1 Introduction

2 Novikov's work

3 Morava's Extraordinary K-theories

4 Bousfield Localization

We have seen that K-theory can't see everything. In some lectures in Chicago in 1973, I asked the following question: can one make sense out of the version of homology theory which K-theory can see? Let me give definitions: suppose given a spectrum E and the resulting homology theory E_*, and suppose that by hook or by crook we have constructed an Adams spectral sequence using E-homology,

$$\mathrm{Ext}_?^{**}(E_*(X), E_*(Y)) \Rightarrow ?,$$

and suppose further that some map $f : X \to X'$ or $g : Y \to Y'$ induces an isomorphism in E-homology. Then f_* or g_* will certainly commute with all the structure you can possibly use in defining $\mathrm{Ext}_?^{**}$, so such an f or g ought to induce an isomorphism of the whole spectral sequence. Therefore it ought to induce an isomorphism of the right-hand side. Therefore the best thing that the spectral sequence could possibly converge to would be not $[X, Y]$, but maps in some category of fractions which you obtain by inverting those maps f or g which induce an isomorphism of E-homology.

Unfortunately, there were some set-theoretical difficulties, which I shall explain later, in setting up the proposed category of fractions; however, Bousfield went away and did it. I think it's the only time I have ever persuaded anyone by aesthetic considerations rather than proper proof, but even **.

I shall now give an exposition of Bousfield's work. The two references are

A.K.Bousfield, "The localization of spectra with respect to homology," *Topology 18* (1979), 257-281.

A.K.Bousfield, "The Boolean algebra of spectra," *Comm. Math. Helv. 54* (1979), 368-377.

Following Bousfield, I suppose given a generalized homology theory E_* satisfying the usual axioms. In addition, I suppose that I am given some set I of stable operations $i : E \to E$; they may be of any degrees they like.

Definitions 4.1 *We say that $E_*(X)$ is I-*torsion *if for all $i \in I$, $x \in E_*(X)$, there exists m such that $i^m x = 0$. Define a class $\mathcal{S}$ of maps of spectra: a map $s : X \to Y$ is in the class $\mathcal{S}$ if the kernel and cokernel of $s_* : E_*(X) \to E_*(Y)$ are I-torsion.*

Example 4.2 $I = \{1\}$.

For any theory E_* I can let $I = \{1\}$. Then "I-torsion" means "zero," so $\mathcal{S}$ is the class of maps s such that $s_* : E_*(X) \to E_*(Y)$ is an isomorphism. These maps are usually called E_*-equivalences. This is the context which Bousfield originally considered. Therefore, the theory I am going to set up will include his; however, I gain a bit of flexibility.

Example 4.3 $E_* = BP_*$.

For each r in the coefficient ring $\pi_*(BP) = \mathbf{Z}[v_1, v_2, \ldots]$ I get an operation $x \mapsto rx$ on E_*; I write r for this operation. I let $I = \{p, v_1, v_2, \ldots, v_n\}$. Of course, if a module M over $\pi_*(BP)$ is I-torsion in the sense above, then the same goes for the whole ideal $(p, v_1, v_2, \ldots v_n)$ generated by these elements: for any $r \in (p, v_1, v_2, \ldots v_n)$ and $x \in M$, there exists e such that $r^e x = 0$. The letter I is suggested by the usual notion of I-torsion for an ideal I.

Now you see what I'm doing; I want to make sure that my definition and construction can be written wholly in terms of BP_*, but nevertheless we have the same effect that Bousfield would get by putting $E_* = \bigvee_{i=1}^{n} K(n)_*$.

I don't think I shall need what follows, but one can check that the set $\mathcal{S}$ is closed under composition and contains identity maps. So we think of it as the analogue of a multiplicative set in algebra; this is the collection of things you propose to use as denominators when you form fractions.

Now I can give some definitions.

Definitions 4.4 *We say that a spectrum L is $\mathcal{S}$-*local *if $s_* : [X, L]_* \to [Y, L]_*$ is an isomorphism for each $s \in \mathcal{S}$. An $\mathcal{S}$-*localization *of X is a map $\eta : X \to L$ such that*

a) $\eta \in \mathcal{S}$

b) L is $\mathcal{S}$-local.

Theorem 4.5 *(Bousfield) Every spectrum X has an $\mathcal{S}$-localization.*

Now that we know that $\mathcal{S}$-localizations exist, we can concentrate on their properties. Since L is $\mathcal{S}$-local, η is terminal among maps in $\mathcal{S}$ with source Y. Since $\eta \in \mathcal{S}$, η is initial among maps from X to $\mathcal{S}$-local spectra. Either property characterizes η up to a canonical isomorphism of its image L.

