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So now I am down to proving vanishing theorems in homological algebra.

Lemma 8. [illegible] F_p. Assume $A \supset B$ is a pair of connected algebras over F_p ($A_t = 0$ for $t < 0$, $A_0 = B_0 = F_p$) and A is free as a left B-module on the generator 1 and other generators of degree $\geq d$. Assume M is a left A-module which is free as a left B-module on generators of degree $\geq \mu$. Then

$$\mathrm{Ext}_A^{s,t}(M, F_p) = 0 \text{ for } t < \mu + ds.$$

Proof — True for $s = 0$. Argue by induction on s. Assume it true for $s-1$. Let M be free as a left B-module on generators m_i of degrees $\geq \mu$. Let C_0 be free as a left A-module on generators c_i corr. m_i. Define $C_0 \xrightarrow{\varepsilon} M$ so that $\varepsilon(c_i) = m_i$, and form the short exact sequence

$$0 \to Z_0 \to C_0 \xrightarrow{\varepsilon} M \to 0.$$

Both C_0 & M are free over B; hence Z_0 is projective over B; since B is conn. and Z_0 is bdd below, & we work over a field, Z_0 is free over B. Also the map ε is iso in degrees $< \mu + d$, so Z_0 is zero in degrees $< \mu + d$. The inductive hyp. gives

$$\mathrm{Ext}_A^{s-1,t}(Z_0, F_p) = 0$$

for $t < \mu + d + (s-1)d$. But the

$$\mathrm{Ext}_A^{s-1,t}(Z_0, F_p) \longrightarrow \mathrm{Ext}_A^{s,t}(M, F_p) \text{ is epi,}$$

[illegible]

This gives a long exact sequence of Ext groups.
The Ext groups ~~vanish~~ with compact supp^t
vanish for $L\otimes pM$ by ind. hyp.
because $p^{e+1}(pM) = 0$, and for $L\otimes(M/pM)$
by the first step. Hence...

(iii) For the next case we assume M fin. gen. in each degree & torsion free.
In this case we have an exact seq

$$0 \to M \xrightarrow{p} M \to M/pM \to 0$$

which gives

$$0 \to L\otimes M \xrightarrow{p} L\otimes M \to L\otimes(M/pM) \to 0.$$

This gives a long exact sequence of Ext groups.

This long exact seq shows

$$Ext^{i+1}_{c}(Z_{(p)}, L\otimes M) \xrightarrow{p} Ext^{i+1}_{c}(Z_{(p)}, L\otimes M)$$

is epi. Everything on right is fin. gen. over $Z_{(p)}$
so Nakayama's Lemma shows $Ext^{i+1}_{c}(Z_{(p)}, L\otimes M) =$

(iv) At this step we assume only that M is finitely generated over $Z_{(p)}$ (taking all degrees together). In this case we have an exact seq

$$0 \to M' \to M \to M'' \to 0$$

in which M' is the torsion subobject and M'' is the torsion-free quotient and step (ii) applies to M'. Applying as in step (iii), If we can work with fin-dim'l v'spaces then we can dualise; ~~and~~ so we get

$$\mathrm{Ext}^{s,t}_{F_p[x]}(F_p, L\otimes\Gamma)$$

$$\cong \mathrm{Ext}^{s,t}_{F_p[x]^*}(L^*\otimes\Gamma^*, F_p).$$

But here $F_p[x]^*$ is well known to be a tensor product of truncated algebras $F_p[y]/y^p$ on generators of degree $2d, 2dp, 2dp^2 \ldots$.

We can apply Lemma 8 taking $A = F_p[x]^*$, B the tensor product of the first h truncated algebras. ~~[illegible]~~ $L^*\otimes\Gamma^*$ is free over B, and we get

$$\mathrm{Ext}^{s,t}_{F_p[x]^*}(L^*\otimes\Gamma^*, F_p) = 0$$

for $t < n + 2d.p^h s$ as claimed.

Now we will lift the assumption on Γ a bit. (ii) We will assume that [illegible] Γ is in each degree fin-gen. over $Z_{(p)}$; but for the next step we assume $p^e\Gamma = 0$. We prove the same conclusion by induction over e. We then use the short exact sequence

$$0 \to p\Gamma \longrightarrow \Gamma \longrightarrow \Gamma/p\Gamma \longrightarrow 0$$

& this is a sequence of C-comodules. Tensoring with L preserves exactness & gives

$$0 \to L\otimes p\Gamma \longrightarrow L\otimes\Gamma \longrightarrow L\otimes\Gamma/p\Gamma \longrightarrow 0.$$

of E_2-terms is zero and elements of $\pi_*(T_{p^{n-1}} \wedge R)$ gain at least 1 in filtration at each step.

Consider then an element $\beta \in \pi_*(T_{p^{n-1}} \wedge R)$.

After s steps it is still in $t = |\beta|$, but of filtration $\geq s$; and the vanishing line shows it is zero if

$$|\beta| < p^n \cdot 2n \cdot s - sq - c.$$

By choosing $p^n \cdot 2n > q$ we can be sure that this will happen for suitable $s = s(\beta)$.

we obtain the conclusion for M.

(v) Every M which is zero in degrees < μ is the direct limit of submodules which are zero in degrees < μ and fin. gen. over $Z_{(p)}$ (taking all degrees together). The result passes to direct limits, because we only have $Z_{(p)}$ on the LHS. This proves Lemma 9,

which proves Theorem 5.

Corollary 10. Suppose R is a ring-spectrum bdd below, with ie $\pi_i(R) = 0$ for $i < 0$, and $\alpha \in \pi_a(R)$ an element which maps to 0 in $E_a(R)$ for $E = X(n+1)$. Then $\exists$ n sufficiently large so that $F_{p^n-1} \wedge Tel(\bar\alpha)$ is contractible at p?

Proof. Consider the map

$$E_*(R) \xrightarrow{\bar\alpha_*} E_*(S^{-a} \wedge R).$$

Clearly this is multiplication by the image of α in $E_a(R)$, i.e. it is zero. Similarly the map

$$E_*(F_{p^n-1} \wedge R) \longrightarrow E_*(F_{p^n-1} \wedge S^{-a} \wedge R).$$

(So, while we're looking at the induced map of the SS's of Thms 1, 5, the induced map

~~is ho pue~~ [illegible] $W \wedge \hat{A}$ [illegible] $W \wedge \hat{A}$ [illegible] well homotopic [illegible]
~~impulse been~~ pulling on W and pulling on f.

It looks as if the products spectra become rapidly larger, but if we can prove anything about them there is no harm in their being large.

(iii) The study of ring-spectra. If R is a ring-spectrum then $\pi_*(R)$ is a ring, and we can take powers α^m in that sense. By putting $R = S^0$ we can recover Nishida's conject.

Hopkins has results for each of these conjects.

Theorem 1. Let X be a finite spectrum and $f : X \longrightarrow S^{-a}X$ a map. Then f is nilpotent under composition, i.e. $\exists m$ s.t. $X \xrightarrow{f} S^{-a}X \xrightarrow{f} \cdots \xrightarrow{f} S^{-am}X$ is $\simeq 0$, iff $MU_*(f) : MU_*(X) \to MU_*(S^{-a}X)$ is nilpotent, i.e. $\exists m$ s.t.

$$MU_*(X) \xrightarrow{f_*} MU_*(S^{-a}X) \to \cdots \to MU_*(S^{-am}X)$$

is 0.

This solves in the affirmative the first problem set by Ravenel, 10.1(a). It identifies Nishida's theorem; if $S^0 \xrightarrow{f} S^{-a}S^0$

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§7. Nilpotence theorems.

In this section we can start to harvest the fruits of our labours.

