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Radon Transform
Second Edition

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Preface to the Second Edition

The first edition of this book has been out of print for some time and I have decided to follow the publisher's kind suggestion to prepare a new edition. Many examples of the explicit inversion formulas and range theorems have been added, and the group-theoretic viewpoint emphasized. For example, the integral geometric viewpoint of the Poisson integral for the disk leads to interesting analogies with the X-ray transform in Euclidean 3-space. To preserve the introductory flavor of the book the short and self-contained Chapter V on Schwartz' distributions has been added. Here §5 provides proofs of the needed results about the Riesz potentials while §§3–4 develop the tools from Fourier analysis following closely the account in Hörmander's books [1963] and [1983]. There is some overlap with my books [1984] and [1994b] which, however, rely heavily on Lie group theory. The present book is much more elementary.

I am indebted to Sine Jensen for a critical reading of parts of the manuscript and to Hilgert and Schlichtkrull for concrete contributions mentioned at specific places in the text. Finally I thank Jan Wetzel and Bonnie Friedman for their patient and skillful preparation of the manuscript.

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Preface to the First Edition

The title of this booklet refers to a topic in geometric analysis which has its origins in results of Funk [1916] and Radon [1917] determining, respectively, a symmetric function on the two-sphere $\mathbf{S}^2$ from its great circle integrals and a function of the plane $\mathbf{R}^2$ from its line integrals. (See references.) Recent developments, in particular applications to partial differential equations, X-ray technology, and radio astronomy, have widened interest in the subject.

These notes consist of a revision of lectures given at MIT in the Fall of 1966, based mosty on my papers during 1959–1965 on the Radon transform and its generalizations. (The term "Radon Transform" is adopted from John [1955].)

The viewpoint for these generalizations is as follows.

The set of points on $\mathbf{S}^2$ and the set of great circles on $\mathbf{S}^2$ are both homogeneous spaces of the orthoginal group $O(3)$. Similarly, the set of points in $\mathbf{R}^2$ and the set of lines in $\mathbf{R}^2$ are both homogeneous spaces of the group $\mathbf{M}(2)$ of rigid motions of $\mathbf{R}^2$. This motivates our general Radon transform definition from [1965a, 1966a] which forms the framwork of Chapter II: Given two homogeneous spaces G/K and G/H of the same group G, the Radon transform $u \to \widehat{u}$ maps functions u on the first space to functions $\widehat{u}$ on the second space. For $\xi \in G/H$, $\widehat{u}(\xi)$ is defined as the (natural) integral of u over the set of points $x \in G/K$ which are incident to ξ in the sense of Chern [1942]. The problem of inverting $u \to \widehat{u}$ is worked out in a few cases.

It happens when G/K is a Euclidean space, and more generally when G/K is a Riemannian symmetric space, that the natural differential operators A on G/K are transferred by $u \to \widehat{u}$ into much more manageable differential operators $\widehat{A}$ on G/H; the connection is $(Au)\widehat{} = \widehat{A}\widehat{u}$. Then the theory of the transform $u \to \widehat{u}$ has significant applications to the study of properties of A.

On the other hand, the applications of the original Radon transform on $\mathbf{R}^2$ to X-ray technology and radio astronomy are based on the fact that for an unknown density u, X-ray attenuation measurements give $\widehat{u}$ directly and therefore yield u via Radon's inversion formula. More precisely, let B be a convex body, $u(x)$ its density at the point x, and suppose a thin beam of X-rays is directed at B along a line ξ. Then the line integral $\widehat{u}(\xi)$ of u along ξ equals $\log(I_o/I)$ where I_o and I, respectively, are the intensities of the beam before hitting B and after leaving B. Thus while the function u is at first unknown, the function $\widehat{u}$ is determined by the X-ray data.

The lecture notes indicated above have been updated a bit by including a short account of some applications (Chapter I, §7), by adding a few cprollaries (Corollaries 2.8 and 2.12, Theorem 6.3 in Chapter I, Corollaries 2.8

and 4.1 in Chapter III), and by giving indications in the bibliographical notes of some recent developments.

An effort has been made to keep the exposition rather elementary. The distribution theory and the theory of Riesz potentials, occasionally needed in Chapter I, is reviewed in some detail in §8 (now Chapter V). Apart from the general homogeneous space framework in Chapter II, the treatment is restricted to Euclidean and isotropic spaces (spaces which are "the same in all directions"). For more general symmetric spaces the treatment is postposed (except for §4 in Chapter III) to another occasion since more machinery from the theorem of semisimple Lie groups is required.

I am indebted to R. Melrose and R. Seeley for helpful suggestions and to F. Gonzalez and J. Orloff for critical reading of parts of the manuscript.

Cambridge, MA 1980

CHAPTER I

THE RADON TRANSFORM ON $\mathbf{R}^N$

§1 Introduction

It was proved by J. Radon in 1917 that a differentiable function on $\mathbf{R}^3$ can be determined explicitly by means of its integrals over the planes in $\mathbf{R}^3$. Let $J(\omega, p)$ denote the integral of f over the hyperplane $\langle x, \omega \rangle = p$, ω denoting a unit vector and $\langle\ ,\ \rangle$ the inner product. Then

$$f(x) = -\frac{1}{8\pi^2} L_x \left(\int_{\mathbf{S}^2} J(\omega, \langle \omega, x \rangle)\, d\omega \right),$$

where L is the Laplacian on $\mathbf{R}^3$ and $d\omega$ the area element on the sphere $\mathbf{S}^2$ (cf. Theorem 3.1).

We now observe that the formula above has built in a remarkable duality: first one integrates over the set of points in a hyperplane, then one integrates over the set of hyperplanes passing through a given point. This suggests considering the transforms $f \to \widehat{f}, \varphi \to \check{\varphi}$ defined below.

The formula has another interesting feature. For a fixed ω the integrand $x \to J(\omega, \langle \omega, x \rangle)$ is a *plane wave*, that is a function constant on each plane perpendicular to ω. Ignoring the Laplacian the formula gives a continuous decomposition of f into plane waves. Since a plane wave amounts to a function of just one variable (along the normal to the planes) this decomposition can sometimes reduce a problem for $\mathbf{R}^3$ to a similar problem for $\mathbf{R}$. This principle has been particularly useful in the theory of partial differential equations.

The analog of the formula above for the line integrals is of importance in radiography where the objective is the description of a density function by means of certain line integrals.

In this chapter we discuss relationships between a function on $\mathbf{R}^n$ and its integrals over k-dimensional planes in $\mathbf{R}^n$. The case $k = n - 1$ will be the one of primary interest. We shall occasionally use some facts about Fourier transforms and distributions. This material will be developed in sufficient detail in Chapter V so the treatment should be self-contained.

Following Schwartz [1966] we denote by $\mathcal{E}(\mathbf{R}^n)$ and $\mathcal{D}(\mathbf{R}^n)$, respectively, the space of complex-valued $\mathcal{C}^\infty$ functions (respectively $\mathcal{C}^\infty$ functions of compact support) on $\mathbf{R}^n$. The space $\mathcal{S}(\mathbf{R}^n)$ of rapidly decreasing functions on $\mathbf{R}^n$ is defined in connection with (6) below. $C^m(\mathbf{R}^n)$ denotes the space of m times continuously differentiable functions. We write $C(\mathbf{R}^n)$ for $C^0(\mathbf{R}^n)$, the space of continuous function on $\mathbf{R}^n$.

For a manifold M, $C^m(M)$ (and $C(M)$) is defined similarly and we write $\mathcal{D}(M)$ for $\mathcal{C}_c^\infty(M)$ and $\mathcal{E}(M)$ for $\mathcal{C}^\infty(M)$.

§2 The Radon Transform of the Spaces $\mathcal{D}(\mathbf{R}^n)$ and $\mathcal{S}(\mathbf{R}^n)$. The Support Theorem

Let f be a function on $\mathbf{R}^n$, integrable on each hyperplane in $\mathbf{R}^n$. Let $\mathbf{P}^n$ denote the space of all hyperplanes in $\mathbf{R}^n$, $\mathbf{P}^n$ being furnished with the obvious topology. The *Radon transform* of f is defined as the function $\widehat{f}$ on $\mathbf{P}^n$ given by

$$\widehat{f}(\xi) = \int_\xi f(x)dm(x)\,,$$

where dm is the Euclidean measure on the hyperplane ξ. Along with the transformation $f \to \widehat{f}$ we consider also the *dual transform* $\varphi \to \check{\varphi}$ which to a continuous function φ on $\mathbf{P}^n$ associates the function $\check{\varphi}$ on $\mathbf{R}^n$ given by

$$\check{\varphi}(x) = \int_{x\in\xi} \varphi(\xi)\, d\mu(\xi)$$

where $d\mu$ is the measure on the compact set $\{\xi \in \mathbf{P}^n : x \in \xi\}$ which is invariant under the group of rotations around x and for which the measure of the whole set is 1 (see Fig. I.1). We shall relate certain function spaces on $\mathbf{R}^n$ and on $\mathbf{P}^n$ by means of the transforms $f \to \widehat{f}, \varphi \to \check{\varphi}$; later we obtain explicit inversion formulas.

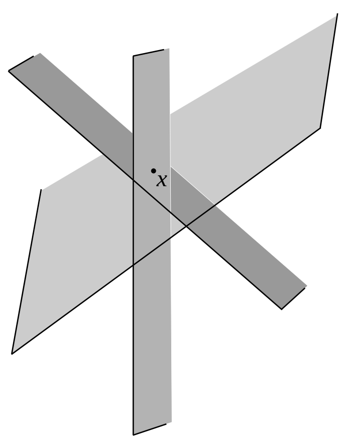

FIGURE I.1.

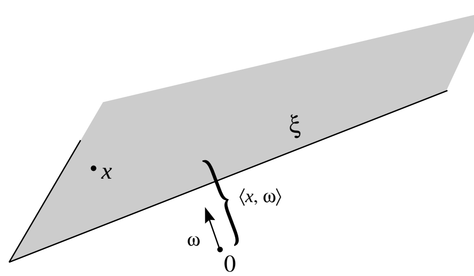

FIGURE I.2.

Each hyperplane $\xi \in \mathbf{P}^n$ can be written $\xi = \{x \in \mathbf{R}^n : \langle x, \omega\rangle = p\}$ where $\langle\ ,\ \rangle$ is the usual inner product, $\omega = (\omega_1, \dots, \omega_n)$ a unit vector and $p \in \mathbf{R}$ (Fig. I.2). Note that the pairs (ω, p) and $(-\omega, -p)$ give the same ξ; the mapping $(\omega, p) \to \xi$ is a double covering of $\mathbf{S}^{n-1} \times \mathbf{R}$ onto $\mathbf{P}^n$. Thus $\mathbf{P}^n$ has a canonical manifold structure with respect to which this covering map is differentiable and regular. We thus identify continuous (differentiable) function on $\mathbf{P}^n$ with continuous (differentiable) functions φ on $\mathbf{S}^{n-1} \times \mathbf{R}$ satisfying the symmetry condition $\varphi(\omega, p) = \varphi(-\omega, -p)$. Writing $\widehat{f}(\omega, p)$ instead of $\widehat{f}(\xi)$ and f_t (with $t \in \mathbf{R}^n$) for the translated function $x \to f(t + x)$ we have

$$\widehat{f_t}(\omega, p) = \int_{\langle x,\omega\rangle = p} f(x + t)\, dm(x) = \int_{\langle y,\omega\rangle = p + \langle t,\omega\rangle} f(y)\, dm(y)$$

so

$$\widehat{f_t}(\omega, p) = \widehat{f}(\omega, p + \langle t, \omega\rangle)\,. \tag{1}$$

Taking limits we see that if $\partial_i = \partial/\partial x_i$

$$(\partial_i f)\widehat{\ }(\omega, p) = \omega_i \frac{\partial \widehat{f}}{\partial p}(\omega, p)\,. \tag{2}$$

Let L denote the Laplacian $\Sigma_i \partial_i^2$ on $\mathbf{R}^n$ and let $\square$ denote the operator

$$\varphi(\omega, p) \to \frac{\partial^2}{\partial p^2}\varphi(\omega, p)\,,$$

which is a well-defined operator on $\mathcal{E}(\mathbf{P}^n) = \mathcal{C}^\infty(\mathbf{P}^n)$. It can be proved that if $\mathbf{M}(n)$ is the group of isometries of $\mathbf{R}^n$, then L (respectively $\square$) generates the algebra of $\mathbf{M}(n)$-invariant differential operators on $\mathbf{R}^n$ (respectively $\mathbf{P}^n$).

Lemma 2.1. *The transforms $f \to \widehat{f}, \varphi \to \check{\varphi}$ intertwine L and $\square$, i.e.,*

$$(Lf)\widehat{\ } = \square(\widehat{f})\,, \quad (\square\varphi)^\vee = L\check{\varphi}\,.$$

Proof. The first relation follows from (2) by iteration. For the second we just note that for a certain constant c

$$\check{\varphi}(x) = c\int_{\mathbf{S}^{n-1}} \varphi(\omega, \langle x, \omega\rangle)\, d\omega\,, \tag{3}$$

where $d\omega$ is the usual measure on $\mathbf{S}^{n-1}$.

The Radon transform is closely connected with the Fourier transform

$$\widetilde{f}(u) = \int_{\mathbf{R}^n} f(x)e^{-i\langle x,\omega\rangle}\, dx \quad u \in \mathbf{R}^n.$$

In fact, if $s \in \mathbf{R}$, ω a unit vector,

$$\widetilde{f}(s\omega) = \int_{-\infty}^{\infty} dr \int_{\langle x,\omega\rangle = r} f(x)e^{-is\langle x,\omega\rangle}\, dm(x)$$

so

$$\widetilde{f}(s\omega) = \int_{-\infty}^{\infty} \widehat{f}(\omega, r) e^{-isr} \, dr \,. \tag{4}$$

This means that the n-dimensional Fourier transform is the 1-dimensional Fourier transform of the Radon transform. From (4), or directly, it follows that the Radon transform of the convolution

$$f(x) = \int_{\mathbf{R}^n} f_1(x - y) f_2(y) \, dy$$

is the convolution

$$\widehat{f}(\omega, p) = \int_{\mathbf{R}} \widehat{f_1}(\omega, p - q) \widehat{f_2}(\omega, q) \, dq \,. \tag{5}$$

We consider now the space $\mathcal{S}(\mathbf{R}^n)$ of complex-valued rapidly decreasing functions on $\mathbf{R}^n$. We recall that $f \in \mathcal{S}(\mathbf{R}^n)$ if and only if for each polynomial P and each integer $m \geq 0$,

$$\sup_x \big| |x|^m P(\partial_1, \ldots, \partial_n) f(x) \big| < \infty \,, \tag{6}$$

$|x|$ denoting the norm of x. We now formulate this in a more invariant fashion.

Lemma 2.2. *A function $f \in \mathcal{E}(\mathbf{R}^n)$ belongs to $\mathcal{S}(\mathbf{R}^n)$ if and only if for each pair $k, \ell \in \mathbb{Z}^+$*

$$\sup_{x \in \mathbf{R}^n} |(1 + |x|)^k (L^\ell f)(x)| < \infty \,.$$

This is easily proved just by using the Fourier transforms.

In analogy with $\mathcal{S}(\mathbf{R}^n)$ we define $\mathcal{S}(\mathbf{S}^{n-1} \times \mathbf{R})$ as the space of $\mathcal{C}^\infty$ functions φ on $\mathbf{S}^{n-1} \times \mathbf{R}$ which for any integers $k, \ell \geq 0$ and any differential operator D on $\mathbf{S}^{n-1}$ satisfy

$$\sup_{\omega \in \mathbf{S}^{n-1}, r \in \mathbf{R}} \left| (1 + |r|^k) \frac{d^\ell}{dr^\ell} (D\varphi)(\omega, r) \right| < \infty \,. \tag{7}$$

The space $\mathcal{S}(\mathbf{P}^n)$ is then defined as the set of $\varphi \in \mathcal{S}(\mathbf{S}^{n-1} \times \mathbf{R})$ satisfying $\varphi(\omega, p) = \varphi(-\omega, -p)$.

Lemma 2.3. *For each $f \in \mathcal{S}(\mathbf{R}^n)$ the Radon transform $\widehat{f}(\omega, p)$ satisfies the following condition: For $k \in \mathbb{Z}^+$ the integral*

$$\int_{\mathbf{R}} \widehat{f}(\omega, p) p^k \, dp$$

can be written as a k^{th} degree homogeneous polynomial in $\omega_1, \ldots, \omega_n$.

Proof. This is immediate from the relation

(8)
$$\int_{\mathbf{R}} \widehat{f}(\omega,p)p^k\,dp = \int_{\mathbf{R}} p^k\,dp \int_{\langle x,\omega\rangle = p} f(x)\,dm(x) = \int_{\mathbf{R}^n} f(x)\langle x,\omega\rangle^k\,dx\,.$$

In accordance with this lemma we define the space

$$\mathcal{S}_H(\mathbf{P}^n) = \left\{ F \in \mathcal{S}(\mathbf{P}^n) : \begin{array}{l} \text{For each } k \in \mathbb{Z}^+, \int_{\mathbf{R}} F(\omega,p)p^k\,dp \\ \text{is a homogeneous polynomial} \\ \text{in } \omega_1,\ldots,\omega_n \text{ of degree } k \end{array} \right\}.$$

With the notation $\mathcal{D}(\mathbf{P}^n) = \mathcal{C}_c^\infty(\mathbf{P}^n)$ we write

$$\mathcal{D}_H(\mathbf{P}^n) = \mathcal{S}_H(\mathbf{P}^n) \cap \mathcal{D}(\mathbf{P}^n)\,.$$

According to Schwartz [1966], p. 249, the Fourier transform $f \to \widetilde{f}$ maps the space $\mathcal{S}(\mathbf{R}^n)$ onto itself. See Ch. V, Theorem 3.1. We shall now settle the analogous question for the Radon transform.

Theorem 2.4. *(The Schwartz theorem) The Radon transform $f \to \widehat{f}$ is a linear one-to-one mapping of $\mathcal{S}(\mathbf{R}^n)$ onto $\mathcal{S}_H(\mathbf{P}^n)$.*

Proof. Since
$$\frac{d}{ds}\widetilde{f}(s\omega) = \sum_{i=1}^n \omega_i(\partial_i \widetilde{f})$$
it is clear from (4) that for each fixed ω the function $r \to \widehat{f}(\omega,r)$ lies in $\mathcal{S}(\mathbf{R})$. For each $\omega_0 \in \mathbf{S}^{n-1}$ a subset of $(\omega_1,\ldots,\omega_n)$ will serve as local coordinates on a neighborhood of ω_0 in $\mathbf{S}^{n-1}$. To see that $\widehat{f} \in \mathcal{S}(\mathbf{P}^n)$, it therefore suffices to verify (7) for $\varphi = \widehat{f}$ on an open subset $N \subset \mathbf{S}^{n-1}$ where ω_n is bounded away from 0 and $\omega_1,\ldots,\omega_{n-1}$ serve as coordinates, in terms of which D is expressed. Since

(9)
$$u_1 = s\omega_1,\ldots,u_{n-1} = s\omega_{n-1}\,, \quad u_n = s(1-\omega_1^2-\cdots-\omega_{n-1}^2)^{1/2}$$

we have
$$\frac{\partial}{\partial\omega_i}(\widetilde{f}(s\omega)) = s\frac{\partial\widetilde{f}}{\partial u_i} - s\omega_i(1-\omega_1^2-\cdots-\omega_{n-1}^2)^{-1/2}\frac{\partial\widetilde{f}}{\partial u_n}\,.$$

It follows that if D is any differential operator on $\mathbf{S}^{n-1}$ and if $k,\ell \in \mathbb{Z}^+$ then

(10)
$$\sup_{\omega\in N, s\in\mathbf{R}} \left|(1+s^{2k})\frac{d^\ell}{ds^\ell}(D\widetilde{f})(\omega,s)\right| < \infty\,.$$

We can therefore apply D under the integral sign in the inversion formula to (4),
$$\widehat{f}(\omega,r) = \frac{1}{2\pi}\int_{\mathbf{R}} \widetilde{f}(s\omega)e^{isr}\,ds$$

and obtain

$$(1+r^{2k})\frac{d^\ell}{dr^\ell}\Big(D_\omega(\widehat{f}(\omega,r))\Big)=\frac{1}{2\pi}\int\Big(1+(-1)^k\frac{d^{2k}}{ds^{2k}}\Big)\Big((is)^\ell D_\omega(\widetilde{f}(s\omega))\Big)e^{isr}ds\,.$$

Now (10) shows that $\widehat{f}\in\mathcal{S}(\mathbf{P}^n)$ so by Lemma 2.3, $\widehat{f}\in\mathcal{S}_H(\mathbf{P}^n)$.

Because of (4) and the fact that the Fourier transform is one-to-one it only remains to prove the surjectivity in Theorem 2.4. Let $\varphi\in\mathcal{S}_H(\mathbf{P}^n)$. In order to prove $\varphi=\widehat{f}$ for some $f\in\mathcal{S}(\mathbf{R}^n)$ we put

$$\psi(s,\omega)=\int_{-\infty}^{\infty}\varphi(\omega,r)e^{-irs}\,dr\,.$$

Then $\psi(s,\omega)=\psi(-s,-\omega)$ and $\psi(0,\omega)$ is a homogeneous polynomial of degree 0 in $\omega_1,\ldots,\omega_n$, hence constant. Thus there exists a function F on $\mathbf{R}^n$ such that

$$F(s\omega)=\int_{\mathbf{R}}\varphi(\omega,r)e^{-irs}\,dr\,.$$

While F is clearly smooth away from the origin we shall now prove it to be smooth at the origin too; this is where the homogeneity condition in the definition of $\mathcal{S}_H(\mathbf{P}^n)$ enters decisively. Consider the coordinate neighborhood $N\subset\mathbf{S}^{n-1}$ above and if $h\in\mathcal{C}^\infty(\mathbf{R}^n-0)$ let $h^*(\omega_1,\ldots,\omega_{n-1},s)$ be the function obtained from h by means of the substitution (9). Then

$$\frac{\partial h}{\partial u_i}=\sum_{j=1}^{n-1}\frac{\partial h^*}{\partial\omega_j}\frac{\partial\omega_j}{\partial u_i}+\frac{\partial h^*}{\partial s}\cdot\frac{\partial s}{\partial u_i}\quad(1\le i\le n)$$

and

$$\begin{aligned}\frac{\partial\omega_j}{\partial u_i}&=\frac{1}{s}(\delta_{ij}-\frac{u_iu_j}{s^2})\quad(1\le i\le n\,,\quad 1\le j\le n-1)\,,\\ \frac{\partial s}{\partial u_i}&=\omega_i\quad(1\le i\le n-1),\frac{\partial s}{\partial u_n}=(1-\omega_1^2-\cdots-\omega_{n-1}^2)^{1/2}\,.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial h}{\partial u_i}&=\frac{1}{s}\frac{\partial h^*}{\partial\omega_i}+\omega_i\left(\frac{\partial h^*}{\partial s}-\frac{1}{s}\sum_{j=1}^{n-1}\omega_j\frac{\partial h^*}{\partial\omega_j}\right)\quad(1\le i\le n-1)\\ \frac{\partial h}{\partial u_n}&=(1-\omega_1^2-\cdots-\omega_{n-1}^2)^{1/2}\left(\frac{\partial h^*}{\partial s}-\frac{1}{s}\sum_{j=1}^{n-1}\omega_j\frac{\partial h^*}{\partial\omega_j}\right)\,.\end{aligned}$$

In order to use this for $h=F$ we write

$$F(s\omega)=\int_{-\infty}^{\infty}\varphi(\omega,r)\,dr+\int_{-\infty}^{\infty}\varphi(\omega,r)(e^{-irs}-1)\,dr\,.$$

By assumption the first integral is independent of ω. Thus using (7) we have for constant $K > 0$

$$\left|\frac{1}{s}\frac{\partial}{\partial \omega_i}(F(s\omega))\right| \leq K \int (1+r^4)^{-1} s^{-1} |e^{-isr} - 1|\, dr \leq K \int \frac{|r|}{1+r^4}\, dr$$

and a similar estimate is obvious for $\partial F(s\omega)/\partial s$. The formulas above therefore imply that all the derivatives $\partial F/\partial u_i$ are bounded in a punctured ball $0 < |u| < \epsilon$ so F is certainly continuous at $u = 0$.

More generally, we prove by induction that

(11)
$$\frac{\partial^q h}{\partial u_{i_1} \ldots \partial u_{i_q}} = \sum_{1 \leq i+j \leq q,\, 1 \leq k_1, \cdots, k_i \leq n-1} A_{j,k_1 \ldots k_i}(\omega, s) \frac{\partial^{i+j} h^*}{\partial \omega_{k_1} \ldots \partial \omega_{k_i} \partial s^j}$$

where the coefficients A have the form

$$A_{j,k_1 \ldots k_i}(\omega, s) = a_{j,k_1 \ldots k_i}(\omega) s^{j-q}\,. \tag{12}$$

For $q = 1$ this is in fact proved above. Assuming (11) for q we calculate

$$\frac{\partial^{q+1} h}{\partial u_{i_1} \ldots \partial u_{i_{q+1}}}$$

using the above formulas for $\partial/\partial u_i$. If $A_{j,k_1 \ldots k_i}(\omega, s)$ is differentiated with respect to $u_{i_{q+1}}$ we get a formula like (12) with q replaced by $q + 1$. If on the other hand the $(i + j)^{\text{th}}$ derivative of h^* in (11) is differentiated with respect to $u_{i_{q+1}}$ we get a combination of terms

$$s^{-1} \frac{\partial^{i+j+1} h^*}{\partial \omega_{k_1} \ldots \partial \omega_{k_{i+1}} \partial s^j}\,,\quad \frac{\partial^{i+j+1} h^*}{\partial \omega_{k_1} \ldots \partial \omega_{k_i} \partial s^{j+1}}$$

and in both cases we get coefficients satisfying (12) with q replaced by $q+1$. This proves (11)–(12) in general. Now

$$F(s\omega) = \int_{-\infty}^{\infty} \varphi(\omega, r) \sum_0^{q-1} \frac{(-isr)^k}{k!}\, dr + \int_{-\infty}^{\infty} \varphi(\omega, r) e_q(-irs)\, dr\,, \tag{13}$$

where

$$e_q(t) = \frac{t^q}{q!} + \frac{t^{q+1}}{(q+1)!} + \cdots$$

Our assumption on φ implies that the first integral in (13) is a polynomial in $u_1, \ldots, u_n$ of degree $\leq q-1$ and is therefore annihilated by the differential operator (11). If $0 \leq j \leq q$, we have

$$|s^{j-q} \frac{\partial^j}{\partial s^j}(e_q(-irs))| = |(-ir)^q (-irs)^{j-q} e_{q-j}(-irs)| \leq k_j r^q\,, \tag{14}$$

where k_j is a constant because the function $t \to (it)^{-p} e_p(it)$ is obviously bounded on $\mathbf{R}\,(p \geq 0)$. Since $\varphi \in \mathcal{S}(\mathbf{P}^n)$ it follows from (11)–(14) that each q^{th} order derivative of F with respect to $u_1, \ldots, u_n$ is bounded in a punctured ball $0 < |u| < \epsilon$. Thus we have proved $F \in \mathcal{E}(\mathbf{R}^n)$. That F is rapidly decreasing is now clear from (7), Lemma 2.2 and (11). Finally, if f is the function in $\mathcal{S}(\mathbf{R}^n)$ whose Fourier transform is F then

$$\widetilde{f}(s\omega) = F(s\omega) = \int_{-\infty}^{\infty} \varphi(\omega, r) e^{-irs}\, dr\,;$$

hence by (4), $\widehat{f} = \varphi$ and the theorem is proved.

To make further progress we introduce some useful notation. Let $S_r(x)$ denote the sphere $\{y : |y - x| = r\}$ in $\mathbf{R}^n$ and $A(r)$ its area. Let $B_r(x)$ denote the open ball $\{y : |y - x| < r\}$. For a continuous function f on $S_r(x)$ let $(M^r f)(x)$ denote the mean value

$$(M^r f)(x) = \frac{1}{A(r)} \int_{S_r(x)} f(\omega)\, d\omega\,,$$

where $d\omega$ is the Euclidean measure. Let K denote the orthogonal group $\mathbf{O}(n)$, dk its Haar measure, normalized by $\int dk = 1$. If $y \in \mathbf{R}^n, r = |y|$ then

$$(15)\quad (M^r f)(x) = \int_K f(x + k \cdot y)\, dk\,.$$

(Fig. I.3) In fact, for x, y fixed both sides represent rotation-invariant functionals on $C(S_r(x))$, having the same value for the function $f \equiv 1$. The rotations being transitive on $S_r(x)$, (15) follows from the uniqueness of such invariant functionals. Formula (3) can similarly be written

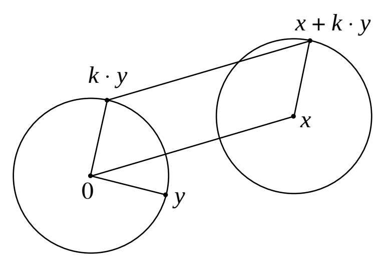

FIGURE I.3.

$$(16)\qquad \check{\varphi}(x) = \int_K \varphi(x + k \cdot \xi_0)\, dk$$

if ξ_0 is some fixed hyperplane through the origin. We see then that if $f(x) = 0(|x|^{-n}), \Omega_k$ the area of the unit sphere in $\mathbf{R}^k$, i.e., $\Omega_k = 2\frac{\pi^{k/2}}{\Gamma(k/2)}$,

$$\begin{aligned}(\widehat{f})^{\vee}(x) &= \int_K \widehat{f}(x + k \cdot \xi_0)\, dk = \int_K \left(\int_{\xi_0} f(x + k \cdot y)\, dm(y) \right)\, dk \\ &= \int_{\xi_0} (M^{|y|} f)(x)\, dm(y) = \Omega_{n-1} \int_0^{\infty} r^{n-2} \left(\frac{1}{\Omega_n} \int_{\mathcal{S}^{n-1}} f(x + r\omega)\, d\omega \right) dr\end{aligned}$$

so

$$(\widehat{f})^\vee(x) = \frac{\Omega_{n-1}}{\Omega_n} \int_{\mathbf{R}^n} |x-y|^{-1} f(y)\, dy\,. \tag{17}$$

We consider now the analog of Theorem 2.4 for the transform $\varphi \to \check{\varphi}$. But $\varphi \in \mathcal{S}_H(\mathbf{P}^n)$ does not imply $\check{\varphi} \in \mathcal{S}(\mathbf{R}^n)$. (If this were so and we by Theorem 2.4 write $\varphi = \widehat{f}, f \in \mathcal{S}(\mathbf{R}^n)$ then the inversion formula in Theorem 3.1 for $n = 3$ would imply $\int f(x)\, dx = 0$.) On a smaller space we shall obtain a more satisfactory result.

Let $\mathcal{S}^*(\mathbf{R}^n)$ denote the space of all functions $f \in \mathcal{S}(\mathbf{R}^n)$ which are orthogonal to all polynomials, i.e.,

$$\int_{\mathbf{R}^n} f(x)P(x)\, dx = 0 \quad \text{for all polynomials } P\,.$$

Similarly, let $\mathcal{S}^*(\mathbf{P}^n) \subset \mathcal{S}(\mathbf{P}^n)$ be the space of φ satisfying

$$\int_{\mathbf{R}} \varphi(\omega, r)p(r)\, dr = 0 \quad \text{for all polynomials } p\,.$$

Note that under the Fourier transform the space $\mathcal{S}^*(\mathbf{R}^n)$ corresponds to the subspace $\mathcal{S}_0(\mathbf{R}^n) \subset \mathcal{S}(\mathbf{R}^n)$ of functions all of whose derivatives vanish at 0.

Corollary 2.5. *The transforms $f \to \widehat{f}, \varphi \to \check{\varphi}$ are bijections of $\mathcal{S}^*(\mathbf{R}^n)$ onto $\mathcal{S}^*(\mathbf{P}^n)$ and of $\mathcal{S}^*(\mathbf{P}^n)$ onto $\mathcal{S}^*(\mathbf{R}^n)$, respectively.*

The first statement is clear from (8) if we take into account the elementary fact that the polynomials $x \to \langle x, \omega\rangle^k$ span the space of homogeneous polynomials of degree k. To see that $\varphi \to \check{\varphi}$ is a bijection of $\mathcal{S}^*(\mathbf{P}^n)$ onto $\mathcal{S}^*(\mathbf{R}^n)$ we use (17), knowing that $\varphi = \widehat{f}$ for some $f \in \mathcal{S}^*(\mathbf{R}^n)$. The right hand side of (17) is the convolution of f with the tempered distribution $|x|^{-1}$ whose Fourier transform is by Chapter V, §5 a constant multiple of $|u|^{1-n}$. (Here we leave out the trivial case $n = 1$.) By Chapter V, (12) this convolution is a tempered distribution whose Fourier transform is a constant multiple of $|u|^{1-n}\widetilde{f}(u)$. But, by Lemma 5.6, Chapter V this lies in the space $\mathcal{S}_0(\mathbf{R}^n)$ since $\widetilde{f}$ does. Now (17) implies that $\check{\varphi} = (\widehat{f})^\vee \in \mathcal{S}^*(\mathbf{R}^n)$ and that $\check{\varphi} \not\equiv 0$ if $\varphi \not\equiv 0$. Finally we see that the mapping $\varphi \to \check{\varphi}$ is surjective because the function

$$((\widehat{f})^\vee)\widetilde{\ }(u) = c|u|^{1-n}\widetilde{f}(u)$$

(where c is a constant) runs through $\mathcal{S}_0(\mathbf{R}^n)$ as f runs through $\mathcal{S}^*(\mathbf{R}^n)$.

We now turn to the space $\mathcal{D}(\mathbf{R}^n)$ and its image under the Radon transform. We begin with a preliminary result. (See Fig. I.4.)

Theorem 2.6. *(The support theorem.) Let $f \in C(\mathbf{R}^n)$ satisfy the following conditions:*

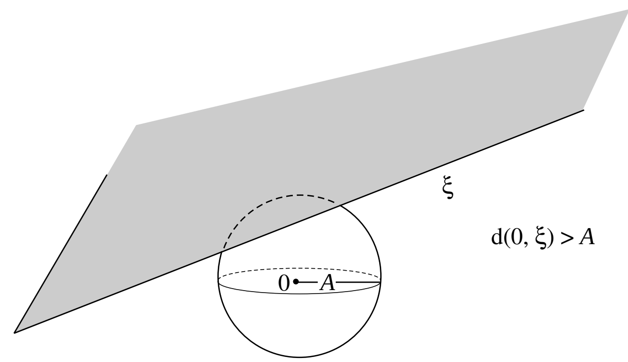

FIGURE I.4.

(i) *For each integer* $k > 0, |x|^k f(x)$ *is bounded.*

(ii) *There exists a constant* $A > 0$ *such that*

$$\widehat{f}(\xi) = 0 \text{ for } d(0, \xi) > A\,,$$

d denoting distance.

Then

$$f(x) = 0 \text{ for } |x| > A\,.$$

Proof. Replacing f by the convolution $\varphi * f$ where φ is a radial $\mathcal{C}^\infty$ function with support in a small ball $B_\epsilon(0)$ we see that it suffices to prove the theorem for $f \in \mathcal{E}(\mathbf{R}^n)$. In fact, $\varphi * f$ is smooth, it satisfies (i) and by (5) it satisfies (ii) with A replaced by $A+\epsilon$. Assuming the theorem for the smooth case we deduce that support $(\varphi * f) \subset B_{A+\epsilon}(0)$ so letting $\epsilon \to 0$ we obtain support $(f) \subset$ Closure $B_A(0)$.

To begin with we assume f is a radial function. Then $f(x) = F(|x|)$ where $F \in \mathcal{E}(\mathbf{R})$ and even. Then $\widehat{f}$ has the form $\widehat{f}(\xi) = \widehat{F}(d(0, \xi))$ where $\widehat{F}$ is given by

$$\widehat{F}(p) = \int_{\mathbf{R}^{n-1}} F((p^2 + |y|^2)^{1/2})\, dm(y)\,, \quad (p \geq 0)$$

because of the definition of the Radon transform. Using polar coordinates in $\mathbf{R}^{n-1}$ we obtain

$$\widehat{F}(p) = \Omega_{n-1} \int_0^\infty F((p^2 + t^2)^{1/2}) t^{n-2}\, dt\,. \tag{18}$$

Here we substitute $s = (p^2 + t^2)^{-1/2}$ and then put $u = p^{-1}$. Then (18) becomes

$$u^{n-3} \widehat{F}(u^{-1}) = \Omega_{n-1} \int_0^u (F(s^{-1}) s^{-n})(u^2 - s^2)^{(n-3)/2}\, ds\,.$$

We write this equation for simplicity

$$h(u) = \int_0^u g(s)(u^2 - s^2)^{(n-3)/2}\, ds\,. \tag{19}$$

This integral equation is very close to Abel's integral equation (Whittaker-Watson [1927], Ch. IX) and can be inverted as follows. Multiplying both sides by $u(t^2 - u^2)^{(n-3)/2}$ and integrating over $0 \le u \le t$ we obtain

$$\begin{aligned}
&\int_0^t h(u)(t^2 - u^2)^{(n-3)/2} u\, du \\
&\quad = \int_0^t \left[\int_0^u g(s)[(u^2 - s^2)(t^2 - u^2)]^{(n-3)/2}\, ds\right] u\, du \\
&\quad = \int_0^t g(s) \left[\int_{u=s}^t u[(t^2 - u^2)(u^2 - s^2)]^{(n-3)/2}\, du\right] ds\,.
\end{aligned}$$

The substitution $(t^2 - s^2)V = (t^2 + s^2) - 2u^2$ gives an explicit evaluation of the inner integral and we obtain

$$\int_0^t h(u)(t^2 - u^2)^{(n-3)/2} u\, du = C \int_0^t g(s)(t^2 - s^2)^{n-2}\, ds\,,$$

where $C = 2^{1-n}\pi^{\frac{1}{2}}\Gamma((n-1)/2)/\Gamma(n/2)$. Here we apply the operator $\frac{d}{d(t^2)} = \frac{1}{2t}\frac{d}{dt}$ $(n-1)$ times whereby the right hand side gives a constant multiple of $t^{-1}g(t)$. Hence we obtain

$$F(t^{-1})t^{-n} = ct\left[\frac{d}{d(t^2)}\right]^{n-1} \int_0^t (t^2 - u^2)^{(n-3)/2} u^{n-2} \widehat{F}(u^{-1})\, du \tag{20}$$

where $c^{-1} = (n-2)!\Omega_n/2^n$. By assumption (ii) we have $\widehat{F}(u^{-1}) = 0$ if $u^{-1} \ge A$, that is if $u \le A^{-1}$. But then (20) implies $F(t^{-1}) = 0$ if $t \le A^{-1}$, that is if $t^{-1} \ge A$. This proves the theorem for the case when f is radial.

We consider next the case of a general f. Fix $x \in \mathbf{R}^n$ and consider the function

$$g_x(y) = \int_K f(x + k \cdot y)\, dk$$

as in (15). Then g_x satisfies (i) and

$$\widehat{g}_x(\xi) = \int_K \widehat{f}(x + k \cdot \xi)\, dk\,, \tag{21}$$

$x + k \cdot \xi$ denoting the translate of the hyperplane $k \cdot \xi$ by x. The triangle inequality shows that

$$d(0, x + k \cdot \xi) \geq d(0, \xi) - |x| , \quad x \in \mathbf{R}^n, k \in K .$$

Hence we conclude from assumption (ii) and (21) that

$$\widehat{g}_x(\xi) = 0 \quad \text{if} \quad d(0, \xi) > A + |x| . \tag{22}$$

But g_x is a radial function so (22) implies by the first part of the proof that

$$\int_K f(x + k \cdot y)\, dk = 0 \quad \text{if} \quad |y| > A + |x| . \tag{23}$$

Geometrically, this formula reads: The surface integral of f over $S_{|y|}(x)$ is 0 if the ball $B_{|y|}(x)$ contains the ball $B_A(0)$. The theorem is therefore a consequence of the following lemma.

Lemma 2.7. *Let $f \in C(\mathbf{R}^n)$ be such that for each integer $k > 0$,*

$$\sup_{x \in \mathbf{R}^n} |x|^k |f(x)| < \infty .$$

Suppose f has surface integral 0 over every sphere S which encloses the unit ball. Then $f(x) \equiv 0$ for $|x| > 1$.

Proof. The idea is to perturb S in the relation

$$\int_S f(s)\, d\omega(s) = 0 \tag{24}$$

slightly, and differentiate with respect to the parameter of the perturbations, thereby obtaining additional relations. (See Fig. I.5.) Replacing, as above, f with a suitable convolution $\varphi * f$ we see that it suffices to prove the lemma for f in $\mathcal{E}(\mathbf{R}^n)$. Writing $S = S_R(x)$ and viewing the exterior of the ball $B_R(x)$ as a union of spheres with center x we have by the assumptions,

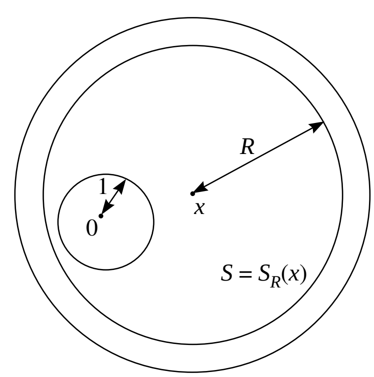

FIGURE I.5.

$$\int_{B_R(x)} f(y)\, dy = \int_{\mathbf{R}^n} f(y)\, dy ,$$

which is a constant. Differentiating with respect to x_i we obtain

$$\int_{B_R(0)} (\partial_i f)(x+y)\,dy = 0\,. \tag{25}$$

We use now the divergence theorem

$$\int_{B_R(0)} (\mathrm{div} F)(y)\,dy = \int_{S_R(0)} \langle F, \mathbf{n}\rangle(s)\,d\omega(s) \tag{26}$$

for a vector field F on $\mathbf{R}^n, \mathbf{n}$ denoting the outgoing unit normal and $d\omega$ the surface element on $S_R(0)$. For the vector field $F(y) = f(x+y)\frac{\partial}{\partial y_i}$ we obtain from (25) and (26), since $\mathbf{n} = R^{-1}(s_1, \ldots, s_n)$,

$$\int_{S_R(0)} f(x+s)s_i\,d\omega(s) = 0\,. \tag{27}$$

But by (24)

$$\int_{S_R(0)} f(x+s)x_i\,d\omega(s) = 0$$

so by adding

$$\int_S f(s)s_i\,d\omega(s) = 0\,.$$

This means that the hypotheses of the lemma hold for $f(x)$ replaced by the function $x_i f(x)$. By iteration

$$\int_S f(s)P(s)\,d\omega(s) = 0$$

for any polynomial P, so $f \equiv 0$ on S. This proves the lemma as well as Theorem 2.6.

Corollary 2.8. *Let $f \in C(\mathbf{R}^n)$ satisfy (i) in Theorem 2.6 and assume*

$$\widehat{f}(\xi) = 0$$

for all hyperplanes ξ disjoint from a certain compact convex set C. Then

$$f(x) = 0 \quad \textit{for} \quad x \notin C\,. \tag{28}$$

In fact, if B is a closed ball containing C we have by Theorem 2.6, $f(x) = 0$ for $x \notin B$. But C is the intersection of such balls so (28) follows.

Remark 2.9. While condition (i) of rapid decrease entered in the proof of Lemma 2.7 (we used $|x|^k f(x) \in L^1(\mathbf{R}^n)$ for each $k > 0$) one may wonder whether it could not be weakened in Theorem 2.6 and perhaps even dropped in Lemma 2.7.

As an example, showing that the condition of rapid decrease can not be dropped in either result consider for $n = 2$ the function

$$f(x, y) = (x + iy)^{-5}$$

made smooth in $\mathbf{R}^2$ by changing it in a small disk around 0. Using Cauchy's theorem for a large semicircle we have $\int_\ell f(x)\, dm(x) = 0$ for every line ℓ outside the unit circle. Thus (ii) is satisfied in Theorem 2.6. Hence (i) cannot be dropped or weakened substantially.

This same example works for Lemma 2.7. In fact, let S be a circle $|z - z_0| = r$ enclosing the unit disk. Then $d\omega(s) = -ir\frac{dz}{z-z_0}$ so, by expanding the contour or by residue calculus,

$$\int_S z^{-5}(z - z_0)^{-1}\, dz = 0\,,$$

(the residue at $z = 0$ and $z = z_0$ cancel) so we have in fact

$$\int_S f(s)\, d\omega(s) = 0\,.$$

We recall now that $\mathcal{D}_H(\mathbf{P}^n)$ is the space of symmetric $\mathcal{C}^\infty$ functions $\varphi(\xi) = \varphi(\omega, p)$ on $\mathbf{P}^n$ of compact support such that for each $k \in \mathbb{Z}^+$, $\int_{\mathbf{R}} \varphi(\omega, p)p^k\, dp$ is a homogeneous kth degree polynomial in $\omega_1, \ldots, \omega_n$. Combining Theorems 2.4, 2.6 we obtain the following characterization of the Radon transform of the space $\mathcal{D}(\mathbf{R}^n)$. This can be regarded as the analog for the Radon transform of the Paley-Wiener theorem for the Fourier transform (see Chapter V).

Theorem 2.10. *(The Paley-Wiener theorem.) The Radon transform is a bijection of $\mathcal{D}(\mathbf{R}^n)$ onto $\mathcal{D}_H(\mathbf{P}^n)$.*

We conclude this section with a variation and a consequence of Theorem 2.6.

Lemma 2.11. *Let $f \in C_c(\mathbf{R}^n)$, $A > 0$, ω_0 a fixed unit vector and $N \subset S$ a neighborhood of ω_0 in the unit sphere $S \subset \mathbf{R}^n$. Assume*

$$\widehat{f}(\omega, p) = 0 \quad \textit{for} \quad \omega \in N,\, p > A\,.$$

Then

$$f(x) = 0 \textit{ in the half-space } \langle x, \omega_0 \rangle > A\,. \tag{29}$$

Proof. Let B be a closed ball around the origin containing the support of f. Let $\epsilon > 0$ and let H_ϵ be the union of the half spaces $\langle x, \omega \rangle > A + \epsilon$ as ω runs through N. Then by our assumption

$$\widehat{f}(\xi) = 0 \quad \text{if} \quad \xi \subset H_\epsilon\,. \tag{30}$$

Now choose a ball B_ϵ with a center on the ray from 0 through $-\omega_0$, with the point $(A+2\epsilon)\omega_0$ on the boundary, and with radius so large that any hyperplane ξ intersecting B but not B_ϵ must be in H_ϵ. Then by (30)

$$\widehat{f}(\xi)=0 \quad \text{whenever} \quad \xi \in \mathbf{P}^n,\ \xi \cap B_\epsilon = \emptyset\,.$$

Hence by Theorem 2.6, $f(x)=0$ for $x \notin B_\epsilon$. In particular, $f(x)=0$ for $\langle x,\omega_0\rangle > A+2\epsilon$; since $\epsilon>0$ is arbitrary, the lemma follows.

Corollary 2.12. *Let N be any open subset of the unit sphere $\mathbf{S}^{n-1}$. If $f \in C_c(\mathbf{R}^n)$ and*

$$\widehat{f}(\omega,p)=0 \quad \textit{for } p \in \mathbf{R}, \omega \in N$$

then

$$f \equiv 0\,.$$

Since $\widehat{f}(-\omega,-p)=\widehat{f}(\omega,p)$ this is obvious from Lemma 2.11.

§3 The Inversion Formula

We shall now establish explicit inversion formulas for the Radon transform $f \to \widehat{f}$ and its dual $\varphi \to \check{\varphi}$.

Theorem 3.1. *The function f can be recovered from the Radon transform by means of the following inversion formula*

$$cf = (-L)^{(n-1)/2}((\widehat{f})^\vee) \quad f \in \mathcal{E}(\mathbf{R}^n)\,, \tag{31}$$

provided $f(x)=0(|x|^{-N})$ for some $N>n$. Here c is the constant

$$c=(4\pi)^{(n-1)/2}\Gamma(n/2)/\Gamma(1/2)\,.$$

Proof. We use the connection between the powers of L and the Riesz potentials in Chapter V, §5. Using (17) we in fact have

$$(\widehat{f})^\vee = 2^{n-1}\pi^{\frac{n}{2}-1}\Gamma(n/2)I^{n-1}f\,. \tag{32}$$

By Chapter V, Proposition 5.7, we thus obtain the desired formula (31).

