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AN EXPOSITION OF CARLSSON'S PROOF

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This will be an expository paper about Carlsson's proof of Segal's Burnside Ring Conjecture. First I should explain what the conjecture says.

Let me begin with a theorem of Atiyah and Segal about equivariant K-theory [4]. Let G be a finite group, with universal bundle $EG \longrightarrow BG$. Let X be a suitable space on which G acts, for example, a finite G-CW-complex. Then one has a projection map

$$EG \times X \longrightarrow X \; .$$

Here I must warn you of a well-known danger; this map is an ordinary homotopy equivalence, but it is not a G-homotopy-equivalence. Still, it gives an induced map

$$K_G(X) \xrightarrow{\alpha} K_G(EG \times X) \; .$$

This map is not yet an isomorphism, but it becomes an isomorphism if you pass to a suitable completion

$$K_G(X)^\wedge \xrightarrow[\cong]{\alpha^\wedge} K_G(EG \times X) \; .$$

In particular, one can take X to be a point P ; on the left $K_G(P)$ gives the representation ring $R(G)$, and on the right $K_G(EG)$ gives the ordinary K-group $K(BG)$; thus one recovers the theorem that the map

$$R(G)^\wedge \xrightarrow{\alpha^\wedge} K(BG)$$

is an isomorphism.

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Segal suggested that one should take the result of Atiyah and Segal, and replace equivariant K-theory by equivariant cohomotopy. So I should next explain about equivariant cohomotopy. (For readers who want more details than seem appropriate here, I have tried to provide an exposition of the prerequisites in [1]; I hope it may be convenient if I give references to it, provided it is understood that the references given in [1] have priority.)

Let G be a finite group. Let X and Y be suitable spaces on which G acts; for definiteness, let us say that they are finite G-CW-complexes with base-points. We define

$$[X, Y]^G$$

to be the set of G-homotopy-classes of G-maps from X to Y, where maps and homotopies preserve the base-points.

We now introduce suspension. In equivariant homotopy theory, you must say how your group G is to act on the suspension coordinates you introduce. So we suppose given a representation of G on a finite-dimensional inner-product space V. We define S^V to be the one-point compactification of V, with base-point at infinity. Thus we get a suspension function

$$S^V\colon [X, Y]^G \longrightarrow [S^V \wedge X, S^V \wedge Y]^G ;$$

it carries a G-map $f\colon X \longrightarrow Y$ to $1 \wedge f$. It can be shown [1 §3, §4] that $[S^V \wedge X, S^V \wedge Y]^G$ attains a common value for all sufficiently large V; this common value gives us the stable group $\{X, Y\}^G$.

We can now define (reduced) equivariant cohomotopy groups by setting

$$\pi_G^q(X) = \{X, S^q\}^G .$$

At least, this is the definition if q is a non-negative integer; if q is negative we modify the definition in an obvious way. For consistency, it will be best if all cohomotopy groups in what follows are reduced groups applied to G-spaces with base-point. Notice also that my gradings q will run over integers only.

Next we want to complete. For the purpose of proving the Segal conjecture, it is sufficient to consider the special case in which G is a p-group [9]. So I will fix on a prime p, and suppose that G is a p-group. I define $\hat{\pi}_G^q(X)$ to be the p-adic completion of $\pi_G^q(X)$. Since $\pi_G^q(X)$ is a finitely-generated abelian group [1 §4], $\hat{\pi}_G^q(X)$ is a compact Hausdorff (abelian) group. In the category of compact Hausdorff groups, inverse limits preserve exactness; so as long as we stay in the category of compact Hausdorff groups, we may use inverse limits freely. For any infinite G-CW-complex X we define

$$\hat{\pi}_G^q(X) = \underset{\alpha}{\underleftarrow{\mathrm{Lim}}}\ \hat{\pi}_G^q(X_\alpha) ,$$

where X_α runs over the finite G-CW-subcomplexes of X.

I will now discuss different but equivalent forms of the Segal conjecture. The discussion that follows is based on contributions by J.P. May, and I am grateful for his help.

THEOREM 1. The following statements are equivalent.

(i) The map

$$\hat{\pi}_G^*(X) \longrightarrow \hat{\pi}_G^*((EG \sqcup P) \wedge X)$$

is an isomorphism (for all X).