Hopkins' first few results are all modelled on Nishida's nilpotence theorem. Nishida says that in the ring $\pi_*(S^0)$, any element α of positive degree, $\alpha \in \pi_q(S^0)$ with $q > 0$, is nilpotent; $\exists\, m$ s.t. $\alpha^m = 0$.

If you want to generalize this, you must look for a more general context in which you can still make sense of the iterated product α^m. Hopkins considers three such contexts.

(i). Composition. We may suppose given a map $X \xrightarrow{f} S^{-q}X$ and form the m iterated composite

$$X \xrightarrow{f} S^{-q}X \xrightarrow{f} S^{-2q}X \to \dots \to S^{-mq}X.$$

If we hope to prove $f^m = 0$, it ~~It~~ is reasonable to suppose X is a finite spectrum. ~~if we hope to prove $f^m = 0$.~~

(ii). Smash-product. After all the result of Barratt and Hilton rests on the fact that the product in $\pi_*(S^0)$ can be defined by smash-product as well as composition. So we might suppose given $X \xrightarrow{f} Y$ and study the m-fold smash product

$$\overset{m}{\bigwedge} X \xrightarrow{\overset{m}{\wedge} f} \overset{m}{\bigwedge} Y.$$

involving the zero map of MU_*.

(Apply Th 2 to $\overset{m}{\Lambda} f$).

What I am complaining about is that Th. 2 concerns a map $f: S^k \to Y$ rather than a map $f: X \to Y$. But for such a result, the condition $MU_*(f) = 0$ is insufficient. Consider the simple example

$$S^0 \cup_p e^1 \xrightarrow{j} S^1.$$

$MU_*(j) = 0$, but no smash power $\overset{m}{\Lambda} j$ is $\simeq 0$

(Consider the induced map in $H_*(\;;F_p))$

Now there are reasonable ways to strengthen the assumption $MU_*(f) = 0$, but we shall have a better result later.

I turn to ring spectra.

is a map with $q > 0$ then certainly $MU_*(f) = 0$.
This is a strikingly satisfactory result, because on the one hand one can feel, one can compute with MU_*, but on the other hand the condition is necessary and sufficient for a statement in homotopy theory.
Hopkins also has sharper results, but this will do for now.

Now I return to the smash-product with one assumption omitted. ([illegible])

__Theorem 2__ Let $f : S^a \to Y$ be a map representing an elt of $\pi_a(Y)$ whose image in $MU_a(Y)$ is 0. (This is the same as saying $f_* : MU_*(S^a) \to MU_*(Y)$ is 0). Then $\exists$ m s.t.

$$\overset{m}{\underset{1}{\wedge}} f : S^{ma} \longrightarrow \overset{m}{\underset{1}{\wedge}} Y \quad \text{is} \simeq 0.$$

This is rather a limited result, but still useful.
When I say it is limited, I am not complaining that the condn is suff without being necessary. We could easily define a version in which the condn is both N & S: in order that $\exists m$ with

$$\overset{m}{\underset{1}{\wedge}} f : S^{ma} \longrightarrow \overset{m}{\underset{1}{\wedge}} Y \quad \text{is} \simeq 0,$$

it is N&S that $\exists$ m with $\overset{m}{\underset{1}{\wedge}} f : S^{ma} \to \overset{m}{\underset{1}{\wedge}} Y$

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Th 4 Let W & X be finite spectra, Y bdd below, $X \xrightarrow{f} Y$ a map.

In order that $\exists m$ s.t. $1_W \wedge \overset{m}{\wedge} f \simeq 0$, it is N & S that $\forall p$, $\forall n$ $(0 \leq n \leq \infty)$

either (i) $K(n)_*(W) = 0$

or (ii) $K(n)_*(\mathrm{Id} \wedge f) : K(n)_* X \to K(n)_* Y$ $\equiv 0$.

Explanation. (i) $K(0)$ means HQ, $\forall p$; $K(\infty)$ means HF_p.

(ii) According to Ravenel (Thm 2.11 p366), for each p there exists such that "$K(n)_* W = 0$" is satisfied ~~[illegible]~~ (with $n_0 = -1$ allowed if [illegible] non

Similarly ~~a~~ for $n \leq n_0$ ~~with~~ & $n_0 = \infty$ allowed if W is contractible at p).

So at p, the condn $K(n)_* f = 0$ is required ~~[illegible]~~ for $n > n_0$.

(iii) The necessity of the condn is seen as follows. The $K(n)_*$ are fields [illegible] $[e, t]$, which is a graded field in the sense [illegible]

[3]

Th 3. Let R be a ring-spectrum. (Htpy-associative with htpy-unit; no commutativity needed; no condns of finiteness type; no hypothesis that R is bounded below.) Then $\alpha \in \pi_* R$ is nilpotent iff its image in $MU_*(R)$ is nilpotent.

This is another ~~[illegible], su~~ very satisfying result.

Now I go back to improve Th. 2. First I must explain that I have become reconciled to Mahowald's terminology. One minor reason is that I now see hope of placing a sensible numerical meaning on it. The major reason will appear later.

Secondly I must explain that Hopkins requires a "nilpotence result 'with parameters'". That is, he studies

$$W \wedge \overset{m}{\bigwedge} X \xrightarrow{1 \wedge \overset{m}{\bigwedge} f} W \wedge \overset{m}{\bigwedge} Y,$$

and aims to prove $1 \wedge \overset{m}{\bigwedge} f \simeq 0$ under his hypotheses bearing partly on W and partly on f.

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~~if its image in $MU_*(R)$ is nilpotent.~~

~~Our sequence~~ ~~This is another entirely satisfactory result.~~

It will be convenient if we lead up for Theorem 13 part. To begin with we will assume R is bdd below: i.e. $\pi_i(R) = 0$ for $i < 0$ [illegible] $\pi_i(\ ) = 0$ for $i < 0$.

Proposition 6. Let $\alpha \in \pi_q(R)$ be an element which becomes nilpotent in $X(n+1)_*(R)$; then α becomes nilpotent in $X(n)_*(R)$.

Proof. Suppose $\alpha \in \pi_q(R)$ becomes nilpotent in $X(n+1)_*(R)$; i.e. $\exists\, m$ s.t. α^m becomes zero in $X(n+1)_*(R)$. Then it is sufft to replace α by α^m; for if we show α^m becomes nilpotent in $X(n)_* R$ that says $(\alpha^m)^{m'}$ becomes zero in $X(n)_* R$, so α becomes nilpotent in $X(n)_* R$. So we can assume α maps to zero in $X(n+1)_* R$.

Then §6 Corollary 10 says that $\exists\, h$ sufficiently large that

$$F_{p^h-1} \wedge Tel(\alpha) \simeq pt \text{ at } p$$

By the main theorem of §5, that says $X(n) \wedge Tel(\alpha) = pt$ at p.

module over it is free. It follows that we have the Künneth formula in the form of an iso

$$K(n)_*U \otimes_{\pi_*K(n)} K(n)_*V \xrightarrow{\cong} K(n)_*(U\wedge V)$$

It then follows that if

$$U\wedge V \xrightarrow{g\wedge h} U'\wedge V'$$

induces the zero map of $K(n)_*$, then either g or h must do so.

In particular, if $1\wedge\bigwedge^n f \simeq 0$, then $1\wedge\bigwedge^n f$ induces the zero map of $K(n)_*$; so either $1: W\longrightarrow W$ or $f: X\longrightarrow Y$ must do so. If $1: W\longrightarrow W$ induces the zero map of $K(n)_*$, then $K(n)_*(W)=0$.

I hope this gives the right general idea; the technical painting of making use a graded field is just too sad culture.

Proof of Thm 2. Let $f : S^a \to Y$ be a map whose image in $\pi U_a(Y)$ is 0.

The map f must map into some finite subspectrum $Y_1 \subset Y_{m_1}$ and then it must go to zero in $\pi U_a(Y_2)$ for some finite subspectrum $Y_2 \supset Y_1$, so we can suppose w.log. that Y is finite, in particular, bdd below. By suspending f and Y if necessary, we can arrange $\pi_i(Y) = 0$ for $i \leq 0$.

