

18.031 Problem Set 1

Due in class Friday, Jan. 22.

We encourage collaboration on homework in this course. However, if you do your homework in a group, be sure it works to your advantage rather than against you. Good grades for homework you have not thought through translate to poor grades on exams. **You must turn in your own writeups of all problems, and, if you do collaborate, you must write on the front of your solution sheet the names of the students you worked with.**

Because the solutions will be available immediately after the problem sets are due, **no extensions will be possible.**

Problems in both parts are keyed closely to the lectures, and numbered to match them. Try the problems as soon as you can after the indicated lecture. Most problem sets correspond to four lectures, through the Monday or Wednesday before the set is due. Each problem set is graded out of 100 points.

0. (18.03) (a) Write down the general solution of the homogeneous linear constant coefficient ordinary differential equation

$$\ddot{x} + 4\dot{x} + 5x = 0.$$

(b) Write a phrase describing each of the six underlined terms on (a).

(c) Express $z = 1 + \sqrt{3}i$ in “polar form,” $z = |z|e^{i\theta}$. Use this to write $\operatorname{Re}((1 + \sqrt{3}i)e^{2it})$ in the form $a \cos(\omega t) + b \sin(\omega t)$ and in the form $A \cos(\omega t - \phi)$ where a, b, A, ϕ , and ω are real with the last three positive.

(d) Suppose $x(t)$ is *any* nonzero sinusoidal function of angular frequency ω . What’s the phase lag of $\dot{x}(t)$ relative to $x(t)$? What’s the relationship between the amplitude of $\dot{x}(t)$ and the amplitude of $x(t)$?

1. (Tue 19 Jan) Open <http://mathlets.org/mathlets/amplitude-and-phase-2nd-order>. You may have seen this in 18.03. You can see how this system is analyzed in the “Theory” tab. The input signal is the position of the blue block at the end of the plunger, y ; the system response is the position of the yellow mass, x . These two measurements are arranged so that when $x = y$ the spring is relaxed. The graphs of these two functions are shown in blue and yellow.

Set the damping constant $b = 0.50$, the spring constant $k = 4.0$, and the input frequency $\omega = 2.0$.

(a) By rolling over the graphing window, measure the amplitude of the system response. What value does this give for the *gain* of this system?

(b) The time lag t_0 is indicated by a small red bar on the graphing window. Its value is read out at the bottom of the Mathlet. What value of the *phase lag* (relative to the input signal) does this give?

(c) Now verify your findings. Begin by replacing the sinusoidal input signal with a complex exponential one. The differential equation is of the form $p(D)x = q(D)y$ where y is the input signal and x is the system response. What are the characteristic polynomials $p(s)$ and $q(s)$? Use the Exponential Response Formula to determine the complex gain $G(i\omega)$. Write

down the exponential system response if the input signal is e^{2it} . Then use the complex gain to determine the gain and the tangent of the phase lag relative to the input signal. Using a calculator, check that the Mathlet's estimate of the phase lag isn't far off.

(e) What would the gain and phase lag be if the input signal were $5 \sin(2t)$ instead of $\cos(2t)$?

(f) Explain in any way you like why this system, with input signal $\cos(\omega t)$, is modeled by the differential equation displayed in yellow at the top of the screen. In particular, why is the right hand side $k \cos(\omega t)$ and not just $\cos(\omega t)$? What would the right hand side be if the input signal were $5 \sin(\omega t)$ instead of $\cos(\omega t)$?

(g) What are the transients of this system? If we demand "rest" initial conditions, so that $x(0) = 0$ and $\dot{x}(0) = 0$, what transient should be added to the sinusoidal solution? Is this system stable?

(h) (Independent of this Mathlet) Consider the population model $\dot{x} - kx = -ax$, where k is the growth rate of some population and a is a harvest rate (measured as a fraction of the current total population). For what values of a is this system stable? Is there a value of a that leads to what a biologist would call a "stable" population – that is, one that neither blows up nor dies off to zero? Is the system stable (in our sense) for that value of a (or those values of a , if any such exist)?

