SEMI-SIMPLICIAL COMPLEXES AND POSTNIKOV SYSTEMS

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Classically, in algebraic topology, one studied simplicial complexes. However, many of the spaces which arise naturally in modern algebraic topology are not simplicial complexes. For example, the loop space of a simplicial complex is not a simplicial complex. This illustrates the fact that simplicial complexes are not adequate to deal with homotopy from a modern point of view. In 1950, the concept of semi-simplicial complex was introduced by Eilenberg and Zilber [1]. At present it seems certain that the category of semi-simplicial complexes and semi-simplicial maps is the most convenient category to work in when studying homotopy problems. Sometimes it is convenient to work with an arbitrary semi-simplicial complex, and sometimes with one satisfying the extension condition of Kan [2].

In this paper part of the theory of semi-simplicial complexes will be outlined, including in particular an outline of the development of homotopy theory for those complexes which satisfy the extension condition. After this is done, the results will be used to describe and discuss Postnikov systems [3].

Much of the material in this paper was presented in a course of lectures at Princeton during 1955–1956, or in the Cartan seminar of 1954–1955 [4].

§1. Semi-simplicial complexes and homotopy

DEFINITIONS 1.1. Let Z^+ denote the set of non-negative integers. Now a *semi-simplicial complex* consists of the following:

(1) A set $X = \bigcup_{q \in Z^+} X_q$, where the X_q are disjoint sets (an element of X_q is called a q-simplex of X);

(2) functions $\partial_i : X_{q+1} \to X_q$, $i = 0, \cdots, q+1$, called *face operators*, and

(3) functions $s_i : X_q \to X_{q+1}$, $i = 0, \cdots, q$ called *degeneracy operators*.

The face and degeneracy operators are assumed to satisfy the relations

$$\partial_i \partial_j = \partial_{j-1} \partial_i \qquad i < j,$$

$$s_i s_j = s_{j+1} s_i \qquad i \leqq j,$$

$$\partial_j s_j = \partial_{j+1} s_j = \text{identity},$$

$$\partial_i s_j = s_{j-1} \partial_i \qquad i < j, \text{ and}$$

$$\partial_i s_j = s_j \partial_{i-1} \qquad i > j+1.$$

We will denote a semi-simplicial complex by its set X of simplexes. A simplex $x \in X_{n+1}$ is called *degenerate* if $x = s_j y$ for some $y \in X_n$ and some degeneracy operator s_j; otherwise x is called *non-degenerate*.

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EXAMPLE 1. Recall that a simplicial complex K is a set whose elements are finite subsets of a given set $\bar{K}$, subject to the condition that if $x \in K$ and y is a non-empty subset of x, then $y \in K$. Sets with 1 element are called vertices, and sets with $(n+1)$ elements are called n-simplexes of K.

Linearly order the elements of $\bar{K}$, i.e., the vertices of K. Now define a semi-simplicial complex $X(K)$ by letting the n-simplexes of $X(K)$ be $(n+1)$-tuples $(a_0, \cdots, a_n)$ of elements of $\bar{K}$ such that $a_0 \leqq \cdots \leqq a_n$, and such that the set $\{a_0, \cdots, a_n\}$ is an r simplex of K for some $r \leqq n$. Define

$$\partial_i(a_0, \cdots, a_n) = (a_0, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n), \text{ and}$$

$$s_i(a_0, \cdots, a_n) = (a_0, \cdots, a_i, a_i, a_{i+1}, \cdots, a_n).$$

EXAMPLE 2. Let Δ_n denote the standard n-simplex, in other words a point of Δ_n is an $(n+1)$-tuple $(t_0, \cdots, t_{n+1})$ of real numbers such that $0 \leqq t_i \leqq 1$, $i = 0, \cdots, n$, and $\sum t_i = 1$. Let A be a topological space. A singular n-simplex of A is a map $U : \Delta_n \to A$. Denote by $S(A)_n$ the set of singular n-simplexes of A, and set $S(A) = \bigcup_{n \in Z+} S(A)_n$. Define

$$\partial_i : S(A)_n \to S(A)_{n-1}$$

by $\partial_i U(t_0, \cdots, t_{n-1}) = U(t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_{n-1})$, and define

$$s_i : (SA)_n \to S(A)_{n+1}$$

by $s_i U(t_0, \cdots, t_{n+1}) = U(t_0, \cdots, t_{i-1}, t_i + t_{i+1}, t_{i+i}, \cdots, t_{n+1})$. One verifies easily that $S(A)$ is a semi-simplicial complex; it is known as the total singular complex of the space A.

DEFINITION 1.2. A semi-simplicial complex X is said to satisfy the *extension condition* if given $x_0, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{n+1} \in X_n$ such that $\partial_i x_j = \partial_{j-1} x_i$ for $i < j$, $i, j \neq k$, there exists $x \in X_{n+1}$ such that $\partial_i x = x_i$ for $i \neq k$. Such a complex will be called a *Kan complex*.

PROPOSITION 1.3. *If A is a topological space, then the total singular complex $S(A)$ satisfies the extension condition.*

The proposition follows from the fact that the union of $(n+1)$ faces Δ_{n+1} is a retract of Δ_{n+1}.

Although it has long been realized that the total singular complex satisfies the extension condition, it was only recently that it was pointed out by D. M. Kan that the extension condition is sufficient for the definition of homotopy groups. In fact, in the category of Kan complexes and semi-simplicial maps, one can treat all problems involving only questions of homotopy type. The original work of Kan in this direction was done on semi-cubical complexes, but it was clear from the outset that one could work equally well with semi-simplicial complexes. At present almost everyone is agreed that for various technical reasons the category of semi-simplicial complexes is more convenient than the category of semi-cubical complexes.

DEFINITION 1.4. Let Δ_n denote the semi simplicial complex whose q-simplexes are $(q+1)$-tuples $(a_0, \cdots, a_q)$ of integers such that $0 \leqq a_0 \leqq \cdots \leqq a_q \leqq n$. Suppose further face and degeneracy operations are defined as in Example 1, by

$$\partial_i(a_0, \cdots, a_n) = (a_0, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n), \text{ and}$$

$$s_i(a_0, \cdots, a_n) = (a_0, \cdots, a_{i-1}, a_i, a_i, a_{i+1}, \cdots, a_n).$$

This is just exactly the standard simplicial complex for the n-simplex. Further let

$$\varepsilon_i : \Delta_{n-1} \to \Delta_n, \text{ and}$$

$$\eta_i : \Delta_n \to \Delta_{n-1}$$

be the simplicial maps such that $\varepsilon_i(j) = j$ for $j < i$, $\varepsilon_i(j) = j+1$, $j \geqq i$; $\eta_i(j) = j$ for $j \leqq i$, and $\eta_i(j) = j - 1$, $j > i$.