These properties are generally exactly the same as the formal properties of localization in the sense of Sullivan.

In principle, the pattern of our construction is as follows: we assume given a definite E_* and I, as above. In terms of these we define the class $\mathcal{S}$; we also define a set of spectra $\{\mathcal{F}_\alpha\}$. Then we define another class $\mathcal{T}$ of maps of spectra.

When I say "class," I mean that it shares the following properties with $\mathcal{S}$:

(i) The statement $t \in \mathcal{T}$ is meaningful. In fact, it is defined here by $t_* : [F_\alpha, X]_* \to [F_\alpha, Y]_*$ is an isomorphism for each α. If the F_α are known, that's a meaningful statement.

(ii) Nevertheless, we cannot place any upper bound on the cardinal number of maps t which may satisfy $t \in \mathcal{T}$. That depends on the size of the spectra X and Y you are willing to consider, just as for $\mathcal{S}$. This has the following implication:

(iii) There is no objection to using the class $\mathcal{T}$ in definitions. A definition is just a way of writing long, complicated statements in a shorter and more comprehensible way; if the long statement is meaningful and provable, then so is the short one.

(iv) Some uses of $\mathcal{T}$ which may appear to be contradictions are not in fact constructive. Suppose for example that some author takes X and forms

$$Z = \operatorname{Holim} Y | X \to Y \text{ in } \mathcal{S}.$$

We may assume that the author intends the usual construction of Holim, for nothing is said to the contrary. Then the cardinal number of stable cells in Z will be at least equal to that in any Y: $|Z| \geq |Y|$. Certainly, if Z is well-constructed, the number of its cells must be some cardinal, however large it is. But if you specify that cardinal, then I can at once construct $X \to Y$ in $\mathcal{S}$ so that $|Y|$ is larger, which is a contradiction. This contradiction shows that Z is not well-constructed.

This difficulty becomes more obvious when the author goes on to argue that the map $X \to Z$ is in $\mathcal{S}$. If so, this argument is circular; what the author is purporting to construct was already among the things assumed known for the purpose of constructing it. This will not do. However, the use we intend of $\mathcal{T}$ is not of this sort; let us see it.

Definition 4.6 *Let C be a spectrum. We say that C is $\mathcal{T}$*-colocal *if $(t : X \to Y) \in \mathcal{T}$ implies that $t_* : [C, X]_* \to [C, Y]_*$ is an isomorphism.*

For example, a point is $\mathcal{T}$-colocal for any $\mathcal{T}$; otherwise this may seem not to be an effective definition: given C, there is no easy way to check through all t in $\mathcal{T}$; but that won't stop us.

Lemma 4.7 *(i) If C is $\mathcal{T}$-colocal and $C' \simeq C$ then C' is $\mathcal{T}$-colocal.*

(ii) If C is $\mathcal{T}$-colocal then $S^n C$ is $\mathcal{T}$-colocal for $n \in \mathbf{Z}$.

(iii) If $A \to B \to C$ is a cofibration and two of the terms are $\mathcal{T}$-colocal, then so is the third.

(iv) If each C_α is $\mathcal{T}$-colocal, then so is $\vee_\alpha C_\alpha$.

(v) Suppose L is the direct limit of some direct system of spectra L_α, indexed on a directed set. If each L_α is $\mathcal{T}$-colocal, then so is L.

All these are provable on the basis that they are statements $P(t) \Rightarrow Q(t)$ and it doesn't matter what t is.

Remark: Clearly (iv) follows from the others: if each C_α is $\mathcal{T}$-colocal, then the finite wedges $C_{\alpha_1} \vee \cdots \vee C_{\alpha_n}$ are $\mathcal{T}$-colocal by induction on n using (iii); $\vee_\alpha C_\alpha$ is the direct limit of the finite wedges, so it is $\mathcal{T}$-colocal by (v). However, it is stated for convenience.

Remark: **

Proof: (i), (ii), and (iv) are trivial, so I leave them to you.

