$

The inverse homomorphism will be defined by means of a clutching function. Let $(E, \underline{\underline{M}}, \alpha)$ represent an element of $K_G(X,A)$. By 2.2 the isomorphism α extends to a G-stable open neighbourhood U of A. Its restriction to $U-A$ is an isomorphism $\tilde{\alpha} : E|U-A \longrightarrow (U-A) \times M$. Now construct the vector bundle

$$E' = (E|X-A) \cup_{\tilde{\alpha}} r(U) \times M$$

over $(X-A) \cup r(U) = X|A$. By construction there is an isomorphism $\alpha' : E'_* \longrightarrow M$. The triple $(E', \underline{\underline{M}}, \alpha')$ represents the required element of $K_G(X|A,*)$.

<u>Lemma</u> 2.6.3. If A is G-contractible to a point $x_0 \in A$, then $K_G(X,A) \cong K_G(X,x_0)$

<u>Proof</u> Let $r: A \longrightarrow x_0$ be the retraction and define homomorphisms

$$K_G(X,A) \longrightarrow K_G(X,x_0) \longrightarrow K_G(X,A)$$

by
$$(E,F,\alpha) \longmapsto (E,F,\alpha|x_0) \longmapsto (E,F,r^*(\alpha|x_0)).$$

By 2.2 we have, since A is G-contractible, $r^*(\alpha|x_0)$ is G-homotopic to α, and therfore $K_G(X,A) \cong K_G(X,x_0)$.

2.7 Bott periodicity

If E is a G-vector-bundle over X then, by deleting the zero section and dividing out by the action of non-zero scalars we obtain a space P(E) called the projective bundle of E. The action of G on E makes P(E) a G-space. The projection $p: P(E) \longrightarrow X$ is a G-map, and induces a ring homomorphism $p': K_G(X) \longrightarrow K_G(P(E))$, so that $K_G(P(E))$ becomes a $K_G(X)$-module.

By construction a point $y \in p^{-1}(x)$ corresponds to a 1-dimensional linear subspace $[y]$ of the fibre E_x of E over X.

Now consider the G-vector bundle p^*E over P(E). The subspace

$$\{y \times e \in P(E) \times E;\ e \in [y]\} \subset p^*E \subset P(E) \times E$$

is a G-stable subspace of p^*E; it is the bundle space of a G-vector bundle H^* over P(E) with fibre C. Let η denote the corresponding element of $K_G(P(E))$.

The periodicity theorem describes the structure of the $K_G(X)$-module $K_G(P(E))$ in a particular case.

Theorem 2.7.1. Let X be a compact G-space, L a G-vector-bundle over X with fibre C, and ρ the corresponding element of $K_G(X)$. Let $\eta \in K_G(P(L \oplus \underline{C}))$ be the element defined above. Then the $K_G(X)$ algebra $K_G(P(L \oplus \underline{C}))$ is generated by η subject to the relation

$$(\eta-1)(\eta-\rho) = 0 .$$

The proof of the periodicity theorem is a straightforward generalisation of the elementary proof for K-theory given by Atiyah and Bott.

We give a number of corollaries of the periodicity theorem for the special case in which L is a G-vector-bundle space $X \times C$ and with group action defined by a representation $\rho : G \to U(1)$.

In this case $P(L \oplus \underline{C}) = X \times S^2$ and $p: X \times S^2 \to X$ is the product projection. Regard S^2 as the Riemann sphere and introduce notation as follows.

$$D_+ = \{z \in C;\ |z| \leqslant 1\},$$

$$D_- = \{z \in C;\ |z| \geqslant 1\} \cup \{\infty\}$$

$$S^2 = D_+ \cup D_-, \quad S^1 = D_+ \cap D_- .$$

The representation ρ defines an action of G on S^2 by

$$g \cdot z = \rho(g) z , \qquad z \in C$$

under which $\{0\}$, $\{\infty\}$ are kept fixed and D_+, D_-, S^1 are G-stable. Let $\alpha : \underline{C}/X \times S^1 \to p^*L|X \times S^1$ be the G-isomorphism

$$x \times z \times c \longmapsto x \times z \times zc,$$

$x \in X$, $z \in S^1$, $c \in L$. Then $H^* = (\underline{C}|X \times D_+) \cup_\alpha (p^*L|X \times D_-)$. In this way the element $\eta \in K_G(X \times S^2)$ is described directly by means of a clutching function.

Corollary 2.7.2. $K_G(X \times S^2) \cong K_G(X)[\eta]$ subject to the relation $(\eta-1)(\eta-\rho) = 0$.

Corollary 2.7.3. $K_G(S^2) \cong R(G)[\eta]$ subject to the relation $(\eta-1)(\eta-\rho) = 0$.

Corollary 2.7.4. Let G act trivially on S^2. Then $K_G(X \times S^2) \cong K_G(X)[\eta]$ subject to the relation $(\eta-1)^2 = 0$.

Corollary 2.7.5. $K_G(X \times D+, X \times S^1) \cong K_G(X)$.

Proof Corollary 2.7.2. follows from the periodicity theorem by taking L as above; Corollary 2.7.3. by taking X a point; Corollary 2.7.4. by taking $L = \underline{C}$, $\rho = 1$. To prove Corollary 2.7.5. consider the exact sequence

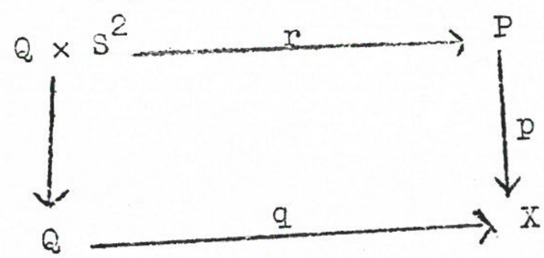
$$K_G(X \times S^2, X \times \{\infty\}) \longrightarrow K_G(X \times S^2) \xrightarrow{i^!} K_G(X)$$

where $i : X \hookrightarrow X \times S^2$ is the embedding $X = X \times \{\infty\}$. The product projection p induces a splitting homomorphism

$$p^! : K_G(X) \longrightarrow K_G(X \times S^2).$$

As in the K-theory case the ring $K_G(X \times S^2, X \times \{\infty\})$ can be 1
identified with the kernel of $i^!$. Since $i^!\eta = \rho$ this kernel is the
ideal generated by $(\eta-\rho)$. Since $\eta(\eta-\rho) = \eta-\rho$ this ideal can be
represented by the multiples $K_G(X)\,(\eta-\rho)$ and is therefore isomorphic
to $K_G(X)$. The result then follows from the isomorphism
$K_G(X \times D_+, X \times S') \cong K_G(X \times S^2, X \times \{\infty\})$.

Remark It is of interest that the periodicity theorem (2.7.1.) can
actually be deduced from Corollary 2.7.2. Let Q be the principle
U(1)-bundle associated to L and $q: Q \longrightarrow X$ the projection. Then
$P = P(L \oplus \underline{C}) = Q \times_{U(1)} S^2$ and there is a commutative diagram of maps



The action of G on the bundle space of L induces an action of G
on Q such that q is a G-map, and such that $q^*L = \underline{C}$. On the other
hand, Q can be regarded as a $\tilde{G}$-space, where $\tilde{G} = U(1)$. The action of
$U(1)$ on S^2 defines an action of $\tilde{G}$ on $Q \times S^2$. Similarly q^*L can be
regarded as a $\tilde{G}$ vector-bundle $\tilde{L}$ with bundle space $Q \times C$ and group
action given by

$$(g \times z)(b \times c) = g.b \times z.c \qquad g \in G,\ z \in U(1),\ b \in Q,\ c \in C.$$

Let $\eta \in K_G(P)$ be the element defined in the periodicity
theorem.

We apply the Corollary 2.7.2. to the $\tilde{G}$-space Q, the representation $\tilde{\rho}: \tilde{G} \longrightarrow U(1)$ given by the product projection, and the element $\tilde{\eta} = r^*\eta \in K_{\tilde{G}}(Q \times S^2)$ to obtain :

$K_{\tilde{G}}(Q \times S^2) \cong K_{\tilde{G}}(Q)[\tilde{\eta}]$ subject to the relation $(\tilde{\eta}-1)(\tilde{\eta}-\tilde{\rho}) = 0.$

Now $K_{\tilde{G}}(Q \times S^2) \cong K_G(P)$ and the element $\tilde{\eta}$ corresponds to the element $\eta \in K_G(P)$. Also $K_{\tilde{G}}(Q) \cong K_G(X)$ and the element $\tilde{\rho} = q^* \rho \in K_{\tilde{G}}(Q)$ corresponds to the element $\rho \in K_G(X)$. Therefore $K_G(P) \cong K_G(X)[\eta]$ subject to the relation $(\eta-1)(\eta-\rho) = 0.$

2.8 <u>The groups $K_G^n(X,A)$</u>

We conclude by defining the rest of the extraordinary cohomology theory K_G^*. Let

$$K_G^{-n}(X) = K_G(X \times D^n, X \times S^{n-1})$$

$$K_G^{-n}(X,A) = K_G(X \times D^n, X \times S^{n-1} \cup A \times D^n)$$

where S^{n-1}, D^n are the unit (n-1)-sphere, n-disc respectively in R^n and where G acts trivially on both. Note that

$$\begin{aligned} K_G^{-m-n}(X,A) &= K_G(X \times D^{m+n}, X \times S^{m+n-1} \cup A \times D^{m+n}) \\ &= K_G(X \times D^m \times D^n, X \times D^m \times S^{n-1} \cup X \times S^{m-1} \times D^n \\ &\qquad \cup A \times D^m \times D^n). \\ &= K_G^{-n}(X \times D^m, X \times S^{m-1} \cup A \times D^{m-1}). \end{aligned}$$

By the periodicity theorem (Corollary 2.7.5.)

$$K_G^{-2}(X) = K_G(X \times D^2, X \times S^1) = K_G(X).$$

Similarly $K_G^{-2}(X,A) \cong K_G(X,A)$ and hence $K_G^n(X,A)$ can be defined for all, not only non-positive, values of n.

As for ordinary K-theory there is a long exact sequence (we omit the proof).

The group $K_G^{-1}(X)$ has an especially simple description. It is the group $K_G^{-1}(X \times I, X \times \{0\} \cup X \times \{1\})$ and therefore an element is represented by a triple $(E, \underline{\underline{M}}, \alpha')$ where E is a G-vector bundle over $X \times I$ and α' is defined by two isomorphisms

$$\alpha_0 : M \longrightarrow E|X \times \{0\},$$
$$\alpha_1 : M \longrightarrow E|X \times \{1\}.$$

Let $\alpha = \alpha_1^{-1}\alpha_0$. Then elements of $K_G^{-1}(X)$ can be represented by pairs $(\underline{\underline{M}}, \alpha)$ where M is a G-module and $\alpha: \underline{\underline{M}} \longrightarrow \underline{\underline{M}}$ is an automorphism of $\underline{\underline{M}} = X \times M$. In this case the coboundary map

$$K_G^{-1}(A) \longrightarrow K_G(X,A)$$

is defined by

$$(\underline{\underline{M}}, \alpha) \longmapsto (\underline{\underline{M}}, \underline{\underline{M}}, \alpha).$$

The multiplication in $K_G(X)$ given by tensor products of G-vector-bundles can be extended (again we omit the proof) to define products in relative groups

$$K_G(X,A) \otimes K_G(Y,B) \longrightarrow K_G(X \times Y, A \times Y \cup X \times B),$$

and also in higher K_G-groups.

The periodicity theorem can be extended to include the relative case and the higher K_G-groups. Thus consider the case in which the G-space X has a base point x_0. There is a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & K_G(X,\{x_0\})[\eta] & \longrightarrow & K_G(X)\ [\eta] & \longrightarrow & K_G(\{x_0\})[\eta] \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & K_G(X \times S^2, \{x_0\} \times S^2) & \longrightarrow & K_G(X \times S^2) & \longrightarrow & K_G(\{x_0\} \times S^2) \end{array}$$

where $K_G(X,\{x_0\})[\eta]$ denotes the ring of polynomials in η and in which the left hand entries are zero because $K_G^{-1}(\{x_0\}) = K_G^{-1}(\{x_0\} \times S^2)$ are zero. By Corollary 2.7.2. the second and third vertical arrows denote isomorphisms. Therefore, so does the first.

<u>Proposition</u> 2.8.1. Let Y be a closed G-subspace of a compact G-space X. Then

$K_G(X \times S^2, Y \times S^2) \cong K_G(X,Y)\ [\eta]$ where $(\eta-1)(\eta-\rho) = 0$ and $K_G(X,Y)[\eta]$ denotes the ring of polynomials in η

<u>Proof</u> Let $\{x_0\}$ be the base point of X/Y. Then

$$\begin{aligned} K_G(X,Y)[\eta] \cong K_G(X/Y,\{x_0\}[\eta] &\cong K_G(X/Y) \times S^2,\{x_0\} \times S^2) \\ &\cong K_G\left(\frac{X \times S^2}{Y \times S^2}, \{x_0\} \times \{0\}\right) \\ &\cong K_G(X \times S^2, Y \times S^2). \end{aligned}$$

<u>Proposition</u> 2.8.2. Let Y be a closed G-subspace if a compact G-subspace X. Then

$K_G^*(X \times S^2, Y \times S^2) = K_G^*(X,Y)\ [\eta]$

where $(\eta-1)(\eta-\rho) = 0$, and where $K_G^*(X,Y)\ [\eta]$ denotes the ring of polynomials in η.

<u>Proof</u> $K_G^{-n}(X \times S^2, Y \times S^2) \cong K_G(X \times S^2 \times D^n, X \times S^2 \times S^{n-1} \cup Y \times S^2 \times D^n)$
$\cong K_G(X \times D^n, X \times S^{n-1} \cup Y \times D^n)\ [\eta]$
$\cong K_G^{-n}(X, Y)\ [\eta]$.

2.9. K_G with compact supports

So far in this lecture we have assumed that all G-spaces X are compact. $K_G(X)$ can also be defined for non-compact spaces, but the simple definition by G-vector-bundles is no longer suitable. One can, however, give a simple definition of K_G-with-compact-supports for the case of a locally compact G-space, and we do this now.

<u>Definition</u> If X is a locally compact G-space, $K_{G,cp}(X)$ is the quotient of the semigroup of triples $(E_0, E_1; \alpha_C)$, where E_0, E_1 are G-vector-bundles on X and $\alpha_C : E_0|X - C \to E_1|X - C$ is an isomorphism over the complement of a relatively compact G-subspace C of X, by the equivalence relation ~, where

$(E_0, E_1; \alpha_C) \sim (F_0, F_1; \beta_D)$ <=> there exists a G-vector-bundle L on a compact neighbourhood A of $\bar{C} \cup \bar{D}$ in X, and a G-isomorphism

$$\theta:(E_0 \oplus F_1)|A \oplus L \to (E_1 \oplus F_0)|A \oplus L$$

such that $\theta|W = \alpha_C|W \oplus \beta_D^{-1}|W \oplus id|W$

where $W = A - (C \cup D)$.