Denote by $\dot{\Delta}_n$ the subcomplex of Δ_n such that a q-simplex is a $(q+1)$-tuple $(a_0, \cdots, a_q)$ such that the set $\{a_0, \cdots, a_q\}$ has at most n elements. In other words $\dot{\Delta}_n$ is the boundary of Δ_n, or the $(n-1)$ skeleton of Δ_n. All simplexes of $\dot{\Delta}_n$ of dimension greater than $(n-1)$ are degenerate.

Definition 1.5. If X, Y are semi simplicial complexes the Cartesian product $X \times Y$ of X and Y is the semi simplicial complex such that

(1) $(X \times Y)_n = \{(a, b) | a \in X_n, b \in Y_n\}$,

(2) $\partial_i : (X \times Y)_n \to (X \times Y)_{n-1}$ is defined by $\partial_i(a, b) = (\partial_i a, \partial_i b)$, and

(3) $s_i : (X \times Y)_n \to (X \times Y)_{n+1}$ is defined by $s_i(a, b) = (s_i a, s_i b)$.

Proposition 1.6. *If A and B are topological spaces, then $S(A \times B) = S(A) \times S(B)$.*

This proposition follows immediately from the fact that a singular simplex in the product space $A \times B$ is uniquely determined by its projections on the factors [5].

Definition 1.7. If X and Y are semi-simplicial complexes and $f : X \to Y$ is a function, then f is a semi-simplicial map (or simply map) if

(1) $f(X_n) \subset Y_n$ for $n \in Z^+$,

(2) $\partial_i f = f \partial_i$, and

(3) $s_i f = f s_i$.

Definition 1.8. If X and Y are semi-simplicial complexes, then the complex of maps from X to Y is the semi-simplicial complex Y^X such that

(1) $(Y^X)_n = \{f | f : X \times \Delta_n \to Y \text{ is a map}\}$,

(2) $\partial_i f = f \circ (1 \times \varepsilon_i)$ for f an n-simplex, and

(3) $s_i f = f \circ (1 \times \eta_i)$ for f an n-simplex.

If A, X, B, and Y are semi-simplicial complexes with $A \subset X$, $B \subset Y$ then $(Y, B)^{(X,A)}$ is the subcomplex of Y^X such that an n-simplex is a map $f : (X \times \Delta_n, A \times \Delta_n) \to (Y, B)$.

Theorem 1.9. *If (X, A) and (Y, B) are pairs of semi-simplicial complexes such that Y and B are Kan complexes, then $(Y, B)^{(X,A)}$ is a Kan complex.*

The proof of this theorem is somewhat long and tedious, but not particularly difficult.

Definition 1.10. Let X be a semi-simplicial complex, a *point* in X is a zero simplex of X, i.e., an element of X_0, and a *path* in X is a 1-simplex, i.e., an element

of X_1. If x is a path in X, then $\partial_1 x$ is the *initial point* or origin of x, and $\partial_0 x$ is the final or *terminal* point of x.

Equivalently a point of X is a map of Δ_0 into X, and a path in X is a map of Δ_1 into X.

Two points a, b of X are in the *same path component* of X if there is a path in X with initial point a and final point b, this will be denoted by $a \sim b$.

Proposition 1.11. *If X is a Kan complex, and a, b, c, are points of X, then*

(i) $a \sim a$,

(ii) *if* $a \sim b$, *then* $b \sim a$, *and*

(iii) *if* $a \sim b$, $b \sim c$, *then* $a \sim c$.

Notation. For any Kan complex X let $\pi_0(X)$ denote the set of path components of X. Further if $x \in X_0$ let $[x] \in \pi_0(X)$ denote the equivalence class of x.

Definition 1.12. If (Y, B) is a pair of Kan complexes, and (X, A) is a pair of semi-simplicial complexes, then, $f, g : (X, A) \to (Y, B)$ are *homotopic* if and only if both f and g are in the same path component of $(Y, B)^{(X,A)}$, i.e., $[f] = [g] \in \pi_0((Y, B)^{(X,A)})$. A homotopy between f and g is a path in $(Y, B)^{(X,A)}$ joining f to g. In other words a homotopy is a map $F : (\Delta_1 \times X, \Delta_1 \times A) \to (Y, B)$ such that $\partial_0 F = g$, $\partial_1 F = f$.

Now using the preceding proposition we have that homotopy between maps of a semi-simplicial pair into a Kan pair is an equivalence relation.

Homotopy Extension Theorem 1.13. *If (X, A) is a semi-simplicial pair, Y a Kan complex, $f : X \to Y$ and $F : \Delta_1 \times A \to Y$ maps such that $\partial_1 F = f|A$, then there exists $\bar{F} : \Delta_1 \times X \to Y$ such that $\bar{F}|\Delta_1 \times A = F$, and $\partial_1 \bar{F} = f$.*

Definition 1.14. Let X be a Kan complex, and x a point of X. Also let x denote the subcomplex of X generated by x, i.e., there is a unique q-simplex $s_0^q x$ in this subcomplex for every positive integer q. Now define.

$$\pi_n(X, x) = \pi_0((X, x)^{(\Delta_n, \dot{\Delta}_n)})$$

for $n > 0$.

Lemma 1.15. *If X is a Kan complex, x a point of X, $f, g : (\Delta_n, \dot{\Delta}_n) \to (X, x)$, and i, j, k distinct integers, then there exists $F : \Delta_{n+1} \to X$ such that $\partial_i F = f$, $\partial_j F = g$, $\partial_q F \in x$ for $q \neq i, j, k$, and if F' is another such map, then $[\partial_k F] = [\partial_k F'] \in \pi_n(X, x)$.*

Definition 1.16. Let X be a Kan complex, x a point of X, $f, g : (\Delta_n, \dot{\Delta}_n) \to (X, x)$ maps, and $F : \Delta_{n+1} \to X$ a map such that $\partial_{n+1} F = f$, $\partial_{n-1} F = g$, and $\partial_i F \in x$ for $i < n - 1$. Define

$$[f] + [g] = [\partial_n F] \in \pi_n(X, x).$$

Proposition 1.17. *Let X be a Kan complex, x a point of X, then*

(1) $\pi_n(X, x)$ *is a group for* $n > 0$ *and*

(2) $\pi_n(X, x)$ *is abelian for* $n > 1$.

The group $\pi_n(X, x)$ is the *n-dimensional homotopy group of X* at the base point x.

The homotopy groups of Kan complexes enjoy all the usual properties of homotopy groups of spaces. In fact the homotopy groups of a topological space are just the homotopy groups of its singular complex. A few of the elementary properties of homotopy groups are summarized in the next proposition. Exact sequences of homotopy groups will not be considered until the next section which will be devoted to fibre spaces.

