Hörmander's Topological Paley–Wiener Theorem

(Informal class notes, S. Helgason)

The space $\mathcal{D} = \mathcal{D}(\mathbf{R}^n) = \mathcal{C}_c^\infty(\mathbf{R}^n)$ is given the inductive limit topology of the spaces $\mathcal{D}_{\overline{B_j(0)}}$ of functions $\varphi \in \mathcal{D}$ with support in the ball $\overline{B_j(0)} = \{x \in \mathbf{R}^n = |x| \leq 1\}$. This topology can be characterized by the following result of Schwartz (Distributions, p. 67).

Theorem 1. *Given two monotonic sequences*

$$\{\epsilon\} : \epsilon_0\,,\, \epsilon_1\,, \ldots \qquad \epsilon_i \to 0$$

$$\{N\} : N_0\,,\, N_1\,, \ldots \qquad N_i \to \infty$$

let $V(\{\epsilon\}\{n\})$ denote the set of functions $\varphi \in \mathcal{D}$ satisfying for each $j \geq 0$ the conditions:

(1) $$|D^\alpha \varphi(x)| \leq \epsilon_j \text{ for } |\alpha| \leq N_j\,, \quad |x| \geq j\,.$$

Then the sets $V(\{\epsilon\}\,,\,\{N\})$ form a fundamental system of neighborhoods of 0 in $\mathcal{D}$.

Let $A \geq 0$ and $\mathcal{D}_A$ the space $\mathcal{D}_{\overline{B_A(0)}}$ topologized by the seminorms

(2) $$\|f\|_m = \sum_{|\alpha| \leq m} \sup_{|x|<A} |(D^\alpha f)(x)|\,.$$

Also let $\mathcal{H}_A = \mathcal{H}_A(\mathbf{C}^n)$ denote the space of holomorphic functions of exponential type A, that is the space of holomorphic functions φ such that for each $N \in \mathbf{Z}^+$

(3) $$|||\varphi|||_N = \sup_{\zeta \in \mathbf{C}^n} (1+|\zeta|)^N e^{-A|\operatorname{Im}\zeta|} |\varphi(\zeta)| < \infty\,.$$

$\operatorname{Im}\zeta$ denoting the imaginary part of ζ. We topologize $\mathcal{H}_A$ with the seminorms $|||\quad|||_N$.

Theorem 2. *The Fourier transform $f \to \widetilde{f}$ where*

$$\widetilde{f}(\zeta) = \int_{\mathbf{R}^n} f(x) e^{-i\langle x,\zeta\rangle}\, dx\,, \quad \zeta \in \mathbf{C}^n$$

is a homeomorphism of $\mathcal{D}_A$ onto $\mathcal{H}_A$.

Proof:

The Paley–Wiener theorem states that

$$\widetilde{\mathcal{D}}_A = \mathcal{H}_A\,.$$

The continuity statements follow easily from the formulas

$$i^{|\beta|}\zeta^\beta \widetilde{f}(\zeta) = \int_{\mathbf{R}^n} (D^\beta f)(x) e^{-i\langle x,\zeta\rangle}\, dx \tag{4}$$

and the inversion

$$(\mathcal{D}^\alpha f)(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} (i\zeta)^\alpha \widetilde{f}(\zeta) e^{i\langle x,\zeta\rangle}\, d\zeta\,. \tag{5}$$

The space $\mathcal{D}_A$ is complete. If $\widetilde{f_i}$ is a Cauchy sequence in $\mathcal{H}_A$, replacing f in (5) by $f_i - f_j$ we see

Proof:

Let $W(\{\delta\}, \{M\})$ denote the set of $u \in \mathcal{D}$ satisfying (6). Given $k \in \mathbf{Z}^+$ the set

$$W_k = \{u \in \mathcal{D}_{\overline{B_k(0)}} : |\widetilde{u}(\zeta)| \leq \delta_k \frac{1}{(1+|\zeta|)^{M_k}} e^{k|\,\mathrm{Im}\,|}\}$$

is by Theorem 2 a neighborhood of 0 in $\mathcal{D}_{\overline{B_k(0)}}$ and is clearly contained in $W(\{0\}\{M\})$. If V is a convex set containing $W(\{\delta\}, \{M\})$ then $V \cap \mathcal{D}_{\overline{B_k(0)}}$ contains the neighborhood W_k of 0 in $\mathcal{D}_{\overline{B_k(0)}}$ so by the definition of inductive limit V is a neighborhood of 0 in $\mathcal{D}$.