One defines similarly $K_{G,cp}(X, A)$, for A locally closed in X, by considering only triples $(E_0, E_1; \alpha_C)$ such that $C \subset X - A$.

<u>Remarks:</u> (i) We leave it to the reader to verify that $K_{G,cp}(X)$ and $K_{G,cp}(X, A)$ are well-defined groups.

(ii) It is immediate that if X and A are compact then

$K_{G,cp}(X, A) \cong K_G(X, A)$.

(iii) $K_{G,cp}(X)$ is a ring-without-unit if X is not compact. If we had defined $K_G(X)$ it would be a $K_G(X)$-module. In any case it is clear that a vector-bundle E on X determines an endomorphism

$$(E_0, E_1; \alpha_C) \longmapsto (E \otimes E_0, E \otimes E_1; id \otimes \alpha_C)$$

of $K_{G,cp}(X)$ which depends additively on E.

(iv) $X \mapsto K_{G,cp}(X)$ is a contravariant functor on the category of locally compact G-spaces and <u>proper</u> maps. If $f:X \to Y$ is proper we write $f^!:K_{G,cp}(Y) \to K_{G,cp}(X)$ for the induced map.

On the other hand $X \mapsto K_{G,cp}(X)$ is also a <u>covariant</u> functor on the category of locally compact G-spaces and <u>open embeddings</u>. If $j:U \to X$ is an open embedding we write $j_!:K_{G,cp}(U) \to K_{G,cp}(X)$ for the induced map.

<u>Proposition 2.9.1.</u> If X is an open G-subspace of the compact G-space Y, and B = Y - X, then

$$K_G(Y, B) \xrightarrow{\cong} K_{G,cp}(X).$$

<u>Proof:</u> Given $(E_0, E_1; \alpha)$ representing an element of $K_G(Y, B)$, extend α to an isomorphism $\tilde{\alpha}$ over a closed G-stable neighbourhood $\tilde{B}$ of B in Y. Then $(E_0|X, E_1|X; \tilde{\alpha}|X \cap \tilde{B})$ represents an element of

$K_{G,cp}(X)$, and it is clear that this process defines a homomorphism $\rho: K_G(Y, B) \to K_{G,cp}(X)$.

Conversely, given $(E_0, E_1; \alpha_C)$ representing an element of $K_{G,cp}(X)$, choose a compact neighbourhood A of $\overline{C}$ in X and a complementary bundle F for $E_1|A$. Say $\psi: E_1|A \oplus F \xrightarrow{\cong} \underline{\underline{M}}$. Then $((E_0|A \oplus F) \underset{\psi\alpha_C|A-C}{\cup} (M \times (Y - C)), \underline{\underline{M}}$; identity) defines an element of $K_G(Y, B)$. It is easy to show that this construction gives a homomorphism inverse to ρ.

<u>Corollary:</u> $K_{G,cp}(X) \cong K_G(X^+, pt)$, where X^+ is the one-point compactification of X.

<u>We shall adopt from now on the convention that</u> $K_G(X, A)$ <u>will always mean</u> $K_{G,cp}(X, A)$. This makes no difference when X and A are compact.

REFERENCES

[1] M.F. Atiyah and R. Bott. On the periodicity theorem for complex vector bundles. Acta Math. 112 (1964), 229-247.

[2] G.D. Mostow. Cohomology of topological groups and solvmanifolds. Ann. Math. 73 (1961), 20-48.

Lecture 3. Further properties of K_G

G. SEGAL

In this lecture we set up the spectral sequence for K_G-theory and use it to deduce corollaries on the behaviour of K_G under localisation. We assume that G is a compact Lie group and that X is a compact G-space.

3.1 Nerves of coverings by G-stable subsets

Let X be a compact G-space, $\underline{\underline{F}} = \{F_\alpha\}_{\alpha \in S}$ a finite covering of X by closed G-stable subsets. For $\sigma \subset S$ define

$$F_\sigma = \bigcap_{\alpha \in \sigma} F_\alpha$$

and let $N_{\underline{\underline{F}}}$ be the nerve of $\underline{\underline{F}}$ considered as a set of subsets of $\underline{\underline{F}}$. There is a map of sets

$$\xi : X \longrightarrow N_{\underline{\underline{F}}}$$

defined by $\xi(x) = \{\alpha \in S;\ x \in F_\alpha\}$. Let $|N_{\underline{\underline{F}}}|$ denote a geometric realisation of $N_{\underline{\underline{F}}}$ and $\nu : |N_{\underline{\underline{F}}}| \longrightarrow N_{\underline{\underline{F}}}$ the map which assigns to each point the unique open simplex containing it.

<u>Definition</u> $W_{\underline{\underline{F}}} = \{(n,x) \in |N_{\underline{\underline{F}}}| \times X;\ \nu(n)$ is a face of $\xi(x)\}$

or equivalently

$$W_{\underline{\underline{F}}} = \bigcup_{\sigma \in N_{\underline{\underline{F}}}} |\bar{\sigma}| \times F_\sigma \subset |N_{\underline{\underline{F}}}| \times X$$

The space $W_{\underline{\underline{F}}}$ is closed in $|N_{\underline{\underline{F}}}| \times X$ and is therefore a compact G-space. There are obvious projections $p_1 : W_{\underline{\underline{F}}} \longrightarrow |N_{\underline{\underline{F}}}|$ and $p_2 : W_{\underline{\underline{F}}} \longrightarrow X$. Notice that the construction $(X,\underline{\underline{F}}) \longmapsto W_{\underline{\underline{F}}} = W(X,\underline{\underline{F}})$ is functorial in the sense that if $(f,\theta) : (X,\underline{\underline{F}} = \{F_\alpha\}_{\alpha \in S}) \quad (Y,\underline{\underline{H}} = \{H_\beta\}_{\beta \in T})$ is a morphism (i.e. $f: X \longrightarrow Y$, $\theta: S \longrightarrow T$ and $F_\alpha \subset H_{\theta\alpha}$ for $\alpha \in S$) then $|\theta| \times f$ maps $W(X,\underline{\underline{F}})$ into $W(Y,\underline{\underline{H}})$.

The proof of the following lemma then depends on the standard argument that different choices of refinement map are continguous:

<u>Lemma</u> 3.1.1. $(f,\theta_0), (f,\theta_1) : (X,\underline{\underline{F}}) \longrightarrow (Y,\underline{\underline{H}})$ are two morphisms then the induced maps from $W(X,\underline{\underline{F}})$ to $W(Y,\underline{\underline{H}})$ are G-homotopic.

From now on we keep $(X,\underline{\underline{F}})$ fixed and drop the subscript $\underline{\underline{F}}$ on $N_{\underline{\underline{F}}}$ and $W_{\underline{\underline{F}}}$. The polyhedron $|N|$ has a finite filtration by its skeletons

$$|N| \supset \ldots\ldots \supset |N^p| \supset \ldots\ldots \supset |N^1| \supset |N^0|.$$

Similarly X has a finite filtration by closed G-stable subsets

$$X = X_0 \supset X_1 \supset \ldots\ldots \supset X_r \supset \ldots\ldots \text{where } X_r = \{x \in X; \dim \xi(x) \geq r\}.$$

<u>Definition</u> $W^p = p_1^{-1}|N^p|$,

$W_r = p_2^{-1}|X_r|$.

We now introduce the K_G-groups of W. We shall use several times the following trivial lemma.

<u>Lemma</u> 3.1.2. Let Y be a compact G-space, $\{Y_\alpha\}_{\alpha \in A}$ a closed G-stable covering, and

$$Y_{\alpha\beta} = Y_\alpha \cap Y_\beta \;,\quad Y'_\alpha = \bigcup_{\beta \neq \alpha} Y_{\alpha\beta} \;,\quad Y' = \bigcup_\alpha Y'_\alpha.$$

Then $\coprod_{\alpha \in A} (Y_\alpha, Y'_\alpha) \longrightarrow Y, Y'$ is a relative homeomorphism

($\coprod$ denotes disjoint union) and induces an isomorphism

$$K_G^*(Y,Y') \longrightarrow \Pi_{\alpha \in A} K_G^*(Y_\alpha, Y'_\alpha)$$

Proposition 3.1.3. The projection $p_2 : W \longrightarrow X$ induces an isomorphism $K_G^*(X) \longrightarrow K_G^*(W)$.

Proof $p_2 : W, W_1 \longrightarrow X, X_1$ is a relative homemmorphism and so $p_2^* : K_G^*(X, X_1) \longrightarrow K_G^*(W, W_1)$ is an isomorphism.

Similarly we have for all $r \geqslant 0$

$$K_G^*(W_r, W_{r+1}) \cong K_G^*(\bigcup_{\dim \sigma \geqslant r} |\bar\sigma| \times F_\sigma, \bigcup_{\dim \sigma \geqslant r+1} |\bar\sigma| \times F_s) \text{ by definition of } W_r$$

$$\cong K_G^*(\bigcup_{\dim \sigma = r} |\bar\sigma| \times F_\sigma, \bigcup_{\dim \sigma = r} |\bar\sigma| \times F'_\sigma) \text{ by definition of } F'_\sigma$$

$$\cong \Pi_{\dim \sigma = r} K_G^*(|\bar\sigma| \times F_\sigma, |\bar\sigma| \times F'_\sigma) \text{ by lemma}$$

$$\underset{p_2^*}{\cong} \Pi_{\dim \sigma = r} K_G^*(F_\sigma, F'_\sigma)$$

$$\cong K_G^*(X_r, X_{r+1}) \qquad \text{by lemma}$$

It now follows by induction that $p_2^* : K_G^*(X, X_r) \longrightarrow K_G^*(W, W_r)$ is an isomorphism for all r. But X_r and W_r are empty for r large. Therefore $p_2^* : K_G^*(X) \longrightarrow K_G^*(W)$ is an isomorphism.

.2 The K_G-spectral sequence

Let us apply the Cartan-Eilenberg method for constructing spectral sequences to the filtration.

$$W \supset \ldots\ldots \supset W^p \supset \ldots\ldots \supset W^o.$$

to get a spectral sequence whose termination is $K_G^*(W) \cong K_G^*(X)$ and whose $E_2^{p,q}$ term is the p^{th} cohomology group of the complex

$$K^q_G(W^o) \longrightarrow K_G^{q+1}(W^1, W^o) \longrightarrow \ldots\ldots \longrightarrow K_G^{p+q}(W^p, W^{p-1}).$$

But the standard argument gives

$$K_G^{p+q}(W^p, W^{p-1}) \cong \prod_{\dim \sigma = p} K_G^{p+q}(|\bar\sigma| \times F_\sigma, |\dot\sigma| \times F_\sigma)$$

$$\cong \prod_{\dim \sigma = p} K_G^q(F_\sigma) \cong C^p(N, \{K_G^q(F_\sigma)\})$$

where $C^p(N, \quad)$ denotes the p th cochain group of the simplicial complex N in the indicated coefficient coefficient system.

In order to check that the coboundary $d_1 : K_G^{p+q}(W^p, W^{p-1}) \to K_G^{p+q+1}(W^{p+1}, W^p)$ the usual coboundary operator on cochains we must be more precise about the identification of $K_G^q(F_\sigma)$ with $K_G^{p+q}(|\bar\sigma| \times F_\sigma, |\dot\sigma| \times F_\sigma)$. For convenience we drop the realisation signs. Choose a sequence

$$\sigma_0 \subset \sigma_1 \subset \ldots\ldots \subset \sigma_p = \sigma$$

of faces of σ of the dimensions indicated. Identify $K_G^{p+q}(\bar\sigma_p \times F_\sigma, \dot\sigma_p \times F_\sigma)$ with $K_G^{p+q-1}(\bar\sigma_{p-1} \times F_\sigma, \dot\sigma_{p-1} \times F_\sigma)$ by the composition of the isomorphism (relative homeomorphism)

$$K_G^{p+q-1}(\bar\sigma_{p-1} \times F_\sigma, \dot\sigma_{p-1} \times F_\sigma) \cong K_G^{p+q-1}(\dot\sigma_p \times F_\sigma, \dot\sigma_p - \mathring{\sigma}_{p-1} \times F_\sigma)$$

and the isomorphism

$$d : K_G^{p+q-1}(\dot\sigma_p \times F_\sigma, (\dot\sigma_p - \mathring{\sigma}_{p-1}) \times F_\sigma) \longrightarrow K_G^{p+q}(\bar\sigma_p \times F_\sigma, \dot\sigma_p \times F_\sigma)$$

given by the cobundary operator of the triple

$$(\bar{\sigma}_p \times F_\sigma,\ \dot{\sigma}_p \times F_\sigma, (\dot{\sigma}_p - \mathring{\sigma}_p) \times F_\sigma).$$

and so on down to $K_G^q(F_\sigma)$. Of course the choice of $\sigma_0 \subset \sigma_1 \subset \dots \subset \sigma_p = \sigma$ defines an orient. tion of σ and different choices define the same identification if and only if they define the same orientation. Now consider the diagram

$$\begin{array}{ccccccc}
K_G^{p+q}(W^p,W^{p-1}) & = & K_G^{p+q}(W^p,W^{p-1}) & = & K_G^{p+q}(W^p,W^{p-1}) & \xrightarrow{d_1} & K_G^{p+q+1}(W^{p+1},W^p) \\
\downarrow\uparrow & & \downarrow & & \downarrow & & \downarrow\uparrow \\
K_G^{p+q}(\bar{\sigma}\times F_\sigma, \dot{\sigma}\times F_\sigma) & \longrightarrow & K_G^{p+q}(\bar{\sigma}\times F_\tau, \dot{\sigma}\times F_\tau) & \xrightarrow{\cong} & K^{p+q}(\dot{\tau}\times F_\tau, (\dot{\tau}-\mathring{\sigma})\times F_\tau) & \xrightarrow{d} & K_G^{p+q+1}(\tau\times F_\tau, \dot{\tau}\times F_\tau) \\
\uparrow\cong & & \uparrow\cong & & & & \uparrow\cong \\
K_G^q(F_\sigma) & \longrightarrow & K_G^q(F_\tau) & = & = & = & K_G^q(F_\tau)
\end{array}$$

where τ is a $(p+1)$-simplex and σ a p-simplex of N. All the rectangles of the diagram obviously commute except the lower right hand one which commutes or anti-commutes or is zero according as σ is a coherently or non-coherently oriented face of τ ,or not a face of τ at all. This shows that the coboundary d_1 is theusual cobounday operator.

Hence we obtain

Theorem 3.2.1. There is a spectral sequence with $E_2^{p,q} = H^p(N, \{K_G^q(F_\sigma)\})$ amd termination $K_G^*(X)$.

It is easy to see that this spectral sequence is moreover functorial in (X,F).