(ii) The map

$$f^*: \hat{\pi}_G^*(Y) \longrightarrow \hat{\pi}_G^*(X)$$

is an isomorphism whenever $f: X \longrightarrow Y$ is a G-map which is also an ordinary homotopy equivalence.

(iii) We have

$$\hat{\pi}_G^*(Z) = 0$$

whenever Z is a G-space which is contractible (though not necessarily G-contractible).

Of these, (i) is the formulation usually considered; the product $(EG \sqcup P) \wedge X$ appears instead of $EG \times X$ because this is the version for reduced cohomotopy of G-spaces with base-point.

All three formulations avoid any finiteness assumption on the G-spaces X, Y, Z, considered; this is possible because our completion process is fairly crude.

PROOF that (i) implies (ii). Consider the following diagram.

$$\begin{array}{ccc} \hat{\pi}^*_G(X) & \xrightarrow{\cong} & \hat{\pi}^*_G((EG \sqcup P) \wedge X) \\ f^* \uparrow & & \uparrow (1 \wedge f)^* \\ \hat{\pi}^*_G(Y) & \xrightarrow{\cong} & \hat{\pi}^*_G((EG \sqcup P) \wedge Y) \end{array}$$

It is sufficient to show that

$$1 \wedge f: (EG \sqcup P) \wedge X \longrightarrow (EG \sqcup P) \wedge Y$$

is a G-homotopy-equivalence; and for this it is sufficient [1 §2] to show that it induces ordinary equivalences of the fixed-point sets. These fixed-point sets reduce to the base-point except for the subgroup $H = 1$, and for that the map

$$1 \wedge f: (EG \sqcup P) \wedge X \longrightarrow (EG \sqcup P) \wedge Y$$

is an ordinary homotopy equivalence.

PROOF that (ii) implies (iii). Take $X = Z$, $Y = P$ (or vice versa).

PROOF that (iii) implies (i). Let us form a cofibering

$$EG \sqcup P \xrightarrow{j} S^0 \longrightarrow \widetilde{EG} .$$

Here j is an ordinary homotopy equivalence and therefore $\widetilde{EG}$ is contractible. Taking the smash product with X, we get a cofibering

$$(EG \sqcup P) \wedge X \xrightarrow{j \wedge 1} X \longrightarrow \widetilde{EG} \wedge X$$

in which $\widetilde{EG} \wedge X$ is contractible. By (iii) we have

$$\hat{\pi}^*_G(\widetilde{EG} \wedge X) = 0 ,$$

and so the exact sequence of the cofibering yields (i).

THEOREM 2 [5, 6] The statements given as equivalent in Theorem 1 are all true.

This is Segal's conjecture, as proved by Carlsson [5, 6]. I will use the discussion above to explain one of the ingredients in Carlsson's proof.

Segal's interests always led him to consider "localisation so as to invert the Euler class", that is, to take a direct limit under multiplication by the Euler class. Carlsson's analysis of what is relevant leads him to consider an inverse limit under multiplication by the Euler class. I shall suggest that most of the inverse limits in Carlsson's proof can best be interpreted as the cohomotopy of infinite complexes. In this case, the relevant infinite complex is as follows. Let V be a representation of G ; let $S^{\infty V}$ be the union $\bigcup_{k\geq 0} S^{kV}$ (under the obvious inclusions). Let X be a G-CW-complex with base-point.

THEOREM 3. If $V \neq 0$, then

$$\hat{\pi}_G^*(S^{\infty V} \wedge X) = 0 .$$

It is clear that this is a special case of the results to be proved; since $V \neq 0$ the space $S^{\infty V}$ is contractible, so $S^{\infty V} \wedge X$ is contractible and $\hat{\pi}_G^*(S^{\infty V} \wedge X) = 0$ by formulation (iii).

Let me now explain the approach to Carlsson's proof which I shall suggest. It seems that Carlsson found his proof by translating the problems from cohomotopy into homotopy. Now that he has found the proof, I suggest that we can follow it better by translating all his statements back from homotopy into cohomotopy, so that we get a proof stated as far as possible in terms of cohomotopy. In particular, I suggest that we take all the statements about "convergence of skeletal filtrations" in [5] or "convergence of singular filtrations" in [6], and interpret them as instances of the Segal conjecture, in one of the forms stated in Theorem 1. Similarly, I suggest that we take all the statements about "convergence of representational filtrations" in [5] or [6], and interpret them as instances of Theorem 3.