Form the spectrum $R = \bigvee_{m \geq 0} (\hat{\wedge}^m Y)$. I write R because this is indeed a ring-spectrum; the maps

$$(\hat{\wedge}^m Y) \wedge (\hat{\wedge}^n Y) \longrightarrow \hat{\wedge}^{m+n} Y$$

give the components for a product map $\mu : R \wedge R \to R$ which is homotopy-associative and has a homotopy-unit. (Notice that $\hat{\wedge}^0 Y = S^0$.)

Now I have a ring-spectrum R, bdd below and a map $S^a \xrightarrow{f} Y \longrightarrow R$ which goes to zero in $\pi U_a(R)$. By Coroll 6, there is an m s.t. $[f]^n = 0$ in $\pi_* R$. By the symbol of R, this is the same as saying that $\hat{\wedge}^m f : S^{ma} \longrightarrow \hat{\wedge}^n Y$ is $\simeq 0$. This proves Thm 2.

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This is true for all n, i.e.

$$X(n) \wedge \mathrm{Tel}(\bar{\alpha}) \simeq pt.$$

By §4 Coroll. 12, α becomes nilpotent in $X(n)_*(R)$.

Coroll. 6 Thm 3 is true if R is bdd below.

Proof. It is clear that if α is nilpotent, then its image in $MU_*(R)$ is nilpotent.

Conversely, assume the image of α in $MU_*(R)$ is nilpotent; say

$$S^{ma} \xrightarrow{\alpha^m} R \longrightarrow MU \wedge R$$

is null homotopic. S^{ma} is a finite spectrum and $MU = \underrightarrow{\text{holim}}\, X(n)$, so α^m is already null homotopic in $X(n) \wedge R$ for some n. That says α becomes nilpotent in $X(n)_* R$. By Prop. 5 we can proceed by induction downwards over n, showing that α becomes nilpotent in $X(i)_* R$ for $i = n, n-1, \ldots 3, 2, 1$. The final conclusion is that α becomes nilpotent in $X(1)_* R = \pi_*(R)$ i.e. α is nilpotent.

~~I think it will be convenient~~ if I turn to clear (next, because this will introduce some useful techniques.

Lemma 8. Suppose $X \xrightarrow{f} S^{-a}X$ induces the zero map $MU_*(X) \xrightarrow[=0]{MU_*(f)} MU_*(S^{-a}X)$. Then for any finite spectrum W there is an integer m such that the composite f^m induces the zero map of $(MU \wedge W)_*$.

One thinks of $(MU \wedge W)_*$ as "MU_* with coefficients in W". This sort of theory does not avoid the insufficiency of MU which [illegible]

Proof. Filter W by skeletons. This gives a spectral sequence, converging in a finite number of steps to $(MU \wedge W)_*(X)$, with E_1-term $\overset{n}{\oplus} MU_*(X)$ with shifts of the grading! The map f induces a map of spectral sequences which is zero on the E_1-term and therefore decreases filtration by 1. The filtration is finite, so there exists an m for which f^m induces the zero map of $(MU \wedge W)_*(X)$.

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Proof of Th 3. ~~As before~~ By passing to a cover, we may suppose given a map $S^a \xrightarrow{f} R$ which goes to zero in $MU_a(R)$.

By ~~Coroll~~ Thm 2, there is an m s.t. $S^{am} \xrightarrow{\overset{m}{\wedge} f} \overset{m}{\wedge} R$ is nullhomotopic.

Compose it with the product map $\overset{m}{\wedge} R \to R$; we see $[f]^m = 0$ in $\pi_a R$. This proves Theorem 3.

In fact, if μ and ν are duality maps we get natural (1-1) correspondences

$$\begin{tikzcd}
{[V, Y\wedge DX\wedge DW]} \arrow[r, leftrightarrow] & {[V\wedge W, Y\wedge DX]} \arrow[d] \\
& {[V\wedge W\wedge X, Y]}
\end{tikzcd}$$

and you check that the composite is induced the way I said.

Now you can check that duality preserves smash products: if

$$f \in [U, Y\wedge DW] \longleftrightarrow [U\wedge W, Y] \ni \bar{f}$$

$$g \in [V, Z\wedge DX] \longleftrightarrow [V\wedge X, Z] \ni \bar{g}$$

then $f\wedge g$ corresponds to $\bar{f}\wedge\bar{g}$, up to the obvious rearrangement of the factors.

Let's go back to our original adjunction

$$[W, Y\wedge DX] \longleftrightarrow [W\wedge X, Y]$$

and substitute $W \mapsto S^0$, $Y \mapsto X$, to get

$$[S^0, X\wedge DX] \longleftrightarrow [S^0\wedge X, X]$$

We see that there is a map $S^0 \to$ [illegible] so that the composite

$$X \cong S^0\wedge X \xrightarrow{\eta\wedge 1} X\wedge DX\wedge X \xrightarrow{1\wedge\epsilon} X\wedge S^0 \cong X$$

Now I introduce Spanier-Whitehead duality. For any finite spectrum X there is a finite spectrum DX which comes with a map $DX \wedge X \xrightarrow{\mu} S^0$ which has a characteristic property I shall explain. I use μ to construct a natural transformation

$$[W, Y \wedge DX] \longrightarrow [W \wedge X, Y].$$

Namely, for any map $W \xrightarrow{g} Y \wedge DX$ I form

$$W \wedge X \xrightarrow{g \wedge 1} Y \wedge DX \wedge X \xrightarrow{1 \wedge \mu} Y \wedge S^0 \simeq Y.$$

The characteristic property is that this natural transformation

$$[W, Y \wedge DX] \longrightarrow [W \wedge X, Y]$$

is iso for all finite W and Y. Of course you can then pass to direct limits on Y and get the same result for any Y. If you like you can generalise on W also, but I don't think we need that.

Notice that if $DX \wedge X \xrightarrow{\mu} S^0$ and $DW \wedge W \xrightarrow{\nu} S^0$ are duality maps, then so is

$$\begin{tikzcd}
DX \wedge DW \wedge W \wedge X \arrow[r, "1 \wedge \nu \wedge 1"] & DX \wedge S^0 \wedge X \arrow[d, phantom, "\wr"] & \\
 & DX \wedge X \arrow[r, "\mu"] & S^0.
\end{tikzcd}$$

$$\begin{tikzcd}
S^a \simeq S^a\wedge S^0 \arrow[r, "1\wedge\eta"] & S^a\wedge X\wedge DX \arrow[r, "f\wedge 1"] \arrow[d] & X\wedge DX \arrow[d] \\
& MU\wedge S^a\wedge X\wedge DX \arrow[r, "1\wedge f\wedge 1"] & MU\wedge X\wedge DX
\end{tikzcd}$$

That is, the element $\alpha \in \pi_a(X\wedge DX)$ corr. to f maps to zero in $MU_a(X\wedge DX)$.

Now Thm B, or even Cor B, says $\alpha^m = 0$ in $\pi_*(X\wedge DX)$, i.e. $f^m \simeq 0$ under composition.

Now we will prove all the ~~[illegible]~~ T V which involve Morava K-theories. First let me recall that for each p there is a ring-spectrum $P(n)$ with $\pi_* P(n) = \pi_*(BP)/(p, v_1, \ldots v_{n-1})$, $P(0) = BP$, and ~~[illegible]~~ a cofibration

$$S^{2(p^n-1)} P(n) \xrightarrow{\bar{v}_n} P(n) \longrightarrow P(n+1)$$

with $\mathrm{Tel}(\bar{v}_n) =_{p} B(n)$.