(iii) Let $t : X \to Y$ be any map in $\mathcal{T}$ and use it to form a fiber sequence $F \to X \xrightarrow{t} Y$. (Fiber sequences are actually the same as cofiber sequences in the category of spectra, but things may be clearer if we don't quote that until it's relevant.) If A is $\mathcal{T}$-colocal, then

$[A,X]_* \xrightarrow{t_*} [A,Y]_*$ is an isomorphism. By the long exact sequence of the fibration, $[A,F]_* = 0$. Similarly, if B is $\mathcal{T}$-colocal, then $[B,F]_* = 0$. In the category of spectra, fiber sequences are the same as cofiber sequences **; hence using the exact sequence of the cofibration, we get $[C,F]_* = 0$. Using the exact sequence of the fibration again, we get $[C,X]_* \xrightarrow{t_*} [C,Y]_*$ is an isomorphism. This holds for all $t \in \mathcal{T}$, so C is $\mathcal{T}$-colocal. Similarly, if any other two are given to be $\mathcal{T}$-colocal, then the third must also be.

(v) This is the crux. You can go a long way around attempting to avoid it; Bousfield tried it, and he got it right when he first wrote it out, but he got it wrong when he wrote it out for publication, so he had to write a correction. With the correct technique we can march straight ahead.

But first we must recall about limits. Usually one begins with the case in which the set of indices is $\{1, 2, 3, \ldots\}$, and the directed system is an increasing sequence

$$L_1 \subseteq L_2 \subseteq \cdots \subseteq L = \bigcup_n L_n.$$

In the category of CW-complexes, the usual construction of the homotopy direct limit is the telescope: inside $L \times [1, \infty)$ you have

$$T = \bigcup_n L_n \times [n, n+1]$$

You can also construct it by taking

$$\coprod_n L_n \times [n, n+1]$$

and identifying $(x_n, n+1) \in L_n \times [n, n+1]$ with $(i_n x_n, n+1) \in L_{n+1} \times [n+1, n+2]$, where i_n is the inclusion. If your direct system has some other set of maps $f_n : L_n \to L_{n+1}$ instead of the inclusions, then you use $f_n x_n$ instead of $i_n x_n$. For CW-complexes with base point, you replace $L_n \times [n, n+1]$ by $\frac{L_n \times [n,n+1]}{\mathrm{pt} \times [n,n+1]}$. In the case of inclusions, there is an obvious map $T \to L$ which induces an isomorphism of homotopy groups, because the homotopy groups on both sides are $\varinjlim \pi_*(L_n)$. Therefore $T \to L$ is an equivalence. If the maps aren't inclusions, then T is the best definition you've got for the limit anyway.

All this goes over to CW-spectra; you do the construction above on each term of the spectrum.

Now we want to look at the case of a more general set of indices i. In this case modify the telescope construction: replace $[1, \infty)$ by the following infinite simplicial complex K. It has vertices corresponding to the indices i, and it has a q-simplex $\sigma(i_0, i_1, \ldots i_q)$ corresponding to each totally-ordered subset

$$i_0 < i_1 < i_2 < \ldots < i_q.$$

We replace $L_n \times [n, n+1]$ by $L_{i_0} \times \sigma(i_0, i_1, \ldots, i_q)$. The rest of the construction carries over. We still have $\pi_*(T) = \varinjlim \pi_*(L_i)$ so if we are in the case of inclusions, the map $T \to L$ is an equivalence.

Now we filter T, taking T^q to be the complex we get using simplices of dimension $\leq q$. Then we get a cofibration

$$T^{q-1} \to T^q \to \bigvee_{i_0 < \cdots < i_q} S^q L_{i_0}.$$

We assume each L_i is $\mathcal{T}$-colocal, so $\bigvee_{i_0 < \cdots < i_q} S^q L_{i_0}$ is $\mathcal{T}$-colocal by (ii) and (iv). By induction on q, starting with $T^{-1} = \text{pt}$ and using (iii), we see that T^q is $\mathcal{T}$-colocal. Now take $X \xrightarrow{t} Y$ in $\mathcal{T}$. We use Milnor's $\lim^0$–$\lim^1$ diagram**.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varprojlim{}^1 [T^q, X]_* & \longrightarrow & [T, X]_* & \longrightarrow & \varprojlim{}^0 [T^q, X]_* & \longrightarrow & 0 \\
 & & \cong \big\downarrow t_* & & \big\downarrow t_* & & \cong \big\downarrow t_* & & \\
0 & \longrightarrow & \varprojlim{}^1 [T^q, Y]_* & \longrightarrow & [T, Y]_* & \longrightarrow & \varprojlim{}^0 [T^q, Y]_* & \longrightarrow & 0
\end{array}$$