3.3 The sheaf $K_G{}^q f$

Let G act trivially on the compact space Y and let $f: X \longrightarrow Y$ be a G-map. Define a presheaf $K_G^q f$ on Y by $(K_G^q f)(U) = K_G^q(f^{-1}\bar{U})$. By the continuity of K_G^* (see 2.5.2), the stalk of $K_G^q f$ at $y \in Y$ is just $K_G^q(f^{-1}(y))$. By definition $K_G^q f$ is a presheaf of R(G)-modules.

Let Cov(Y) be the set of finite open coverings of Y. To each $\underline{U} \in \text{Cov}(Y)$ is associated a finite closed covering $\bar{\underline{U}}$. Let Cov'(Y) be the subset of Cov(Y) consisting of coverings $\underline{U}$ such that $N_{\underline{U}} = N_{\underline{U}'}$. It is easy to see that Cov'(Y) is cofinal in Cov(Y). Similarly if Y has finite covering dimension n then the set of coverings $\underline{U}$ such that $N_{\underline{U}} = N_{\underline{U}'}$ and $\dim \underline{U} = n$ is cofinal in Cov(Y).

For each $\underline{U} \in \text{Cov}(Y)$, Theorem 3.2.1. gives a spectral sequence

$$\sum_{\underline{U}} : \quad H^p(N_{\bar{\underline{U}}}, \quad \{K_G^q(f^{-1}\bar{U}_\sigma)\}) \Longrightarrow K_G^*(X).$$

If V is a refinement of $\underline{U}$ there is a well defined morphism of spectral sequences $\sum_{\underline{U}} \longrightarrow \sum_{\underline{V}}$. Now $\underrightarrow{\lim}$ is an exact functor; apply it to the family $\{\Sigma_{\underline{U}}\}_{\underline{U} \in \text{Cov}'(Y)}$ and the result is a spectral sequence of R(G)-modules

$$H^p(Y, K_G^q f) \Longrightarrow K_G^*(X)$$

where now $K_G^q f$ denotes the sheaf associated to the above presheaf. If Y has finite covering dimension n then we may take the limit over coverings of dimension n. In this case we have a direct limit of convergent spectral sequences; each of which give rise to a filtration of length n.

Therefore if Y has finite covering dimension the spectral sequence $H^p(Y, K_G^q f) \Longrightarrow K_G^*(X)$ is convergent.

Consider the case in which $Y = X/G$ and $f: X \longrightarrow X/G$ is the identification map. Then the stalk of $K_G^q f$ at xG is

$$K_G^q(G/G_x) \cong K_{G_x}^q(\text{point}) \cong R(G_x) \otimes K^q(\text{point}).$$

Therefore the stalk is $R(G_x)$ at xG for q even, and 0 for q odd.

When $G=1$, f is the identity map and $K_G^q f \cong Z$ for q even. Then $H^*(X,Z) \Longrightarrow K^*(X)$ is the spectral sequence constructed in [1],§2. For general G we obtain a spectral sequence

$$H^p(X/G, K_G^q f) \Longrightarrow K_G^*(X).$$

3.4 Localisation

The ring $R(G)$ is a sub-ring of the ring of complex valued functions on the set of conjugacy classes of G. We use the same notation for a conjugacy class γ in G, for the corresponding evaluation map $\gamma : R(G) \longrightarrow C$, and for the prime ideal γ in $R(G)$ which is the kernel of this evaluation map.

Now $K_G(X)$ is an $R(G)$-module. Localising with respect to the prime ideal γ of $R(G)$ we obtain a $R(G)_\gamma$-module $K_G(X)_\gamma$. We shall apply the K_G-spectral sequence to give a geometric interpretation of $K_G(X)_\gamma$.

Definition If γ is a conjugacy class in G let

$$X^\gamma = \{x \in X;\ G_x \cap \gamma \neq \varphi\}$$
$$= \bigcup_{g \in \gamma} X^g \quad \text{where } X^g \text{ is the fixed point set of } g.$$

Clearly $gx \in X^\gamma$ if and only if $x \in X^\gamma$. Therefore X^γ, is a G-space.

The inclusion $i : X^{\gamma} \longrightarrow X$ defines homomorphisms

$$i^{!} : K_G^*(X) \longrightarrow K_G^*(X^{\gamma})$$

$$i^{!}_{\gamma} : K_G^*(X)_{\gamma} \longrightarrow K_G^*(X^{\gamma})_{\gamma}$$

Theorem 3.4.1. If X/G has finite covering dimension then $i^{!}_{\gamma}$ is an isomorphism.

Proof We first observe that localisation is an exact functor and therefore the spectral sequence

$$H^*(X/G, K_G^* f) \Longrightarrow K_G^*(X)$$

defined by $f : X \to X/G$ can be localised to give a spectral sequence

$$H^*(X/G, K_G^* f)_{\gamma} \Longrightarrow K_G^*(X)_{\gamma}.$$

The inclusion $i : X^{\gamma} \longrightarrow X$ induces a homomorphism of spectral sequences.

$$\begin{array}{ccccc} H^*(X/G, K_G^* f_{\gamma}) & = & H^*(X/G, K_G^* f)_{\gamma} & \Longrightarrow & K_G^*(X)_{\gamma} \\ & & \downarrow i^*_{\gamma} & & \downarrow i^{!}_{\gamma} \\ H^*(X/G, K_G^* f_{\gamma} | X^{\gamma}/G) & = & H^*(X_p^{\gamma}/G, K_G^* f | X^{\gamma}/G)_{\gamma} & \Longrightarrow & K_G^*(X^{\gamma})_{\gamma} \end{array}$$

Since $X|G$ has finite covering dimension the spectral sequences converge and it is therefore sufficient to prove that i^*_{γ} is an isomorphism. We assume the following lemma which is proved in the appendix.

Lemma 3.4.2. Let H be a closed subgroup of a compact Lie group G, and γ a conjugary class of G. If $H \cap \gamma = \emptyset$ then there exists a character χ of G such that $\chi(\gamma) \neq 0$ but $\bar{\chi} | H = 0$

Since H is a closed subgroup of G, R(H) is a R(G)-module. If $X(\gamma) \neq 0$ but $\chi|H = 0$ for some $\chi \in R(G)$ then $R(H)\gamma = 0$.

At a point $f(x) \in X|G$ the stalk of $K_G^q f\gamma$ is $R(G_x)\gamma$ for q even, 0 for q odd. If $x \notin X^\gamma$ then $G_x \cap \gamma = \emptyset$. Therefore, by the lemma, $R(G_x)\gamma = 0$. Thus $K_G^* f\gamma$ is zero at all points off X^γ/G. It follows that i_γ^* is an isomorphism.

Corollary 3.4.3. Suppose that $g \in G$ acts without fixed points. Then $K_G(X)_{[g]} = 0$.

Proof Apply the theorem with $\gamma = [g]$ the conjugacy class determined by g.

5 Finiteness theorems

If H is a closed subgroup of G then R(H) is a finite R(G)-module (see the appendix). This fact can be used to prove that under suitable hypotheses the E_2-terms of the spectral sequence

$$H^p(X/G, K_G^q f) \Longrightarrow K_G^*(X)$$

are finite R(G)-modules. This is for example the case if X is a compact differentiable G-manifold, or more generally if X is locally G-contractible (i.e. each orbit has a neighbourhood of which it is a G-deformation retract). If in addition X/G has finite covering dimension then the spectral sequence can be used to deduce that $K_G^*(X)$ is a finite R(G)-module. In particular we have

Theorem 3.5.1 If X is a compact differentiable G-manifold then $K_G^*(X)$ is a finite R(G)-module.

For the proof and generalisations of this theorem we refer to a forthcoming paper of G. Segal. It is used only in 7.2.5.

REFERENCES

[1] M. F. Atiyah and F. Hirzebruch. Vector bundles and homogeneous spaces. Proceedings of Tucson Symposium on Differential Geometry, A.M.S. 1961.

[2] A. Grothendieck and J. Dieudonné. Eléments de géometrie algébrique. Publ. Math. Inst. Hautes Études Scientifiques. 4(1960).

[3] A. Grothendieck. Théorèmes de finitude....
Bull Soc. Math. France 84(1956), 1-7.

Lecture 4. Integrality theorems I.

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In this lecture we prove a Thom isomorphism theorem for K_G-theory when G is abelian, and use it to obtain an integrality theorem (4.3.1) for differentiable G-manifolds. We assume that X is a compact G-space.

4.1. The Thom homomorphism.

Let X be a compact G-space, F a G-vector-bundle on X with n-dimensional fibres, and $s: X \to F$ a G-invariant section.

At each point $x \in X$ there is a linear map

$$s(x) \wedge : \lambda^i F_x \to \lambda^{i+1} F_x.$$

These maps combine to give a sequence

$$(\Lambda_{-1}F, s) : 0 \to \lambda^0 F \to \dots \to \lambda^k F \to \lambda^{k+1} F \to \dots \to \lambda^n F \to 0$$

which is exact over all points $x \in X$ at which $s(x) \neq 0$.

In particular, suppose that Y is a closed G-subspace of X, and that there is given a never-zero G-invariant section s of $F|Y$. Then the above construction defines an element of $K_G(X, Y)$. (Recall that not only does a triple (E_0, E_1, α) define an element of $K_G(X, Y)$ but so also does a complex (E_i, d_i) of G-vector-bundles on X which is acyclic over Y. For details see [3].)

Now observe that if $p: E \to X$ is a G-vector-bundle with n-dimensional fibres then the G-vector-bundle p^*E on E has a canonical G-invariant section δ (the diagonal map $E \to E \times_X E = p^*E$) which is never zero on the complement of the zero-section of E. So we obtain a complex of G-vector-bundles $(\Lambda_{-1}p^*E,\ \delta)$ on the locally compact space E which is acyclic on the complement of a compact set and so defines a canonical element λ_E in $K_G(E)$.

For the applications it is convenient to compactify E by regarding it as the "affine part" of the projective bundle $P(E \oplus \underline{C}) = E \cup P(E)$ on X. (E is embedded in $P(E \oplus \underline{C})$ by $\xi \longmapsto (\xi, 1)$.) Then $K_G(E) = K_G(P(E \oplus \underline{C}), P(E))$.

But the complex $(\Lambda_{-1}p^*E,\ \delta)$ on E does not extend to $P(E \oplus \underline{C})$ in the most obvious way. To be precise, a point $\xi \in P(E_x \oplus C)$ does not determine a homomorphism $C \to E_x$, but it does determine, by definition, an inclusion $H^*{}_\xi \to E_x \oplus C$, and so an inclusion $C \to (E_x \oplus C) \otimes H_\xi$ where H^* is the co-Hopf bundle on $P(E \oplus \underline{C})$ as defined in 2.7. That is, if $\tilde{p}:P(E \oplus \underline{C}) \to X$ is the projection, the bundle $\tilde{p}^*(E \oplus \underline{C}) \otimes H$ on $P(E \oplus \underline{C})$ has a canonical never-zero section whose projection into $\tilde{p}^*E \otimes H$ we call $\tilde{\delta}$. If $\xi \in P(E_x) \subset P(E_x \oplus C)$ then $H^*_\xi \subset E_x \subset E_x \oplus C$, so $\tilde{\delta}$ is certainly never zero on P(E). But $H^* \to \tilde{p}^*(E \oplus \underline{C}) \to \underline{C}$ induces a canonical trivialization $\underline{C} \to H|E$, and so an isomorphism $\theta:p^*E \to p^*E \otimes (H|E)$; and it is easy to see that $\theta \circ \delta = \tilde{\delta}|E$. So the complex $(\Lambda_{-1}p^*E,\ \delta)$ on E is to be regarded as the restriction of the complex $(\Lambda_{-1}(\tilde{p}^*E \otimes H),\ \tilde{\delta})$ on $P(E \oplus \underline{C})$. Finally observe that $\lambda^k(\tilde{p}^*E \otimes H) \cong (\lambda^k\tilde{p}^*E) \otimes H^k \cong (\tilde{p}^*\lambda^kE) \otimes H^k$. In summary:

<u>Proposition 4.1.1.</u> There is a canonical element λ_E in $K_G(E)$. If $i:X \to E$ is the zero-section, then $i^!\lambda_E = \sum_{k=0}^{n} (-)^k\lambda^kE = \lambda_{-1}E$.

If $K_G(E)$ is regarded as $K_G(P(E \oplus \underline{\underline{C}}), P(E))$, then the image of λ_E in $K_G(P(E \oplus \underline{\underline{C}}))$ is $\sum_{k=0}^{n} (-)^k \, \tilde{p}^!(\lambda^k E) \cdot \eta^{-k}$.

<u>Corollary:</u> The restriction of $\sum_{k=0}^{n} (-)^k \, \tilde{p}^!(\lambda^k E) \cdot \eta^{-k}$ to $K_G(P(E))$ is zero.

<u>Definition:</u> If A is a closed G-subspace of X, the homomorphism of $K_G^*(X)$-modules

$$\varphi_E : K_G^*(X, A) \to K_G^*(E, E|p^{-1}A)$$

defined by $\varphi_E(u) = (-1)^n \, (p^! u) \cdot \lambda_E$ is called the Thom homomorphism.

<u>Remarks:</u>

(i) $p^! u$ is actually an element of $K_G^*(E, E|p^{-1}A)$, but this does not matter in view of the remark (iii) in 2.9.1.

(ii) $i^! \circ \varphi_E : K_G^*(X, A) \to K_G^*(X, A)$ is clearly multiplication by $(-1)^q \lambda_{-1}[E] \in K_G(X)$.

(iii) $\varphi_E : K_G^*(X) \to K_G^*(P(E \oplus C), P(E))$ is seen by the above discussion to be even a homomorphism of $K_G^*(P(E \oplus \underline{\underline{C}}))$-modules.

(iv) The periodicity theorem of 2.7 is precisely the statement that $\varphi_L : K_G^*(X) \to K_G^*(L) = K_G^*(P(L \oplus \underline{\underline{C}}), P(L))$ is an isomorphism when L is a G-line bundle. For $P(L) \cong X$, and because of the existence of the section $X \cong P(L) \to P(L \oplus \underline{\underline{C}})$ [the one called i_∞ in the proof of the periodicity theorem. Observe $i_\infty^*(H^*) \cong L$.] we know that the top line of the diagram

$$\begin{array}{ccccc}
K_G^*(P(L \oplus \underline{\underline{C}}), X) & \to & K_G^*(P(L \oplus \underline{\underline{C}})) & \to & K_G^*(X) \\
\uparrow \varphi_L & & \uparrow \cong & & \| \\
K_G^*(X) & \underset{\mu}{\to} & K_G^*(X)[\eta]/(\eta - 1)(\eta - \rho) & \overset{\nu}{\to} & K_G^*(X)
\end{array}$$

is a short exact sequence. Define μ by $a \mapsto -(\eta - \rho)a$, ν by $f(\eta) \mapsto f(\rho)$. Then the diagram commutes, and one can show that the lower line is a short exact sequence, so φ_L is an isomorphism.