For n odd one can give a more geometric proof of (31). We start with some general useful facts about the mean value operator M^r. It is a familiar fact that if $f \in C^2(\mathbf{R}^n)$ is a radial function, i.e., $f(x) = F(r), r = |x|$, then

$$(Lf)(x) = \frac{d^2F}{dr^2} + \frac{n-1}{r}\frac{dF}{dr}\,. \tag{33}$$

This is immediate from the relations

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial^2 f}{\partial r^2}\left(\frac{\partial r}{\partial x_i}\right)^2 + \frac{\partial f}{\partial r}\frac{\partial^2 r}{\partial x_i^2}\,.$$

Lemma 3.2. *(i)* $LM^r = M^r L$ *for each* $r > 0$.

(ii) For $f \in C^2(\mathbf{R}^n)$ *the mean value* $(M^r f)(x)$ *satisfies the "Darboux equation"*

$$L_x\big((M^r f)(x)\big) = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right)(M^r f(x))\,,$$

that is, the function $F(x,y) = (M^{|y|}f)(x)$ *satisfies*

$$L_x(F(x,y)) = L_y(F(x,y))\,.$$

Proof. We prove this group theoretically, using expression (15) for the mean value. For $z \in \mathbf{R}^n, k \in K$ let T_z denote the translation $x \to x + z$ and R_k the rotation $x \to k \cdot x$. Since L is invariant under these transformations, we have if $r = |y|$,

$$\begin{aligned}
(LM^r f)(x) &= \int_K L_x(f(x + k\cdot y))\,dk = \int_K (Lf)(x + k\cdot y)\,dk = (M^r Lf)(x)\\
&= \int_K [(Lf)\circ T_x \circ R_k](y)\,dk = \int_K [L(f\circ T_x\circ R_k)](y)\,dk\\
&= L_y\left(\int_K f(x + k\cdot y)\right)\,dk
\end{aligned}$$

which proves the lemma.

Now suppose $f \in \mathcal{S}(\mathbf{R}^n)$. Fix a hyperplane ξ_0 through 0, and an isometry $g \in \mathbf{M}(n)$. As k runs through $\mathbf{O}(n)$, $gk\cdot\xi_0$ runs through the set of hyperplanes through $g\cdot 0$, and we have

$$\check{\varphi}(g\cdot 0) = \int_K \varphi(gk\cdot\xi_0)\,dk$$

and therefore

$$\begin{aligned}
(\widehat{f})^\vee(g\cdot 0) &= \int_K \left(\int_{\xi_0} f(gk\cdot y)\,dm(y)\right)dk\\
&= \int_{\xi_0} dm(y)\int_K f(gk\cdot y)\,dk = \int_{\xi_0} (M^{|y|}f)(g\cdot 0)\,dm(y)\,.
\end{aligned}$$

Hence

$$((\widehat{f}))^{\vee}(x) = \Omega_{n-1} \int_0^{\infty} (M^r f)(x) r^{n-2}\, dr\,, \tag{34}$$

where Ω_{n-1} is the area of the unit sphere in $\mathbf{R}^{n-1}$. Applying L to (34), using (33) and Lemma 3.2, we obtain

$$L((\widehat{f})^{\vee}) = \Omega_{n-1} \int_0^{\infty} \left(\frac{d^2F}{dr^2} + \frac{n-1}{r}\frac{dF}{dr} \right) r^{n-2}\, dr \tag{35}$$

where $F(r) = (M^r f)(x)$. Integrating by parts and using

$$F(0) = f(x),\ \lim_{r\to\infty} r^k F(r) = 0\,,$$

we get

$$L((\widehat{f})^{\vee}) = \begin{cases} -\Omega_{n-1} f(x) & \text{if} \quad n = 3\,, \\ -\Omega_{n-1}(n-3)\int_0^{\infty} F(r) r^{n-4}\, dr & \quad (n > 3)\,. \end{cases}$$

More generally,

$$L_x \left(\int_0^{\infty} (M^r f)(x) r^k\, dr \right) = \begin{cases} -(n-2) f(x) & \text{if } k = 1\,, \\ -(n-1-k)(k-1)\int_0^{\infty} F(r) r^{k-2} dr, & (k > 1)\,. \end{cases}$$

If n is odd the formula in Theorem 3.1 follows by iteration. Although we assumed $f \in \mathcal{S}(\mathbf{R}^n)$ the proof is valid under much weaker assumptions.

Remark 3.3. The condition $f(x) = 0(|x|^{-N})$ for some $N > n$ cannot in general be dropped. In [1982] Zalcman has given an example of a smooth function f on $\mathbf{R}^2$ satisfying $f(x) = 0(|x|^{-2})$ on all lines with $\widehat{f}(\xi) = 0$ for all lines ξ and yet $f \not\equiv 0$. The function is even $f(x) = 0(|x|^{-3})$ on each line which is not the x-axis. See also Armitage and Goldstein [1993].

Remark 3.4. It is interesting to observe that while the inversion formula requires $f(x) = 0(|x|^{-N})$ for *one* $N > n$ the support theorem requires $f(x) = 0(|x|^{-N})$ for *all* N as mentioned in Remark 2.9.

We shall now prove a similar inversion formula for the dual transform $\varphi \to \check{\varphi}$ on the subspace $\mathcal{S}^*(\mathbf{P}^n)$.

Theorem 3.5. *We have*

$$c\varphi = (-\square)^{(n-1)/2}(\check{\varphi})\widehat{\ }, \quad \varphi \in \mathcal{S}^*(\mathbf{P}^n)\,,$$

where c is the constant $(4\pi)^{(n-1)/2}\Gamma(n/2)/\Gamma(1/2)$.

Here $\square$ denotes as before the operator $\frac{d^2}{dp^2}$ and its fractional powers are again defined in terms of the Riesz' potentials on the 1-dimensional p-space.

If n is odd our inversion formula follows from the odd-dimensional case in Theorem 3.1 if we put $f = \check{\varphi}$ and take Lemma 2.1 and Corollary 2.5 into account. Suppose now n is even. We claim that

$$((-L)^{\frac{n-1}{2}} f)\widehat{} = (-\Box)^{\frac{n-1}{2}} \widehat{f} \qquad f \in \mathcal{S}^*(\mathbf{R}^n)\,. \tag{36}$$

By Lemma 5.6 in Chapter V, $(-L)^{(n-1)/2} f$ belongs to $\mathcal{S}^*(\mathbf{R}^n)$. Taking the 1-dimensional Fourier transform of $((-L)^{(n-1)/2} f)\widehat{}$ we obtain

$$\big((-L)^{(n-1)/2} f\big)\widetilde{}(s\omega) = |s|^{n-1} \widetilde{f}(s\omega)\,.$$

On the other hand, for a fixed ω, $p \to \widehat{f}(\omega, p)$ is in $\mathcal{S}^*(\mathbf{R})$. By the lemma quoted, the function $p \to ((-\Box)^{(n-1)/2} \widehat{f})(\omega, p)$ also belongs to $\mathcal{S}^*(\mathbf{R})$ and its Fourier transform equals $|s|^{n-1} \widetilde{f}(s\omega)$. This proves (36). Now Theorem 3.5 follows from (36) if we put in (36)

$$\varphi = \widehat{g}\,, \quad f = (\widehat{g})^\vee\,, \quad g \in \mathcal{S}^*(\mathbf{R}^n)\,,$$

because, by Corollary 2.5, $\widehat{g}$ belongs to $\mathcal{S}^*(\mathbf{P}^n)$.

Because of its theoretical importance we now prove the inversion theorem (3.1) in a different form. The proof is less geometric and involves just the one variable Fourier transform.

Let $\mathcal{H}$ denote the Hilbert transform

$$(\mathcal{H}F)(t) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{F(p)}{t-p}\, dp \qquad F \in \mathcal{S}(\mathbf{R})$$

the integral being considered as the Cauchy principal value (see Lemma 3.7 below). For $\varphi \in \mathcal{S}(\mathbf{P}^n)$ let $\Lambda\varphi$ be defined by

$$(\Lambda\varphi)(\omega, p) = \begin{cases} \frac{d^{n-1}}{dp^{n-1}} \varphi(\omega, p) & n \text{ odd,} \\ \mathcal{H}_p \frac{d^{n-1}}{dp^{n-1}} \varphi(\omega, p) & n \text{ even.} \end{cases} \tag{37}$$

Note that in both cases $(\Lambda\varphi)(-\omega, -p) = (\Lambda\varphi)(\omega, p)$ so $\Lambda\varphi$ is a function on $\mathbf{P}^n$.

Theorem 3.6. *Let Λ be as defined by (37). Then*

$$cf = (\Lambda \widehat{f})^\vee\,, \quad f \in \mathcal{S}(\mathbf{R}^n)\,,$$

where as before

$$c = (-4\pi)^{(n-1)/2} \Gamma(n/2)/\Gamma(1/2)\,.$$

Proof. By the inversion formula for the Fourier transform and by (4)

$$f(x) = (2\pi)^{-n} \int_{\mathbf{S}^{n-1}} d\omega \int_0^\infty \bigg(\int_{-\infty}^{\infty} e^{-isp} \widehat{f}(\omega, p)\, dp \bigg) e^{is\langle x, \omega\rangle} s^{n-1}\, ds$$

which we write as

$$f(x) = (2\pi)^{-n}\int_{\mathbf{S}^{n-1}} F(\omega,x)\,d\omega = (2\pi)^{-n}\int_{\mathbf{S}^{n-1}} \tfrac{1}{2}(F(\omega,x)+F(-\omega,x))\,d\omega\,.$$

Using $\widehat{f}(-\omega,p) = \widehat{f}(\omega,-p)$ this gives the formula

$$f(x) = \tfrac{1}{2}(2\pi)^{-n}\int_{\mathbf{S}^{n-1}} d\omega \int_{-\infty}^{\infty} |s|^{n-1} e^{is\langle x,\omega\rangle}\,ds \int_{-\infty}^{\infty} e^{-isp}\widehat{f}(\omega,p)\,dp\,. \tag{38}$$

If n is odd the absolute value on s can be dropped. The factor s^{n-1} can be removed by replacing $\widehat{f}(\omega,p)$ by $(-i)^{n-1}\frac{d^{n-1}}{dp^{n-1}}\widehat{f}(\omega,p)$. The inversion formula for the Fourier transform on $\mathbf{R}$ then gives

$$f(x) = \tfrac{1}{2}(2\pi)^{-n}(2\pi)^{+1}(-i)^{n-1}\int_{\mathbf{S}^{n-1}} \left\{\frac{d^{n-1}}{dp^{n-1}}\widehat{f}(\omega,p)\right\}_{p=\langle x,\omega\rangle} d\omega$$

as desired.

In order to deal with the case n even we recall some general facts.

Lemma 3.7. *Let S denote the Cauchy principal value*

$$S : \psi \to \lim_{\epsilon\to 0}\int_{|x|\geq\epsilon} \frac{\psi(x)}{x}\,dx\,.$$

Then S is a tempered distribution and $\widetilde{S}$ is the function

$$\widetilde{S}(s) = -\pi i\,\mathrm{sgn}(s) = \begin{cases} -\pi i & s \geq 0 \\ \pi i & s < 0 \end{cases}.$$

Proof. It is clear that S is tempered. Also $xS = 1$ so

$$2\pi\delta = \widetilde{1} = (xS)\widetilde{} = i(\widetilde{S})'\,.$$

But $\mathrm{sgn}' = 2\delta$ so $\widetilde{S} = -\pi i\,\mathrm{sgn} + C$. But $\widetilde{S}$ and sgn are odd so $C = 0$.

This implies

$$(\mathcal{H}F)\widetilde{}(s) = \mathrm{sgn}(s)\widetilde{F}(s)\,. \tag{39}$$

For n even we write in (38), $|s|^{n-1} = \mathrm{sgn}(s)s^{n-1}$ and then (38) implies

$$f(x) = c_0\int_{\mathbf{S}^{n-1}} d\omega \int_{\mathbf{R}} \mathrm{sgn}(s)e^{is\langle x,\omega\rangle}\,ds \int_{\mathbf{R}} \frac{d^{n-1}}{dp^{n-1}}\widehat{f}(\omega,p)e^{-isp}\,dp\,, \tag{40}$$

where $c_0 = \frac{1}{2}(-i)^{n-1}(2\pi)^{-n}$. Now we have for each $F \in \mathcal{S}(\mathbf{R})$ the identity

$$\int_{\mathbf{R}} \mathrm{sgn}(s)e^{ist}\left(\int_{\mathbf{R}} F(p)e^{-ips}\,dp\right)ds = 2\pi(\mathcal{H}F)(t)\,.$$

In fact, if we apply both sides to $\widetilde{\psi}$ with $\psi \in \mathcal{S}(\mathbf{R})$, the left hand side is by (39)

$$\begin{aligned}
&\int_{\mathbf{R}} \Big(\int_{\mathbf{R}} \operatorname{sgn}(s) e^{ist} \widetilde{F}(s)\, ds \Big) \widetilde{\psi}(t)\, dt \\
&\quad = \int_{\mathbf{R}} \operatorname{sgn}(s) \widetilde{F}(s) 2\pi \psi(s)\, ds = 2\pi (\mathcal{H}F)\widetilde{\ }(\psi) = 2\pi (\mathcal{H}F)(\widetilde{\psi})\,.
\end{aligned}$$

Putting $F(p) = \frac{d^{n-1}}{dp^{n-1}} \widehat{f}(\omega, p)$ in (40) Theorem 3.6 follows also for n even.

For later use we add here a few remarks concerning $\mathcal{H}$. Let $F \in \mathcal{D}$ have support contained in $(-R, R)$. Then

$$-i\pi(\mathcal{H}F)(t) = \lim_{\epsilon \to 0} \int_{\epsilon < |t-p|} \frac{F(p)}{t-p}\, dp = \lim_{\epsilon \to 0} \int_I \frac{F(p)}{t-p}\, dp$$

where $I = \{p : |p| < R, \epsilon < |t-p|\}$. We decompose this last integral

$$\int_I \frac{F(p)}{t-p}\, dp = \int_I \frac{F(p) - F(t)}{t-p}\, dp + F(t) \int_I \frac{dp}{t-p}\,.$$

The last term vanishes for $|t| > R$ and all $\epsilon > 0$. The first term on the right is majorized by

$$\int_{|p|<R} \Big| \frac{F(t) - F(p)}{t-p} \Big|\, dp \leq 2R \sup |F'|\,.$$

Thus by the dominated convergence theorem

$$\lim_{|t| \to \infty} (\mathcal{H}F)(t) = 0\,.$$

Also if $J \subset (-R, R)$ is a compact subset the mapping $F \to \mathcal{H}F$ is continuous from $\mathcal{D}_J$ into $\mathcal{E}(\mathbf{R})$ (with the topologies in Chapter V, §1).

§4 The Plancherel Formula

We recall that the functions on $\mathbf{P}^n$ have been identified with the functions φ on $\mathbf{S}^{n-1} \times \mathbf{R}$ which are even: $\varphi(-\omega, -p) = \varphi(\omega, p)$. The functional

$$(41) \qquad \varphi \to \int_{\mathbf{S}^{n-1}} \int_{\mathbf{R}} \varphi(\omega, p)\, d\omega\, dp \qquad \varphi \in C_c(\mathbf{P}^n)\,,$$

is therefore a well defined measure on $\mathbf{P}^n$, denoted $d\omega\, dp$. The group $\mathbf{M}(n)$ of rigid motions of $\mathbf{R}^n$ acts transitively on $\mathbf{P}^n$: it also leaves the measure $d\omega\, dp$ invariant. It suffices to verify this latter statement for the translations

T in $\mathbf{M}(n)$ because $\mathbf{M}(n)$ is generated by them together with the rotations around 0, and these rotations clearly leave $d\omega\, dp$ invariant. But

$$(\varphi \circ T)(\omega, p) = \varphi(\omega, p + q(\omega, T))$$

where $q(\omega, T) \in \mathbf{R}$ is independent of p so

$$\iint (\varphi \circ T)(\omega, p)\, d\omega\, dp = \iint \varphi(\omega, p + q(\omega, T))\, d\omega\, dp = \iint \varphi(\omega, p)\, dp\, d\omega\,,$$

proving the invariance.

In accordance with (49)–(50) in Ch. V the fractional power $\square^k$ is defined on $\mathcal{S}(\mathbf{P}^n)$ by

$$(-\square^k)\varphi(\omega, p) = \frac{1}{H_1(-2k)} \int_{\mathbf{R}} \varphi(\omega, q)|p - q|^{-2k-1}\, dq \tag{42}$$

and then the 1-dimensional Fourier transform satisfies

$$((-\square)^k \varphi)\widetilde{\ }(\omega, s) = |s|^{2k} \widetilde{\varphi}(\omega, s)\,. \tag{43}$$

Now, if $f \in \mathcal{S}(\mathbf{R}^n)$ we have by (4)

$$\widehat{f}(\omega, p) = (2\pi)^{-1} \int \widetilde{f}(s\omega) e^{isp}\, ds$$

and

$$(-\square)^{\frac{n-1}{4}} \widehat{f}(\omega, p) = (2\pi)^{-1} \int_{\mathbf{R}} |s|^{\frac{n-1}{2}} \widetilde{f}(s\omega) e^{isp}\, ds\,. \tag{44}$$

Theorem 4.1. *The mapping* $f \to \square^{\frac{n-1}{4}} \widehat{f}$ *extends to an isometry of* $L^2(\mathbf{R}^n)$ *onto the space* $L^2_e(\mathbf{S}^{n-1} \times \mathbf{R})$ *of even functions in* $L^2(\mathbf{S}^{n-1} \times \mathbf{R})$, *the measure on* $\mathbf{S}^{n-1} \times \mathbf{R}$ *being*

$$\tfrac{1}{2}(2\pi)^{1-n}\, d\omega\, dp\,.$$

Proof. By (44) we have from the Plancherel formula on $\mathbf{R}$

$$(2\pi) \int_{\mathbf{R}} |(-\square)^{\frac{n-1}{4}} \widehat{f}(\omega, p)|^2\, dp = \int_{\mathbf{R}} |s|^{n-1} |\widetilde{f}(s\omega)|^2\, ds$$

so by integration over $\mathbf{S}^{n-1}$ and using the Plancherel formula for $f(x) \to \widetilde{f}(s\omega)$ we obtain

$$\int_{\mathbf{R}^n} |f(x)|^2\, dx = \tfrac{1}{2}(2\pi)^{1-n} \int_{\mathbf{S}^{n-1} \times \mathbf{R}} |\square^{\frac{n-1}{4}} \widehat{f}(\omega, p)|^2\, d\omega\, dp\,.$$

It remains to prove that the mapping is surjective. For this it would suffice to prove that if $\varphi \in L^2(\mathbf{S}^{n-1} \times \mathbf{R})$ is even and satisfies

$$\int_{\mathbf{S}^{n-1}} \int_{\mathbf{R}} \varphi(\omega, p)(-\square)^{\frac{n-1}{4}} \widehat{f}(\omega, p)\, d\omega\, dp = 0$$

for all $f \in \mathcal{S}(\mathbf{R}^n)$ then $\varphi = 0$. Taking Fourier transforms we must prove that if $\psi \in L^2(\mathbf{S}^{n-1} \times \mathbf{R})$ is even and satisfies

$$\int_{\mathbf{S}^{n-1}} \int_{\mathbf{R}} \psi(\omega, s)|s|^{\frac{n-1}{2}} \widetilde{f}(s\omega)\, ds\, d\omega = 0 \tag{45}$$

for all $f \in \mathcal{S}(\mathbf{R}^n)$ then $\psi = 0$. Using the condition $\psi(-\omega, -s) = \psi(\omega, s)$ we see that

$$\begin{aligned} &\int_{\mathbf{S}^{n-1}} \int_{-\infty}^{0} \psi(\omega, s)|s|^{\frac{1}{2}(n-1)} \widetilde{f}(s\omega)\, ds\, d\omega \\ = &\int_{\mathbf{S}^{n-1}} \int_{0}^{\infty} \psi(\omega, t)|t|^{\frac{1}{2}(n-1)} \widetilde{f}(t\omega)\, dt\, d\omega \end{aligned}$$

so (45) holds with $\mathbf{R}$ replaced with the positive axis $\mathbf{R}^+$. But then the function

$$\Psi(u) = \psi\left(\frac{u}{|u|}, |u|\right)|u|^{-\frac{1}{2}(n-1)}, \quad u \in \mathbf{R}^n - \{0\}$$

satisfies

$$\int_{\mathbf{R}^n} \Psi(u)\widetilde{f}(u)\, du = 0\,, \quad f \in \mathcal{S}(\mathbf{R}^n)$$

so $\Psi = 0$ almost everywhere, whence $\psi = 0$.

If we combine the inversion formula in Theorem 3.6 with (46) below we obtain the following version of the Plancherel formula

$$c \int_{\mathbf{R}^n} f(x)g(x)\, dx = \int_{\mathbf{P}^n} (\Lambda \widehat{f})(\xi)\widehat{g}(\xi)\, d\xi\,.$$

§5 Radon Transform of Distributions

It will be proved in a general context in Chapter II (Proposition 2.2) that

$$\int_{\mathbf{P}^n} \widehat{f}(\xi)\varphi(\xi)\, d\xi = \int_{\mathbf{R}^n} f(x)\check{\varphi}(x)\, dx \tag{46}$$

for $f \in C_c(\mathbf{R}^n), \varphi \in C(\mathbf{P}^n)$ if $d\xi$ is a suitable fixed $\mathbf{M}(n)$-invariant measure on $\mathbf{P}^n$. Thus $d\xi = \gamma\, d\omega\, dp$ where γ is a constant, independent of f and φ. With applications to distributions in mind we shall prove (46) in a somewhat stronger form.

Lemma 5.1. *Formula (46) holds (with $\widehat{f}$ and $\check{\varphi}$ existing almost anywhere) in the following two situations:*

(a) $f \in L^1(\mathbf{R}^n)$ *vanishing outside a compact set;* $\varphi \in C(\mathbf{P}^n)$.

(b) $f \in C_c(\mathbf{R}^n)$, φ *locally integrable.*

Also $d\xi = \Omega_n^{-1}\, d\omega\, dp$.

Proof. We shall use the Fubini theorem repeatedly both on the product $\mathbf{R}^n \times \mathbf{S}^{n-1}$ and on the product $\mathbf{R}^n = \mathbf{R} \times \mathbf{R}^{n-1}$. Since $f \in L^1(\mathbf{R}^n)$ we have for each $\omega \in \mathbf{S}^{n-1}$ that $\widehat{f}(\omega, p)$ exists for almost all p and

$$\int_{\mathbf{R}^n} f(x)\, dx = \int_{\mathbf{R}} \widehat{f}(\omega, p)\, dp\,.$$

We also conclude that $\widehat{f}(\omega, p)$ exists for almost all $(\omega, p) \in \mathbf{S}^{n-1} \times \mathbf{R}$. Next we consider the measurable function

$$(x, \omega) \to f(x)\varphi(\omega, \langle \omega, x\rangle) \;\text{ on }\; \mathbf{R}^n \times \mathbf{S}^{n-1}\,.$$

We have

$$\begin{aligned}
&\int_{\mathbf{S}^{n-1}\times\mathbf{R}^n} |f(x)\varphi(\omega, \langle \omega, x\rangle)|\, d\omega\, dx \\
&\quad = \int_{\mathbf{S}^{n-1}} \left(\int_{\mathbf{R}^n} |f(x)\varphi(\omega, \langle \omega, x\rangle)|\, dx \right) d\omega \\
&\quad = \int_{\mathbf{S}^{n-1}} \left(\int_{\mathbf{R}} |f|\widehat{\ }(\omega, p) |\varphi(\omega, p)|\, dp \right) d\omega\,,
\end{aligned}$$

which in both cases is finite. Thus $f(x) \cdot \varphi(\omega, \langle \omega, x\rangle)$ is integrable on $\mathbf{R}^n \times \mathbf{S}^{n-1}$ and its integral can be calculated by removing the absolute values above. This gives the left hand side of (46). Reversing the integrations we conclude that $\check{\varphi}(x)$ exists for almost all x and that the double integral reduces to the right hand side of (46).

The formula (46) dictates how to define the Radon transform and its dual for distributions (see Chapter V). In order to make the definitions formally consistent with those for functions we would require $\widehat{S}(\varphi) = S(\check{\varphi})$, $\check{\Sigma}(f) = \Sigma(\widehat{f})$ if S and Σ are distributions on $\mathbf{R}^n$ and $\mathbf{P}^n$, respectively. But while $f \in \mathcal{D}(\mathbf{R}^n)$ implies $\widehat{f} \in \mathcal{D}(\mathbf{P}^n)$ a similar implication does not hold for φ; we do not even have $\check{\varphi} \in \mathcal{S}(\mathbf{R}^n)$ for $\varphi \in \mathcal{D}(\mathbf{P}^n)$ so $\widehat{S}$ cannot be defined as above even if S is assumed to be tempered. Using the notation $\mathcal{E}$ (resp. $\mathcal{D}$) for the space of $\mathcal{C}^\infty$ functions (resp. of compact support) and $\mathcal{D}'$ (resp. $\mathcal{E}'$) for the space of distributions (resp. of compact support) we make the following definition.

Definition. For $S \in \mathcal{E}'(\mathbf{R}^n)$ we define the functional $\widehat{S}$ by

$$\widehat{S}(\varphi) = S(\check{\varphi}) \qquad \text{for } \varphi \in \mathcal{E}(\mathbf{P}^n)\,;$$

for $\Sigma \in \mathcal{D}'(\mathbf{P}^n)$ we define the functional $\check{\Sigma}$ by

$$\check{\Sigma}(f) = \Sigma(\widehat{f}) \qquad \text{for } f \in \mathcal{D}(\mathbf{R}^n)\,.$$

Lemma 5.2. *(i) For each* $\Sigma \in \mathcal{D}'(\mathbf{P}^n)$ *we have* $\check{\Sigma} \in \mathcal{D}'(\mathbf{R}^n)$.

(ii) For each $S \in \mathcal{E}'(\mathbf{R}^n)$ *we have* $\widehat{S} \in \mathcal{E}'(\mathbf{P}^n)$.

Proof. For $A > 0$ let $\mathcal{D}_A(\mathbf{R}^n)$ denote the set of functions $f \in \mathcal{D}(\mathbf{R}^n)$ with support in the closure of $B_A(0)$. Similarly let $\mathcal{D}_A(\mathbf{P}^n)$ denote the set of functions $\varphi \in \mathcal{D}(\mathbf{P}^n)$ with support in the closure of the "ball"

$$\beta_A(0) = \{\xi \in \mathbf{P}^n : d(0, \xi) < A\}.$$

The mapping of $f \to \widehat{f}$ from $\mathcal{D}_A(\mathbf{R}^n)$ to $\mathcal{D}_A(\mathbf{P}^n)$ being continuous (with the topologies defined in Chapter V, §1) the restriction of $\check{\Sigma}$ to each $\mathcal{D}_A(\mathbf{R}^n)$ is continuous so (i) follows. That $\widehat{S}$ is a distribution is clear from (3). Concerning its support select $R > 0$ such that S has support inside $B_R(0)$. Then if $\varphi(\omega, p) = 0$ for $|p| \leq R$ we have $\check{\varphi}(x) = 0$ for $|x| \leq R$ whence $\widehat{S}(\varphi) = S(\check{\varphi}) = 0$.

Lemma 5.3. *For* $S \in \mathcal{E}'(\mathbf{R}^n), \Sigma \in \mathcal{D}'(\mathbf{P}^n)$ *we have*

$$(LS)\widehat{\ } = \Box \widehat{S}, \quad (\Box \Sigma)^{\vee} = L\check{\Sigma}.$$

Proof. In fact by Lemma 2.1,

$$(LS)\widehat{\ }(\varphi) = (LS)(\check{\varphi}) = S(L\check{\varphi}) = S((\Box\varphi)^{\vee}) = \widehat{S}(\Box\varphi) = (\Box\widehat{S})(\varphi).$$

The other relation is proved in the same manner.

We shall now prove an analog of the support theorem (Theorem 2.6) for distributions. For $A > 0$ let $\beta_A(0)$ be defined as above and let supp denote support.

Theorem 5.4. *Let* $T \in \mathcal{E}'(\mathbf{R}^n)$ *satisfy the condition*

$$\operatorname{supp} \widehat{T} \subset C\ell(\beta_A(0)), \quad (C\ell = \textit{ closure}).$$

Then

$$\operatorname{supp}(T) \subset C\ell(B_A(0)).$$

Proof. For $f \in \mathcal{D}(\mathbf{R}^n)$, $\varphi \in \mathcal{D}(\mathbf{P}^n)$ we can consider the "convolution"

$$(f \times \varphi)(\xi) = \int_{\mathbf{R}^n} f(y)\varphi(\xi - y)\, dy,$$

where for $\xi \in \mathbf{P}^n$, $\xi - y$ denotes the translate of the hyperplane ξ by $-y$. Then

$$(f \times \varphi)^\vee = f * \check{\varphi}\,.$$

In fact, if ξ_0 is any hyperplane through 0,

$$\begin{aligned} (f \times \varphi)^\vee(x) &= \int_K dk \int_{\mathbf{R}^n} f(y)\varphi(x + k\cdot\xi_0 - y)\,dy \\ &= \int_K dk \int_{\mathbf{R}^n} f(x-y)\varphi(y + k\cdot\xi_0)\,dy = (f * \check{\varphi})(x)\,. \end{aligned}$$

By the definition of $\widehat{T}$, the support assumption on $\widehat{T}$ is equivalent to

$$T(\check{\varphi}) = 0$$

for all $\varphi \in \mathcal{D}(\mathbf{P}^n)$ with support in $\mathbf{P}^n - C\ell(\beta_A(0))$. Let $\epsilon > 0$, let $f \in \mathcal{D}(\mathbf{R}^n)$ be a symmetric function with support in $C\ell(B_\epsilon(0))$ and let $\varphi = \mathcal{D}(\mathbf{P}^n)$ have support contained in $\mathbf{P}^n - C\ell(\beta_{A+\epsilon}(0))$. Since $d(0, \xi - y) \leq d(0,\xi) + |y|$ it follows that $f \times \varphi$ has support in $\mathbf{P}^n - C\ell(\beta_A(0))$; thus by the formulas above, and the symmetry of f,

$$(f * T)(\check{\varphi}) = T(f * \check{\varphi}) = T((f \times \varphi)^\vee) = 0\,.$$

But then

$$(f * T)\widehat{\ }(\varphi) = (f * T)(\check{\varphi}) = 0\,,$$

which means that $(f*T)\widehat{\ }$ has support in $C\ell(\beta_{A+\epsilon}(0))$. But now Theorem 2.6 implies that $f * T$ has support in $C\ell(B_{A+\epsilon})(0)$. Letting $\epsilon \to 0$ we obtain the desired conclusion, $\operatorname{supp}(T) \subset C\ell(B_A(0))$.

We can now extend the inversion formulas for the Radon transform to distributions. First we observe that the Hilbert transform $\mathcal{H}$ can be extended to distributions T on $\mathbf{R}$ of compact support. It suffices to put

$$\mathcal{H}(T)(F) = T(-\mathcal{H}F)\,, \quad F \in \mathcal{D}(\mathbf{R})\,.$$

In fact, as remarked at the end of §3, the mapping $F \longrightarrow \mathcal{H}F$ is a continuous mapping of $\mathcal{D}(\mathbf{R})$ into $\mathcal{E}(\mathbf{R})$. In particular $\mathcal{H}(T) \in \mathcal{D}'(\mathbf{R})$.

Theorem 5.5. *The Radon transform $S \longrightarrow \widehat{S}$ $\;(S \in \mathcal{E}'(\mathbf{R}^n))$ is inverted by the following formula*

$$cS = (\Lambda\widehat{S})^\vee\,, \quad S \in \mathcal{E}'(\mathbf{R}^n)\,,$$

where the constant $c = (-4\pi)^{(n-1)/2}\Gamma(n/2)/\Gamma(1/2)$.

In the case when n is odd we have also

$$c\,S = L^{(n-1)/2}((\widehat{S})^\vee)\,.$$

Remark 5.6. Since $\widehat{S}$ has compact support and since Λ is defined by means of the Hilbert transform the remarks above show that $\Lambda\widehat{S} \in \mathcal{D}'(\mathbf{P}^n)$ so the right hand side is well defined.

Proof. Using Theorem 3.6 we have

$$(\Lambda\widehat{S})^\vee(f) = (\Lambda\widehat{S})(\widehat{f}) = \widehat{S}(\Lambda\widehat{f}) = S((\Lambda\widehat{f})^\vee) = cS(f)\,.$$

The other inversion formula then follows, using the lemma.

In analogy with β_A we define the "sphere" σ_A in $\mathbf{P}^n$ as

$$\sigma_A = \{\xi \in \mathbf{P}^n : d(0,\xi) = A\}\,.$$

From Theorem 5.5 we can then deduce the following complement to Theorem 5.4.

Corollary 5.7. *Suppose n is odd. Then if $S \in \mathcal{E}'(\mathbf{R}^n)$,*

$$\operatorname{supp}(\widehat{S}) \in \sigma_R \Rightarrow \operatorname{supp}(S) \subset S_R(0)\,.$$

To see this let $\epsilon > 0$ and let $f \in \mathcal{D}(\mathbf{R}^n)$ have $\operatorname{supp}(f) \subset B_{R-\epsilon}(0)$. Then $\operatorname{supp}\widehat{f} \in \beta_{R-\epsilon}$ and since Λ is a differential operator, $\operatorname{supp}(\Lambda\widehat{f}) \subset \beta_{R-\epsilon}$. Hence

$$cS(f) = S((\Lambda\widehat{f})^\vee) = \widehat{S}(\Lambda\widehat{f}) = 0$$

so $\operatorname{supp}(S) \cap B_{R-\epsilon}(0) = \emptyset$. Since $\epsilon > 0$ is arbitrary,

$$\operatorname{supp}(S) \cap B_R(0) = \emptyset\,.$$

On the other hand by Theorem 5.4, $\operatorname{supp}(S) \subset \overline{B_R(0)}$. This proves the corollary.

Let M be a manifold and $d\mu$ a measure such that on each local coordinate patch with coordinates $(t_1, \ldots, t_n)$ the Lebesque measure $dt_1, \ldots, dt_n$ and $d\mu$ are absolutely continuous with respect to each other. If h is a function on M locally integrable with respect to $d\mu$ the distribution $\varphi \to \int \varphi h\, d\mu$ will be denoted T_h.

Proposition 5.8. *(a) Let $f \in L^1(\mathbf{R}^n)$ vanish outside a compact set. Then the distribution T_f has Radon transform given by*

$$\widehat{T_f} = T_{\widehat{f}}\,. \tag{47}$$

(b) Let φ be a locally integrable function on $\mathbf{P}^n$. Then

$$(T_\varphi)^\vee = T_{\check{\varphi}}\,. \tag{48}$$

Proof. The existence and local integrability of $\widehat{f}$ and $\check{\varphi}$ was established during the proof of Lemma 5.1. The two formulas now follow directly from Lemma 5.1.

As a result of this proposition the smoothness assumption can be dropped in the inversion formula. In particular, we can state the following result.

Corollary 5.9. *(n odd.) The inversion formula*

$$cf = L^{(n-1)/2}((\widehat{f})^{\vee}),$$

$c = (-4\pi)^{(n-1)/2}\Gamma(n/2)/\Gamma(1/2)$, *holds for all* $f \in L^1(\mathbf{R}^n)$ *vanishing outside a compact set, the derivative interpreted in the sense of distributions.*

Examples. If μ is a measure (or a distribution) on a closed submanifold S of a manifold M the distribution on M given by $\varphi \to \mu(\varphi|S)$ will also be denoted by μ.

(a) Let δ_0 be the delta distribution $f \to f(0)$ on $\mathbf{R}^n$. Then

$$\widehat{\delta_0}(\varphi) = \delta_0(\check{\varphi}) = \Omega_n^{-1} \int_{S^{n-1}} \varphi(\omega, 0)\, d\omega$$

so

$$\widehat{\delta_0} = \Omega_n^{-1} m_{\mathbf{S}^{n-1}} \tag{49}$$

the normalized measure on $\mathbf{S}^{n-1}$ considered as a distribution on $\mathbf{S}^{n-1} \times \mathbf{R}$.

(b) Let ξ_0 denote the hyperplane $x_n = 0$ in $\mathbf{R}^n$, and δ_{ξ_0} the delta distribution $\varphi \to \varphi(\xi_0)$ on $\mathbf{P}^n$. Then

$$(\delta_{\xi_0})^{\vee}(f) = \int_{\xi_0} f(x)\, dm(x)$$

so

$$(\delta_{\xi_0})^{\vee} = m_{\xi_0}, \tag{50}$$

the Euclidean measure of ξ_0.

(c) Let χ_B be the characteristic function of the unit ball $B \subset \mathbf{R}^n$. Then by (47),

$$\widehat{\chi}_B(\omega, p) = \begin{cases} \frac{\Omega_{n-1}}{n-1}(1-p^2)^{(n-1)/2} & , \ |p| \le 1 \\ 0 & , \ |p| > 1 \end{cases}.$$

(d) Let Ω be a bounded convex region in $\mathbf{R}^n$ whose boundary is a smooth surface. We shall obtain a formula for the volume of Ω in terms of the areas of its hyperplane sections. For simplicity we assume n odd. The characteristic function χ_Ω is a distribution of compact support and $(\chi_\Omega)\widehat{}$ is thus well defined. Approximating χ_Ω in the L^2-norm by a sequence $(\psi_n) \subset \mathcal{D}(\Omega)$ we see from Theorem 4.1 that $\partial_p^{(n-1)/2}\widehat{\psi}_n(\omega, p)$ converges in the L^2-norm on $\mathbf{P}^n$. Since

$$\int \widehat{\psi}(\xi)\varphi(\xi)\,d\xi = \int \psi(x)\check{\varphi}(x)\,dx$$

it follows from Schwarz' inequality that $\widehat{\psi}_n \longrightarrow (\chi_\Omega)\widehat{}$ in the sense of distributions and accordingly $\partial^{(n-1)/2}\widehat{\psi}_n$ converges as a distribution to $\partial^{(n-1)/2}((\chi_\Omega)\widehat{}\,)$. Since the L^2 limit is also a limit in the sense of distributions this last function equals the L^2 limit of the sequence $\partial^{(n-1)/2}\widehat{\psi}_n$. From Theorem 4.1 we can thus conclude the following result:

Theorem 5.10. *Let $\Omega \subset \mathbf{R}^n$ (n odd) be a convex region as above and $V(\Omega)$ its volume. Let $A(\omega, p)$ denote the $(n-1)$-dimensional area of the intersection of Ω with the hyperplane $\langle x, \omega\rangle = p$. Then*

$$V(\Omega) = \tfrac{1}{2}(2\pi)^{1-n}\int_{\mathbf{S}^{n-1}}\int_{\mathbf{R}}\left|\frac{\partial^{(n-1)/2}A(\omega,p)}{\partial p^{(n-1)/2}}\right|^2 dp\,d\omega\,. \tag{51}$$

§6 Integration over d-planes. X-ray Transforms. The Range of the d-plane Transform

Let d be a fixed integer in the range $0 < d < n$. We define the *d-dimensional Radon transform* $f \to \widehat{f}$ by

$$\widehat{f}(\xi) = \int_\xi f(x)\,dm(x) \qquad \xi \quad \text{a } d\text{-plane}\,. \tag{52}$$

Because of the applications to radiology indicated in § 7,b) the 1-dimensional Radon transform is often called the *X-ray transform*. Since a hyperplane can be viewed as a disjoint union of parallel d-planes parameterized by $\mathbf{R}^{n-1-d}$ it is obvious from (4) that the transform $f \to \widehat{f}$ is injective. Similarly we deduce the following consequence of Theorem 2.6.

Corollary 6.1. *Let $f, g \in C(\mathbf{R}^n)$ satisfy the rapid decrease condition: For each $m > 0$, $|x|^m f(x)$ and $|x|^m g(x)$ are bounded on $\mathbf{R}^n$. Assume for the d-dimensional Radon transforms*

$$\widehat{f}(\xi) = \widehat{g}(\xi)$$

whenever the d-plane ξ lies outside the unit ball. Then

$$f(x) = g(x) \quad \textit{for} \quad |x| > 1\,.$$

We shall now generalize the inversion formula in Theorem 3.1. If φ is a continuous function on the space of d-planes in $\mathbf{R}^n$ we denote by $\check{\varphi}$ the point function

$$\check{\varphi}(x) = \int_{x \in \xi} \varphi(\xi)\, d\mu(\xi)\,,$$

where μ is the unique measure on the (compact) space of d-planes passing through x, invariant under all rotations around x and with total measure 1. If σ is a fixed d-plane through the origin we have in analogy with (16),

$$\check{\varphi}(x) = \int_K \varphi(x + k \cdot \sigma)\, dk\,. \tag{53}$$

Theorem 6.2. *The d-dimensional Radon transform in $\mathbf{R}^n$ is inverted by the formula*

$$cf = (-L)^{d/2}((\widehat{f})^\vee)\,, \tag{54}$$

where $c = (4\pi)^{d/2}\Gamma(n/2)/\Gamma((n-d)/2)$. Here it is assumed that $f(x) = 0(|x|^{-N})$ for some $N > n$.

Proof. We have in analogy with (34)

$$\begin{aligned}
(\widehat{f})^\vee(x) &= \int_K \left(\int_\sigma f(x + k \cdot y)\, dm(y) \right) dk \\
&= \int_\sigma dm(y) \int_K f(x + k \cdot y)\, dk = \int_\sigma (M^{|y|} f)(x)\, dm(y)\,.
\end{aligned}$$

Hence

$$(\widehat{f})^\vee(x) = \Omega_d \int_0^\infty (M^r f)(x) r^{d-1}\, dr$$

so using polar coordinates around x,

$$(\widehat{f})^\vee(x) = \frac{\Omega_d}{\Omega_n} \int_{\mathbf{R}^n} |x - y|^{d-n} f(y)\, dy\,. \tag{55}$$

The theorem now follows from Proposition 5.7 in Chapter V.

As a consequence of Theorem 2.10 we now obtain a generalization, characterizing the image of the space $\mathcal{D}(\mathbf{R}^n)$ under the d-dimensional Radon transform.

The set $\mathbf{G}(d,n)$ of d-planes in $\mathbf{R}^n$ is a manifold, in fact a homogeneous space of the group $\mathbf{M}(n)$ of all isometries of $\mathbf{R}^n$. Let $\mathbf{G}_{d,n}$ denote the manifold of all d-dimensional subspaces (d-planes through 0) of $\mathbf{R}^n$. The parallel translation of a d-plane to one through 0 gives a mapping π of $\mathbf{G}(d,n)$ onto $\mathbf{G}_{d,n}$. The inverse image $\pi^{-1}(\sigma)$ of a member $\sigma \in \mathbf{G}_{d,n}$ is naturally identified with the orthogonal complement $\sigma^\perp$. Let us write

$$\xi = (\sigma, x'') = x'' + \sigma \ \text{ if } \ \sigma = \pi(\xi) \ \text{ and } \ x'' = \sigma^\perp \cap \xi\,.$$

(See Fig. I.6.) Then (52) can be written

(56)
$$\widehat{f}(x''+\sigma)=\int_\sigma f(x'+x'')\,dx'\,.$$

For $k\in\mathbb{Z}^+$ we consider the polynomial

(57)
$$P_k(u)=\int_{\mathbf{R}^n} f(x)\langle x,u\rangle^k\,dx\,.$$

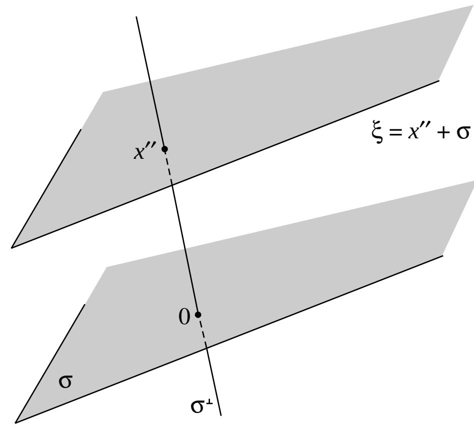

FIGURE I.6.

If $u=u''\in\sigma^\perp$ this can be written

$$\int_{\mathbf{R}^n} f(x)\langle x,u''\rangle^k\,dx=\int_{\sigma^\perp}\int_\sigma f(x'+x'')\langle x'',u''\rangle^k\,dx'\,dx''$$

so the polynomial

$$P_{\sigma,k}(u'')=\int_{\sigma^\perp}\widehat{f}(x''+\sigma)\langle x'',u''\rangle^k\,dx''$$

is the restriction to $\sigma^\perp$ of the polynomial P_k.

In analogy with the space $\mathcal{D}_H(\mathbf{P}^n)$ in No. 2 we define the space $\mathcal{D}_H(\mathbf{G}(d,n))$ as the set of $\mathcal{C}^\infty$ functions

$$\varphi(\xi)=\varphi_\sigma(x'')=\varphi(x''+\sigma)\quad(\text{if }\ \xi=(\sigma,x''))$$

on $\mathbf{G}(d,n)$ of compact support satisfying the following condition.

(H) : *For each $k\in\mathbb{Z}^+$ there exists a homogeneous k^{th} degree polynomial P_k on $\mathbf{R}^n$ such that for each $\sigma\in\mathbf{G}_{d,n}$ the polynomial*

$$P_{\sigma,k}(u'')=\int_{\sigma^\perp}\varphi(x''+\sigma)\langle x'',u''\rangle^k\,dx''\,,\quad u''\in\sigma^\perp\,,$$

coincides with the restriction $P_k|\sigma^\perp$.

Theorem 6.3. *The d-dimensional Radon transform is a bijection of $\mathcal{D}(\mathbf{R}^n)$ onto $\mathcal{D}_H(\mathbf{G}(d,n))$.*

Proof. For $d=n-1$ this is Theorem 2.10. We shall now reduce the case of general $d\le n-2$ to the case $d=n-1$. It remains just to prove the surjectivity in Theorem 6.3.

We shall actually prove a stronger statement.

Theorem 6.4. *Let $\varphi \in \mathcal{D}(\mathbf{G}(d,n))$ have the property: For each pair $\sigma, \tau \in \mathbf{G}_{d,n}$ and each $k \in \mathbb{Z}^+$ the polynomials*

$$\begin{aligned} P_{\sigma,k}(u) &= \int_{\sigma^\perp} \varphi(x''+\sigma)\langle x'', u\rangle^k \, dx'' \qquad u \in \mathbf{R}^n \\ P_{\tau,k}(u) &= \int_{\tau^\perp} \varphi(y''+\tau)\langle y'', u\rangle^k \, dy'' \qquad u \in \mathbf{R}^n \end{aligned}$$

agree for $u \in \sigma^\perp \cap \tau^\perp$. Then $\varphi = \widehat{f}$ for some $f \in \mathcal{D}(\mathbf{R}^n)$.

Proof. Let $\varphi = \mathcal{D}(\mathbf{G}(d,n))$ have the property above. Let $\omega \in \mathbf{R}^n$ be a unit vector. Let $\sigma, \tau \in \mathbf{G}_{d,n}$ be perpendicular to ω. Consider the $(n-d-1)$-dimensional integral

$$\Psi_\sigma(\omega, p) = \int_{\langle \omega, x''\rangle = p,\, x'' \in \sigma^\perp} \varphi(x''+\sigma) d_{n-d-1}(x'')\,, \quad p \in \mathbf{R}\,. \tag{58}$$

We claim that

$$\Psi_\sigma(\omega,p) = \Psi_\tau(\omega,p)\,.$$

To see this consider the moment

$$\begin{aligned} &\int_{\mathbf{R}} \Psi_\sigma(\omega,p)p^k \, dp \\ &= \int_{\mathbf{R}} p^k \left(\int \varphi(x''+\sigma) d_{n-d-1}(x'') \right) dp = \int_{\sigma^\perp} \varphi(x''+\sigma)\langle x'', \omega\rangle^k \, dx'' \\ &= \int_{\tau^\perp} \varphi(y''+\tau)\langle y'', \omega\rangle^k \, dy'' = \int_{\mathbf{R}} \Psi_\tau(\omega,p)p^k \, dp\,. \end{aligned}$$

Thus $\Psi_\sigma(\omega,p) - \Psi_\tau(\omega,p)$ is perpendicular to all polynomials in p; having compact support it would be identically 0. We therefore put $\Psi(\omega,p) = \Psi_\sigma(\omega,p)$. Observe that Ψ is smooth; in fact for ω in a neighborhood of a fixed ω_0 we can let σ depend smoothly on ω so by (58), $\Psi_\sigma(\omega,p)$ is smooth.