<u>Lemma 8</u> (i) $\langle B(n)\rangle = \langle K(n)\rangle$

(ii) $\langle P(n)\rangle = \langle K(n)\rangle \vee \langle P(n+1)\rangle$

(iii) $\langle BP\rangle = \langle K(0)\rangle \vee \langle K(1)\rangle \vee \cdots \vee \langle K(n-1)\rangle \vee \langle P(n)\rangle$.

(All this was known to Ravenel).

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is the identity.

We now check that in the other direction our natural transformation carries

$$W \wedge X \xrightarrow{f} Y$$

to

$$W \simeq W \wedge S^0 \xrightarrow{1 \wedge \eta} W \wedge X \wedge DX \xrightarrow{f \wedge 1} Y \wedge DX.$$

By construction ε corresponds under the natural transformation to $DX \simeq S^0 \wedge DX$, so the composite

$$DX \simeq DX \wedge S^0 \xrightarrow{1 \wedge \eta} DX \wedge X \wedge DX \xrightarrow{\varepsilon \wedge 1} S^0 \wedge DX \simeq DX$$

is the identity.

Now it follows that $X \wedge DX$ is a ring-spectrum, with identity $S^0 \xrightarrow{\eta} X \wedge DX$ and product map

$$X \wedge DX \wedge X \wedge DX \xrightarrow{1 \wedge \varepsilon \wedge 1} X \wedge S^0 \wedge DX \simeq X \wedge DX$$

Moreover, the multiplication of maps $S^q \xrightarrow{\alpha} X \wedge DX$ corresponds to composition of maps $S^q \wedge X \to X$.

In Thm 1 I have a map $S^a X \xrightarrow{f} X$, and originally I assume $MU_*(f) = 0$, but by Lemma 8, after replacing f by some iterate, I can assume that it induces the zero map of $(MU \wedge DX)_*$. In particular, We repeat several of the tricks [illegible] [illegible] [illegible]

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Proof of Thm 4. Suppose $W \wedge X \xrightarrow{f} Y$ are [illegible] as w. We can dualize: the dual of $1: W \to W$ is $\eta: S^0 \to W \wedge DW$ and the dual of f is $\bar{f}: S^0 \to Y \wedge DX$.

We can use as in proving Thm 1:

$$\begin{tikzcd}
S^0 \wedge S^0 \arrow[r, "\eta\wedge\eta'"] & W \wedge DW \wedge X \wedge DX \arrow[r, "1\wedge f\wedge 1"] \arrow[d] & W \wedge DW \wedge Y \wedge DX \arrow[d] \\
 & H(n) \wedge W \wedge DW \wedge X \wedge DX \arrow[r, "1\wedge 1\wedge f\wedge 1"] & H(n) \wedge W \wedge DW \wedge Y \wedge DX
\end{tikzcd}$$

represents an elt of the $H(n)_*$-homology of $W \wedge DW \wedge X \wedge DX$. The map $W \wedge X \xrightarrow{1\wedge f} W \wedge Y$ induces zero map on $H(n)_*$, and does so on homology. So if you smash it with the identity map of $DW \wedge DX$. Then the $\perp$ [illegible] elt of $\alpha \in \pi_0(W \wedge DW \wedge Y \wedge DX)$ which will be zero in $H(n)_*$ for all p & all n.

<u>Proofs</u>. (i) A result of Johnson - Wilson says

$$\pi_* K(n) \otimes_{\pi_* B(n)} B(n)_*(X) \xrightarrow{\cong} K(n)_*(X).$$

So $B(n)_*(X) = 0 \Rightarrow K(n)_*(X) = 0$

~~(ii)~~ A result of Würgler says

$$B(n)_*(X) \xrightarrow{\cong} P(n)_* K(n) \square_{K(n)_* K(n)} K(n)_* X.$$

So $K(n)_*(X) = 0 \Rightarrow B(n)_*(X) = 0$.

Strictly speaking, Würgler writes for $p > 2$, but his results work for $p = 2$ if you take extra care. Also Ravenel [illegible]
This proves (i).

(ii) Ravenel's result ~~[illegible]~~ applied to the cofibre as

$$\begin{aligned} \langle P(n) \rangle &= \langle C_{\bar{v}_n} \rangle \vee \langle Tel(\bar{v}_n) \rangle \\ &= \langle P(n+1) \rangle \vee B(n) \\ &= \langle P(n+1) \rangle \vee K(n). \end{aligned}$$

(iii) $\langle BP \rangle = \langle P(0) \rangle$

$= \langle K(1) \rangle \vee \langle P(1) \rangle$
induct on n.

$K(n) \wedge \mathrm{Tel}\, \bar{F}$ is contractible.
So by the lemma we have $BP \wedge \mathrm{Tel}\, \bar{F}$ is
contractible. Since this is true for all p,
it shows that $MU \wedge \mathrm{Tel}\, \bar{F}$ is contractible.
This shows that α becomes nilpotent in
$MU_0(R)$. Therefore α is nilpotent in
$\pi_0(R)$ by Th B; that is
there is an m s.t. the

$$\eta \wedge f^m : S^0 \wedge S^0 \longrightarrow (W \wedge DW) \wedge \bigwedge^{m}(Y \wedge DX))$$

is $\simeq 0$. Dualizing back, we have

$$1 \wedge f^m : W \wedge \bigwedge^{n} X \longrightarrow W \wedge \bigwedge^{n} Y \text{ is}$$

$\simeq 0$.

Now $W \wedge DW$ is a wq-spectrum, and $\bigvee_{m \geq 0} \bigwedge_1^m (Y \wedge DX)$ is a wq-spectrum,

so $(W \wedge DW) \wedge \left(\bigvee_{m \geq 0} \bigwedge_1^m Y \wedge DX \right)$ is a wq-spectrum R. Now, unfortunately, R is not bounded below. ~~Since~~ ~~with apply Theorem 5~~ However, ~~we can complete proof.~~ $W \wedge DW \wedge Y \wedge DX$ is bounded below. Therefore we can find an n so large that $P(n) \wedge W \wedge DW \wedge Y \wedge DX$ is equivalent to $K(\infty) \wedge W \wedge DW \wedge Y \wedge DX$ in any required range. In particular, we can choose n so that α maps to 0 in $P(n)_* (W \wedge DW \wedge Y \wedge DX)$.

Now it follows that $P(n) \wedge \mathrm{Tel}\, J$ is contractible. But also as

$$Y\wedge\cdots\wedge Y \xrightarrow{1\wedge\cdots\wedge 1\wedge f} Y\wedge\cdots\wedge Y\wedge Z \longrightarrow \cdots Y\wedge Z\wedge\cdots\wedge Z \xrightarrow{f\wedge 1} Z\wedge\cdots\wedge Z;$$

for the separate factors, the cofibres

$(\bigwedge^{i} Y)\wedge C_f \wedge (\bigwedge^{j} Z)$ lie in $\mathcal{C}$

by Remark 1; so $C_{\text{defining map}}$ lies in $\mathcal{C}$.

Theorem 9 Such a class $\mathcal{C}$ is either ϕ, or the class of contractible spectra, or else it is for some $n<\infty$ the class C_n of spectra X s.t. $K(i)_*(X)=0$ for $i<n$

(and X satisfies the p-local & finiteness conditions.)

Proof. If X is finite & noncontractible we see from the AHSS that

$$K(i)_*(X)\neq 0 \quad \text{for } i>>0.$$

So apart from the two trivial cases, we may find an n s.t. $\exists\ Y\in\mathcal{C}$ with $K(n)_*(Y)\neq 0$, but all $X\in\mathcal{C}$ have $K(i)_*(X)=0$ for $i<n$.

Conversely, take any X s.t. $K(i)_*(X)=0$

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For later use we must give one consequence of the nilpotence theorem.

Suppose given a class $\mathcal{C}$ of p-local spectra which are up to equivalence finite in the sense appropriate to p-local spectra; and assume $\mathcal{C}$ is closed under the following operations from §4.