The outside vertical arrows are isomorphisms, so by the five lemma, the middle one is also. This holds for all $t \in \mathcal{T}$, so T is $\mathcal{T}$-colocal. This proves Lemma 4.7. □

All right, now we'll backtrack and suppose given a set of spectra $\{F_\alpha\}$. This is a definite set and under our control. Let $t : X \to Y$ be a map; we write "$t \in \mathcal{T}$" to mean that $[F_\alpha, X]_* \xrightarrow{t_*} [F_\alpha, Y]_*$ is an isomorphism for each F_α, as I said. We introduce Class$\{F_\alpha\}$, the class of spectra obtained from the class $\{F_\alpha\}$ by iterating the constructions (i)–(v) of Lemma 4.7.

Remark: If you want to generate the same class by using less operations, it is sufficient to use the operations (i)–(iv). [proof omitted]

Remark: On the other hand, if you want to use more operations, then Class$\{F_\alpha\}$ is also closed under the following operations.

(vi) If $X \in \text{Class}\{F_\alpha\}$ and Y is arbitrary, then $X \wedge Y \in \text{Class}\{F_\alpha\}$ (ideals).

(vii) If $X \vee Y \in \text{Class}\{F_\alpha\}$, then $X \in \text{Class}\{F_\alpha\}$ (closed under summands).

[proof omitted]

Corollary 4.8 *If $X \in \text{Class}\{F_\alpha\}$, then X is $\mathcal{T}$-colocal.*

Actually, this is true with "if and only if," but we won't prove it until later (see Corollary 4.10). This is one reason that the class of $\mathcal{T}$-colocal spectra is useful, although of course it does contain some very large spectra.

Proof of Corollary 4.8: We want that each F_α is $\mathcal{T}$-colocal (which is trivial from the definitions), and then Lemma 4.7 does the job. □

Proposition 4.9 *For any X there is a cofibration $X \to Y \to Z$ such that $[F_\alpha, Y]_* = 0$ for all α and $Z \in \text{Class}\{F_\alpha\}$.*

Proof: $\{F_\alpha\}$ is a definite set of definite spectra, so there is a cardinal σ such that each spectrum F_α has at most σ stable cells (in the applications σ is usually $\aleph_0$). We can assume that σ is infinite.

The ordinals of cardinality $\leq \sigma$ form an initial segment of the ordinals, and our construction will be by transfinite induction up it, i.e., rising to the first ordinal Ω of cardinality $> \sigma$. I don't like that much, even when σ is $\aleph_0$, but if you have a set-theoretic difficulty then you have to adopt a set-theoretic solution. The whole point is that our induction stops before some cardinal known a priori, say 2^σ.

We take $Y_0 = X$ and construct Y_i by transfinite induction for ordinals $i \leq \Omega$. For successive ordinals, say $i+1$, we suppose Y_i is constructed and Y_i/X lies in $\text{Class}\{F_\alpha\}$. The homotopy classes of maps

$$S^n F_\alpha \to Y_i$$

form a set; for each one take a representative** map

$$S^{n_\beta} F'_{\alpha\beta} \xrightarrow{q_\beta} Y_i$$

where $F'_{\alpha\beta}$ is a cofinal subspectrum in F_α. We have the cofibration

$$\bigvee_\beta S^{n_\beta} F'_{\alpha\beta} \xrightarrow{\{q_\beta\}} Y_i \longrightarrow Y_{i+1}.$$

Then equally we have a cofibration

$$\bigvee_\beta S^{n_\beta} F'_{\alpha\beta} \xrightarrow{\{q_\beta\}} Y_i/X \longrightarrow Y_{i+1}/X,$$

so $Y_{i+1}/X \in \text{Class}\{F_\alpha\}$; we just used the spectra F_α and more instances of (i)–(iv).

For a limit ordinal, say j, suppose that for $i < j$, Y_i has been constructed and $Y_i/X \in \text{Class}\{F_\alpha\}$. We just let $Y_j = \bigcup_{i<j} Y_i$. Then $Y_j/X = \bigcup_{i<j} Y_i/X$, and it lies in $\text{Class}\{F_\alpha\}$ by another instance of (v).

This constructs $Y = Y_\Omega = \bigcup_{i<\Omega} Y_i$. Consider any map $f : S^n F\alpha \to Y$. $S^n F\alpha$ has at most σ stable cells, and each one maps into some Y_i; therefore f maps to Y_j for some j of cardinality $\leq \sigma$. Then f becomes null-homotopic in Y_{j+1}. This means that $[F_\alpha, Y]_* = 0$, which proves Proposition 4.9. □

At this point I will omit to give any categorical formulation of what we have done; I will just harvest some consequences**.