In the next section we show that φ_E is an isomorphism for any G-vector-bundle E which is a sum of G-line-bundles.

4.2 Generalization of the periodity theorem.

In 4.1 we explained how a G-vector-bundle E on X can be identified with the open subset of the compact space $P(E \oplus \underline{\underline{C}})$ complementary to the closed subspace $P(E)$. This can be generalized in the following way:

Suppose given two G-vector-bundles E_1, E_2 on X. Choose a metric on E_2. Lift E_1 to the G-vector-bundle $\tilde{E}_1 = E_1 \times_X P(E_2)$ on $P(E_2)$. Then $E_1 \times_X P(E_2)$ can be identified with the open subset of $P(E_1 \oplus E_2)$ complementary to the closed subspace $P(E_1)$. In fact the embedding is simply

$$(\xi_1, \xi_2) \longmapsto (\xi_1 \|\xi_2\|, \xi_2).$$

By replacing everywhere $\underline{\underline{C}}$ by E_2 in the argument of 4.1 - which makes it, if anything, clearer - one discovers that when $K_G(\tilde{E}_1)$ is identified with $K_G(P(E_1 \oplus E_2), P(E_1))$ the canonical element $\lambda_{\tilde{E}_1}$ in $K_G(\tilde{E}_1)$ maps to $\sum_{k=0}^{n_1} (-)^k \, \tilde{p}^!(\lambda^k E_1) \cdot \eta^{-k}$ in $K_G(P(E_1 \oplus E_2))$. And $\varphi_{\tilde{E}_1} : K_G(P(E_2)) \to K_G(P(E_1 \oplus E_2), P(E_1))$ is furthermore a homomorphism of $K_G(P(E_1 \oplus E_2))$-modules.

We can now prove a generalization of the periodicity theorem.

Theorem 4.2.1 Let X be a compact G-space, and $E = L_1 \oplus \ldots \oplus L_n$, where L_j $(1 \leq j \leq n)$ are G-line-bundles on X. Let $\eta \in K_G^*(P(E))$ be the element determined by the co-Hopf bundle H^*. Then $K_G^*(P(E))$ is generated as an algebra over $K_G^*(X)$ by η subject to the single relation $\theta_n(\eta) = \sum_{k=0}^{n} (-)^k \lambda^k \cdot \eta^{n-k} = 0$.

Remarks: Write $A = K_G^*(X)$, and let $\rho_j \in K_G(X)$ be the element determined by L_j. Then

(i) $\theta_n(x) = \prod_{j=1}^{n} (x - \rho_j)$ is a polynomial identity in $A[x]$, as is seen very easily by induction.

(ii) By the corollary to Proposition 4.1.1 we know we can define a homomorphism of rings $\alpha_n : A[x]/(\theta_n(x)) \to K_G^*(P(E))$ by $x \longmapsto \eta$.

Proof: We proceed by induction. The case $n = 1$ is trivial, as then $P(E) \cong X$. So it suffices to show that if

$\alpha_n : A[x]/(\theta(x)) \xrightarrow{\cong} K_G^*(P(E))$ then

$\alpha_{n+1} : A[y]/(\theta(y)\cdot(y - \rho)) \xrightarrow{\cong} K_G^*(P(E \oplus L))$ where E is a G-vector-bundle on X and L is a G-line-bundle. It is obvious that the co-Hopf bundle on $P(E \oplus L)$ restricts to the co-Hopf bundle on $P(E)$.

Now the bundle $\tilde{p} : P(E \oplus L) \to X$ has a canonical section i, so we have a split short exact sequence

$$0 \to K_G^*(P(E \oplus L), P(L)) \xrightarrow{r!} K_G^*(P(E \oplus L)) \xrightarrow{i!} K_G^*(X) \to 0$$

of $K_G^*(P(E \oplus L))$-modules.

On the other hand by the periodicity theorem the Thom homomorphism $\varphi_L : K_G^*(X) \to K_G^*(L \times_X P(E)) \cong K_G^*(P(E \oplus L), P(L))$, which we have seen to be a homomorphism of $K_G^*(P(E \oplus L))$-modules, is an isomorphism. Consider the diagram:

$$\begin{array}{ccccccccc}
0 \to & A[x]/(\theta(x)) & \xrightarrow{\mu} & A[y]/(\theta(y)\cdot(y-\rho)) & \xrightarrow{\nu} & A & \to 0 \\
 & \downarrow \alpha_n & & \downarrow \alpha_{n+1} & & \| & \\
0 \to & K_G^*(P(E)) & \xrightarrow{r! \circ \varphi_{\tilde{L}}} & K_G^*(P(E \oplus L)) & \xrightarrow{i!} & K_G^*(X) & \to 0
\end{array}$$

where μ is defined by $f(x) \longmapsto (-)^n(y-\rho)f(y)$,

and ν by $h(y) \longmapsto h(\rho)$.

The upper line is also a short exact sequence, and the diagram commutes by construction, so α_{n+1} is an isomorphism.

An immediate corollary is:

<u>Theorem 4.2.2</u> The Thom homomorphism $\varphi_E : K_G^*(X) \to K_G^*(E)$ is an isomorphism whenever E is a sum of G-line-bundles.

<u>Remark:</u> The restriction is unnecessary, as we shall see in Lecture 6.

<u>Proof:</u> Consider this time the diagram

$$\begin{array}{ccccccccc}
0 \to & A & \xrightarrow[\mu]{} & A[y]/(\theta_{n+1}(y)) & \xrightarrow[\nu]{} & A[x]/(\theta_n(x)) & \to 0 \\
 & \downarrow \varphi_E & & \downarrow \cong & & \downarrow \cong & \\
0 \to & K_G^*(E) & \longrightarrow & K_G^*(P(E \oplus \underline{\underline{C}})) & \xrightarrow[\psi]{} & K_G^*(P(E)) & \to 0
\end{array}$$

where μ is defined by $a \longmapsto (-1)^n a \cdot \sum_{k=0}^{n} (-)^k \lambda^k \cdot y^{-k} = (-1)^n a \cdot \theta_n(y)$

and ν by $f(y) \longmapsto f(x)$.

The upper line is exact by inspection.

The lower line is exact because ψ is now known to be surjective.

And the diagram commutes. So φ_E is an isomorphism.

4.3. The Gysin homomorphism

Let X be a differentiable manifold, TX the total space of the tangent bundle of X. If G acts differentiably on X then the differential induces an action of G on TX, and we may consider the group (with compact supports!) $K_G(TX)$.

Let $f: Y \to X$ be a differentiable G-map of compact G-manifolds. We wish to define, in a functorial manner, a homomorphism

$$f_! : K_G(TY) \to K_G(TX)$$

which is analogous to the Gysin homomorphism for cohomology theory.

There is an equivariant embedding h of Y in some real G-vector space V. For the proof, which is very similar to the proof in 2.4, see Palais [2].

Therefore the map $f: Y \to X$ can be factored as the composition of G-maps

$$Y \xrightarrow{i} N \xrightarrow{j} X \times V \xrightarrow{p} X$$

where $ji : Y \to X \times V$ is given by $f \times h$, $j : N \to X \times V$ is the embedding of a G-invariant tubular neighbourhood N of $ji(Y)$ in $X \times V$, $i : Y \to N$ is the map of Y onto the zero section of the normal bundle of Y in $X \times V$ and $p : X \times V \to X$ is the product projection. It therefore suffices to define $i_!$, $j_!$ and $p_!$.

<u>Definition of $i_!$</u> : Consider the diagram of projection maps

$$\begin{array}{ccc} TY & \longleftarrow & TN \\ \big\downarrow p & & \big\downarrow \\ Y & \longleftarrow & N \end{array}$$

Here TN is regarded as a real vector bundle over TY with $N_y \oplus N_y$ over each point of $p^{-1}(y)$. Therefore TN can be identified in a natural way with the total space of the G-vector-bundle $p^*(N \otimes_R C)$ over TY. The (relative) Thom homomorphism

$$\phi : K_G(TY) \longrightarrow K_G(TN)$$

is the required homomorphism $i_!$. By Proposition 4.2.1, $i^! i_!$ is multiplication by $(-1)^n \lambda_{-1}(N \otimes_R C)$ where n is the fibre dimension of N. Note that $n = \dim X - \dim Y + \dim V$.

<u>Definition of $j_!$</u> : Let $(TN)^+$ denote the one-point compactification of TN. There is a G-map $r: (T(X \times V))^+ \longrightarrow (TN)^+$ obtained by collapsing everything outside TN to a point.
The homomorphism

$$r^! : K_G(TN) \longrightarrow K_G(T(X \times V))$$

is the required homomorphism $j_!$.

<u>Definition of $p_!$</u> : The Thom homomorphism for the G-vector-bundle <u>$V \otimes C$</u> is a homomorphism

$$\phi : K_G(TX) \longrightarrow K_G(TX \times (V \otimes C)) = K_G(T(X \times V))$$

The required homomorphism $p_!$ is the homomorphism

$$\phi^{-1} : K_G(T(X \times V)) \longrightarrow K_G(TX)$$

and is defined whenever ϕ is an isomorphism.

By the corollary to Theorem 4.2.2, ϕ is an isomorphism whenever G is abelian, and therefore $p_!$ is defined in this case.

<u>Theorem</u> 4.3.1. Let $f : Y \longrightarrow X$ be a differentiable G-map of compact G-manifolds and suppose G is abelian. There is a natural homomorphism

$$f_! : K_G(TY) \longrightarrow K_G(TX)$$

such that

$$(fg)_! = f_! \, g_!$$

and

$$f_!(y.f^!(x)) = f_!(y)\, x \quad \text{for} \quad x \in K_G(TX),\ y \in K_G(TY).$$

Moreover if f is an embedding of codimension n with normal bundle N then $f^! f_!$ is multiplication by $(-1)^n \lambda_{-1}(N \otimes C)$. Euler class.

<u>Proof</u> Let i,j,p be the maps defined above and define $f_! = p_! j_! i_!$. The rest of the proof now follows as for K-theory (Atiyah-Hirzebruch [1]). The main step is to show that $f_!$ is independent of the choice of V : two embeddings in V and V' define embeddings in $V \oplus V'$ which are equivariantly homotopic.

<u>Corollary</u> 4.3.2. Let Y be a compact G-manifold, and $f: Y \to \{point\}$ the constant map. Then $f_!(u) \in R(G)$ for all $u \in K_G(TY)$

In the next lecture we shall, in particular cases, obtain alternative expressions for $f_!(u)$ as an element of $R(G) \otimes C$. The statement that in fact $f_!(u) \in R(G)$ is then called an integrality theorem.

REFERENCES

[1] M. F. Atiyah and F. Hirzebruch. The Riemann- Roch theorem for analytic embeddings. Topology 1(1962) 151-166

[2] R. Palais. Theory of G-spaces. Memoirs A. M. S.

[3] M. F. Atiyah, R. Bott and A. Shapiro. Clifford Modules. Topology 3, Suppl.1,(1964),3-38.

Lecture 5. Integrality theorems II

M. F. ATIYAH

In this lecture we apply Theorem 4.3.1. in the case when G is a cyclic group. It is then possible to obtain more concrete results and to see clearly the relationship with the usual integrality theorems for differentiable manifolds. We assume that X is a compact differentiable G-manifold.

5.1 Differentiable actions of cyclic groups

For the purpose of this lecture a cyclic group G will mean either a finite cyclic group with generator $g \in G$ or a compact Lie group in which powers of some fixed element $g \in G$ are dense. Typical cyclic groups of the second type are the torus $T = S^1 \times \ldots \times S^1$, or more generally the product of a torus and a finite cyclic group.

Let X be a differentable G-manifold, where $G = \{g^k\}$ is cyclic with generator $g : X \longrightarrow X$. The fixed point set of g is denoted by X^g. For each point $x \in X^g$ there is an automorphism $dg : T_x \longrightarrow T_x$ of the tangent space T_x to X at x. The subspace

$$T_x^g = \{v \in T_x, \quad dg(v) = v\} \subset T_x$$

is identified in a natural way (see the appendix) with the tangent space at x of the differentiable manifolds X^g. Thus $T_x \cong T_x^g \oplus N_x^g$, where N_x^g is the subspace of T_x spanned by eigenvectors of dg with eigenvalues $\neq 1$. We denote by N^g the normal bundle $\bigcup_{x \in X^g} N_x^g$ of X^g in X and by $N^g \otimes C$ the complexification of N^g.

The action of G on $N^g \otimes C$ defines an element $\lambda_{-1}(N^g \otimes C) \in K_G(X^g)$. We use the same notation for the corresponding element of $K_G(X^g)_g$.

<u>Proposition</u> 5.1.1. $\lambda_{-1}(N^g \otimes C)$ is invertible in $K_G(X^g)_g$.

<u>Proof</u>. First suppose that X^g has n connedted components with base points x_i $(i = 1,\ldots,n)$. Since the action of G on X^g is trivial, the embedding $f_i : \{x_i\} \longrightarrow X^g$ is a G-map. Consider the exact sequence

$$\begin{array}{ccccccc}
0 \longrightarrow & I(X^g) \otimes R(G) & \longrightarrow & K(X^g) \otimes R(G) & \xrightarrow{\varepsilon} & \bigoplus_i K(\{x_i\}) \otimes R(G) \\
& \| & & \| & & \| \\
0 \longrightarrow & I_G(X^g) & \longrightarrow & K_G(X^g) & \xrightarrow{\varepsilon} & \bigoplus_i K_G(\{x_i\})
\end{array}$$

where $\varepsilon = \bigoplus_i f_i^!$. The ideal $I_G(X^g)$ is nilpotent and therefore $u \in K_G(X^g)$ is invertible if and only if $\varepsilon(u)$ is invertible, <u>i.e.</u> if and only if $f_i^!(u)$ is invertible for $i = 1,\ldots,n$. It is therefore sufficient to consider the case $n = 1$. Then $K_G(\{x_i\})_g = R(G)_g$ is a local ring A with maximal ideal $\underline{m}$.

If G is finite then $R(G) \cong Z[x]/(x^n - 1)$ and the prime ideal (g) is the kernel of the homomorphism $R(G) \longrightarrow C$ taking x to ζ, a primitive n th root of unity. Then $R(G)_g = A \cong Q(\zeta)$. If $G = (S^1)^n$ is a torus then $R(G)$ is the ring of polynomials in $e^{\pm i\theta_j}$ $(1 \leq j \leq n)$ with integral coefficients, and $R(G)_g = A$ is the field of quotients of such polynomials with rational coefficients.

To prove that $f_1^!(u)$ is invertible it is sufficient to prove that $f_1^!(u) \notin \underline{m}$, i.e. that the evaluation of $f_1^!(u)$ at g is non-zero. Now let $u_t = \lambda_t(N^g \otimes C)$. Then

$$f_1^!(u_t)(g) = \lambda_t(N^g_{X_1} \otimes C)(g) = \det(1 + t dg)$$

where $dg : N^g_{X_1} \longrightarrow N^g_{X_1}$ is the automorphism induced by dg.