Writing

$$\langle x'', \omega\rangle^k = \sum_{|\alpha|=k} p_\alpha(x'')\omega^\alpha\,, \qquad \omega^\alpha = \omega_1^{\alpha_1} \dots \omega_n^{\alpha_n}$$

we have

$$\int_{\mathbf{R}} \Psi(\omega,p)p^k \, dp = \sum_{|\alpha|=k} A_\alpha \omega^\alpha\,,$$

where

$$A_\alpha = \int_{\sigma^\perp} \varphi(x''+\sigma)p_\alpha(x'') \, dx''\,.$$

Here A_α is independent of σ if $\omega \in \sigma^\perp$; in other words, viewed as a function of ω, A_α has for each σ a constant value as ω varies in $\sigma^\perp \cap S_1(0)$. To see that this value is the same as the value on $\tau^\perp \cap S_1(0)$ we observe that there exists a $\rho \in \mathbf{G}_{d,n}$ such that $\rho^\perp \cap \sigma^\perp \neq 0$ and $\rho^\perp \cap \tau^\perp \neq 0$. (Extend the 2-plane spanned by a vector in $\sigma^\perp$ and a vector in $\tau^\perp$ to an $(n-d)$-plane.) This shows that A_α is constant on $S_1(0)$ so $\Psi \in \mathcal{D}_H(\mathbf{P}^n)$. Thus by Theorem 2.10,

$$\Psi(\omega, p) = \int_{\langle x,\omega\rangle = p} f(x)\, dm(x) \tag{59}$$

for some $f \in \mathcal{D}(\mathbf{R}^n)$. It remains to prove that

$$\varphi(x'' + \sigma) = \int_\sigma f(x' + x'')\, dx'\,. \tag{60}$$

But as x'' runs through an arbitrary hyperplane in $\sigma^\perp$ it follows from (58) and (59) that both sides of (60) have the same integral. By the injectivity of the $(n-d-1)$-dimensional Radon transform on $\sigma^\perp$ equation (60) follows. This proves Theorem 6.4.

Theorem 6.4 raises the following elementary question: If a function f on $\mathbf{R}^n$ is a polynomial on each k-dimensional subspace, is f itself a polynomial? The answer is no for $k = 1$ but yes if $k > 1$. See Proposition 6.13 below, kindly communicated by Schlichtkrull.

We shall now prove another characterization of the range of $\mathcal{D}(\mathbf{R}^n)$ under the d-plane transform (for $d \leq n-2$). The proof will be based on Theorem 6.4.

Given any $d+1$ points $(x_0, \dots, x_d)$ in general position let $\xi(x_0, \dots, x_d)$ denote the d-plane passing through them. If $\varphi \in \mathcal{E}(\mathbf{G}(d,n))$ we shall write $\varphi(x_0, \dots, x_d)$ for the value $\varphi(\xi(x_0, \dots, x_d))$. We also write $V(\{x_i - x_0\}_{i=1,d})$ for the volume of the parallelepiped spanned by vectors $(x_i - x_0), (1 \leq i \leq d)$. The mapping

$$(\lambda_1, \dots, \lambda_d) \to x_0 + \Sigma_{i=1}^d \lambda_i (x_i - x_0)$$

is a bijection of $\mathbf{R}^d$ onto $\xi(x_0, \dots, x_d)$ and

$$\widehat{f}(x_0, \dots, x_d) = V(\{x_i - x_0\}_{i=1,d}) \int_{\mathbf{R}^d} f(x_0 + \Sigma_i \lambda_i (x_i - x_0))\, d\lambda\,. \tag{61}$$

The range $\mathcal{D}(\mathbf{R}^n)$ can now be described by the following alternative to Theorem 6.4. Let x_i^k denote the k^{th} coordinate of x_i.

Theorem 6.5. *If $f \in \mathcal{D}(\mathbf{R}^n)$ then $\varphi = \widehat{f}$ satisfies the system*

$$(\partial_{i,k}\partial_{j,\ell} - \partial_{j,k}\partial_{i,\ell})\big(\varphi(x_0, \dots, x_d)/V(\{x_i - x_0\}_{i=1,d})\big) = 0\,, \tag{62}$$

where

$$0 \leq i, j \leq d\,,\ 1 \leq k, \ell \leq n\,, \partial_{i,k} = \partial/\partial x_i^k\,.$$

Conversely, if $\varphi \in \mathcal{D}(\mathbf{G}(d,n))$ satisfies (62) then $\varphi = \widehat{f}$ for some $f \in \mathcal{D}(\mathbf{R}^n)$.

The validity of (62) for $\varphi = \widehat{f}$ is obvious from (61) just by differentiation under the integral sign. For the converse we first prove a simple lemma.

Lemma 6.6. *Let $\varphi \in \mathcal{E}(\mathbf{G}(d,n))$ and $A \in \mathbf{O}(n)$. Let $\psi = \varphi \circ A$. Then if $\varphi(x_0, \dots, x_d)$ satisfies (62) so does the function*

$$\psi(x_0, \dots, x_d) = \varphi(Ax_0, \dots, Ax_d)\,.$$

Proof. Let $y_i = Ax_i$ so $y_i^\ell = \Sigma_p a_{\ell p} x_i^p$. Then, if $D_{i,k} = \partial/\partial y_i^k$,

$$(\partial_{i,k}\partial_{j,\ell} - \partial_{j,k}\partial_{i,\ell}) = \Sigma_{p,q=1}^n a_{pk}a_{q\ell}(D_{i,p}D_{j,q} - D_{i,q}D_{j,p})\,. \tag{63}$$

Since A preserves volumes, the lemma follows.

Suppose now φ satisfies (62). We write $\sigma = (\sigma_1, \dots, \sigma_d)$ if (σ_j) is an orthonormal basis of σ. If $x'' \in \sigma^\perp$, the $(d+1)$-tuple

$$(x'', x'' + \sigma_1, \dots, x'' + \sigma_d)$$

represents the d-plane $x'' + \sigma$ and the polynomial

$$\begin{aligned}(64)\ P_{\sigma,k}(u'') &= \int_{\sigma^\perp} \varphi(x''+\sigma)\langle x'', u''\rangle^k \, dx'' \\ &= \int_{\sigma^\perp} \varphi(x'', x''+\sigma_1, \dots, x''+\sigma_d)\langle x'', u''\rangle^k \, dx''\,, \quad u'' \in \sigma^\perp\,,\end{aligned}$$

depends only on σ. In particular, it is invariant under orthogonal transformations of $(\sigma_1, \dots, \sigma_d)$. In order to use Theorem 6.4 we must show that for any $\sigma, \tau \in \mathbf{G}_{d,n}$ and any $k \in \mathbb{Z}^+$,

$$P_{\sigma,k}(u) = P_{\tau,k}(u) \ \text{ for } \ u \in \sigma^\perp \cap \tau^\perp\,, \quad |u| = 1\,. \tag{65}$$

The following lemma is a basic step towards (65).

Lemma 6.7. *Assume $\varphi \in \mathbf{G}(d,n)$ satisfies (62). Let*

$$\sigma = (\sigma_1, \dots, \sigma_d), \tau = (\tau_1, \dots, \tau_d)$$

be two members of $\mathbf{G}_{d,n}$. Assume

$$\sigma_j = \tau_j \quad \text{for } 2 \leq j \leq d\,.$$

Then

$$P_{\sigma,k}(u) = P_{\tau,k}(u) \quad \text{for } u \in \sigma^\perp \cap \tau^\perp\,, \quad |u| = 1\,.$$

Proof. Let $e_i (1 \leq i \leq n)$ be the natural basis of $\mathbf{R}^n$ and $\epsilon = (e_1, \dots, e_d)$. Select $A \in \mathbf{O}(n)$ such that

$$\sigma = A\epsilon\,, \quad u = Ae_n\,.$$

Let

$$\eta = A^{-1}\tau = (A^{-1}\tau_1, \dots, A^{-1}\tau_d) = (A^{-1}\tau_1, e_2, \dots, e_d)\,.$$

The vector $E = A^{-1}\tau_1$ is perpendicular to e_j $(2 \le j \le d)$ and to e_n (since $u \in \tau^\perp$). Thus

$$E = a_1 e_1 + \sum_{d+1}^{n-1} a_i e_i \qquad (a_1^2 + \sum_i a_i^2 = 1)\,.$$

In (64) we write $P^\varphi_{\sigma,k}$ for $P_{\sigma,k}$. Putting $x'' = Ay$ and $\psi = \varphi \circ A$ we have

$$P^\varphi_{\sigma,k}(u) = \int_{\epsilon^\perp} \varphi(Ay, A(y+e_1), \dots, A(y+e_d))\langle y, e_n\rangle^k \; dy = P^\psi_{\epsilon,k}(e_n)$$

and similarly

$$P^\varphi_{\tau,k}(u) = P^\psi_{\eta,k}(e_n)\,.$$

Thus, taking Lemma 6.6 into account, we have to prove the statement:

$$P_{\epsilon,k}(e_n) = P_{\eta,k}(e_n)\,, \tag{66}$$

where $\epsilon = (e_1, \dots, e_d), \eta = (E, e_2, \dots, e_d)$, E being any unit vector perpendicular to e_j $(2 \le j \le d)$ and to e_n. First we take

$$E = E_t = \sin t\, e_1 + \cos t\, e_i \quad (d < i < n)$$

and put $\epsilon_t = (E_t, e_2, \dots, e_d)$. We shall prove

$$P_{\epsilon_t,k}(e_n) = P_{\epsilon,k}(e_n)\,. \tag{67}$$

Without restriction of generality we can take $i = d+1$. The space $\epsilon_t^\perp$ consists of the vectors

$$x_t = (-\cos t\, e_1 + \sin t\, e_{d+1})\lambda_{d+1} + \sum_{i=d+2}^{n} \lambda_i e_i\,, \quad \lambda_i \in \mathbf{R}\,. \tag{68}$$

Putting $P(t) = P_{\epsilon_t,k}(e_n)$ we have

$$P(t) = \int_{\mathbf{R}^{n-d}} \varphi(x_t, x_t + E_t, x_t + e_2, \dots, x_t + e_d)\lambda_n^k \, d\lambda_n \dots d\lambda_{d+1}\,. \tag{69}$$

In order to use (62) we replace φ by the function

$$\psi(x_0,\dots,x_d)=\varphi(x_0,\dots,x_d)/V(\{x_i-x_0\}_{i=1,d}).$$

Since the vectors in (69) span volume 1 replacing φ by ψ in (69) does not change $P(t)$. Applying $\partial/\partial t$ we get (with $d\lambda=d\lambda_n\dots d\lambda_{d+1}$),

$$\begin{aligned}P'(t) &= \int_{\mathbf{R}^{n-d}}\Bigg[\sum_{j=0}^{d}\lambda_{d+1}(\sin t\,\partial_{j,1}\psi+\cos t\,\partial_{j,d+1}\psi)\\ &+ \cos t\,\partial_{1,1}\psi-\sin t\,\partial_{1,d+1}\psi\Bigg]\lambda_n^k\,d\lambda\,.\end{aligned} \tag{70}$$

Now φ is a function on $\mathbf{G}(d,n)$. Thus for each $i\neq j$ it is invariant under the substitution

$$y_k=x_k\ (k\neq i),\ y_i=sx_i+(1-s)x_j=x_j+s(x_i-x_j)\,,\quad s>0$$

whereas the volume changes by the factor s. Thus

$$\psi(y_0,\dots,y_d)=s^{-1}\psi(x_0,\dots,x_d)\,.$$

Taking $\partial/\partial s$ at $s=1$ we obtain

$$\psi(x_0,\dots,x_d)+\sum_{k=1}^{n}(x_i^k-x_j^k)(\partial_{i,k}\psi)(x_0,\dots,x_d)=0\,. \tag{71}$$

Note that in (70) the derivatives are evaluated at

$$(x_0,\dots,x_d)=(x_t,x_t+E_t,x_t+e_2,\dots,x_t+e_d)\,. \tag{72}$$

Using (71) for $(i,j)=(1,0)$ and $(i,j)=(0,1)$ and adding we obtain

$$\sin t\,(\partial_{0,1}\psi+\partial_{1,1}\psi)+\cos t\,(\partial_{0,d+1}\psi+\partial_{1,d+1}\psi)=0\,. \tag{73}$$

For $i\geq 2$ we have

$$x_i-x_0=e_i,\ x_i-x_1=-\sin t\,e_1-\cos t\,e_{d+1}+e_i\,,$$

and this gives the relations (for $j=0$ and $j=1$)

$$\psi(x_0,\dots,x_d)+(\partial_{i,i}\psi)(x_0,\dots,x_d)=0\,, \tag{74}$$

$$\psi-\sin t\ (\partial_{i,1}\psi)-\cos t\ (\partial_{i,d+1}\psi)+\partial_{i,i}\psi=0\,. \tag{75}$$

Thus by (73)–(75) formula (70) simplifies to

$$P'(t)=\int_{\mathbf{R}^{n-d}}[\cos t\ (\partial_{1,1}\psi)-\sin t\ (\partial_{1,d+1}\psi)]\,\lambda_n^k\,d\lambda\,.$$

In order to bring in 2^{nd} derivatives of ψ we integrate by parts in λ_n,

$$(76)\quad (k+1)P'(t) = \int_{\mathbf{R}^{n-d}} -\frac{\partial}{\partial \lambda_n} \left[\cos t\ (\partial_{1,1}\psi) - \sin t\ (\partial_{1,d+1}\psi)\right] \lambda_n^{k+1}\, d\lambda\,.$$

Since the derivatives $\partial_{j,k}\psi$ are evaluated at the point (72) we have in (76)

$$(77)\qquad \frac{\partial}{\partial \lambda_n}(\partial_{j,k}\psi) = \sum_{i=0}^{d} \partial_{i,n}(\partial_{j,k}\psi)$$

and also, by (68) and (72),

$$(78)\qquad \frac{\partial}{\partial \lambda_{d+1}}(\partial_{j,k}\psi) = -\cos t \sum_{0}^{d} \partial_{i,1}(\partial_{j,k}\psi) + \sin t \sum_{0}^{d} \partial_{i,d+1}(\partial_{j,k}\psi)\,.$$

We now plug (77) into (76) and then invoke equations (62) for ψ which give

$$(79)\quad \sum_{0}^{d} \partial_{i,n}\partial_{1,1}\psi = \partial_{1,n}\sum_{0}^{d}\partial_{i,1}\psi\,, \quad \sum_{0}^{d}\partial_{i,n}\partial_{1,d+1}\psi = \partial_{1,n}\sum_{0}^{d}\partial_{i,d+1}\psi\,.$$

Using (77) and (79) we see that (76) becomes

$$-(k+1)P'(t) =$$
$$\int_{\mathbf{R}^{n-d}} [\partial_{1,n}(\cos t\, \Sigma_i \partial_{i,1}\psi - \sin t\, \Sigma_i \partial_{i,d+1}\psi)]\, (x_t, x_t+E_t, \ldots, x_t+e_d)\lambda_n^{k+1}\, d\lambda$$

so by (78)

$$(k+1)P'(t) = \int_{\mathbf{R}^{n-d}} \frac{\partial}{\partial \lambda_{d+1}}(\partial_{1,n}\psi)\lambda_n^{k+1}\, d\lambda\,.$$

Since $d+1 < n$, the integration in λ_{d+1} shows that $P'(t) = 0$, proving (67).

This shows that without changing $P_{\epsilon,k}(e_n)$ we can pass from $\epsilon = (e_1, \ldots, e_d)$ to

$$\epsilon_t = (\sin\ t\, e_1 + \cos\ t\, e_{d+1}, e_2, \ldots, e_d)\,.$$

By iteration we can replace e_1 by

$$\sin t_{n-d-1} \ldots \sin t_1 e_1 + \sin t_{n-d-1} \ldots \sin t_2 \cos t_1 e_{d+1} + \cdots + \cos t_{n-d-1} e_{n-1},$$

but keeping $e_2, \ldots, e_d$ unchanged. This will reach an arbitrary E so (66) is proved.

We shall now prove (65) in general. We write σ and τ in orthonormal bases, $\sigma = (\sigma_1, \dots, \sigma_d), \tau = (\tau_1, \dots, \tau_d)$. Using Lemma 6.7 we shall pass from σ to τ without changing $P_{\sigma,k}(u)$, u being fixed.

Consider τ_1. If two members of σ, say σ_j and σ_k, are both not orthogonal to τ_1 that is ($\langle \sigma_j, \tau_1 \rangle \neq 0, \langle \sigma_k, \tau_1 \rangle \neq 0$) we rotate them in the (σ_j, σ_k)-plane so that one of them becomes orthogonal to τ_1. As remarked after (63) this has no effect on $P_{\sigma,k}(u)$. We iterate this process (with the same τ_1) and end up with an orthogonal frame $(\sigma_1^*, \dots, \sigma_d^*)$ of σ in which at most one entry σ_i^* is not orthogonal to τ_1. In this frame we replace this σ_i^* by τ_1. By Lemma 6.7 this change of σ does not alter $P_{\sigma,k}(u)$.

We now repeat this process with $\tau_2, \tau_3 \dots$, etc. Each step leaves $P_{\sigma,k}(u)$ unchanged (and u remains fixed) so this proves (65) and the theorem.

We consider now the case $d = 1, n = 3$ in more detail. Here $f \to \widehat{f}$ is the X-ray transform in $\mathbf{R}^3$. We also change the notation and write ξ for x_0, η for x_1 so $V(\{x_1 - x_0\})$ equals $|\xi - \eta|$. Then Theorem 6.5 reads as follows.

Theorem 6.8. *The X-ray transform $f \to \widehat{f}$ in $\mathbf{R}^3$ is a bijection of $\mathcal{D}(\mathbf{R}^3)$ onto the space of $\varphi \in \mathcal{D}(\mathbf{G}(1,3))$ satisfying*

$$\left(\frac{\partial}{\partial \xi_k}\frac{\partial}{\partial \eta_\ell} - \frac{\partial}{\partial \xi_\ell}\frac{\partial}{\partial \eta_k}\right)\left(\frac{\varphi(\xi,\eta)}{|\xi-\eta|}\right) = 0\,, \quad 1 \leq k, \ell \leq 3\,. \tag{80}$$

Now let $\mathbf{G}'(1,3) \subset \mathbf{G}(1,3)$ denote the open subset consisting of the *non-horizontal* lines. We shall now show that for $\varphi \in \mathcal{D}(\mathbf{G}(1,n))$ (and even for $\varphi \in \mathcal{E}(\mathbf{G}'(1,n))$) the validity of (80) for $(k,\ell) = (1,2)$ implies (80) for general (k,ℓ). Note that (71) (which is also valid for $\varphi \in \mathcal{E}(\mathbf{G}'(1,n))$) implies

$$\frac{\varphi(\xi,\eta)}{|\xi-\eta|} + \sum_1^3 (\xi_i - \eta_i)\frac{\partial}{\partial \xi_i}\left(\frac{\varphi(\xi,\eta)}{|\xi-\eta|}\right) = 0\,.$$

Here we apply $\partial/\partial \eta_k$ and obtain

$$\left(\sum_{i=1}^3 (\xi_i - \eta_i)\frac{\partial^2}{\partial \xi_i \partial \eta_k} - \frac{\partial}{\partial \xi_k} + \frac{\partial}{\partial \eta_k}\right)\left(\frac{\varphi(\xi,\eta)}{|\xi-\eta|}\right) = 0\,.$$

Exchanging ξ and η and adding we derive

$$\sum_{i=1}^3 (\xi_i - \eta_i)\left(\frac{\partial^2}{\partial \xi_i \partial \eta_k} - \frac{\partial^2}{\partial \xi_k \partial \eta_i}\right)\left(\frac{\varphi(\xi,\eta)}{|\xi-\eta|}\right) = 0 \tag{81}$$

for $k = 1,2,3$. Now assume (80) for $(k,\ell) = (1,2)$. Taking $k = 1$ in (81) we derive (80) for $(k,\ell) = (1,3)$. Then taking $k = 3$ in (81) we deduce (80) for $(k,\ell) = (3,2)$. This verifies the claim above.

We can now put this in a simpler form. Let $\ell(\xi,\eta)$ denote the line through the points $\xi \neq \eta$. Then the mapping

$$(\xi_1, \xi_2, \eta_1, \eta_2) \to \ell((\xi_1, \xi_2, 0), (\eta_1, \eta_2, -1))$$

is a bijection of $\mathbf{R}^4$ onto $\mathbf{G}'(1,3)$. The operator

$$(82) \qquad \Lambda = \frac{\partial^2}{\partial\xi_1\partial\eta_2} - \frac{\partial^2}{\partial\xi_2\partial\eta_1}$$

is a well defined differential operator on the dense open set $\mathbf{G}'(1,3)$. If $\varphi \in \mathcal{E}(\mathbf{G}(1,3))$ we denote by ψ the restriction of the function $(\xi,\eta) \to \varphi(\xi,\eta)/|\xi-\eta|$ to $\mathbf{G}'(1,3)$. Then we have proved the following result.

Theorem 6.9. *The X-ray transform $f \to \widehat{f}$ is a bijection of $\mathcal{D}(\mathbf{R}^3)$ onto the space*

$$(83) \qquad \{\varphi \in \mathcal{D}(\mathbf{G}(1,3)) : \Lambda\psi = 0\}\,.$$

We shall now rewrite the differential equation (83) in Plücker coordinates. The line joining ξ and η has Plücker coordinates $(p_1,p_2,p_3,q_1,q_2,q_3)$ given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \end{vmatrix} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}\,, \qquad q_i = \begin{vmatrix} \xi_i & 1 \\ \eta_i & 1 \end{vmatrix}$$

which satisfy

$$(84) \qquad p_1q_1 + p_2q_2 + p_3q_3 = 0\,.$$

Conversely, each ratio $(p_1 : p_2 : p_3 : q_1 : q_2 : q_3)$ determines uniquely a line provided (84) is satisfied. The set $\mathbf{G}'(1,3)$ is determined by $q_3 \neq 0$. Since the common factor can be chosen freely we fix q_3 as 1. Then we have a bijection $\tau : \mathbf{G}'(1,3) \to \mathbf{R}^4$ given by

$$(85) \qquad x_1 = p_2 + q_2,\ x_2 = -p_1 - q_1,\ x_3 = p_2 - q_2,\ x_4 = -p_1 + q_1$$

with inverse

$$(p_1,p_2,p_3,q_1,q_2) = \big(\tfrac{1}{2}(-x_2-x_4), \tfrac{1}{2}(x_1+x_3), \tfrac{1}{4}(-x_1^2-x_2^2+x_3^2+x_4^2), \tfrac{1}{2}(-x_2+x_4), \tfrac{1}{2}(x_1-x_3)\big).$$

Theorem 6.10. *If $\varphi \in \mathcal{D}(\mathbf{G}(1,3))$ satisfies (83) then the restriction $\varphi|\mathbf{G}'(1,3)$ (with $q_3 = 1$) has the form*

$$(86) \qquad \varphi(\xi,\eta) = |\xi-\eta|\ u(p_2+q_2, -p_1-q_1, p_2-q_2, -p_1+q_1)$$

where u satisfies

$$(87) \qquad \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} - \frac{\partial^2 u}{\partial x_4^2} = 0\,.$$

On the other hand, if u satisfies (87) then (86) defines a function φ on $\mathbf{G}'(1,3)$ which satisfies (80).

Proof. First assume $\varphi \in \mathcal{D}(\mathbf{G}(1,3))$ satisfies (83) and define $u \in \mathcal{E}(\mathbf{R}^4)$ by

$$u(\tau(\ell)) = \varphi(\ell)(1 + q_1^2 + q_2^2)^{-\frac{1}{2}}\,, \tag{88}$$

where $\ell \in \mathbf{G}'(1,3)$ has Plücker coordinates $(p_1, p_2, p_3, q_1, q_2, 1)$. On the line ℓ consider the points ξ, η for which $\xi_3 = 0, \eta_3 = -1$ (so $q_3 = 1$). Then since

$$p_1 = -\xi_2,\; p_2 = \xi_1,\; q_1 = \xi_1 - \eta_1,\; q_2 = \xi_2 - \eta_2$$

we have

$$\frac{\varphi(\xi,\eta)}{|\xi - \eta|} = u(\xi_1 + \xi_2 - \eta_2,\, -\xi_1 + \xi_2 + \eta_1,\, \xi_1 - \xi_2 + \eta_2,\, \xi_1 + \xi_2 - \eta_1)\,. \tag{89}$$

Now (83) implies (87) by use of the chain rule.

On the other hand, suppose $u \in \mathcal{E}(\mathbf{R}^4)$ satisfies (87). Define φ by (88). Then $\varphi \in \mathcal{E}(\mathbf{G}'(1,3))$ and by (89),

$$\Lambda\left(\frac{\varphi(\xi,\eta)}{|\xi - \eta|}\right) = 0\,.$$

As shown before the proof of Theorem 6.9 this implies that the whole system (80) is verified.

We shall now see what implications Ásgeirsson's mean-value theorem (Theorem 4.5, Chapter V) has for the range of the X-ray transform. We have from (85),

$$\int_0^{2\pi} u(r\cos\varphi, r\sin\varphi, 0, 0)\,d\varphi = \int_0^{2\pi} u(0, 0, r\cos\varphi, r\sin\varphi)\,d\varphi\,. \tag{90}$$

The first points $(r\cos\varphi, r\sin\varphi, 0, 0)$ correspond via (85) to the lines with

$$(p_1, p_2, p_3, q_1, q_2, q_3) \;=\; (-\tfrac{r}{2}\sin\varphi, \tfrac{r}{2}\cos\varphi, -\tfrac{r^2}{4}, -\tfrac{r}{2}\sin\varphi, \tfrac{r}{2}\cos\varphi, 1)$$

containing the points

$$\begin{aligned}(\xi_1, \xi_2, \xi_3) &= (\tfrac{r}{2}\cos\varphi, \tfrac{r}{2}\sin\varphi, 0)\\ (\eta_1, \eta_2, \eta_3) &= (\tfrac{r}{2}(\sin\varphi + \cos\varphi), +\tfrac{r}{2}(\sin\varphi - \cos\varphi), -1)\end{aligned}$$

with $|\xi - \eta|^2 = 1 + \frac{r^2}{4}$. The points $(0, 0, r\cos\varphi, r\sin\varphi)$ correspond via (85) to the lines with

$$(p_1, p_2, p_3, q_1, q_2, q_3) = (-\tfrac{r}{2}\sin\varphi, \tfrac{r}{2}\cos\varphi, \tfrac{r^2}{4}, \tfrac{r}{2}\sin\varphi, -\tfrac{r}{2}\cos\varphi, 1)$$

containing the points

$$\begin{aligned}(\xi_1, \xi_2, \xi_3) &= (\tfrac{r}{2}\cos\varphi, \tfrac{r}{2}\sin\varphi, 0)\\ (\eta_1, \eta_2, \eta_3) &= (\tfrac{r}{2}(\cos\varphi - \sin\varphi), \tfrac{r}{2}(\cos\varphi + \sin\varphi), -1)\end{aligned}$$

with $|\xi-\eta|^2 = 1+\frac{r^2}{4}$. Thus (90) takes the form

$$(91)\quad \int_0^{2\pi} \varphi(\tfrac{r}{2}\cos\theta, \tfrac{r}{2}\sin\theta, 0, \tfrac{r}{2}(\sin\theta+\cos\theta), \tfrac{r}{2}(\sin\theta-\cos\theta), -1)\, d\theta$$

$$= \int_0^{2\pi} \varphi(\tfrac{r}{2}\cos\theta, \tfrac{r}{2}\sin\theta, 0, \tfrac{r}{2}(\cos\theta-\sin\theta), \tfrac{r}{2}(\cos\theta+\sin\theta), -1)\, d\theta\,.$$

The lines forming the arguments of φ in these integrals are the two families of generating lines for the hyperboloid (see Fig. I.7)

$$x^2+y^2 = \tfrac{r^2}{4}(z^2+1)\,.$$

Definition. A function $\varphi \in \mathcal{E}(\mathbf{G}'(1,3))$ is said to be a *harmonic line function* if

$$\Lambda\left(\frac{\varphi(\xi,\eta)}{|\xi-\eta|}\right) = 0\,.$$

Theorem 6.11. *A function $\varphi \in \mathcal{E}(\mathbf{G}'(1,3))$ is a harmonic line function if and only if for each hyperboloid of revolution H of one sheet and vertical axis the mean values of φ over the two families of generating lines of H are equal. (The variable of integration is the polar angle in the equatorial plane of H.).*

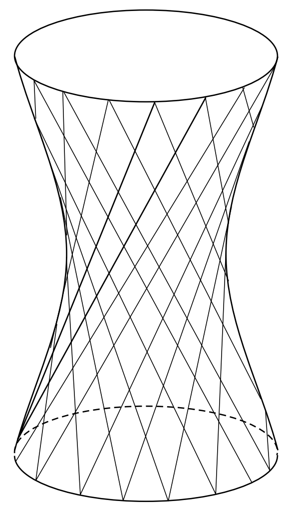

FIGURE I.7.

The proof of (91) shows that φ harmonic implies the mean value property for φ. The converse follows since (90) (with $(0,0)$ replaced by an arbitrary point in $\mathbf{R}^2$) is equivalent to (87) (Chapter V, Theorem 4.5).

Corollary 6.12. *Let $\varphi \in \mathcal{D}(\mathbf{G}(1,3))$. Then φ is in the range of the X-ray transform if and only if φ has the mean value property for arbitrary hyperboloid of revolution of one sheet (and arbitrary axis).*

We conclude this section with the following result due to Schlichtkrull mentioned in connection with Theorem 6.4.

Proposition 6.13. *Let f be a function on $\mathbf{R}^n$ and $k \in \mathbb{Z}^+$, $1<k<n$. Assume that for each k-dimensional subspace $E_k \subset \mathbf{R}^n$ the restriction $f|E_k$ is a polynomial on E_k. Then f is a polynomial on $\mathbf{R}^n$.*

For $k=1$ the result is false as the example $f(x,y) = xy^2/(x^2+y^2)$, $f(0,0)=0$ shows. We recall now the Lagrange interpolation formula. Let $a_0,\ldots,a_m$ be distinct numbers in $\mathbf{C}$. Then each polynomial $P(x)$ $(x \in \mathbf{R})$

of degree $\leq m$ can be written

$$P(x) = P(a_0)Q_0(x) + \cdots + P(a_m)Q_m(x)\,,$$

where

$$Q_i(x) = \prod_{j=0}^{m}(x-a_j)/(x-a_i)\prod_{j\neq i}(a_i-a_j)\,.$$

In fact, the two sides agree at $m+1$ distinct points. This implies the following result.

Lemma 6.14. *Let $f(x_1,\ldots,x_n)$ be a function on $\mathbf{R}^n$ such that for each i with $x_j (j \neq i)$ fixed the function $x_i \to f(x_1,\ldots,x_n)$ is a polynomial. Then f is a polynomial.*

For this we use Lagrange's formula on the polynomial $x_1 \longrightarrow f(x_1,x_2,\ldots,x_n)$ and get

$$f(x_1,\ldots,x_n) = \sum_{j=0}^{m} f(a_j,x_2,\ldots,x_m)Q_j(x_1)\,.$$

The lemma follows by iteration.

For the proposition we observe that the assumption implies that f restricted to each 2-plane E_2 is a polynomial on E_2. For a fixed $(x_2,\ldots,x_n)$ the point $(x_1,\ldots,x_n)$ is in the span of $(1,0,\ldots,0)$ and $(0,x_2,\ldots,x_n)$ so $f(x_1,\ldots,x_n)$ is a polynomial in x_1. Now the lemma implies the result.

§7 Applications

A. Partial differential equations.

The inversion formula in Theorem 3.1 is very well suited for applications to partial differential equations. To explain the underlying principle we write the inversion formula in the form

$$f(x) = \gamma\, L_x^{\frac{n-1}{2}}\left(\int_{\mathbf{S}^{n-1}} \widehat{f}(\omega,\langle x,\omega\rangle)\,d\omega\right). \tag{92}$$

where the constant γ equals $\frac{1}{2}(2\pi i)^{1-n}$. Note that the function $f_\omega(x) = \widehat{f}(\omega,\langle x,\omega\rangle)$ is a *plane wave with normal* ω, that is, it is constant on each hyperplane perpendicular to ω.

Consider now a differential operator

$$D = \sum_{(k)} a_{k_1\ldots k_n}\partial_1^{k_1}\ldots\partial_n^{k_n}$$

with constant coefficients $a_{k_1,\ldots,k_n}$, and suppose we want to solve the differential equation

$$Du = f \tag{93}$$

where f is a given function in $\mathcal{S}(\mathbf{R}^n)$. To simplify the use of (92) we assume n to be odd. We begin by considering the differential equation

$$Dv = f_\omega\,, \tag{94}$$

where f_ω is the plane wave defined above and we look for a solution v which is also a plane wave with normal ω. But a plane wave with normal ω is just a function of one variable; also if v is a plane wave with normal ω so is the function Dv. The differential equation (94) (with v a plane wave) is therefore an *ordinary* differential equation with constant coefficients. Suppose $v = u_\omega$ is a solution and assume that this choice can be made smoothly in ω. Then the function

$$u = \gamma\, L^{\frac{n-1}{2}} \int_{\mathbf{S}^{n-1}} u_\omega\, d\omega \tag{95}$$

is a solution to the differential equation (93). In fact, since D and $L^{\frac{n-1}{2}}$ commute we have

$$Du = \gamma L^{\frac{n-1}{2}} \int_{\mathbf{S}^{n-1}} Du_\omega\, d\omega = \gamma L^{\frac{n-1}{2}} \int_{\mathbf{S}^{n-1}} f_\omega\, d\omega = f\,.$$

This method only assumes that the plane wave solution u_ω to the ordinary differential equation $Dv = f_\omega$ exists and can be chosen so as to depend smoothly on ω. This cannot always be done because D might annihilate all plane waves with normal ω. (For example, take $D = \partial^2/\partial x_1 \partial x_2$ and $\omega = (1,0)$.) However, if this restriction to plane waves is never 0 it follows from a theorem of Trèves [1963] that the solution u_ω can be chosen depending smoothly on ω. Thus we can state

Theorem 7.1. *Assuming the restriction D_ω of D to the space of plane waves with normal ω is $\neq 0$ for each ω formula (95) gives a solution to the differential equation $Du = f \quad (f \in \mathcal{S}(\mathbf{R}^n))$.*

The method of plane waves can also be used to solve the Cauchy problem for hyperbolic differential equations with constant coefficients. We illustrate this method by means of the wave equation $\mathbf{R}^n$,

$$Lu = \frac{\partial^2 u}{\partial t^2},\ u(x,0) = f_0(x),\ u_t(x,0) = f_1(x)\,, \tag{96}$$

f_0, f_1 being given functions in $\mathcal{D}(\mathbf{R}^n)$.

Lemma 7.2. *Let $h \in C^2(\mathbf{R})$ and $\omega \in \mathbf{S}^{n-1}$. Then the function*

$$v(x,t) = h(\langle x, \omega\rangle + t)$$

satisfies $Lv = (\partial^2/\partial t^2)v$.

The proof is obvious. It is now easy, on the basis of Theorem 3.6, to write down the unique solution of the Cauchy problem (96).

Theorem 7.3. *The solution to (96) is given by*

$$u(x,t) = \int_{\mathbf{S}^{n-1}} (Sf)(\omega, \langle x, \omega\rangle + t)\, d\omega \tag{97}$$

where

$$Sf = \begin{cases} c(\partial^{n-1}\widehat{f}_0 + \partial^{n-2}\widehat{f}_1)\,, & n \text{ odd} \\ c\,\mathcal{H}(\partial^{n-1}\widehat{f}_0 + \partial^{n-2}\widehat{f}_1)\,, & n \text{ even}\,. \end{cases}$$

Here $\partial = \partial/\partial p$ and the constant c equals

$$c = \tfrac{1}{2}(2\pi i)^{1-n}\,.$$

Lemma 7.2 shows that (97) is annihilated by the operator $L - \partial^2/\partial t^2$ so we just have to check the initial conditions in (96).

(a) If $n > 1$ is odd then $\omega \to (\partial^{n-1}\widehat{f}_0)(\omega, \langle x, \omega\rangle)$ is an even function on $\mathbf{S}^{n-1}$ but the other term in Sf, that is the function $\omega \to (\partial^{n-2}\widehat{f}_1)(\omega, \langle x, \omega\rangle)$, is odd. Thus by Theorem 3.6, $u(x, 0) = f_0(x)$. Applying $\partial/\partial t$ to (97) and putting $t = 0$ gives $u_t(x, 0) = f_1(x)$, this time because the function $\omega \to (\partial^n\widehat{f}_0)(\omega, \langle x, \omega\rangle)$ is odd and the function $\omega \to (\partial^{n-1}\widehat{f}_1)(\omega, \langle x, \omega\rangle)$ is even.

(b) If n is even the same proof works if we take into account the fact that $\mathcal{H}$ interchanges odd and even functions on $\mathbf{R}$.

Definition. For the pair $f = \{f_0, f_1\}$ we refer to the function Sf in (97) as the *source*.

In the terminology of Lax-Philips [1967] the wave $u(x,t)$ is said to be

(a) *outgoing* if $u(x,t) = 0$ in the *forward cone* $|x| < t$;

(b) *incoming* if $u(x,t) = 0$ in the *backward cone* $|x| < -t$.

The notation is suggestive because "outgoing" means that the function $x \to u(x,t)$ vanishes in larger balls around the origin as t increases.

Corollary 7.4. *The solution $u(x,t)$ to (96) is*

(i) outgoing if and only if $(Sf)(\omega, s) = 0$ for $s > 0$, all ω.

(ii) incoming if and only if $(Sf)(\omega, s) = 0$ for $s < 0$, all ω.

Proof. For (i) suppose $(Sf)(\omega, s) = 0$ for $s > 0$. For $|x| < t$ we have $\langle x, \omega\rangle + t \geq -|x| + t > 0$ so by (97) $u(x,t) = 0$ so u is outgoing. Conversely, suppose $u(x,t) = 0$ for $|x| < t$. Let $t_0 > 0$ be arbitrary and let $\varphi(t)$ be a smooth function with compact support contained in (t_0, ∞).

Then if $|x| < t_0$ we have

$$\begin{aligned} 0 &= \int_{\mathbf{R}} u(x,t)\varphi(t)\,dt = \int_{\mathbf{S}^{n-1}} d\omega \int_{\mathbf{R}} (Sf)(\omega, \langle x, \omega\rangle + t)\varphi(t)\,dt \\ &= \int_{\mathbf{S}^{n-1}} d\omega \int_{\mathbf{R}} (Sf)(\omega, p)\varphi(p - \langle x, \omega\rangle)\,dp\,. \end{aligned}$$

Taking arbitrary derivative $\partial^k/\partial x_{i_1} \dots \partial x_{i_k}$ at $x = 0$ we deduce

$$\int_{\mathbf{R}} \left(\int_{\mathbf{S}^{n-1}} (Sf)(\omega, p)\omega_{i_1} \dots \omega_{i_k}\,d\omega \right) (\partial^k \varphi)(p)\,dp = 0$$

for each k and each $\varphi \in \mathcal{D}(t_0, \infty)$. Integrating by parts in the p variable we conclude that the function

$$p \to \int_{\mathbf{S}^{n-1}} (Sf)(\omega, p)\omega_{i_1} \dots \omega_{i_k}\,d\omega\,, \quad p \in \mathbf{R} \tag{98}$$

has its k^{th} derivative $\equiv 0$ for $p > t_0$. Thus it equals a polynomial for $p > t_0$. However, if n is odd the function (98) has compact support so it must vanish identically for $p > t_0$.

On the other hand, if n is even and $F \in \mathcal{D}(\mathbf{R})$ then as remarked at the end of §3, $\lim_{|t|\to\infty} (\mathcal{H}F)(t) = 0$. Thus we conclude again that expression (98) vanishes identically for $p > t_0$.

Thus in both cases, if $p > t_0$, the function $\omega \to (Sf)(\omega, p)$ is orthogonal to all polynomials on $\mathbf{S}^{n-1}$, hence must vanish identically.

One can also solve (96) by means of the *Fourier transform*

$$\widetilde{f}(\zeta) = \int_{\mathbf{R}^n} f(x) e^{-i\langle x, \zeta\rangle}\,dx\,.$$

Assuming the function $x \to u(x,t)$ in $\mathcal{S}(\mathbf{R}^n)$ for a given t we obtain

$$\widetilde{u}_{tt}(\zeta, t) + \langle \zeta, \zeta\rangle \widetilde{u}(\zeta, t) = 0\,.$$

Solving this ordinary differential equation with initial data given in (96) we get

$$\widetilde{u}(\zeta, t) = \widetilde{f}_0(\zeta)\cos(|\zeta| t) + \widetilde{f}_1(\zeta)\frac{\sin(|\zeta| t)}{|\zeta|}\,. \tag{99}$$

The function $\zeta \to \sin(|\zeta| t)/|\zeta|$ is entire of exponential type $|t|$ on $\mathbf{C}^n$ (of at most polynomial growth on $\mathbf{R}^n$). In fact, if $\varphi(\lambda)$ is even, holomorphic on **C** and satisfies the exponential type estimate (13) in Theorem 3.3, Ch. V, then the same holds for the function Φ on $\mathbf{C}^n$ given by $\Phi(\zeta) = \Phi(\zeta_1, \dots, \zeta_n) = \varphi(\lambda)$ where $\lambda^2 = \zeta_1^2 + \cdots + \zeta_n^2$. To see this put

$$\lambda = \mu + iv\,, \quad \zeta = \xi + i\eta \qquad \mu, \nu \in \mathbf{R}\,, \quad \xi\,, \eta \in \mathbf{R}^n\,.$$

Then

$$\mu^2 - \nu^2 = |\xi|^2 - |\eta|^2\,, \quad \mu^2\nu^2 = (\xi \cdot \eta)^2\,,$$

so

$$|\lambda|^4 = (|\xi|^2 \,-\, |\eta|^2)^2 + 4(\xi \cdot \eta)^2$$

and

$$2|\mathrm{Im}\,\lambda|^2 = |\eta|^2 - |\xi|^2 + \left[(|\xi|^2 - |\eta|^2)^2 + 4(\xi \cdot \eta)^2\right]^{1/2}\,.$$

Since $|(\xi \cdot \eta)| \leq |\xi|\,|\eta|$ this implies $|\mathrm{Im}\,\lambda| \leq |\eta|$ so the estimate (13) follows for Φ. Thus by Theorem 3.3, Chapter V there exists a $T_t \in \mathcal{E}'(\mathbf{R}^n)$ with support in $\overline{B_{|t|}(0)}$ such that

$$\frac{\sin(|\zeta|t)}{|\zeta|} = \int_{\mathbf{R}^n} e^{-i\langle\zeta, x\rangle}\, dT_t(x)\,.$$

Theorem 7.5. *Given $f_0, f_1 \in \mathcal{E}(\mathbf{R}^n)$ the function*

$$u(x,t) = (f_0 * T_t')(x) + (f_1 * T_t)(x) \tag{100}$$

satisfies (96). Here T_t' stands for $\partial_t(T_t)$.

Note that (96) implies (100) if f_0 and f_1 have compact support. The converse holds without this support condition.

Corollary 7.6. *If f_0 and f_1 have support in $B_R(0)$ then u has support in the region*

$$|x| \leq |t| + R\,.$$

In fact, by (100) and support property of convolutions (Ch. V, §2), the function $x \to u(x,t)$ has support in $B_{R+|t|}(0)^-$. While Corollary 7.6 implies that for $f_0, f_1 \in \mathcal{D}(\mathbf{R}^n)$ u has support in a suitable solid cone we shall now see that Theorem 7.3 implies that if n is odd u has support in a conical shell (see Fig. I.8).

Corollary 7.7. *Let n be odd. Assume f_0 and f_1 have support in the ball $B_R(0)$.*

(i) Huygens' Principle. The solution u to (96) has support in the conical shell

$$|t| - R \leq |x| \leq |t| + R\,, \tag{101}$$

which is the union for $|y| \leq R$ of the light cones,

$$C_y = \{(x,t) : |x - y| = |t|\}\,.$$

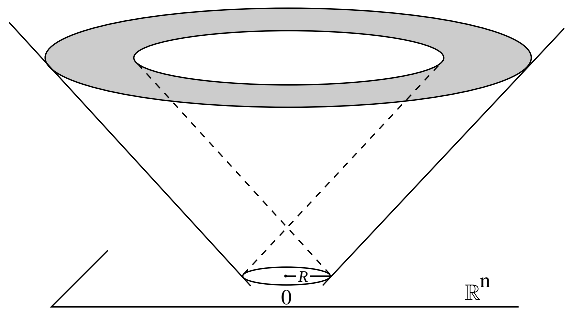

FIGURE I.8.

(ii) The solution to (96) is outgoing if and only if

$$\widehat{f}_0(\omega,p) = \int_p^\infty \widehat{f}_1(\omega,s)\,ds\,, \quad p>0\,,\ \textit{all}\ \omega \tag{102}$$

and incoming if and only if

$$\widehat{f}_0(\omega,p) = -\int_{-\infty}^p \widehat{f}_1(\omega,s)\,ds\,, \quad p<0\,,\ \textit{all}\ \omega\,.$$

Note that Part (ii) can also be stated: The solution is outgoing (incoming) if and only if

$$\int_\pi f_0 = \int_{H_\pi} f_1 \qquad \left(\int_\pi f_0 = -\int_{H_\pi} f_1\right)$$

for an arbitrary hyperplane $\pi (0 \notin \pi)$ H_π being the halfspace with boundary π which does not contain 0.

To verify (i) note that since n is odd, Theorem 7.3 implies

$$u(0,t) = 0 \quad \text{for} \quad |t| \geq R\,. \tag{103}$$

If $z \in \mathbf{R}^n$, $F \in \mathcal{E}(\mathbf{R}^n)$ we denote by F^z the translated function $y \to F(y+z)$. Then u^z satisfies (96) with initial data f_0^z, f_1^z which have support contained in $B_{R+|z|}(0)$. Hence by (103)

$$u(z,t) = 0 \quad \text{for} \quad |t| > R + |z|\,. \tag{104}$$

The other inequality in (101) follows from Corollary 7.6.

For the final statement in (i) we note that if $|y| \leq R$ and $(x,t) \in C_y$ then $|x-y| = t$ so $|x| \leq |x-y| + |y| \leq |t| + R$ and $|t| = |x-y| \leq |x| + R$ proving (101). Conversely, if (x,t) satisfies (101) then $(x,t) \in C_y$ with $y = x - |t|\frac{x}{|x|} = \frac{x}{|x|}(|x|-t)$ which has norm $\leq R$.

For (ii) we just observe that since $\widehat{f}_i(\omega,p)$ has compact support in p, (102) is equivalent to (i) in Corollary 7.4.

Thus (102) implies that for $t>0$, $u(x,t)$ has support in the thinner shell $|t| \leq |x| \leq |t| + R$.

B. X-ray Reconstruction.

The classical interpretation of an X-ray picture is an attempt at reconstructing properties of a 3-dimensional body by means of the X-ray projection on a plane.