(i) If $C \in \mathcal{C}$ & $C \simeq C'$ then $C' \in \mathcal{C}$.
(ii) If $C \in \mathcal{C}$ then $S^n C \in \mathcal{C}$ for $n \in Z$.
(iii) If $A \to B \to C$ is a cofibering and two of the terms are in $\mathcal{C}$ then so is the third.
(vii) If $X \vee Y \in \mathcal{C}$ then $X \in \mathcal{C}$.

Remark 1. We can have the benefit of operation (v) of §4: if $X \in \mathcal{C}$ then $X \wedge Y \in \mathcal{C}$.

Proof As in §4. We can build up Y from p-local spheres $S^n_{(p)}$ by a finite number of cofiberings; this builds up $X \wedge Y$ from suspensions $X \wedge S^n$ by a finite number of cofiberings.

Remark 2. Let $X \xrightarrow{f} Y \longrightarrow C_f$ be a cofibering with $C_f \in \mathcal{C}$; then $C_{f \wedge f \wedge \cdots \wedge f} \in \mathcal{C}$.

Proof It is standard that for any 2 composable maps

$$U \xrightarrow{g} V \xrightarrow{h} W$$

we have a cofibering

$$C_g \to C_{hg} \longrightarrow C_h.$$

In particular, if C_g & $C_h \in \mathcal{C}$, then $C_{hg} \in \mathcal{C}$. Now decompose $f \wedge f \wedge \cdots \wedge f$

On the other hand, $K(i)_*(X) = 0$ for $i < n$. Therefore Theorem 3 applies, and $\exists f$ s.t.

$$X \wedge \hat{\Lambda}^{m}_{1} Z \xrightarrow{1 \wedge \hat{\Lambda}^{m}_{1} f} X \wedge S^0$$

is $\simeq 0$. So X is a direct summand in $X \wedge C_{\hat{\Lambda}^{m}_{1} f}$.

But $Y \in \mathcal{C}$, so $C_f = Y \wedge DY \in \mathcal{C}$, by Remark 1 alone, so $C_{\hat{\Lambda}^{m}_{1} f} \in \mathcal{C}$ by Remark 2, so $X \wedge C_{\hat{\Lambda}^{m}_{1} f} \in \mathcal{C}$, by Remark 1 so its direct summand X lies in $\mathcal{C}$ by operation (vii). This completes the proof.

Corollary 10. All the spectra in $C_n - C_{n+1}$ are Bousfield-equivalent.
This is essentially Ravenel's Class Invariance Conjecture 10.4.

Proof Let $Y \in C_n - C_{n+1}$

Then the class of spectra X with $\langle X \rangle \leq \langle Y \rangle$ is closed under operations (i) (ii) (iii) (vii), so by the proof above it for $i < n$; we will show $X \in \mathcal{C}$.

Since $K(n)_*(Y) \neq 0$, Ravenel's result shows $K(i)_*(Y) \neq 0$ for $i \geq n$. Consider

$$Y \simeq S^0 \wedge Y \xrightarrow{\eta \wedge 1} Y \wedge DY \wedge Y \xrightarrow{1 \wedge \varepsilon} Y \wedge S^0 \simeq Y.$$

This composite be id map of $K(i)_*(Y)$, which is non-zero for $i \geq n$; in particular,

$$K(i)_*(S^0) \xrightarrow{\eta_*} K(i)_*(Y \wedge DY)$$

must be non-zero; since $K(i)_*(S^0)$ is a graded field, η_* must be mono. Use η to find a cofibering

$$Z \xrightarrow{f} S^0 \xrightarrow{\eta} Y \wedge DY = C_f.$$

We see that

$$K(i)_* Z \xrightarrow{f_*} K(i)_* S^0$$

must be zero for $i \geq n$.

§8. At this point we have to construct ~~studying~~ some finite complexes and some periodicity maps on them, and we are going to do this by the methods of J. Smith. He uses the representation-theory of the symmetric groups, so we are now going to have a holiday and do some algebra. [illegible]

If G is any finite group, $R(G)$ will be the representation ring of G, that is the Grothendieck group which is a free Z-module on the irred. repns as generators. $R(G)$ is self-dual over Z; the inner product is given by

$$(V, W) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(V, W)$$

$$= \int_G \chi_V(g) \, \chi_W(g) \; ;$$

since this is bilinear it extends to virtual repns, i.e. elts of $R(G)$. We have

$$R(G \times H) = R(G) \otimes R(H).$$

The classical way to obtain repns of Σ_n is to induce up from representations of $\Sigma_i \times \Sigma_j$, $i+j=n$.

We therefore introduce a graded [illegible]

$$A = \bigoplus_{n \geq 0} R(\Sigma_n) \; ;$$

it is certainly a graded ring; the multiplication we write $\$_x$: ~~possibly because it is a Good Thing.~~

contains all of C_n; in particular, if $Y \in C_n - C_{n+1}$ & $X \in C_n - C_{n+1}$ then $\langle Y\rangle \geq \langle X\rangle$, $\langle X\rangle \geq \langle Y\rangle$.

~~Theorem 9 is actually rather better than Ravenel's Class Invariance Conjecture 10.4, so I will not bother to state it, except for one special case, which Hopkins proved earlier and found useful.~~

Corollary 11. Let X be a finite spectrum such that $H_*(X;Q) \neq 0$; then $\langle X\rangle = \langle S^0\rangle$.

Proof. Both $X_{(p)}$ and $S^0_{(p)}$ lie in $C_0 - C_1$, so $\langle X_{(p)}\rangle = \langle S^0_{(p)}\rangle$ by Corollary 10; so

$$\langle X\rangle = \bigvee_p \langle X_{(p)}\rangle = \bigvee_p \langle S^0_{(p)}\rangle = \langle S^0\rangle.$$

~~This Corollary was explicitly stated by Ravenel as part of his Class Invariance Conjecture 10.4; as the proof of it is essentially Corollary 10, and Theorem 9 is better, I will~~

and $\psi\,\varepsilon_n = \sum_{i+j=n} \varepsilon_i \otimes \varepsilon_j$ $(\varepsilon_0 = 1)$.

Topologists will thus recognise this $H_*(BU)$ Hopf algebra. We have an isomorphism

$$\bigoplus_n R(\Sigma_n) \cong H_*(BU)$$

mapping 1_n on the generators corresponding to $H_{2n}(BU(1))$. We also have an isomorphism

$$\bigoplus_n R(\Sigma_n) \cong H^*(BU)$$

mapping ε_n on the Chern classes c_n [illegible] H_{2n} [illegible] $(BU(1))$.

<u>Addendum.</u> These isomorphisms turn the inner product in $\bigoplus_n R(\Sigma_n)$ into the Kronecker product between $H^*(BU)$ and $H_*(BU)$.

The only potential issue in the proof lies in verifying the Hopf algebra condition, that $\otimes$ and ψ are compatible. This took quite a long time to emerge, because the old-time algebraists could consider restriction on one $\Sigma_i \times \Sigma_j$ but their natural definition was ~~restriction to all $\Sigma_i \times \Sigma_j$ with~~ considering restriction on all $\Sigma_i \times \Sigma_j$ with $i+j=n$.

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It becomes a graded ring by using

$\Sigma_i \times \Sigma_j \longrightarrow \Sigma_{i+j}$ to define

$$R(\Sigma_i) \otimes R(\Sigma_j) \xrightarrow{\text{ind}} R(\Sigma_{i+j}).$$

Of course there are several embeddings of $\Sigma_i \times \Sigma_j$ in Σ_{i+j}, but they are all conjugate, so the product map is well defined. This product is associative and commutative, and has $1 \in R(\Sigma_0)$ as a unit.