PROPOSITION 1.18. *Let (X, x), (Y, y), (Z, z) be Kan complexes with base point, then*

(1) *if $f: (X, x) \to (Y, y)$ is a map, f induces a homomorphism $f^{\#}: \pi_n(X, x) \to \pi_n(Y, y)$ for $n > 0$.*

(2) *if $f: (X, x) \to (Y, y)$ and $g: (Y, y) \to (Z, z)$ are maps then $(gf)^{\#} = g^{\#}f^{\#}: \pi_n(X, x) \to \pi_n(Z, z)$ for $n > 0$,*

(3) *if $f, g: (X, x) \to (Y, y)$ are maps such that f, g belong to the same component of $(Y, y)^{(X,x)}$, then $f^{\#} = g^{\#}$.*

(4) *if $f: (X, x) \to (X, x)$ is the identity map, then $f^{\#}$ is the identity homomorphism, and*

(5) *if $f: (X, x) \to (y, y)$, then $f^{\#}$ is the zero homomorphism.*

Now following Eilenberg and Zilber ([1]) we will outline the proof that any Kan complex has a minimal subcomplex which is equivalent to the original complex as far as homotopy is concerned.

DEFINITION 1.19. Let X be a Kan complex. The complex X is *minimal* if whenever $x, y\ X_q$ are such that $\partial_i x = \partial_i y$ for $i \neq k$, then $\partial_k x = \partial_k y$. Two maps $f, g: \Delta_q \to X$ are *compatible* if $f|\dot{\Delta}_q = g|\dot{\Delta}_q$, and f, g are homotopic if there exists $F: \Delta_1 \times \Delta_q \to X$ such that $F|(0) \times \Delta_q = f$, $F|(1) \times \Delta_q = g$, and $F(\sigma \times \tau) = f(\tau)$ for $\tau \in \dot{\Delta}_q$.

LEMMA 1.20. *The Kan complex X is minimal if and only if for each compatible homotopic pair of maps $f, g: \Delta_q \to X$ we have $f = g$.*

DEFINITION 1.21. Let X be a semi-simplicial complex and A a subcomplex of X. Then A is a deformation retract of X if there is a map $F: \Delta_1 \times X \to X$ such that $F(\sigma \times \tau) = \tau$ for $\tau \in A$, $F(s_0^q(0) \times \tau) = \tau$, and $F(s_0^q(1) \times \tau) \in A$ for any $\tau \in X$.

THEOREM 1.22. *If X is a Kan complex, then there is a minimal subcomplex M of X which is a deformation retract of X, and if M' is another such complex, then M' is isomorphic to M.*

§2. Fibre spaces

Now having developed a little of the theory of semi-simplicial complexes, we now turn to the study of fibre spaces. It is here that the study of Postnikov systems naturally arises.

DEFINITIONS 2.1 A *fibre space* (or semi-simplicial fibre space) is a triple (E, p, B) where E and B are semi-simplicial complexes, and $p: E \to B$ is a semi-simplicial map such that if $x \in B_{q+1}$, $y_0, \cdots, y_{k-1}, y_{k+1}, \cdots, y_{q+1} \in E_q$ are such that $p(y_i) = \partial_i x$ and $\partial_i y_j = \partial_{j-1} y_i$ for $i < j$, $i, j \neq k$, then there exists $y \in E_{q+1}$ such that $\partial_i y = y_i$ for $i \neq k$, and $p(y) = x$. The map p is called a fibre map.

A fibre map $p: E \to B$ is *minimal* if $y, y' \in E_{q+1}$ are such that $p(y) = p(y')$ and $\partial_i y = \partial_i y'$ for $i \neq k$, then $\partial_k y = \partial_k y'$. The fibre space (E, p, B) is *minimal* if the fibre map p is minimal and the complex B is minimal.

Let $b \in B_0$, and let F be the counter image in E of the complex generated by b. The complex F is called the fibre over b.

PROPOSITION 2.2. *Let (E, p, B) be a fibre space.*

(1) *If F is the fibre over a point of B, then F is a Kan complex.*

(2) *The complex E is a Kan complex if and only if B is a Kan complex.*

DEFINITION 2.3. Let (E, p, B) be a fibre space, where B is a Kan complex, b a point of B and a a point of F the fibre over b.

For $q \geqq 2$ define

$$\partial^{\#}: \pi_q(B, b) \to \pi_{q-1}(F, a).$$

Recall that $\alpha \in \pi_q(B, b)$ is represented by $x \in B_q$ such that $\partial_i x = s_0^{q-1} b$ for all i. Since p is a fibre map, there exists $y \in E_q$ such that $p(y) = x$ and $\partial_i y = s_0^{q-1} a$ for $i > 0$. Then $\partial_0 y \in F_{q-1}$ and represents an element of $\pi_{q-1}(F, a)$. Checking independence of representative, define $\partial^{\#}([x]) = [\partial_0 y]$.

THEOREM 2.4. *Let (E, p, B) be a fibre space, and suppose B is a Kan complex. Let $b \in B_0$, F be the fibre over b, and $a \in F_0$, then*

(1) *$\partial^{\#}: \pi_q(B, b) \to \pi_{q-1}(F, a)$ is a homomorphism for $q \geqq 2$, and*

(2) *The sequence*

$$\cdots \to \pi_q(F, a) \xrightarrow{i^{\#}} \pi_q(E, b) \xrightarrow{p^{\#}} \pi_q(B, b) \xrightarrow{\partial^{\#}} \pi_{q-1}(F, a) \to \cdots$$

is exact, where $i: F \to E$ is the inclusion map.

DEFINITIONS 2.5. Let X be a semi-simplicial complex. Define a new semi-simplicial complex $X^{(n)}$ as follows:

(1) A q-simplex of $X^{(n)}$ is an equivalence class of q-simplexes of X, where two q simplexes x, x' are equivalent if their faces of dimension less than or equal to n agree, i.e., $x, x': \Delta_q^n \to X$ and $x|\Delta_q^n = x'|\Delta_q^n$ where Δ_q^n is the n-skeleton of Δ_q.

(2) The face and degeneracy operations in $X^{(n)}$ are induced by those in X.

Let $X^{(\infty)} = X$, and let $p_k^n: X^{(n)} \to X^{(k)}$ be the natural map for $n \geqq k$, where $\infty \geqq k$ for every k. When there is no danger of confusion, p_k^n will be abbreviated by p.

THEOREM 2.6. *Let X be a Kan complex, then*

(1) *$X^{(n)}$ is a Kan complex for every n,*

(2) *$(X^{(n)}, p, X^{(k)})$ is a fibre space for $n \geqq k$, and*

(3) *if x is a point of X, then $\pi_q(X^{(n)}, x) = 0$ for $q > n$, and*

$$p^{\#}: \pi_q(X^{(n)}, x) \xrightarrow{\approx} \pi_q(X^{(k)}, x) \quad \text{for} \quad q \leqq k.$$

In the context of complexes satisfying the extension condition, the proof of the preceding theorem is very easy. This theorem contains many classical results. For example, consider the case $(X, p, X^{(k)})$. We then have that $p^{\#}: \pi_q(X, x) \to \pi_q(X^{(k)}, x)$ for $q \leqq k$, and $\pi_q(X^{(k)}, x) = 0$ for $q > k$. In other words the fibre spaces $(X, p, X^{(k)})$ are the precise analogue of the construction (II) of Cartan and Serre [8].