In modern X-ray technology the picture is given a more refined mathematical interpretation. Let $B \subset \mathbf{R}^3$ be a body (for example a part of a human body) and let $f(x)$ denote its density at a point x. Let ξ be a line in $\mathbf{R}^3$ and suppose a thin beam of X-rays is directed at B along ξ. Let I_0 and I respectively, denote the intensity of the beam before entering B and after leaving B (see Fig. I.9). As the X-ray traverses the distance Δx along ξ it will undergo the relative intensity loss $\Delta I/I = f(x)\,\Delta x$. Thus $dI/I = -f(x)\,dx$ whence

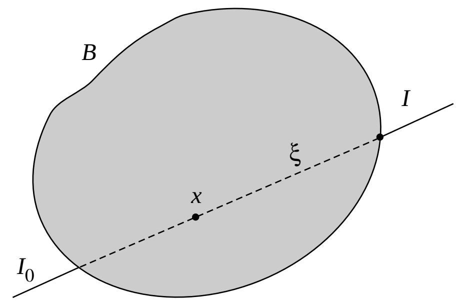

FIGURE I.9.

$$\log(I_0/I) = \int_\xi f(x)\,dx\,, \tag{105}$$

the integral $\widehat{f}(\xi)$ of f along ξ. Since the left hand side is determined by the X-ray picture, the *X-ray reconstruction problem* amounts to the determination of the function f by means of its line integrals $\widehat{f}(\xi)$. *The inversion formula in Theorem 3.1 gives an explicit solution of this problem.*

If $B_0 \subset B$ is a convex subset (for example the heart) it may be of interest to determine the density of f outside B_0 using only X-rays which do not intersect B_0. *The support theorem (Theorem 2.6, Cor. 2.8 and Cor. 6.1) implies that f is determined outside B_0 on the basis of the integrals $\widehat{f}(\xi)$ for which ξ does not intersect B_0.* **Thus the density outside the heart can be determined by means of X-rays which bypass the heart.**

In practice one can of course only determine the integrals $\widehat{f}(\xi)$ in (105) for *finitely* many directions. A compensation for this is the fact that only an approximation to the density f is required. One then encounters the mathematical problem of selecting the directions so as to optimize the approximation.

As before we represent the line ξ as the pair $\xi = (\omega, z)$ where $\omega \in \mathbf{R}^n$ is a unit vector in the direction of ξ and $z = \xi \cap \omega^\perp$ ($\perp$ denoting orthogonal complement). We then write

$$\widehat{f}(\xi) = \widehat{f}(\omega, z) = (P_\omega f)(z)\,. \tag{106}$$

The function $P_\omega f$ is the X-ray picture or the *radiograph* in the direction ω. Here f is a function on $\mathbf{R}^n$ vanishing outside a ball B around the origin and for the sake of Hilbert space methods to be used it is convenient to assume in addition that $f \in L^2(B)$. Then $f \in L^1(\mathbf{R}^n)$ so by the Fubini theorem we have: for each $\omega \in \mathbf{S}^{n-1}$, $P_\omega f(z)$ is defined for almost all $z \in \omega^\perp$. Moreover, we have in analogy with (4),

$$\widetilde{f}(\zeta) = \int_{\omega^\perp} (P_\omega f)(z) e^{-i\langle z,\zeta\rangle}\, dz \quad (\zeta \in \omega^\perp)\,. \tag{107}$$

Proposition 7.8. *An object is determined by any infinite set of radiographs.*

In other words, a compactly supported function f is determined by the functions $P_\omega f$ for any infinite set of ω.

Proof. Since f has compact support $\widetilde{f}$ is an analytic function on $\mathbf{R}^n$. But if $\widetilde{f}(\zeta) = 0$ for $\zeta \in \omega^\perp$ we have $\widetilde{f}(\eta) = \langle \omega, \eta\rangle g(\eta) \quad (\eta \in \mathbf{R}^n)$ where g is also analytic. If $P_{\omega_1} f, \ldots, P_{\omega_k} f \ldots$ all vanish identically for an infinite set $\omega_1, \ldots, \omega_k \ldots$ we see that for each k

$$\widetilde{f}(\eta) = \prod_{i=1}^{k} \langle \omega_i, \eta\rangle g_k(\eta)\,,$$

where g_k is analytic. But this would contradict the power series expansion of $\widetilde{f}$ which shows that for a suitable $\omega \in \mathbf{S}^{n-1}$ and integer $r \geq 0$, $\lim_{t\to 0} \widetilde{f}(t\omega)t^{-r} \neq 0$.

If only finitely many radiographs are used we get the opposite result.

Proposition 7.9. *Let $\omega_1, \ldots, \omega_k \in \mathbf{S}^{n-1}$ be an arbitrary finite set. Then there exists a function $f \in \mathcal{D}(\mathbf{R}^n), f \not\equiv 0$ such that*

$$P_{\omega_i} f \equiv 0 \quad \textit{for all} \quad 1 \leq i \leq k\,.$$

Proof. We have to find $f \in \mathcal{D}(\mathbf{R}^n), f \not\equiv 0$, such that $\widetilde{f}(\zeta) = 0$ for $\zeta \in \omega_i^\perp (1 \leq i \leq k)$. For this let D be the constant coefficient differential operator such that

$$(Du)\widetilde{\ }(\eta) = \prod_{1}^{k} \langle \omega_i, \eta\rangle \widetilde{u}(\eta) \quad \eta \in \mathbf{R}^n\,.$$

If $u \not\equiv 0$ is any function in $\mathcal{D}(\mathbf{R}^n)$ then $f = Du$ has the desired property.

We next consider the problem of *approximate reconstruction* of the function f from a finite set of radiographs $P_{\omega_1} f, \ldots, P_{\omega_k} f$.

Let N_j denote the null space of P_{ω_j} and let P_j the orthogonal projection of $L^2(B)$ on the plane $f + N_j$; in other words

$$P_j g = Q_j(g - f) + f\,, \tag{108}$$

where Q_j is the (linear) projection onto the subspace $N_j \subset L^2(B)$. Put $P = P_k \dots P_1$. Let $g \in L^2(B)$ be arbitrary (the initial guess for f) and form the sequence $P^m g, m = 1, 2, \dots$. Let $N_0 = \cap_1^k N_j$ and let P_0 (resp. Q_0) denote the orthogonal projection of $L^2(B)$ on the plane $f + N_0$ (subspace N_0). We shall prove that the sequence $P^m g$ converges to the projection $P_0 g$. This is natural since by $P_0 g - f \in N_0$, $P_0 g$ and f have the same radiographs in the directions $\omega_1, \dots, \omega_k$.

Theorem 7.10. *With the notations above,*

$$P^m g \longrightarrow P_0 g \quad \textit{as} \quad m \longrightarrow \infty$$

for each $g \in L^2(B)$.

Proof. We have, by iteration of (108)

$$(P_k \dots P_1)g - f = (Q_k \dots Q_1)(g - f)$$

and, putting $Q = Q_k \dots Q_1$ we obtain

$$P^m g - f = Q^m (g - f)\,.$$

We shall now prove that $Q^m g \longrightarrow Q_0 g$ for each g; since

$$P_0 g = Q_0(g - f) + f$$

this would prove the result. But the statement about Q^m comes from the following general result about abstract Hilbert space.

Theorem 7.11. *Let $\mathcal{H}$ be a Hilbert space and Q_i the projection of $\mathcal{H}$ onto a subspace $N_i \subset \mathcal{H}(1 \le i \le k)$. Let $N_0 = \cap_1^k N_i$ and $Q_0 : \mathcal{H} \longrightarrow N_0$ the projection. Then if $Q = Q_k \dots Q_1$*

$$Q^m g \longrightarrow Q_0 g \quad \textit{for each} \quad g \in \mathcal{H}, .$$

Since Q is a contraction ($\|Q\| \le 1$) we begin by proving a simple lemma about such operators.

Lemma 7.12. *Let $T : \mathcal{H} \longrightarrow \mathcal{H}$ be a linear operator of norm ≤ 1. Then*

$$\mathcal{H} = C\ell((I - T)\mathcal{H}) \oplus \ \textit{Null space}\ (I - T)$$

is an orthogonal decomposition, $C\ell$ denoting closure, and I the identity.

Proof. If $Tg = g$ then since $\|T^*\| = \|T\| \le 1$ we have

$$\|g\|^2 = (g, g) = (Tg, g) = (g, T^* g) \le \|g\| \|T^* g\| \le \|g\|^2$$

so all terms in the inequalities are equal. Hence

$$\|g - T^* g\|^2 = \|g\|^2 - (g, T^* g) - (T^* g, g) + \|T^* g\|^2 = 0$$

so $T^* g = g$. Thus $I - T$ and $I - T^*$ have the same null space. But $(I - T^*)g = 0$ is equivalent to $(g, (I - T)\mathcal{H}) = 0$ so the lemma follows.

Definition. An operator T on a Hilbert space $\mathcal{H}$ is said to have *property S* if

$$\|f_n\| \leq 1, \|Tf_n\| \longrightarrow 1 \ \text{ implies } \|(I-T)f_n\| \longrightarrow 0\,. \tag{109}$$

Lemma 7.13. *A projection, and more generally a finite product of projections, has property* (S).

Proof. If T is a projection then

$$\|(I-T)f_n\|^2 = \|f_n\|^2 - \|Tf_n\|^2 \leq 1 - \|Tf_n\|^2 \longrightarrow 0$$

whenever

$$\|f_n\| \leq 1 \text{ and } \|Tf_n\| \longrightarrow 1\,.$$

Let T_2 be a projection and suppose T_1 has property (S) and $\|T_1\| \leq 1$. Suppose $f_n \in \mathcal{H}$ and $\|f_n\| \leq 1, \|T_2T_1f_n\| \longrightarrow 1$. The inequality implies $\|T_1f_n\| \leq 1$ and since

$$\|T_1f_n\|^2 = \|T_2T_1f_n\|^2 + \|(I-T_2)(T_1f_n)\|^2$$

we also deduce $\|T_1f_n\| \longrightarrow 1$. Writing

$$(I - T_2T_1)f_n = (I-T_1)f_n + (I-T_2)T_1f_n$$

we conclude that T_2T_1 has property (S). The lemma now follows by induction.

Lemma 7.14. *Suppose T has property (S) and $\|T\| \leq 1$. Then for each $f \in \mathcal{H}$*

$$T^nf \longrightarrow \pi f \quad \text{as} \quad n \longrightarrow \infty\,,$$

where π is the projection onto the fixed point space of T.

Proof. Let $f \in \mathcal{H}$. Since $\|T\| \leq 1, \|T^nf\|$ decreases monotonically to a limit $\alpha \geq 0$. If $\alpha = 0$ we have $T^nf \longrightarrow 0$. By Lemma 7.12 $\pi T = T\pi$ so $\pi f = T^n\pi f = \pi T^nf$ so $\pi f = 0$ in this case. If $\alpha > 0$ we put $g_n = \|T^nf\|^{-1}(T^nf)$. Then $\|g_n\| = 1$ and $\|Tg_n\| \to 1$. Since T has property (S) we deduce

$$T^n(I-T)f = \|T^nf\|(I-T)g_n \longrightarrow 0,.$$

Thus $T^nh \longrightarrow 0$ for all h in the range of $I-T$. If g is in the closure of this range then given $\epsilon > 0$ there exist $h \in (I-T)\mathcal{H}$ such that $\|g-h\| < \epsilon$. Then

$$\|T^ng\| \leq \|T^n(g-h)\| + T^nh\| < \epsilon + \|T^nh\|$$

whence $T^ng \longrightarrow 0$. On the other hand, if h is in the null space of $I-T$ then $Th = h$ so $T^nh \longrightarrow h$. Now the lemma follows from Lemma 7.12.

In order to deduce Theorem 7.11 from Lemmas 7.13 and 7.14 we just have to verify that N_0 is the fixed point space of Q. But if $Qg = g$ then

$$\|g\| = \|Q_k \dots Q_1 g\| \leq \|Q_{k-1} \dots Q_1 g\| \leq \dots \leq \|Q_1 g\| \leq \|g\|$$

so equality signs hold everywhere. But the Q_i are projections so the norm identities imply

$$g = Q_1 g = Q_2 Q_1 g = \dots = Q_k \dots Q_1 g$$

which shows $g \in N_0$. This proves Theorem 7.11.

Bibliographical Notes

§§*1-2.* The inversion formulas

(i) $f(x) = \frac{1}{2}(2\pi i)^{1-n} L_x^{(n-1)/2} \int_{\mathbf{S}^{n-1}} J(\omega, \langle \omega, x \rangle)\, , d\omega$ (n odd)

(ii) $f(x) = \frac{1}{2}(2\pi i)^{-n} L_x^{(n-2)/2} \int_{\mathbf{S}^{n-1}} d\omega \int_{-\infty}^{\infty} \frac{dJ(\omega,p)}{p-(\omega,x)}$ (n even)

for a function $f \in \mathcal{D}(X)$ in terms of its plane integrals $J(\omega, p)$ go back to Radon [1917] and John [1934], [1955]. According to Bockwinkel [1906] the case $n = 3$ had been proved before 1906 by H.A. Lorentz, but fortunately, both for Lorentz and Radon, the transformation $f(x) \to J(\omega, p)$ was not baptized "Lorentz transformation". In John [1955] the proofs are based on the Poisson equation $Lu = f$. Other proofs, using distributions, are given in Gelfand-Shilov [1960]. See also Nievergelt [1986]. The dual transforms, $f \to \widehat{f}, \varphi \to \check{\varphi}$, the unified inversion formula and its dual,

$$cf = L^{(n-1)/2}((\widehat{f})^{\vee})\,, \quad c\varphi = \square^{(n-1)/2}((\check{\varphi})\widehat{\ })$$

were given by the author in [1964]. The second proof of Theorem 3.1 is from the author's paper [1959]. It is valid for constant curvature spaces as well. The version in Theorem 3.6 is also proved in Ludwig [1966].

The support theorem, the Paley-Wiener theorem and the Schwartz theorem (Theorems 2.4,2.6, 2.10) are from Helgason [1964], [1965a]. The example in Remark 2.9 was also found by D.J. Newman, cf. Weiss' paper [1967], which gives another proof of the support theorem. See also Droste [1983]. The local result in Corollary 2.12 goes back to John [1935]; our derivation is suggested by the proof of a similar lemma in Flensted-Jensen [1977], p. 81. Another proof is in Ludwig [1966].See Palamodov and Denisjuk [1988] for a related inversion formula.

The simple geometric Lemma 2.7 is from the authors paper [1965a] and is extended to hyperbolic spaces in [1980b]. In the Proceedings containing

[1966a] the author raised the problem (p. 174) to extend Lemma 2.7 to each complete simply connected Riemannian manifold M of negative curvature. If in addition M is analytic this was proved by Quinto [1993b] and Grinberg and Quinto [1998]. This is an example of injectivity and support results obtained by use of the techniques of microlocal analysis and wave front sets. As further samples involving very general Radon transforms we mention Boman [1990], [1992], [1993], Boman and Quinto [1987], [1993], Quinto [1983], [1992], [1993b], [1994a], [1994b], Agranovsky and Quinto [1996], Gelfand, Gindikin and Shapiro [1979].

Corollary 2.8 is derived by Ludwig [1966] in a different way. There he proposes alternative proofs of the Schwartz- and Paley-Wiener theorems by expanding $\widehat{f}(\omega, p)$ in spherical harmonics in ω. However, the principal point—the smoothness of the function F in the proof of Theorem 2.4—is overlooked. Theorem 2.4 is from the author's papers [1964] [1965a].

Since the inversion formula is rather easy to prove for odd n it is natural to try to prove Theorem 2.4 for this case by showing directly that if $\varphi \in \mathcal{S}_H(\mathbf{P}^n)$ then the function $f = cL^{(n-1)/2}(\check{\varphi})$ for n odd belongs to $\mathcal{S}(\mathbf{R}^n)$ (in general $\check{\varphi} \notin \mathcal{S}(\mathbf{R}^n)$). This approach is taken in Gelfand-Graev-Vilenkin [1966], pp. 16-17. However, this method seems to offer some unresolved technical difficulties. For some generalizations see Kuchment and Lvin [1990], Aguilar, Ehrenpreis and Kuchment [1996] and Katsevich [1997]. Cor. 2.5 is stated in Semyanisty [1960].

§5. The approach to Radon transforms of distributions adopted in the text is from the author's paper [1966a]. Other methods are proposed in Gelfand-Graev-Vilenkin [1966] and in Ludwig [1966]. See also Ramm [1995].

§6. The d-plane transform and Theorem 6.2 are from Helgason [1959], p. 284. Formula (55) was already proved by Fuglede [1958]. The range characterization for the d-plane transform in Theorem 6.3 is from the 1980-edition of this book and was used by Kurusa [1991] to prove Theorem 6.5, which generalizes John's range theorem for the X-ray transform in $\mathbf{R}^3$ [1938]. The geometric range characterization (Corollary 6.12) is also due to John [1938]. Papers devoted to the d-plane range question for $\mathcal{S}(\mathbf{R}^n)$ are Gelfand-Gindikin and Graev [1982], Grinberg [1987], Richter [1986] and Gonzalez [1991]. This last paper gives the range as the kernel of a single 4^{th} order differential operator on the space of d-planes. As shown by Gonzalez, the analog of Theorem 6.3 fails to hold for $\mathcal{S}(\mathbf{R}^n)$. An L^2-version of Theorem 6.3 was given by Solmon [1976], p. 77. Proposition 6.13 was communicated to me by Schlichtkrull.

Some difficulties with the d-plane transform on $L^2(\mathbf{R}^n)$ are pointed out by Smith and Solmon [1975] and Solmon [1976], p. 68. In fact, the function $f(x) = |x|^{-\frac{1}{2}n}(\log|x|)^{-1} \quad (|x| \geq 2)$, 0 otherwise, is square integrable on $\mathbf{R}^n$ but is not integrable over any plane of dimension $\geq \frac{n}{2}$. Nevertheless, see for example Rubin [1998a], Strichartz [1981] for L^p-extensions of the d-plane transform.

§7. The applications to partial differential equations go in part back to Herglotz [1931]; see John [1955]. Other applications of the Radon transform to partial differential equations with constant coefficients can be found in Courant-Lax [1955], Gelfand-Shapiro [1955], John [1955], Borovikov [1959], Gårding [1961] and Ludwig [1966]. Our discussion of the wave equation (Theorem 7.3 and Corollary 7.4) is closely related to the treatment in Lax-Phillips [1967], Ch. IV, where however, the dimension is assumed to be odd. Applications to general elliptic equations were given by John [1955].

While the Radon transform on $\mathbf{R}^n$ can be used to "reduce" partial differential equations to ordinary differential equations one can use a Radon type transform on a symmetric space X to "reduce" an invariant differential operator D on X to a partial differential operator with constant coefficients. For an account of these applications see the author's monograph [1994b], Chapter V.

While the applications to differential equations are perhaps the most interesting to mathematicians, the tomographic applications of the X-ray transform have revolutionized medicine. These applications originated with Cormack [1963], [1964] and Hounsfield [1973]. For the approximate reconstruction problem, including Propositions 7.8 and 7.9 and refinements of Theorems 7.10, 7.11 see Smith, Solmon and Wagner [1977], Solmon [1976] and Hamaker and Solmon [1978]. Theorem 7.11 is due to Halperin [1962], the proof in the text to Amemiya and Ando [1965]. For an account of some of those applications see e.g. Deans [1983], Natterer [1986] and Ramm and Katsevich [1996]. Applications in radio astronomy appear in Bracewell and Riddle [1967].

CHAPTER II

A DUALITY IN INTEGRAL GEOMETRY. GENERALIZED RADON TRANSFORMS AND ORBITAL INTEGRALS

§1 Homogeneous Spaces in Duality

The inversion formulas in Theorems 3.1, 3.5, 3.6 and 6.2, Ch. I suggest the general problem of determining a function on a manifold by means of its integrals over certain submanifolds. In order to provide a natural framework for such problems we consider the Radon transform $f \to \widehat{f}$ on $\mathbf{R}^n$ and its dual $\varphi \to \check{\varphi}$ from a group-theoretic point of view, motivated by the fact that the isometry group $\mathbf{M}(n)$ acts transitively both on $\mathbf{R}^n$ and on the hyperplane space $\mathbf{P}^n$. Thus

$$\mathbf{R}^n = \mathbf{M}(n)/\mathbf{O}(n)\,, \quad \mathbf{P}^n = \mathbf{M}(n)/\mathbb{Z}_2 \times \mathbf{M}(n-1)\,, \tag{1}$$

where $\mathbf{O}(n)$ is the orthogonal group fixing the origin $0 \in \mathbf{R}^n$ and $\mathbb{Z}_2 \times \mathbf{M}(n-1)$ is the subgroup of $\mathbf{M}(n)$ leaving a certain hyperplane ξ_0 through 0 stable. ($\mathbb{Z}_2$ consists of the identity and the reflection in this hyperplane.)

We observe now that a point $g_1\mathbf{O}(n)$ in the first coset space above lies on a plane $g_2(\mathbb{Z}_2 \times \mathbf{M}(n-1))$ in the second if and only if these cosets, considered as subsets of $\mathbf{M}(n)$, have a point in common. In fact

$$\begin{aligned} g_1 \cdot 0 \subset g_2 \cdot \xi_0 \quad &\Leftrightarrow \quad g_1 \cdot 0 = g_2 h \cdot 0 \text{ for some } h \in \mathbb{Z}_2 \times \mathbf{M}(n-1) \\ &\Leftrightarrow \quad g_1 k = g_2 h \text{ for some } k \in \mathbf{O}(n)\,. \end{aligned}$$

This leads to the following general setup.

Let G be a locally compact group, X and Ξ two left coset spaces of G,

$$X = G/K\,, \quad \Xi = G/H\,, \tag{2}$$

where K and H are closed subgroups of G. Let $L = K \cap H$. We assume that the subset $KH \subset G$ is *closed*. This is automatic if one of the groups K or H is compact.

Two elements $x \in X$, $\xi \in \Xi$ are said to be *incident* if as cosets in G they intersect. We put (see Fig. II.1)

$$\begin{aligned} \check{x} &= \{\xi \in \Xi : x \text{ and } \xi \text{ incident}\} \\ \widehat{\xi} &= \{x \in X : x \text{ and } \xi \text{ incident}\}\,. \end{aligned}$$

Let $x_0 = \{K\}$ and $\xi_0 = \{H\}$ denote the origins in X and Ξ, respectively. If $\Pi : G \to G/H$ denotes the natural mapping then since $\check{x}_0 = K \cdot \xi_0$ we

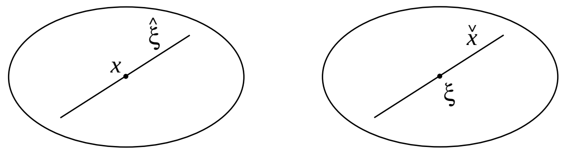

FIGURE II.1.

have

$$\Pi^{-1}(\Xi - \check{x}_0) = \{g \in G : gH \notin KH\} = G - KH\,.$$

In particular $\Pi(G - KH) = \Xi - \check{x}_0$ so since Π is an open mapping, $\check{x}_0$ is a closed subset of Ξ. This proves

Lemma 1.1. *Each $\check{x}$ and each $\widehat{\xi}$ is closed.*

Using the notation $A^g = gAg^{-1}$ $(g \in G, A \subset G)$ we have the following lemma.

Lemma 1.2. *Let $g, \gamma \in G$, $x \in gK$, $\xi = \gamma H$. Then*

$$\check{x} \text{ is an orbit of } K^g,\ \widehat{\xi} \text{ is an orbit of } H^\gamma\,,$$

and

$$\check{x} = K^g/L^g\,, \quad \widehat{\xi} = H^\gamma/L^\gamma\,.$$

Proof. By definition

$$\check{x} = \{\delta H \,:\, \delta H \cap gK \neq \emptyset\} = \{gkH \,:\, k \in K\} \tag{3}$$

which is the orbit of the point gH under gKg^{-1}. The subgroup fixing gH is $gKg^{-1} \cap gHg^{-1} = L^g$. Also (3) implies

$$\check{x} = g \cdot \check{x}_0 \qquad \widehat{\xi} = \gamma \cdot \widehat{\xi}_0\,,$$

where the dot $\cdot$ denotes the action of G on X and Ξ.

Lemma 1.3. *Consider the subgroups*

$$\begin{aligned} K_H &= \{k \in K \,:\, kH \cup k^{-1}H \subset HK\} \\ H_K &= \{h \in H \,:\, hK \cup h^{-1}K \subset KH\}\,. \end{aligned}$$

The following properties are equivalent:

(a) $K \cap H = K_H = H_K$.

(b) *The maps $x \to \check{x}\,(x \in X)$ and $\xi \to \widehat{\xi}\,(\xi \in \Xi)$ are injective.*

We think of property (a) as a kind of *transversality* of K and H.

Proof. Suppose $x_1 = g_1K$, $x_2 = g_2K$ and $\check{x}_1 = \check{x}_2$. Then by (3) $g_1 \cdot \check{x}_0 = g_1 \cdot \check{x}_0$ so $g \cdot \check{x}_0 = \check{x}_0$ if $g = g_1^{-1}g_2$. In particular $g \cdot \xi_0 \subset \check{x}_0$ so $g \cdot \xi_0 = k \cdot \xi_0$ for some $k \in K$. Hence $k^{-1}g = h \in H$ so $h \cdot \check{x}_0 = \check{x}_0$, that is $hK \cdot \xi_0 = K \cdot \xi_0$. As a relation in G, this means $hKH = KH$. In particular $hK \subset KH$. Since $h \cdot \check{x}_0 = \check{x}_0$ implies $h^{-1} \cdot \check{x}_0 = \check{x}_0$ we have also $h^{-1}K \subset KH$ so by (b) $h \in K$ which gives $x_1 = x_2$.

On the other hand, suppose the map $x \to \check{x}$ is injective and suppose $h \in H$ satisfies $h^{-1}K \cup hK \subset KH$. Then

$$hK \cdot \xi_0 \subset K \cdot \xi_0 \text{ and } h^{-1}K \cdot \xi_0 \subset K \cdot \xi_0 \,.$$

By Lemma 1.2, $h \cdot \check{x}_0 \subset \check{x}_0$ and $h^{-1} \cdot \check{x}_0 \subset \check{x}_0$. Thus $h \cdot \check{x}_0 = \check{x}_0$ whence by the assumption, $h \cdot x_0 = x_0$ so $h \in K$.

Thus we see that under the transversality assumption a) the elements ξ can be viewed as the subsets $\widehat{\xi}$ of X and the elements x as the subsets $\check{x}$ of Ξ. We say X and Ξ are *homogeneous spaces in duality*.

The maps are also conveniently described by means of the following *double fibration*

(4)

$$\begin{array}{ccc} & G/L & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle \pi} \\ G/K & & G/H \end{array}$$

where $p(gL) = gK$, $\pi(\gamma L) = \gamma H$. In fact, by (3) we have

$$\check{x} = \pi(p^{-1}(x)) \qquad \widehat{\xi} = p(\pi^{-1}(\xi))\,.$$

We now prove some group-theoretic properties of the incidence, supplementing Lemma 1.3.

Theorem 1.4. *(i) We have the identification*

$$G/L = \{(x, \xi) \in X \times \Xi \,:\, x \text{ and } \xi \text{ incident}\}$$

via the bijection $\tau : gL \to (gK, gH)$.

(ii) The property

$$KHK = G$$

is equivariant to the property:

Any two $x_1, x_2 \in X$ *are incident to some* $\xi \in \Xi$. *A similar statement holds for* $HKH = G$.

(iii) The property

$$HK \cap KH = K \cup H$$

is equivalent to the property:

For any two $x_1 \neq x_2$ in X there is at most one $\xi \in \Xi$ incident to both. *By symmetry, this is equivalent to the property:*

For any $\xi_1 \neq \xi_2$ in Ξ there is at most one $x \in X$ incident to both.

Proof. (i) The map is well-defined and injective. The surjectivity is clear because if $gK \cap \gamma H \neq \emptyset$ then $gk = \gamma h$ and $\tau(gkL) = (gK, \gamma H)$.

(ii) We can take $x_2 = x_0$. Writing $x_1 = gK$, $\xi = \gamma H$ we have

$$\begin{aligned} x_0, \xi \text{ incident } &\Leftrightarrow \gamma h &=& \quad k \quad (\text{some } h \in H, k \in K) \\ x_1, \xi \text{ incident } &\Leftrightarrow \gamma h_1 &=& \quad g_1 k_1 \quad (\text{some } h_1 \in H, k_1 \in K) \end{aligned}$$

Thus if x_0, x_1 are incident to ξ we have $g_1 = kh^{-1}h_1k_1^{-1}$. Conversely if $g_1 = k'h'k''$ we put $\gamma = k'h'$ and then x_0, x_1 are incident to $\xi = \gamma H$.

(iii) Suppose first $KH \cap HK = K \cup H$. Let $x_1 \neq x_2$ in X. Suppose $\xi_1 \neq \xi_2$ in Ξ are both incident to x_1 and x_2. Let $x_i = g_iK$, $\xi_j = \gamma_jH$. Since x_i is incident to ξ_j there exist $k_{ij} \in K$, $h_{ij} \in H$ such that

$$g_ik_{ij} = \gamma_jh_{ij} \qquad i = 1, 2\,;\ j = 1, 2\,. \tag{5}$$

Eliminating g_i and γ_j we obtain

$$k_{2\,2}^{-1}k_{2\,1}h_{2\,1}^{-1}h_{1\,1} = h_{2\,2}^{-1}h_{1\,2}k_{1\,2}^{-1}k_{1\,1}\,. \tag{6}$$

This being in $KH \cap HK$ it lies in $K \cup H$. If the left hand side is in K then $h_{2\,1}^{-1}h_{1\,1} \in K$ so

$$g_2K = \gamma_1h_{2\,1}K = \gamma_1h_{1\,1}K = g_1K\,,$$

contradicting $x_2 \neq x_1$. Similarly if expression (6) is in H then $k_{1\,2}^{-1}k_{1\,1} \in H$ so by (5) we get the contradiction

$$\gamma_2H = g_1k_{1\,2}H = g_1k_{1\,1}H = \gamma_1H\,.$$

Conversely, suppose $KH \cap HK \neq K \cup H$. Then there exist h_1, h_2, k_1, k_2 such that $h_1k_1 = k_2h_2$ and $h_1k_1 \notin K \cup H$. Put $x_1 = h_1K, \xi_2 = k_2H$. Then $x_1 \neq x_0, \xi_0 \neq \xi_2$, yet both ξ_0 and ξ_2 are incident to both x_0 and x_1.

Examples

(i) *Points outside hyperplanes.* We saw before that if in the coset space representation (1) $\mathbf{O}(n)$ is viewed as the isotropy group of 0 and $\mathbb{Z}_2\mathbf{M}(n-1)$ is viewed as the isotropy group of a hyperplane *through* 0 then the abstract incidence notion is equivalent to the naive one: $x \in \mathbf{R}^n$ is incident to $\xi \in \mathbf{P}^n$ if and only if $x \in \xi$.

On the other hand we can also view $\mathbb{Z}_2\mathbf{M}(n-1)$ as the isotropy group of a hyperplane ξ_δ at a distance $\delta > 0$ from 0. (This amounts to a different embedding of the group $\mathbb{Z}_2\mathbf{M}(n-1)$ into $\mathbf{M}(n)$.) Then we have the following generalization.

Proposition 1.5. *The point $x \in \mathbf{R}^n$ and the hyperplane $\xi \in \mathbf{P}^n$ are incident if and only if distance $(x, \xi) = \delta$.*

Proof. Let $x = gK$, $\xi = \gamma H$ where $K = \mathbf{O}(n), H = \mathbb{Z}_2\mathbf{M}(n-1)$. Then if $gK \cap \gamma H \neq \emptyset$, we have $gk = \gamma h$ for some $k \in K, h \in H$. Now the orbit $H \cdot 0$ consists of the two planes ξ'_δ and ξ''_δ parallel to ξ_δ at a distance δ from ξ_δ. The relation

$$g \cdot 0 = \gamma h \cdot 0 \in \gamma \cdot (\xi'_\delta \cup \xi''_\delta)$$

together with the fact that g and γ are isometries shows that x has distance δ from $\gamma \cdot \xi_\delta = \xi$.

On the other hand if distance $(x, \xi) = \delta$ we have $g \cdot 0 \in \gamma \cdot (\xi'_\delta \cup \xi''_\delta) = \gamma H \cdot 0$ which means $gK \cap \gamma H \neq \emptyset$.

(ii) *Unit spheres.* Let σ_0 be a sphere in $\mathbf{R}^n$ of radius one passing through the origin. Denoting by Σ the set of all *unit* spheres in $\mathbf{R}^n$ we have the dual homogeneous spaces

$$\mathbf{R}^n = \mathbf{M}(n)/\mathbf{O}(n)\,; \quad \Sigma = \mathbf{M}(n)/\mathbf{O}^*(n) \tag{7}$$

where $\mathbf{O}^*(n)$ is the set of rotations around the center of σ_0. Here a point $x = g\mathbf{O}(n)$ is incident to $\sigma_0 = \gamma\mathbf{O}^*(n)$ if and only if $x \in \sigma$.

§2 The Radon Transform for the Double Fibration

With K, H and L as in §1 we assume now that K/L and H/L have positive measures $d\mu_0 = dk_L$ and $dm_0 = dh_L$ invariant under K and H, respectively. This is for example guaranteed if L is compact.

Lemma 2.1. *Assume the transversality condition (a). Then there exists a measure on each $\check{x}$ coinciding with $d\mu_0$ on $K/L = \check{x}_0$ such that whenever $g \cdot \check{x}_1 = \check{x}_2$ the measures on $\check{x}_1$ and $\check{x}_2$ correspond under g. A similar statement holds for dm on $\widehat{\xi}$.*

Proof. If $\check{x} = g \cdot \check{x}_0$ we transfer the measure $d\mu_0 = dk_L$ over on $\check{x}$ by the map $\xi \to g \cdot \xi$. If $g \cdot \check{x}_0 = g_1 \cdot \check{x}_0$ then $(g \cdot x_0)^\vee = (g_1 \cdot x_0)^\vee$ so by Lemma 1.3, $g \cdot x_0 = g_1 \cdot x_0$ so $g = g_1 k$ with $k \in K$. Since $d\mu_0$ is K-invariant the lemma follows.

The measures defined on each $\check{x}$ and $\widehat{\xi}$ under condition (a) are denoted by $d\mu$ and dm, respectively.

Definition. The Radon transform $f \to \widehat{f}$ and its dual $\varphi \to \check{\varphi}$ are defined by

$$\widehat{f}(\xi) = \int_{\widehat{\xi}} f(x)\, dm(x)\,, \qquad \check{\varphi}(x) = \int_{\check{x}} \varphi(\xi)\, d\mu(\xi)\,. \tag{8}$$

whenever the integrals converge. Because of Lemma 1.1, this will always happen for $f \in C_c(X)$, $\varphi \in C_c(\Xi)$.

In the setup of Proposition 1.5, $\widehat{f}(\xi)$ is the integral of f over the two hyperplanes at distance δ from ξ and $\check{\varphi}(x)$ is the average of φ over the set of hyperplanes at distance δ from x. For $\delta = 0$ we recover the transforms of Ch. I, §1.

Formula (8) can also be written in the group-theoretic terms,

$$\widehat{f}(\gamma H) = \int_{H/L} f(\gamma h K)\, dh_L\,, \quad \check{\varphi}(gK) = \int_{K/L} \varphi(gkH)\, dk_L\,. \tag{9}$$

Note that (9) serves as a definition even if condition (a) in Lemma 1.3 is not satisfied. In this abstract setup the spaces X and Ξ have equal status. The theory in Ch. I, in particular Lemma 2.1, Theorems 2.4, 2.10, 3.1 thus raises the following problems:

Principal Problems:

A. Relate function spaces on X and on Ξ by means of the transforms $f \to \widehat{f}$, $\varphi \to \check{\varphi}$. In particular, determine their ranges and kernels.

B. Invert the transforms $f \to \widehat{f}$, $\varphi \to \check{\varphi}$ on suitable function spaces.

C. In the case when G is a Lie group so X and Ξ are manifolds let $\mathbf{D}(X)$ and $\mathbf{D}(\Xi)$ denote the algebras of G-invariant differential operators on X and Ξ, respectively. Is there a map $D \to \widehat{D}$ of $\mathbf{D}(X)$ into $\mathbf{D}(\Xi)$ and a map $E \to \check{E}$ of $\mathbf{D}(\Xi)$ into $\mathbf{D}(X)$ such that

$$(Df)^{\widehat{}} = \widehat{D}\widehat{f}\,, \quad (E\varphi)^{\vee} = \check{E}\check{\varphi}\,?$$

Although weaker assumptions would be sufficient, we assume now that the groups G, K, H and L all have bi-invariant Haar measures dg, dk, dh and $d\ell$. These will then generate invariant measures dg_K, dg_H, dg_L, dk_L, dh_L on G/K, G/H, G/L, K/L, H/L, respectively. This means that

$$\int_G F(g)\, dg = \int_{G/K} \left(\int_K F(gk)\, dk \right) dg_K \tag{10}$$

and similarly dg and dh determine dg_H, etc. Then

$$\int_{G/L} Q(gL)\,dg_L = c\int_{G/K} dg_K \int_{K/L} Q(gkL)\,dk_L \tag{11}$$

for $Q \in C_c(G/L)$ where c is a constant. In fact, the integrals on both sides of (11) constitute invariant measures on G/L and thus must be proportional. However,

$$\int_G F(g)\,dg = \int_{G/L}\left(\int_L F(g\ell)\,d\ell\right) dg_L \tag{12}$$

and

$$\int_K F(k)\,dk = \int_{K/L}\left(\int_L F(k\ell)\,d\ell\right) dk_L\,. \tag{13}$$

Using (13) on (10) and combining with (11) we see that the constant c equals 1.

We shall now prove that $f \to \widehat{f}$ and $\varphi \to \check{\varphi}$ are adjoint operators. We write dx for dg_K and $d\xi$ for dg_H.

Proposition 2.2. *Let $f \in C_c(X)$, $\varphi \in C_c(\Xi)$. Then $\widehat{f}$ and $\check{\varphi}$ are continuous and*

$$\int_X f(x)\check{\varphi}(x)\,dx = \int_\Xi \widehat{f}(\xi)\varphi(\xi)\,d\xi\,.$$

Proof. The continuity statement is immediate from (9). We consider the function

$$P = (f\circ p)(\varphi\circ\pi)$$

on G/L. We integrate it over G/L in two ways using the double fibration (4). This amounts to using (11) and its analog with G/K replaced by G/H with $Q = P$. Since $P(gk\,L) = f(gK)\varphi(gkH)$ the right hand side of (11) becomes

$$\int_{G/K} f(gK)\check{\varphi}(gK)\,dg_K\,.$$

If we treat G/H similarly, the lemma follows.

The result shows how to define the Radon transform and its dual for measures and, in case G is a Lie group, for distributions.

Definition. Let s be a measure on X of compact support. Its Radon transform is the functional $\widehat{s}$ on $C_c(\Xi)$ defined by

$$\widehat{s}(\varphi) = s(\check{\varphi})\,. \tag{14}$$

Similarly $\check{\sigma}$ is defined by

$$\check{\sigma}(f) = \sigma(\widehat{f})\,, \qquad f \in C_c(X) \tag{15}$$

if σ is a compactly supported measure on Ξ.

Lemma 2.3. *(i) If s is a compactly supported measure on X, $\widehat{s}$ is a measure on Ξ.*

(ii) If s is a bounded measure on X and if $\check{x}_0$ has finite measure then $\widehat{s}$ as defined by (14) is a bounded measure.

Proof. (i) The measure s can be written as a difference $s = s^+ - s^-$ of two positive measures, each of compact support. Then $\widehat{s} = \widehat{s^+} - \widehat{s^-}$ is a difference of two positive *functionals* on $C_c(\Xi)$.

Since a positive functional is necessarily a measure, $\widehat{s}$ is a measure.

(ii) We have

$$\sup_x |\check{\varphi}(x)| \leq \sup_\xi |\varphi(\xi)|\, \mu_0(\check{x}_0)$$

so for a constant K,

$$|\widehat{s}(\varphi)| = |s(\check{\varphi})| \leq K \sup |\check{\varphi}| \leq K \mu_0(\check{x}_0) \sup |\varphi| \,,$$

and the boundedness of $\widehat{s}$ follows.

If G is a Lie group then (14), (15) with $f \in \mathcal{D}(X)\,, \quad \varphi \in \mathcal{D}(\Xi)$ serve to define the Radon transform $s \to \widehat{s}$ and the dual $\sigma \to \check{\sigma}$ for distributions s and σ of compact support. We consider the spaces $\mathcal{D}(X)$ and $\mathcal{E}(X)$ $(= \mathcal{C}^\infty(X))$ with their customary topologies (Chapter V,§1). The duals $\mathcal{D}'(X)$ and $\mathcal{E}'(X)$ then consist of the distributions on X and the distributions on X of compact support, respectively.

Proposition 2.4. *The mappings*

$$\begin{array}{rcl} f \in \mathcal{D}(X) & \to & \widehat{f} \in \mathcal{E}(\Xi) \\ \varphi \in \mathcal{D}(\Xi) & \to & \check{\varphi} \in \mathcal{E}(X) \end{array}$$

are continuous. In particular,

$$\begin{array}{rcl} s \in \mathcal{E}'(X) & \Rightarrow & \widehat{s} \in \mathcal{D}'(\Xi) \\ \sigma \in \mathcal{E}'(\Xi) & \Rightarrow & \check{\sigma} \in \mathcal{D}'(X)\,. \end{array}$$

Proof. We have

$$\widehat{f}(g \cdot \xi_0) = \int_{\widehat{\xi}_0} f(g \cdot x)\, dm_0(x)\,. \tag{16}$$

Let g run through a local cross section through e in G over a neighborhood of ξ_0 in Ξ. If $(t_1, \ldots, t_n)$ are coordinates of g and $(x_1, \ldots, x_m)$ the coordinates of $x \in \widehat{\xi}_0$ then (16) can be written in the form

$$\widehat{F}(t_1, \ldots, t_n) = \int F(t_1, \ldots, t_n\,;\, x_1, \ldots, x_m)\, dx_1 \ldots dx_m\,.$$

Now it is clear that $\widehat{f} \in \mathcal{E}(\Xi)$ and that $f \to \widehat{f}$ is continuous, proving the proposition.

The result has the following refinement.

Proposition 2.5. *Assume K compact. Then*

(i) $f \to \widehat{f}$ *is a continuous mapping of* $\mathcal{D}(X)$ *into* $\mathcal{D}(\Xi)$.

(ii) $\varphi \to \check{\varphi}$ *is a continuous mapping of* $\mathcal{E}(\Xi)$ *into* $\mathcal{E}(X)$.

A self-contained proof is given in the author's book [1994b], Ch. I, § 3. The result has the following consequence.

Corollary 2.6. *Assume K compact. Then* $\mathcal{E}'(X)^{\widehat{}} \subset \mathcal{E}'(\Xi)$, $\mathcal{D}'(\Xi)^{\vee} \subset \mathcal{D}'(X)$.

In Chapter I we have given solutions to Problems A, B, C in some cases. Further examples will be given in § 4 of this chapter and Chapter III will include their solution for the antipodal manifolds for compact two-point homogeneous spaces.

The variety of the results for these examples make it doubtful that the individual results could be captured by a general theory. Our abstract setup in terms of homogeneous spaces in duality is therefore to be regarded as a framework for examples rather than as axioms for a general theory.

Nevertheless, certain general features emerge from the study of these examples. If $\dim X = \dim \Xi$ and $f \to \widehat{f}$ is injective the range consists of functions which are either arbitrary or at least subjected to rather weak conditions. As the difference $\dim \Xi - \dim X$ increases more conditions are imposed on the functions in the range. (See the example of the d-plane transform in $\mathbf{R}^n$.) In the case when G is a Lie group there is a group-theoretic explanation for this. Let $\mathbf{D}(G)$ denote the algebra of left-invariant differential operators on G. Since $\mathbf{D}(G)$ is generated by the left invariant vector fields on G, the action of G on X and on Ξ induces homomorphisms

$$\lambda : \mathbf{D}(G) \longrightarrow E(X)\,, \tag{17}$$

$$\Lambda : \mathbf{D}(G) \longrightarrow E(\Xi)\,, \tag{18}$$

where for a manifold M, $E(M)$ denotes the algebra of all differential operators on M. Since $f \to \widehat{f}$ and $\varphi \to \check{\varphi}$ commute with the action of G we have for $D \in \mathbf{D}(G)$,

$$(\lambda(D)f)^{\widehat{}} = \Lambda(D)\widehat{f}\,, \quad (\Lambda(D)\varphi)^{\vee} = \lambda(D)\check{\varphi}\,. \tag{19}$$

Therefore $\Lambda(D)$ annihilates the range of $f \to \widehat{f}$ if $\lambda(D) = 0$. In some cases, including the case of the d-plane transform in $\mathbf{R}^n$, the range is characterized as the null space of these operators $\Lambda(D)$ (with $\lambda(D) = 0$).

§3 Orbital Integrals

As before let $X = G/K$ be a homogeneous space with origin $o = (K)$. Given $x_0 \in X$ let G_{x_0} denote the subgroup of G leaving x_0 fixed, i.e., the isotropy subgroup of G at x_0.

Definition. A *generalized sphere* is an orbit $G_{x_0} \cdot x$ in X of some point $x \in X$ under the isotropy subgroup at some point $x_0 \in X$.

Examples. (i) If $X = \mathbf{R}^n$, $G = \mathbf{M}(n)$ then the generalized spheres are just the spheres.

(ii) Let X be a locally compact subgroup L and G the product group $L \times L$ acting on L on the right and left, the element $(\ell_1, \ell_2) \in L \times L$ inducing action $\ell \to \ell_1 \ell \, \ell_2^{-1}$ on L. Let ΔL denote the diagonal in $L \times L$. If $\ell_0 \in L$ then the isotropy subgroup of ℓ_0 is given by

$$(L \times L)_{\ell_0} = (\ell_0, e)\Delta L(\ell_0^{-1}, e) \tag{20}$$

and the orbit of ℓ under it by

$$(L \times L)_{\ell_0} \cdot \ell = \ell_0(\ell_0^{-1}\ell)^L \, .$$

that is the left translate by ℓ_0 of the conjugacy class of the element $\ell_0^{-1}\ell$. Thus the *generalized spheres in the group L are the left (or right) translates of its conjugacy classes*.

Coming back to the general case $X = G/K = G/G_0$ we assume that G_0, and therefore each G_{x_0}, is unimodular. But $G_{x_0} \cdot x = G_{x_0}/(G_{x_0})_x$ so $(G_{x_0})_x$ unimodular implies the orbit $G_{x_0} \cdot x$ has an invariant measure determined up to a constant factor. We can now consider the following general problem (following Problems A, B, C above).

D. *Determine a function f on X in terms of its integrals over generalized spheres.*

Remark 3.1. In this problem it is of course significant how the invariant measures on the various orbits are normalized.

(a) If G_0 is compact the problem above is rather trivial because each orbit $G_{x_0} \cdot x$ has finite invariant measure so $f(x_0)$ is given as the limit as $x \to x_0$ of the average of f over $G_{x_0} \cdot x$.

(b) Suppose that for each $x_0 \in X$ there is a G_{x_0}-invariant open set $C_{x_0} \subset X$ containing x_0 in its closure such that for each $x \in C_{x_0}$ the isotropy group $(G_{x_0})_x$ is compact. The invariant measure on the orbit $G_{x_0} \cdot x \, (x_0 \in X, x \in C_{x_0})$ can then be consistently normalized as follows: Fix a Haar measure dg_0 on G_0. If $x_0 = g \cdot o$ we have $G_{x_0} = gG_0g^{-1}$ and can carry dg_0 over to a measure dg_{x_0} on G_{x_0} by means of the conjugation $z \to gzg^{-1} \quad (z \in G_0)$.

Since dg_0 is bi-invariant, dg_{x_0} is independent of the choice of g satisfying $x_0 = g \cdot o$, and is bi-invariant. Since $(G_{x_0})_x$ is compact it has a unique Haar measure $dg_{x_0,x}$ with total measure 1 and now dg_{x_0} and $dg_{x_0,x}$ determine canonically an invariant measure μ on the orbit $G_{x_0} \cdot x = G_{x_0}/(G_{x_0})_x$. We can therefore state Problem D in a more specific form.

D′. *Express $f(x_0)$ in terms of integrals*

$$\int_{G_{x_0} \cdot x} f(p)\, d\mu(p) \qquad x \in C_{x_0}\,. \tag{21}$$

For the case when X is an *isotropic Lorentz manifold* the assumptions above are satisfied (with C_{x_0} consisting of the "timelike" rays from x_0) and we shall obtain in Ch. IV an explicit solution to Problem D' (Theorem 4.1, Ch. IV).