Since no eigenvector of dg has eigenvalue 1 we have $f_1^!(u_1)(g) \neq 0$ and $f_1^!(u_1) \notin m$. Therefore $\lambda_{-1}(N^g \otimes C)$ is invertible in $K_G(X^g)_g$.

Remark Since G acts trivially on X^g,

$$K_G(X^g)_g = K(X^g) \otimes R(G)_g$$

$$= K(X^g) \otimes A.$$

We introduce the following notation : $\Lambda \subset A$ where

(i) if G is cyclic of order n, $\Lambda = Z[\zeta], \zeta^n = 1$.

(ii) if $G = (S^1)^n$, Λ = the subring of trigonometric polynomials with integer coefficients.

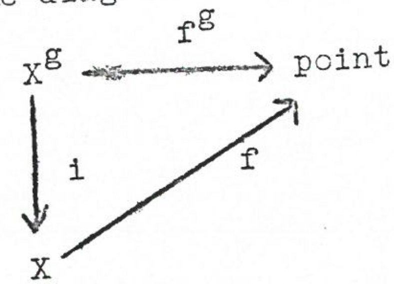
Theorem 5.1.2. Let G be a cyclic group with generator g, X a G-manifold with fixed point set X^g, $i : X^g \longrightarrow X$ the embedding and $f^g : X^g \longrightarrow$ point, $f : X \longrightarrow$ point, the constant maps. Let N^g be the normal bundle of X^g in X and $u \in K_G(TX)$. Then, regarded as elements in $A = R(G)_g$

$$(-1)^n f_!(u) = f_!^g\left(\frac{i^! u}{\lambda_{-1}(N^g \otimes C)}\right),$$

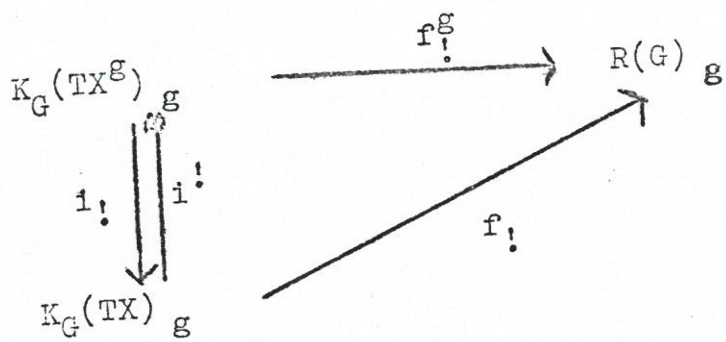
where n is the codimension of X^g in X.

Corollary 5.1.3. $f^g_!\left(\dfrac{i^! u}{\lambda_{-1}(N^g \otimes C)}\right) \in \Lambda.$

Proof Consider the diagram



By Theorem 4.3.1. there is a corresponding diagram



such that $f_! i_!(v) = f^g_!(v)$ for all $v \in K_G(TX^g)_g$.

By 3.4.1. $i^!$ is an isomorphsim with inverse $i_!\left(\dfrac{1}{\lambda_{-1}(N^g \otimes C)}\right)$

and therefore, using 4.3.1,

$$(-1)^{n_g} f^g_!\left(\frac{i^! u}{\lambda_{-1}(N^g \otimes C)}\right) = f_! \; i_!\left(\frac{i^! u}{\lambda_{-1}(N^g \otimes C)}\right) = f_!(u).$$

5.2 Actions with a finite number of fixed points

Suppose that X^g consists of a finite number of points $p \in X$.

Then $$f^g_!\left(\frac{i^! u}{\lambda_{-1}(N^g \otimes C)}\right) = \sum_p \frac{i^!_p u}{\lambda_{-1}(T_p \otimes C)} \quad \in R(G)$$

where $i_p : \{p\} \longrightarrow X$ is the embedding, T_p is the tangent space to X at p, and $u \in K_G(TX)$. Let $0 \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_n \longrightarrow 0$ be a sequence of G-vector-bundles over T_X which is exact off the zero section of TX. This defines an element $u \in K_G(TX)$ such that

$$\mathrm{tr}\, g| \; f^g_! \left(\frac{i^! u}{\lambda_{-1}(N^g \otimes C)} \right) = \sum_p \left(\frac{\sum_i (-1)^i \operatorname{trace} g|E_{i,p}}{\mathrm{tr}\, g| \, \lambda_{-1}(T_p \otimes C)} \right)$$

$$= \sum_p \left(\frac{\sum_i (-1)^i \operatorname{trace} g|E_{i,p}}{\det(1 - dg_p)} \right).$$

Theorem 5.1. then implies

<u>Proposition</u> 5.2.1. Under the above hypotheses

$$\sum_p \left(\frac{\sum_i (-1)^i \operatorname{trace} g|E_{i,p}}{\det(1 - dg_p)} \right) \in \Lambda .$$

Proposition 5.2.1. can be regarded as a set of conditions on the linear maps dg_p, $g|E_{i,p}$ which must be satisfied if the points p can occur as fixed points of an action of G on a differentiable manifold.

5.3 <u>Trivial actions</u>

In order to show how Theorem 5.1.2. links up with the usual integrality theorems for characteristic classes we consider the case $G = 1$. Since $X^g = X$ this is the opposite extreme to the case in which X^g is finite.

If $G = 1$ then $\Lambda = Z$ and the map

$$f_! \; : \; K(TX) \longrightarrow K(T\{x_0\}) = Z, \quad x_0 = \text{point}$$

is always defined. On the other hand it can be calculated by cohomology methods. The Chern character defines a ring homomorphism

$$ch \; : \; K(TX) \longrightarrow H^*(TX, Q) \quad .$$

from $K(TX)$ to the ring $H^*(TX,Q)$ of rational cohomology with compact supports. There is a diagram

$$\begin{array}{ccc} K(TX) & \xrightarrow{\quad f_! \quad} & K(T\{x_0\}) = Z \\ \Big\downarrow ch & & \Big\downarrow ch = j \\ H^*(TX,Q) & \xrightarrow{\quad f^* \quad} & H^*(T\{x_0\},Q) = Q \end{array}$$

in which $j : Z \longrightarrow Q$ is the usual homomorphism and $f_*(u) = (u)[TX]$.

Recall the construction of $f_!$ in 4.3. The essential step was to consider an embedding $i : X \longrightarrow V$ with real n-dimensional normal bundle N. In this case

$$i^! \, i_! \; = \; \text{multiplication by } \; (-1)^n \lambda_{-1}(N \otimes C)$$

$$i^* \, i_* \; = \; \text{multiplication by Chern class } \; c_n(N \otimes C).$$

Consider a formal factorisation $c(N \otimes C) = \prod_{j=1}^{n} (1-x_j^2)$, so that the Pontrjagin class $p(N) = \prod_{j=1}^{n} (1+x_j^2)$. Then

$$i^* ch\, i_!(u) = ch\, i^! i_!(u) = ch(u) \quad (-1)^n [ch\, \lambda_{-1}(N \otimes C)]$$

$$= (-1)^n\, ch\,(u) \prod_{i=1}^{n} (1-e^{x_j})(1-e^{-x_j}),$$

$$i^* i_* ch(u) = (-1)^n\, ch\,(u) \prod_{i=1}^{n} (-x_j)(x_j),$$

and therefore

$$i^* ch\, i_!(u) = i^* i_* \left[ch(u) \left(\prod_{i=1}^{n} \left(\frac{x_j}{1-e^{-x_j}}\right)\left(\frac{-x_j}{1-e^{x_j}}\right)\right)^{-1}\right].$$

We cannot of course conclude directly that

$$ch\, i_!(u) = i_* \left[ch(u) \left(\prod_{i=1}^{n} \left(\frac{x_j}{1-e^{-x_j}}\right)\left(\frac{-x_j}{1-e^{x_j}}\right)\right)^{-1}\right]$$

However, an exactly similar calculation on the classifying space for $N \otimes C$ shows that this formula is true, and that

$$ch\ f_!\,(u) = f_*\big(ch(u). \quad \tau(X)\big)\,[TX]$$

where $\tau(X)$ is a polynomial in the Pontrjagin classes $p_i(X) \in H^{4i}(X,Q)$ of X. If $p(X) = \Pi(1 + y_j^2)$ then

$$c(T \otimes C) = \Pi(1 - y_j^2) = (\Pi(1 - x_j^2))^{-1} \quad \text{and}$$

$$\tau(X) = \Pi\left(\frac{y_j}{1-e^{-y_j}}\right)\left(\frac{-y_j}{1-e^{y_j}}\right)$$

$$= \Pi\left(\frac{x_j}{1-e^{-x_j}}\right)^{-1}\left(\frac{-x_j}{1-e^{x_j}}\right)^{-1}.$$

We therefore obtain

Theorem 5.3.1. If $u \in K(TX)$ then

$$(\mathrm{ch}\, u.\ \tau(X))\,[TX] \text{ is an integer.}$$

Corollary 5.3.2. If X is orientable and $\phi : H^*(X,Q) \longrightarrow H^*(TX,Q)$ is the Thom isomorphism then $(\phi^{-1}\, \mathrm{ch}\, u.\ \tau(X))\,[X]$ is an integer.

This is the customary integrality theorem. To obtain more concrete results it is necessary to choose an element $u \in K(TX)$. Different integrality theorems correspond to different choices of u.

5.4 The integrality theorem

We now seek a formula which combines the two extreme cases discussed in 5.2 and 5.3. We keep the notations of Theorem 5.1.2. Since G acts trivially on X^g we have $K_G(X^g) = K(X^g) \otimes R(G)$ and therefore a Chern character homomorphism

$$\mathrm{ch} \;:\; K_G(X^g) \longrightarrow H^*(X^g, A)$$

where $A = (R(G) \otimes C)_g$. Similarly if $\{x_0\}$ is a single point

$$\mathrm{ch} \;:\; K_G(T\{x_0\}) \longrightarrow H^*(T\{x_0\}, A) \;=\; A$$

maps onto the subring $\Lambda \subset A$. By 5.1.2.

$$\mathrm{ch}\, f_!(u) \;=\; \mathrm{ch}\, f_!^{\,g}\left(\frac{i^! u}{\lambda_{-1}(N^g \otimes C)}\right).$$

On the one hand $\mathrm{ch}\, f_!(u) \in \Lambda$. On the other hand, since ch is multiplicative, and G acts trivially on X^g (see 5.3)

$$\mathrm{ch}\; f_!^g\left(\frac{i^! u}{\lambda_{-1}(N^g \otimes C)}\right) = \left\{\frac{\mathrm{ch}\, i^! u}{\mathrm{ch}\, \lambda_{-1}(N^g \otimes C)}\ \tau(X^g)\right\}[TX^g].$$

Consider the G-vector-bundle N^g. Every representation of the cyclic group is a direct sum of one-dimensional representations which are determined by the eigenvalues of g. It follows that

$$N^g = N_\pi \oplus \sum_\theta N_\theta$$

where N_π is the subspace of N^g spanned by eigenvectors with eigenvalue -1 and N_θ is the (real) subspace of N^g spanned by eigenvectors with eigenvalues $e^{\pm i\theta}$, $0<\theta<\pi$. Then N_θ has a natural complex structure and

$$N_\theta \otimes C = N_\theta^+ \oplus N_\theta^-$$

where N_θ^+, N_θ^- are (complex) G-vector-bundles on which g operates by multiplication by $e^{i\theta}$, $e^{-i\theta}$ respectively. Similarly $N_\pi \otimes C$ is a G-vector-bundle on which g operates by multiplication by -1.

Consider formal factorisations

$$p(N_\pi) = \prod_j (1 + y^2_{\pi,j})$$

$$c(N_\theta) = \prod_j (1 + y_{\theta,j})$$

Then $$c(N^g \otimes C) = \prod_j (1 - y^2_{\pi,j}) \prod_{0<\theta<\pi} (1 + y_{\theta,j})(1 - y_{\theta,j})$$

and

$$\begin{aligned} ch\,(\lambda_{-1}(N_\theta^+)) &= \sum_r (-1)^r \; ch \; \lambda^r(N_\theta^+) \\ &= \sum_r (-1)^r \sum e^{ir\theta} \; e^{y_{\theta,i_1} + \dots + y_{\theta,i_r}} \\ &= \prod_j (1 - e^{i\theta} e^{y_{\theta,j}}). \end{aligned}$$

milarly $\quad ch(\lambda_{-1}(N_\theta^-)) = \prod_j (1-e^{-i\theta}e^{y_{\theta,j}})$

$$ch(N_\pi \otimes C) = \prod_j (1+e^{y_{\theta,j}})(1+e^{-y_{\pi,j}})$$

d therefore

$$ch\,(\lambda_{-1}(N^g \otimes C)) = \prod_{0<\theta\leqslant\pi} \prod_j (1-e^{i\theta}e^{y_{\theta,j}})(1-e^{-i\theta}e^{-y_{\theta,j}})$$

$$= \prod_{0<\theta\leqslant\pi} \prod_j (-1)(e^{\frac{1}{2}(i\theta+y_{\theta,j})}-e^{-\frac{1}{2}(i\theta+y_{\theta,j})})^2 .$$

bstituting this result in the formula for $ch\ f_!(u)$ we have

heorem 5.4.1. Let G be a cyclic group with generator g, X a -manifold with fixed point set X^g, $i:X^g \longrightarrow X$ the inclusion and N^g the normal bundle of X^g in X. Let $y_{\theta,j}$ be defined as above and let $u \in K_G(TX)$. Then

$$\left\{ \frac{ch\, i^! u \qquad \tau(X^g)}{\prod_j (e^{\frac{1}{2}y_{\pi,j}} + e^{-\frac{1}{2}y_{\pi,j}})^2 \prod_{0<\theta<\pi} \prod_j (-1)(e^{\frac{1}{2}(i\theta+y_{\theta,j})}-e^{-\frac{1}{2}(i\theta+y_{\theta,j})})^2} \right\}[TX^g] \in \Lambda.$$

Again, to obtain more concrete results, it is necessary to choose an element $u \in K_G(TX)$. This is done in 5.5 ; for details see [1] .