2. (Wed 20 Jan) Continue with the same applet. Select the [Bode plot]. You can see the graphs of the gain $g(\omega)$ and of the phase gain relative to the input signal, $-\phi(\omega)$, both as a function of ω .

(a) Still with $b = 0.5$ and $k = 4.0$, you can see that $\omega = 2.0$ is pretty close to a resonant peak. What is the exact resonant frequency ω_r (for which $g(\omega)$ is maximal)? Calculate this by writing down $g(\omega)$ and setting the derivative to zero.

(b) Determine the frequency $\omega_{\pi/2}$ at which $\phi(\omega_{\pi/2}) = \pi/2$. Hint: if $\phi = \pi/2$, what is $\text{Re}(G(\omega))$?

(c) The transfer or system function of this system is $G(s) = k/p(s)$, displayed on the Mathlet if you select [Nyquist plot]. Where are the poles of $G(s)$? Suppose that the dashpot is made weaker, so b becomes smaller. What happens to the poles? Where are they when $b = 0$? Please comment on limitations of the mathematical model in this case.

(d) A certain two-spring system has transfer function given by

$$G(s) = \frac{128}{(s^2 + b_1s + 2)(s^2 + b_2s + 64)}.$$

Suppose first that $b_1 = 2$ and $b_2 = 20$. Sketch the pole diagram.

(e) Write down a differential equation in "input/output standard form" $p(D)x = q(D)y$ for which this $W(s)$ is the transfer function (still with $b_1 = 2$ and $b_2 = 20$).

(f) Now suppose that b_1 and b_2 are both small (on the order of 0.1 say). Make a rough sketch of the new pole diagram. Also make a rough sketch of the graph of the gain $|G(i\omega)|$ as a function of ω in the range $\omega = 0$ up to $\omega = 10$. Explain the relationship between these two sketches.

(g) In class we determined the equation $p(D)x = q(D)y$ controlling a series RLC circuit, choosing the impressed voltage V as input signal and the voltage drop across the resistor,

V_R , as system response. Now work out the corresponding equations if we continue to take V as the input signal but now choose the current I or the voltage drops V_C or V_L as system response. In each of these three cases, work out the complex gain $G(i\omega)$. For each case, the gain $|G(i\omega)|$ behaves, for ω large, like a constant times some power of ω : $a\omega^k$. Work out what a and k are for each of the four choices of system response (I, V_C, V_R, V_L).

3. (Thu 21 Jan) This problem will use the table of Laplace transforms posted on the class website. It will also use the Mathlet <http://mathlets.org/mathlets/poles-and-vibrations>.

(a) Let $p(D) = D + 2I$. Assuming rest initial conditions, use the Laplace transform to solve $p(D)x = e^t$. Identify in your solution the exponential solution and the transient.

(b) Solve $\ddot{x} + 2\dot{x} + 2x = \cos(2t)$ with initial condition $x(0) = 1, \dot{x}(0) = 1$, using Laplace transform. Again, in your solution identify the sinusoidal solution and the transient. What if you had had a different initial condition – would the sinusoidal part change?

(c) (i) Compute $\mathcal{L}(e^{at} \cos(bt))$ and $\mathcal{L}(e^{at} \sin(bt))$ using the s -shift law (see the Laplace table).

(ii) Compute the same Laplace transforms by writing $e^{at} \cos(bt)$ and $e^{at} \sin(bt)$ as a linear combination of complex exponentials.

(d) (i) Where are the poles of each of the following Laplace transforms?

$$\mathcal{L}(e^{at}), \quad \mathcal{L}(e^{at} \cos(bt)), \quad \mathcal{L}(e^{at} \sin(bt)).$$

(ii) Let $p(D) = (D + 2I)^2 + I$ and $q(D) = D$. Where are the poles of the transfer functions of the systems controlled by each of the following systems differential equations? (In both systems we consider $f(t)$ to be the input and $x(t)$ the system response.)

$$p(D)x = f, \quad p(D)x = q(D)f.$$