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GROUP REPRESENTATIONS AND SYMMETRIC SPACES

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1. Introduction.

In this lecture I shall discuss some special instances of the following three general problems concerning a homogeneous space G/H, H being a closed subgroup of a Lie group G.

(A) Determine the algebra $\mathbf{D}(G/H)$ of all differential operators on G/H which are invariant under G.

(B) Determine the functions on G/H which are eigenfunctions of each $D \in \mathbf{D}(G/H)$.

(C) For each joint eigenspace for the operators in $\mathbf{D}(G/H)$ study the natural representation of G on this eigenspace ; in particular, when is it irreducible and what representations of G are so obtained ?

Here we shall deal with the case of a symmetric space X of the noncompact type and with the case of the space Ξ of horocycles in X. We refer to [6] for proofs of most of the results reported here.

2. The eigenfunctions of the Laplacian on the non-Euclidean disk.

Let X denote the open unit disk in the plane equipped with the Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{[1 - (x^2 + y^2)]^2} \cdot$$

The corresponding Laplace-Beltrami operator is given by

$$\Delta = [1 - (x^2 + y^2)]^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \cdot$$

We shall begin by stating some recent results about the eigenfunctions of Δ. Let B denote the boundary of X and $P(z, b)$ the Poisson kernel

$$P(z, b) = \frac{1 - |z|^2}{|z - b|^2} \qquad z \in X, \quad b \in B \ .$$

It is then easily verified that if $\mu \in \mathbf{C}$ then $\Delta_z(P(z, b)^\mu) = 4\mu(\mu - 1)P(z, b)^\mu$ so for any measure m on B the function $z \to \int_B P(z, b)^\mu \, dm(b)$ is an eigenfunction of Δ. If $\mu \in \mathbf{R}$ and $m \geqslant 0$ this gives all the positive eigenfunctions of Δ (cf. [1],

[7]). More generally one can take m to be a distribution on B and even more generally, an *analytic functional* on B, that is a continuous linear functional on the space of analytic functions on the boundary B with the customary topology.

THEOREM 1. — *The eigenfunctions of the Laplace-Beltrami operator on the non-Euclidean disk are precisely the functions*

$$f(z) = \int_B P(z, b)^{\mu} \, dT(b) \tag{1}$$

where $\mu \in \mathbf{C}$ *and* T *is an analytic functional on* B.

The functional T is related to the boundary behaviour of f. Assuming, as we can, that μ in (1) satisfies $\operatorname{Re} \mu \geqslant 1/2$ we have as $|z| \to 1$

$$c_{\mu}(1 - |z|^2)^{\mu - 1} f(z) \to T \qquad c_{\mu} = \Gamma(\mu)^2/\Gamma(2\mu - 1) \tag{2}$$

in the sense that the Fourier series of the left hand side converge formally for $|z| \to 1$ to the Fourier series of T. (For $\operatorname{Re} \mu = 1/2$ a minor modification of (2) is necessary).

The case $\mu = 1$ in Theorem 1 is closely related to Köthe's Cauchy kernel representation of holomorphic functions by analytic functionals, [9]. For Eisenstein series a result analogous to (2) was proved by John Lewis in his thesis.

It is well known that the eigenspaces of the Laplacian on a sphere are irreducible under the action of the rotation group. The analogous statement for X is in general false : The largest connected group G of isometries of X does not act irreducibly on the space of harmonic functions ($\mu = 1$). In fact, the constants form an invariant subspace. However we have the following result.