Corollary 4.10 *X is $\mathcal{T}$-colocal $\Leftrightarrow$ $X \in \text{Class}\{F_\alpha\}$.*

Proof: We have $\Leftarrow$ by Corollary 4.8. So suppose X is $\mathcal{T}$-colocal, and form the fibration $X \to Y \to Z$ from Proposition 4.9. Then $Z \in \text{Class}\{F_\alpha\}$, so Z is $\mathcal{T}$-colocal by Corollary 4.8. Since X and Z are $\mathcal{T}$-colocal, so is Y by Lemma 4.7(iii). $[F_\alpha, Y]_* = 0$ for all α, i.e., the maps $\text{pt} \to Y \to \text{pt}$ are in $\mathcal{T}$. This says $[Y, \text{pt}]_* \to [Y, Y]_* \to [Y, \text{pt}]_*$ are isomorphisms, so $[Y, Y]_* = 0$, whence $Y \simeq \text{pt}$. Therefore the map $Z \to SX$ in the cofibration is an equivalence. But $Z \in \text{Class}\{F_\alpha\}$, so $X \in \text{Class}\{F_\alpha\}$, using operations (i) and (ii). □

Now let's backtrack and suppose given E_* and I. Let σ be an infinite cardinal such that $|\pi_*(E)| \leq \sigma$, $|I| \leq \sigma$. The whole idea is to avoid set-theoretic difficulties by using σ to put an a priori bound on the length of our inductions and the number of cases we have to consider.

It is easy to show that for any finite spectrum K we have $|E_*(K)| \leq \sigma$. The same conclusion holds if K has fewer than σ cells, for each element $x \in E_*(K)$ must come from some finite subspectrum $K_\alpha \subset K$; there are at most σ of those, and each yields at most σ elements.

Lemma 4.11 *Let X be any spectrum such that $E_*(X)$ is I-torsion. Then any closed subspectrum $A \subset X$ with at most σ stable cells is contained in a closed subspectrum $B \subset X$ with at most σ stable cells and with $E_*(B)$ I-torsion.*

Proof: We make a construction by induction. Suppose that A_n is constructed with at most σ stable cells, starting from $A_0 = A$. Then there are at most σ pairs $(x \in E_*(A_n), i \in I)$ and each x comes from a finited subcomplex $K \subset A_n$. Since $E_*(X)$ is I-torsion, there exists m such that $i^m x = 0$ in $E_*(X)$, and in fact this already holds in some finite subcomplex L, $K \subset L \subset X$. By throwing in at most σ such finite subcomplexes L we can construct $A_n \subset A_{n+1} \subset X$ so that A_{n+1} has at most σ stable cells, and for each pair (x, i), an equation $i^m x = 0$ holds in A_{n+1}. This completes the induction, so take $B = \bigcup_n A_n$. □

We are now ready to choose $\{F_\alpha\}$. Let $\{F_\alpha\}$ provide at least one representative for each homotopy-equivalence class of spectra F with at most σ stable cells and $E_*(F)$ I-torsion. Clearly we can do this with a set; you can place a bound on its cardinality if you like.

Corollary 4.12 *Any spectrum X such that $E_*(X)$ is I-torsion is a direct limit of closed subspectra $X_\alpha \subset X$, each equivalent to some F_α.*

Proof: X is the direct limit of closed subspectra X_β which have $\leq \beta$ stable cells. By Lemma 4.11 the ones such that ($E_*(X_\beta)$ is I-torsion) are cofinal, so X is the direct limit of those. □

Let $\mathcal{T}$ be defined in terms of the $\{F_\alpha\}$ and use Corollary 4.8.

Corollary 4.13 *With this set $\{F_\alpha\}$,*

$$\begin{aligned} X \text{ is } \mathcal{T}\text{-colocal} &\iff X \in \text{Class}\{F_\alpha\} \\ &\iff E_*(X) \text{ is } I\text{-torsion.} \end{aligned}$$

Proof: The first $\Leftrightarrow$ follows from Corollary 4.10, so we move on to the second $\Rightarrow$: the property "$E_*(X)$ is I-torsion" is true for each F_α and is preserved under each of the operations (i)–(v) used in constructing Class$\{F_\alpha\}$. The second $\Leftarrow$ is immediate from Corollary 4.12. □

Proof of Theorem 4.5: Let E_*, I be as assumed; let $\{F_\alpha\}$ be as constructed for Corollary 4.12; and let $\mathcal{T}$ be defined for this $\{F_\alpha\}$ as above. By Proposition 4.9 we have a cofibration $X \xrightarrow{\eta} Y \to Z$.