(c) If in Example (ii) above L is a semisimple Lie group Problem D is a basic step (Gelfand-Graev [1955], Harish-Chandra [1957]) in proving the Plancherel formula for the Fourier transform on L.

§4 Examples of Radon Transforms for Homogeneous Spaces in Duality

In this section we discuss some examples of the abstract formalism and problems set forth in the preceding sections §1–§2.

A. The Funk Transform.

This case goes back to Funk [1916] (preceding Radon's paper [1917]) where he proved that a symmetric function on $\mathbf{S}^2$ is determined by its great circle integrals. This is carried out in more detail and in greater generality in Chapter III, §1. Here we state the solution of Problem B for $X = \mathbf{S}^2$, Ξ the set of all great circles, both as homogeneous spaces of $\mathbf{O}(3)$. Given $p \geq 0$ let $\xi_p \in \Xi$ have distance p from the North Pole o, $H_p \subset \mathbf{O}(3)$ the subgroup leaving ξ_p invariant and $K \subset \mathbf{O}(3)$ the subgroup fixing o. Then in the double fibration

$$\begin{array}{ccc} & \mathbf{O}(3)/(K \cap H_p) & \\ \swarrow & & \searrow \\ X = \mathbf{O}(3)/K & & \Xi = \mathbf{O}(3)/H_p \end{array}$$

$x \in X$ and $\xi \in \Xi$ are incident if and only if $d(x, \xi) = p$. The proof is the same as that of Proposition 1.5. In order to invert the Funk transform $f \to \widehat{f}$ $(= \widehat{f}_0)$ we invoke the transform $\varphi \to \check{\varphi}_p$. Note that $(\widehat{f})^\vee_p(x)$ is the

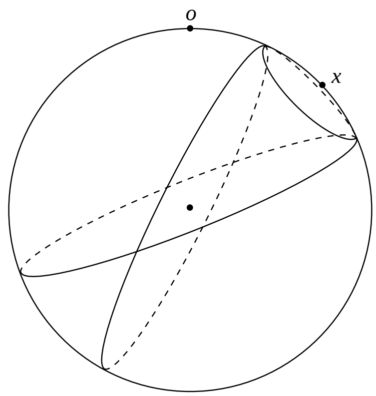

FIGURE II.2.

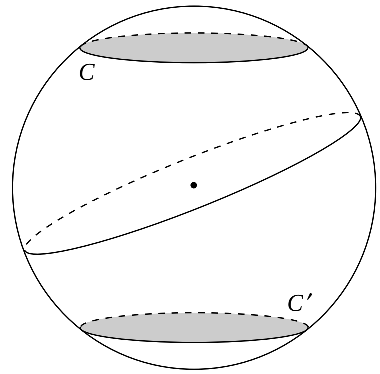

FIGURE II.3.

average of the integrals of f over the great circles ξ at distance p from x (see Figure II.2). As a special case of Theorem 1.11, Chapter III, we have the following inversion.

Theorem 4.1. *The Funk transform $f \to \widehat{f}$ is (for f even) inverted by*

$$f(x) = \frac{1}{2\pi}\left\{\frac{d}{du}\int_0^u (\widehat{f})^\vee_{\cos^{-1}(v)}(x)v(u^2-v^2)^{-\frac{1}{2}}\,dv\right\}_{u=1}. \tag{22}$$

Another inversion formula is

$$f = -\frac{1}{4\pi}LS((\widehat{f})^\vee) \tag{23}$$

(Theorem 1.15, Chapter III), where L is the Laplacian and S the integral operator given by (66)–(68), Chapter III. While (23) is short the operator S is only given in terms of a spherical harmonics expansion. Also Theorem 1.17, Ch. III shows that if f is even and if all its derivatives vanish on the equator then f vanishes outside the "arctic zones" C and C' if and only if $\widehat{f}(\xi) = 0$ for all great circles ξ disjoint from C and C' (Fig. II.3).

The Hyperbolic Plane $\mathbf{H}^2$.

This remarkable object enters into several fields in mathematics. In particular, it offers at least two interesting cases of Radon transforms. We take $\mathbf{H}^2$ as the disk $D : |z| < 1$ with the Riemannian structure

$$\langle u, v\rangle_z = \frac{(u,v)}{(1-|z|^2)^2}, \quad ds^2 = \frac{|dz|^2}{(1-|z|^2)^2} \tag{24}$$

if u and v are any tangent vectors at $z \in D$. Here (u, v) denotes the usual inner product on $\mathbf{R}^2$. The Laplace-Beltrami operator for (24) is given by

$$L = (1-x^2-y^2)^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

The group $G = \mathbf{SU}(1,1)$ of matrices

$$\left\{ \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \,:\, |a|^2 - |b|^2 = 1 \right\}$$

acts transitively on the unit disk by

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \cdot z = \frac{az+b}{\overline{b}z+\overline{a}} \tag{25}$$

and leaves the metric (24) invariant. The length of a curve $\gamma(t)$ $(\alpha \leq t \leq \beta)$ is defined by

$$L(\gamma) = \int_\alpha^\beta (\langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)})^{1/2}\, dt\,. \tag{26}$$

If $\gamma(\alpha) = o$, $\gamma(\beta) = x \in \mathbf{R}$ and $\gamma_o(t) = tx \quad (0 \leq t \leq 1)$ then (26) shows easily that $L(\gamma) \geq L(\gamma_o)$ so γ_o is a geodesic and the distance d satisfies

$$d(o,x) = \int_0^1 \frac{|x|}{1-t^2x^2}\, dt = \tfrac{1}{2} \log \frac{1+|x|}{1-|x|}\,. \tag{27}$$

Since G acts conformally on D the *geodesics* in $\mathbf{H}^2$ are the circular arcs in $|z| < 1$ perpendicular to the boundary $|z| = 1$.

We consider now the following subgroups of G:

$$\begin{aligned}
K &= \{k_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : 0 \leq \theta < 2\pi\} \\
M &= \{k_0, k_\pi\}\,, \qquad M' = \{k_0, k_\pi\,, k_{-\frac{\pi}{2}}, k_{\frac{\pi}{2}}\} \\
A &= \{a_t = \begin{pmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{ch} t \end{pmatrix} \,:\, t \in \mathbf{R}\}, \\
N &= \{n_x = \begin{pmatrix} 1+ix, & -ix \\ ix, & 1-ix \end{pmatrix} \,:\, x \in \mathbf{R}\} \\
\Gamma &= C\mathbf{SL}(2,\mathbb{Z})C^{-1}\,,
\end{aligned}$$

where C is the transformation $w \to (w-i)/(w+i)$ mapping the upper half-plane onto the unit disk.

The orbit $A \cdot o$ is the horizontal diameter and the orbits $N \cdot (a_t \cdot o)$ are the circles tangential to $|z| = 1$ at $z = 1$. Thus $NA \cdot o$ is the entire disk D so we see that $G = NAK$ and also $G = KAN$.

B. The X-ray Transform in $\mathbf{H}^2$.

The (unoriented) geodesics for the metric (24) were mentioned above. Clearly the group G permutes these geodesics transitively (Fig. II.4). Let

Ξ be the set of all these geodesics. Let o denote the origin in $\mathbf{H}^2$ and ξ_o the horizontal geodesic through o. Then

(28)
$$X = G/K\,, \quad \Xi = G/M'A\,.$$

We can also fix a geodesic ξ_p at distance p from o and write

(29)
$$X = G/K\,, \quad \Xi = G/H_p\,,$$

where H_p is the subgroup of G leaving ξ_p stable. Then for the homogeneous spaces (29), x and ξ are incident if and only if $d(x,\xi) = p$. The transform $f \to \widehat{f}$ is inverted by means of the dual transform $\varphi \to \check{\varphi}_p$ for (29). The inversion below is a special case of Theorem 1.10, Chapter III, and is the analog of (22). Note however the absence of v in the integrand. Observe also that the metric ds is renormalized by the factor 2 (so curvature is -1).

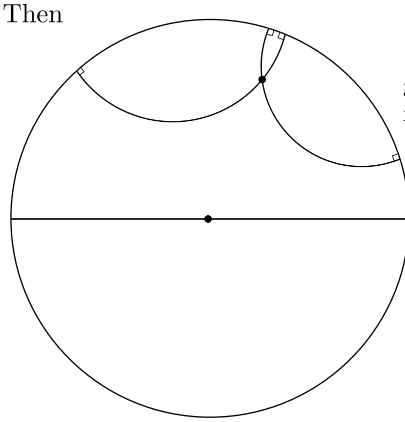

FIGURE II.4.

Theorem 4.2. *The X-ray transform in* $\mathbf{H}^2$ *with the metric*

$$ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}$$

is inverted by

(30)
$$f(z) = \frac{1}{\pi}\left\{\frac{d}{du}\int_0^u (\widehat{f})^\vee_{\operatorname{lm} v}(z)(u^2-v^2)^{-\frac{1}{2}}\,dv\right\}_{u=1}\,,$$

where $\operatorname{lm} v = \cosh^{-1}(v^{-1})$.

Another inversion formula is

(31)
$$f = -\frac{1}{4\pi} LS((\widehat{f})^\vee)\,,$$

where S is the operator of convolution on $\mathbf{H}^2$ with the function $x \to \coth(d(x,o)) - 1$, (Theorem 1.14, Chapter III).

C. The Horocycles in $\mathbf{H}^2$.

Consider a family of geodesics with the same limit point on the boundary B. The *horocycles* in $\mathbf{H}^2$ are by definition the orthogonal trajectories of such families of geodesics. Thus the horocycles are the circles tangential to $|z| = 1$ from the inside (Fig. II.5).

One such horocycle is $\xi_0 = N \cdot o$, the orbit of the origin o under the action of N. Since $a_t \cdot \xi$ is the horocycle with diameter $(\tanh t, 1)$ G acts transitively on the set Ξ of horocycles. Now we take $\mathbf{H}^2$ with the metric (24). Since $G = KAN$ it is easy to see that MN is the subgroup leaving ξ_o invariant. Thus we have here

(32)
$$X = G/K\,, \quad \Xi = G/MN\,.$$

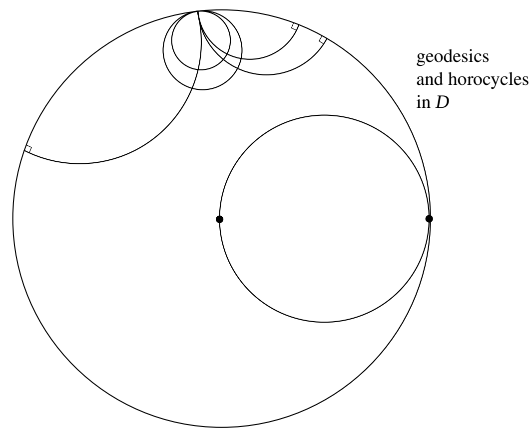

FIGURE II.5.

Furthermore each horocycle has the form $\xi = ka_t \cdot \xi_0$ where $kM \in K/M$ and $t \in \mathbf{R}$ are unique. Thus $\Xi \sim K/M \times A$, which is also evident from the figure.

We observe now that the maps

$$\psi : t \to a_t \cdot o\,, \quad \varphi : x \to n_x \cdot o$$

of $\mathbf{R}$ onto γ_0 and ξ_0, respectively, are isometries. The first statement follows from (27) because

$$d(o, a_t) = d(o, \tanh t) = t\,.$$

For the second we note that

$$\varphi(x) = x(x+i)^{-1}\,, \quad \varphi'(x) = i(x+i)^{-2}$$

so

$$\langle \varphi'(x), \varphi'(x) \rangle_{\varphi(x)} = (x^2+1)^{-4}(1 - |x(x+i)^{-1}|^2)^{-2} = 1\,.$$

Thus we give A and N the Haar measures $d(a_t) = dt$ and $d(n_x) = dx$.

Geometrically, the Radon transform on X relative to the horocycles is defined by

$$\widehat{f}(\xi) = \int_\xi f(x)\, dm(x)\,, \tag{33}$$

where dm is the measure on ξ induced by (24). Because of our remarks about φ, (33) becomes

$$\widehat{f}(g \cdot \xi_0) = \int_N f(gn \cdot o)\, dn\,, \tag{34}$$

so the geometric definition (33) coincides with the group-theoretic one in (9). The dual transform is given by

$$\check{\varphi}(g \cdot o) = \int_K \varphi(gk \cdot \xi_o)\, dk\,, \qquad (dk = d\theta/2\pi)\,. \tag{35}$$

In order to invert the transform $f \to \widehat{f}$ we introduce the non-Euclidean analog of the operator Λ in Chapter I, §3. Let T be the distribution on $\mathbf{R}$ given by

$$T\varphi = \tfrac{1}{2}\int_{\mathbf{R}} (\operatorname{sh} t)^{-1}\varphi(t)\, dt\,, \quad \varphi \in \mathcal{D}(\mathbf{R})\,, \tag{36}$$

considered as the Cauchy principal value, and put $T' = dT/dt$. Let Λ be the operator on $\mathcal{D}(\Xi)$ given by

$$(\Lambda\varphi)(ka_t \cdot \xi_0) = \int_{\mathbf{R}} \varphi(ka_{t-s} \cdot \xi_0)e^{-s}\, dT'(s)\,. \tag{37}$$

Theorem 4.3. *The Radon transform $f \to \widehat{f}$ for horocycles in $\mathbf{H}^2$ is inverted by*

$$f = \frac{1}{\pi}(\Lambda\widehat{f})^{\vee}\,, \quad f \in \mathcal{D}(\mathbf{H}^2)\,. \tag{38}$$

We begin with a simple lemma.

Lemma 4.4. *Let τ be a distribution on $\mathbf{R}$. Then the operator $\widetilde{\tau}$ on $\mathcal{D}(\Xi)$ given by the convolution*

$$(\widetilde{\tau}\varphi)(ka_t \cdot \xi_0) = \int_{\mathbf{R}} \varphi(ka_{t-s} \cdot \xi_0)\, d\tau(s)$$

is invariant under the action of G.

Proof. To understand the action of $g \in G$ on $\Xi \sim (K/M) \times A$ we write $gk = k'a_{t'}n'$. Since each $a \in A$ normalizes N we have

$$gka_t \cdot \xi_0 = gka_tN \cdot o = k'a_{t'}n'a_tN \cdot o = k'a_{t+t'} \cdot \xi_0\,.$$

Thus the action of g on $\Xi \simeq (K/M) \times A$ induces this fixed translation $a_t \to a_{t+t'}$ on A. This translation commutes with the convolution by τ so the lemma follows.

Since the operators Λ, $\widehat{\ }$, $^{\vee}$ in (38) are all G-invariant it suffices to prove the formula at the origin o. We first consider the case when f is K-invariant, i.e., $f(k \cdot z) \equiv f(z)$. Then by (34)

$$\widehat{f}(a_t \cdot \xi_0) = \int_{\mathbf{R}} f(a_tn_x \cdot o)\, dx\,. \tag{39}$$

Because of (27) we have

$$|z| = \tanh\, d(o,z)\,, \quad \cosh^2\, d(o,z) = (1-|z|^2)^{-1}\,. \tag{40}$$

Since

$$a_t n_x \cdot o = (\mathrm{sh}\, t - ix\, e^t)/(\mathrm{ch}\, t - ix\, e^t)$$

(40) shows that the distance $s = d(o, a_t n_x \cdot o)$ satisfies

$$\mathrm{ch}^2 s = \mathrm{ch}^2 t + x^2 e^{2t}\,. \tag{41}$$

Thus defining F on $[1,\infty)$ by

$$F(\mathrm{ch}^2 s) = f(\tanh\, s)\,, \tag{42}$$

we have

$$F'(\mathrm{ch}^2 s) = f'(\tanh\, s)(2\mathrm{sh}\, s\, \mathrm{ch}^3 s)^{-1}$$

so, since $f'(0) = 0$, $\lim_{u\to 1} F'(u)$ exists. The transform (39) now becomes (with $xe^t = y$)

$$e^t \widehat{f}(a_t \cdot \xi_0) = \int_{\mathbf{R}} F(\mathrm{ch}^2 t + y^2)\, dy\,. \tag{43}$$

We put

$$\varphi(u) = \int_{\mathbf{R}} F(u + y^2)\, dy$$

and invert this as follows:

$$\begin{aligned} \int_{\mathbf{R}} \varphi'(u+z^2)\, dz &= \int_{\mathbf{R}^2} F'(u + y^2 + z^2)\, dy\, dz \\ &= 2\pi \int_0^\infty F'(u + r^2) r\, dr = \pi \int_0^\infty F'(u+\rho)\, d\rho\,, \end{aligned}$$

so

$$-\pi F(u) = \int_{\mathbf{R}} \varphi'(u + z^2)\, dz\,.$$

In particular,

$$\begin{aligned} f(o) &= -\frac{1}{\pi} \int_{\mathbf{R}} \varphi'(1+z^2)\, dz = -\frac{1}{\pi} \int_{\mathbf{R}} \varphi'(\mathrm{ch}^2\tau)\mathrm{ch}\tau\, d\tau\,, \\ &= -\frac{1}{\pi} \int_{\mathbf{R}} \int_{\mathbf{R}} F'(\mathrm{ch}^2 t + y^2)\, dy\, \mathrm{ch}\, t\, dt \end{aligned}$$

so

$$f(o) = -\frac{1}{2\pi} \int_{\mathbf{R}} \frac{d}{dt}(e^t \widehat{f}(a_t \cdot \xi_0)) \frac{dt}{\mathrm{sh}\, t}\,.$$

Since $(e^t \widehat{f})(a_t \cdot \xi_0)$ is even (cf. (43)) its derivative vanishes at $t = 0$ so the integral is well defined. With T as in (36), the last formula can be written

$$f(o) = \frac{1}{\pi} T_t'(e^t \widehat{f}(a_t \cdot \xi_0))\,, \tag{44}$$

the prime indicating derivative. If f is not necessarily K-invariant we use (44) on the average

$$f^{\natural}(z) = \int_K f(k \cdot z)\, dk = \frac{1}{2\pi} \int_0^{2\pi} f(k_\theta \cdot z)\, d\theta\,.$$

Since $f^{\natural}(o) = f(o)$, (44) implies

$$f(o) = \frac{1}{\pi} \int_{\mathbf{R}} [e^t (f^{\natural})\widehat{\ }(a_t \cdot \xi_0)]\, dT'(t)\,. \tag{45}$$

This can be written as the convolution at $t = 0$ of $(f^{\natural})\widehat{\ }(a_t \cdot \xi_0)$ with the image of the distribution $e^t T'_t$ under $t \to -t$. Since T' is even the right hand side of (45) is the convolution at $t = 0$ of $\widehat{f^{\natural}}$ with $e^{-t} T'_t$. Thus by (37)

$$f(o) = \frac{1}{\pi} (\Lambda \widehat{f^{\natural}})(\xi_0)\,.$$

Since Λ and $\widehat{\ }$ commute with the K action this implies

$$f(o) = \frac{1}{\pi} \int_K (\Lambda \widehat{f})(k \cdot \xi_0) = \frac{1}{\pi} (\Lambda \widehat{f})^{\vee}(o)$$

and this proves the theorem.

Theorem 4.3 is of course the exact analog to Theorem 3.6 in Chapter I, although we have not specified the decay conditions for f needed in generalizing Theorem 4.3.

D. The Poisson Integral as a Radon Transform.

Here we preserve the notation introduced for the hyperbolic plane $\mathbf{H}^2$. Now we consider the homogeneous spaces

$$X = G/MAN\,, \quad \Xi = G/K\,. \tag{46}$$

Then Ξ is the disk $D : |z| < 1$. On the other hand, X is identified with the boundary $B : |z| = 1$, because when G acts on B, MAN is the subgroup fixing the point $z = 1$. Since $G = KAN$, each coset $gMAN$ intersects eK. Thus each $x \in X$ is incident to each $\xi \in \Xi$. Our abstract Radon transform (9) now takes the form

$$\begin{aligned} \widehat{f}(gK) &= \int_{K/L} f(gkMAN)\, dk_L = \int_B f(g \cdot b)\, db\,, \\ &= \int_B f(b) \frac{d(g^{-1} \cdot b)}{db}\, db\,. \end{aligned} \tag{47}$$

Writing g^{-1} in the form

$$g^{-1} : \zeta \to \frac{\zeta - z}{-\overline{z}\zeta + 1}\,, \quad g^{-1} \cdot e^{i\theta} = e^{i\varphi}\,,$$

we have

$$e^{i\varphi} = \frac{e^{i\theta} - z}{-\overline{z}e^{i\theta} + 1}\,, \qquad \frac{d\varphi}{d\theta} = \frac{1 - |z|^2}{|z - e^{i\theta}|}\,,$$

and this last expression is the classical Poisson kernel. Since $gK = z$, (47) becomes the classical Poisson integral

$$\widehat{f}(z) = \int_B f(b)\frac{1 - |z|^2}{|z - b|^2}\,db\,. \tag{48}$$

Theorem 4.5. *The Radon transform $f \to \widehat{f}$ for the homogeneous spaces (46) is the classical Poisson integral (48). The inversion is given by the classical Schwarz theorem*

$$f(b) = \lim_{z \to b} \widehat{f}(z)\,, \quad f \in C(B)\,, \tag{49}$$

solving the Dirichlet problem for the disk.

We repeat the geometric proof of (49) from our booklet [1981] since it seems little known and is considerably shorter than the customary solution in textbooks of the Dirichlet problem for the disk. In (49) it suffices to consider the case $b = 1$. Because of (47),

$$\begin{aligned} \widehat{f}(\tanh\, t) &= \widehat{f}(a_t \cdot 0) = \frac{1}{2\pi}\int_0^{2\pi} f(a_t \cdot e^{i\theta})\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f\left(\frac{e^{i\theta} + \tanh\, t}{\tanh\, t\, e^{i\theta} + 1}\right)\,d\theta\,. \end{aligned}$$

Letting $t \to +\infty$, (49) follows by the dominated convergence theorem.

The range question A for $f \to \widehat{f}$ is also answered by classical results for the Poisson integral; for example, the classical characterization of the Poisson integrals of bounded functions now takes the form

$$L^\infty(B)\widehat{\ } = \{\varphi \in L^\infty(\Xi) \,:\, L\varphi = 0\}\,. \tag{50}$$

The range characterization (50) is of course quite analogous to the range characterization for the X-ray transform described in Theorem 6.9, Chapter I. Both are realizations of the general expectations at the end of §2 that when $\dim X < \dim \Xi$ the range of the transform $f \to \widehat{f}$ should be given as the kernel of some differential operators. The analogy between (50) and Theorem 6.9 is even closer if we recall Gonzalez' theorem [1990b] that if we view the X-ray transform as a Radon transform between two homogeneous spaces of $\mathbf{M}(3)$ (see next example) then the range (83) in Theorem 6.9, Ch. I, can be described as the null space of a differential operator which is invariant under $\mathbf{M}(3)$. Furthermore, the dual transform $\varphi \to \check{\varphi}$ maps $\mathcal{E}(\Xi)$ on $\mathcal{E}(X)$. (See Corollary 4.7 below.)

Furthermore, John's mean value theorem for the X-ray transform (Corollary 6.12, Chapter I) now becomes the exact analog of Gauss' mean value theorem for harmonic functions.

What is the dual transform $\varphi \to \check{\varphi}$ for the pair (46)? The invariant measure on $MAN/M = AN$ is the functional

$$\varphi \to \int_{AN} \varphi(an \cdot o)\, da\, dn\,. \tag{51}$$

The right hand side is just $\check{\varphi}(b_0)$ where $b_0 = eMAN$. If $g = a'n'$ the measure (51) is seen to be invariant under g. Thus it is a constant multiple of the surface element $dz = (1-x^2-y^2)^{-2}\, dx\, dy$ defined by (24). Since the maps $t \to a_t \cdot o$ and $x \to n_x \cdot o$ were seen to be isometries, this constant factor is 1. Thus the measure (51) is invariant under each $g \in G$. Writing $\varphi_g(z) = \varphi(g \cdot z)$ we know $(\varphi_g)^\vee = \check{\varphi}_g$ so

$$\check{\varphi}(g \cdot b_0) = \int_{AN} \varphi_g(an)\, da\, dn = \check{\varphi}(b_0)\,.$$

Thus the dual transform $\varphi \to \check{\varphi}$ assigns to each $\varphi \in \mathcal{D}(\Xi)$ its integral over the disk.

Table II.1 summarizes the various results mentioned above about the Poisson integral and the X-ray transform. The inversion formulas and the ranges show subtle analogies as well as strong differences. The last item in the table comes from Corollary 4.7 below for the case $n = 3$, $d = 1$.

E. The d-plane Transform.

We now review briefly the d-plane transform from a group theoretic standpoint. As in (1) we write

$$X = \mathbf{R}^n = \mathbf{M}(n)/\mathbf{O}(n)\,, \quad \Xi = \mathbf{G}(d,n) = \mathbf{M}(n)/(\mathbf{M}(d) \times \mathbf{O}(n-d))\,, \tag{52}$$

where $\mathbf{M}(d) \times \mathbf{O}(n-d)$ is the subgroup of $\mathbf{M}(n)$ preserving a certain d-plane ξ_0 through the origin. Since the homogeneous spaces

$$\mathbf{O}(n)/\mathbf{O}(n) \cap (\mathbf{M}(d) \times \mathbf{O}(n-d)) = \mathbf{O}(n)/(\mathbf{O}(d) \times \mathbf{O}(n-d))$$

and

$$(\mathbf{M}(d) \times \mathbf{O}(n-d))/\mathbf{O}(n) \cap (\mathbf{M}(d) \times \mathbf{O}(n-d)) = \mathbf{M}(d)/\mathbf{O}(d)$$

have unique invariant measures the group-theoretic transforms (9) reduce to the transforms (52), (53) in Chapter I. The range of the d-plane transform is described by Theorem 6.3 and the equivalent Theorem 6.5 in Chapter I. It was shown by Richter [1986a] that the differential operators in

	Poisson Integral	*X-ray Transform*
Coset spaces	$X = \mathbf{SU}(1,1)/MAN$ $\Xi = \mathbf{SU}(1,1)/K$	$X = \mathbf{M}(3)/\mathbf{O}(3)$ $\Xi = \mathbf{M}(3)/(\mathbf{M}(1) \times \mathbf{O}(2))$
$f \to \widehat{f}$	$\widehat{f}(z) = \int_B f(b) \frac{1-\|z\|^2}{\|z-b\|^2} db$	$\widehat{f}(\ell) = \int_\ell f(p)\, dm(p)$
$\varphi \to \check{\varphi}$	$\check{\varphi}(x) = \int_\Xi \varphi(\xi)\, d\xi$	$\check{\varphi}(x) =$ average of φ over set of ℓ through x
Inversion	$f(b) = \lim_{z \to b} \widehat{f}(z)$	$f = \frac{1}{\pi}(-L)^{1/2}((\widehat{f})^\vee)$
Range of $f \to \widehat{f}$	$L^\infty(X)\widehat{\ } =$ $\{\varphi \in L^\infty(\Xi) : L\varphi = 0\}$	$\mathcal{D}(X)\widehat{\ } =$ $\{\varphi \in \mathcal{D}(\Xi) : \Lambda(\|\xi - \eta\|^{-1}\varphi) = 0\}$
Range characterization	Gauss' mean value theorem	Mean value property for hyperboloids of revolution
Range of $\varphi \to \check{\varphi}$	$\mathcal{E}(\Xi)^\vee = \mathbf{C}$	$\mathcal{E}(\Xi)^\vee = \mathcal{E}(X)$

TABLE II.1. Analogies between the Poisson Integral and the X-ray Transform.

Theorem 6.5 could be replaced by $\mathbf{M}(n)$-induced second order differential operators and then Gonzalez [1990b] showed that the whole system could be replaced by a single fourth order $\mathbf{M}(n)$-invariant differential operator on Ξ.

Writing (52) for simplicity in the form

$$X = G/K\,, \quad \Xi = G/H \tag{53}$$

we shall now discuss the range question for the dual transform $\varphi \to \check{\varphi}$ by invoking the d-plane transform on $\mathcal{E}'(X)$.

Theorem 4.6. *Let $\mathcal{N}$ denote the kernel of the dual transform on $\mathcal{E}(\Xi)$. Then the range of $S \to \widehat{S}$ on $\mathcal{E}'(X)$ is given by*

$$\mathcal{E}'(X)\widehat{\ } = \{\Sigma \in \mathcal{E}'(\Xi) \,:\, \Sigma(\mathcal{N}) = 0\}\,.$$

The inclusion $\subset$ is clear from the definitions (14),(15) and Proposition 2.5. The converse is proved by the author in [1983a] and [1994b], Ch. I, §2 for $d = n-1$; the proof is also valid for general d.

For Fréchet spaces E and F one has the following classical result. A continuous mapping $\alpha : E \to F$ is surjective if the transpose ${}^t\alpha : F' \to E'$ is injective and has a closed image. Taking $E = \mathcal{E}(\Xi)$, $F = \mathcal{E}(X)$, α as the dual transform $\varphi \to \check{\varphi}$, the transpose ${}^t\alpha$ is the Radon transform on $\mathcal{E}'(X)$. By Theorem 4.6, ${}^t\alpha$ does have a closed image and by Theorem 5.5, Ch. I (extended to any d) ${}^t\alpha$ is injective. Thus we have the following result (Hertle [1984] for $d = n-1$) expressing the surjectivity of α.

Corollary 4.7. *Every $f \in \mathcal{E}(\mathbf{R}^n)$ is the dual transform $f = \check{\varphi}$ of a smooth d-plane function φ.*

F. Grassmann Manifolds.

We consider now the (affine) Grassmann manifolds $\mathbf{G}(p,n)$ and $\mathbf{G}(q,n)$ where $p + q = n-1$. If $p = 0$ we have the original case of points and hyperplanes. Both are homogeneous spaces of the group $\mathbf{M}(n)$ and we represent them accordingly as coset spaces

$$X = \mathbf{M}(n)/H_p\,, \quad \Xi = \mathbf{M}(n)/H_q\,. \tag{54}$$

Here we take H_p as the isotropy group of a p-plane x_0 through the origin $0 \in \mathbf{R}^n$, H_q as the isotropy group of a q-plane ξ_0 through 0, *perpendicular to* x_0. Then

$$H_p \sim \mathbf{M}(p) \times \mathbf{O}(n-p)\,, \quad H_q = \mathbf{M}(q) \times \mathbf{O}(n-q)\,.$$

Also

$$H_q \cdot x_0 = \{x \in X \,:\, x \perp \xi_0, x \cap \xi_0 \neq \emptyset\}\,,$$

the set of p-planes intersecting ξ_0 orthogonally. It is then easy to see that

$$x \text{ is incident to } \xi \Leftrightarrow x \perp \xi\,, \quad x \cap \xi \neq \emptyset\,.$$

Consider as in Chapter I, §6 the mapping

$$\pi \,:\, \mathbf{G}(p,n) \to \mathbf{G}_{p,n}$$

given by parallel translating a p-plane to one such through the origin. If $\sigma \in \mathbf{G}_{p,n}$, the fiber $F = \pi^{-1}(\sigma)$ is naturally identified with the Euclidean space $\sigma^\perp$. Consider the linear operator $\Box_p$ on $\mathcal{E}(\mathbf{G}(p,n))$ given by

$$(\Box_p f)|F = L_F(f|F)\,. \tag{55}$$

Here L_F is the Laplacian on F and bar denotes restriction. Then one can prove that $\Box_p$ is a differential operator on $\mathbf{G}(p,n)$ which is invariant under the action of $\mathbf{M}(n)$. Let $f \to \widehat{f}$, $\varphi \to \check{\varphi}$ be the Radon transform and its dual corresponding to the pair (54). Then $\widehat{f}(\xi)$ represents the integral of f over all p-planes x intersecting ξ under a right angle. For n odd this is inverted as follows (Gonzalez [1984, 1987]).

Theorem 4.8. *Let* $p, q \in \mathbb{Z}^+$ *such that* $p+q+1=n$ *is odd. Then the transform* $f \to \widehat{f}$ *from* $\mathbf{G}(p,n)$ *to* $\mathbf{G}(q,n)$ *is inverted by the formula*

$$C_{p,q} f = ((\square_q)^{(n-1)/2} \widehat{f})^\vee\,, \quad f \in \mathcal{D}(\mathbf{G}(p,n))$$

where $C_{p,q}$ *is a constant.*

If $p=0$ this reduces to Theorem 3.6, Ch. I.

G. Half-lines in a Half-plane.

In this example X denotes the half-plane $\{(a,b) \in \mathbf{R}^2 \,:\, a>0\}$ viewed as a subset of the plane $\{(a,b,1) \in \mathbf{R}^3\}$. The group G of matrices

$$(\alpha,\beta,\gamma) = \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \in \mathbf{GL}(3,\mathbf{R})\,, \quad \alpha > 0$$

acts transitively on X with the action

$$(\alpha,\beta,\gamma) \odot (a,b) = (\alpha a, \beta a + b + \gamma)\,.$$

This is the restriction of the action of $\mathbf{GL}(3,\mathbf{R})$ on $\mathbf{R}^3$. The isotropy group of the point $x_0 = (1,0)$ is the group

$$K = \{(1, \beta - \beta) \,:\, \beta \in \mathbf{R}\}\,.$$

Let Ξ denote the set of half-lines in X which end on the boundary $\partial X = 0 \times \mathbf{R}$. These lines are given by

$$\xi_{v,w} = \{(t, v+tw) \,:\, t>0\}$$

for arbitrary $v, w \in \mathbf{R}$. Thus Ξ can be identified with $\mathbf{R} \times \mathbf{R}$. The action of G on X induces a transitive action of G on Ξ which is given by

$$(\alpha,\beta,\gamma)\Diamond(v,w) = (v+\gamma, \frac{w+\beta}{\alpha})\,.$$

(Here we have for simplicity written (v,w) instead of $\xi_{v,w}$.) The isotropy group of the point $\xi_0 = (0,0)$ (the x-axis) is

$$H = \{(\alpha,0,0) \,:\, \alpha > 0\} = \mathbf{R}^\times_+\,,$$

the multiplicative group of the positive real numbers. Thus we have the identifications

$$X = G/K\,, \quad \Xi = G/H\,. \tag{56}$$

The group $K \cap H$ is now trivial so the Radon transform and its dual for the double fibration in (56) are defined by

$$\widehat{f}(gH) = \int_H f(ghK)\, dh\,, \tag{57}$$

$$\check{\varphi}(gK) = \chi(g) \int_K \varphi(gkH)\, dk\,, \tag{58}$$

where χ is the homomorphism $(\alpha, \beta, \gamma) \to \alpha^{-1}$ of G onto $\mathbf{R}_+^\times$. The reason for the presence of χ is that we wish Proposition 2.2 to remain valid even if G is not unimodular. In (57) and (58) we have the Haar measures

$$dk_{(1,\beta-\beta)} = d\beta\,, \quad dh_{(\alpha,0,0)} = d\alpha/\alpha\,. \tag{59}$$

Also, if $g = (\alpha, \beta, \gamma)$, $h = (a, 0, 0)$, $k = (1, b, -b)$ then

$$gH = (\gamma, \beta/\alpha)\,, \qquad ghK = (\alpha a, \beta a + \gamma)$$
$$gK = (\alpha, \beta + \gamma)\,, \qquad gkH = (-b + \gamma, \tfrac{b+\beta}{\alpha})$$

so (57)–(58) become

$$\widehat{f}(\gamma, \beta/\alpha) = \int_{\mathbf{R}^+} f(\alpha a, \beta a + \gamma) \frac{da}{a}$$
$$\check{\varphi}(\alpha, \beta + \gamma) = \alpha^{-1} \int_{\mathbf{R}} \varphi(-b + \gamma, \tfrac{b+\beta}{\alpha})\, db\,.$$

Changing variables these can be written

$$\widehat{f}(v, w) = \int_{\mathbf{R}^+} f(a, v + aw) \frac{da}{a}\,, \tag{60}$$

$$\check{\varphi}(a, b) = \int_{\mathbf{R}} \varphi(b - as, s)\, ds \qquad a > 0\,. \tag{61}$$

Note that in (60) the integration takes place over all points on the line $\xi_{v,w}$ and in (61) the integration takes place over the set of lines $\xi_{b-as,s}$ all of which pass through the point (a, b). This is an *a posteriori* verification of the fact that our incidence for the pair (56) amounts to $x \in \xi$.

From (60)–(61) we see that $f \to \widehat{f}, \varphi \to \check{\varphi}$ are adjoint relative to the measures $\frac{da}{a}\, db$ and $dv\, dw$:

$$\int_{\mathbf{R}} \int_{\mathbf{R}_+^\times} f(a, b) \check{\varphi}(a, b) \frac{da}{a}\, db = \int_{\mathbf{R}} \int_{\mathbf{R}} \widehat{f}(v, w) \varphi(v, w)\, dv\, dw\,. \tag{62}$$

The proof is a routine computation.

We recall (Chapter V) that $(-L)^{1/2}$ is defined on the space of rapidly decreasing functions on $\mathbf{R}$ by

$$((-L)^{1/2}\psi)^{\sim}\ (\tau) = |\tau| \widetilde{\psi}(\tau) \tag{63}$$

and we define Λ on $\mathcal{S}(\Xi)(=\mathcal{S}(\mathbf{R}^2))$ by having $(-L)^{1/2}$ only act on the second variable:

$$(\Lambda\varphi)(v,w) = ((-L)^{1/2}\varphi(v,\cdot))(w)\,. \tag{64}$$

Viewing $(-L)^{1/2}$ as the Riesz potential I^{-1} on $\mathbf{R}$ (Chapter V, §5) it is easy to see that if $\varphi_c(v,w) = \varphi(v,\frac{w}{c})$ then

$$\Lambda\varphi_c = |c|^{-1}(\Lambda\varphi)_c\,. \tag{65}$$

The Radon transform (57) is now inverted by the following theorem.

Theorem 4.9. *Let $f \in \mathcal{D}(X)$. Then*

$$f = \frac{1}{2\pi}(\Lambda\widehat{f})^{\vee}\,.$$

Proof. In order to use the Fourier transform $F \to \widetilde{F}$ on $\mathbf{R}^2$ and on $\mathbf{R}$ we need functions defined on all of $\mathbf{R}^2$. Thus we define

$$f^*(a,b) = \begin{cases} \frac{1}{a}f(\frac{1}{a}\,,\,\frac{-b}{a}) & a>0\,,\\ 0 & a\leq 0\,.\end{cases}$$

Then

$$\begin{aligned} f(a,b) &= \frac{1}{a}f^*\left(\frac{1}{a}\,,\,-\frac{b}{a}\right)\\ &= a^{-1}(2\pi)^{-2}\iint \widetilde{f^*}(\xi,\eta)e^{i(\frac{\xi}{a}-\frac{b\eta}{a})}\,d\xi\,d\eta\\ &= (2\pi)^{-2}\iint \widetilde{f^*}(a\xi+b\eta,\eta)e^{i\xi}\,d\xi\,d\eta\\ &= a(2\pi)^{-2}\iint |\xi|\widetilde{f^*}((a+ab\eta)\xi,a\eta\xi)e^{i\xi}\,d\xi\,d\eta\,. \end{aligned}$$

Next we express the Fourier transform in terms of the Radon transform. We have

$$\begin{aligned} \widetilde{f^*}((a+ab\eta)\xi,a\eta\xi) &= \iint f^*(x,y)e^{-ix(a+ab\eta)\xi}e^{-iya\eta\xi}\,dx\,dy\\ &= \int_{\mathbf{R}}\int_{x\geq 0}\frac{1}{x}f\left(\frac{1}{x}\,,\,-\frac{y}{x}\right)e^{-ix(a+ab\eta)\xi}e^{-iya\eta\xi}\,dx\,dy\\ &= \int_{\mathbf{R}}\int_{x\geq 0} f\left(\frac{1}{x}\,,\,b+\frac{1}{\eta}+\frac{z}{x}\right)e^{iza\eta\xi}\,\frac{dx}{x}\,dz\,. \end{aligned}$$

This last expression is

$$\int_{\mathbf{R}}\widehat{f}(b+\eta^{-1},z)e^{iza\eta\xi}\,dz = (\widehat{f})^{\sim}(b+\eta^{-1},-a\eta\xi)\,,$$

where $\sim$ denotes the 1-dimensional Fourier transform (in the second variable). Thus

$$f(a,b) = a(2\pi)^{-2} \iint |\xi|(\widehat{f})^{\sim}(b+\eta^{-1}, -a\eta\xi)e^{i\xi}\, d\xi\, d\eta\,.$$

However $\widetilde{F}(c\xi) = |c|^{-1}(F_c)^{\sim}(\xi)$ so by (65)

$$\begin{aligned} f(a,b) &= a(2\pi)^{-2} \iint |\xi|((\widehat{f})_{a\eta})^{\sim}(b+\eta^{-1}, -\xi)e^{i\xi}\, d\xi |a\eta|^{-1}\, d\eta \\ &= (2\pi)^{-1} \int \Lambda((\widehat{f})_{a\eta})(b+\eta^{-1}, -1)|\eta|^{-1}\, d\eta \\ &= (2\pi)^{-1} \int |a\eta|^{-1}(\Lambda\widehat{f})_{a\eta}(b+\eta^{-1}, -1)|\eta|^{-1}\, d\eta \\ &= a^{-1}(2\pi)^{-1} \int (\Lambda\widehat{f})(b+\eta^{-1}, -(a\eta)^{-1})\eta^{-2}\, d\eta \end{aligned}$$

so

$$\begin{aligned} f(a,b) &= (2\pi)^{-1} \int_{\mathbf{R}} (\Lambda\widehat{f})(b-av, v)\, dv \\ &= (2\pi)^{-1}(\Lambda\widehat{f})^{\vee}(a,b)\,. \end{aligned}$$

proving the theorem.

Remark 4.10. It is of interest to compare this theorem with Theorem 3.6, Ch. I. If $f \in \mathcal{D}(X)$ is extended to all of $\mathbf{R}^2$ by defining it 0 in the left half plane then Theorem 3.6 does give a formula expressing f in terms of its integrals over half-lines in a strikingly similar fashion. Note however that while the operators $f \to \widehat{f}, \varphi \to \check{\varphi}$ are in the two cases defined by integration over the same sets (points on a half-line, half-lines through a point) the measures in the two cases are different. Thus it is remarkable that the inversion formulas look exactly the same.

H. Theta Series and Cusp Forms.

Let G denote the group $\mathbf{SL}(2,\mathbf{R})$ of 2×2 matrices of determinant one and Γ the *modular group* $\mathbf{SL}(2,\mathbb{Z})$. Let N denote the unipotent group $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ where $n \in \mathbf{R}$ and consider the homogeneous spaces

$$X = G/N\,, \qquad \Xi = G/\Gamma\,. \tag{66}$$

Under the usual action of G on $\mathbf{R}^2$, N is the isotropy subgroup of $(1,0)$ so X can be identified with $\mathbf{R}^2 - (0)$, whereas Ξ is of course 3-dimensional.

In number theory one is interested in decomposing the space $L^2(G/\Gamma)$ into G-invariant irreducible subspaces. We now give a rough description of this by means of the transforms $f \to \widehat{f}$ and $\varphi \to \check{\varphi}$.

As customary we put $\Gamma_\infty = \Gamma \cap N$; our transforms (9) then take the form

$$\widehat{f}(g\Gamma) = \sum_{\Gamma/\Gamma_\infty} f(g\gamma N)\,, \quad \check{\varphi}(gN) = \int_{N/\Gamma_\infty} \varphi(gn\Gamma)\, dn_{\Gamma_\infty}\,.$$

Since N/Γ_∞ is the circle group, $\check{\varphi}(gN)$ is just the constant term in the Fourier expansion of the function $n\Gamma_\infty \to \varphi(gn\Gamma)$. The null space $L^2_d(G/\Gamma)$ in $L^2(G/\Gamma)$ of the operator $\varphi \to \check{\varphi}$ is called the space of *cusp forms* and the series for $\widehat{f}$ is called *theta series*. According to Prop. 2.2 they constitute the orthogonal complement of the image $C_c(X)\widehat{\ }$.

We have now the G-invariant decomposition

$$L^2(G/\Gamma) = L^2_c(G/\Gamma) \oplus L^2_d(G/\Gamma)\,, \tag{67}$$

where ($-$ denoting closure)

$$L^2_c(G/\Gamma) = (C_c(X)\widehat{\ })^- \tag{68}$$

and as mentioned above,

$$L^2_d(G/\Gamma) = (C_c(X)\widehat{\ })^\perp\,. \tag{69}$$

It is known (cf. Selberg [1962], Godement [1966]) that the representation of G on $L^2_c(G/\Gamma)$ is the *continuous* direct sum of the irreducible representations of G from the principal series whereas the representation of G on $L^2_d(G/\Gamma)$ is the *discrete* direct sum of irreducible representations each occurring with finite multiplicity.

In conclusion we note that the determination of a function in $\mathbf{R}^n$ in terms of its integrals over unit spheres (John [1955]) can be regarded as a solution to the first half of Problem B in §2 for the double fibration (4).

Bibliographical Notes

The Radon transform and its dual for a double fibration

$$\begin{array}{ccc} & Z = G/(K \cap H) & \\ \swarrow & & \searrow \\ X = G/K & & \Xi = G/H \end{array} \tag{70}$$

was introduced in the author's paper [1966a]. The results of §1–§2 are from there and from [1994b]. The definition uses the concept of *incidence* for $X = G/K$ and $\Xi = G/H$ which goes back to Chern [1942]. Even when the elements of Ξ can be viewed as subsets of X and vice versa (Lemma 1.3) it can be essential for the inversion of $f \to \widehat{f}$ not to restrict the incidence to the naive one $x \in \xi$. (See for example the classical case $X = \mathbf{S}^2, \Xi =$ set of great circles where in Theorem 4.1 a more general incidence is essential.) The double fibration in (1) was generalized in Gelfand, Graev and Shapiro [1969], by relaxing the homogeneity assumption.

For the case of geodesics in constant curvature spaces (Examples A, B in §4) see notes to Ch. III.

The proof of Theorem 4.3 (a special case of the author's inversion formula in [1964], [1965b]) makes use of a method by Godement [1957] in another context. Another version of the inversion (38) for $\mathbf{H}^2$ (and $\mathbf{H}^n$) is given in Gelfand-Graev-Vilenkin [1966]. A further inversion of the horocycle transform in $\mathbf{H}^2$ (and $\mathbf{H}^n$), somewhat analogous to (30) for the X-ray transform, is given by Berenstein and Tarabusi [1994].

The analogy suggested above between the X-ray transform and the horocycle transform in $\mathbf{H}^2$ goes even further in $\mathbf{H}^3$. There the 2-dimensional transform for totally geodesic submanifolds has *the same* inversion formula as the horocycle transform (Helgason [1994b], p. 209).

For a treatment of the horocycle transform on a Riemannian symmetric space see the author's monograph [1994b], Chapter II, where Problems A, B, C in §2 are discussed in detail along with some applications to differential equations and group representations. See also Quinto [1993a] and Gonzalez and Quinto [1994] for new proofs of the support theorem.

Example G is from Hilgert's paper [1994], where a related Fourier transform theory is also established. It has a formal analogy to the Fourier analysis on $\mathbf{H}^2$ developed by the author in [1965b] and [1972].

CHAPTER III

THE RADON TRANSFORM ON TWO-POINT HOMOGENEOUS SPACES

Let X be a complete Riemannian manifold, x a point in X and X_x the tangent space to X at x. Let Exp_x denote the mapping of X_x into X given by $\mathrm{Exp}_x(u) = \gamma_u(1)$ where $t \to \gamma_u(t)$ is the geodesic in X through x with tangent vector u at $x = \gamma_u(0)$.

A connected submanifold S of a Riemannian manifold X is said to be *totally geodesic* if each geodesic in X which is tangential to S at a point lies entirely in S.

The totally geodesic submanifolds of $\mathbf{R}^n$ are the planes in $\mathbf{R}^n$. Therefore, in generalizing the Radon transform to Riemannian manifolds, it is natural to consider integration over totally geodesic submanifolds. In order to have enough totally geodesic submanifolds at our disposal we consider in this section Riemannian manifolds X which are *two-point homogeneous* in the sense that for any two-point pairs $p, q \in X$ $p', q' \in X$, satisfying $d(p, q) = d(p', q')$, (where $d =$ distance), there exists an isometry g of X such that $g \cdot p = p',\ g \cdot q = q'$. We start with the subclass of Riemannian manifolds with the richest supply of totally geodesic submanifolds, namely the spaces of constant curvature.