THEOREM 2. — *For* $\mu \in \mathbf{C}$ *let* V_{μ} *denote the space of eigenfunctions of* Δ *for the eigenvalue* $4\mu(\mu - 1)$ *with the topology induced by that of* $C^{\infty}(X)$. *Then* G *acts irreducibly on* V_{μ} *if and only if* μ *is not an integer.*

3. The Fourier transform on a symmetric space X. Spherical functions.

In order to motivate the definition I restate the Fourier inversion formula for $\mathbf{R}^n$ in a suggestive form. If $f \in L^1(\mathbf{R}^n)$ and (,) denotes the inner product on $\mathbf{R}^n$ the Fourier transform $\widetilde{f}$ is defined by

$$\widetilde{f}(\lambda\omega) = \int_{\mathbf{R}^n} f(x) \, e^{-i\lambda(x, \omega)} \, dx \qquad \lambda \geqslant 0 , |\omega| = 1 ,$$

and if for example $f \in C_c^{\infty}(\mathbf{R}^n)$ we have

$$f(x) = (2\pi)^{-n} \iint_{\mathbf{R}^+ \times \mathbf{S}^{n-1}} \widetilde{f}(\lambda\omega) \, e^{i\lambda(x, \omega)} \lambda^{n-1} \, d\lambda \, d\omega \tag{3}$$

where $\mathbf{R}^+$ denotes the set of nonnegative reals and $d\omega$ is the surface element on $\mathbf{S}^{n-1}$.

Now consider a symmetric space X of the noncompact type, that is a coset space $X = G/K$ where G is a connected semisimple Lie group with finite center and K a maximal compact subgroup. We fix an Iwasawa decomposition $G = KAN$

of G, A and N being abelian and nilpotent, respectively. The horocycles in X are the orbits in X of the subgroups of G conjugate to N ; the group G permutes the horocycles transitively and the set Ξ of all horocycles is naturally identified with the coset space G/MN where M is the centralizer of A in K. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$, $\mathfrak{m}$ denote the respective Lie algebras of the groups introduced and log the inverse of the map exp : $\mathfrak{a} \to A$. It is clear from the above that each $\xi \in \Xi$ can be written $\xi = kaMN$, where $kM \in K/M$ and $a \in A$ are unique. Here the coset kM is called the *normal* to ξ and a the *complex distance* from the origin o in X to ξ. If $x \in X$, $b \in B (= K/M)$ there exists exactly one horocycle, denoted $\xi(x , b)$, through x with normal b. Let $a(x , b) \in A$ denote the complex distance from o to $\xi(x , b)$ and put $A(x , b) = \log a(x , b)$. This element of $\mathfrak{a}$ is the symmetric space analog of the inner product (x , ω) in $\mathbf{R}^n$. Denoting by $\mathfrak{a}^*$ the dual space of $\mathfrak{a}$ and defining $\rho \in \mathfrak{a}^*$ by $\rho(H) = ½ \operatorname{Tr}(\operatorname{ad} H \mid \mathfrak{n})$, where ad is adjoint representation and $\mid$ restriction, we can define the *Fourier transform* $\tilde{f}$ of a function $f \in C_c^\infty(X)$ by

$$(4) \qquad \tilde{f}(\lambda , b) = \int_X f(x)\, e^{(-i\lambda+\rho)(A(x,b))}\, dx, \qquad \lambda \in \mathfrak{a}^* , \ b \in B ,$$

dx denoting the volume element on X, suitably normalized. The inversion formula for this Fourier transtorm is

$$(5) \qquad f(x) = w^{-1} \int_{\mathfrak{a}^*} \int_B \tilde{f}(\lambda , b)\, e^{(i\lambda+\rho)(A(x,b))}\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\, db ,$$

where w is the order of the Weyl group W of X, db the normalized K-invariant measure on B and $\mathbf{c}(\lambda)$ Harish-Chandra's function which can be expressed explicitly in terms of Γ-functions as we shall explain later in more detail.

A *spherical function* on X is by definition a K-invariant eigenfunction φ of each G-invariant differential operator on X, normalized by $\varphi(o) = 1$. By a simple reformulation of a theorem of Harish-Chandra the spherical functions are just the functions

$$(6) \qquad \varphi_\lambda(x) = \int_B e^{(i\lambda+\rho)(A(x,b))}\, db$$

λ being arbitrary in the complex dual $\mathfrak{a}_c^*$; also $\varphi_\lambda = \varphi_\mu$ if and only if $\lambda = s\mu$ for some $s \in W$. The $\mathbf{c}$-function arises in Harish-Chandra's work from a study of the behaviour of $\varphi_\lambda(x)$ for large x ; roughly speaking, $\varphi_\lambda(a)$ behaves for large a in the Weyl chamber A^+ as $\Sigma_{s\in W}\, \mathbf{c}(s\lambda)\, e^{(is\lambda-\rho)(\log a)}$ if $\lambda \in \mathfrak{a}^*$.