With this in mind, we can study the organisation of Carlsson's proof. His proof is inductive; we may assume as our inductive hypothesis that Theorem 2 is true for p-groups G' with $|G'| < |G|$. The inductive step falls into three parts.

(a) Proof (from the inductive hypothesis) that $\hat{\pi}_G^*(S^{\infty V}) = 0$ for one particular V. It is essential to the programme that the representation V chosen should have two properties: $V^G = 0$ and $V^H > 0$ for $H < G$. No other property of V is needed. There is an easy choice of a representation with these properties: take the reduced regular representation.

(b) Proof (from (a)) that $\hat{\pi}_G^*(S^{\infty V} \wedge X) = 0$ (for the same V, but for a general X).

(c) Proof (from (b) plus the inductive hypothesis) of Theorem 2, the Segal conjecture for G.

Of these parts, (a) is by far the most substantial, and I will leave it to last; the other two are comparatively easy. I begin with a lemma which is easily proved by standard methods of generalised cohomology theory.

LEMMA 4. We may infer $\hat{\pi}_G^*(W \wedge X) = 0$ provided we know $\hat{\pi}_H^*(W) = 0$ whenever X has a G-cell (other than the base-point) of type $G/H \times E^n$, $G/H \times S^{n-1}$.

PROOF. First we remark that

$$\hat{\pi}_G^q(W \wedge (G/H \sqcup P) \wedge S^n) = \hat{\pi}_H^{q-n}(W) .$$

(If W is a finite complex see [1 §5]; otherwise pass to the limit from finite subcomplexes.) So the result would follow by the Five Lemma if X were built up from the base-point by a finite number of cofibrations

$$G/H \wedge S^n \longrightarrow X' \longrightarrow X'' .$$

Unfortunately, it is one of the standard snags of the subject that not every finite G-CW-complex can be built up in this way. However, there is a standard way of overcoming this snag, for which see [1, end of §5]; this works and proves the result when X is finite. The general case follows by passing to limits.

PROOF of step (b). We apply Lemma 4 with $W = S^{\infty V}$. For $H = G$ we have $\hat{\pi}_G^*(S^{\infty V}) = 0$ by assumption. For $H < G$ we have $V^H \neq 0$; so the map

$$S^{kV} \longrightarrow S^{(k+1)V}$$

is H-nullhomotopic, and for the limit we have

$$\hat{\pi}_H^*(S^{\infty V}) = 0 .$$

So Lemma 4 gives $\hat{\pi}_G^*(S^{\infty V} \wedge X) = 0$.

PROOF of step (c). We prove the Segal conjecture in form (iii). Let Z be a G-space which is contractible. Take the cofibering

$$S^0 \longrightarrow S^{\infty V} \longrightarrow S^{\infty V}/S^0$$

and form the smash product with Z; we get a cofibering

$$Z \longrightarrow S^{\infty V} \wedge Z \longrightarrow (S^{\infty V}/S^0) \wedge Z .$$

In its exact cohomotopy sequence, we have

$$\hat{\pi}_G^*(S^{\infty V} \wedge Z) = 0$$

because we are assuming the conclusion of step (b). We prove

$$\hat{\pi}_G^*((S^{\infty V}/S^0) \wedge Z) = 0$$

by applying Lemma 4 with W replaced by Z, X replaced by $S^{\infty V}/S^0$. We have $V^G = 0$, so $X = S^{\infty V}/S^0$ has no G-cells of the form $G/G \times E^n$, $G/G \times S^{n-1}$ (other than the base-point). Thus we only need to know

$$\hat{\pi}_H^*(Z) = 0 \quad \text{for} \quad H < G ,$$

and this follows from the inductive hypothesis which we assume.

The exact sequence now gives $\hat{\pi}_G^*(Z) = 0$.

In step (c), I have derived both the statement and the proof from those of Proposition 5, p.16 of [5] (compare Proposition II.5 of [6]) by a process of eliminating inessentials ("wring the water out"). I suggest that it is just as easy to take this step in cohomotopy as in homotopy.