First, $Z \in \text{Class}\{F_\alpha\}$ by 4.9, so $E_*(Z)$ is I-torsion by 4.13. It follows that $\ker \eta_*$ and $\operatorname{cok} \eta_*$ are I-torsion, so $\eta \in \mathcal{S}$.

Secondly, I claim that for any spectrum W such that $E_*(W)$ is I-torsion, then $[W, Y]_* = 0$. In fact W is $\mathcal{T}$-colocal by 4.13, and the maps pt $\to Y \to$ pt lie in $\mathcal{T}$ as we have said, so $[W, Y]_* \cong [W, pt]_* = 0$.

Thirdly, I claim that Y is $\mathcal{S}$-local. For suppose $U \xrightarrow{s} V$ is a map such that $\ker s_*$ and $\operatorname{cok} s_*$ are I-torsion. Form the cofibration $U \xrightarrow{s} V \longrightarrow W$; we see that $E_*(W)$ is I-torsion. By the second stage of the proof, $[W, Y]_* = 0$. Applying $[-, Y]_*$ to the cofibration, we get

$$s^* : [U, Y]_* \xleftarrow{\cong} [V, Y]_*,$$

which proves that Y is $\mathcal{S}$-local. This proves Theorem 4.5. □

Now I can explain about the category of fractions. Let $\mathcal{C}$ be the category of all spectra. The objects form a class; they may form a set, but there is no harm in that. We will define a new category $\mathcal{S}^{-1}\mathcal{C}$. The objects are the same as those of $\mathcal{C}$; to define the maps from X to Y in $\mathcal{S}^{-1}\mathcal{C}$, we choose localizations $X \xrightarrow{\xi} X'$, $Y \xrightarrow{\eta} Y'$, and define the maps to be $[X', Y']$. Of course, one could choose different localizations $X \longrightarrow X''$, $Y \longrightarrow Y''$, but then there is a canonical equivalence $X' \simeq X''$, and a canonical equivalence $Y' \simeq Y''$, so there is a canonical way to identify $[X', Y']$ with $[X'', Y'']$.

Composition and identity maps are obvious.

There is a function λ from $\mathcal{C}$ to $\mathcal{S}^{-1}\mathcal{C}$ which is the identity on objects. Viz, given $X \xrightarrow{f} Y$ there is a unique way to fill in the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{\lambda f} & Y' \end{array}$$

Exercise: λ carries every map $s \in \mathcal{S}$ to an invertible map in $\mathcal{S}^{-1}\mathcal{C}$; it is initial among functions which have this property.

This statement characterizes $\mathcal{S}^{-1}\mathcal{C}$ as the desired category of fractions.

Now I must explain about Bousfield equivalence.

Definition 4.14 *Two pairs (E,I) and (F,J) are* Bousfield-equivalent *if $\mathcal{S}(E,I) = \mathcal{S}(F,J)$.*

If this is so, then the whole of the rest of the theory of Bousfield localization is the same for both, because the theory only depends on $\mathcal{S}$.

Of course, Bousfield just defined equivalence of spectra because he was considering the case $I = \{1\}$.

If you have a map $f : X \to Y$, and you want to see if it lies in $\mathcal{S}(E.I)$ or $\mathcal{S}(F,J)$, you form the cofibration $X \xrightarrow{f} Y \to Z$. In this way you translate "Bousfield equivalence" into the following equivalent condition: $E_*(Z)$ is I-torsion iff $F_*(Z)$ is J-torsion. With $I = \{1\}, J = \{1\}$ this says that $E \wedge Z$ is contractible iff $F \wedge Z$ is contractible.

Some of Ravenel's problems implicitly concern the theory of Bousfield classes. In any case, Mike Hopkins finds it convenient to process information by turning it into information about Bousfield classes, manipulating it and taking it out again. So we had better prepare a little for such exercises.

Bousfield classes are partially ordered; we write "$(E,I) \geq (F,J)$" if $E_*(Z)$ is I-torsion implies that $F_*(Z)$ is J-torsion. Notice that this works the right way around: if $E = F$ and $I \supset J$, then $E_*(Z)$ is I-torsion implies that $F_*(Z)$ is J-torsion, so $(E,I) \geq (F,J)$.