If f in (4) is K-invariant, then $\tilde{f}$ is independent of b and by use of (6) formula (5) reduces to Harish-Chandra's inversion formula for the spherical Fourier transform. On the other hand the general formula (5) can be derived quite easily from this special case, [5].

It is of course of interest to characterize the images of various function spaces on X under the Fourier transform $f \to \tilde{f}$. In this regard we have the following result (where $\mathfrak{a}_+^*$ denotes the positive Weyl chamber in $\mathfrak{a}^*$).

THEOREM 3. – *The Fourier transform* $f \to \tilde{f}$ *extends to an isometry of* $L^2(X)$ *onto* $L^2(\mathfrak{a}_+^* \times B)$ (*with the measure* $|\mathbf{c}(\lambda)|^{-2}\, d\lambda\, db$).

A point $x \in X$ is called *regular* if the geodesic (ox) has stabilizer of minimum dimension. Since $(K/M) \times A^+$ is by the "polar coordinate representation" identified with the set X' of all regular points in X, Theorem 3 shows that "*X is self-dual under the Fourier transform*". The c-function is given by

$$\mathbf{c}(\lambda) = c_0 \prod_{\alpha \in P^+} \frac{\Gamma(\langle i\lambda, \alpha_0 \rangle)\, 2^{-i\langle \lambda, \alpha_0 \rangle}}{\Gamma\left(\frac{1}{2}\left(\frac{1}{2} m_\alpha + 1 + \langle i\lambda, \alpha_0 \rangle\right)\right) \Gamma\left(\frac{1}{2}\left(\frac{1}{2} m_\alpha + m_{2\alpha} + \langle i\lambda, \alpha_0 \rangle\right)\right)}$$

where c_0 is a constant. Here P^+ is the set of positive roots which are not integral multiples of other positive roots, m_α and $m_{2\alpha}$ are the multiplicities, $\langle\ ,\ \rangle$ the inner product on $\mathfrak{a}_c^*$ and $\alpha_0 = \alpha / \langle \alpha, \alpha \rangle$. This formula was proved by Harish-Chandra and Bhanu-Murthy in special cases and by Gindikin and Karpelevič [2] in general. Every detail in this formula has turned out to be conceptually significant : the location of the singularities for the Paley-Wiener theorem, the asymptotic behaviour for the Fourier transform of rapidly decreasing functions, the Radon transform on X is inverted by a differential operator (not just a pseudo-differential operator) if and only if $\mathbf{c}^{-1}$ is a polynomial ; using the formula for $\mathbf{c}$ one shows [6] that this happens exactly when all Cartan subgroups of G are conjugate. Finally, we shall now see that the numerator and the denominator have their individual importance. We have in fact the following generalization of Theorem 2.

Let X have rank 1, i.e. $\dim A = 1$, let Δ denote the Laplace-Beltrami operator on X and for $\lambda \in \mathfrak{a}_c^*$ $c_\lambda \in \mathbf{C}$ the eigenvalue given by $\Delta\varphi_\lambda = c_\lambda \varphi_\lambda$. Let $\mathcal{E}_\lambda$ be the eigenspace of Δ for the eigenvalue c_λ, this space taken with the topology induced by the usual topology of $C^\infty(X)$.