We can now turn to step (a). Carlsson's method rests on catching $\hat{\pi}_G^*(S^{\infty V})$ in an exact sequence

(5) $$\cdots \longrightarrow B^* \longrightarrow \hat{\pi}_G^*(S^{\infty V}) \longrightarrow C^* \longrightarrow \cdots .$$

This is the "fundamental exact sequence" of [5 p.27], [6 section III]. In this exact sequence, the group C^* is calculated by exploiting the inductive hypothesis, while the group B^* is calculated by reduction to ordinary homotopy theory plus the Adams spectral sequence. This subdivides step (a) into two major parts.

The behaviour of the "fundamental exact sequence" (5) depends on whether G is an elementary abelian p-group or not. If G is not an elementary abelian p-group then both B^* and C^* turn out to be zero, so we have $\hat{\pi}_G^*(S^{\infty V}) = 0$. If G is an elementary abelian p-group then both B^* and C^* turn out to be non-zero; so further argument is needed. Carlsson's original approach was to cite [2] to show that if G is an elementary abelian p-group then the Segal conjecture is true in its non-equivariant form, and [8, 9] to show that this implies the equivariant form. For various reasons, I think it better to show that the boundary map in the "fundamental exact sequence" (5) is an isomorphism; see [10].

We will next construct the "fundamental exact sequence" (5). We begin with a cofibering which was used in the proof of Theorem 1, that is

$$EG \sqcup P \xrightarrow{j} S^0 \longrightarrow \widetilde{EG} = P \cup CEG .$$

But now we use it in a different way; we form the following exact sequence.

$$\ldots \{S^{kV}, (EG \sqcup P) \wedge S^q\}^G \longrightarrow \{S^{kV}, S^q\}^G \longrightarrow \{S^{kV}, \widetilde{EG} \wedge S^q\}^G \longrightarrow \ldots$$

Of course, this involves us in considering $\{X, Y\}^G$ when Y is no longer a finite G-CW-complex but a G-CW-complex with finite skeletons; still, this causes no trouble. The sequence makes sense even if q is negative, by an obvious modification of the definitions. The groups are finitely-generated abelian groups [1 §4]. It follows [3 p.108] that we may pass to p-adic completions and obtain an exact sequence

$$\ldots \longrightarrow {}^\wedge\{S^{kV}, EG \sqcup P\}^{qG} \longrightarrow {}^\wedge\{S^{kV}, S^0\}^{qG} \longrightarrow {}^\wedge\{S^{kV}, \widetilde{EG}\}^{qG} \longrightarrow \ldots$$

of compact Hausdorff groups. We can now pass to an inverse limit over k and obtain the following exact sequence.

$$\ldots \longrightarrow \underset{k}{\overset{\longleftarrow}{\mathrm{Lim}}}\ {}^\wedge\{S^{kV}, EG \sqcup P\}^{qG} \longrightarrow$$

$$\longrightarrow \underset{k}{\overset{\longleftarrow}{\mathrm{Lim}}}\ {}^\wedge\{S^{kV}, S^0\}^{qG} \longrightarrow$$

$$\longrightarrow \underset{k}{\overset{\longleftarrow}{\mathrm{Lim}}}\ {}^\wedge\{S^{kV}, EG\}^{qG} \longrightarrow \ldots$$

This gives the required exact sequence

$$\ldots \longrightarrow B^q \longrightarrow \hat{\pi}^q_G(S^{\infty V}) \longrightarrow C^q \longrightarrow \ldots .$$

We will first consider the calculation of C^*. If X is a G-CW-complex, let us define the "singular set" or non-free part of X by

$$\mathrm{Sing}(X) = \bigcup_{H>1} X^H .$$

It is easy to show that the restriction map

$$[X, Y]^G \longrightarrow [\mathrm{Sing}\ X, \mathrm{Sing}\ Y]^G$$

is a (1-1) correspondence if the space Y is contractible; this is a remark of tom Dieck [7]. In particular, we have

$$[S^W \wedge S^{kV}, S^W \wedge \widetilde{EG} \wedge S^q]^G$$
$$\overset{\cong}{\longleftrightarrow} [\mathrm{Sing}(S^W \wedge S^{kV}), S^W \wedge S^q]^G .$$