DEFINITION 2.7. Let X be a Kan complex, and x a point of X. Let $E_n(X, x)$ denote the fibre over the point x of $p: X \to X^{(n-1)}$. The complex $E_n(X, x)$ is the n^{th} *Eilenberg subcomplex of* X based at x, and is that subcomplex of X consisting of simplexes whose faces of dimension less than n are at the base point x, [9].

PROPOSITION 2.7′. *Let X be a Kan complex with base point x. We then have*
(1) $\pi_q(E_n(X, x), x) = 0$ *for* $q < n$, *and*
(2) $i^{\#} : \pi_q(E_n(X, x)x) \approx \pi_q(X; x)$ *for* $q \geqq n$, *where* $i : E_n(X, x) \to X$ *is the inclusion map.*

DEFINITION 2.8. If X is a Kan complex, let χ^n be the fibre space $(X^{(n+1)}, p, X^{(n)})$. The sequence of fibre spaces $\chi = (\chi^0, \chi^1, \cdots, \chi^n, \cdots)$ is by definition the *natural Postnikov system of* X, [3].

DEFINITION 2.9. Let X be a connected Kan complex with base point x. Then X is an *Eilenberg-MacLane complex of type* (π, n) if and only if $\pi_q(X, x) = 0$ for $q \neq n$, and $\pi_n(X, x) = \pi$.

THEOREM 2.10. *If X is a Kan complex with base point x, χ is the natural Postnikov system of X, and $F^{(n+1)}$ is the fibre of the map $p : X^{(n+1)} \to X^{(n)}$ which is the n^{th} term of χ, then $F^{(n+1)}$ is an Eilenberg-MacLane complex of type $(\pi_{n+1}(X, x), n + 1)$.*

With this theorem we see that any Kan complex may be constructed in some sense from Eilenberg-MacLane complexes. The process of so doing will be studied further later. However, before doing so we want to consider a generalization of the preceding which applies to a fibre map. It may well at this stage to point out that if X is a semi-simplicial complex and x the complex of a point, then the unique map $f : X \to x$ is a fibre map if and only if X is a Kan complex.

DEFINITION 2.11. Let $p : E \to B$ be a fibre map, e a point of E, $b = p(e)$ and F the fibre over b. Suppose that B and F are connected and that B is a Kan complex (recall that this means E is connected and a Kan complex). Define a new semi-simplicial complex $E^{(n)}$ as follows:
(1) A q-simplex of $E^{(n)}$ is an equivalence class of q-simplexes of E where two q-simplexes x, x' are equivalent if
(i) $p(x) = p(x')$, and
(ii) $x|\Delta_q^n = x'|\Delta_q^n$.
(2) The face and degeneracy operations in $E^{(n)}$ are induced by those in E.

Let $E^{(\infty)} = E$, and let $p_k^n : E^{(n)} \to E^{(k)}$ be the natural map for $n \geqq k$, where $\infty \leqq k$.

THEOREM 2.12. *Let (E, p, B) be a fibre space of connected Kan complexes, e a point of E, $b = p(e)$, and F the fibre over b. Then*
(1) $E^{(n)}$ *is a Kan complex for every* n,
(2) $E^{(0)} = B$ if $E_0 = \{ e \}$,
(3) $(E^{(n)}, p, E^{(k)})$ *is a fibre space for* $n \geqq k$,
(4) $\pi_q(E, e) \overset{\approx}{\to} \pi_q(E^{(n)}, e)$ *for* $q \leqq n$,
(5) $\pi_q(E^{(n)}, e) \overset{\approx}{\to} \pi_q(B, b)$ *for* $q > n + 1$, *and*
(6) *the fibre of* $p : E^{(n)} \to B$ *is* $F^{(n)}$, *the* n^{th} *term in the Postnikov system for the fibre* F.

DEFINITION 2.13. Let (E, p, B) be a fibre space of connected Kan complexes, and let ε^n be the fibre space $(E^{(n+1)}, p, E^{(n)})$. The sequence of fibre spaces $\varepsilon = (\varepsilon^0, \varepsilon^1, \cdots, \varepsilon^n, \cdots,)$ is by definition the natural Postnikov system of (E, p, B).

THEOREM 2.14. *If (E, p, B) is a fibre space of connected Kan complexes, e a point of E, $b = p(e)$, and F the fibre over b, then the fibre over b in ε^n, the n^{th} term of the Postnikov system of the fibre space (E, p, B) is an Eilenberg-MacLane complex of type $(\pi_{n+1}(F, e), n + 1)$.*

Consequently as a result of this theorem we see that a fibre space may be constructed by giving a base complex B, and then "adding" in the homotopy groups of the fibre one at a time. In fact one has given a fibre space (E, p, B) an infinite sequence of fibre spaces, $E \to \cdots \to E^{(n+1)} \to E^{(n)} \to \cdots \to E^{(1)} B$, and the special case of this were B is the complex of a point gives exactly the ordinary Postnikov system of E.

Just as any Kan complex has a minimal complex, which is a deformation retract, any fibre space has a minimal fibre space. The situation however is somewhat better than this as will be seen by the following theorems.

THEOREM 2.15. *Let (E, p, B) be a fibre space of connected Kan complexes, and let B' be a minimal subcomplex of B which is equivalent to B. Define $E' = p^{-1}(B')$. Then*
(1) (E', p, B') *is a fibre space, and*
(2) *there exists a commutative diagram*

$$\begin{array}{ccc} E \times \Delta_1 & \xrightarrow{F} & E \\ \downarrow{\scriptstyle p \times 1} & & \downarrow{\scriptstyle p} \\ B \times \Delta_1 & \xrightarrow{f} & B \end{array}$$

such that $F(\sigma, \tau) = \sigma$ *for* $\sigma \in E'$,
$F(\sigma, s_0^q\, 0) = \sigma$ *for* $\sigma \in E_q$, *and*
$F(\sigma, s_0^q\, 1) \in E_q'$ *for* $\sigma \in E_q$.