Now I'll stick to the case $I = \{1\}$ for a bit. The sphere spectrum S^0 is the highest; $S^0 \wedge Z \simeq \text{pt}$ says $Z \simeq \text{pt}$, which implies $F \wedge Z \simeq \text{pt}$ for any F. With $E = S^0$, the E-equivalences are the truest equivalences. Localization changes nothing: every X is S^0-local, and $X \xrightarrow{1} X$ will serve as the localization. The point spectrum pt is the lowest; $\text{pt} \wedge Z \simeq \text{pt}$ is always true, and any one statement implies it. With $E = \text{pt}$, all maps are E-equivalences; the only E-local spectrum is the point, and localization destroys everything. X is Bousfield-equivalent to the point iff X is homotopy equivalent to a point.

The wedge-sum operation $\vee_\alpha X_\alpha$ (where the sum may be infinite) passes to Bousfield classes, and gives the least upper bound with the order relation:

$$(\bigvee_\alpha X_\alpha) \wedge Z \simeq \bigvee_\alpha (X_\alpha \wedge Z),$$

which is contractible iff $X_\alpha \wedge Z \simeq \mathrm{pt}$ for all α.

The smash product also passes to Bousfield classes. Suppose $E_1 \equiv E_2$; then $E_1 \wedge F \wedge Z \simeq \mathrm{pt}$ iff $E_2 \wedge F \wedge Z \simeq \mathrm{pt}$, by the characteristic** property applied to $F \wedge Z$; this says that $E_1 \wedge F \equiv E_{2 \wedge F}$. However, we don't say that this operation gives the greatest lower bound; however, it may be useful.

Example 4.15 $<S^0> =_B \bigvee_p <S^0_{(p)}>$.

Proof: $\pi_*(S^0_{(p)} \wedge X) = \pi_*(X)_{(p)}$, so $\pi_*(S^0_{(p)} \wedge X) = 0$ for all p is the same as $\pi_*(X)_{(p)} = 0$ for all p, which by well-known algebra is true iff $\pi_*(X) = 0$. □

Corollary 4.16 *If E is any spectrum, then*

$$<E> =_B \bigvee_p <E_{(p)}> .$$

Proof: Take the previous equivalence and apply $- \wedge E$. This is the expression, in this context, of the fact that it is good enough to hold at p for all p. □

Finally I give a little lemma. Last Easter I noticed Mike Hopkins using the same handy argument repeatedly. By August he had noticed that Ravenel had already codified the result which the argument proves, and here it is.

Suppose given a map $S^n X \xrightarrow{f} X$. Use it to form a cofiber sequence $S^n X \xrightarrow{f} X \longrightarrow C_f$ and also the direct limit T of

$$X \xrightarrow{S^{-n} f} S^{-n} X \xrightarrow{S^{-2n} f} S^{-2n} X \longrightarrow \cdots$$

Lemma 4.17 *(Ravenel 1.34)* $<X> = <C_f> \vee <T>$.

Proof: (i) Suppose $X \wedge Z$ is contractible. Then $S^n X \wedge Z \xrightarrow{f \wedge 1} X \wedge Z \longrightarrow C_f \wedge Z$ is a cofibration with two terms contractible, which shows that $C_f \wedge Z$ is contractible. Also, $T \wedge Z$ is the limit of

$$X \wedge Z \xrightarrow{S^{-n} f \wedge 1} S^{-n} X \wedge Z \longrightarrow \cdots$$

which shows $T \wedge Z$ is contractible.

(ii) Suppose C_f is contractible. Then the map

$$S^n X \wedge Z \xrightarrow{f \wedge 1} X \wedge Z$$

is an equivalence. So all the maps of the directed system

$$X \wedge Z \xrightarrow{S^{-n} f \wedge 1} S^{-n} X \wedge Z \xrightarrow{S^{-2n} f \wedge 1} S^{-2n} X \wedge Z \longrightarrow \cdots$$

are equivalences. So the map from the first term to the limit $X \wedge Z \to T \wedge Z$ is an equivalence. So if we suppose that $T \wedge Z$ is contractible, we conclude that $X \wedge Z$ is contractible. □

Corollary 4.18 *If f is nilpotent, i.e., if there exists m such that $X \xrightarrow{f} S^{-n}X \xrightarrow{f} S^{-2n}X \xrightarrow{f} \cdots \xrightarrow{f} S^{-mn}X$ is null-homotopic, then $<X> = <C_f>$.*

Proof: If f is nilpotent, then $T = \varinjlim(X \xrightarrow{f} S^{-n}X \xrightarrow{f} \cdots)$ has $\pi_*(T) = 0$ and is contractible. So $<X> = <C_f> \vee <T> = <C_f>$. □

Example 4.19 $<CP^2> = <S^0>$. *Proof:* $\eta^4 = 0$.