We may pass to an attained limit over W, and find

$$\{S^{kV}, \widetilde{EG} \wedge S^q\}^G$$
$$\overset{\cong}{\longleftrightarrow} \underset{W}{\underrightarrow{\mathrm{Lim}}} [\mathrm{Sing}(S^W \wedge S^{kV}), S^W \wedge S^q]^G .$$

Introducing the remaining limits, we find

(6) $$C^q \cong \underset{k}{\underleftarrow{\mathrm{Lim}}} \wedge \underset{W}{\underrightarrow{\mathrm{Lim}}} [\mathrm{Sing}(S^W \wedge S^{kV}), S^W \wedge S^q]^G .$$

(At least, this is the formula if $q \geq 0$; if $q < 0$ we modify the formula accordingly.)

It is reasonable to suppose that we should obtain a spectral sequence convergent to C^* by filtering the construction $\mathrm{Sing}(X)$ appropriately. When we examine matters more closely, we find that we have to "thicken up" the construction $\mathrm{Sing}(X)$, replacing it by an equivalent but larger construction, before we can have the right sub-constructions inside it. This can be shown by examples; one sees what to do by studying particular groups G such as the cyclic group Z_8 , the elementary abelian group $Z_2 \times Z_2$ and the dihedral group D_8 . I will omit this and get on with the work.

I shall need a construction due to Quillen [12]. From our p-group G we can construct a simplicial complex $Q(G)$. It has one vertex $v(H)$ for each subgroup $H > 1$ in G . It has an r-simplex $\sigma(H_0, H_1, \ldots, H_r)$ for each chain of subgroups

$$1 < H_0 < H_1 < \ldots < H_r \subset G ;$$

the vertices of this r-simplex are

$$v(H_0), v(H_1), \ldots, v(H_r) .$$

The r-simplexes $\sigma(H_0, H_1, \ldots, H_r)$ with $H_r < G$ form a sub-complex $\tilde{Q}(G) \subset Q(G)$, which we may think of as the "reduced $Q(G)$ ". The whole complex $Q(G)$ is then the cone $\tilde{Q}(G) * v(G)$.

G acts on the set of subgroups H by conjugation; g sends H to gHg^{-1}. Therefore G acts on $Q(G)$ and $\tilde{Q}(G)$, which become G-spaces.

I define the "thickened singular set"

$$\mathrm{Thing}(X) \subset Q(G) \times X$$

to be the union of

$$\sigma(H_0, H_1, \ldots, H_r) \times X^{H_r}$$

over all simplexes of $Q(G)$. Using the projection

$$Q(G) \times X \xrightarrow{\pi} X$$

we get a G-map

$$\mathrm{Thing}(X) \xrightarrow{\pi} \mathrm{Sing}(X) .$$

PROPOSITION 7. The map

$$\mathrm{Thing}(X) \xrightarrow{\pi} \mathrm{Sing}(X)$$

is a G-homotopy-equivalence.

SKETCH PROOF. Consider first the pair

$$G/H \times E^n, \ G/H \times S^{n-1} .$$

In this case the pair

$$\mathrm{Thing}(G/H \times E^n), \ \mathrm{Thing}(G/H \times S^{n-1})$$

becomes

$$\phi, \phi \quad \text{if} \quad H = 1 ,$$

or

$$G \times_H (Q(H) \times E^n), \ G \times_H (Q(H) \times S^{n-1}) \qquad \text{if} \quad H > 1 .$$

Here $Q(H)$ is a cone $\tilde{Q}(H) * v(H)$, so the projection

$$Q(H) \times E^n, \ Q(H) \times S^{n-1} \longrightarrow E^n, \ S^{n-1}$$

is an H-equivalence, and the result follows in this case.

Any finite G-CW-complex X can be built up by induction, using pushouts

$$\begin{array}{ccc} G/H \times E^n & \longrightarrow & X'' \\ \uparrow & & \uparrow \\ G/H \times S^{n-1} & \longrightarrow & X' \end{array} .$$

Both Thing and Sing preserve such pushouts; so the result follows by induction when X is a finite G-CW-complex. Then it follows when X is general by limits.