THEOREM 2.16. *Let (E, p, B) be a fibre space of connected Kan complexes, and suppose B is minimal. Then there exists $E' \subset E$ such that*
(1) (E', p, B) *is a minimal fibre space, and*
(2) *there exists a commutative diagram*

$$\begin{array}{ccc} E \times \Delta_1 & \xrightarrow{F} & E \\ \downarrow{\scriptstyle p \times 1} & & \downarrow{\scriptstyle p} \\ B \times \Delta_1 & \xrightarrow{f} & B \end{array}$$

such that $F(\sigma, \tau) = \sigma$ *for* $\sigma \in E'$, $\tau \in \Delta_1$,
$F(\sigma, s_0^q 0) = \sigma$ *for* $\sigma \in E_q$
$F(\sigma, s_0^q 1) \in E_q'$ *for* $\sigma \in E_q$, *and*
$f(\sigma, \tau) = \sigma$ *for* $\sigma \in B$, $\tau \in \Delta_1$.

Henceforth when we speak of a minimal subcomplex of a complex X we will usually mean one which is a deformation retract of X, and similarly in the case of fibre spaces minimal sub fibre spaces will usually mean one which is a deformation retract of the original.

THEOREM 2.17. *Let* (E, p, B) *be a fibre space of connected Kan complexes, and let* (E', p, B') *and* (E'', p, B'') *be minimal fibre spaces which are deformation retracts of* (E, p, B), *then* (E', p, B') *and* (E'', p, B'') *are isomorphic.*

PROPOSITION 2.18. *If* (E, p, B) *is a minimal fibre space of connected complexes, and* $\varepsilon^n = (E^{(n+1)}, p, E^{(n)})$ *is the* n^{th} *term in the natural Postnikov system* ε, *then* $p : E^{(n+1)} \to E^{(n)}$ *is a minimal fibre map.*

We have now reduced the problem of studying either complexes or fibre spaces with the extension condition to the problem of studying minimal ones. Further we have seen that all of these things are put together out of Eilenberg-MacLane complexes. Consequently we wish to make this process more explicit, to see how unique it is, and to see its relationship without homotopy type. These questions will be dealt with in inverse order.

DEFINITION 2.19. If (E, p, B) and (E', p, B') are fibre spaces then a map of the first into the second is a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\bar{f}} & E' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{f} & B' \end{array}$$

of semi-simplicial complexes. Such a mapping $(\bar{f}, f)$ is homotopic to a mapping $(\tilde{g}, g)$ if there exists a commutative diagram of semi simplicial complexes

$$\begin{array}{ccc} E \times \Delta_1 & \xrightarrow{\tilde{F}} & E' \\ \downarrow{\scriptstyle p \times 1} & & \downarrow{\scriptstyle p'} \\ B \times \Delta_1 & \xrightarrow{F} & B' \end{array}$$

such that

(1) $\tilde{F}(\sigma, s_0^q 0) = \bar{f}(\sigma)$, and
(2) $\tilde{F}(\sigma, s_0^q 1) = \tilde{g}(\sigma)$.

DEFINITIONS 2.20. Two semi simplicial complexes X and Y have the same *homotopy type* if and only if there exist maps $f : X \to Y$ and $g : Y \to X$ such that fg is homotopic to the identity map of Y and gf is homotopic to the identity map of X.

Two fibre spaces (E, p, B) and (E', p', B') have the same *homotopy type* if and only if there exist maps $(\bar{f}, f) : (E, p, B) \to (E', p', B')$ and $(\tilde{g}, g) : (E', p', B') \to (E, p, B)$ such that $(\bar{f}, f)(\tilde{g}, g)$ is homotopic to the identity map of (E', p', B') and $(\tilde{g}, g)(\bar{f}, f)$ is homotopic to the identity map of (E, p, B).

LEMMA 2.21. *If* X *and* Y *are connected minimal complexes such that* $\pi_q(X) = \pi_q(Y) = 0$ *for* $q \neq n$, *and* $\varphi : \pi_n(X) \to \pi_n(Y)$ *is a homomorphism, then there is a unique map* $f : X \to Y$ *such that* $f^\# = \varphi : \pi_n(X) \to \pi_n(Y)$.

COROLLARY. *If* X *and* Y *are connected minimal complexes such that* $\pi_n(X) \cong \pi_n(Y)$ *and* $\pi_q(X) = \pi_q(Y) = 0$ *for* $q \neq n$, *then* X *is isomorphic with* Y.

Thus we see that two minimal complexes of type (π, n) are isomorphic. Later we will see how to prove the existence of such complexes after the fashion of Eilenberg-MacLane.

PROPOSITION 2.22. *If* X *and* Y *are connected minimal complexes and* $f : X \to Y$ *is a map such that* $f^\# : \pi_q(X) \to \pi_q(Y)$ *for all* q, *then* f *is an isomorphism.*

THEOREM 2.23. *Let* X *and* Y *be connected Kan complexes. The following conditions are equivalent:*

(1) X *and* Y *have the same homotopy type,*
(2) *there is a map* $f : X \to Y$ *such that*

$$f^\# : \pi_q(X) \xrightarrow{\approx} \pi_q(Y) \text{ for all } q, \text{ and}$$

(3) X *and* Y *have isomorphic minimal subcomplexes.*

The fact that conditions 1 and 2 in the preceding theorem are equivalent is in the framework of CW-complexes a well-known theorem of J. H. C. Whitehead [10]. We now give the analogue of this theorem for fibre spaces.

THEOREM 2.24. *Let* $(E, p, B,)$ *and* (E', p', B') *be fibre spaces of connected Kan complexes. The following conditions are equivalent:*

(1) (E, p, B) *and* (E', p', B') *have the same homotopy type,*
(2) *there is a map* $(\bar{f}, f) : (E, p, B) \to (E', p', B')$ *such that at least two of the following conditions are verified.*

$$\text{(i) } f^\# : \pi_q(E) \xrightarrow{\approx} \pi_q(E') \text{ for all } q$$

$$\text{(ii) } f^\# : \pi_q(B) \xrightarrow{\approx} \pi_q(B') \text{ for all } q$$

$$\text{(iii) } f^\# : \pi_q(F) \xrightarrow{\approx} \pi_q(F') \text{ for all } q$$

where F *and* F' *are the fibres in the respective fibre spaces, and*
(3) (E, p, B) *and* (E', p', B') *have isomorphic minimal sub fibre spaces.*

With this theorem we complete our study of fibre spaces from an elementary point of view. Now we pass on to study them in more detail using cohomology and twisted Cartesian products. The notion of twisted Cartesian product is not an invariant one, but any principal fibre space, or any minimal fibre space may be given such a structure. Further any Postnikov system which is minimal may be constructed as a series of twisted Cartesian products.