Proving the converse amounts to proving that given $V(\{\epsilon\}, \{N\})$ there exist sequences $\{\delta\}$ $\{M\}$ such that

$$W(\{\delta\}\{M\}) \subset V(\{\epsilon\}, \{N\}).$$

For this we shift the path of integration in

$$u(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} \widetilde{u}(\xi) e^{i\langle x,\xi\rangle}\, d\xi \tag{7}$$

to another one, in which the two weight factors in (3) are comparable. We write

$$\begin{aligned}
&x = (x_1, \ldots, x_n), \quad && x' = (x_1, \ldots, x_{n-1}) \\
&\zeta = (\zeta_1, \ldots, \zeta_n), \quad && \zeta' = (\zeta_1, \ldots, \zeta_{n-1}) \\
&\xi = (\xi_1, \ldots, \xi_n), \quad && \xi' = (\xi_1, \ldots, \xi_{n-1}) \\
&\zeta = \xi + i\eta && \xi, \eta \in \mathbf{R}^n.
\end{aligned}$$

Then

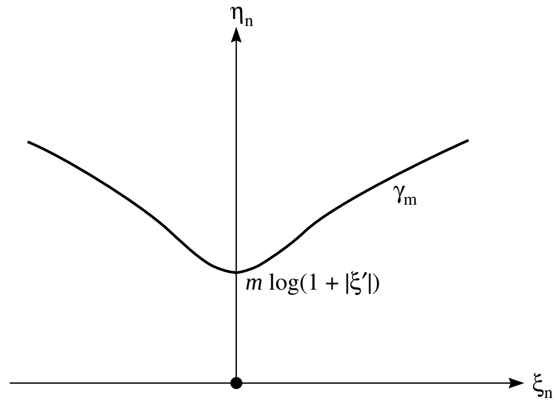
$$\int_{\mathbf{R}^n} \widetilde{u}(\xi) e^{i\langle x,\xi\rangle}\, d\xi = \int_{\mathbf{R}^{n-1}} e^{i\langle x',\xi'\rangle}\, d\xi' \int_{\mathbf{R}} e^{ix_n\xi_n} \widetilde{u}(\xi', \xi_n)\, d\xi_n. \tag{8}$$

In the last integral we shift from $\mathbf{R}$ to the contour in $\mathbf{C}$ given by

$$\gamma_m : \zeta_n = \xi_n + im \log(2 + [|\xi'|^2 + \xi_n^2]^{1/2}), \tag{9}$$

m being arbitrary. We claim that, by Cauchy's theorem

$$\int_{\mathbf{R}} e^{ix_n\xi_n} \widetilde{u}(\xi', \xi_n)\, d\xi_n = \int_{\gamma_m} e^{ix_n\zeta_n} \widetilde{u}(\xi', \zeta_n)\, d\zeta_n. \tag{10}$$



For this we must estimate the right integrand in the "strip" between the ξ_n-axis and the curve γ_m.