We may now replace Sing by Thing and write

(8) $$C^q \cong \langle \underset{k}{\mathrm{Lim}} \wedge \underset{W}{\mathrm{Lim}} \rangle [\mathrm{Thing}(S^W \wedge S^{kV}), S^W \wedge S^q]^G .$$

I shall assume, without a proper discussion, that we can obtain a spectral sequence convergent to C^* by filtering the construction "Thing". In fact, I shall filter it according to the skeletons of $Q(G)$.

Spectral sequences are usually computed by determing suitable relative groups, that is, the groups corresponding to "one skeleton mod the next". In this case it seems better to begin by determining suitable absolute groups.

Take an r-simplex

$$\sigma(H_0, H_1, \ldots, H_r) .$$

The subgroup of G which preserves this simplex is

$$N = N(H_0) \cap N(H_1) \cap \ldots \cap N(H_r) .$$

The G-orbit of this r-simplex consists of the simplexes

$$\sigma(gH_0g^{-1}, gH_1g^{-1}, \ldots, gH_rg^{-1})$$

where g runs over G/N . Corresponding to this part of $Q(G)$ we have an absolute group, namely

(9) $$D^q = \langle \underset{k}{\mathrm{Lim}} \wedge \underset{W}{\mathrm{Lim}} \rangle [T(S^W \wedge S^{kV}), S^W \wedge S^q]^G$$

where T is the construction

$$T(X) = \bigvee_{g \in G/N} (\sigma(gH_0g^{-1}, gH_1g^{-1}, \ldots, gH_rg^{-1}) \sqcup P) \wedge X^{gH_rg^{-1}} .$$

LEMMA 10. The group D^q is

(i) $\hat{\pi}^q(S^0)$ if $H_r = G$

(ii) 0 if $H_r < G$.

Maps induced by inclusions between the groups (i) are iso.

PROOF. We have

$$D^q = \langle \underset{k}{\underline{\mathrm{Lim}}} \wedge \underset{W}{\underline{\mathrm{Lim}}} \rangle \, [(G \sqcup P) \wedge_N S^{W^H} \wedge S^{kV^H}, S^W \wedge S^q]^G$$

where $H = H_r$. This gives

$$D^q = \langle \underset{k}{\underline{\mathrm{Lim}}} \wedge \underset{W}{\underline{\mathrm{Lim}}} \rangle \, [S^{W^H} \wedge S^{kV^H}, S^{W^H} \wedge S^q]^{N/(H\cap N)} .$$

Since W^H can be arbitrarily large among representations of $N/(H \cap N)$, this gives

$$D^q = \langle \underset{k}{\underline{\mathrm{Lim}}} \wedge \{S^{kV}, S^q\}^{N/(H\cap N)}$$

$$= \langle \underset{k}{\underline{\mathrm{Lim}}} \; \hat{\pi}^q_{N/(H\cap N)}(S^{kV^H})$$

$$= \hat{\pi}^q_{N/(H\cap N)}(S^{\infty V^H}) .$$

Now we have to distinguish two cases.

(i) If $H = G$ then $V^H = 0$ and we get $\hat{\pi}^q(S^0)$.

(ii) If $H < G$ then $V^H > 0$; also $H \cap N > 1$ ($H \cap N$ contains at least the centre of H_0) and so $|N/(H \cap N)| < |G|$; thus we can apply the inductive hypothesis and get

$$\hat{\pi}^q_{N/(H\cap N)}(S^{\infty V^H}) = 0 .$$

I shall assume without proper discussion that we can proceed from this input to the final answer, along the lines of the Eilenberg-Steenrod uniqueness proof, using Mayer-Vietoris sequences or introducing relative groups as we prefer. The outcome seems predictable.

LEMMA 11. The E_2 term of our spectral sequence is

$$H^*\left(\frac{Q(G)}{G}, \frac{\tilde{Q}(G)}{G}; \hat{\pi}^*(S^0)\right) = \tilde{H}^*\left(\frac{Q(G)}{G}; \hat{\pi}^*(S^0)\right) .$$

If G is an elementary abelian group then the complex $\frac{\tilde{Q}(G)}{G}$

is the "Tits building"; it has cohomology only in the top dimension, where we get the "Steinberg module" [15]. (The top dimension is $n - 2$, where n is the rank of G.)

LEMMA 12. If G is not an elementary abelian group then $\frac{\tilde{Q}(G)}{G}$ is contractible.