§3. Twisted Cartesian products and monoid complexes

DEFINITION 3.1. A *twisted Cartesian product* is a triple (F, B, E) such that $F, B, E)$ are semi-simplicial complexes with $E_q = \{(a, b)\} | a \in F_q$ and $b \in B_q.\}$ Defining $p : E \to B$ by $p(a, b) = b$ and $i_b : F \to E$ by $i_b(a) = (a, s_0^q b)$ for b a point of B and $a \in F_q$ we assume further

(1) p is a semi-simplicial map,
(2) i_b is a semi-simplicial map for any point b in B, and
(3) $\partial_i(a, b) = (\partial_i a, \partial_i b)$ for $i > 0$, and

$$s_i(a, b) = (s_i a, s_i b) \text{ for } i \geqq 0.$$

F is called the fibre of the twisted Cartesian product, B the base, and E the total space or total complex. Usually, but not always, the map p will be a fibre map as defined earlier. Notice that E is the Cartesian product of F and B if and only if $\partial_0(a, b) = (\partial_0 a, \partial_0 b)$ for (a, b) a positive dimensional simplex of E.

PROPOSITION 3.2. *Let (F, B, E) be a twisted Cartesian product, and define $\tau : E_{q+1} \to F_r$ by the equation $\partial_0(a, b) = (\tau(a, b), \partial_0 b)$, then τ satisfies the identities.*

(1) $\tau(\partial_1 a, \partial_1 b) = \tau(\tau(a, b), \partial_0 b)$,
(2) $\tau(\partial_{i+1} a, \partial_{i+1} b) = \partial_i \tau(a, b)$ *for* $i > 0$
(3) $\tau(s_0 a, s_0 b) = a$,
(4) $\tau(s_{i+1} a, s_{i+1} b) = s_i \tau(a, b)$, *and*
(5) $\tau(a, s_0^q b) = \partial_0 a$ *for* b *a point of* B.

Further if F and B are semi simplicial complexes and $\tau : (F \times B)_{q+1} \to F_q$ is a function satisfying identities 1 through 5 above, then one can define a unique twisted Cartesian product (F, B, E) so that in E $\partial_0(a, b) = (\tau(a, b), \partial_0 b)$.

The function τ of the preceding proposition is known as a *twisting function, and* the proposition establishes a one to one correspondence between twisting functions $\tau : F \times B \to F$ and twisted Cartesian products (F, B, E).

DEFINITIONS 3.3. A semi-simplical complex Γ is a *monoid complex* if
(1) Γ_q is a monoid with identity for each q, and
(2) $\partial_i : \Gamma_{q+1} \to \Gamma_q$ and $s_i : \Gamma_q \to \Gamma_{q+1}$
are homomorphisms of monoids with identity elements. We will denote by e_q or 1_q the identity of Γ_q.

Γ is a *group complex* if Γ is a monoid complex and each Γ_q is a group. When each Γ_q is abelian, Γ will be called an abelian monoid complex, or an abelian group complex as the case may be. When Γ is a group complex and $x \in \Gamma_q$, the inverse of x will be denoted by $\bar{x}$.

Notice that if G is a topological space and there is given a map of $G \times G \to G$ which makes G into a monoid with identity, then the total singular complex of G is a monoid complex which is abelian if and only if G is abelian. Further if G is a topological group, then the total singular complex of G is a group complex.

THEOREM 3.4. *If Γ is a group complex, then Γ is a Kan complex.*

A proof of this fact may be found in [4].

DEFINITION 3.5. A *monoid complex with homotopy* is a monoid complex which is a Kan complex. In this case $\pi_q(\Gamma, e_0)$ will be denoted by $\pi_q(\Gamma)$.

PROPOSITION 3.6. *If Γ is a monoid complex with homotopy, then $\pi_1(\Gamma)$ is abelian, and if $x, y \in \Gamma_q$ are elements such that $\partial_i x = \partial_i y = e_{q-1}$ for $i = 0, \cdots, q$, then $[x], [y] \in \pi_q(\Gamma)$ and $[x][y] = [xy]$.*

The preceding proposition gives the analogue of the classical results that the group operations in the homotopy groups of a topological group come from the group operation in the group, and that the fundamental group of a topological group is abelian.

Now for group complexes we wish to define homotopy groups in an alternative fashion.

DEFINITION 3.7. If Γ is a group complex, define

$$\tilde{\pi}_q(\Gamma) = \cap_{j=0}^{q-1} \text{kernel } \partial_i : \Gamma_q \to \Gamma_{q-1}, \text{ and}$$
$$\tilde{\pi}(\Gamma) = \textstyle\sum_q \tilde{\pi}_q(\Gamma).$$

PROPOSITION 3.8. *If Γ is a group complex, then image $\partial_{q+1} : \tilde{\pi}_{q+1}(\Gamma) \to \Gamma_q$ is a normal subgroup of Γ_q contained in kernel $\partial_q : \tilde{\pi}_q(\Gamma) \to \tilde{\pi}_{q-1}(\Gamma)$.*

DEFINITION 3.9. For any group complex Γ, consider $\tilde{\pi}(\Gamma)$ as a chain complex (not necessarily abelian) with respect to the last face operator. Define

$$\pi'_q(\Gamma) = H_q(\tilde{\pi}(\Gamma)).$$

PROPOSITION 3.10. *If Γ is a group complex, then*
(1) $\pi'_q(\Gamma) = \pi_q(\Gamma)$ *for all* q, *and*
(2) Γ *is minimal if and only if* $\partial_{q+1} : \tilde{\pi}_{q+1}(\Gamma) \to \tilde{\pi}_q(\Gamma)$ *is zero for all* q.

DEFINITIONS 3.11. A twisted Cartesian product (Γ, B, E) is principal if
(1) Γ is a monoid complex, and
(2) the function $f : \Gamma \times E \to E$ defined by $f(a', (a, b)) = (a'a, b)$ is a semi-simplicial map.

PROPOSITION 3.12. *If (Γ, B, E) is a principal twisted Cartesian product and τ its twisting function, then $\tau(a,b) = \partial_0 a \tau(e_q, b)$. Defining $\tau' : B_q \to \Gamma_{q-1}$ by $\tau', (b) = \tau(e_q,b)$ w° have*

(1) $\tau'(\partial_1 b) = \partial_0 \tau'(b) \tau'(\partial_0) b)$,
(2) $\tau' \partial_{i+1} = \partial_i \tau'$ *for* $i > 0$,
(3) $\tau'(s_0)b = e_q$ *for* $b \in B_q$, *and*
(4) $\tau' s_{i+1} = s_i \tau'$.

Further if $\tau' : B_{q+1} \to \Gamma_q$ is a function satisfying the preceding identities, then there is a unique twisted Cartesian product (Γ, B, E) such that $\partial_0(a, b) = (\partial_0 a \tau'(b), \partial_0 b)$.

PROPOSITION 3.13. *Let (F^1, B^1, B^2) and (F^2, B^2, B^3) be twisted Cartesian products, such that B^2 and B^3 have a single vertex, with twisting functions τ_1, and τ_2. Denote by $i_1 : F^1 \to B^2$ the inclusion map. Then there are twisted Cartesian products $(F^2, F^1, F^{1,2})$ with twisting function τ^1 and $(F^{1,2}, B^1, B^3)$ with twisting function τ^2, where $\tau^1(a, b) = \tau_1(a, i_1(b))$ and $\tau^2(a, b, c) = (\tau_2(a, b, c), \tau_1(b, c))$.*

This proposition is the analogue of the well-known theorem that if $B^3 \to B^2$ is a fibre map and $B^2 \to B^1$ is a fibre map, then $B^3 \to B^1$ by composition is a fibre map.