5.5 <u>Construction of elements</u> $u \in K_G(TX)$

In order to define elements $u \in K_G(TX)$ we seek a sequence

$$0 \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots\cdots \longrightarrow E_n \longrightarrow 0$$

of G-vector-bundles E_i over TX which is exact off the zero section of TX. In order to do this in a systematic way we suppose that X has an H structure: that is, there is a principal H-bundle P over X, a real representation space V of H and an isomorphism

$$TX \cong P \times_H V.$$

Now suppose that there exist complex representation spaces $M_0, M_1, \ldots\ldots, M_n$ of H such that there is an H-map $V \otimes M_{i-1} \longrightarrow M_i$ for each $i = 1, \ldots\ldots, n$ and such that the sequence

$$0 \longrightarrow M_0 \xrightarrow{v} M_1 \xrightarrow{v} \ldots\ldots \xrightarrow{v} M_n \longrightarrow 0$$

defined by $v \in V$ is exact for all $v \neq 0$. Then if $p: TX \longrightarrow X$ is the projection map there is a sequence

$$0 \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \ldots\ldots \longrightarrow E_n \longrightarrow 0$$

as required with $E_i = p^*(P \times_H M_i)$. If in addition X is a G-manifold with H-structure (i.e. G acts on P in a manner consistent with the action of H and the differential action of G on TX) then E_i is a G-vector-bundle and the sequence defines an element $u \in K_G(TX)$.

The Thom isomorphism

$$\phi : H^*(X, \tilde{Q}) \longrightarrow H^*(TX, Q)$$

is defined, where $\tilde{Q}$ denotes untwisted or twisted coefficients according as X is orientable or non-orientable (the manifold TX is always orientable). If $s: X \longrightarrow TX$ is the zero section, and dim $X = 2k$, $(-1)^k\ s^*\phi$ is multiplication by the Euler class, or twisted Euler class, $e(T)$ of X. Therefore

$$\phi^{-1}\ \mathrm{ch}\ u = \frac{\sum_i (-1)^i\ \mathrm{ch}\ (P \times_H M_i)}{(-1)^k\ e\ (T)},$$

whenever the right hand side is well defined.

The following are the simplest examples of this construction. For convenience we assume that dim $V = 2k$.

(i) Almost complex structure In this case $H = GL(k,C)$, $V = C^k$, $M_i = \lambda^i V$ and $V \otimes \lambda^{i-1} V \longrightarrow \lambda^i V$ is exterior multiplication. X is automatically orientable and $\sum_i (-1)^i\ \mathrm{ch}\ (P \times_H M_i) = \mathrm{ch}\ \lambda_{-1} T = \prod_{j=1}^{k} (1-e^{y_j})$ where T is the complex tangent bundle of X and $c(T) = \prod_{j=1}^{k} (1+y_j)$.

Thus $$\phi^{-1}\ \mathrm{ch}\ (u) = \prod_{j=1}^{k} \left(\frac{1-e^{y_j}}{-y_j} \right)$$

and $$\phi\ \mathrm{ch}\ (u).\ \tau(X) = \prod_{j=1}^{k} \left(\frac{-y_j}{1-e^{-y_j}} \right)$$

is the Todd characteristic of X. In particular Theorem 5.3.1 implies that the Todd genus $T(X)$ of X is an integer.

(ii) Spin structure In this case $H = \mathrm{Spin}\ (2k)$ and $V = R^{2k}$. Consider the homomorphism $V \otimes M_0 \longrightarrow M_1$ where M_0, M_1 are $\frac{1}{2}$-spin representations and multiplication is Clifford multiplication. In this case $p(X) = \prod_{j=1}^{k} (1+y_j^2)$ and

$$\mathrm{ch}(P \times_H M_0) - \mathrm{ch}(P \times_H M_1) = \prod_{j=1}^{k} (e^{\frac{1}{2}y_j} - e^{-\frac{1}{2}y_j}).$$

X is automatically orientable and

$$\phi^{-1}\ \mathrm{ch}\ u_*\tau(X) = \left(\prod_{j=1}^{k} \frac{e^{\frac{1}{2}y_j} - e^{-\frac{1}{2}y_j}}{-y_j} \right) \tau(X)$$

$$= \prod_{j=1}^{k} \frac{-y_j}{e^{\frac{1}{2}y_j} - e^{-\frac{1}{2}y_j}} = \prod_{j=1}^{k} \frac{-\frac{1}{2}y_j}{\sinh \frac{1}{2}y_j}.$$

In particular Theorem 5.3.1 implies that the $\hat{A}$-genus of X is an integer.

We remark that the sequence

$$0 \longrightarrow Z_2 \longrightarrow \mathrm{Spin}\,(2k) \longrightarrow SO(2k) \longrightarrow 0$$

can be used to show that an oriented 2k-manifold X admits a Spin structure if and only if $w_2(X) \in H^2(X, Z_2)$ is zero.

(iii) Orientation

In this case $H = SO(2k)$ and $V = R^{2k}$. Let Λ^* denote the exterior algebra of R^{2k}, and consider the duality operator $* : \lambda^q R^{2k} \longrightarrow \lambda^{2k-q} R^{2q}$. Define $\alpha : \Lambda^* \longrightarrow \Lambda^*$ by $\alpha(\omega^q) = (-1)^{\frac{1}{2}q(q-1)} * \omega$.

Then $\alpha^2 = (-1)^k$ and

$$\Lambda^* \otimes_R C = M_0 \oplus M_1$$

where M_0 is spanned by eigenvectors with eigenvalues $+i^k$, M_1 by eigenvectors with eigenvalues $-i^k$. Define

$$R^{2k} \otimes M_0 \longrightarrow M_1$$

by $v \otimes w \longmapsto v(w) = v \wedge \omega - v \vee w$ where $\wedge$ is exterior product and $\vee$ is the adjoint inner product. Then

$$v(v(w)) = v \wedge (v \wedge \omega - v \vee \omega) - v \vee (v \wedge \omega - v \vee \omega)$$
$$= -\|v\|^2 w$$

and therefore v is an isomorphism for $v \neq 0$. In this case $p(X) = \prod_{j=1}^{k} (1+y_j^2)$ and

$$ch\,(P \times_H M_0) - ch(P \times_H M_1) = \prod_{j=1}^{k} (e^{-y_j} - e^{y_j}).$$

X is orientable and

$$\phi^{-1} \quad ch\, u.\ \tau(X) = \left(\prod_{j=1}^{k} \frac{e^{-y_j} - e^{y_j}}{-y_j} \right) \tau(X)$$

$$= \prod_{j=1}^{k} \frac{y_j}{\tanh \frac{1}{2} y_j} .$$

In particular Theorem 5.3.1. implies that $\prod_{j=1}^{k} \frac{y_j}{\tanh \frac{1}{2} y_j}$ is an integer when evaluated on the 2k dimensional fundamental cycle of X. Now

$$\left(\prod_{j=1}^{k} \frac{y_j}{\tanh \frac{1}{2} y_j} \right) [X] = \frac{1}{2^k} \left(\prod_{j=1}^{k} \frac{2y_j}{\tanh y_j} \right) [X] = \left(\prod_{j=1}^{k} \frac{y_j}{\tanh y_j} \right) [X]$$

so that this agrees with the usual definition of the L-genus of X.

We remark that the sequence

$$0 \longrightarrow SO(2k) \longrightarrow O(2k) \longrightarrow Z_2 \longrightarrow 0$$

can be used to show that a 2k-manifold X admits an orientation if and only if $w_1(X) \in H^1(X, Z_2)$ is zero.

5.6 <u>A special case</u>

We apply 5.4.1. in the special case where $u \in K_G(TX)$ is the element given by example (iii) of 5.5. We consider a formal factorisation

$$p(X^g) = \prod_j (1 + y_j^2)$$

$$e(T^g) = \prod_j y_j$$

$$c(T^g \otimes C) = \prod_j (1 + y_j)(1 - y_j).$$

Then, in the notation of 5.4, if $s: X^g \longrightarrow TX^g$ is the zero section,

$$s^* \operatorname{ch} i_! u = \prod_j (e^{-y_j} - e^{y_j}) . \prod_j (-e^{-y_{\pi,j}} + e^{y_{\pi,j}}) \prod_\theta \prod_j (e^{-(i\theta + y_{\theta,j})} - e^{(i\theta + y_{\theta,j})}).$$

Therefore $\phi^{-1}\left(\dfrac{\operatorname{ch} i_! u}{\operatorname{ch} \lambda_{-1}(N^g \otimes C)} \tau(X^g) \right)$ is a product of terms of three types

(i) $$\prod_j \left(\frac{e^{-y_j} - e^{y_j}}{-y_j} \right) . \tau(X^g) = \prod_j \frac{y_j}{\tanh \frac{1}{2} y_j}$$

(ii)
$$\frac{\prod_j (-e^{-y_{\pi,j}} + e^{y_{\pi,j}})}{\prod_j (e^{\frac{1}{2}y_{\pi,j}} + e^{-\frac{1}{2}y_{\pi,j}})^2} = \left(\prod_j \frac{y_{\pi,j}}{\tanh \frac{1}{2}y_{\pi,j}}\right)^{-1} \cdot \prod_j (y_{\pi,j})$$

(iii)
$$\prod_{0<\theta<\pi} \frac{\prod_j (e^{-(i\theta+y_{\theta,j})} - e^{(i\theta+y_{\theta,j})})}{\prod_j (-1)(e^{\frac{1}{2}(i\theta+y_{\theta,j})} - e^{-\frac{1}{2}(i\theta+y_{\theta,j})})^2} = \prod_{0<\theta<\pi} \prod_j \left(\frac{1}{\tanh \frac{1}{2}(y_j+i\theta)}\right).$$

Now define

(i) $\mathcal{L}(X^g) = \prod_j \frac{y_j}{\tanh \frac{1}{2}y_j}$ = polynomial in Pontrjagin classes

of X^g with rational coeffecients,

(ii) $\mathcal{L}(N_\pi) = \prod_j \frac{y_{\pi,j}}{\tanh \frac{1}{2}y_{\pi,j}}$ = polynomial in Pontradjagin

classes of N_π with rational coefficients,

(iii) $\mathcal{M}(N_\theta) = \prod_j \frac{1}{\tanh \frac{1}{2}(y_j+i\theta)}$ = polynomial in Chern classes

with complex coefficients,

and let $e(N_\pi)$ be the untwisted or twisted coefficients Euler class of N_π according as X^g is orientable or non-orientable. Then Theorem 5.4.1. implies:

Theorem 5.6.1. Under the above hypotheses

$$\{\mathcal{L}(X^g).\,\mathcal{L}(N_\pi)^{-1} \prod_{0<\theta<\pi} \mathcal{M}(N_\theta).\, e(N_\pi)\}\,[X^g] \in \Lambda$$

In the case $G=Z_2$ the eigenvalues of g are all ± 1 and therefore $N_\pi = N^g$ is the whole normal bundle of X^g. Hence

Corollary 5.6.2. Suppose $G=Z_2$. Then

$$\{\mathcal{L}(X^g).\,\mathcal{L}(N^g)^{-1}.\, e(N^g)\}\ [X^g] \in \Lambda$$

Corollary 5.6.3. Suppose that the characteristic classes of N_π and N_θ are zero for all θ. Then

$$\{\mathcal{L}(X^g).\ \prod_{0<\theta\leq\pi} \cot\mathrm{anh}\ \tfrac{1}{2}i\theta\ \}\ [X^g] \in \Lambda.$$

We conclude with one more concrete application of Corollary 5.6.3.

Proposition 5.6.4. Let G be cyclic of order p^r, p odd. Then G cannot have precisely one fixed point.

Proof Suppose that X^g consisted of a single point. Then $N_\pi = 0$, $(X^g) = 1$, and the eigenvalues of g have the form $\eta^{r_1},\ldots\ldots,\eta^{r_k}$ where $\eta = e^{\frac{2\pi i}{n}}$, $n = p^r$ and $\eta \neq -1$ since p odd. Then by Corollary 5.6.3 with $\theta = \dfrac{2\pi r_j}{n}$

$$\tau = \prod_j \frac{1+\eta^{r_j}}{1-\eta^{r_j}} \in Z[\eta].$$

Therefore
$$\tau \prod_j (1-\eta^{r_j}) = \prod_j (1+\eta^{r_j}) = \prod_j (2-(1-\eta^{r_j})).$$

The left hand side belongs to the ideal in $Z[\eta]$ generated by $1-\eta$. On the other hand the right hand side is a power of 2 modulo this ideal. Since $Z[\eta]/(\eta-1) \cong Z_p$ this is a contradiction.

REFERENCE

[1] R. Palais. Seminar on the Atiyah-Singer index theorem, with contributions by A. Borel, F. E. Browder and R. Solovay. Annals of Mathematics Studies 57, Princeton University Press, 1965.

Lecture 6. The analytic index

M. F. ATIYAH

In this lecture we outline the proofs of two theorems - the determination of $K_G^*(P(E))$ and the Thom isomosphism $K_G^*(X) \cong K_G^*(E)$ - which have so far been proved only for E a direct sum of G-vector-bundles with fibre C. As a corollary we obtain the determination of $K_G^*(X \times P(V))$ and the Thom isomorphism $K_G^*(X) \cong K_G^*(X \times V)$ for all G-modules V; so far this had been done only for G abelian. We assume that X is a compact G-space.

6.1 The G-splitting principle

We wish to extend to G-vector-bundles the familiar splitting principle for vector bundles. This extension depends on the following theorem. The proof involves differential operators and is outlined very briefly on 6.2.

Theorem 6.1.1. Let X be a compact G-space, E a G-vector-bundle over X, and $p: P(E) \to X$ the projective bundle of E. There is a natural homomorphism

$$p_! : K_G(P(E)) \longrightarrow K_G(X)$$

such that $p_! p^! =$ identity. In particular $p^!$ is a monomorphism.

We give a number of applications of Theorem 6.1.1. Another application is given in 7.2.

Let E be a G-vector-bundle with fibre C^n, P the associated principal bundle with fibre $GL(n,C)$, and $F(E)$ the fibre bundle over X obtained by dividing out P by the action of the triangular matrices of $GL(n,C)$. The action of G on E makes $F(E)$ a G-space and the projection $\pi : F(E) \longrightarrow X$ is a G-map. $F(E)$ is called the <u>flag bundle</u> of E. By construction a point $y \in \pi^{-1}(x)$ corresponds to a flag of linear subspaces

$$0 = M_0 \subset M_1 \subset \cdots\cdots \subset M_n = E_x$$

of the fibre E_x of E over X. There are G-vector-bundles $L_1, \ldots\ldots L_n$ over $F(E)$ with fibre C such that $\pi^*E = L_1 \oplus \cdots\cdots \oplus L_n$.

<u>Proposition</u> 6.1.2. There is a natural homomorphism

$$\pi_! : K_G(F(E)) \longrightarrow K_G(X)$$

such that $\pi_! \pi^! =$ identity. In particular $\pi^!$ is a momomorphism.