PROOF (after [12]). Let Φ be the Fitting subgroup of G, that is, in this case [16 p.141], the kernel of the projection from G to its maximal elementary abelian quotient. By assumption we have $\Phi > 1$. The crucial properties of Φ are

(i) Φ is normal,

(ii) if $H < G$ then $\langle\Phi, H\rangle < G$, where $\langle\Phi, H\rangle$ is the subgroup generated by Φ and H [16 p.52].

We will define a G-homotopy

$$h: I \times \tilde{Q}(G) \longrightarrow \tilde{Q}(G)$$

of the identity map. We subdivide $I \times \tilde{Q}(G)$ into a finite simplical complex in the usual way, so that we have one simplex with vertices

$$0 \times v(H_0),\ 0 \times v(H_1),\ \ldots,\ 0 \times v(H_i),\ 1 \times v(H_i),\ \ldots,\ 1 \times v(H_r)$$

for each chain

$$1 < H_0 < H_1 < \ldots < H_r < G$$

and each i. We define h to be the simplicial map which carries

$$0 \times v(H) \quad \text{to} \quad v(H)$$

$$1 \times v(H) \quad \text{to} \quad v(\langle\Phi, H\rangle) .$$

Clearly this is a G-map, and passes to the quotient to give a deformation retraction from $\tilde{Q}(G)/G$ to Y/G, where Y is the subcomplex of $\tilde{Q}(G)$ consisting of simplexes $\sigma(H_0, H_1, \ldots, H_r)$ with $\Phi \subset H_0$. Clearly Y is a G-cone $v(\Phi) * Z$, and Y/G is a cone $v(\Phi) * (Z/G)$.

This proof would still work if we replaced Φ by any non-trivial subgroup $K \subset \Phi$ normal in G ; this would give the first homotopy less work and the second one more. With the details chosen, the action of G on Y is actually trivial (every subgroup H with $\Phi \subset H \subset G$ is normal); so $Y/G = Y$.

Since $\tilde{Q}(G)/G$ is contractible, its reduced cohomology is zero, and the spectral sequence shows that $C^* = 0$. This completes my exposition of this part of the argument.

The reader will have noticed that I have not given complete details. Before doing work to write a careful and detailed version of such a proof, I would prefer to give thought to the statement of the conclusions; what do we really want from this argument?

Two more points are in order. First, the argument involves manipulations with unstable groups before one passes to a limit over suspension; the world of possible suspensions is not the same for $N/(H \cap N)$ as for G . Secondly, we have not used much about $S^{\infty V}$; so far it could be replaced by any G-CW-complex Y such that $Y^G = S^0$ and Y^H is contractible for $H < G$. The precise choice of $S^{\infty V}$ will become relevant in what follows.

Let us turn to the calculation of the group B^* . Since we have already dealt with the "non-free part" of our problem, we may hope that we have only to deal with problems about G-free G-CW-complexes, for which the reduction to ordinary homotopy theory is understood.

We wish to determine

$$B^q = \underset{k}{\underleftarrow{\mathrm{Lim}}}{}^{\wedge} \{S^{kV}, EG \sqcup P\}^{qG} .$$

Let Y_α run over the finite G-CW-subcomplexes of $EG \sqcup P$; then we have

$$B^q = \underset{k}{\underleftarrow{\mathrm{Lim}}}{}^{\wedge} \underset{\alpha}{\underrightarrow{\mathrm{Lim}}} \{S^{kV}, Y_\alpha\}^{qG} .$$

Here the limit over α is attained; and in fact, in what follows, all the limits over α which matter will be attained in each degree. We thus obtain

$$B^q = \langle \underset{k}{\mathrm{Lim}}\ \underset{\alpha}{\mathrm{Lim}} \rangle \wedge \{S^{kV}, Y_\alpha\}^{qG} .$$

By G-S-duality [1 §8] the G-free object Y_α has a G-S-dual $D_G Y_\alpha$; it is important that $D_G Y_\alpha$ is also a G-free object, and that this duality is an n-duality, where the dimension n is an ordinary integer and not an element of $RO(G)$. We thus obtain

$$B^q = \langle \underset{k}{\mathrm{Lim}}\ \underset{\alpha}{\mathrm{Lim}} \rangle \wedge \{S^{kV} \wedge D_G Y_\alpha, S^n\}^{qG} .$$