PROPOSITION 3.14. *If (Γ, B, E) is a principal twisted, Cartesian product where Γ is a group complex, then $E \to B$ is a fibre map.*

DEFINITION 3.15. Let (F, B, E) and (F', B', E') be twisted Cartesian products. A map of (F, B, E) into (F', B', E') is a map $f : F \to F'$, a map $g : B \to B'$ and a map $h : E \to E'$ such that $h(a, b) = (f(a), g(b))$. Any such map h is said to be compatible with the map $f : F \to F'$ of the fibres.

DEFINITION 3.16. A principal twisted Cartesian product (Γ, B, E) is of type (W) if B_0 has a single element and $\partial_0 : e_{q+1} \times B_{q+1} \to E_r$ is one to one.

THEOREM 3.17. *Let (Γ, B, E) and (Γ', B', E') be principal twisted Cartesian products, the second being of type (W), and suppose $f : \Gamma \to \Gamma'$ be a map of monoid*

Complexes, then there is a unique map $\bar{f}: (\Gamma, B, E) \to (\Gamma', B', E')$ *compatible with* f.

COROLLARY 3.18. *If* (Γ, B, E) and (Γ, B', E') *are twisted Cartesian products of type* (W) *there is a unique isomorphism between them compatible with the identity map* $i: \Gamma \to \Gamma$.

THEOREM 3.19. *If* Γ *is a monoid complex, then there is a twisted Cartesian product* $(\Gamma, \bar{W}(\Gamma), W(\Gamma))$ *of type* (W), *and* $W(\Gamma)$ *is acyclic.*

This theorem was originally proved by MacLane [11], and is an extension of work of Eilenberg and MacLane who gave an explicit description of $\bar{W}(\Gamma)$[6], without introducing $W(\Gamma)$. More details concerning the W-construction $(\Gamma, \bar{W}(\Gamma), W(\Gamma))$ may be found in [4]. Geometrically one thinks of Γ as corresponding to a topological group, $\bar{W}(\Gamma)$ to its classifying space, and $W(\Gamma)$ as the contractible fibre bundle over $\bar{W}(\Gamma)$ with fibre Γ.

THEOREM 3.20. *Let* Γ *be a connected monoid complex. Then*

(1) *if* Γ *is minimal,* Γ *is group complex and* $(\Gamma, \bar{W}(\Gamma), W(\Gamma))$ *is a minimal fibre space,*

(2) *if* Γ *is a group complex, then* $\bar{W}(\Gamma)$ *and* $W(\Gamma)$ *are Kan complexes, and*

(3) *if* Γ *is abelian there is a unique map of the twisted Cartesian product* $(\Gamma \times \Gamma, \bar{W}(\Gamma) \times \bar{W}(\Gamma), W(\Gamma) \times W(\Gamma))$ *into* $(\Gamma, \bar{W}(\Gamma), W(\Gamma))$ *compatible with the multiplication map* $\Gamma \times \Gamma \to \Gamma$, *and this map makes* $\bar{W}(\Gamma)$ *and* $W(\Gamma)$ *into abelian monoid complexes.*

DEFINITION 3.21. If Γ is an abelian monoid complex, let $\bar{W}(\Gamma)^0 = \Gamma$, $\bar{W}(\Gamma)^1 = \bar{W}(\Gamma)$, and $\bar{W}(\Gamma)^{n+1} = \bar{W}(\bar{W}(\Gamma)^n)$.

DEFINITION 3.22. If π is a group, let $K(\pi, 0)$ be the group complex such that $K(\pi, 0)_0 = \pi$ and $\partial_i: K(\pi, 0)_{q+1} \to K(\pi, 0)_q$ s_i: $K(\pi, 0)_q \to K(\pi, 0)_{q+1}$ are isomorphisms.

THEOREM 3.23. *If* π *is an abelian group, then* $\bar{W}(K(\pi, 0))^n = K(\pi, n)$.

Recall that $K(\pi, n)$ was the unique minimal complex with its n-dimensional homotopy group π and all others zero. Then the preceding theorem ([6]) gives the existence of such complexes. There is also a well-known explicit description of $K(\pi, n)$ by letting $K(\pi, n)_q = Z^n(\Delta_q, \pi)$ (for details concerning this see [6]).

Now we want to reconstruct Postnikov systems, but before doing so it is necessary to introduce the notion of induced twisted Cartesian product.

DEFINITION 3.24. Let (F, B, E) be a twisted Cartesian product and $f: X \to B$ a map. Define (F, X, E_f) to be the twisted Cartesian product with twisting function $\tau' = \tau(i \times f)$ where $i \times f: F \times F \to F \times B$ and $\tau: F \times B \to F$ is the twisting function of the twisted Cartesian product (F, B, E).

Henceforth we will assume familiarity with the homology and cohomology theory of semi simplicial complexes.

NOTATION. If X is a semi simplicial complex $C(X)_N$ denotes the normalized chain complex of X. Further if π is an abelian group we will denote by $C^q(X; \pi)$ the group of normalized q-cochains of X, i.e., an element of $C^q(X; \pi)$ is a function on X_q with values in π which vanishes on degenerate q-simplexes. Let $Z^q(X; \pi)$ be the sub-group of $C^q(X; \pi)$ consisting of cocycles, i.e. such that if $x \in X_{q+1}$ and $f \in Z^q(X; \pi)$, then $\sum(-1)^i f(\partial_i x) = 0$.

Now we are in a position to state the well-known theorem of Eilenberg-MacLane concerning mappings into $K(\pi, n)$.

THEOREM 3.25. *Let* X *be a semi-simplicial complex. For any map* $f: X \to W(K(\pi, n))$ *let* $\bar{f}$ *be the n-cochain of* X *which is* $f|X_n$. *Then we have*

(1) *the correspondence* $f \to \bar{f}$ *between maps of* $X \to W(K(\pi, n))$ *is one to one,*

(2) $f: X \to K(\pi, n)$ *if and only if* $\bar{f} \in Z^n(X; \pi)$, *and*

(3) *the correspondence induces a natural isomorphism between homotopy classes of maps of* X *into* $K(\pi, n)$ *and* $H^n(X; \pi)$.

In other words $(W(K(\pi, n)^X)_0 = C^n(X; \pi)$ $(K(\pi, n)^X)_0 = Z^n(X; \pi)$, and $\pi_0(K(\pi, n)^X) = H^n(X; \pi)$. Notice that since $K(\pi, n)$, $W(K(\pi, n)$ and $K(\pi, n+1)$ are group complexes, the space of mapping of X into one of these complexes is a group complex, and the above isomorphisms are isomorphisms of groups. Further the map $W(K(\pi, n)) \to K(\pi, n+1)$ just induces the map $\delta: C^n(X; \pi) \to C^{n+1}(X; \pi)$.