The function $\zeta_n \to \widetilde{u}(\xi', \zeta_n)$ satisfies

$$|\widetilde{u}(\xi', \zeta_n)| \leq C_N \frac{e^{A|\operatorname{Im}\, \zeta_n|}}{(1+|\zeta_n|)^N} \tag{11}$$

for some A, all N, the constant C_N depending only on N . On the vertical line joining $(\xi_n, 0)$ to (ξ_n, η_n), $\widetilde{u}(\xi', \zeta_n)$ (with ξ' fixed) decays faster than any power of $|\zeta_n|^{-1}$. Secondly,

$$|e^{ix_n\zeta_n}| \leq e^{|x_n||\eta_n|}\,,$$

which is bounded by a polynomial in $|\zeta_n|$. Also on γ_m

$$\left|\frac{d\zeta_n}{d\xi_n}\right| = \left|1 + im\frac{1}{2+|\xi|}\ \frac{\partial(|\xi|)}{\partial\xi_n}\right| \leq 1 + m \quad (m > 0) \tag{12}$$

thus (10) follows from Cauchy's theorem in *one* variable. Putting

$$\Gamma_m = \{\zeta \in \mathbf{C}^n | \zeta' \in \mathbf{R}^{n-1}\,,\, \zeta_n \in \gamma_m\}$$

and $d\zeta = d\xi_1 \dots d\xi_{n-1}\, d\zeta_n$ we thus have for each $m > 0$

$$u(x) = (2\pi)^{-n} \int\limits_{\Gamma_m} \widetilde{u}(\zeta) e^{i\langle x,\zeta\rangle}\, d\zeta\,. \tag{13}$$

Now suppose the sequences $\{\epsilon\}$, $\{N\}$ and $V(\{\epsilon\},\{N\})$ are given as in Theorem 1. We have to construct sequences $\{\delta\}$ $\{M\}$ such that (6) implies

(1). By rotational invariance we may assume $x = (0, \ldots, 0, x_n)$ with $x_n > 0$. For each n-tuple α we have

$$(14) \qquad (D^\alpha u)(x) = (2\pi)^{-n} \int_{\Gamma_m} \widetilde{u}(\zeta)(i\zeta)^\alpha e^{i\langle x,\zeta\rangle}\, d\zeta\,.$$

Starting with positive sequences $\{\delta\}$, $\{M\}$ we shall modify them successively such that (6) $\Rightarrow$ (1). Note that for $\zeta \in \Gamma_m$

$$(15) \qquad e^{k|\,\mathrm{Im}\ \zeta|} \leq (2 + |\xi|)^{km}$$

$$(16) \qquad |\zeta^\alpha| \leq |\zeta|^{|\alpha|} \leq ([|\xi|^2 + m^2(\log(2 + |\xi|))^2]^{1/2})^{|\alpha|}\,.$$

For (1) with $j = 0$ we take $x_n = |x| \geq 0$, $|\alpha| \leq N_0$ so

$$(17) \qquad |e^{i\langle x,\zeta\rangle}| = e^{-\langle x,\mathrm{Im}\ \zeta|} \leq 1 \quad \text{for } \zeta \in \Gamma_m\,.$$

Thus if u satisfies (6) we have by (12), (15), (16)

$$(18) \quad |(D^\alpha u)(x)| \leq \sum_0^\infty \delta_k \int_{\mathbf{R}^n} (1 + [|\xi|^2 + m^2(\log(2 + |\xi|))^2]^{1/2})^{N_0 - M_k} (2 + |\xi|)^{km}(1 + m)\, d\xi\,.$$

We can choose sequences $\{\delta\}$, $\{M\}$ (all δ_k, $M_k > 0$) such that this expression is $\leq \epsilon_0$. This then verifies (1) for $j = 0$. We now fix δ_0 and M_0. Next we want to prove (1) for $j = 1$ by shrinking the terms in $\delta_1, \delta_2, \ldots$ and increasing the terms in $M_1, M_2, \ldots$ (δ_0, M_0 having been fixed).

Now we have $x_n = |x| \geq 1$ so (17) is replaced by

$$(19) \qquad |e^{i\langle x,\zeta\rangle}| = e^{-\langle x,\mathrm{Im}\ \zeta\rangle} \leq (2 + |\xi|)^{-m} \text{ for } \zeta \in \Gamma_m$$

so in the integrals in (18) the factor $(2+|\xi|)^{km}$ is replaced by $(2+|\xi|)^{(k-1)m}$.