Now we actually can reduce to ordinary cohomotopy; by a theorem on changing groups [1 §5] we get

$$B^q = \langle \underset{k}{\mathrm{Lim}}\ \underset{\alpha}{\mathrm{Lim}} \rangle \left\{ \frac{S^{kV} \wedge D_G Y_\alpha}{G}, S^n \right\}^q .$$

We have an Adams spectral sequence for computing the ordinary cohomotopy group $\{X, S^n\}^q$; in our case it involves the ordinary homology group

$$H_*\left(\frac{S^{kV} \wedge D_G Y_\alpha}{G}; F_p \right) .$$

Here we have a Thom isomorphism

$$H_*\left(\frac{S^{kV} \wedge D_G Y_\alpha}{G}; F_p \right) \cong H_*\left(\frac{D_G Y_\alpha}{G}; F_p \right) .$$

Of course, this does not commute with Steenrod operations. However, we can write

$$H_*\left(\frac{S^{kV} \wedge D_G Y_\alpha}{G}; F_p \right) = \phi_k\ H_*\left(\frac{D_G Y_\alpha}{G}; F_p \right)$$

where we are supposed to know how to write $a(\phi_k h)$ in the form $\phi_k h'$. We have

$$\frac{D_G Y_\alpha}{G} = D(Y_\alpha/G)$$

by a result on G-S-duality, Theorem 8.5 of [1]. So

$$H_*\left(\frac{D_G Y_\alpha}{G}; F_p \right) \cong H^*(Y_\alpha/G; F_p) .$$

Passing to a limit over α which is attained in each degree, I get a convergent Adams spectral sequence

$$\mathrm{Ext}_A^{**}(\phi_k H^*(BG;\, F_p),\, F_p)$$

$$\Longrightarrow \underset{\alpha}{\underleftarrow{\mathrm{Lim}}}\ \widehat{\left\{ \frac{S^{kV} \wedge D_G Y_\alpha}{G},\, S^k \right\}}^{*} .$$

Since we work in the category of compact Hausdorff (abelian) group groups, inverse limits preserve exactness; the inverse limit of an exact couple is an exact couple, and the inverse limit of convergent Adams spectral sequences is a convergent spectral sequence. Ext_A^{**} carries direct limits of the first variable into inverse limits. So we get a convergent spectral sequence

$$\mathrm{Ext}_A^{**}(\underset{k}{\underrightarrow{\mathrm{Lim}}}\ \phi_k H^*(BG;\, F_p),\, F_p)$$

$$\Longrightarrow \underset{k}{\underleftarrow{\mathrm{Lim}}}\ \underset{\alpha}{\underrightarrow{\mathrm{Lim}}}\ \widehat{\left\{ \frac{S^{kV} \wedge D_G Y_\alpha}{G},\, S^n \right\}}^{*} = B^* .$$

It remains to identify $\underset{k}{\underrightarrow{\mathrm{Lim}}}\ \phi_k H^*(BG;\, F_p)$.

Here I think I shall resume my previous lazy ways; like the axiomatic method, the method of omitting details has many advantages, and they are the same as the advantages of theft over honest toil. Anyway, you can all guess the answer; $\underset{k}{\underrightarrow{\mathrm{Lim}}}\ \phi_k H^*(BG;\, F_p)$ must surely be the result of localising $H^*(BG;\, F_p)$ so as to invert the Euler class of V .

LEMMA 13. If G is not an elementary abelian p-group then the Euler class $e(V)$ is nilpotent.

PROOF. If H is a proper subgroup of G then $V^H \neq 0$; in other words, the restriction of V to H is the direct sum of the trivial representation 1 and something else. So the restriction of $e(V)$ to H is zero. By assumption, any elementary abelian p-subgroup E is proper, so the restriction of $e(V)$ to E is zero. Then $e(V)$ is nilpotent by the theorem of Quillen [11, 13].

Carlsson obtains the same conclusion using the work of Serre [14].

We now conclude that if G is not elementary abelian, then the relevant localisation of $H^*(BG; F_p)$ is zero. Therefore its Ext groups are zero, and the spectral sequence gives $B^* = 0$.

If G is an elementary abelian p-group, then we reach an Ext group which is calculated in [2].

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