<u>Proof</u> The proposition is true for $n=1$ since then $F(E) = X$. We therefore assume that the proposition is true for $n-1$, $n>1$. There is a commutative diagram

$$\begin{array}{ccc} F(E') & = & F(E) \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ P(E) & \xrightarrow[p]{} & X \end{array}$$

where E' is the G-vector bundle over $P(E)$ defined by the exact sequence

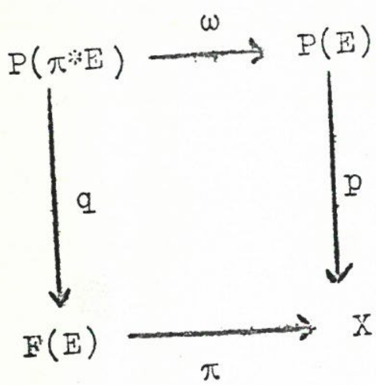
$$0 \longrightarrow H^* \longrightarrow p^*E \longrightarrow E' \longrightarrow 0.$$

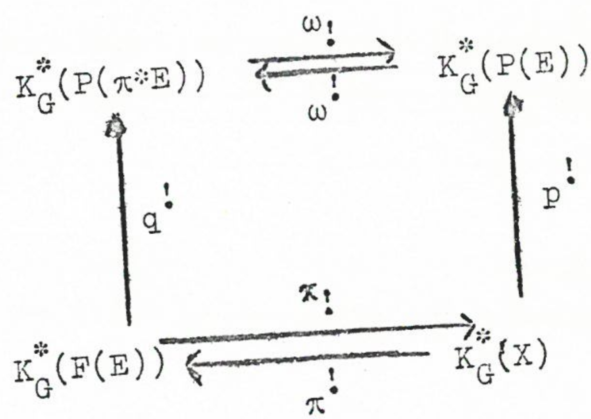
By the induction hypothesis $\pi'_!$ is defined. By Theorem 6.1.1. $p_!$ is defined. We define $\pi_! = p_! \pi'_!$

<u>Proposition</u> 6.1.3. $K_G^*(P(E))$ is generated as an algebra over $K_G^*(X)$ by the element η (see 2.7) subject to the single relation

$$\sum_{i=0}^{n} (-1)^i \eta^{n-i} \lambda^i(E) = 0 .$$

<u>Proof</u> The case in which $E = L_1 \oplus \cdots\cdots \oplus L_n$, where each L_i is a G-vector-bundle with fibre C, has been proved in Theorem 4.2.1. Consider the diagrams





where p,q are projection maps of projective bundles and π, ω are projection maps of flag bundles. Then $K_G^*(P(\pi^*E))$ is generated over $K_G^*(F(E))$ by an element η' subject to the relation (Theorem 4.2.1.)

$$\sum_{i=0}^{n} (-1)^i \eta'^{n-i} . q^! \lambda^i(\pi^*E) = 0,$$

where $\eta' = \omega^! \eta$.

Thus $\omega^! [\sum_{i=0}^{n} (-1)^i \eta^{n-i} \cdot p^! \lambda^i(E)] = 0$ and, since $\omega^!$ is a monomorphism (Proposition 6.1.2.),

$$\sum_{i=0}^{n} (-1)^i \eta^{n-i} \cdot \lambda^i(E) = 0.$$

Since $\omega^!$ is a monomorphism this is essentially the only relation between $1, \eta, \ldots, \eta^n$; since $\omega_!$ is an epimorphism, every element of $K_G^*(P(E))$ is a linear combination of $1, \eta, \ldots, \eta^{n-1}$. The result follows.

Proposition 6.1.3. now implies the Thom isomorphism theorem; by exactly the same argument as in Theorem 4.2.2:

<u>Theorem</u> 6.1.4. Let X be a compact G-space and E a G-vector-bundle over X. Then $K_G^*(E)$ is a free $K_G^*(X)$-module with generator λ_V. In particular the Thom homomorphism $\phi : K_G(X) \rightarrow K_G(E)$ is an isomorphism.

<u>Proof</u> Let $i : P(E) \longrightarrow P(E \oplus \underline{C})$ be the inclusion. By Proposition 6.1.3 $i^!$ is an epimorphism and there is an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_G^*(E) & \longrightarrow & K_G^*(P(E \oplus \underline{C})) & \xrightarrow{i^!} & K_G^*(P(E)) & \longrightarrow & 0 \\ & & & & \| & & \| & & \\ & & & & K_G^*(X)[\eta'] & \longrightarrow & K_G^*(X)[\eta] & \longrightarrow & 0 \end{array}$$

subject to the relations $i^! \eta' = \eta$, $\sum_{i=0}^{n} (-1)^i \eta^{n-i} \lambda^i(E) = 0$, and

$$\sum_{i=0}^{n+1} (-1)^i \eta'^{n+1-i} \lambda^i(E \oplus \underline{\underline{C}}) = (\eta-1) \sum_{i=0}^{n} (-1)^i \eta'^{n-i} \lambda^i(E) = 0.$$

Therefore $K_G^*(X) = \ker i^!$ is isomorphic to the free $K_G^*(X)$-module generated by $\sum_{i=0}^{n} (-1)^i \eta^{n-i} \lambda^i(E)$, or, taking the corresponding element in $K_G^*(E)$, to the free $K_G^*(X)$-module generated by $\sum_{i=0}^{n} (-1)^i \lambda^i(E) = \lambda_E$.

6.2 Families of differential operators

Let $p : Y \longrightarrow X$ be a fibre bundle with fibre a compact differentiable manifold M and with structure group a subgroup of the group of diffeomorphisms of M. Then the differential $dp : TY \to p^*TX$ has maximal rank. The kernel Tp of dp is a real vector bundle with fibre dimension dim M called the bundle along the fibres of Y. Suppose that G acts on $p: Y \longrightarrow X$ compatibly with the structure group, and recall that by means of a metric we may identify Tp with its dual.

Proposition 6.2.1. There is a natural homomorphism of $K_G(X)$-modules

$$\text{ind} : K_G(Tp) \longrightarrow K_G(X)$$

which reduces, when X is a point, to the analytic index.

We shall not prove the proposition here, but make some remarks on the case when X is a point. Let $\pi : TM \longrightarrow M$ be the projection and consider differentiable complex vector bundles E, F over M. A partial differential operator, $d: \Gamma(E) \longrightarrow \Gamma(F)$ of order k defines a symbol

$$\sigma : \pi^*E \longrightarrow \pi^*F.$$

<u>Definition</u> d is elliptic if σ is an isomorphism off the zero section of TM.

Thus an elliptic differential operator d defines an element $\sigma(d) = (\pi^*E, \pi^*F, \sigma)$ of $K(TM)$. On the other hand the elliptic condition implies that ker d and coker d are finite dimensional. The index ind d of d is defined by

$$\text{ind } d = \dim \ker d - \dim \text{coker } d.$$

In fact not every element $d \in K(TM)$ arises from a differntial operator. This is however the case once we consider the larger class of pseudo-differntial operators defined by Seeley [4]. The same argument then defines a homomorphism

$$\text{ind} : K(TM) \longrightarrow Z .$$

Full details can be found in Palais [3] which also contains the analysis needed for the more general case.

If X is a point and M is a G-manifold then an elliptic differntial operator d which respects the action of G on $\Gamma(E)$, $\Gamma(F)$ defines on the one hand an element

$$\sigma(d) \in K_G(TM)$$

and on the other hand an element

$$\text{ind } d = \ker d - \text{coker } d \in R(G)$$

where now ker d and coker d are representation spaces for G. This is the required homomorphism

$$\text{ind} : K_G(TM) \longrightarrow K_G(\text{point}) = R(G).$$

The extension to a general G-space X and G-fibre-bundle $p: Y \longrightarrow X$ involves families $d = \{d_x\}$ of elliptic differential operators. Any such is homotopic to an operator for which $\ker d_x$ and $\operatorname{coker} d_x$ have constant dimension and so define G-vector-bundles ker d and coker d. Then

$\operatorname{ind} d = \ker d - \operatorname{coker} d \in K_G(X)$ and there is a homomorphism

$$\operatorname{ind} : K_G(Tp) \longrightarrow K_G(X).$$

If also Tp is a complex vector bundle then the composition of ind with the Thom homomorphism $\phi : K_G(Y) \longrightarrow K_G(Tp)$ defines a natural homomorphsim

$$p_! : K_G(Y) \longrightarrow K_G(X).$$

Note that we do not claim that $p_!$ satisfies any particular properties in relation to $p^!$. This is done in 6.3 for fibre bundles $p: Y \longrightarrow X$ which satisfy an additional restriction.

6.3 Families of differential operators over complex mainfolds

Let M be a compact complex manifold, and O_M the sheaf of germs of holomorphic functions on M. If M is connected then, Liouville's theorem, $H^0(M, O_M) \cong C$. We say that M is regular if M is connected and if also $H^i(M, O_M) = 0$ for $i > 0$ (Warning: the word regular is also used to mean simply $H^1(M, O_M) = 0$).

Lemma 6.3.1. Let V be a complex vector space. The complex manifolds P(V) and F(V) are regular.

Proof (see [2], §15) : The theorem of Dolbeault implies that dim $H^i(M,O_M)$ is equal to the number $h^{0,i}$ of linearly independent harmonic forms of type (0,i). On the other hand P(V) and F(V) are Kähler manifolds whose complex cohomology ring is generated by elements in $H^2(P(V),C)$ and $H^2(F(V),C)$. These elements are Chern classes of complex analytic line bundles and therefore of type (1,1). It follows that $h^{p,q} = 0$ for $p \neq q$ and hence that $h^{0,i} = 0$ for $i > 0$.

Now assume that $p : Y \longrightarrow X$ is a fibre bundle with fibre a regular complex manifold M, and that the structure group is compatible with the group of complex analytic homeomorphisms of M. Let T_p denote the dual complex vector bundle along the fibre. There is a differential operator (for details see [2] §§15 and 25)

$$\bar{\partial} : \Gamma(\lambda^r \bar{T}_p) \longrightarrow \Gamma(\lambda^{r+1} \bar{T}_p)$$

defined by differentiation with respect to $\bar{Z}$ coordinates. The corresponding complex of differential operators

$$0 \rightarrow \Gamma(\lambda^0 \bar{T}_p) \xrightarrow{\bar{\partial}} \cdots\cdots \rightarrow \Gamma(\lambda^{m-1} \bar{T}_p) \xrightarrow{\bar{\partial}} \Gamma(\lambda^m \bar{T}_p) \rightarrow 0$$

is elliptic. Its symbol is the element of $K_G(\bar{T}_p) = K_G(T_p)$ defined by the exterior product sequence

$$0 \longrightarrow \pi^*\lambda^0 \bar{T}_p \longrightarrow \cdots\cdots \longrightarrow \pi^*\lambda^{m-1} \bar{T}_p \longrightarrow \pi^*\lambda^m \bar{T}_p \longrightarrow 0$$

Here $\pi : \bar{T}_p \longrightarrow X$ denotes the complex vector bundle conjugate to T_p ; the underlying bundle spaces are of course the same.

At each point $x \in X$ the corresponding differential operator d_x has kernel the space of harmonic forms of M of type (o,r), r even, and cokernel the space of harmonic forms of M of type (o,r), r odd. Since M is regular there are isomorphisms

$$\ker \; d_x \cong C \,, \quad \operatorname{coker} \; d_x = 0,$$
$$\ker \; \{d_x\} \cong \underline{\underline{C}} \,, \quad \operatorname{coker} \; \{d_x\} = 0.$$

If $p: Y \to X$ is a G-fibre-bundle and G is contained in the structure group, then the image of $\{d_x\}$ under the homomorphism

$$\operatorname{ind} : K_G(T_p) \to K_G(X) \quad \text{is the element} \quad 1 \in K_G(X).$$

Recall the definition of the Thom homomorphism $\phi: K_G(Y) \to K_G(\overline{T}_p)$. The construction of $\{d_x\}$ implies that its symbol is $\phi(1)$ and hence that

$$p_!(1) = \operatorname{ind} (\phi(1)) = 1 \in K_G(X).$$

Since $p_!$ is a homomorphism of $K_G(X)$-modules, we conclude that under the above assumptions $p_! p^! =$ identity. This proves Theorem 6.1.1 (taking $Y = P(E)$) and Proposition 6.1.2. (taking $Y=F(E)$).

6.4 Remarks on the index theorem

In 4.3. we defined a homomorphism

$$f_! : K_G(TY) \to K_G(\text{point}) = R(G)$$

for any abelian group G. In 6.2 we defined a homomorphism

$$\operatorname{ind} : K_G(TY) \to K_G(\text{point}) \in R(G)$$

for any G. Note that the definiton of $f_!$ depends on topological methods, the definition of ind on analytic methods.

Theorem 6.4.1. $f_!$ and ind are equal when both are defined.

When G is the identity this is the Atiyah-Singer index theorem, which is proved in [1]. An alternative proof, which will appear in a forthcoming paper of Atiyah and Singer, is valid for arbitrary compact Lie groups G.

Theorem 6.4.1. shows that we can use the Thom isomorphism theorem (6.1.4) to extend the definition of $f_!$ to non-abelian groups G without inconsistency. It also gives an explanation of the integers and representations which occur in the integrality theorems; these now appear as indexes of differential operators.

REFERENCES

[1] M.F.Atiyah and I.M.Singer. The index of elliptic operators on compact manifolds. Bull. Amer. Math. Soc. 69(1963)422-433.

[2] F. Hirzebruch. Topological methods in algebraic geometry (translated by R.L.E. Schwarzenberger) Springer-Verlag. To appear

[3] R. Palais. Seminar on the Atiyah-Singer index theorem, with contributions by A. Borel, F.E. Browder and R. Solovay. Annals of Maths Studies 57, Princeton University Press.

[4] R. Seeley. Integro-differential operators on vector bundles. Trans. Amer. Math, Soc. 117(1965)167-204.

Lecture 7. Completions

G. SEGAL

In this lecture we state a result concerning the completion $K_G(X)^\wedge$ of the $R(G)$-module $K_G(X)$ with respect to the $I(G)$-adic topology. We assume that X is a compact G-space.

7.1 Completions of R(G)-modules

Let A be a ring, I an ideal of A, and M an A-module. The I-adic topology of M is that for which the submodules I^nM of M form a basis of neighbourhoods of $0 \in M$. There are inclusions

$$\cdots\cdots \subset I^nM \subset I^{n-1}M \subset \cdots\cdots \subset I^2M \subset IM \subset M$$

and hence homomorphisms

$$\cdots\cdots \to M/I^nM \to M/I^{n-1}M \to \cdots\cdots \to M/I^2M \to M/IM.$$

The completion $M^\wedge$ of M with respect to the I-adic topology is defined by

$$M^\wedge = \varprojlim_n M/I^nM.$$

Clearly $M^\wedge$ is a $A^\wedge$-module.

There is a natural map $M \to M^\wedge$ with kernel $\bigcap_{n=0}^{\infty} I^nM$.

Remark M is a Hausdorff space if and only if $\bigcap_{n=0}^{\infty} I^nM$ is empty.