Dually, we consider $\Sigma_i \times \Sigma_j \longrightarrow \Sigma_{i+j}$ which gives

$$R(\Sigma_{i+j}) \xrightarrow{\text{res}} R(\Sigma_i) \otimes R(\Sigma_j);$$

if we do this for all i, j s.t. $i+j=n$ we get a coproduct map

$$R(\Sigma_n) \xrightarrow{\psi} \bigoplus_{i+j=n} R(\Sigma_i) \otimes R(\Sigma_j).$$

This coproduct is coassoc. and cocomm., and has the obvious counit.

Thm 1. $A = \bigoplus_{n \geq 0} R(\Sigma_n)$ is a Hopf algebra.

It is a poly. algebra $\mathbb{Z}[1_1, 1_2, \ldots, 1_n, \ldots]$ where 1_n means the trivial [illegible] and $\psi\, 1_n = \sum_{i+j=n} 1_i \otimes 1_j$ $(1_0 = 1)$.

It is also a poly. algebra $\mathbb{Z}[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \ldots]$ where ε_n means the alternating 1-dim'l rep'n of Σ_n

If we take f to be the constant map

$G/H \longrightarrow G/G = pt,$

then $f^* = \text{res}$: $R_H \longleftarrow R_G$

and $f_* = \text{ind}$: $R_H \longrightarrow R_G$.

Now suppose given a pull-back diagram of finite G-sets,

$$\begin{tikzcd} U \arrow[r, "e"] \arrow[d, "p"] & W \arrow[d, "a"] \\ V \arrow[r, "f"] & X \end{tikzcd}$$

~~Under these~~ Then we have

$$\begin{tikzcd} K_G(U) \arrow[d, "p_*"] & K_G(W) \arrow[l, "e^*"'] \arrow[d, "a_*"] \\ H_G(V) & K_G(X) \arrow[l, "f^*"'] \end{tikzcd}$$

This can give a very convenient way of handling things which classically were done by the double coset formula.

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The simplest proof uses ~~[illegible]~~ a little extra flexibility.

Let X be a finite G-set. Then $K_G(X)$ means the ~~[illegible]~~ Grothendieck group of G-vector-bundles on X; that is, for each $x \in X$ we have a vector-space V_x, and G acts on their union over the action on X.

Clearly we have $K_G(X) \cong \bigoplus_\alpha K_G(X_\alpha)$, where the X_α are the G-orbits in X. (Our vector bundles can have different dimensions on different orbits.) For a single orbit G/H we have

$$K_G(G/H) \cong R(H).$$

Now let $f: X \longrightarrow Y$ be a map of G-sets. This gives an obvious map $f^*: K_G(X) \longleftarrow K_G(Y)$ — that is, the form the pull back of vector bundles. But it also gives a "transfer" map

$$\int_f : K_G(X) \longrightarrow K_G(Y);$$

that is, given $\{V_x\}$ on X, we associate to each $y \in Y$ the sum $\bigoplus_{fx=y} V_x$.

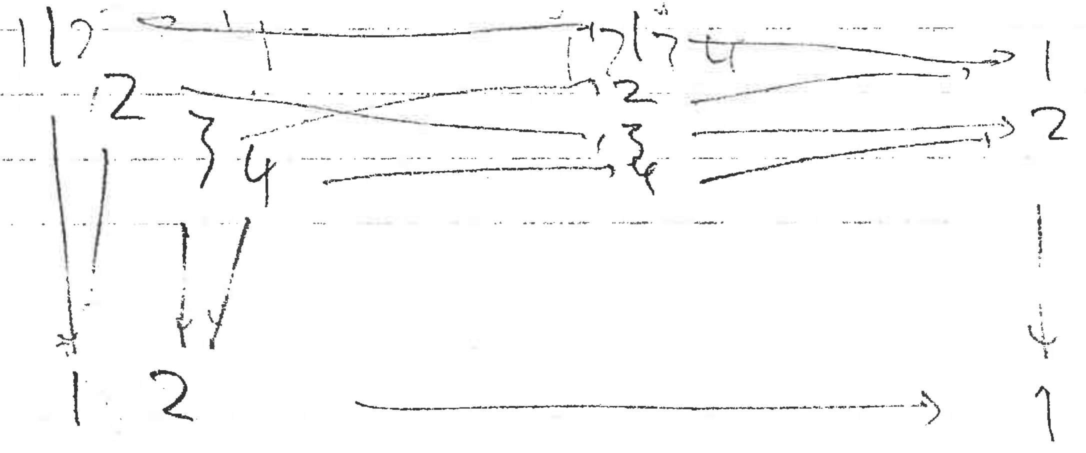
With $G = \Sigma_n$, the various groups there are obtained by applying $H_G(-)$ to the following finite G-sets:

$\{123\cdots n\}^{\{1234\}}$ $\{123\cdots n\}^{\{1234\}}$ $\{123\cdots n\}^{\{12\}}$

$\{123\cdots n\}^{\{12\}}$ $\{123\cdots n\}^{\{12\}}$

Here the exponent-sets $\{12\}$ or $\{1234\}$ index the copies of A in A^2 or A^4.

Our maps Φ, Ψ are then obtained by applying f_*, or f^* to the diagram of G-sets obtained from the following diagram of exponents.



Well, that's a pullback diagram, and it

In particular, consider the map

$$\{1,2\} \longrightarrow \{1\}$$

and the resulting map

$$(\{1, 2, \dots, n\})^{\{1,2\}} \xrightarrow{p} \{1, 2, \dots, n\}^{\{1\}} = pt .$$

The LHS is a Σ_n-set of size 2^n and it splits up into orbits according to the size of the maps, i inverse to 1 and j inverse to 2; i.e., this Σ-set

is $\displaystyle\coprod_{i+j=n} \Sigma_n / \Sigma_i \times \Sigma_j$.

Thus we get induced maps,

$$\bigoplus_{i+j=n} R(\Sigma_i) \otimes R(\Sigma_j) \underset{\psi = p^*}{\overset{\Phi = p_*}{\rightleftarrows}} R(\Sigma_n)$$

Here the indexing 1, 2 corr to the first & second factors of the tensor product

Well, now, what is the diagram we would like to see? During it the same way as the former, we want

$$\begin{tikzcd}
(A^4)_n \arrow[d, "\varphi\otimes\varphi"'] & (A^4)_n \arrow[l, "1\otimes\tau\otimes 1"'] & (A\otimes A)_n \arrow[l, "\psi\otimes\psi"'] \arrow[d, "\varphi"] \\
(A\otimes A)_n & & A_n \arrow[ll, "\psi"']
\end{tikzcd}$$

$$\psi c_n = \sum_{i+j=n} c_i \otimes c_j$$

$$\psi b_n = \sum_{i+j=n} b_i \otimes b_j,$$ and copr respectively preserve products.

Notice that we have to prove A is a Hopf algebra before we can use this argument.

Now I claim that

$$(\theta^* h^*, \theta_* h_*) = (h^*, h_*).$$

Well, this is true ~~of~~ on gen's and they are both accurate because

$$(c_n, b_n) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n \geq 2. \end{cases}$$

I claim if this true for (c_i, x) $\forall i$ & (c_j, y) $\forall j$ then it's true for $(c_n, xy$

In fact,

$$(\theta^* c_n, \theta_* xy) = (\psi\theta^* c_n, \theta_* x \cdot \theta_* y)$$

$$= \sum_{i+j=n} (\theta^* c_i, \theta_* x)(\theta^* c_j, \theta$$

$$= \sum_{i+j=n} (c_i, x)(c_j, y)$$

$$= (c_n, xy).$$

It follows, then, that this is true for (c_n, x) $\forall$ $x \in H_*(BU)$. But now the same shows a pull back diagram when you take its product with itself n times. So applying the diagram which encodes the double coset formula, we get the required commutative diagram.