While §1, which constitutes most of this chapter, is elementary, §2–§5 will involve a bit of Lie group theory.

§1 Spaces of Constant Curvature. Inversion and Support Theorems

Let X be a simply connected complete Riemannian manifold of dimension $n \geq 2$ and constant sectional curvature.

Lemma 1.1. *Let $x \in X$, V a subspace of the tangent space X_x. Then* $\mathrm{Exp}_x(V)$ *is a totally geodesic submanifold of X.*

Proof. For this we choose a specific embedding of X into $\mathbf{R}^{n+1}$, and assume for simplicity the curvature is $\epsilon(= \pm 1)$. Consider the quadratic form

$$B_\epsilon(x) = x_1^2 + \cdots + x_n^2 + \epsilon x_{n+1}^2$$

and the quadric Q_ϵ given by $B_\epsilon(x) = \epsilon$. The orthogonal group $\mathbf{O}(B_\epsilon)$ acts transitively on Q_ϵ. The form B_ϵ is positive definite on the tangent space $\mathbf{R}^n \times (0)$ to Q_ϵ at $x^0 = (0, \ldots, 0, 1)$; by the transitivity B_ϵ induces a positive definite quadratic form at each point of Q_ϵ, turning Q_ϵ into a

Riemannian manifold, on which $\mathbf{O}(B_\epsilon)$ acts as a transitive group of isometries. The isotropy subgroup at the point x^0 is isomorphic to $\mathbf{O}(n)$ and its acts transitively on the set of 2-dimensional subspaces of the tangent space $(Q_\epsilon)_{x^0}$. It follows that all sectional curvatures at x^0 are the same, namely ϵ, so by homogeneity, Q_ϵ has constant curvature ϵ. In order to work with connected manifolds, we replace Q_{-1} by its intersection Q_{-1}^+ with the half-space $x_{n+1} > 0$. Then Q_{+1} and Q_{-1}^+ are simply connected complete Riemannian manifolds of constant curvature. Since such manifolds are uniquely determined by the dimension and the curvature it follows that we can identify X with Q_{+1} or Q_{-1}^+.

The geodesic in X through x^0 with tangent vector $(1, 0, \ldots, 0)$ will be left point-wise fixed by the isometry

$$(x_1, x_2, \ldots, x_n, x_{n+1}) \to (x_1, -x_2, \ldots, -x_n, x_{n+1}) .$$

This geodesic is therefore the intersection of X with the two-plane $x_2 = \cdots = x_n = 0$ in $\mathbf{R}^{n+1}$. By the transitivity of $\mathbf{O}(n)$ all geodesics in X through x^0 are intersections of X with two-planes through 0. By the transitivity of $\mathbf{O}(Q_\epsilon)$ it then follows that the geodesics in X are precisely the nonempty intersections of X with two-planes through the origin.

Now if $V \subset X_{x^0}$ is a subspace, $\mathrm{Exp}_{x^0}(V)$ is by the above the intersection of X with the subspace of $\mathbf{R}^{n+1}$ spanned by V and x^0. Thus $\mathrm{Exp}_{x^0}(V)$ is a quadric in $V + \mathbf{R}x^0$ and its Riemannian structure induced by X is the same as induced by the restriction $B_\epsilon|(V + \mathbf{R}x^0)$. Thus, by the above, the geodesics in $\mathrm{Exp}_{x^0}(V)$ are obtained by intersecting it with two-planes in $V + \mathbf{R}x^0$ through 0. Consequently, the geodesics in $\mathrm{Exp}_{x^0}(V)$ are geodesics in X so $\mathrm{Exp}_{x^0}(V)$ is a totally geodesic submanifold of X. By the homogeneity of X this holds with x^0 replaced by an arbitrary point $x \in X$. The lemma is proved.

In accordance with the viewpoint of Ch. II we consider X as a homogeneous space of the identity component G of the group $\mathbf{O}(Q_\epsilon)$. Let K denote the isotropy subgroup of G at the point $x^0 = (0, \ldots, 0, 1)$. Then K can be identified with the special orthogonal group $\mathbf{SO}(n)$. Let k be a fixed integer, $1 \leq k \leq n-1$; let $\xi_0 \subset X$ be a fixed totally geodesic submanifold of dimension k passing through x^0 and let H be the subgroup of G leaving ξ_0 invariant. We have then

$$X = G/K , \quad \Xi = G/H , \tag{1}$$

Ξ denoting the set of totally geodesic k-dimensional submanifolds of X. Since $x^0 \in \xi_0$ it is clear that the abstract incidence notion boils down to the naive one, in other words: The cosets $x = gK$ $\xi = \gamma H$ have a point in common if and only if $x \in \xi$. In fact

$$x \in \xi \Leftrightarrow x^0 \in g^{-1}\gamma \cdot \xi_0 \Leftrightarrow g^{-1}\gamma \cdot \xi_0 = k \cdot \xi_0 \quad \text{for some } k \in K .$$

A. The Hyperbolic Space

We take first the case of negative curvature, that is $\epsilon = -1$. The transform $f \to \widehat{f}$ is now given by

$$\widehat{f}(\xi) = \int_\xi f(x)\, dm(x) \tag{2}$$

ξ being any k-dimensional totally geodesic submanifold of X $(1 \le k \le n-1)$ with the induced Riemannian structure and dm the corresponding measure. From our description of the geodesics in X it is clear that any two points in X can be joined by a unique geodesic. Let d be a distance function on X, and for simplicity we write o for the origin x^o in X. Consider now geodesic polar-coordinates for X at o; this is a mapping

$$\mathrm{Exp}_o Y \to (r, \theta_1, \ldots, \theta_{n-1})\,,$$

where Y runs through the tangent space $X_o, r = |Y|$ (the norm given by the Riemannian structure) and $(\theta_1, \ldots, \theta_{n-1})$ are coordinates of the unit vector $Y/|Y|$. Then the Riemannian structure of X is given by

$$ds^2 = dr^2 + (\sinh\, r)^2\, d\sigma^2\,, \tag{3}$$

where $d\sigma^2$ is the Riemannian structure

$$\sum_{i,j=1}^{n-1} g_{ij}(\theta_1, \cdots, \theta_{n-1})\, d\theta_i\, d\theta_j$$

on the unit sphere in X_o. The surface area $A(r)$ and volume $V(r) = \int_o^r A(t)\, dt$ of a sphere in X of radius r are thus given by

$$A(r) = \Omega_n (\sinh\, r)^{n-1}\,, \quad V(r) = \Omega_n \int_o^r \sinh^{n-1} t\, dt \tag{4}$$

so $V(r)$ increases like $e^{(n-1)r}$. This explains the growth condition in the next result where $d(o, \xi)$ denotes the distance of o to the manifold ξ.

Theorem 1.2. *(The support theorem.) Suppose $f \in C(X)$ satisfies*

(i) For each integer $m > 0$, $f(x)e^{md(o,x)}$ is bounded.

(ii) There exists a number $R > 0$ such that

$$\widehat{f}(\xi) = 0 \quad \textit{for } d(o, \xi) > R\,.$$

Then

$$f(x) = 0 \quad \textit{for } d(o, x) > R\,.$$

Taking $R \to 0$ we obtain the following consequence.

Corollary 1.3. *The Radon transform $f \to \widehat{f}$ is one-to-one on the space of continuous functions on X satisfying condition (i) of "exponential decrease".*

Proof of Theorem 1.2. Using smoothing of the form

$$\int_G \varphi(g) f(g^{-1} \cdot x)\, dg$$

($\varphi \in \mathcal{D}(G)$, dg Haar measure on G) we can (as in Theorem 2.6, Ch. I) assume that $f \in \mathcal{E}(X)$.

We first consider the case when f in (2) is a radial function. Let P denote the point in ξ at the minimum distance $p = d(o, \xi)$ from o, let $Q \in \xi$ be arbitrary and let

$$q = d(o, Q)\,, \quad r = d(P, Q)\,.$$

Since ξ is totally geodesic $d(P, Q)$ is also the distance between P and Q in ξ. Consider now the totally geodesic plane π through the geodesics oP and oQ as given by Lemma 1.1 (Fig. III.1). Since a totally geodesic submanifold contains the geodesic joining any two of its points, π contains the geodesic PQ. The angle oPQ being $90°$ (see e.g. [DS], p. 77) we conclude by hyperbolic trigonometry, (see e.g. Coxeter [1957])

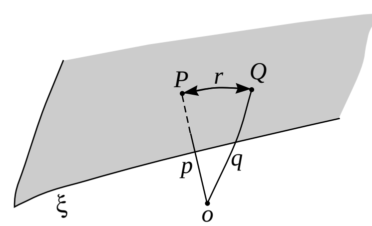

FIGURE III.1.

$$\cosh q = \cosh p \cosh r\,. \tag{5}$$

Since f is radial it follows from (5) that the restriction $f|\xi$ is constant on spheres in ξ with center P. Since these have area $\Omega_k (\sinh r)^{k-1}$ formula (2) takes the form

$$\widehat{f}(\xi) = \Omega_k \int_0^\infty f(Q)(\sinh r)^{k-1}\, dr\,. \tag{6}$$

Since f is a radial function it is invariant under the subgroup $K \subset G$ which fixes o. But K is not only transitive on each sphere $S_r(o)$ with center o, it is for each fixed k transitive on the set of k-dimensional totally geodesic submanifolds which are tangent to $S_r(o)$. Consequently, $\widehat{f}(\xi)$ depends only on the distance $d(o, \xi)$. Thus we can write

$$f(Q) = F(\cosh q)\,, \quad \widehat{f}(\xi) = \widehat{F}(\cosh p)$$

for certain 1-variable functions F and $\widehat{F}$, so by (5) we obtain

$$\widehat{F}(\cosh p) = \Omega_k \int_0^\infty F(\cosh p \, \cosh r)(\sinh r)^{k-1}\, dr\,. \tag{7}$$

Writing here $t = \cosh p$, $s = \cosh r$ this reduces to

$$\widehat{F}(t) = \Omega_k \int_1^\infty F(ts)(s^2-1)^{(k-2)/2}\, ds\,. \tag{8}$$

Here we substitute $u = (ts)^{-1}$ and then put $v = t^{-1}$. Then (8) becomes

$$v^{-1}\widehat{F}(v^{-1}) = \Omega_k \int_0^v \{F(u^{-1})u^{-k}\}(v^2-u^2)^{(k-2)/2}\, du\,.$$

This integral equation is of the form (19), Ch. I so we get the following analog of (20), Ch. I:

$$F(u^{-1})u^{-k} = cu\left(\frac{d}{d(u^2)}\right)^k \int_o^u (u^2-v^2)^{(k-2)/2}\widehat{F}(v^{-1})\, dv\,, \tag{9}$$

where c is a constant. Now by assumption (ii) $\widehat{F}(\cosh p) = 0$ if $p > R$. Thus

$$\widehat{F}(v^{-1}) = 0 \quad \text{if } 0 < v < (\cosh R)^{-1}\,.$$

From (9) we can then conclude

$$F(u^{-1}) = 0 \quad \text{if } u < (\cosh R)^{-1}$$

which means $f(x) = 0$ for $d(o,x) > R$. This proves the theorem for f radial.

Next we consider an arbitrary $f \in \mathcal{E}(X)$ satisfying (i), (ii) . Fix $x \in X$ and if dk is the normalized Haar measure on K consider the integral

$$F_x(y) = \int_K f(gk\cdot y)\, dk\,, \quad y \in X\,,$$

where $g \in G$ is an element such that $g\cdot o = x$. Clearly, $F_x(y)$ is the average of f on the sphere with center x, passing through $g\cdot y$. The function F_x satisfies the decay condition (i) and it is radial. Moreover,

$$\widehat{F}_x(\xi) = \int_K \widehat{f}(gk\cdot\xi)\, dk\,. \tag{10}$$

We now need the following estimate

$$d(o, gk\cdot\xi) \geq d(o,\xi) - d(o, g\cdot o)\,. \tag{11}$$

For this let x_o be a point on ξ closest to $k^{-1}g^{-1}\cdot o$. Then by the triangle inequality

$$\begin{aligned} d(o, gk\cdot\xi) = d(k^{-1}g^{-1}\cdot o, \xi) &\geq d(o, x_o) - d(o, k^{-1}g^{-1}\cdot o) \\ &\geq d(o,\xi) - d(o, g\cdot o)\,. \end{aligned}$$

Thus it follows by (ii) that

$$\widehat{F}_x(\xi) = 0 \;\text{ if }\; d(o,\xi) > d(o,x) + R\,.$$

Since F_x is radial this implies by the first part of the proof that

$$\int_K f(gk \cdot y)\,dk = 0 \tag{12}$$

if

$$d(o,y) > d(o, g\cdot o) + R\,. \tag{13}$$

But the set $\{gk \cdot y : k \in K\}$ is the sphere $S_{d(o,y)}(g\cdot o)$ with center $g \cdot o$ and radius $d(o,y)$; furthermore, the inequality in (13) implies the inclusion relation

$$B_R(o) \subset B_{d(o,y)}(g\cdot o) \tag{14}$$

for the balls. But considering the part in $B_R(o)$ of the geodesic through o and $g\cdot o$ we see that conversely relation (14) implies (13). Theorem 1.2 will therefore be proved if we establish the following lemma.

Lemma 1.4. *Let $f \in C(X)$ satisfy the conditions:*

(i) For each integer $m > 0$, $f(x)e^{m\,d(o,x)}$ is bounded.

(ii) There exists a number $R > 0$ such that the surface integral

$$\int_S f(s)\,d\omega(s) = 0\,,$$

whenever the spheres S encloses the ball $B_R(o)$.

Then

$$f(x) = 0 \;\text{ for }\; d(o,x) > R\,.$$

Proof. This lemma is the exact analog of Lemma 2.7, Ch. I, whose proof, however, used the vector space structure of $\mathbf{R}^n$. By using a special model of the hyperbolic space we shall nevertheless adapt the proof to the present situation. As before we may assume f is smooth, i.e., $f \in \mathcal{E}(X)$.

Consider the unit ball $\{x \in \mathbf{R}^n : \sum_1^n x_i^2 < 1\}$ with the Riemannian structure

$$ds^2 = \rho(x_1,\ldots,x_n)^2(dx_1^2 + \cdots + dx_n^2) \tag{15}$$

where

$$\rho(x_1,\ldots,x_n) = 2(1 - x_1^2 - \ldots - x_n^2)^{-1}\,.$$

This Riemannian manifold is well known to have constant curvature -1 so we can use it for a model of X. This model is useful here because the spheres in X are the ordinary Euclidean spheres inside the ball. This fact is obvious for the spheres Σ with center 0. For the general statement it suffices to prove that if T is the geodesic symmetry with respect to a point (which we can take on the x_1-axis) then $T(\Sigma)$ is a Euclidean sphere. The unit disk D in the x_1x_2-plane is totally geodesic in X, hence invariant under T. Now the isometries of the non-Euclidean disk D are generated by the complex conjugation $x_1 + ix_2 \to x_1 - ix_2$ and fractional linear transformations so they map Euclidean circles into Euclidean circles. In particular $T(\Sigma \cap D) = T(\Sigma) \cap D$ is a Euclidean circle. But T commutes with the rotations around the x_1-axis. Thus $T(\Sigma)$ is invariant under such rotations and intersects D in a circle; hence it is a Euclidean sphere.

After these preliminaries we pass to the proof of Lemma 1.4. Let $S = S_r(y)$ be a sphere in X enclosing $B_r(o)$ and let $B_r(y)$ denote the corresponding ball. Expressing the exterior $X - B_r(y)$ as a union of spheres in X with center y we deduce from assumption (ii)

$$\int_{B_r(y)} f(x)\,dx = \int_X f(x)\,dx\,, \tag{16}$$

which is a constant for small variations in r and y. The Riemannian measure dx is given by

$$dx = \rho^n\,dx_o\,, \tag{17}$$

where $dx_o = dx_1 \dots dx_n$ is the Euclidean volume element. Let r_o and y_o, respectively, denote the Euclidean radius and Euclidean center of $S_r(y)$. Then $S_{r_o}(y_o) = S_r(y), B_{r_o}(y_o) = B_r(y)$ set-theoretically and by (16) and (17)

$$\int_{B_{r_0}(y_0)} f(x_0)\rho(x_0)^n\,dx_o = \text{const.}\,, \tag{18}$$

for small variations in r_o and y_o; thus by differentiation with respect to r_o,

$$\int_{S_{r_0}(y_0)} f(s_0)\rho(s_0)^n\,d\omega_o(s_o) = 0\,, \tag{19}$$

where $d\omega_o$ is the Euclidean surface element. Putting $f^*(x) = f(x)\rho(x)^n$ we have by (18)

$$\int_{B_{r_o}(y_o)} f^*(x_o)\,dx_o = \text{const.}\,,$$

so by differentiating with respect to y_o, we get

$$\int_{B_{r_o}(o)} (\partial_i f^*)(y_o + x_o)\,dx_o = 0\,.$$

Using the divergence theorem (26), Chapter I, §2, on the vector field $F(x_o) = f^*(y_o+x_o)\partial_i$ defined in a neighborhood of $B_{r_o}(0)$ the last equation implies

$$\int_{S_{r_o}(0)} f^*(y_o + s)s_i \, d\omega_o(s) = 0$$

which in combination with (19) gives

$$\int_{S_{r_o}(y_o)} f^*(s)s_i \, d\omega_o(s) = 0 \,. \tag{20}$$

The Euclidean and the non-Euclidean Riemannian structures on $S_{r_o}(y_o)$ differ by the factor ρ^2. It follows that $d\omega = \rho(s)^{n-1}\, d\omega_o$ so (20) takes the form

$$\int_{S_r(y)} f(s)\rho(s)s_i \, d\omega(s) = 0 \,. \tag{21}$$

We have thus proved that the function $x \to f(x)\rho(x)x_i$ satisfies the assumptions of the theorem. By iteration we obtain

$$\int_{S_r(y)} f(s)\rho(s)^k s_{i_1} \ldots s_{i_k} \, d\omega(s) = 0 \,. \tag{22}$$

In particular, this holds with $y = 0$ and $r > R$. Then $\rho(s) =$ constant and (22) gives $f \equiv 0$ outside $B_R(o)$ by the Weierstrass approximation theorem. Now Theorem 1.2 is proved.

Now let L denote the Laplace-Beltrami operator on X. (See Ch. IV, §1 for the definition.) Because of formula (3) for the Riemannian structure of X, L is given by

$$L = \frac{\partial^2}{\partial r^2} + (n-1)\coth r \frac{\partial}{\partial r} + (\sinh r)^{-2} L_S \tag{23}$$

where L_S is the Laplace-Beltrami operator on the unit sphere in X_0. We consider also for each $r \geq 0$ the mean value operator M^r defined by

$$(M^r f)(x) = \frac{1}{A(r)} \int_{S_r(x)} f(s)\, d\omega(s) \,.$$

As we saw before this can also be written

$$(M^r f)(g \cdot o) = \int_K f(gk \cdot y)\, dk \tag{24}$$

if $g \in G$ is arbitrary and $y \in X$ is such that $r = d(o, y)$. If f is an analytic function one can, by expanding it in a Taylor series, prove from (24) that M^r is a certain power series in L (cf. Helgason [1959], pp. 270-272). In particular we have the commutativity

$$M^r L = L M^r \,. \tag{25}$$

This in turn implies the "Darboux equation"

$$L_x(F(x,y)) = L_y(F(x,y)) \tag{26}$$

for the function $F(x,y) = (M^{d(o,y)}f)(x)$. In fact, using (24) and (25) we have if $g \cdot o = x$, $r = d(o,y)$

$$\begin{aligned} L_x(F(x,y)) &= (LM^r f)(x) = (M^r Lf)(x) \\ &= \int_K (Lf)(gk \cdot y)\, dk = \int_K (L_y(f(gk \cdot y)))\, dk \end{aligned}$$

the last equation following from the invariance of the Laplacian under the isometry gk. But this last expression is $L_y(F(x,y))$.

We remark that the analog of Lemma 2.13 in Ch. IV which also holds here would give another proof of (25) and (26).

For a fixed integer $k (1 \leq k \leq n-1)$ let Ξ denote the manifold of all k-dimensional totally geodesic submanifolds of X. If φ is a continuous function on Ξ we denote by $\check{\varphi}$ the point function

$$\check{\varphi}(x) = \int_{x \in \xi} \varphi(\xi)\, d\mu(\xi)\,,$$

where μ is the unique measure on the (compact) space of ξ passing through x, invariant under all rotations around x and having total measure one.

Theorem 1.5. *(The inversion formula.) For k even let Q_k denote the polynomial*

$$Q_k(z) = [z + (k-1)(n-k)][z + (k-3)(n-k+2)] \dots [z + 1 \cdot (n-2)]$$

of degree $k/2$. The k-dimensional Radon transform on X is then inverted by the formula

$$cf = Q_k(L)\big((\widehat{f})^\vee\big)\,, \quad f \in \mathcal{D}(X)\,.$$

Here c is the constant

$$c = (-4\pi)^{k/2}\Gamma(n/2)/\Gamma((n-k)/2)\,. \tag{27}$$

The formula holds also if f satisfies the decay condition (i) in Corollary 4.1.

Proof. Fix $\xi \in \Xi$ passing through the origin $o \in X$. If $x \in X$ fix $g \in G$ such that $g \cdot o = x$. As k runs through K, $gk \cdot \xi$ runs through the set of totally geodesic submanifolds of X passing through x and

$$\check{\varphi}(g \cdot o) = \int_K \varphi(gk \cdot \xi)\, dk\,.$$

Hence

$$(\widehat{f})^\vee(g \cdot o) = \int_K \left(\int_\xi f(gk \cdot y)\, dm(y) \right) dk = \int_\xi (M^r f)(g \cdot o)\, dm(y)\,,$$

where $r = d(o, y)$. But since ξ is totally geodesic in X, it has also constant curvature -1 and two points in ξ have the same distance in ξ as in X. Thus we have

$$(\widehat{f})^\vee(x) = \Omega_k \int_0^\infty (M^r f)(x)(\sinh\, r)^{k-1}\, dr\,. \tag{28}$$

We apply L to both sides and use (23). Then

$$(L(\widehat{f})^\vee)(x) = \Omega_k \int_0^\infty (\sinh\, r)^{k-1} L_r(M^r f)(x)\, dr\,, \tag{29}$$

where L_r is the "radial part" $\frac{\partial^2}{\partial r^2} + (n-1)\coth r \frac{\partial}{\partial r}$ of L. Putting now $F(r) = (M^r f)(x)$ we have the following result.

Lemma 1.6. *Let m be an integer $0 < m < n = \dim X$. Then*

$$\int_0^\infty \sinh^m r L_r F\, dr =$$
$$(m+1-n)\left[m \int_0^\infty \sinh^m r F(r)\, dr + (m-1)\int_0^\infty \sinh^{m-2} r F(r)\, dr\right].$$

If $m = 1$ the term $(m-1)\int_0^\infty \sinh^{m-2} r F(r)\, dr$ should be replaced by $F(0)$.

This follows by repeated integration by parts.

From this lemma combined with the Darboux equation (26) in the form

$$L_x(M^r f(x)) = L_r(M^r f(x)) \tag{30}$$

we deduce

$$[L_x + m(n-m-1)] \int_0^\infty \sinh^m r (M^r f)(x)\, dr$$
$$= -(n-m-1)(m-1)\int_0^\infty \sinh^{m-2} r (M^r f)(p)\, dr\,.$$

Applying this repeatedly to (29) we obtain Theorem 1.5.

B. The Spheres and the Elliptic Spaces

Now let X be the unit sphere $\mathbf{S}^n(0) \subset \mathbf{R}^{n+1}$ and Ξ the set of k-dimensional totally geodesic submanifolds of X. Each $\xi \in \Xi$ is a k-sphere. We shall now invert the Radon transform

$$\widehat{f}(\xi) = \int_\xi f(x)\, dm(x)\,, \quad f \in \mathcal{E}(X)$$

where dm is the measure on ξ given by the Riemannian structure induced by that of X. In contrast to the hyperbolic space, each geodesic X through a point x also passes through the antipodal point A_x. As a result, $\widehat{f} = (f \circ A)\widehat{}$ and our inversion formula will reflect this fact. Although we state our result for the sphere, it is really a result for the *elliptic space*, that is the sphere with antipodal points identified. The functions on this space are naturally identified with symmetric functions on the sphere.

Again let

$$\check{\varphi}(x) = \int_{x \in \xi} \varphi(\xi)\, d\mu(\xi)$$

denote the average of a continuous function on Ξ over the set of ξ passing through x.

Theorem 1.7. *Let k be an integer, $1 \leq k < n = \dim X$.*

(i) The mapping $f \to \widehat{f}$ $(f \in \mathcal{E}(X))$ has kernel consisting of the skew function (the functions f satisfying $f + f \circ A = 0$).

(ii) Assume k even and let P_k denote the polynomial

$$P_k(z) = [z-(k-1)(n-k)][z-(k-3)(n-k+2)]\dots[z-1(n-2)]$$

of degree $k/2$. The k-dimensional Radon transform on X is then inverted by the formula

$$c(f + f \circ A) = P_k(L)((\widehat{f})^\vee)\,, \quad f \in \mathcal{E}(X)$$

where c is the constant in (27).

Proof. We first prove (ii) in a similar way as in the noncompact case. The Riemannian structure in (3) is now replaced by

$$ds^2 = dr^2 + \sin^2 r\, d\sigma^2\,;$$

the Laplace-Beltrami operator is now given by

$$L = \frac{\partial^2}{\partial r^2} + (n-1)\cot r \frac{\partial}{\partial r} + (\sin r)^{-2} L_S \tag{31}$$

instead of (23) and

$$(\widehat{f})^\vee(x) = \Omega_k \int_0^\pi (M^r f)(x) \sin^{k-1} r\, dr\,.$$

For a fixed x we put $F(r) = (M^r f)(x)$. The analog of Lemma 1.6 now reads as follows.

Lemma 1.8. *Let m be an integer, $0 < m < n = \dim X$. Then*

$$\int_0^\pi \sin^m r L_r F\, dr =$$

$$(n-m-1)\left[m\int_0^\pi \sin^m rF(r)\,dr - (m-1)\int_0^\pi \sin^{m-2} rF(r)\,dr\right].$$

If $m = 1$, the term $(m-1)\int_0^\pi \sin^{m-2} rF(r)\,dr$ should be replaced by $F(o) + F(\pi)$.

Since (30) is still valid the lemma implies

$$\begin{aligned}\big[L_x - m(n-m-1)\big]&\int_0^\pi \sin^m r(M^r f)(x)\,dr \\ = \quad -(n-m-1)(m-1)&\int_0^\pi \sin^{m-2} r(M^r f)(x)\,dr\end{aligned}$$

and the desired inversion formula follows by iteration since

$$F(0) + F(\pi) = f(x) + f(Ax)\,.$$

In the case when k is even, Part (i) follows from (ii). Next suppose $k = n-1$, n even. For each ξ there are exactly two points x and Ax at maximum distance, namely $\frac{\pi}{2}$, from ξ and we write

$$\widehat{f}(x) = \widehat{f}(Ax) = \widehat{f}(\xi)\,.$$

We have then

$$\widehat{f}(x) = \Omega_n(M^{\frac{\pi}{2}} f)(x)\,. \tag{32}$$

Next we recall some well-known facts about spherical harmonics. We have

$$L^2(X) = \sum_0^\infty \mathcal{H}_s\,, \tag{33}$$

where the space $\mathcal{H}_s$ consist of the restrictions to X of the homogeneous harmonic polynomials on $\mathbf{R}^{n+1}$ of degree s.

(a) $Lh_s = -s(s+n-1)h_s \quad (h_s \in \mathcal{H}_s)$ for each $s \geq 0$. This is immediate from the decomposition

$$L_{n+1} = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}L$$

of the Laplacian L_{n+1} of $\mathbf{R}^{n+1}$ (cf. (23)). Thus the spaces $\mathcal{H}_s$ are precisely the eigenspaces of L.

(b) Each $\mathcal{H}_s$ contains a function ($\not\equiv 0$) which is invariant under the group K of rotations around the vertical axis (the x_{n+1}-axis in $\mathbf{R}^{n+1}$). This function φ_s is nonzero at the North Pole o and is uniquely determined by the condition $\varphi_s(o) = 1$. This is easily seen since by (31) φ_s satisfies the ordinary differential equation

$$\frac{d^2\varphi_s}{dr^2} + (n-1)\cot r\frac{d\varphi_s}{dr} = -s(s+n-1)\varphi_s\,, \quad \varphi_s'(o) = 0\,.$$

It follows that $\mathcal{H}_s$ is irreducible under the orthogonal group $\mathbf{O}(n+1)$.

(c) Since the mean value operator $M^{\pi/2}$ commutes with the action of $\mathbf{O}(n+1)$ it acts as a scalar c_s on the irreducible space $\mathcal{H}_s$. Since we have

$$M^{\pi/2}\varphi_s = c_s\varphi_s\,, \quad \varphi_s(o) = 1\,,$$

we obtain

$$c_s = \varphi_s\left(\tfrac{\pi}{2}\right). \tag{34}$$

Lemma 1.9. *The scalar $\varphi_s(\pi/2)$ is zero if and only if s is odd.*

Proof. Let H_s be the K-invariant homogeneous harmonic polynomial whose restriction to X equals φ_s. Then H_s is a polynomial in $x_1^2 + \cdots + x_n^2$ and x_{n+1} so if the degree s is odd, x_{n+1} occurs in each term whence $\varphi_s(\pi/2) = H_s(1, 0, \ldots 0, 0) = 0$. If s is even, say $s = 2d$, we write

$$H_s = a_0(x_1^2 + \cdots + x_n^2)^d + a_1 x_{n+1}^2(x_1^2 + \cdots + x_n^2)^{d-1} + \cdots + a_d x_{n+1}^{2d}\,.$$

Using $L_{n+1} = L_n + \partial^2/\partial x_{n+1}^2$ and formula (31) in Ch. I the equation $L_{n+1}H_s \equiv 0$ gives the recursion formula

$$a_i(2d-2i)(2d-2i+n-2) + a_{i+1}(2i+2)(2i+1) = 0$$

($0 \leq i < d$). Hence $H_s(1, 0 \ldots 0)$, which equals a_0, is $\neq 0$; Q.e.d.

Now each $f \in \mathcal{E}(X)$ has a uniformly convergent expansion

$$f = \sum_0^\infty h_s \quad (h_s \in \mathcal{H}_s)$$

and by (32)

$$\widehat{f} = \Omega_n \sum_0^\infty c_s h_s\,.$$

If $\widehat{f} = 0$ then by Lemma 1.9, $h_s = 0$ for s even so f is skew. Conversely $\widehat{f} = 0$ if f is skew so Theorem 1.7 is proved for the case $k = n-1$, n even.

If k is odd, $0 < k < n-1$, the proof just carried out shows that $\widehat{f}(\xi)=0$ for all $\xi \in \Xi$ implies that f has integral 0 over every $(k+1)$-dimensional sphere with radius 1 and center o. Since $k+1$ is even and $< n$ we conclude by (ii) that $f + f \circ A = 0$ so the theorem is proved.

As a supplement to Theorems 1.5 and 1.7 we shall now prove an inversion formula for the Radon transform for general k (odd or even).

Let X be either the hyperbolic space $\mathbf{H}^n$ or the sphere $\mathbf{S}^n$ and Ξ the space of totally geodesic submanifolds of X of dimension k $(1 \leq k \leq n-1)$. We then generalize the transforms $f \to \widehat{f}, \varphi \to \check{\varphi}$ as follows. Let $p \geq 0$. We put

$$(35) \qquad \widehat{f}_p(\xi) = \int_{d(x,\xi)=p} f(x)\, dm(x)\,, \quad \check{\varphi}_p(x) = \int_{d(x,\xi)=p} \varphi(\xi)\, d\mu(\xi)\,,$$

where dm is the Riemannian measure on the set in question and $d\mu$ is the average over the set of ξ at distance p from x. Let ξ_p be a fixed element of Ξ at distance p from 0 and H_p the subgroup of G leaving ξ_p stable. It is then easy to see that in the language of Ch. II, §1

$$(36) \qquad x = gK\,, \quad \xi = \gamma H_p \text{ are incident } \Leftrightarrow d(x,\xi) = p\,.$$

This means that the transforms (35) are the Radon transform and its dual for the double fibration

$$\begin{array}{ccc} & G/(K \cap H_p) & \\ \swarrow & & \searrow \\ G/K & & G/H_p \end{array}$$

For $X = \mathbf{S}^2$ the set $\{x : d(x,\xi) = p\}$ is two circles on $\mathbf{S}^2$ of length $2\pi \cos p$. For $X = \mathbf{H}^2$, the non-Euclidean disk, ξ a diameter, the set $\{x : d(x,\xi) = p\}$ is a pair of circular arcs with the same endpoints as ξ. Of course $\widehat{f}_0 = \widehat{f}\,, \quad \check{\varphi}_0 = \check{\varphi}$.

We shall now invert the transform $f \to \widehat{f}$ by invoking the more general transform $\varphi \to \check{\varphi}_p$. Consider $x \in X, \xi \in \Xi$ with $d(x,\xi) = p$. Select $g \in G$ such that $g \cdot o = x$. Then $d(o, g^{-1}\xi) = p$ so $\{kg^{-1} \cdot \xi : k \in K\}$ is the set of $\eta \in \Xi$ at distance p from o and $\{gkg^{-1} \cdot \xi : k \in K\}$ is the set of $\eta \in \Xi$ at distance p from x. Hence

$$\begin{aligned} (\widehat{f})^\vee_p(g \cdot o) &= \int_K \widehat{f}(gkg^{-1} \cdot \xi)\, dk = \int_K dk \int_\xi f(gkg^{-1} \cdot y)\, dm(y) \\ &= \int_\xi \left(\int_K f(gkg^{-1} \cdot y)\, dk \right) dm(y) \end{aligned}$$

so

$$(37) \qquad (\widehat{f})^\vee_p(x) = \int_\xi (M^{d(x,y)} f)(x)\, dm(y)\,.$$

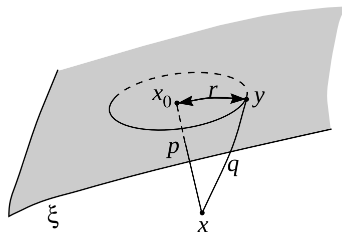

FIGURE III.2.

Let $x_0 \in \xi$ be a point at minimum distance (i.e., p) from x and let (Fig. III.2)

(38)
$$r = d(x_0, y)\,, \quad q = d(x, y)\,, \quad y \in \xi\,.$$

Since $\xi \subset X$ is totally geodesic, $d(x_o, y)$ is also the distance between x_o and y in ξ. In (37) the integrand $(M^{d(x,y)}f)(x)$ is constant in y on each sphere in ξ with center x_o.

Theorem 1.10. *The k-dimensional totally geodesic Radon transform $f \to \widehat{f}$ on the hyperbolic space $\mathbf{H}^n$ is inverted by*

$$f(x) = c\left[\left(\frac{d}{d(u^2)}\right)^k \int_0^u (\widehat{f})^\vee_{\operatorname{lm} v}(x)(u^2 - v^2)^{\frac{k}{2}-1}\,dv\right]_{u=1},$$

where $c^{-1} = (k-1)!\Omega_{k+1}/2^{k+1}$, $\operatorname{lm} v = \cosh^{-1}(v^{-1})$.

Proof. Applying geodesic polar coordinates in ξ with center x_0 we obtain from (37)–(38),

$$(\widehat{f})^\vee_p(x) = \Omega_k \int_0^\infty (M^q f)(x) \sinh^{k-1} r\,dr\,. \tag{39}$$

Using the cosine relation on the right-angled triangle (xx_0y) we have by (38) and $d(x_0, x) = p$,

$$\cosh q = \cosh p \cosh r\,. \tag{40}$$

With x fixed we define F and $\widehat{F}$ by

$$F(\cosh q) = (M^q f)(x)\,, \quad \widehat{F}(\cosh p) = (\widehat{f})^\vee_p(x)\,. \tag{41}$$

Then by (39),

$$\widehat{F}(\cosh p) = \Omega_k \int_0^\infty F(\cosh p \cosh r) \sinh^{k-1} r\,dr\,. \tag{42}$$

Putting here $t = \cosh p$, $s = \cosh r$ this becomes

$$\widehat{F}(t) = \Omega_k \int_1^\infty F(ts)(s^2 - 1)^{\frac{k}{2}-1}\,ds\,,$$

which by substituting $u = (ts)^{-1}$, $v = t^{-1}$ becomes

$$v^{-1}\widehat{F}(v^{-1}) = \Omega_k \int_0^v F(u^{-1})u^{-k}(v^2 - u^2)^{\frac{k}{2}-1}\,du\,.$$

This is of the form (19), Ch. I, §2 and is inverted by

$$F(u^{-1})u^{-k} = cu\left(\frac{d}{d(u^2)}\right)^k \int_0^u (u^2 - v^2)^{\frac{k}{2}-1}\widehat{F}(v^{-1})\,dv\,, \tag{43}$$

where $c^{-1} = (k-1)!\,\Omega_{k+1}/2^{k+1}$. Defining $\operatorname{lm} v$ by $\cosh(\operatorname{lm} v) = v^{-1}$ and noting that $f(x) = F(\cosh 0)$ the theorem follows by putting $u = 1$ in (43).

For the sphere $X = \mathbf{S}^n$ we can proceed in a similar fashion. We assume f symmetric ($f(s) \equiv f(-s)$) because $\widehat{f} \equiv 0$ for f odd. Now formula (37) takes the form

$$(\widehat{f})_p^{\vee}(x) = 2\Omega_k \int_0^{\frac{\pi}{2}} (M^q f)(x) \sin^{k-1} r\,dr\,, \tag{44}$$

(the factor 2 and the limit $\pi/2$ coming from the symmetry assumption). This time we use spherical trigonometry on the triangle (xx_0y) to derive

$$\cos q = \cos p \cos r\,.$$

We fix x and put

$$F(\cos q) = (M^q f)(x)\,, \quad \widehat{F}(\cos p) = (\widehat{f})_p^{\vee}(x)\,. \tag{45}$$

and

$$v = \cos p\,, \quad u = v\cos r\,.$$

Then (44) becomes

$$v^{k-1}\widehat{F}(v) = 2\Omega_k \int_0^v F(u)(v^2 - u^2)^{\frac{k}{2}-1}\,du\,, \tag{46}$$

which is inverted by

$$F(u) = \frac{c}{2}u\left(\frac{d}{d(u^2)}\right)^k \int_0^u (u^2 - v^2)^{\frac{k}{2}-1} v^k \widehat{F}(v)\,dv\,,$$

c being as before. Since $F(1) = f(x)$ this proves the following analog of Theorem 1.10.

Theorem 1.11. *The k-dimensional totally geodesic Radon transform $f \to \widehat{f}$ on $\mathbf{S}^n$ is for f symmetric inverted by*

$$f(x) = \frac{c}{2}\left[\left(\frac{d}{d(u^2)}\right)^k \int_0^u (\widehat{f})^{\vee}_{\cos^{-1}(v)}(x) v^k (u^2 - v^2)^{\frac{k}{2}-1}\,dv\right]_{u=1}$$

where

$$c^{-1} = (k-1)!\Omega_{k+1}/2^{k+1}\,.$$

Geometric interpretation

In Theorems 1.10–1.11, $(\widehat{f})_p^\vee(x)$ is the average of the integrals of f over the k-dimensional totally geodesic submanifolds of X which have distance p from x.

We shall now look a bit closer at the geometrically interesting case $k = 1$. Here the transform $f \to \widehat{f}$ is called the *X-ray transform*.

We first recall a few facts about the *spherical transform* on the constant curvature space $X = G/K$, that is the hyperbolic space $\mathbf{H}^n = Q_-^+$ or the sphere $\mathbf{S}^n = Q_+$. A *spherical function* φ on G/K is by definition a K-invariant function which is an eigenfunction of the Laplacian L on X satisfying $\varphi(o) = 1$. Then the eigenspace of L containing φ consists of the functions f on X satisfying the functional equation

$$\int_K f(gk \cdot x)\, dk = f(g \cdot o)\varphi(x) \tag{47}$$

([GGA], p. 64). In particular, the spherical functions are characterized by

$$\int_K \varphi(gk \cdot x)\, dk = \varphi(g \cdot o)\varphi(x) \quad \varphi \not\equiv 0\,. \tag{48}$$

Consider now the case $\mathbf{H}^2$. Then the spherical functions are the solutions $\varphi_\lambda(r)$ of the differential equation

$$\frac{d^2\varphi_\lambda}{dr^2} + \coth r \frac{d\varphi_\lambda}{dr} = -(\lambda^2 + \tfrac{1}{4})\varphi_\lambda\,, \quad \varphi_\lambda(o) = 1\,. \tag{49}$$

Here $\lambda \in \mathbf{C}$ and $\varphi_{-\lambda} = \varphi_\lambda$. The function φ_λ has the integral representation

$$\varphi_\lambda(r) = \frac{1}{\pi}\int_0^\pi (\operatorname{ch} r - \operatorname{sh} r \cos\theta)^{-i\lambda + \frac{1}{2}}\, d\theta\,. \tag{50}$$

In fact, already the integrand is easily seen to be an eigenfunction of the operator L in (23) (for $n = 2$) with eigenvalue $-(\lambda^2 + 1/4)$.

If f is a radial function on X its *spherical transform* $\widetilde{f}$ is defined by

$$\widetilde{f}(\lambda) = \int_X f(x)\varphi_{-\lambda}(x)\, dx \tag{51}$$

for all $\lambda \in \mathbf{C}$ for which this integral exists. The continuous radial functions on X form a commutative algebra $C_c^\natural(X)$ under convolution

$$(f_1 \times f_2)(g \cdot o) = \int_G f_1(gh^{-1} \cdot o) f_2(h \cdot o)\, dh \tag{52}$$

and as a consequence of (48) we have

$$(f_1 \times f_2)^\sim(\lambda) = \widetilde{f}_1(\lambda)\widetilde{f}_2(\lambda)\,. \tag{53}$$

In fact,

$$\begin{aligned}(f_1 \times f_2)^{\sim}(\lambda) &= \int_G f_1(h\cdot o)\left(\int_G f_2(g\cdot o)\varphi_{-\lambda}(hg\cdot o)\,dg\right)dh\\ &= \int_G f_1(h\cdot o)\left(\int_G f_2(g\cdot o)\right)\left(\int_K \varphi_{-\lambda}(hkg\cdot o)\,dk\,dg\right)dh\\ &= \widetilde{f_1}(\lambda)\widetilde{f_2}(\lambda)\,.\end{aligned}$$

We know already from Corollary 1.3 that the Radon transform on $\mathbf{H}^n$ is injective and is inverted in Theorem 1.5 and Theorem 1.10. For the case $n = 2, k = 1$ we shall now obtain another inversion formula based on (53).

The spherical function $\varphi_\lambda(r)$ in (50) is the classical Legendre function $P_v(\cosh r)$ with $v = i\lambda - \frac{1}{2}$ for which we shall need the following result ([Prudnikov, Brychkov and Marichev], Vol. III, 2.17.8(2)).

Lemma 1.12.

$$(54)\qquad 2\pi\int_0^\infty e^{-pr}P_v(\cosh r)\,dr = \pi\frac{\Gamma(\frac{p-v}{2})\Gamma(\frac{p+v+1}{2})}{\Gamma(1+\frac{p+v}{2})\Gamma(\frac{1+p-v}{2})}\,,$$

for

$$(55)\qquad Re(p-v) > 0\,,\quad Re(p+v) > -1\,.$$

We shall require this result for $p = 0, 1$ and λ real. In both cases, conditions (55) are satisfied.

Let τ and σ denote the functions

$$(56)\qquad \tau(x) = \sinh\, d(o,x)^{-1}\,,\quad \sigma(x) = \coth(d(o,x)) - 1\,,\quad x \in X\,.$$

Lemma 1.13. *For $f \in \mathcal{D}(X)$ we have*

$$(57)\qquad (\widehat{f})^\vee(x) = \pi^{-1}(f\times\tau)(x)\,.$$

Proof. In fact, the right hand side is

$$\int_X \sinh\, d(x,y)^{-1}f(y)\,dy = \int_0^\infty dr(\sinh\, r)^{-1}\int_{S_r(x)} f(y)\,dw(y)$$

so the lemma follows from (28).

Similarly we have

$$(58)\qquad Sf = f\times\sigma\,,$$

where S is the operator

$$(59)\qquad (Sf)(x) = \int_X (\coth(d(x,y)) - 1)f(y)\,dy\,.$$

Theorem 1.14. *The operator $f \to \widehat{f}$ is inverted by*

$$LS((\widehat{f})^\vee) = -4\pi f\,, \quad f \in \mathcal{D}(X)\,. \tag{60}$$

Proof. The operators $\widehat{\ }$, $^\vee$, S and L are all G-invariant so it suffices to verify (60) at o. Let $f^\natural(x) = \int_K f(k \cdot x)\,dk$. Then

$$(f \times \tau)^\natural = f^\natural \times \tau\,,\ \ (f \times \sigma)^\natural = f^\natural \times \sigma\,,\ \ (Lf)(o) = (Lf^\natural)(o)\,.$$

Thus by (57)–(58)

$$\begin{aligned} LS((\widehat{f})^\vee)(o) &= L(S((\widehat{f})^\vee))^\natural(o) = \pi^{-1}L(f \times \tau \times \sigma)^\natural(o) \\ &= LS(((f^\natural)\widehat{\ })^\vee))(o)\,. \end{aligned}$$

Now, if (60) is proved for a radial function this equals $cf^\natural(o) = cf(o)$. Thus (60) would hold in general. Consequently, it suffices to prove

$$L(f \times \tau \times \sigma) = -4\pi^2 f\,, \quad f \text{ radial in } \mathcal{D}(X)\,. \tag{61}$$

Since f, $\tau\varphi_\lambda$ (λ real) and σ are all integrable on X, we have by the proof of (53)

$$(f \times \tau \times \sigma)^\sim(\lambda) = \widetilde{f}(\lambda)\widetilde{\tau}(\lambda)\widetilde{\sigma}(\lambda)\,. \tag{62}$$

Since $\coth r - 1 = e^{-r}/\sinh r$, and since $dx = \sinh r\,dr\,d\theta$, $\widetilde{\tau}(\lambda)$ and $\widetilde{\sigma}(\lambda)$ are given by the left hand side of (54) for $p = 0$ and $p = 1$, respectively. Thus

$$\begin{aligned} \widetilde{\tau}(\lambda) &= \pi\frac{\Gamma(\frac14 - \frac{i\lambda}{2})\Gamma(\frac{i\lambda}{2} + \frac14)}{\Gamma(\frac{i\lambda}{2} + \frac34)\Gamma(\frac34 - \frac{i\lambda}{2})}\,, \\ \widetilde{\sigma}(\lambda) &= \pi\frac{\Gamma(\frac34 - \frac{i\lambda}{2})\Gamma(\frac{i\lambda}{2} + \frac34)}{\Gamma(\frac{i\lambda}{2} + \frac54)\Gamma(\frac54 - \frac{i\lambda}{2})}\,. \end{aligned}$$

Using the identity $\Gamma(x+1) = x\Gamma(x)$ on the denominator of $\widetilde{\sigma}(\lambda)$ we see that

$$\widetilde{\tau}(\lambda)\widetilde{\sigma}(\lambda) = 4\pi^2(\lambda^2 + \tfrac14)^{-1}\,. \tag{63}$$

Now

$$L(f \times \tau \times \sigma) = (Lf \times \tau \times \sigma)\,, \quad f \in \mathcal{D}^\natural(X)\,,$$

and by (49), $(Lf)^\sim(\lambda) = -(\lambda^2 + \frac14)\widetilde{f}(\lambda)$. Using the decomposition $\tau = \varphi\tau + (1-\varphi)\tau$ where φ is the characteristic function of a ball $B(0)$ we see that $f \times \tau \in L^2(X)$ for $f \in \mathcal{D}^\sharp(X)$. Since $\sigma \in L'(X)$ we have $f \times \tau \times \sigma \in L^2(X)$. By the Plancherel theorem, the spherical transform is injective on $L^2(X)$ so we deduce from (62)–(63) that (60) holds with the constant $-4\pi^2$.