In the sum we separate out the term with $k = 0$. Here M_0 has been fixed but now we have the factor $(2 + |\xi|)^{-m}$ which assures that this $k = 0$ term is $< \frac{\epsilon_1}{2}$ for a sufficiently large m which we now fix. In the remaining terms in (18) (for $k > 0$) we can now increase $1/\delta_k$ and M_k such that the sum is $< \epsilon_1/2$. Thus (1) holds for $j = 1$ and it will remain valid for $j = 0$. We now fix this choice of δ_1 and M_1.

Now the inductive process is clear. We assume $\delta_0, \delta_1, \ldots, \delta_{j-1}$ and $M_0, M_1, \ldots, M_{j-1}$ having been fixed by this shrinking of the δ_i and enlarging of the M_i.

We wish to prove (1) for this j by increasing $1/\delta_k$, M_k for $k \geq j$. Now we have $x_n = |x| \geq j$ and (19) is replaced by

$$|e^{i\langle x,\zeta\rangle}| = e^{-\langle x,\mathrm{Im}\,\zeta\rangle} \leq (2+|\xi|)^{-jm} \tag{20}$$

and since $|\alpha| \leq N_j$, (18) is replaced by

$$(21) \qquad |(D^\alpha f)(x)|$$
$$\leq \sum_{k=0}^{j-1} \delta_k \int\limits_{\mathbf{R}^n} (1+[|\xi|^2 + m^2(\log(2+|\xi|))^2]^{1/2})^{N_j - M_k} (2+|\xi|)^{(k-j)m}(1+m)\, d\xi$$
$$+ \sum_{k \geq j} \delta_k \int\limits_{\mathbf{R}^n} (1+[|\xi|^2 + m^2(\log(2+|\xi|))^2]^{1/2})^{N_j - M_k} (2+|\xi|)^{(k-j)m}(1+m)\, d\xi\,.$$

In the first sum the M_k have been fixed but the factor $(2+|\xi|)^{(k-j)m}$ decays exponentially. Thus we can fix m such that the first sum is $< \frac{\epsilon_j}{2}$.

In the latter sum the $1/\delta_k$ and the M_k can be increased so that the total sum is $< \frac{\epsilon_j}{2}$. This implies the validity of (1) for this particular j and it remains valid for $0, 1, \ldots j-1$. Now we fix δ_j and M_j.

This completes the induction. With this construction of $\{\delta\}$, $\{M\}$ we have proved that $W(\{\delta\},\{M\}) \subset V(\{\epsilon\},\{N\})$. This proves Theorem 3.

Differential Operators with Constant Coefficients

The description of the topology of $\mathcal{D}$ in terms of the range $\widetilde{\mathcal{D}}$ given in Theorem 3 has important consequences for solvability of differential equations on $\mathbf{R}^n$ with constant coefficients.

Theorem 4. *Let $D \neq 0$ be a differential operator on $\mathbf{R}^n$ with constant coefficients. Then the mapping $f \to \mathcal{D}f$ is a homeomorphism of $\mathcal{D}$ onto $D\mathcal{D}$.*

Proof: This proof was shown to me by Hörmander in 1972. A related proof appears in Ehrenpries, *loc. cit.*

It is clear from Theorem 2 that the mapping $f \to Df$ is injective on $\mathcal{D}$. The continuity is also obvious.

For the continuity of the inverse we need the following simple lemma.

Lemma 5. *Let $P \neq 0$ be a polynomial of degree m, F an entire function on $\mathbf{C}^n$ and $G = PF$. Then*

$$|F(\zeta)| \leq C \sup_{|z|\leq 1} |G(z+\zeta)|, \quad \zeta \in \mathbf{C}^n\,,$$

where C is a constant.