This proves that $A = \bigoplus_{n \geq 0} R(\Sigma_n)$ is a Hopf algebra.

Remark. $(\varepsilon_n, 1_n) = \begin{cases} 1 \text{ if } n=1 \\ 0 \text{ if } n \geq 2 \end{cases}$.

Proof. ε_1 and 1_1 are both the one only 1-dimensional rep of Σ_1. For $n \geq 2$, ε_n and 1_n are the inequivalent irred. reps of Σ_n.

Now I take a map of algebras

$$\begin{tikzcd} H_*(BU) \arrow[r, "\theta_*"] \arrow[d, equal] & A \\ \mathbb{Z}[b_1, b_2, \dots b_n \dots] & \end{tikzcd}$$

mapping the canonical b_n on 1_n, and a map of algebras

$$\begin{tikzcd} H^*(BU) \arrow[r, "\theta^*"] \arrow[d, equal] & A \\ \mathbb{Z}[c_1, c_2, \dots c_n] & \end{tikzcd}$$

mapping the Chern class c_n on ε_n. Both these maps commute with coproducts, because on the left they are given by the same formulae that we have in A, viz

So this would say the inner product between $H^*(BU)$ and $H_*(BU)$ has determinant ab. We know it has determinant 1, so ab = 1; that is, θ^* and θ_* are iso. This proves Theorem 1.

Omit — Of course, professional algebraists are under no obligation to know or love $H^*(BU)$ or $H_*(BU)$. However, they are supposed to admit the existence of the relation between the representation theory of the symmetric group Σ_n and the general linear groups GL_N, or $U(N)$ if you prefer compact groups. So I had better point out that this relationship is the same again.

First let me summarise what we may suppose known about $RU(N)$. The maximal torus T in $U(n)$ consists of diagonal matrices

$$\begin{bmatrix} z_1 & 0 & & 0 \\ 0 & z_2 & & 0 \\ & & \ddots & \\ 0 & 0 & & z_N \end{bmatrix} \quad \text{with } |z_r| = 1 \ \forall r.$$

Representations of the compact group $U(n)$ are distinguished ~~at~~ by their characters χ; characters take equal values at conjugate points,

$$\chi(ghg^{-1}) = \chi(h);$$

every point $g \in U(N)$ is conjugate to a point $t \in T$; Moreover, if argument on the other side shows that it is true for (u, x), all $u \in H^*(BU)$, all $x \in H_*^*(BU)$. In fact, assume it true for u & v; then

$$(\theta^* uv, \theta_* x) = (\theta^* u . \theta^* v, \theta_* x)$$
$$= \sum_i (\theta^* u, \theta_* x_i')(\theta^* v, \theta_* x_i'')$$
where $\psi x = \sum_i x_i' \otimes x_i''$
$$= \sum_i (u, x_i')(v, x_i'')$$
$$= (uv, x).$$

Now the pairing of $H^{2n}(BU)$ & $H_{2n}(BU)$ to Z is non-singular i.e. $H^{2n}(BU) \xrightarrow{\cong} \mathrm{Hom}$, $H_{2n}(BU) \xrightarrow{\cong} \mathrm{Hom}$. It follows then that ~~θ^* and~~ θ_*, are mono. Specifically, of course, $H^{2n}(BU)$ and $H_{2n}(BU)$ ~~[illegible]~~ have the same rank as $R(\Sigma_n)$, namely the number of partitions of n.

So $\mathrm{Im}\,\theta^*$ and $\mathrm{Im}\,\theta_x$ must be subgroups of finite index, say indices a, b.

The inner product on $R(\Sigma_n)$ is unimodular;

Next let me explain why we have $R(G \times H) \cong R(G) \otimes R(H)$.

Let V be a representation of $G \times H$, and W a ~~an irreducible~~ representation of H. Then we have $\mathrm{Hom}_{kH}(W, V)$, and it is a representation of G. We have an obvious map

$$\mathrm{Hom}_{kH}(W, V) \otimes W \longrightarrow V,$$

and if we run W over a set of representatives for the isom. classes of irreducible reps of H, the map

$$\bigoplus_{(W)} \mathrm{Hom}_{kH}(W, V) \otimes W \longrightarrow V$$

is iso. This leads to the result

$$R(G \times H) \cong R(G) \otimes R(H).$$

Another way to see the same conclusion is to say that $\mathrm{Hom}_H(W, V)$ is bilinear in V and W, and hence we have

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suffit to give the value of χ at a general point

$$\begin{pmatrix} z_1 & 0 & & 0 \\ 0 & z_2 & & 0 \\ & & \ddots & \\ 0 & & & z_N \end{pmatrix}$$ in T.

In effect, this describes $RU(N)$ in the following terms. The map

$$RU(N) \longrightarrow RT$$ is mono;

by using characters, we can identify

$$\underbrace{RT}_{\,} = \underbrace{\mathbb{Z}[z_1, z_1^{-1}, z_2, z_2^{-1}, \dots z_N z_N^{-1}]}$$

with the ring of Laurent polynomials. For each $\pi \in \Sigma_N$, the elements

$$\begin{pmatrix} z_1 & & \\ & z_2 & \\ & & \ddots \\ & & & z_N \end{pmatrix}, \quad \begin{pmatrix} z_{\pi 1} & & & \\ & z_{\pi 2} & & \\ & & \ddots & \\ & & & z_{\pi N} \end{pmatrix}$$

of T are conjugate in U; therefore the image of $RU(N)$ lies in the subring

RT^{Σ_N} of elements invariant under Σ_N.

(In fact, the standard theory shows

$$RU(N) \xrightarrow{\cong} RT^{\Sigma_N}.)$$

For example, if we take W = ...,
$T(V) = \lambda^n V$.

Now we can give a construction which has been the centre of the subject for a long time. Let V be a complex vector space, and let W be a repn of Σ_n. Then Σ_n acts on $V^{\otimes n}$ by permuting the coordinates, and we can form

$$\mathrm{Hom}_{\Sigma_n}(W, V^{\otimes n}).$$

This gives us a functor $T(V)$ of V.
From a functor is we form a representation.
Certainly, we can take $V = \mathbb{C}^N$ and get a representation of $GL(N, \mathbb{C})$; but in fact we get a representation of the monoid $\mathrm{Hom}(V, V)$ of all $N \times N$ matrices; moreover, we get one of these for each N, and there is some relationship between the reps we get for different values of N.

However, we can neglect good information, ~~and take~~ we take $V = \mathbb{C}^N$; we regard $V^{\otimes n}$ as a representation of $U(N) \times \Sigma_n$; and we form

$$\mathrm{Hom}_{\Sigma_n}(W, V^{\otimes n}) = V^{\otimes n}/W$$

regarding it as a repn of $U(N)$.

<u>Remark</u>. Then $V^{\otimes n}/W \in Z[z_1, z_2 \cdots z_N]^{\Sigma_N}_n$;

power to elts $x \in R(G \times H)$, $y \in R(H)$
to define a slant product

$$x/y \in R(G).$$

This has the property that if

$$x = u \otimes v \quad \text{for some } u \in R(G), v \in R(H)$$

then $\cancel{u \otimes v}$

$$x/y = (u \otimes v)/y = u(v,y).$$

We may need to record some properties of this.

Suppose $H \supset K$, so $G \times H \supset G \times K$

$x \in R(G \times H)$, $\quad z \in R(K)$

Then $x/\operatorname{ind} z = (\operatorname{res} x)/z$

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$$\sigma_r = \sum_{i_1 < i_2 < \cdots < i_r} z_{i_1} z_{i_2} \cdots z_{i_r}.$$

[illegible] character of the rth exterior power operation λ^r (as is well known and easy to calculate), so we have

$$RU(N)^+ = \mathbb{Z}[\lambda^1 \lambda^2 \cdots \lambda^N]$$

where the corresponding l_n in $RU(N)^+_n$ ~~Cones~~ equals λ^r in degree r.