THEOREM 4. — *Let* $\mathbf{e}(\lambda)^{-1}$ *denote the denominator in the expression for* $\mathbf{c}(\lambda)$. *Then the natural representation of* G *on* $\mathcal{E}_\lambda$ *is irreducible if and only if*

$$\mathbf{e}(\lambda)\, \mathbf{e}(-\lambda) \neq 0 .$$

4. The conical distribution on Ξ.

The spaces

$$X = G/K, \ \Xi = G/MN$$

have many analogies reminiscent of the duality between points and hyperplanes in $\mathbf{R}^n$. For example, we have the following natural identifications for the orbit spaces of K on X, MN on Ξ,

$$K\backslash G/K = A/W, \quad MN\backslash G/MN = A \times W . \tag{7}$$

In the spirit of this analogy we define the counterparts to the spherical functions.

DEFINITION. — A distribution on Ξ is called a *conical distribution* if it is an MN-invariant eigendistribution of each G-invariant differential operator on Ξ.

Since by (6) the set of spherical functions is parametrized by $\mathfrak{a}_c^*/W$, the identifications (7) suggest that the set of conical distributions should somehow be parametrized by $\mathfrak{a}_c^* \times W$. We shall now explain how this turns out to be essentially so. The Bruhat decomposition for G implies that Ξ decomposes into finitely many disjoint orbits under MNA

$$\Xi = \bigcup_{s \in W} \Xi_s \,, \quad \Xi_s = MNA \cdot \xi_s \,.$$

There is a natural measure $d\nu$ on the orbit Ξ_s and if $\lambda \in \mathfrak{a}_c^*$ we consider the functional

$$\Phi_{\lambda,s} : \varphi \to \int_{\Xi_s} \varphi(\xi)\, e^{(is\lambda + s\rho)\,(\log a(\xi))}\, d\nu(\xi) \,, \quad \varphi \in C_c^\infty(\Xi) \,,$$

where $a(\xi)$ denotes the A-component of $\xi \in \Xi_s$. Since Ξ_s is not in general closed there is no guarantee of convergence. However one does have absolute convergence for all $\varphi \in C_c^\infty(\Xi)$ if and only if $\mathrm{Re}(\langle i\lambda , \alpha \rangle) > 0$ for all

$$\alpha \in P^+ \cap s^{-1} P^- \qquad (P^- = -P^+) \,.$$

If this is the case, $\Phi_{\lambda,s}$ is a conical distribution. One would now like to obtain a meromorphic continuation of the distribution-valued function $\lambda \to \Phi_{\lambda,s}$ because then all the values and "residues" of this extension would still be conical distributions. Remarkably enough it turns out that the singularities are the same as those in the numerator for the c-function except that one restricts the product to $P^+ \cap s^{-1} P^-$. Thus we have

THEOREM 5. – *Let* $s \in W$, $\alpha_0 = \alpha / \langle \alpha , \alpha \rangle$, $\mathrm{d}_s(\lambda) = \prod_{\alpha \in P^+ \cap s^{-1} P^-} \Gamma(\langle i\lambda , \alpha_0 \rangle)$. *Then the mapping*

$$\lambda \to \Psi_{\lambda,s} = \frac{1}{\mathrm{d}_s(\lambda)}\, \Phi_{\lambda,s}$$

extends to an entire function on $\mathfrak{a}_c^*$.

The residues of $\Phi_{\lambda,s}$, that is the values of $\Psi_{\lambda,s}$ at the removable singularities λ_0, have the following geometric interpretation. The closure of Ξ_s in Ξ is a union of Ξ_s and some other orbits $\Xi_{s'}$. Then the residue $\mathrm{Res}_{\lambda=\lambda_0} \Phi_{\lambda,s}$ is a linear combination of certain transversal derivatives of the various $\Psi_{\lambda,s'}$ constructed from the other orbits in the closure.

We have now to each $(\lambda , s) \in \mathfrak{a}_c^* \times W$ associated a conical distribution $\Psi_{\lambda,s}$. One can now prove ([6], Ch. III) that essentially all the conical distributions arise in this manner. This is done by transferring the differential equations for the conical distributions to differential equations on X by means of the dual to the Radon transform. Under this transform a conical distribution is sent into a C^∞ function so the differential equations become easier to handle. Thus our preliminary guess that the set of conical distributions can be parametrized by $\mathfrak{a}_c^* \times W$ is essentially verified.