It is easy to write down an analog of (60) for $\mathbf{S}^2$. Let o denote the North Pole and put

$$\tau(x) = \sin d(o,x)^{-1} \quad x \in \mathbf{S}^2 \,.$$

Then in analogy with (57) we have

$$(\widehat{f})^\vee(x) = \pi^{-1}(f \times \tau)(x)\,, \tag{64}$$

where $\times$ denotes the convolution on $\mathbf{S}^2$ induced by the convolution on G. The spherical functions on G/K are the functions

$$\varphi_n(x) = P_n(\cos\, d(o,x)) \quad n \geq 0\,,$$

where P_n is the Legendre polynomial

$$P_n(\cos\theta) = \frac{1}{2\pi}\int_0^{2\pi}(\cos\theta + i\sin\theta\cos u)^n\,du\,.$$

Since $P_n(\cos(\pi-\theta)) = (-1)^n P_n(\cos\theta)$, the expansion of τ into spherical functions

$$\tau(x) \sim \sum_{n=0}^{\infty}(4n+1)\widetilde{\tau}(2n)P_{2n}(\cos\, d(o,x))$$

only involves even indices. The factor $(4n+1)$ is the dimension of the space of spherical harmonics containing φ_{2n}. Here the Fourier coefficient $\widetilde{\tau}(2n)$ is given by

$$\widetilde{\tau}(2n) = \frac{1}{4\pi}\int_{\mathbf{S}^2}\tau(x)\varphi_{2n}(x)\,dx\,,$$

which, since $dx = \sin\theta\, d\theta\, d\varphi$, equals

$$\frac{1}{4\pi}2\pi\int_0^{\pi}P_{2n}(\cos\theta)\,d\theta = \frac{\pi}{2^{4n-1}}\binom{2n}{n}^2\,, \tag{65}$$

by *loc. cit.*, Vol. 2, 2.17.6 (11). We now define the functional σ on $\mathbf{S}^2$ by the formula

$$\sigma(x) = \sum_0^{\infty}(4n+1)a_{2n}P_{2n}(\cos\, d(o,x))\,, \tag{66}$$

where

$$a_{2n} = \frac{2^{4n}\pi}{\binom{2n}{n}^2 n(2n+1)}\,. \tag{67}$$

To see that (66) is well-defined note that

$$\begin{aligned}\binom{2n}{n} &= 2^n 1\cdot 3\cdots(2n-1)/n! \geq 2^n 1\cdot 2\cdot 4\cdots(2n-2)/n! \\ &\geq 2^{2n-1}/n\end{aligned}$$

so a_{2n} is bounded in n. Thus σ is a distribution on $\mathbf{S}^2$. Let S be the operator

$$Sf = f \times \sigma\,. \tag{68}$$

Theorem 1.15. *The operator $f \to \widehat{f}$ is inverted by*

$$LS((\widehat{f})^\vee) = -4\pi f\,. \tag{69}$$

Proof. Just as is the case with Theorem 1.14 it suffices to prove this for f K-invariant and there it is a matter of checking that the spherical transforms on both sides agree. For this we use (64) and the relation

$$L\varphi_{2n} = -2n(2n+1)\varphi_{2n}\,.$$

Since

$$(\tau \times \sigma)^\sim(2n) = \widetilde{\tau}(2n)a_{2n}\,.$$

the identity (69) follows.

A drawback of (69) is of course that (66) is not given in closed form. We shall now invert $f \to \widehat{f}$ in a different fashion on $\mathbf{S}^2$. Consider the spherical coordinates of a point $(x_1, x_2, x_3) \in \mathbf{S}^2$.

$$x_1 = \cos\varphi \sin\theta\,, \quad x_2 = \sin\varphi\sin\theta\,, \quad x_3 = \cos\theta \tag{70}$$

and let $k_\varphi = K$ denote the rotation by the angle φ around the x_3-axis. Then f has a Fourier expansion

$$f(x) = \sum_{n\in\mathbb{Z}} f_n(x)\,, \quad f_n(x) = \frac{1}{2\pi}\int_0^{2\pi} f(k_\varphi \cdot x)e^{-in\varphi}\,d\varphi\,.$$

Then

$$f_n(k_\varphi \cdot x) = e^{in\varphi} f_n(x)\,, \quad \widehat{f}_n(k_\varphi \cdot \gamma) = e^{in\varphi}\widehat{f}_n(\gamma) \tag{71}$$

for each great circle γ. In particular, f_n is determined by its restriction $g = f_n|_{x_1=0}$, i.e.,

$$g(\cos\theta) = f_n(0, \sin\theta, \cos\theta)\,.$$

Since f_n is even, (70) implies $g(\cos(\pi - \theta)) = (-1)^n g(\cos\theta)$, so

$$g(-u) = (-1)^n g(u)\,.$$

Let Γ be the set of great circles whose normal lies in the plane $x_1 = 0$. If $\gamma \in \Gamma$ let x_γ be the intersection of γ with the half-plane $x_1 = 0, x_2 > 0$ and let α be the angle from o to x_γ, (Fig. III.3). Since f_n is symmetric,

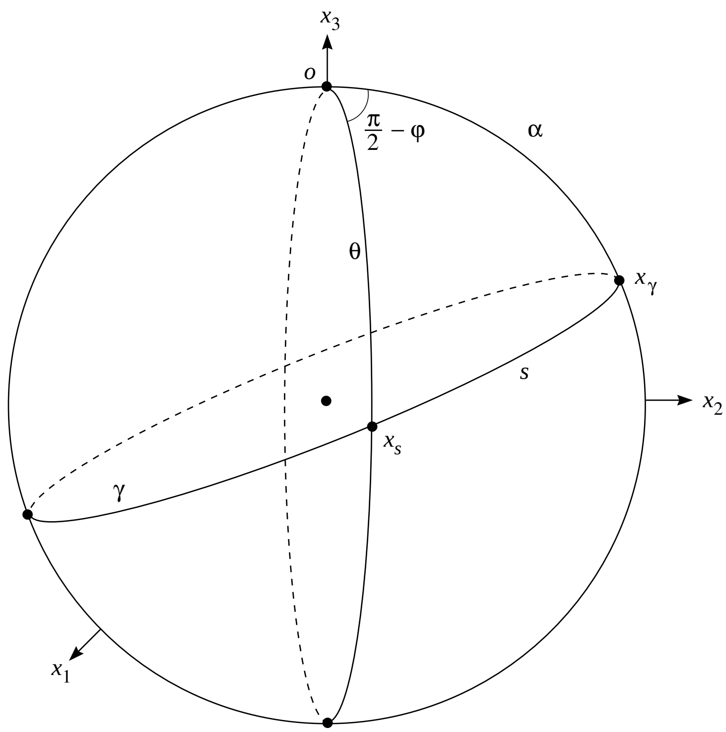

FIGURE III.3.

$$\widehat{f}_n(\gamma) = 2\int_0^\pi f_n(x_s)\,ds\,, \tag{72}$$

where x_s is the point on γ at distance s from x_γ (with $x_1(x_s) \geq 0$). Let φ and θ be the coordinates (70) of x_s. Considering the right angled triangle $x_s o x_\gamma$ we have

$$\cos\theta = \cos s\,\cos\alpha$$

and since the angle at o equals $\pi/2 - \varphi$, (71) implies

$$g(\cos\alpha) = f_n(x_\gamma) = e^{in(\pi/2-\varphi)} f_n(x_s)\,.$$

Writing

$$\widehat{g}(\cos\alpha) = \widehat{f}_n(\gamma) \tag{73}$$

equation (72) thus becomes

$$\widehat{g}(\cos\alpha) = 2(-i)^n \int_0^\pi e^{in\varphi} g(\cos\theta)\,ds\,.$$

Put $v = \cos\alpha\,,\quad u - v\cos s$, so

$$du = v(-\sin s)\,ds = -(v^2-u^2)^{1/2}\,ds\,.$$

Then

$$\widehat{g}(v) = 2(-i)^n \int_{-v}^{v} e^{in\varphi(u,v)} g(u)(v^2 - u^2)^{-\frac{1}{2}}\, du\,, \tag{74}$$

where the dependence of φ on u and v is indicated (for $v \neq 0$).

Now $-u = v\cos(\pi - s)$ so by the geometry, $\varphi(-u, v) = -\varphi(u, v)$. Thus (74) splits into two Abel-type Volterra equations

$$\widehat{g}(v) \;=\; 4(-1)^{n/2} \int_0^v \cos(n\varphi(u,v)) g(u)(v^2 - u^2)^{-\frac{1}{2}}\, du\,, \quad n \text{ even} \tag{75}$$

$$\widehat{g}(v) \;=\; 4(-1)^{(n-1)/2} \int_0^v \sin(n\varphi(u,v)) g(u)(v^2 - u^2)^{-\frac{1}{2}}\, du\,, \quad n \text{ odd}\,. \tag{76}$$

For $n = 0$ we derive the following result from (43) and (75).

Proposition 1.16. *Let $f \in C^2(\mathbf{S}^2)$ be symmetric and K-invariant and $\widehat{f}$ its X-ray transform. Then the restriction $g(\cos d(o,x)) = f(x)$ and the function $\widehat{g}(\cos d(o,\gamma)) = \widehat{f}(\gamma)$ are related by*

$$\widehat{g}(v) = 4 \int_0^v g(u)(v^2 - u^2)^{-\frac{1}{2}}\, du \tag{77}$$

and its inversion

$$2\pi g(u) = \frac{d}{du} \int_0^u \widehat{g}(v)(u^2 - v^2)^{-\frac{1}{2}}\, v dv\,. \tag{78}$$

We shall now discuss the analog for $\mathbf{S}^n$ of the support theorem (Theorem 1.2) relative to the X-ray transform $f \to \widehat{f}$.

Theorem 1.17. *Let C be a closed spherical cap on $\mathbf{S}^n$, C' the cap on $\mathbf{S}^n$ symmetric to C with respect to the origin $0 \in \mathbf{R}^{n+1}$. Let $f \in C(\mathbf{S}^n)$ be symmetric and assume*

$$\widehat{f}(\gamma) = 0 \tag{79}$$

for every geodesic γ which does not enter the "arctic zones" C and C'. (See Fig. II.3.)

(i) If $n \geq 3$ then $f \equiv 0$ outside $C \cup C'$.

(ii) If $n = 2$ the same conclusion holds if all derivatives of f vanish on the equator.

Proof. (i) Given a point $x \in \mathbf{S}^n$ outside $C \cup C'$ we can find a 3-dimensional subspace ξ of $\mathbf{R}^{n+1}$ which contains x but does not intersect $C \cup C'$. Then $\xi \cap \mathbf{S}^n$ is a 2-sphere and f has integral 0 over each great circle on it. By Theorem 1.7, $f \equiv 0$ on $\xi \cap \mathbf{S}^n$ so $f(x) = 0$.

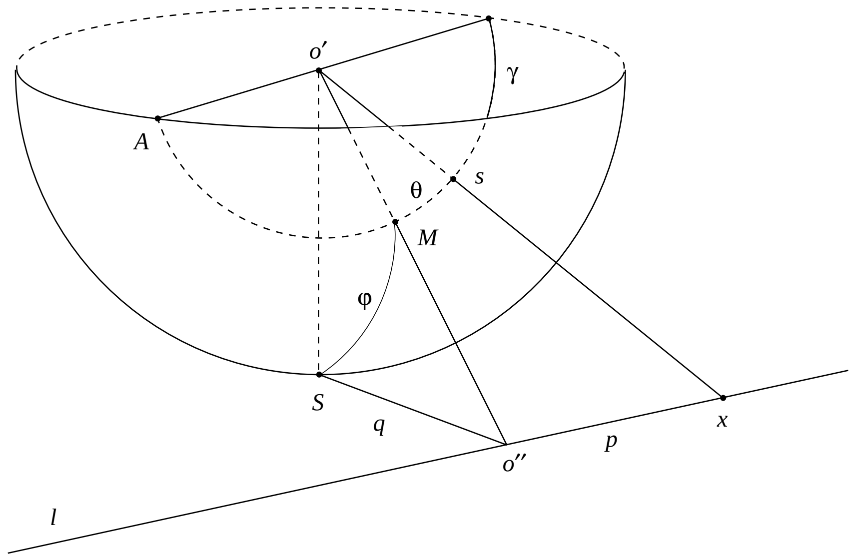

FIGURE III.4.

(ii) If f is K-invariant our statement follows quickly from Proposition 1.16. In fact, if C has spherical radius β, (79) implies $\widehat{g}(v) = 0$ for $0 < v < \cos\beta$ so by (78) $g(u) = 0$ for $0 < u < \cos\beta$ so $f \equiv 0$ outside $C \cup C'$.

Generalizing this method to f_n in (71) by use of (75)–(76) runs into difficulties because of the complexity of the kernel $e^{in\varphi(u,v)}$ in (74) near $v = 0$. However, if f is assumed $\equiv 0$ in a belt around the equator the theory of the Abel-type Volterra equations used on (75)–(76) does give the conclusion of (ii). The reduction to the K-invariant case which worked very well in the proof of Theorem 1.2 does not apply in the present compact case.

A better method, due to Kurusa, is to consider only the lower hemisphere $\mathbf{S}^2_-$ of the unit sphere and its tangent plane π at the South Pole S. The central projection μ from the origin is a bijection of $\mathbf{S}^2_-$ onto π which intertwines the two Radon transforms as follows: If γ is a (half) great circle on $\mathbf{S}^2_-$ and ℓ the line $\mu(\gamma)$ in π we have (Fig. III.4)

$$\cos d(S,\gamma)\widehat{f}(\gamma) = 2\int_\ell (f \circ \mu^{-1})(x)(1+|x|^2)^{-1}\,dm(x)\,. \tag{80}$$

The proof follows by elementary geometry: Let on Fig. III.4, $x = \mu(s)$, φ and θ the lengths of the arcs SM, Ms. The plane $o'So''$ is perpendicular to ℓ and intersects the semi-great circle γ in M. If $q = |So''|, p = |o''x|$ we have for $f \in C(\mathbf{S}^2)$ symmetric,

$$\widehat{f}(\gamma) = 2\int_\gamma f(s)d\theta = 2\int_\ell (f \circ \mu^{-1})(x)\frac{d\theta}{dp}\,dp\,.$$

Now

$$\tan\varphi = q\,, \quad \tan\theta = \frac{p}{(1+q^2)^{1/2}}\,, \quad |x|^2 = p^2 + q^2\,.$$

so

$$\frac{dp}{d\theta} = (1+q^2)^{1/2}(1+\tan^2\theta) = (1+|x|^2)/(1+q^2)^{1/2}\,.$$

Thus

$$\frac{dp}{d\theta} = (1+|x|^2)\cos\varphi$$

and since $\varphi = d(S,\gamma)$ this proves (80). Considering the triangle $o'xS$ we obtain

$$|x| = \tan d(S,s)\,. \tag{81}$$

Thus the vanishing of all derivatives of f on the equator implies rapid decrease of $f \circ \mu^{-1}$ at ∞.

Now if $\varphi > \beta$ we have by assumption, $\widehat{f}(\gamma) = 0$ so by (80) and Theorem 2.6 in Chapter I,

$$(f \circ \mu^{-1})(x) = 0 \quad \text{for } |x| > \tan\beta\,,$$

whence by (81),

$$f(s) = 0 \quad \text{for } d(S,s) > \beta\,.$$

Remark 1.18. Because of the example in Remark 2.9 in Chapter I the vanishing condition in (ii) cannot be dropped.

There is a generalization of (80) to d-dimensional totally geodesic submanifolds of $\mathbf{S}^n$ as well as of $\mathbf{H}^n$ (Kurusa [1992], [1994], Berenstein-Casadio Tarabusi [1993]). This makes it possible to transfer the range characterizations of the d-plane Radon transform in $\mathbf{R}^n$ (Chapter I, §6) to the d-dimensional totally geodesic Radon transform in $\mathbf{H}^n$. In addition to the above references see also Berenstein-Casadio Tarabusi-Kurusa [1997], Gindikin [1995] and Ishikawa [1997].

C. The Spherical Slice Transform

We shall now briefly consider a variation on the Funk transform and consider integrations over circles on $\mathbf{S}^2$ passing through the North Pole. This Radon transform is given by $f \to \widehat{f}$ where f is a function on $\mathbf{S}^2$,

$$\widehat{f}(\gamma) = \int_\gamma f(s)\,dm(s)\,, \tag{82}$$

γ being a circle on $\mathbf{S}^2$ passing through N and dm the arc-element on γ.

It is easy to study this transform by relating it to the X-ray transform on $\mathbf{R}^2$ by means of stereographic projection from N.

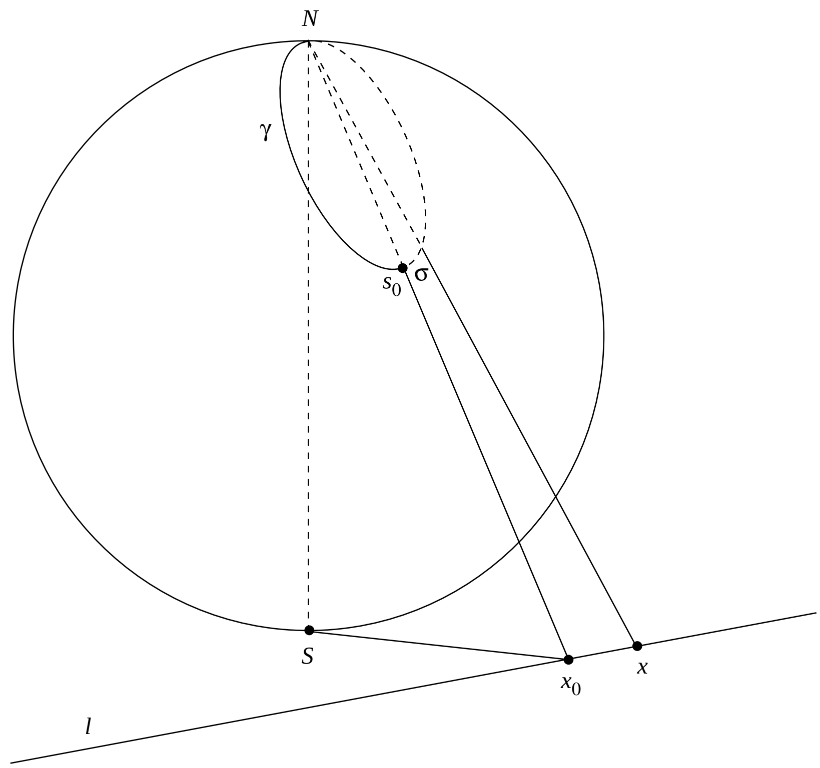

FIGURE III.5.

We consider a two-sphere $\mathbf{S}^2$ of *diameter* 1, lying on top of its tangent plane $\mathbf{R}^2$ at the South Pole. Let $\nu : \mathbf{S}^2 - N \to \mathbf{R}^2$ be the stereographic projection. The image $\nu(\gamma)$ is a line $\ell \subset \mathbf{R}^2$. (See Fig. III.5.) The plane through the diameter NS perpendicular to ℓ intersects γ in s_0 and ℓ in x_0. Then Ns_0 is a diameter in γ, and in the right angle triangle NSx_0, the line Ss_0 is perpendicular to Nx_0. Thus, d denoting the Euclidean distance in $\mathbf{R}^3$, and $q = d(S, x_0)$, we have

$$d(N, s_0) = (1 + q^2)^{-1/2}\,, \quad d(s_0, x_0) = q^2(1 + q^2)^{-1/2}\,. \tag{83}$$

Let σ denote the circular arc on γ for which $\nu(\sigma)$ is the segment (x_0, x) on ℓ. If θ is the angle between the lines Nx_0, Nx then

$$\sigma = (2\theta) \cdot \tfrac{1}{2}(1 + q^2)^{-1/2}\,, \quad d(x_0, x) = \tan\theta (1 + q^2)^{1/2}\,. \tag{84}$$

Thus, $dm(x)$ being the arc-element on ℓ,

$$\begin{aligned}\frac{dm(x)}{d\sigma} &= \frac{dm(x)}{d\theta}\cdot\frac{d\theta}{d\sigma} = (1+q^2)^{1/2}\cdot(1+\tan^2\theta)(1+q^2)^{1/2} \\ &= (1+q^2)\left(1+\frac{d(x_0,x)^2}{1+q^2}\right) = 1+|x|^2\,.\end{aligned}$$

Hence we have

$$\widehat{f}(\gamma) = \int_\ell (f\circ\nu^{-1})(x)(1+|x|^2)^{-1}\,dm(x)\,, \tag{85}$$

a formula quite similar to (80).

If f lies on $C^1(\mathbf{S}^2)$ and vanishes at N then $f\circ\nu^{-1} = 0(x^{-1})$ at ∞. Also of $f\in\mathcal{E}(\mathbf{S}^2)$ and all its derivatives vanish at N then $f\circ\nu^{-1}\in\mathcal{S}(\mathbf{R}^2)$. As in the case of Theorem 1.17 (ii) we can thus conclude the following corollaries of Theorem 3.1, Chapter I and Theorem 2.6, Chapter I.

Corollary 1.19. *The transform $f\to\widehat{f}$ is one-to-one on the space $C_0^1(\mathbf{S}^2)$ of C^1-functions vanishing at N.*

In fact, $(f\circ\nu^{-1})(x)/(1+|x|^2) = 0(|x|^{-3})$ so Theorem 3.1, Chapter I applies.

Corollary 1.20. *Let B be a spherical cap on $\mathbf{S}^2$ centered at N. Let $f\in\mathcal{C}^\infty(\mathbf{S}^2)$ have all its derivatives vanish at N. If*

$$\widehat{f}(\gamma) = 0 \textit{ for all } \gamma \textit{ through } N\,, \quad \gamma\subset B$$

then $f\equiv 0$ on B.

In fact $(f\circ\nu^{-1})(x) = 0(|x|^{-k})$ for each $k\geq 0$. The assumption on $\widehat{f}$ implies that $(f\circ\nu^{-1})(x)(1+|x|^2)^{-1}$ has line integral 0 for all lines outside $\nu(B)$ so by Theorem 2.6, Ch. I, $f\circ\nu^{-1}\equiv 0$ outside $\nu(B)$.

Remark 1.21. In Cor. 1.20 the condition of the vanishing of all derivatives at N cannot be dropped. This is clear from Remark 2.9 in Chapter I where the rapid decrease at ∞ was essential for the conclusion of Theorem 2.6.

If according to Remark 3.3, Ch. I $g\in\mathcal{E}(\mathbf{R}^2)$ is chosen such that $g(x) = 0(|x|^{-2})$ and all its line integrals are 0, the function f on $\mathbf{S}^2 - N$ defined by

$$(f\circ\nu^{-1})(x) = (1+|x|^2)g(x)$$

is bounded and by (85), $\widehat{f}(\gamma) = 0$ for all γ. This suggests, but does not prove, that the vanishing condition at N in Cor. 1.19 cannot be dropped.

§2 Compact Two-point Homogeneous Spaces. Applications

We shall now extend the inversion formula in Theorem 1.7 to compact two-point homogeneous spaces X of dimension $n > 1$. By virtue of Wang's classification [1952] these are also the compact symmetric spaces of rank one (see Matsumoto [1971] and Szabo [1991] for more direct proofs), so their geometry can be described very explicitly. Here we shall use some geometric and group theoretic properties of these spaces ((i)–(vii) below) and refer to Helgason ([1959], p. 278, [1965a], §5–6 or [DS], Ch. VII, §10) for their proofs.

Let U denote the group $I(X)$ of isometries X. Fix an origin $o \in X$ and let K denote the isotropy subgroup U_o. Let $\mathfrak{k}$ and $\mathfrak{u}$ be the Lie algebras of K and U, respectively. Then $\mathfrak{u}$ is semisimple. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ and $\mathfrak{u}$ with respect to the Killing form B of $\mathfrak{u}$. Changing the distance function on X by a constant factor we may assume that the differential of the mapping $u \to u \cdot o$ of U onto X gives an isometry of $\mathfrak{p}$ (with the metric of $-B$) onto the tangent space X_o. This is the canonical metric X which we shall use.

Let L denote the diameter of X, that is the maximal distance between any two points. If $x \in X$ let A_x denote the set of points in X of distance L from x. By the two-point homogeneity the isotropy subgroup U_x acts transitively on A_x; thus $A_x \subset X$ is a submanifold, *the antipodal manifold* associated to x.

(i) *Each A_x is a totally geodesic submanifold of X; with the Riemannian structure induced by that of X it is another two-point homogeneous space.*

(ii) *Let Ξ denote the set of all antipodal manifolds in X; since U acts transitively on Ξ, the set Ξ has a natural manifold structure. Then the mapping $j : x \to A_x$ is a one-to-one diffeomorphism; also $x \in A_y$ if and only if $y \in A_x$.*

(iii) *Each geodesic in X has period $2L$. If $x \in X$ the mapping $\mathrm{Exp}_x : X_x \to X$ gives a diffeomorphism of the ball $B_L(0)$ onto the open set $X - A_x$.*

Fix a vector $H \in \mathfrak{p}$ of length L (i.e., $L^2 = -B(H, H)$). For $Z \in \mathfrak{p}$ let T_Z denote the linear transformation $Y \to [Z, [Z, Y]]$ of $\mathfrak{p}$, $[\,,\,]$ denoting the Lie bracket in $\mathfrak{u}$. For simplicity, we now write Exp instead of Exp_o. A point $Y \in \mathfrak{p}$ is said to be *conjugate* to o if the differential $d\mathrm{Exp}$ is singular at Y.

The line $\mathfrak{a} = \mathbf{R}H$ is a maximal abelian subspace of $\mathfrak{p}$. The eigenvalues of T_H are 0, $\alpha(H)^2$ and possibly $(\alpha(H)/2)^2$ where $\pm\alpha$ (and possibly $\pm\alpha/2$) are the roots of $\mathfrak{u}$ with respect to $\mathfrak{a}$. Let

$$\mathfrak{p} = \mathfrak{a} + \mathfrak{p}_\alpha + \mathfrak{p}_{\alpha/2} \tag{86}$$

be the corresponding decomposition of $\mathfrak{p}$ into eigenspaces; the dimensions $q = \dim(\mathfrak{p}_\alpha)$, $p = \dim(\mathfrak{p}_{\alpha/2})$ are called the *multiplicities* of α and $\alpha/2$, respectively.

(iv) *Suppose H is conjugate to o. Then* $\mathrm{Exp}(\mathfrak{a}+\mathfrak{p}_\alpha)$, *with the Riemannian structure induced by that of X, is a sphere, totally geodesic in X, having o and* $\mathrm{Exp}H$ *as antipodal points and having curvature $\pi^2 L^2$. Moreover*

$$A_{\mathrm{Exp}H} = \mathrm{Exp}(\mathfrak{p}_{\alpha/2})\,.$$

(v) *If H is not conjugate to o then $\mathfrak{p}_{\alpha/2} = 0$ and*

$$A_{\mathrm{Exp}H} = \mathrm{Exp}\ \mathfrak{p}_\alpha\,.$$

(vi) *The differential at Y of* Exp *is given by*

$$d\mathrm{Exp}_Y = d\tau(\exp Y) \circ \sum_0^\infty \frac{T_Y^k}{(2k+1)!}\,,$$

where for $u \in U$, $\tau(u)$ is the isometry $x \to u \cdot x$.

(vii) *In analogy with (23) the Laplace-Beltrami operator L on X has the expression*

$$L = \frac{\partial^2}{\partial r^2} + \frac{1}{A(r)} A'(r) \frac{\partial}{\partial r} + L_{S_r}\,,$$

where L_{S_r} is the Laplace-Beltrami operator on $S_r(o)$ and $A(r)$ its area.

(viii) *The spherical mean-value operator M^r commutes with the Laplace-Beltrami operator.*

Lemma 2.1. *The surface area $A(r)$ $(0 < r < L)$ is given by*

$$A(r) = \Omega_n \lambda^{-p}(2\lambda)^{-q} \sin^p(\lambda r)\sin^q(2\lambda r)$$

where p and q are the multiplicities above and $\lambda = |\alpha(H)|/2L$.

Proof. Because of (iii) and (vi) the surface area of $S_r(o)$ is given by

$$A(r) = \int_{|Y|=r} \det\Big(\sum_0^\infty \frac{T_Y^k}{(2k+1)!}\Big)\, d\omega_r(Y)\,,$$

where $d\omega_r$ is the surface on the sphere $|Y| = r$ in $\mathfrak{p}$. Because of the two-point homogeneity the integrand depends on r only so it suffices to evaluate it for $Y = H_r = \frac{r}{L}H$. Since the nonzero eigenvalues of T_{H_r} are $\alpha(H_r)^2$ with multiplicity q and $(\alpha(H_r)/2)^2$ with multiplicity p, a trivial computation gives the lemma.

We consider now Problems A, B and C from Chapter II, §2 for the homogeneous spaces X and Ξ, which are acted on transitively by the same group U. Fix an element $\xi_o \in \Xi$ passing through the origin $o \in X$. If $\xi_o = A_0$, then an element $u \in U$ leaves ξ_o invariant if and only if it lies in the isotropy subgroup $K' = U_o$; we have the identifications

$$X = U/K\,, \quad \Xi = U/K'$$

and $x \in X$ and $\xi \in \Xi$ are incident if and only if $x \in \xi$.

On Ξ we now choose a Riemannian structure such that the diffeomorphism $j : x \to A_x$ from (ii) is an isometry. Let L and Λ denote the Laplacians on X and Ξ, respectively. With $\check{x}$ and $\widehat{\xi}$ defined as in Ch. II, §1, we have

$$\widehat{\xi} = \xi\,, \quad \check{x} = \{j(y) : y \in j(x)\}\,;$$

the first relation amounts to the incidence description above and the second is a consequence of the property $x \in A_y \Leftrightarrow y \in A_x$ listed under (ii).

The sets $\check{x}$ and $\widehat{\xi}$ will be given the measures $d\mu$ and dm, respectively, induced by the Riemannian structures of Ξ and X. The Radon transform and its dual are then given by

$$\widehat{f}(\xi) = \int_\xi f(x)\,dm(x)\,, \quad \check{\varphi}(x) = \int_{\check{x}} \varphi(\xi)\,d\mu(\xi)\,.$$

However

$$\check{\varphi}(x) = \int_{\check{x}} \varphi(\xi)\,d\mu(\xi) = \int_{y \in j(x)} \varphi(j(y))\,d\mu(j(y)) = \int_{j(x)} (\varphi \circ j)(y)\,dm(y)$$

so

$$\check{\varphi} = (\varphi \circ j)\widehat{\ } \circ j\,. \tag{87}$$

Because of this correspondence between the transforms $f \to \widehat{f}$, $\varphi \to \check{\varphi}$ it suffices to consider the first one. Let $\mathbf{D}(X)$ denote the algebra of differential operators on X, invariant under U. It can be shown that $\mathbf{D}(X)$ is generated by L. Similarly $\mathbf{D}(\Xi)$ is generated by Λ.

Theorem 2.2. *(i) The mapping $f \to \widehat{f}$ is a linear one-to-one mapping of $\mathcal{E}(X)$ onto $\mathcal{E}(\Xi)$ and*

$$(Lf)\widehat{\ } = \Lambda \widehat{f}\,.$$

(ii) Except for the case when X is an even-dimensional elliptic space

$$f = P(L)((\widehat{f})^\vee)\,, \quad f \in \mathcal{E}(X)\,,$$

where P is a polynomial, independent of f, explicitly given below, (90)–(93). In all cases

$$\text{degree } P = \tfrac{1}{2} \text{ dimension of the antipodal manifold}\,.$$

Proof. (Indication.) We first prove (ii). Let dk be the Haar measure on K such that $\int dk = 1$ and let Ω_X denote the total measure of an antipodal manifold in X. Then $\mu(\check{o}) = m(A_o) = \Omega_X$ and if $u \in U$,

$$\check{\varphi}(u \cdot o) = \Omega_X \int_K \varphi(uk \cdot \xi_o)\, dk\,.$$

Hence

$$(\widehat{f})^\vee(u \cdot o) = \Omega_X \int_K \left(\int_{\xi_o} f(uk \cdot y)\, dm(y) \right) dk = \Omega_X \int_{\xi_o} (M^r f)(u \cdot o)\, dm(y)\,,$$

where r is the distance $d(o, y)$ in the space X between o and y. If $d(o, y) < L$ there is a unique geodesic in X of length $d(o, y)$ joining o to y and since ξ_0 is totally geodesic, $d(o, y)$ is also the distance in ξ_0 between o and y. Thus using geodesic polar coordinates in ξ_0 in the last integral we obtain

(88) $$(\widehat{f})^\vee(x) = \Omega_X \int_0^L (M^r f)(x) A_1(r)\, dr\,,$$

where $A_1(r)$ is the area of a sphere of radius r in ξ_0. By Lemma 2.1 we have

(89) $$A_1(r) = C_1 \sin^{p_1}(\lambda_1 r) \sin^{q_1}(2\lambda_1 r)\,,$$

where C_1 and λ_1 are constants and p_1, q_1 are the multiplicities for the antipodal manifold. In order to prove (ii) on the basis of (88) we need the following complete list of the compact symmetric spaces of rank one and their corresponding antipodal manifolds:

X		A_0
Spheres	$\mathbf{S}^n (n = 1, 2, \ldots)$	point
Real projective spaces	$\mathbf{P}^n(\mathbf{R}) (n = 2, 3, \ldots)$	$\mathbf{P}^{n-1}(\mathbf{R})$
Complex projective spaces	$\mathbf{P}^n(\mathbf{C}) (n = 4, 6, \ldots)$	$\mathbf{P}^{n-2}(\mathbf{C})$
Quaternian projective spaces	$\mathbf{P}^n(\mathbf{H}) (n = 8, 12, \ldots)$	$\mathbf{P}^{n-4}(\mathbf{H})$
Cayley plane	$\mathbf{P}^{16}(\mathbf{Cay})$	$\mathbf{S}^8$

Here the superscripts denote the real dimension. For the lowest dimensions, note that

$$\mathbf{P}^1(\mathbf{R}) = \mathbf{S}^1\,, \quad \mathbf{P}^2(\mathbf{C}) = \mathbf{S}^2\,, \quad \mathbf{P}^4(\mathbf{H}) = \mathbf{S}^4\,.$$

For the case $\mathbf{S}^n$, (ii) is trivial and the case $X = \mathbf{P}^n(\mathbf{R})$ was already done in Theorem 1.7. The remaining cases are done by classification starting with (88). The mean value operator M^r still commutes with the Laplacian L

$$M^r L = L M^r$$

and this implies

$$L_x((M^r f)(x)) = L_r((M^r f)(x))\,,$$

where L_r is the radial part of L. Because of (vii) above and Lemma 2.1 it is given by

$$L_r = \frac{\partial^2}{\partial r^2} + \lambda\{p\cot(\lambda r) + 2q\cot(2\lambda r)\}\frac{\partial}{\partial r}\,.$$

For each of the two-point homogeneous spaces we prove (by extensive computations) the analog of Lemma 1.8. Then by the pattern of the proof of Theorem 1.5, part (ii) of Theorem 2.2 can be proved. The full details are carried out in Helgason ([1965a] or [GGA], Ch. I, §4).

The polynomial P is explicitly given in the list below. Note that for $\mathbf{P}^n(\mathbf{R})$ the metric is normalized by means of the Killing form so it differs from that of Theorem 1.7 by a nontrivial constant.

The polynomial P is now given as follows:

For $X = \mathbf{P}^n(\mathbf{R})$, n odd

$$\begin{aligned} P(L) &= c\left(L - \tfrac{(n-2)1}{2n}\right)\left(L - \tfrac{(n-4)3}{2n}\right)\cdots\left(L - \tfrac{1(n-2)}{2n}\right) \\ c &= \tfrac{1}{4}(-4\pi^2 n)^{\frac{1}{2}(n-1)}\,. \end{aligned} \tag{90}$$

For $X = \mathbf{P}^n(\mathbf{C})$, $n = 4, 6, 8, \ldots$

$$\begin{aligned} P(L) &= c\left(L - \tfrac{(n-2)2}{2(n+2)}\right)\left(L - \tfrac{(n-4)4}{2(n+2)}\right)\cdots\left(L - \tfrac{2(n-2)}{2(n+2)}\right) \\ c &= (-8\pi^2(n+2))^{1-\frac{n}{2}}\,. \end{aligned} \tag{91}$$

For $X = \mathbf{P}^n(\mathbf{H})$, $n = 8, 12, \ldots$

$$\begin{aligned} P(L) &= c\left(L - \tfrac{(n-2)4}{2(n+8)}\right)\left(L - \tfrac{(n-4)6}{2(n+8)}\right)\cdots\left(L - \tfrac{4(n-2)}{2(n+8)}\right) \\ c &= \tfrac{1}{2}[-4\pi^2(n+8)]^{2-n/2}\,. \end{aligned} \tag{92}$$

For $X = \mathbf{P}^{16}(\mathbf{Cay})$

$$P(L) = c\left(L - \tfrac{14}{9}\right)^2\left(L - \tfrac{15}{9}\right)^2\,, \quad c = 3^6\pi^{-8}2^{-13}\,. \tag{93}$$

That $f \to \widehat{f}$ is injective follows from (ii) except for the case $X = \mathbf{P}^n(\mathbf{R})$, n even. But in this exceptional case the injectivity follows from Theorem 1.7.

For the surjectivity we use once more the fact that the mean-value operator M^r commutes with the Laplacian (property (viii)). We have

$$\widehat{f}(j(x)) = c(M^L f)(x)\,, \tag{94}$$

where c is a constant. Thus by (87)

$$(\widehat{f})^\vee(x) = (\widehat{f}\circ j)\widehat{\ }(j(x)) = cM^L(\widehat{f}\circ j)(x)$$

so

$$(\widehat{f})^{\vee} = c^2 M^L M^L f \,. \tag{95}$$

Thus if X is not an even-dimensional projective space f is a constant multiple of $M^L P(L) M^L f$ which by (94) shows $f \to \widehat{f}$ surjective. For the remaining case $\mathbf{P}^n(\mathbf{R})$, n even, we use the expansion of $f \in \mathcal{E}(\mathbf{P}^n(\mathbf{R}))$ in spherical harmonics

$$f = \sum_{k,m} a_{km} S_{km} \quad (k \text{ even}) \,.$$

Here $k \in \mathbb{Z}^+$, and $S_{km}(1 \leq m \leq d(k))$ is an orthonormal basis of the space of spherical harmonics of degree k. Here the coefficients a_{km} are rapidly decreasing in k. On the other hand, by (32) and (34),

$$\widehat{f} = \Omega_n M^{\frac{\pi}{2}} f = \Omega_n \sum_{k,m} a_{km} \varphi_k \left(\tfrac{\pi}{2}\right) S_{km} \quad (k \text{ even}) \,. \tag{96}$$

The spherical function φ_k is given by

$$\varphi_k(s) = \frac{\Omega_{n-1}}{\Omega_n} \int_0^{\pi} (\cos\theta + i \sin\theta \cos\varphi)^k \sin^{n-2}\varphi \, d\varphi$$

so $\varphi_{2k}(\frac{\pi}{2}) \sim k^{-\frac{n-1}{2}}$. Thus every series $\sum_{k,m} b_{k,m} S_{2k,m}$ with $b_{2k,m}$ rapidly decreasing in k can be put in the form (96). This verifies the surjectivity of the map $f \to \widehat{f}$.

It remains to prove $(Lf)\widehat{\ } = \Lambda \widehat{f}$. For this we use (87), (vii), (41) and (94). By the definition of Λ we have

$$(\Lambda\varphi)(j(x)) = L(\varphi \circ j)(x) \,, \qquad x \in X, \varphi \in \mathcal{E}(X) \,.$$

Thus

$$(\Lambda\widehat{f})(j(x)) = (L(\widehat{f} \circ j))(x) = cL(M^L f)(x) = cM^L(Lf)(x) = (Lf)\widehat{\ }(j(x)) \,.$$

This finishes our indication of the proof of Theorem 2.2.

Corollary 2.3. *Let X be a compact two-point homogeneous space and suppose f satisfies*

$$\int_\gamma f(x)\, ds(x) = 0$$

for each (closed) geodesic γ in X, ds being the element of arc-length. Then

(i) If X is a sphere, f is skew.

(ii) If X is not a sphere, $f \equiv 0$.

Taking a convolution with f we may assume f smooth. Part (i) is already contained in Theorem 1.7. For Part (ii) we use the classification; for $X = \mathbf{P}^{16}(\mathbf{Cay})$ the antipodal manifolds are totally geodesic spheres so using Part (i) we conclude that $\widehat{f} \equiv 0$ so by Theorem 2.2, $f \equiv 0$. For the remaining cases $\mathbf{P}^n(\mathbf{C})$ $(n = 4, 6, \dots)$ and $\mathbf{P}^n(\mathbf{H}), (n = 8, 12, \dots)$ (ii) follows similarly by induction as the initial antipodal manifolds, $\mathbf{P}^2(\mathbf{C})$ and $\mathbf{P}^4(\mathbf{H})$, are totally geodesic spheres.

Corollary 2.4. *Let B be a bounded open set in $\mathbf{R}^{n+1}$, symmetric and star-shaped with respect to 0, bounded by a hypersurface. Assume for a fixed k $(1 \leq k < n)$*

$$\text{Area } (B \cap P) = \text{ constant} \tag{97}$$

for all $(k+1)$-planes P through 0. Then B is an open ball.

In fact, we know from Theorem 1.7 that if f is a symmetric function on $X = \mathbf{S}^n$ with $\widehat{f}(\mathbf{S}^n \cap P)$ constant (for all P) then f is a constant. We apply this to the function

$$f(\theta) = \rho(\theta)^{k+1} \quad \theta \in \mathbf{S}^n$$

if $\rho(\theta)$ is the distance from the origin to each of the two points of intersection of the boundary of B with the line through 0 and θ; f is well defined since B is symmetric. If $\theta = (\theta_1, \dots, \theta_k)$ runs through the k-sphere $\mathbf{S}^n \cap P$ then the point

$$x = \theta r \quad (0 \leq r < \rho(\theta))$$

runs through the set $B \cap P$ and

$$\text{Area } (B \cap P) = \int_{\mathbf{S}^n \cap P} d\omega(\theta) \int_0^{\rho(\theta)} r^k \, dr \, .$$

It follows that Area $(B \cap P)$ is a constant multiple of $\widehat{f}(\mathbf{S}^n \cap P)$ so (97) implies that f is constant. This proves the corollary.

§3 Noncompact Two-point Homogeneous Spaces

Theorem 2.2 has an analog for noncompact two-point homogeneous spaces which we shall now describe. By Tits' classification [1955], p. 183, of homogeneous manifolds L/H for which L acts transitively on the tangents to L/H it is known, in principle, what the noncompact two-point homogeneous spaces are. As in the compact case they turn out to be symmetric. A direct proof of this fact was given by Nagano [1959] and Helgason [1959]. The theory of symmetric spaces then implies that the noncompact two-point homogeneous spaces are the Euclidean spaces and the noncompact spaces $X = G/K$ where G is a connected semisimple Lie group with finite center and real rank one and K a maximal compact subgroup.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the direct decomposition of the Lie algebra of G into the Lie algebra $\mathfrak{k}$ of K and its orthogonal complement $\mathfrak{p}$ (with respect to the Killing form of $\mathfrak{g}$). Fix a 1-dimensional subspace $\mathfrak{a} \subset \mathfrak{p}$ and let

$$\mathfrak{p} = \mathfrak{a} + \mathfrak{p}_\alpha + \mathfrak{p}_{\alpha/2} \tag{98}$$

be the decomposition of $\mathfrak{p}$ into eigenspaces of T_H (in analogy with (86)). Let ξ_o denote the totally geodesic submanifold $\operatorname{Exp}(\mathfrak{p}_{\alpha/2})$; in the case $\mathfrak{p}_{\alpha/2} = 0$ we put $\xi_o = \operatorname{Exp}(\mathfrak{p}_\alpha)$. By the classification and duality for symmetric spaces we have the following complete list of the spaces G/K. In the list the superscript denotes the real dimension; for the lowest dimensions note that

$$\mathbf{H}^1(\mathbf{R}) = \mathbf{R}\,, \quad \mathbf{H}^2(\mathbf{C}) = \mathbf{H}^2(\mathbf{R})\,, \quad \mathbf{H}^4(\mathbf{H}) = \mathbf{H}^4(\mathbf{R})\,.$$

X		ξ_0
Real hyperbolic spaces	$\mathbf{H}^n(\mathbf{R})(n = 2, 3, \ldots)$,	$\mathbf{H}^{n-1}(\mathbf{R})$
Complex hyperbolic spaces	$\mathbf{H}^n(\mathbf{C})(n = 4, 6, \ldots)$,	$\mathbf{H}^{n-2}(\mathbf{C})$
Quaternian hyperbolic spaces	$\mathbf{H}^n(\mathbf{H})(n = 8, 12, \ldots)$,	$\mathbf{H}^{n-4}(\mathbf{H})$
Cayley hyperbolic spaces	$\mathbf{H}^{16}(\mathbf{Cay})$,	$\mathbf{H}^8(\mathbf{R})$.

Let Ξ denote the set of submanifolds $g \cdot \xi_0$ of X as g runs through G; Ξ is given the canonical differentiable structure of a homogeneous space. Each $\xi \in \Xi$ has a measure m induced by the Riemannian structure of X and the Radon transform on X is defined by

$$\widehat{f}(\xi) = \int_\xi f(x)\, dm(x)\,, \quad f \in C_c(X)\,.$$

The dual transform $\varphi \to \check{\varphi}$ is defined by

$$\check{\varphi}(x) = \int_{\xi \ni x} \varphi(\xi)\, d\mu(\xi)\,, \quad \varphi \in C(\Xi)\,,$$

where μ is the invariant average on the set of ξ passing through x. Let L denote the Laplace-Beltrami operator on X, Riemannian structure being that given by the Killing form of $\mathfrak{g}$.

Theorem 3.1. *The Radon transform $f \to \widehat{f}$ is a one-to-one mapping of $\mathcal{D}(X)$ into $\mathcal{D}(\Xi)$ and, except for the case $X = \mathbf{H}^n(\mathbf{R})$, n even, is inverted by the formula $f = Q(L)((\widehat{f})^\vee)$. Here Q is given by*

$X = \mathbf{H}^n(\mathbf{R}), n$ *odd:*

$$Q(L) = \gamma\left(L + \tfrac{(n-2)1}{2n}\right)\left(L + \tfrac{(n-4)3}{2n}\right)\cdots\left(L + \tfrac{1(n-2)}{2n}\right).$$

$X = \mathbf{H}^n(\mathbf{C})$:

$$Q(L) = \gamma\left(L + \tfrac{(n-2)2}{2(n+2)}\right)\left(L + \tfrac{(n-4)4}{2(n+2)}\right)\cdots\left(L + \tfrac{2(n-2)}{2(n+2)}\right).$$

$X = \mathbf{H}^n(\mathbf{H})$:

$$Q(L) = \gamma\left(L + \tfrac{(n-2)4}{2(n+8)}\right)\left(L + \tfrac{(n-4)6}{2(n+8)}\right)\cdots\left(L + \tfrac{4(n-2)}{2(n+8)}\right).$$

$X = \mathbf{H}^{16}(\boldsymbol{Cay})$:

$$Q(L) = \gamma\left(L + \tfrac{14}{9}\right)^2\left(L + \tfrac{15}{9}\right)^2.$$

The constants γ are obtained from the constants c in (90)–(93) by multiplication by the factor Ω_X which is the volume of the antipodal manifold in the compact space corresponding to X. This factor is explicitly determined for each X in [GGA], Chapter I, §4.

§4 The X-ray Transform on a Symmetric Space

Let X be a complete Riemannian manifold of dimension > 1 in which any two points can be joined by a unique geodesic. The *X-ray transform* on X assigns to each continuous function f on X the integrals

$$\widehat{f}(\gamma) = \int_\gamma f(x)\,ds(x)\,, \qquad (99)$$

γ being any complete geodesic in X and ds the element of arc-length. In analogy with the X-ray reconstruction problem on $\mathbf{R}^n$ (Ch.I, §7) one can consider the problem of inverting the X-ray transform $f \to \widehat{f}$. With d denoting the distance in X and $o \in X$ some fixed point we now define two subspaces of $C(X)$. Let

$$\begin{aligned} F(X) &= \{f \in C(X) : \sup_x d(o,x)^k |f(x)| < \infty \text{ for each } k \geq 0\} \\ \mathfrak{F}(X) &= \{f \in C(X) : \sup_x e^{kd(o,x)} |f(x)| < \infty \text{ for each } k \geq 0\}\,. \end{aligned}$$

Because of the triangle inequality these spaces do not depend on the choice of o. We can informally refer to $F(X)$ as the space of continuous *rapidly decreasing functions* and to $\mathfrak{F}(X)$ as the space of continuous *exponentially decreasing functions*. We shall now prove the analog of the support theorem (Theorem 2.6, Ch. I, Theorem 1.2, Ch. III) for the X-ray transform on a symmetric space of the noncompact type. This general analog turns out to be a direct corollary of the Euclidean case and the hyperbolic case, already done.