It is sufficient to consider $RU(N)^+$ instead of $RU(N)$, because the ~~[illegible]~~ representation λ^N is 1-dimensional and hence invertible; it is just the determinant representation $A \mapsto \det A$. Its character is $z_1 z_2 \cdots z_N$, and multiplying by a suff[icient] power of that moves any elt of $\mathbb{Z}[z_1 z_1^{-1} \cdots z_N z_N^{-1}]$ into $\mathbb{Z}[z_1 z_2 \cdots z_N]$.

The left-hand side $V^{\otimes n}/W$, regard this as an element of $RU(N)^+_n$ and as one of $RU(N-1)^+_n$, what is the relation between the two.

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it is a polynomial rather than a Laurent polynomial, and it is homogeneous of degree n.

In fact, if we take a linear map $\mathbb{C}^N \to \mathbb{C}^N$, or equivalently a matrix A, then $A^{\otimes n} : V^{\otimes n} \longrightarrow V^{\otimes n}$ is given by a matrix whose entries are homog. polynomials of degree n in the entries a_{ij} of A. This means the same for $T(A)$, because the remaining parts of our construction are essentially linear. In particular, $\mathrm{Tr}\, A$ is a homog. polynomial of degree n in the entries a_{ij} of A. In particular, this is true for the diagonal matrix

$$\begin{bmatrix} z_1 & 0 & & 0 \\ 0 & z_2 & & 0 \\ & & \ddots & \\ 0 & 0 & & z_n \end{bmatrix}.$$

For convenience, let us define

$$RU(N)^+_n = \mathbb{Z}[z_1 \cdots z_N]^{\Sigma_n}_n ,$$

where the + indicates that we use polynomials rather than Laurent polynomials, and the subscript n indicates degree n.

Here we have

$$\mathbb{Z}[z_1 \ldots z_N]^{\Sigma_n} = \mathbb{Z}[\sigma_1, \sigma_2, \ldots \sigma_N],$$

where σ_r means the r^{th} ~~el~~ elementary symmetric function,

~~Equivalently, we have~~

~~$\mathbb{Z}[\sigma_1, \sigma_2, \ldots \sigma_N] \longrightarrow \mathbb{Z}[\sigma_1, \sigma_2, \ldots \sigma_{N-1}]$~~

~~by $\sigma_r \mapsto \sigma_r$ for $r < N$~~

~~$\sigma_N \longmapsto 0$~~

Let us define RU^+ by

$$RU^+_n = \varprojlim_N RU(N)^+_n ;$$

(in each degree this is an attained limit, by the remark above.) With

$$RU^+ = \mathbb{Z}[\lambda^1, \lambda^2, \ldots \lambda^N \ldots] ;$$

we can think of its elements as symmetric functions in a large number of variables, where you are always ready to increase the number of variables until it is enough for the purpose in hand.

At this stage we have constructed a map

$$A = \bigoplus_{n \geq 0} R\Sigma_n \longrightarrow RU^+ = \mathbb{Z}[\lambda^1 \lambda^2 \ldots \lambda^n \ldots]$$

~~For example, if we shift to Σ_n [illegible]~~

~~[illegible]~~

~~$V^{\otimes n}/\Sigma_n = \lambda^n$, [illegible] λ^n [illegible]~~

~~perm representation~~, it is well known

Characters.

Remark. If the first is $f(z_1, z_2 \dots z_{N-1}, z_N)$, then the second is $f(z_1, z_2 \dots z_{N-1}, 0)$.

Proof. We use the fact that Tr is defined on all matrices, not just on square ones. Take

$$P = T \begin{bmatrix} z_1 & 0 & & 0 & 0 \\ 0 & z_2 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & z_{N-1} & 0 \end{bmatrix}$$

$$Q = T \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \\ 0 & 0 & & 0 \end{bmatrix}$$

& use $\mathrm{Tr}(PQ) = \mathrm{Tr}(QP)$.

We get a homo. of graded rings

$$RU(N)^f_\chi \longrightarrow RU(N-1)^+_\chi$$

by $f(z_1 \dots z_N) \longmapsto f(z_1 \dots z_{N-1}, 0)$,

equivalently it is $\mathbb{Z}[\lambda^1 \dots \lambda^N] \to \mathbb{Z}[\lambda^1 \dots \lambda^{N-1}]$

by $\lambda^r \longrightarrow \lambda^r$ for $r < N$

$\lambda^N \longrightarrow 0$.

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also must be A_n.
It follows that the map

$$A \xrightarrow{\cong} \mathbb{Z}\{\lambda^1 \lambda^2 \dots \lambda^r \dots\}$$

is also iso.

Remark The map $A \to RU^+$ carries the coproduct in A to the coproduct in RU^+.
The coproduct in RU^+ is defined by restriction of representations, using the inclusion

$$~~RU~~ \quad U(N) \times U(M) \longrightarrow U(N+M)$$

and passing to the limit. In other words, it is the map of algebras which carries λ^n

to $\sum_{i+j=n} \lambda^i \otimes \lambda^j$, where λ^0 is [illegible] unit.

The proof amounts to the following.
Let $U(N), U(~~M~~)$ act on $V = \mathbb{C}^N$, $W = \mathbb{C}^M$.
~~Then~~ $(V \oplus W)^{\otimes n}$ splits as the sum of a lot of parts, of which a typical one is ~~obtained~~ $V^{\otimes i} \otimes W^{\otimes j}$ (with $i+j=n$) has action with the cosets of Σ_i in Σ_n. It follows then, that if

X, Y are representations of Σ_i, Σ_j

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(and easy to calculate) basis is one to n^{th} elementary symmetric function of $z_1, z_2, \dots z_N$

and this map carries ε_n to λ^n. We wish to see how much structure this map preserves.

Remark. This map carries the product in A to the product in $R(U)^+$.

Proof $V^{\otimes i+j}/\mathrm{ind}(x\otimes y)$ $(x\in RS_i, y\in RS_j)$

$$= \mathrm{res}\, V^{\otimes i+j}/x\otimes y$$

$$= (V^{\otimes i}\otimes V^{\otimes j})/x\otimes y$$

$$= (V^{\otimes i}/x)(V^{\otimes j}/y).$$

At this point we get a second proof that the map

$$\begin{tikzcd}
\mathbb{Z}[\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \dots] \arrow[r] \arrow[dr, "\cong"'] & A \arrow[d] \\
& \mathbb{Z}[\lambda^1, \lambda^2 \dots \lambda^n \dots]
\end{tikzcd}$$

is iso; in fact

so in each degree n, $\mathbb{Z}[\varepsilon_1, \varepsilon_2 \dots \varepsilon_n \dots]_n$ is isomorphic to a direct summand of A_n ~~of the same rank as A_n~~ which is of the same rank as A_n

But the first is the dual of

P_{ij}/Υ, and the second is the dual of

$V^{\otimes i}\otimes W^{\otimes j}/\mathrm{res}\,\Upsilon$,

where "res" means restriction from Σ_{i+j} to $\Sigma_i \times \Sigma_j$.

Then $P_{ij}/\Upsilon \cong V^{\otimes i}\otimes W^{\otimes j}/\mathrm{res}\,\Upsilon$.

This gives us a second proof that A is a Hopf algebra. In fact, the map

$$A \xrightarrow{\alpha} RU^+$$

preserves both product & coproduct. The product & coproduct in RU^+ satisfy the nec. condn, & α is iso, therefore they did so in A.

The reason that this is the same thing as before is as follows.

We said $\xi_\nu \longleftrightarrow c_\nu \longrightarrow$ νth esf of x's in H^*BT

$\xi_\nu \longleftrightarrow \lambda_\nu \longrightarrow$ νth esf of $z_1 z_2 \dots z_n$