In this case $M^\wedge$ is the completion of M with respect to the metric

$$\| x \| = e^{-v(x)}$$

$$v(x) = \begin{cases} \infty & \text{if} \quad x = 0 \\ n & \text{if} \quad x \in I^n M,\ x \notin I^{n+1} M. \end{cases}$$

A homomorphism $u : M' \longrightarrow M$ induces a diagram of homomorphisms

$$\begin{array}{ccc} M'/I^n M' & \longrightarrow & M'/I^{n-1} M' \\ \downarrow & & \downarrow \\ M/I^n M & \longrightarrow & M/I^{n-1} M \end{array}$$

and hence a homomorphism $u^\wedge : M'^\wedge \longrightarrow M^\wedge$ of $A^\wedge$-modules.

Now consider the ring $R(G)$. There is a homomorphism

$$\varepsilon : R(G) \longrightarrow Z$$

induced by the map which assigns to each representation its dimension. We define $I(G) = \ker \varepsilon$ and consider the $I(G)$-adic completion $R(G)^\wedge$ of $R(G)$. For further detailed description of $R(G)^\wedge$ we refer to [3], when G is connected and to [1], when G is finite.

We recall (see the appendix) that $R(G)$ is a Noetherian ring and that (see 3.5) if X is a compact differentiable G-manifold then $K_G(X)$ is a module of finite type over $R(G)$. The importance of these facts is due partly to the fact ([1],§3) that completion is an exact functor for modules of finite type over a Noetherian ring.

7.2 Generalisation of Atiyah's theorem $R(G)^\wedge = K(B_G)$.

In this section we discuss the question : can one associate naturally to a compact G-space X a space X_G such that $K_G(X) \cong K(X_G)$?

If G acts freely on X we know the answer is yes, because then $K_G(X) \cong K(X/G)$. If G does not act freely the answer appears to be no, but one can define a functor $X \longmapsto X_G$ so that $K_G(X)$ and $K(X_G)$ become isomorphic when completed with respect to their natural topologies as filtered modules.

Consider the case X = point. It is known that $R(G)^\wedge \cong K(E_G/G)$, where E_G is a contractible space on which G acts freely. (See [1] for finite G, [3] for connected G, [2] for general G.) This suggests the general solution: choose a free G-space E(X) and a G-map $E(X) \to X$ which is a homotopy equivalence over X (not necessarily a G-homotopy equivalence). Define $X_G = E(X)/G$. Then we would like to prove an isomorphism of the type:

$$K_G(X)^\wedge \xrightarrow{\cong} K_G(E(X))^\wedge = K(X_G)^\wedge.$$

In particular, if X is a free G-space one can take E(X) = X.

We cannot give a full discussion of this matter here, for two reasons. The first is that a compact group cannot act freely on a compact contractible space (at any rate if the space is also locally contractible), so E(X) cannot be compact unless X is free. But we have not defined K_G for non-compact spaces.

The second reason is that we have not defined the filtration on $K_G(X)$. However it is known that the filtration-topology on $R(G) = K_G(\text{point})$ coincides with the $I(G)$-adic topology as defined in 7.1. Furthermore $K_G(X)$ is a filtered module over the filtered ring $R(G)$, and it seems at any rate plausible that when $K_G(X)$ is a finite $R(G)$-module the filtration-topology of $K_G(X)$ will be induced by that of $R(G)$ and so will coincide with the $I(G)$-adic topology. We suggest this as motivation for the following theorem.

Let $\underline{E}(X)$ be the category of compact free G-spaces over X. (In other words, an object of $\underline{E}(X)$ is a pair $(E, p_E : E \to X)$, where E is a compact free G-space and p_E is a G-map.)

Theorem 7.2.1 If $K^*_G(X)$ is a finite $R(G)$-module, then (i) $K^*_G(X)\hat{} \xrightarrow{\cong} \varprojlim_E K^*_G(E)\hat{} = \varprojlim_E K^*(E/G)\hat{}$; (ii) $R^1\varprojlim_E K^*_G(E)\hat{} = 0$; where E runs through the category $\underline{E}(X)$, and all completions refer to the $I(G)$-adic topology.

Remark (ii) is included for the benefit of those familiar with the behaviour of generalized cohomology theories who see that it would be reasonable to expect a short exact sequence (cf [5])

$$0 \to R^1 \varprojlim K^{q-1}_G(E)\hat{} \to K^q_G(X)\hat{} \to \varprojlim K^q_G(E)\hat{} \to 0.$$

Before proceeding with the proof observe that it is permissible, and will be convenient, to redefine $\underline{E}(X)$ as the categor of compact free G-spaces over X and G-<u>fibre-homotopy-classes</u> of maps.

Recall also that a functor $\theta : C_0 \to C$ is called <u>cofinal</u> if

(i) each object of C admits a morphism into an object $\theta(E)$, $E \in C_0$;

(ii) each diagram $P \begin{smallmatrix} \nearrow \theta(E_1) \\ \searrow \theta(E_2) \end{smallmatrix}$ in C can be extended to a commutative diagram

$$\begin{array}{ccccc} & & \theta(E_1) & & \\ & \nearrow & & \searrow{\scriptstyle \theta(f_1)} & \\ P & & & & \theta(E) \\ & \searrow & & \nearrow{\scriptstyle \theta(f_2)} & \\ & & \theta(E_2) & & \end{array}$$

It is then a trivial exercise to show that the natural map $\theta^* : \varprojlim_{C} F \longrightarrow \varprojlim_{C_0} F \circ \theta$ is an isomorphism for any contravariant functor from C to abelian groups.

We prove Theorem 7.2.1 in several stages.

<u>Proposition 7.2.2</u> Theorem 7.2.1 is equivalent to:

If $K_G^*(X)$ is a finite R(G)-module, then

(i) $K_G^*(X)^\wedge \xrightarrow{\cong} \varprojlim_{P} K_G^*(P \times X)^\wedge$; (ii) $R^1 \varprojlim_{P} K_G^*(P \times X)^\wedge = 0$;

where P runs through the category of compact free G-spaces and G-homotopy-classes of maps.

<u>Proof</u>: We have to see that the sub-category of projections $(P \times X \to X)$ is cofinal in the category $\underline{E}(X)$. But any $(E \to X)$ in $\underline{E}(X)$ admits an obvious map into $(E \times X \xrightarrow{pr_2} X)$. On the other hand, given $E \begin{smallmatrix} \nearrow P_1 \times X \\ \searrow P_2 \times X \end{smallmatrix}$ one can complete to the diagram

$$\begin{array}{ccccc} & \nearrow & P_1 \times X & \searrow & \\ E & & & & (P_1 * P_2) \times X, \\ & \searrow & P_2 \times X & \nearrow & \end{array}$$

which is commutative up to G-fibre-homotopy. ("$*$" denotes "join".)

Now choose a fixed embedding $\varphi : G \to U$ of G in a unitary group. φ induces a functor $\varphi_* : \{G\text{-spaces}\} \to \{U\text{-spaces}\}$ by $X \longmapsto U \times_G X$. And by 1.3.3 $K_U^*(\varphi_* X) \cong K_G^*(X)$. But the I(G)-adic topology on any R(G)-module coincides with the I(U)-adic topology (see appendix), so also $K_U^*(\varphi_* X)\hat{} \cong K_G^*(X)\hat{}$. And φ_* takes free G-spaces to free U-spaces. And when regarded as a functor from $\underline{E}(X)$ to $\underline{E}(\varphi_* X)$ φ_* is cofinal, as X admits a G-map into $U \times_G X$, etc. Thus theorem 7.2.1 for a G-space X $\Longleftrightarrow$ theorem 7.2.1 for the U-space $\varphi_* X$, i.e.

<u>Proposition 7.2.3</u> It is sufficient to prove 7.2.1 for the unitary groups U(n).

<u>Proposition 7.2.4</u> It is sufficient to prove 7.2.1 when G is a torus $T^n = T$.

<u>Proof:</u> Recall that we defined in 6.1 a natural homomorphism

$$p_! : K_G(F \times X) \to K_G(X) \quad \text{such that } p_! p^! = \text{identity},$$

where F was the flag manifold of C^n, $F = U(n)/T$. Apply this when $G = U(n)$, observing that if X is a U(n)-space then $U(n)/_T \times X \cong U(n) \times_T X$. One obtains a natural homomorphism $K_T(X) \cong K_{U(n)} (U(n) \times_T X) \xrightarrow{p_!} K_{U(n)}(X)$ which will be left-inverse to the "restriction" $p^!: K_{U(n)}(X) \to K_T(X)$. Now consider the diagram

$$\begin{array}{ccc} K^*_{U(n)}(X)^\wedge & \xrightarrow{\alpha} & \varprojlim K^*_{U(n)}(E)^\wedge \\ p^! \downarrow \uparrow p_! & & \tilde{p}^! \downarrow \uparrow \tilde{p}_! \\ K^*_T(X)^\wedge & \xrightarrow[\beta]{} & \varprojlim K^*_T(E)^\wedge \end{array}$$

where $p_! p^! = \mathrm{id}$ and $\tilde{p}_! p^! = \mathrm{id}$. It is immediate that β isomorphic $\Longrightarrow$ α isomorphic.

Similarly $R^1 \varprojlim K^*_{U(n)}(E)^\wedge$ is a direct summand of $R^1 \varprojlim K^*_T(E)^\wedge$, which proves the second part.

<u>Proof when G is a torus T</u>

First observe that if $G = H_1 \times H_2$ and if P_1, P_2 are free H_1, H_2 spaces respectively then $P_1 \times P_2$ is a free G-space and that free G-spaces of this form are cofinal in the category of all free G-spaces. (In fact any free G-space P admits the diagonal map into $P/H_1 \times P/H_2$, etc.)

Applying this to the torus T one finds by induction using 7.2.2 that when G = T 7.2.1 is implied by

<u>Proposition 7.2.5</u> If $K^*_T(X)$ is a finite R(T)-module, then

(i) $K^*_T(X)^\wedge \xrightarrow{\cong} \varprojlim_P K^*_T(X \times P)^\wedge$ (ii) $R^1 \varprojlim_P K^*_T(X \times P)^\wedge$

when P runs through the category of compact free S^1-spaces and S^1-homotopy-classes of maps. (One supposes given a fixed homomorphism $\theta: T \to S^1$ by which S^1-spaces become T-spaces.)

<u>Proof of Proposition 7.2.5</u> Regard S^1 as the group of complex numbers of modules 1. Then S^1 acts naturally on S^{2n-1}, the unit

sphere in C^n. Furthermore the set of spheres S^{2n-1} with inclusion maps is cofinal in the category of free S^1-spaces. For any free S^1-space P can be regarded as the unit sphere bundle of a complex line-bundle L. But L can be regarded as a subbundle of a trivial bundle with fibre C^n. The resulting C-map $L \to C^n$ induces an S^1-map $P \to S^{2n-1}$. But any two S^1-maps $P \to S^{2n-1}$, $P \to S^{2m-1}$ certainly become homotopic in $S^{2(n+m)-1} \cong S^{2n-1} * S^{2m-1}$. So we have to show:

(i) $K_T^*(X)^\wedge \xrightarrow{\cong} \varprojlim_n K_T^*(X \times S^{2n-1})^\wedge$ (ii) $R^1\varprojlim_n K_T^*(X \times S^{2n-1})^\wedge = 0$.

Consider the long exact sequence

$$\cdots \to K_T^*(X \times D^{2n}, X \times S^{2n-1}) \to K_T^*(X \times D^{2n}) \to K_T^*(X \times S^{2n-1}) \to \cdots$$

Using the Thom isomorphism this becomes

$$\cdots \to K_T^*(X) \xrightarrow{\alpha} K_T^*(X) \to K_T^*(X \times S^{2n-1}) \to \cdots$$

where α is multiplication by $\lambda_{-1}[\underline{C}^n] = (1 - \theta)^n$.

Because $K_T^*(X)$ is a finite R(T)-module this sequence remains exact when completed. (This is the only point at which one uses finiteness.) We deduce exact sequences

(i) $0 \to A/(1-\theta)^n A \to K_T^*(X \times S^{2n-1})^\wedge \to B_n \to 0$

(ii) $0 \to B_n \to A_n \xrightarrow{(1-\theta)^n} (1-\theta)^n A \to 0$

where $A = K_T^*(X)^\wedge$, and $A_n = A$, but the map from A_{n+1} to A_n is multiplication by $(1 - \theta)$. Now observe that $(1-\theta)^n A \subset I(T)^n.A$, so $\varprojlim_n (1-\theta)^n A = 0$. For the same reason $\varprojlim_n A/(1-\theta)^n A = A$. $R^1\varprojlim A/(1-\theta)^n A = 0$ because the maps in the sequence are surjective. Finally recall that

$$0 \to \varprojlim A_n \to \Pi A_n \xrightarrow{\varphi} \Pi A_n \to R^1\varprojlim A_n \to 0,$$

where $\varphi(\{a_i\}) = \{a_i + (1 - \theta)a_{i+1}\}$. (See [5].)

But φ is an isomorphism on the complete module ΠA_n because it has an inverse: $\{a_i\} \longmapsto \{\sum_{j\geq 0} (1 - \theta)^j a_{i+j}\}$.

So $\varprojlim A_n = R^1\varprojlim A_n = 0$.

This completes the proof, as (i) induces

$$0 \to \varprojlim A/(1 - \theta)^n A \to \varprojlim K_T^*(X \times S^{2n-1})^\wedge \to \varprojlim B_n \to$$
$$R^1\varprojlim A/(1 - \theta)^n A \to R^1\varprojlim K_T^*(X \times S^{2n-1})^\wedge \to R^1\varprojlim B_n \to 0,$$

while from (ii)

$$0 \to \varprojlim B_n \to \varprojlim A_n \to \varprojlim (1 - \theta)^n A \to R^1\varprojlim B_n \to R^1\varprojlim A_n,$$

showing that $\varprojlim B_n = R^1\varprojlim B_n = 0$.

<u>Remark 7.2.6</u> The construction of the cofinal sequence of free S^1-spaces is of course a particular case of Milnor's construction of universal bundles [4]. In fact the sequence G, $G * G$, $G * G * G$, ... is cofinal in the category of compact free G-spaces, so 7.2.1 is equivalent to

(i) $K_G^*(X)^\wedge \xrightarrow{\cong} \varprojlim_n K^*(X_G^n)^\wedge$ (ii) $R^1 \varprojlim_n K^*(X_G^n)^\wedge = 0$

where $X_G^n = (X \times E_G^n)/G$ and $E_G^n = G * \ldots * G$ (n copies). When X is a point, this is precisely Atiyah's result.

REFERENCES.

[1] M.F. Atiyah. Characters and cohomology of finite groups. Publ. Math. Inst. Hautes Etudes Scientifiques 9 (1961).

[2] M.F. Atiyah. Finiteness theorems for compact Lie groups. (unpublished).

[3] M.F. Atiyah and F. Hirzebruch. Vector bundles and homogenous spaces. Proceedings of Tucson Symposium on Differential Geometry, A.M.S., 1961.

[4] J. Milnor. Construction of universal bundles I, II. Ann. Math. 63(1956), 272-284 and 430-436.

[5] J. Milnor. Axiomatic homology theory. Pacific J. Math. 12 (1962) 337-341.