Mike Hopkins wants to run part of his argument the other way. Now a priori the argument seems to fail unless X is a finite spectrum: if T is contractible then the injection $X \to T$ is null-homotopic, and if this null-homotopy takes place in some finite stage of the telescope, say $S^{-mn}X$, then $f^m \simeq 0$; but if X is infinite there is no reason why the null-homotopy shouldn't use the whole spectrum. And, of course, we have to use infinite spectra.

The way forward is that we can say something for those maps which are determined by their values on some finite subspectrum.

Examples 4.20 $<HP^2> = <S^0>$. *Proof:* $\gamma^4 = 0$.
$<CP^2 \wedge HP^2> = <S^0>$.

Let R be a ring-spectrum. It needs to be homotopy-associative, with unit $S^0 \xrightarrow{\eta} R$ up to homotopy, but no commutativity is needed. As we say, for each element $\alpha \in \pi_q(R)$ we can construct the map

$$S^q \wedge R \xrightarrow{\alpha \wedge 1} R \wedge R \xrightarrow{\mu} R;$$

this we interpret as an operation, "multiply by α in the coefficient ring." Write $\overline{\alpha} = \mu(\alpha \wedge 1)$. Then $\overline{\alpha}$ is a map of right R-module spectra, and this gives a one-to-one correspondence between elements of $\pi_*(R)$ and maps of right R-module spectra; the correspondence the other way is that $\overline{\alpha}$ satisfies**

$$S^q = S^q \wedge S^0 \xrightarrow{1 \wedge \eta} S^q \wedge R \xrightarrow{\overline{\alpha}} R.$$

Composition of maps $\overline{\alpha}$ corresponds to multiplication of elements $\alpha \in \pi_*(R)$; i.e., $\overline{\alpha\beta} = \overline{\alpha}\overline{\beta}$.

Lemma 4.21 *α is nilpotent in $\pi_*(R)$ iff* $\mathrm{Tel}(\overline{\alpha}) \simeq \mathrm{pt}$.

Proof: If $\alpha^m = 0$, then $\overline{\alpha}^m = 0$, so $\mathrm{Tel}(\overline{\alpha})$ is contractible. If $\mathrm{Tel}(\overline{\alpha})$ is contractible, then $S^0 \xrightarrow{\eta} R \longrightarrow \mathrm{Tel}(\overline{\alpha})$ is null-homotopic, and since S^0 is finite, this null-homotopy must take place in a finite part of the telescope; i.e., there exists m such that $\alpha^m = 0$. □

Actually this is not yet the form in which we want to use the lemma. We want to suppose given two ring-spectra E and R, and apply the lemma to $E \wedge R$—which of course is a ring-spectrum again. But we don't want to do it with a general element of $\pi_*(E \wedge R) = E_*(R)$, but with an element of the following special form:

$$S^q \xrightarrow{\alpha} R \simeq S^0 \wedge R \longrightarrow E \wedge R.$$

Basically, we want to take an element $\alpha \in \pi_q(R)$ and then take its image in $E_q(R)$ under the E-Hurewicz homomorphism. Easy diagram-chasing shows that the map

$$\overline{\overline{\alpha}} : S^q \wedge E \wedge R \longrightarrow E \wedge R$$

which we get is, up to a rearrangement of where you put S^q, the smash product of $1 : E \to E$ and $S^q \wedge R \xrightarrow{\overline{\alpha}} (R)$. So $\mathrm{Tel}(\overline{\overline{\alpha}}) = E \wedge \mathrm{Tel}(\overline{\alpha})$.

Corollary 4.22 *$\alpha \in \pi_q(R)$ becomes nilpotent in $E_q(R)$ iff $E \wedge \mathrm{Tel}(\overline{\alpha})$ is contractible.*

Of course, you can prove this more directly by cutting out some of the way I built it up.

Corollary 4.23 *Suppose E and F are ring-spectra and $<E> = <F>$; then $\alpha \in \pi_Q(R)$ becomes nilpotent in $E_*(R)$ iff it becomes nilpotent in $F_*(R)$.*