THEOREM 3.26. *Let* X *be a connected minimal complex,* $\pi_n = \pi_n(X)$, *and* k^{n+2} *a cocycle representing the obstruction to a cross section of the fibre map* $p: X^{(n+1)} \to X^{(n)}$. *Then*

(1) $X^{(1)} = \bar{W}(K(\pi_0, 0))$, *and*

(2) if $f^{n+2}: X^{(n)} \to K(\pi_{n+1}, n+2)$ *is the mapping corresponding to* k^{n+2}, *there is an isomorphism between* $X^{(n+1)}$ *and the total space of the twisted Cartesian product* $(K(\pi_n, n), X^{(n)}, W_f n+2)$ *induced by* f^{n+2} *from the twisted Cartesian product* $(K(\pi_n, n), K(\pi_n, n+1), W(K(\pi_n, n)))$, *and this isomorphism makes the fibre spaces* $(X^{(n+1)}, p, X^{(n)})$ *into a twisted Cartesian product* $(K(\pi_n, n), X^{(n)}, X^{(n+1)})$.

Now it is clear how one can construct any minimal complex. Suppose there is given an infinite sequence $(\pi_1, \pi_2, \cdots, \pi_n, \cdots)$ of groups such that π_i is abelian for $i > 1$, then let $X^{(1)} = \bar{W}(K(\pi_1, 0))$. Suppose k^3 is a 3-cocycle on $X^{(1)}$ with coefficients in π_2, we have $f^3: X^{(1)} \to K(\pi_2, 3)$ and an induced twisted Cartesian product $(K(\pi_2, 2), X^{(1)}, X^{(2)})$. Now if k^4 is a 4-cocycle on $X^{(2)}$ with coefficients in π_3 we have $f^4: X^{(2)} \to K(\pi_3, 4)$ and a twisted Cartesian product $(K(\pi_3, 3), X^{(2)}, X^{(3)})$ etc.

As always, a theorem such as the preceding one is a special case of a more general theorem involving a fibre map instead of the special fibre map into a point. We, therefore proceed to the general case.

THEOREM 3.27. *Let* (E, p, B) *be a minimal fibre space with connected base and fibre, let* F *denote the fibre, and* $\pi_n = \pi_n(F)$. *Suppose further that* k^{n+2} *is a cocycle representing the obstruction to a cross section of the fibre map* $p: E^{(n+1)} \to E^{(n)}$. *We then have*

(1) $E^{(0)} = B$, *and*

(2) *if* $f^{n+2}: E^{(n)} \to K(\pi_{n+1}, n+2)$

is the mapping corresponding to k^{n+2}, *there is an isomorphism between* $E^{(n+1)}$ *and the total space of the twisted Cartesian product* $(K(\pi_{n+1}, n+1), E^{(n)}, W_f n+2)$ *induced by* f^{n+2}, *and this isomorphism makes the fibre space* $(E^{(n+1)}, p, E^{(n)})$ *into a twisted Cartesian product.*

COROLLARY 3.28. *Any minimal fibre space* (E, p, B) *with connected fibre and base may be given the structure of a twisted Cartesian product.*

This corollary says only that the structure of a twisted Cartesian product may be given to any minimal fibre space. The way of doing this is by no means unique. In fact suppose that each fibre space $(E^{(n+1)}, p, E^{(n)})$ in the preceding theorem has been given the structure of a twisted Cartesian product. We then have the twisted Cartesian product $(K(\pi_{n+1}, n+1), E^{(n)}, E^{(n+1)})$. Suppose τ is its twisting function define $k^{(n+2}(X) = \tau(X)$ for X an $n+2$ simplex of $E^{(n)}$, then k^{n+2} is a cocycle which is the obstruction to a cross section. Further define $\tilde{k}^{n+1}(a, x)$ to be a, for (a, x) an $(n+1)$ simplex of $E^{(n+1)}$. Then $\delta\tilde{k}^{n+1} = p^*(k^{n+2})$. Therefore we have k^{n+2} chosen and an $(n+1)$ chain $\tilde{k}^{n+1} \in C^{n+1}(E^{(n+1)}; \pi_{n+1})$ whose coboundary is the cochain $p^*(k^{n+2})$: It is not difficult to see that in order to make $(E^{(n+1)}, p, E^{(n)})$ into a twisted Cartesian product it suffices to choose k^{n+2} and $\tilde{k}^{n+1}$ so that $k^{n+2} \in Z^{n+2}(E^{(n)}; \pi_{n+1})$ is the obstruction to a cross section and $\tilde{k}^{n+1} \in c^{n+1}(E^{(n+1}; \pi_{n+1})$ has the property that $\delta\tilde{k}^{n+1} = p^*k^{n+2}$. In other words so that with the obvious notation we have a commutative diagram

$$\begin{array}{ccc} E^{(n+1)} & \xrightarrow{\tilde{k}^{n+1}} & W(K(\pi_{n+1}, n+1)) \\ \downarrow & & \downarrow \\ E^{(n)} & \xrightarrow{k^{n\ 2}} & (K\pi_{n+1}, n+2). \end{array}$$

There are many applications of the preceding theory, but we will not go into them here. Instead we will content ourselves with one not particularly surprising result which uses only a small part of the preceding theory. Namely, if one has an abelian group complex, then all of its k-invariants are zero.

THEOREM 3.29. *Let* Γ *be a connected abelian group complex, and let* $\pi_n = \pi_n(\Gamma)$. *Then* Γ *has the homotopy type of the infinite Cartesian product* $X_{n=1}^{\infty} K(\pi_n, n)$.

To prove this theorem it suffices to produce a mapping of $H_n(\Gamma) \to \pi_n(\Gamma)$ so that the composite of this map with the natural map of $\pi_n(\Gamma) \to H_n(\Gamma)$ is the identity. For then we can choose an n-cocycle $f^n \in Z^n(\Gamma, \pi_n)$ corresponding to this map, and this determines a map $f^n : \Gamma \to K(\pi_n, n)$ which maps the n-dimensional homotopy group isomorphically. The fact that we can choose such a map follows easily from the following proposition.

PROPOSITION 3.30. *Let* Γ *be a connected abelian group complex, and define* $\delta : \Gamma_n \to \Gamma_{n-1}$ *by* $\delta x = \sum(-1)i\, \partial_i x$, *then* $\pi_n(\Gamma)$ *is the kernel of* $\partial : \Gamma_n \to \Gamma_{n-1}$ *modulo the image of* $\partial : \Gamma_{n+1} \to \Gamma_n$.

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