5. Eigenspaces of invariant differential operators on Ξ.

Let $\mathrm{D}(\Xi)$ denote the algebra of all G-invariant differential operators on Ξ. For $\lambda \in \mathfrak{a}_c^*$ let the eigenvalue $\gamma_\lambda(D)$ be determined by $D\Psi_{\lambda,s} = \gamma_\lambda(D)\, \Psi_{\lambda,s}$ for all $D \in \mathrm{D}(\Xi)$. As indicated by the notation, the eigenvalue is independent of $s \in W$. Let $\mathcal{D}'(\Xi)$ denote the set of all distributions on Ξ and put

$$\mathcal{D}'_\lambda = \{ \Psi \in \mathcal{D}'(\Xi) \mid D\Psi = \gamma_\lambda(D)\, \Psi \quad \text{for} \quad D \in \mathrm{D}(\Xi) \} \,.$$

Each eigendistribution of the operators in $\mathbf{D}(\Xi)$ lies in one of the spaces $\mathcal{D}'_\lambda$. Let τ_λ denote the natural representation of G on the distribution space $\mathcal{D}'_\lambda$ (strong distribution topology).

THEOREM 6. — *The representation τ_λ is irreducible if and only if* $\mathbf{e}(\lambda)\,\mathbf{e}(-\lambda) \neq 0$.

This is proved by relating the representation τ_λ to the Hilbert space representation π_λ of G induced by the one-dimensional representation $man \to e^{i\lambda(\log a)}$ of MAN. According to Harish-Chandra [3], Theorem 5, irreducibility of π_λ is equivalent to the (algebraic) irreducibility of the representation $d\pi_\lambda$ of $\mathfrak{g}$ on the space of K-finite vectors in the Hilbert space and a criterion for the algebraic irreducibility of $d\pi_\lambda$ is given by Kostant [8], p. 63. For X of rank one an entirely different proof of Theorem 6 is given in [6].

The distributions $\Psi_{\lambda,s}$ all belong to $\mathcal{D}'_\lambda$ and play the role of *extreme weight vectors* for the infinite-dimensional representation τ_λ.

If the irreducibility condition for τ_λ is satisfied the representations τ_λ and $\tau_{s\lambda}$ are equivalent for each $s \in W$. The intertwining operator realizing this equivalence is a kind of a convolution operator by means of the conical distribution $\Psi_{s\lambda,s^{-1}}$. More precisely, the map $S \to \Psi$ given by

$$\Psi(\varphi) = \int_{K/M} \left(\int_A e^{(i\lambda+\rho)(\log a)} \varphi(kaMN)\, da \right) dS(kM)$$

is a bijection of $\mathcal{D}'(B)$ onto $\mathcal{D}'_\lambda$, Thus τ_λ can be regarded as a representation of G on $\mathcal{D}'(B)$. If $S_{s\lambda,s^{-1}} \in \mathcal{D}'(B)$ corresponds to $\Psi_{s\lambda,s^{-1}}$ then the convolution operator $S \to S \times S_{s\lambda,s^{-1}}$ sets up the equivalence between τ_λ and τ_{s_λ}. The relationship of these results with the work of Knapp, Kunze, Schiffmann, Stein and Zhelobenko on intertwining operators is explained in [6], Ch. III, § 6.

For $\lambda \in \mathfrak{a}^*$ the intertwining operator is very simply described in terms of the Fourier transform $\tilde{f}(\lambda, b)$. In fact, the space $\{\tilde{f}(\lambda, .) \mid f \in C_c^\infty(X)\}$ is dense in $L^2(B)$ as well as in $\mathcal{D}'(B)$ and the operator $\tilde{f}(\lambda, .) \to \tilde{f}(s\lambda, .)$ extends to an isometry of $L^2(B)$ onto itself and induces the operator which intertwines τ_λ and $\tau_{s\lambda}$.

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