Proof: Suppose first $n = 1$ and that $P(z) = \sum_0^m a_k z^k (a_m \neq 0)$. Let $Q(z) = z^m \sum_0^m \overline{a}_k z^{-k}$. Then, by the maximum principle,

$$|a_m F(0)| = |Q(0)F(0)| \leq \max_{|z|=1} |Q(z)F(z)| = \max_{|z|=1} |P(z)F(z)| . \tag{22}$$

For general n let A be an $n \times n$ complex matrix, mapping the ball $|\zeta| < 1$ in $\mathbf{C}^n$ into itself and such that

$$P(A\zeta) = a\zeta_1^m + \sum_0^{m-1} P_k(\zeta_2, \ldots, \zeta_n)\zeta_1^k, \quad a \neq 0 .$$

Let

$$F_1(\zeta) = F(A\zeta), \quad G_1(\zeta) = G(A\zeta), \quad P_1(\zeta) = P(A\zeta) .$$

Then

$$G_1(\zeta_1 + z, \zeta_2, \ldots, \zeta_n) = F_1(\zeta_1 + z, \zeta_2, \ldots, \zeta_n) P_1(\zeta_1 + z, \zeta_2, \ldots, \zeta_n)$$

and the polynomial

$$z \to P_1(\zeta_1 + z, \ldots, \zeta_n)$$

has leading coefficient a. Thus by (22)

$$|aF_1(\zeta)| \leq \max_{|z|=1} |G_1(\zeta_1 + z, \zeta_2, \ldots, \zeta_n)| \leq \max_{\substack{z \in \mathbf{C}^n \\ |z| \leq 1}} |G_1(\zeta + z)| .$$

Hence by the choice of A

$$|aF(\zeta)| \leq \sup_{\substack{z \in \mathbf{C}^n \\ |z| \leq 1}} |G(\zeta + z)|$$

proving the lemma.

For Theorem 4 it remains to prove that if V is a convex neighborhood of 0 in $\mathcal{D}$ then there exists a convex neighborhood W of 0 in $\mathcal{D}$ such that

$$f \in \mathcal{D},\ Df \in W \Rightarrow f \in V . \tag{23}$$

We take V as the neighborhood $W(\{\delta\}, \{M\})$. We shall show that if $W = W(\{\epsilon\}, \{M\})$ (same $\{M\}$) then (26) holds provided the ϵ_j in $\{\epsilon\}$ are small enough. We write $u = Df$ so $\widetilde{u}(\zeta) = P(\zeta)\widetilde{f}(\zeta)$ where P is a polynomial. By Lemma 5

$$|\widetilde{f}(\zeta)| \leq C \sup_{|z| \leq 1} |\widetilde{u}(\zeta + z)| . \tag{24}$$

But $|z| \leq 1$ implies

$$(1 + |z + \zeta|)^{-M_j} \leq 2^{M_j}(1 + |\zeta|)^{-M_j}, \quad |\operatorname{Im}(z + \zeta)| \leq |\operatorname{Im} \zeta| + 1 ,$$

so if $C2^{M_j} e^j \epsilon_j \leq \delta_j$ then (23) holds.

Q.e.d.

Corollary 6. *Let $D \neq 0$ be a differential operator on $\mathbf{R}^n$ with constant (complex) coefficients. Then*

$$D\,\mathcal{D}' = \mathcal{D}' . \tag{25}$$

In particular, there exists a distribution T on $\mathbf{R}^n$ such that

$$DT = \delta . \tag{26}$$

Definition A distribution T satisfying (26) is called a *fundamental solution* for D.

To verify (25) let $L \in \mathcal{D}'$ and consider the functional $D^*u \to L(u)$ on $D^*\mathcal{D}$ ($*$ denoting adjoint). Because of Theorem 2 this functional is well defined and by Theorem 4 it is continuous. By the Hahn-Banach theorem it extends to a distribution $S \in \mathcal{D}'$. Thus $S(D^*u) = Lu$ so $DS = L$, as claimed.

Corollary 7. *Given $f \in \mathcal{D}$ there exists a smooth function u on $\mathbf{R}^n$ such that*

$$Du = f .$$

In fact, if T is a fundamental solution one can put $u = f * T$.