Corollary 4.1. *Let X be a symmetric space of the noncompact type, B any ball in M.*

(i) If a function $f \in \mathfrak{F}(X)$ satisfies

$$\widehat{f}(\xi) = 0 \quad \textit{whenever } \xi \cap B = \emptyset\,, \quad \xi \textit{ a geodesic,} \qquad (100)$$

then

$$f(x) = 0 \quad \textit{for } x \notin B\,. \tag{101}$$

In particular, the X-ray transform is one-to-one on $\mathfrak{F}(X)$.

(ii) If X *has rank greater than one statement (i) holds with* $\mathfrak{F}(X)$ *replaced by* $F(X)$.

Proof. Let o be the center of B, r its radius, and let γ be an arbitrary geodesic in X through o.

Assume first X has rank greater than one. By a standard conjugacy theorem for symmetric spaces γ lies in a 2-dimensional, flat, totally geodesic submanifold of X. Using Theorem 2.6, Ch. I on this Euclidean plane we deduce $f(x) = 0$ if $x \in \gamma, d(o,x) > r$. Since γ is arbitrary (101) follows.

Next suppose X has rank one. Identifying $\mathfrak{p}$ with the tangent space X_o let $\mathfrak{a}$ be the tangent line to γ. We can then consider the eigenspace decomposition (98). If $\mathfrak{b} \subset \mathfrak{p}_\alpha$ is a line through the origin then $S = \mathrm{Exp}(\mathfrak{a} + \mathfrak{b})$ is a totally geodesic submanifold of X (cf. (iv) in the beginning of §2). Being 2-dimensional and not flat, S is necessarily a hyperbolic space. From Theorem 1.2 we therefore conclude $f(x) = 0$ for $x \in \gamma$, $d(o,x) > r$. Again (101) follows since γ is arbitrary.

§5 Maximal Tori and Minimal Spheres in Compact Symmetric Spaces

Let $\mathfrak{u}$ be a compact semisimple Lie algebra, θ an involutive automorphism of $\mathfrak{u}$ with fixed point algebra $\mathfrak{k}$. Let U be the simply connected Lie group with Lie algebra $\mathfrak{u}$ and $\mathrm{Int}(\mathfrak{u})$ the adjoint group of $\mathfrak{u}$. Then θ extends to an involutive automorphism of U and $\mathrm{Int}(\mathfrak{u})$. We denote these extensions also by θ and let K and K_θ denote the respective fixed point groups under θ. The symmetric space $X_\theta = \mathrm{Int}(\mathfrak{u})/K_\theta$ is called the *adjoint space* of $(\mathfrak{u}, \theta)$ (Helgason [1978], p. 327), and is covered by $X = U/K$, this latter space being simply connected since K is automatically connected.

The flat totally geodesic submanifolds of X_θ of maximal dimension are permuted transitively by $\mathrm{Int}(\mathfrak{u})$ according to a classical theorem of Cartan. Let E_θ be one such manifold passing through the origin eK_θ in X_θ and let H_θ be the subgroup of $\mathrm{Int}(\mathfrak{u})$ preserving E_θ. We then have the pairs of homogeneous spaces

$$X_\theta = \mathrm{Int}(\mathfrak{u})/K_\theta\,, \quad \Xi_\theta = \mathrm{Int}(\mathfrak{u})/H_\theta\,. \tag{102}$$

The corresponding Radon transform $f \to \widehat{f}$ from $C(X_\theta)$ to $C(\Xi_\theta)$ amounts to

$$\widehat{f}(E) = \int_E f(x)\, dm(x)\,, \quad E \in \Xi_\theta\,, \tag{103}$$

E being any flat totally geodesic submanifold of X_θ of maximal dimension and dm the volume element. If X_θ has rank one, E is a geodesic and we are in the situation of Corollary 2.3. The transform (103) is often called the *flat Radon transform.*

Theorem 5.1. *Assume X_θ is irreducible. Then the flat Radon transform is injective.*

For a proof see Grinberg [1992].

The sectional curvatures of the space X lie in an interval $[0, \kappa]$. The space X contains totally geodesic spheres of curvature κ and all such spheres S of maximal dimension are conjugate under U (Helgason [1966b]). Fix one such sphere S_0 through the origin eK and let H be the subgroup of U preserving S_0. Then we have another double fibration

$$X = U/K\,, \quad \Xi = U/H$$

and the accompanying Radon transform

$$\widehat{f}(S) = \int_S f(x)\, d\sigma(x)\,.$$

$S \in \Xi$ being arbitrary and $d\sigma$ being the volume element on S.

It is proved by Grinberg [1994] that injectivity holds in many cases although the general question is not fully settled.

Bibliographical Notes

As mentioned earlier, it was shown by Funk [1916] that a function f on the two-sphere, symmetric with respect to the center, can be determined by the integrals of f over the great circles. When f is rotation-invariant (relative to a vertical axis) he gave an explicit inversion formula, essentially (78) in Proposition 1.16.

The Radon transform on hyperbolic and on elliptic spaces corresponding to k-dimensional totally geodesic submanifolds was defined in the author's paper [1959]. Here and in [1990] are proved the inversion formulas in Theorems 1.5, 1.7, 1.10 and 1.11. See also Semyanisty [1961] and Rubin [1998b]. The alternative version in (60) was obtained by Berenstein and Casadio Tarabusi [1991] which also deals with the case of $\mathbf{H}^k$ in $\mathbf{H}^n$ (where the regularization is more complex). Still another interesting variation of Theorem 1.10 (for $k = 1, n = 2$) is given by Lissianoi and Ponomarev [1997]. By calculating the dual transform $\check{\varphi}_p(z)$ they derive from (30) in Chapter II an inversion formula which has a formal analogy to (38) in Chapter II. The underlying reason may be that to each geodesic γ in $\mathbf{H}^2$ one can associate a pair of horocycles tangential to $|z| = 1$ at the endpoints of γ having the same distance from o as γ.

The support theorem (Theorem 1.2) was proved by the author ([1964], [1980b]) and its consequence, Cor. 4.1, pointed out in [1980d]. Interesting generalizations are contained in Boman [1991], Boman and Quinto [1987], [1993]. For the case of $\mathbf{S}^{n-1}$ see Quinto [1983] and in the stronger form of Theorem 1.17, Kurusa [1994]. The variation (82) of the Funk transform has also been considered by Abouelaz and Daher [1993] at least for K-invariant functions. The theory of the Radon transform for antipodal manifolds in compact two-point homogeneous spaces (Theorem 2.2) is from Helgason [1965a]. R. Michel has in [1972] and [1973] used Theorem 2.2 in establishing certain infinitesimal rigidity properties of the canonical metrics on the real and complex projective spaces. See also Guillemin [1976] and A. Besse [1978], Goldschmidt [1990], Estezet [1988].

CHAPTER IV

ORBITAL INTEGRALS AND THE WAVE OPERATOR FOR ISOTROPIC LORENTZ SPACES

In Chapter II, §3 we discussed the problem of determining a function on a homogeneous space by means of its integrals over generalized spheres. We shall now solve this problem for the *isotropic Lorentz spaces* (Theorem 4.1 below). As we shall presently explain these spaces are the Lorentzian analogs of the two-point homogeneous spaces considered in Chapter III.

§1 Isotropic Spaces

Let X be a manifold. A *pseudo-Riemannian structure* of signature (p, q) is a smooth assignment $y \to g_y$ where $y \in X$ and g_y is a symmetric nondegenerate bilinear form on $X_y \times X_y$ of signature (p, q). This means that for a suitable basis $Y_1, \ldots, Y_{p+q}$ of X_y we have

$$g_y(Y) = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_{p+q}^2$$

if $Y = \sum_1^{p+q} y_i Y_i$. If $q = 0$ we speak of a *Riemannian* structure and if $p = 1$ we speak of a *Lorentzian* structure. Connected manifolds X with such structures g are called pseudo-Riemannian (respectively Riemannian, Lorentzian) manifolds.

A manifold X with a pseudo-Riemannian structure g has a differential operator of particular interest, the so-called Laplace-Beltrami operator. Let $(x_1, \ldots, x_{p+q})$ be a coordinate system on an open subset U of X. We define the functions g_{ij}, g^{ij}, and $\overline{g}$ on U by

$$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right), \quad \sum_j g_{ij} g^{jk} = \delta_{ik}, \quad \overline{g} = |\det(g_{ij})|.$$

The *Laplace-Beltrami operator* L is defined on U by

$$Lf = \frac{1}{\sqrt{\overline{g}}}\left(\sum_k \frac{\partial}{\partial x_k}\left(\sum_i g^{ik}\sqrt{\overline{g}}\frac{\partial f}{\partial x_i}\right)\right)$$

for $f \in \mathcal{C}^\infty(U)$. It is well known that this expression is invariant under coordinate changes so L is a differential operator on X.

An *isometry* of a pseudo-Riemannian manifold X is a diffeomorphism preserving g. It is easy to prove that L is *invariant* under each isometry φ, that is $L(f \circ \varphi) = (Lf) \circ \varphi$ for each $f \in \mathcal{E}(X)$. Let $I(X)$ denote the group of all isometries of X. For $y \in X$ let $I(X)_y$ denote the subgroup of $I(X)$

fixing y (the isotropy subgroup at y) and let H_y denote the group of linear transformations of the tangent space X_y induced by the action of $I(X)_y$. For each $a \in \mathbf{R}$ let $\sum_a(y)$ denote the "sphere"

$$\Sigma_a(y) = \{Z \in X_y : g_y(Z, Z) = a\,, \quad Z \neq 0\}\,. \tag{1}$$

Definition. The pseudo-Riemannian manifold X is called *isotropic* if for each $a \in \mathbf{R}$ and each $y \in X$ the group H_y acts transitively on $\sum_a(y)$.

Proposition 1.1. *An isotropic pseudo-Riemannian manifold X is homogeneous; that is, $I(X)$ acts transitively on X.*

Proof. The pseudo-Riemannian structure on X gives an affine connection preserved by each isometry $g \in I(X)$. Any two points $y, z \in X$ can be joined by a curve consisting of finitely many geodesic segments $\gamma_i (1 \leq i \leq p)$. Let g_i be an isometry fixing the midpoint of γ_i and reversing the tangents to γ_i at this point. The product $g_p \cdots g_1$ maps y to z, whence the homogeneity of X.

A. The Riemannian Case

The following simple result shows that the isotropic spaces are natural generalizations of the spaces considered in the last chapter.

Proposition 1.2. *A Riemannian manifold X is isotropic if and only if it is two-point homogeneous.*

Proof. If X is two-point homogeneous and $y \in X$ the isotropy subgroup $I(X)_y$ at y is transitive on each sphere $S_r(y)$ in X with center y so X is clearly isotropic. On the other hand if X is isotropic it is homogeneous (Prop. 1.1) hence complete; thus by standard Riemannian geometry any two points in X can be joined by means of a geodesic. Now the isotropy of X implies that for each $y \in X, r > 0$, the group $I(X)_y$ is transitive on the sphere $S_r(y)$, whence the two-point homogeneity.

B. The General Pseudo-Riemannian Case

Let X be a manifold with pseudo-Riemannian structure g and curvature tensor R. Let $y \in X$ and $S \subset X_y$ a 2-dimensional subspace on which g_y is nondegenerate. The curvature of X along the section S spanned by Z and Y is defined by

$$K(S) = -\frac{g_p(R_p(Z,Y)Z,Y)}{g_p(Z,Z)g_p(Y,Y) - g_p(Z,Y)^2}$$

The denominator is in fact $\neq 0$ and the expression is independent of the choice of Z and Y.

We shall now construct isotropic pseudo-Riemannian manifolds of signature (p,q) and constant curvature. Consider the space $\mathbf{R}^{p+q+1}$ with the flat pseudo-Riemannian structure

$$B_e(Y) = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_{p+q}^2 + e\,y_{p+q+1}^2\,, \quad (e = \pm 1)\,.$$

Let Q_e denote the quadric in $\mathbf{R}^{p+q+1}$ given by

$$B_e(Y) = e\,.$$

The orthogonal group $\mathbf{O}(B_e)$ ($= \mathbf{O}(p,q+1)$ or $\mathbf{O}(p+1,q)$) acts transitively on Q_e; the isotropy subgroup at $o = (0,\dots,0,1)$ is identified with $\mathbf{O}(p,q)$.

Theorem 1.3. *(i) The restriction of B_e to the tangent spaces to Q_e gives a pseudo-Riemannian structure g_e on Q_e of signature (p,q).*

(ii) We have

(2) $$Q_{-1} \cong \mathbf{O}(p,q+1)/\mathbf{O}(p,q) \qquad \textit{(diffeomorphism)}$$

and the pseudo-Riemannian structure g_{-1} on Q_{-1} has constant curvature -1.

(iii) We have

(3) $$Q_{+1} = \mathbf{O}(p+1,q)/\mathbf{O}(p,q) \qquad \textit{(diffeomorphism)}$$

and the pseudo-Riemannian structure g_{+1} on Q_{+1} has constant curvature $+1$.

(iv) The flat space $\mathbf{R}^{p+q}$ with the quadratic form $g_o(Y) = \sum_1^p y_i^2 - \sum_{p+1}^{p+q} y_j^2$ and the spaces

$$\mathbf{O}(p,q+1)/\mathbf{O}(p,q)\,, \quad \mathbf{O}(p+1,q)/\mathbf{O}(p,q)$$

are all isotropic and (up to a constant factor on the pseudo-Riemannian structure) exhaust the class of pseudo-Riemannian manifolds of constant curvature and signature (p,q) except for local isometry.

Proof. If s_o denotes the linear transformation

$$(y_1,\dots,y_{p+q},y_{p+q+1}) \to (-y_1,\dots,-y_{p+q},y_{p+q+1})$$

then the mapping $\sigma: g \to s_o g s_o$ is an involutive automorphism of $\mathbf{O}(p,q+1)$ whose differential $d\sigma$ has fixed point set $\mathfrak{o}(p,q)$ (the Lie algebra of $\mathbf{O}(p,q)$). The (-1)-eigenspace of $d\sigma$, say $\mathfrak{m}$, is spanned by the vectors

(4) $$Y_i = E_{i,p+q+1} + E_{p+q+1,i} \qquad (1 \le i \le p)\,,$$

(5) $$Y_j = E_{j,p+q+1} - E_{p+q+1,j} \qquad (p+1 \le j \le p+q)\,.$$

Here E_{ij} denotes a square matrix with entry 1 where the i^{th} row and the j^{th} column meet, all other entries being 0.

The mapping $\psi : g\mathbf{O}(p,q) \to g \cdot o$ has a differential $d\psi$ which maps $\mathfrak{m}$ bijectively onto the tangent plane $y_{p+q+1} = 1$ to Q_{-1} at o and $d\psi(X) = X{\cdot}o$ $(X \in \mathfrak{m})$. Thus

$$d\psi(Y_k) = (\delta_{1k}, \ldots, \delta_{p+q+1,k})\,, \quad (1 \leq k \leq p+q)\,.$$

Thus

$$B_{-1}(d\psi(Y_k)) = 1 \quad \text{if} \quad 1 \leq k \leq p \text{ and } -1 \text{ if } p+1 \leq k \leq p+q\,,$$

proving (i). Next, since the space (2) is symmetric its curvature tensor satisfies

$$R_o(X,Y)(Z) = [[X,Y],Z]\,,$$

where [,] is the Lie bracket. A simple computation then shows for $k \neq \ell$

$$K(\mathbf{R}Y_k + \mathbf{R}Y_\ell) = -1 \quad (1 \leq k, \ell \leq p+q)$$

and this implies (ii). Part (iii) is proved in the same way. For (iv) we first verify that the spaces listed are isotropic. Since the isotropy action of $\mathbf{O}(p,q+1)_o = \mathbf{O}(p,q)$ on $\mathfrak{m}$ is the ordinary action of $\mathbf{O}(p,q)$ on $\mathbf{R}^{p+q}$ it suffices to verify that $\mathbf{R}^{p+q}$ with the quadratic form g_o is isotropic. But we know $\mathbf{O}(p,q)$ is transitive on $g_e = +1$ and on $g_e = -1$ so it remains to show $\mathbf{O}(p,q)$ transitive on the cone $\{Y \neq 0 : g_e(Y) = 0\}$. By rotation in $\mathbf{R}^p$ and in $\mathbf{R}^q$ it suffices to verify the statement for $p = q = 1$. But for this case it is obvious. The uniqueness in (iv) follows from the general fact that a symmetric space is determined locally by its pseudo-Riemannian structure and curvature tensor at a point (see e.g. [DS], pp. 200–201). This finishes the proof.

The spaces (2) and (3) are the pseudo-Riemannian analogs of the spaces $\mathbf{O}(p,1)/\mathbf{O}(p)$, $\mathbf{O}(p+1)/\mathbf{O}(p)$ from Ch. III, §1. But the other two-point homogeneous spaces listed in Ch. III, §2–§3 have similar pseudo-Riemannian analogs (indefinite elliptic and hyperbolic spaces over **C**, **H** and **Cay**). As proved by Wolf [1967], p. 384, each non-flat isotropic pseudo-Riemannian manifold is locally isometric to one of these models.

We shall later need a lemma about the connectivity of the groups $\mathbf{O}(p,q)$. Let $I_{p,q}$ denote the diagonal matrix (d_{ij}) with

$$d_{ii} = 1 \quad (1 \leq i \leq p)\,,\, d_{jj} = -1 \quad (p+1 \leq j \leq p+q)$$

so a matrix g with transpose ${}^t g$ belongs to $\mathbf{O}(p,q)$ if and only if

$$ {}^t g I_{p,q} g = I_{p,q}\,. \tag{6}$$

If $y \in \mathbf{R}^{p+q}$ let

$$y^T = (y_1, \ldots, y_p, 0 \ldots 0), y^S = (0, \ldots, 0, y_{p+1}, \ldots, y_{p+q})$$

and for $g \in \mathbf{O}(p,q)$ let g_T and g_S denote the matrices

$$\begin{array}{ll} (g_T)_{ij} = g_{ij} & (1 \le i,j \le p)\,, \\ (g_S)_{k\ell} = g_{k\ell} & (p+1 \le k,\ell \le p+q) \end{array} .$$

If $g_1, \ldots, g_{p+q}$ denote the column vectors of the matrix g then (6) means for the scalar products

$$\begin{array}{ll} g_i^T \cdot g_i^T - g_i^S \cdot g_i^S = 1\,, & 1 \le i \le p\,, \\ g_j^T \cdot g_j^T - g_j^S \cdot g_j^S = -1\,, & p+1 \le j \le p+q\,, \\ g_j^T \cdot g_k^T = g_j^S \cdot g_k^S\,, & j \ne k\,. \end{array}$$

Lemma 1.4. *We have for each* $g \in \mathbf{O}(p,q)$

$$|\det(g_T)| \ge 1\,, \quad |\det(g_S)| \ge 1\,.$$

The components of $\mathbf{O}(p,q)$ *are obtained by*

(7) $\det g_T \ge 1$, $\det g_S \ge 1$; *(identity component)*

(8) $\det g_T \le -1$, $\det g_S \ge 1$;

(9) $\det g_T \ge -1$, $\det g_S \le -1$,

(10) $\det g_T \le -1$, $\det g_S \le -1$.

Thus $\mathbf{O}(p,q)$ *has four components if* $p \ge 1, q \ge 1$, *two components if* p *or* $q = 0$.

Proof. Consider the Gram determinant

$$\det \begin{pmatrix} g_1^T \cdot g_1^T & g_1^T \cdot g_2^T & \cdots & g_1^T \cdot g_p^T \\ g_2^T \cdot g_1^T & \cdot & & \\ \vdots & & & \\ g_p^T \cdot g_1^T & \cdots & & g_p^T \cdot g_p^T \end{pmatrix},$$

which equals $(\det g_T)^2$. Using the relations above it can also be written

$$\det \begin{pmatrix} 1 + g_1^S \cdot g_1^S & g_1^S \cdot g_2^S & \cdots & g_1^S \cdot g_p^S \\ g_2^S \cdot g_1^S & \cdot & \cdots & \\ \vdots & & & \\ g_p^S \cdot g_1^S & & & 1 + g_p^S \cdot g_p^S \end{pmatrix},$$

which equals 1 plus a sum of lower order Gram determinants each of which is still positive. Thus $(\det g_T)^2 \ge 1$ and similarly $(\det g_S)^2 \ge 1$. Assuming now $p \ge 1, q \ge 1$ consider the decomposition of $\mathbf{O}(p,q)$ into the four pieces (7), (8), (9), (10). Each of these is $\ne \emptyset$ because (8) is obtained from (7) by multiplication by $I_{1,p+q-1}$ etc. On the other hand, since the functions $g \to$

$\det(g_T)$, $g \to \det(g_S)$ are continuous on $\mathbf{O}(p,q)$ the four pieces above belong to different components of $\mathbf{O}(p,q)$. But by Chevalley [1946], p. 201, $\mathbf{O}(p,q)$ is homeomorphic to the product of $\mathbf{O}(p,q) \cap \mathbf{U}(p+q)$ with a Euclidean space. Since $\mathbf{O}(p,q) \cap \mathbf{U}(p+q) = \mathbf{O}(p,q) \cap \mathbf{O}(p+q)$ is homeomorphic to $\mathbf{O}(p) \times \mathbf{O}(q)$ it just remains to remark that $\mathbf{O}(n)$ has two components.

C. The Lorentzian Case

The isotropic Lorentzian manifolds are more restricted than one might at first think on the basis of the Riemannian case. In fact there is a theorem of Lichnerowicz and Walker [1945] (see Wolf [1967], Ch. 12) which implies that an isotropic Lorentzian manifold has constant curvature. Thus we can deduce the following result from Theorem 1.3.

Theorem 1.5. *Let X be an isotropic Lorentzian manifold (signature $(1,q)$, $q \geq 1$). Then X has constant curvature so (after a multiplication of the Lorentzian structure by a positive constant) X is locally isometric to one of the following:*

$$\mathbf{R}^{1+q}(\textit{flat, signature } (1,q)),$$
$$Q_{-1} = \mathbf{O}(1,q+1)/\mathbf{O}(1,q): \; y_1^2 - y_2^2 - \cdots - y_{q+2}^2 = -1,$$
$$Q_{+1} = \mathbf{O}(2,q)/\mathbf{O}(1,q): \; y_1^2 - y_2^2 - \cdots - y_{q+1}^2 + y_{q+2}^2 = 1,$$

the Lorentzian structure being induced by $y_1^2 - y_2^2 - \cdots \mp y_{q+2}^2$.

§2 Orbital Integrals

The orbital integrals for isotropic Lorentzian manifolds are analogs to the spherical averaging operator M^r considered in Ch. I, §1, and Ch. III, §1. We start with some geometric preparation.

For manifolds X with a Lorentzian structure g we adopt the following customary terminology: If $y \in X$ the cone

$$C_y = \{Y \in X_y : g_y(Y,Y) = 0\}$$

is called the *null cone* (or the *light cone*) in X_y with vertex y. A nonzero vector $Y \in X_y$ is said to be *timelike, isotropic* or *spacelike* if $g_y(Y,Y)$ is positive, 0, or negative, respectively. Similar designations apply to geodesics according to the type of their tangent vectors.

While the geodesics in $\mathbf{R}^{1+q}$ are just the straight lines, the geodesics in Q_{-1} and Q_{+1} can be found by the method of Ch. III, §1.

Proposition 2.1. *The geodesics in the Lorentzian quadrics Q_{-1} and Q_{+1} have the following properties:*

(i) The geodesics are the nonempty intersections of the quadrics with two-planes in $\mathbf{R}^{2+q}$ *through the origin.*

(ii) For Q_{-1} *the spacelike geodesics are closed, for* Q_{+1} *the timelike geodesics are closed.*

(iii) The isotropic geodesics are certain straight lines in $\mathbf{R}^{2+q}$*.*

Proof. Part (i) follows by the symmetry considerations in Ch. III, §1. For Part (ii) consider the intersection of Q_{-1} with the two-plane

$$y_1 = y_4 = \cdots = y_{q+2} = 0\,.$$

The intersection is the circle $y_2 = \cos t$, $y_3 = \sin t$ whose tangent vector $(0, -\sin t, \cos t, 0, \ldots, 0)$ is clearly spacelike. Since $\mathbf{O}(1, q+1)$ permutes the spacelike geodesics transitively the first statement in (ii) follows. For Q_{+1} we intersect similarly with the two-plane

$$y_2 = \cdots = y_{q+1} = 0\,.$$

For (iii) we note that the two-plane $\mathbf{R}(1, 0, \ldots, 0, 1) + \mathbf{R}(0, 1, \ldots, 0)$ intersects Q_{-1} in a pair of straight lines

$$y_1 = t, y_2 \pm 1, y_3 = \cdots = y_{q+1} = 0, y_{q+2} = t$$

which clearly are isotropic. The transitivity of $\mathbf{O}(1, q+1)$ on the set of isotropic geodesics then implies that each of these is a straight line. The argument for Q_{+1} is similar.

Lemma 2.2. *The quadrics* Q_{-1} *and* Q_{+1} $(q \geq 1)$ *are connected.*

Proof. The q-sphere being connected, the point $(y_1, \ldots, y_{q+2})$ on $Q_{\mp 1}$ can be moved continuously on $Q_{\mp 1}$ to the point

$$(y_1, (y_2^2 + \cdots + y_{q+1}^2)^{1/2}, 0, \ldots, 0, y_{q+2})$$

so the statement follows from the fact that the hyperboloids $y_1^2 - y_1^2 \mp y_3^2 = \mp 1$ are connected.

Lemma 2.3. *The identity components of* $\mathbf{O}(1, q+1)$ *and* $\mathbf{O}(2, q)$ *act transitively on* Q_{-1} *and* Q_{+1}*, respectively, and the isotropy subgroups are connected.*

Proof. The first statement comes from the general fact (see e.g [DS], pp. 121–124) that when a separable Lie group acts transitively on a connected manifold then so does its identity component. For the isotropy groups we use the description (7) of the identity component. This shows quickly that

$$\begin{aligned} \mathbf{O}_o(1, q+1) \cap \mathbf{O}(1, q) &= \mathbf{O}_o(1, q)\,, \\ \mathbf{O}_o(2, q) \cap \mathbf{O}(1, q) &= \mathbf{O}_o(1, q) \end{aligned}$$

the subscript o denoting identity component. Thus we have

$$\begin{aligned} Q_{-1} &= \mathbf{O}_o(1, q+1)/\mathbf{O}_o(1,q)\,, \\ Q_{+1} &= \mathbf{O}_o(2,q)/\mathbf{O}_o(1,q)\,, \end{aligned}$$

proving the lemma.

We now write the spaces in Theorem 1.5 in the form $X = G/H$ where $H = \mathbf{O}_o(1,q)$ and G is either $G^0 = \mathbf{R}^{1+q} \cdot \mathbf{O}_o(1,q)$ (semi-direct product) $G^- = \mathbf{O}_o(1,q+1)$ or $G^+ = \mathbf{O}_o(2,q)$. Let o denote the origin $\{H\}$ in X, that is

$$\begin{aligned} &o = (0,\ldots,0) && \text{if } X = \mathbf{R}^{1+q} \\ &o = (0,\ldots,0,1) && \text{if } X = Q_{-1} \text{ or } Q_{+1}\,. \end{aligned}$$

In the cases $X = Q_{-1}, X = Q_{+1}$ the tangent space X_o is the hyperplane $\{y_1,\ldots,y_{q+1},1\} \subset \mathbf{R}^{2+q}$.

The timelike vectors at o fill up the "interior" C_o^o of the cone C_o. The set C_o^o consists of two components. The component which contains the timelike vector

$$v_o = (-1,0,\ldots,0)$$

will be called the *solid retrograde cone* in X_o. It will be denoted by D_o. The component of the hyperboloid $g_o(Y,Y) = r^2$ which lies in D_o will be denoted $S_r(o)$. If y is any other point of X we define $C_y, D_y, S_r(y) \subset X_y$ by

$$C_y = g \cdot C_o\,,\ D_y = g \cdot D_o\,,\ S_r(y) = g \cdot S_r(o)$$

if $g \in G$ is chosen such that $g \cdot o = y$. This is a valid definition because the connectedness of H implies that $h \cdot D_o \subset D_o$. We also define

$$B_r(y) = \{Y \in D_y : 0 < g_y(Y,Y) < r^2\}\,.$$

If Exp denotes the exponential mapping of X_y into X, mapping rays through 0 onto geodesics through y we put

$$\begin{aligned} \mathbf{D}_y = \operatorname{Exp} D_y\,, \quad \mathbf{C}_y = \operatorname{Exp} C_y \\ \mathbf{S}_r(y) = \operatorname{Exp} S_r(y)\,, \quad \mathbf{B}_r(y) = \operatorname{Exp} B_r(y)\,. \end{aligned}$$

Again $\mathbf{C}_y$ and $\mathbf{D}_y$ are respectively called the *light cone* and *solid retrograde cone* in X with vertex y. For the spaces $X = Q_+$ we always assume $r < \pi$ in order that Exp will be one-to-one on $B_r(y)$ in view of Prop. 2.1(ii).

Figure IV.1 illustrates the situation for Q_{-1} in the case $q = 1$. Then Q_{-1} is the hyperboloid

$$y_1^2 - y_2^2 - y_3^2 = -1$$

and the y_1-axis is vertical. The origin o is

$$o = (0,0,1)$$

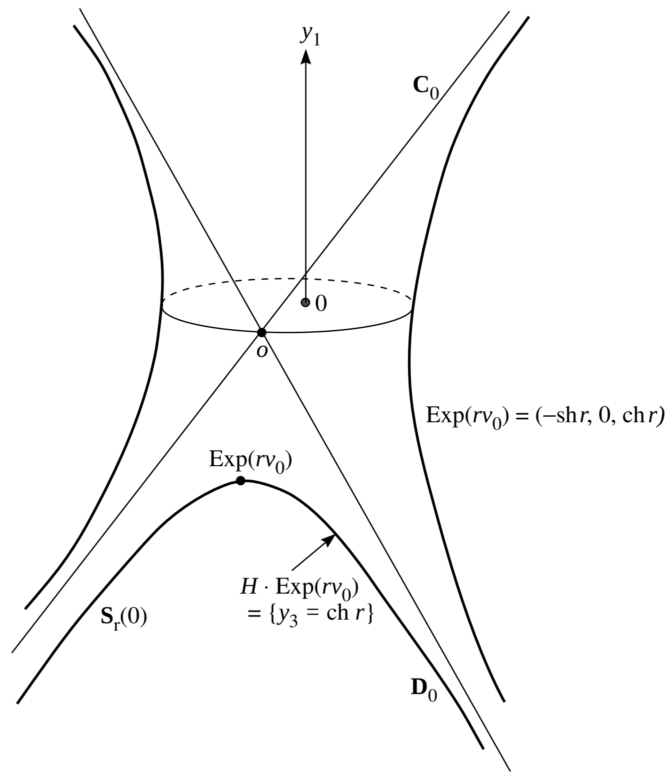

FIGURE IV.1.

and the vector $v_o = (-1, 0, 0)$ lies in the tangent space

$$(Q_{-1})_o = \{y : y_3 = 1\}$$

pointing downward. The mapping $\psi : gH \to g \cdot o$ has differential $d\psi : \mathfrak{m} \to (Q_{-1})_o$ and

$$d\psi(E_{1\,3} + E_{3\,1}) = -v_o$$

in the notation of (4). The geodesic tangent to v_o at o is

$$t \to \operatorname{Exp}(tv_o) = \exp(-t(E_{1\,3} + E_{3\,1})) \cdot o = (-\sinh t, 0, \cosh t)$$

and this is the section of Q_{-1} with the plane $y_2 = 0$. Note that since H preserves each plane $y_3 =$ const., the "sphere" $\mathbf{S}_r(o)$ is the plane section $y_3 = \cosh r, y_1 < 0$ with Q_{-1}.

Lemma 2.4. *The negative of the Lorentzian structure on $X = G/H$ induces on each $\mathbf{S}_r(y)$ a Riemannian structure of constant negative curvature $(q > 1)$.*

Proof. The manifold X being isotropic the group $H = \mathbf{O}_o(1,q)$ acts transitively on $\mathbf{S}_r(o)$. The subgroup leaving fixed the geodesic from o with tangent vector v_o is $\mathbf{O}_o(q)$. This implies the lemma.

Lemma 2.5. *The timelike geodesics from y intersect $\mathbf{S}_r(y)$ under a right angle.*

Proof. By the group invariance it suffices to prove this for $y = o$ and the geodesic with tangent vector v_o. For this case the statement is obvious.

Let $\tau(g)$ denote the translation $xH \to gxH$ on G/H and for $Y \in \mathfrak{m}$ let T_Y denote the linear transformation $Z \to [Y,[Y,Z]]$ of $\mathfrak{m}$ into itself. As usual, we identify $\mathfrak{m}$ with $(G/H)_o$.

Lemma 2.6. *The exponential mapping* $\mathrm{Exp} : \mathfrak{m} \to G/H$ *has differential*

$$d\mathrm{Exp}_Y = d\tau(\exp Y) \circ \sum_0^\infty \frac{T_Y^n}{(2n+1)!} \qquad (Y \in \mathfrak{m})\,.$$

For the proof see [DS], p. 215.

Lemma 2.7. *The linear transformation*

$$A_Y = \sum_0^\infty \frac{T_Y^n}{(2n+1)!}$$

has determinant given by

$$\det A_Y = \left\{\frac{\sinh(g(Y,Y))^{1/2}}{(g(Y,Y))^{1/2}}\right\}^q \quad \textit{for } Q_{-1}$$

$$\det A_Y = \left\{\frac{\sin(g(Y,Y))^{1/2}}{(g(Y,Y))^{1/2}}\right\}^q \quad \textit{for } Q_{+1}$$

for Y timelike.

Proof. Consider the case of Q_{-1}. Since $\det(A_Y)$ is invariant under H it suffices to verify this for $Y = cY_1$ in (4), where $c \in \mathbf{R}$. We have $c^2 = g(Y,Y)$ and $T_{Y_1}(Y_j) = Y_j \quad (2 \le j \le q+1)$. Thus T_Y has the eigenvalue 0 and $g(Y,Y)$; the latter is a q-tuple eigenvalue. This implies the formula for the determinant. The case Q_{+1} is treated in the same way.

From this lemma and the description of the geodesics in Prop. 2.1 we can now conclude the following result.

Proposition 2.8. *(i) The mapping* $\mathrm{Exp} : \mathfrak{m} \to Q_{-1}$ *is a diffeomorphism of D_o onto $\mathbf{D}_o$.*

(ii) The mapping $\mathrm{Exp} : \mathfrak{m} \to Q_{+1}$ *gives a diffeomorphism of $B_\pi(o)$ onto $\mathbf{B}_\pi(o)$.*

Let dh denote a bi-invariant measure on the unimodular group H. Let $u \in \mathcal{D}(X)$, $y \in X$ and $r > 0$. Select $g \in G$ such that $g \cdot o = y$ and select $x \in \mathbf{S}_r(o)$. Consider the integral

$$\int_H u(gh \cdot x)\, dh\,.$$

Since the subgroup $K \subset H$ leaving x fixed is compact it is easy to see that the set

$$C_{g,x} = \{h \in H : gh \cdot x \in \text{ support } (u)\}$$

is compact; thus the integral above converges. By the bi-invariance of dh it is independent of the choice of g (satisfying $g \cdot o = y$) and of the choice of $x \in \mathbf{S}_r(o)$. In analogy with the Riemannian case (Ch. III, §1) we thus define the operator M^r (*the orbital integral*) by

$$(M^r u)(y) = \int_H u(gh \cdot x)\, dh\,. \tag{11}$$

If g and x run through suitable compact neighborhoods, the sets $C_{g,x}$ are enclosed in a fixed compact subset of H so $(M^r u)(y)$ depends smoothly on both r and y. It is also clear from (11) that the operator M^r is invariant under the action of G: if $m \in G$ and $\tau(m)$ denotes the transformation $nH \to mnH$ of G/H onto itself then

$$M^r(u \circ \tau(m)) = (M^r u) \circ \tau(m)\,.$$

If dk denotes the normalized Haar measure on K we have by standard invariant integration

$$\int_H u(h \cdot x)\, dh = \int_{H/K} d\dot{h} \int_K u(hk \cdot x)\, dk = \int_{H/K} u(h \cdot x)\, d\dot{h}\,,$$

where $d\dot{h}$ is an H-invariant measure on H/K. But if $d\mathbf{w}^r$ is the volume element on $\mathbf{S}_r(o)$ (cf. Lemma 2.4) we have by the uniqueness of H-invariant measures on the space $H/K \approx \mathbf{S}_r(o)$ that

$$\int_H u(h \cdot x)\, dh = \frac{1}{A(r)} \int_{\mathbf{S}_r(o)} u(z)\, d\mathbf{w}^r(z)\,, \tag{12}$$

where $A(r)$ is a positive scalar. But since g is an isometry we deduce from (12) that

$$(M^r u)(y) = \frac{1}{A(r)} \int_{\mathbf{S}_r(y)} u(z)\, d\mathbf{w}^r(z)\,.$$

Now we have to determine $A(r)$.

Lemma 2.9. *For a suitable fixed normalization of the Haar measure dh on H we have*

$$A(r) = r^q\,, \quad (\sinh r)^q\,, \quad (\sin r)^q$$

for the cases

$$\mathbf{R}^{1+q}\,, \quad \mathbf{O}(1,q+1)/\mathbf{O}(1,q)\,, \quad \mathbf{O}(2,q)/\mathbf{O}(1,q)\,,$$

respectively.

Proof. The relations above show that $dh = A(r)^{-1}\,d\mathbf{w}^r\,dk$. The mapping $\mathrm{Exp} : D_o \to \mathbf{D}_o$ preserves length on the geodesics through o and maps $S_r(o)$ onto $\mathbf{S}_r(o)$. Thus if $z \in S_r(o)$ and Z denotes the vector from 0 to z in X_o the ratio of the volume of elements of $\mathbf{S}_r(o)$ and $S_r(o)$ at z is given by $\det(d\mathrm{Exp}_Z)$. Because of Lemmas 2.6–2.7 this equals

$$1, \left(\frac{\sinh r}{r}\right)^q, \left(\frac{\sin r}{r}\right)^q$$

for the three respective cases. But the volume element $d\omega^r$ on $S_r(o)$ equals $r^q d\omega^1$. Thus we can write in the three respective cases

$$dh = \frac{r^q}{A(r)}\,d\omega^1\,dk\,, \quad \frac{\sinh^q r}{A(r)}\,d\omega^1\,dk\,, \quad \frac{\sin^q r}{A(r)}\,d\omega^1\,dk\,.$$

But we can once and for all normalize dh by $dh = d\omega^1\,dk$ and for this choice our formulas for $A(r)$ hold.

Let $\Box$ denote the *wave operator* on $X = G/H$, that is the Laplace-Beltrami operator for the Lorentzian structure g.

Lemma 2.10. *Let $y \in X$. On the solid retrograde cone $\mathbf{D}_y$, the wave operator $\Box$ can be written*

$$\Box = \frac{\partial^2}{\partial r^2} + \frac{1}{A(r)}\,\frac{dA}{dr}\,\frac{\partial}{\partial r} - L_{\mathbf{S}_r(y)}\,,$$

where $L_{\mathbf{S}_r(y)}$ is the Laplace-Beltrami operator on $\mathbf{S}_r(y)$.

Proof. We can take $y = o$. If $(\theta_1, \dots, \theta_q)$ are coordinates on the "sphere" $S_1(o)$ in the flat space X_o then $(r\theta_1, \dots, r\theta_q)$ are coordinates on $S_r(o)$. The Lorentzian structure on D_o is therefore given by

$$dr^2 - r^2\,d\theta^2\,,$$

where $d\theta^2$ is the Riemannian structure of $S_1(o)$. Since A_Y in Lemma 2.7 is a diagonal matrix with eigenvalues 1 and $r^{-1}A(r)^{1/q}$ (q-times) it follows from Lemma 2.6 that the image $\mathbf{S}_r(o) = \mathrm{Exp}(S_r(o))$ has Riemannian structure

$r^2\, d\theta^2$, $\sinh^2 r\, d\theta^2$ and $\sin^2 r\, d\theta^2$ in the cases $\mathbf{R}^{1+q}$, Q_{-1} and Q_{+1}, respectively. By the perpendicularity in Lemma 2.5 it follows that the Lorentzian structure on $\mathbf{D}_o$ is given by

$$dr^2 - r^2\, d\theta^2\,, \quad dr^2 - \sinh^2 r\, d\theta^2\,, \quad dr^2 - \sin^2 r\, d\theta^2$$

in the three respective cases. Now the lemma follows immediately.

The operator M^r is of course the Lorentzian analog to the spherical mean value operator for isotropic Riemannian manifolds. We shall now prove that in analogy to the Riemannian case (cf. (41), Ch. III) the operator M^r commutes with the wave operator $\square$.

Theorem 2.11. *For each of the isotropic Lorentz spaces $X = G^-/H$, G^+/H or G^0/H the wave operator $\square$ and the orbital integral M^r commute:*

$$\square M^r u = M^r \square u \qquad \text{for } u \in \mathcal{D}(X)\,.$$

(For G^+/H we assume $r < \pi$.)

Given a function u on G/H we define the function $\widetilde{u}$ on G by $\widetilde{u}(g) = u(g \cdot o)$.

Lemma 2.12. *There exists a differential operator $\widetilde{\square}$ on G invariant under all left and all right translations such that*

$$\widetilde{\square}\widetilde{u} = (\square u)^{\sim} \quad \text{for } u \in \mathcal{D}(X)\,.$$

Proof. We consider first the case $X = G^-/H$. The bilinear form

$$K(Y, Z) = \tfrac{1}{2}\operatorname{Tr}(YZ)$$

on the Lie algebra $\mathfrak{o}(1, q+1)$ of G^- is nondegenerate; in fact K is nondegenerate on the complexification $\mathfrak{o}(q+2, \mathbf{C})$ consisting of all complex skew symmetric matrices of order $q+2$. A simple computation shows that in the notation of (4) and (5)

$$K(Y_1, Y_1) = 1\,, \quad K(Y_j, Y_j) = -1 \quad (2 \leq j \leq q+1)\,.$$

Since K is symmetric and nondegenerate there exists a unique left invariant pseudo-Riemannian structure $\widetilde{K}$ on G^- such that $\widetilde{K}_e = K$. Moreover, since K is invariant under the conjugation $Y \to gYg^{-1}$ of $\mathfrak{o}(1, q+1)$, $\widetilde{K}$ is also right invariant. Let $\widetilde{\square}$ denote the corresponding Laplace-Beltrami operator on G^-. Then $\widetilde{\square}$ is invariant under all left and right translations on G^-. Let $u = \mathcal{D}(X)$. Since $\widetilde{\square}\widetilde{u}$ is invariant under all right translations from H there is a unique function $v \in \mathcal{E}(X)$ such that $\widetilde{\square}\widetilde{u} = \widetilde{v}$. The mapping $u \to v$ is a differential operator which at the origin must coincide with $\square$, that is $\widetilde{\square}\widetilde{u}(e) = \square u(o)$. Since, in addition, both $\square$ and the operator $u \to v$ are invariant under the action of G^- on X it follows that they coincide. This proves $\widetilde{\square}\widetilde{u} = (\square u)^{\sim}$.

The case $X = G^+/H$ is handled in the same manner. For the flat case $X = G^0/H$ let

$$Y_j = (0, \dots, 1, \dots, 0)\,,$$

the j^{th} coordinate vector on $\mathbf{R}^{1+q}$. Then $\square = Y_1^2 - Y_2^2 - \cdots - Y_{q+1}^2$. Since $\mathbf{R}^{1+q}$ is naturally embedded in the Lie algebra of G^0 we can extend Y_j to a left invariant vector field $\widetilde{Y}_j$ on G^0. The operator

$$\widetilde{\square} = \widetilde{Y}_1^2 - \widetilde{Y}_2^2 - \cdots - \widetilde{Y}_{q+1}^2$$

is then a left and right invariant differential operator on G^0 and again we have $\widetilde{\square}\widetilde{u} = (\square u)^{\sim}$. This proves the lemma.

We can now prove Theorem 2.11. If $g \in G$ let $L(g)$ and $R(g)$, respectively, denote the left and right translations $\ell \to g\ell$, and $\ell \to \ell g$ on G. If $\ell \cdot o = x, x \in \mathbf{S}_r(o)$ $(r > 0)$ and $g \cdot o = y$ then

$$(M^r u)(y) = \int_H \widetilde{u}(gh\ell)\,dh$$

because of (11). As g and ℓ run through sufficiently small compact neighborhoods the integration takes place within a fixed compact subset of H as remarked earlier. Denoting by subscript the argument on which a differential operator is to act we shall prove the following result.

Lemma 2.13.

$$\widetilde{\square}_\ell \left(\int_H \widetilde{u}(gh\ell)\,dh \right) = \int_H (\widetilde{\square}\widetilde{u})(gh\ell)\,dh = \widetilde{\square}_g \left(\int_H \widetilde{u}(gh\ell)\,dh \right).$$

Proof. The first equality sign follows from the left invariance of $\widetilde{\square}$. In fact, the integral on the left is

$$\int_H (\widetilde{u} \circ L(gh))(\ell)\,dh$$

so

$$\widetilde{\square}_\ell \left(\int_H \widetilde{u}(gh\ell)\,dh \right) = \int_H \left[\widetilde{\square}(\widetilde{u} \circ L(gh)) \right] (\ell)\,dh$$
$$= \int_H \left[(\widetilde{\square}\widetilde{u}) \circ L(gh) \right] (\ell)\,dh = \int_H (\widetilde{\square}\widetilde{u})(gh\ell)\,dh\,.$$

The second equality in the lemma follows similarly from the right invariance of $\widetilde{\square}$. But this second equality is just the commutativity statement in Theorem 2.11.

Lemma 2.13 also implies the following analog of the Darboux equation in Lemma 3.2, Ch. I.

Corollary 2.14. *Let $u \in \mathcal{D}(X)$ and put*

$$U(y,z) = (M^r u)(y) \quad \textit{if } z \in \mathbf{S}_r(o)\,.$$

Then

$$\Box_y(U(y,z)) = \Box_z(U(y,z))\,.$$

Remark 2.15. In $\mathbf{R}^n$ the solutions to the Laplace equation $Lu = 0$ are characterized by the spherical mean-value theorem $M^r u = u$ (all r). This can be stated equivalently: $M^r u$ is a constant in r. In this latter form the mean value theorem holds for the solutions of the wave equation $\Box u = 0$ in an isotropic Lorentzian manifold: *If u satisfies $\Box u = 0$ and if u is suitably small at ∞ then $(M^r u)(o)$ is constant in r.* For a precise statement and proof see Helgason [1959], p. 289. For $\mathbf{R}^2$such a result had also been noted by Ásgeirsson.

§3 Generalized Riesz Potentials

In this section we generalize part of the theory of Riesz potentials (Ch. V, §5) to isotropic Lorentz spaces.

Consider first the case

$$X = Q_{-1} = G^-/H = \mathbf{O}_o(1,n)/\mathbf{O}_o(1,n-1)$$

of dimension n and let $f \in \mathcal{D}(X)$ and $y \in X$. If $z = \mathrm{Exp}_y Y$ $(Y \in D_y)$ we put $r_{yz} = g(Y,Y)^{1/2}$ and consider the integral

$$(I_-^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_{\mathbf{D}_y} f(z) \sinh^{\lambda-n}(r_{yz})\, dz\,, \tag{13}$$

where dz is the volume element on X, and

$$H_n(\lambda) = \pi^{(n-2)/2} 2^{\lambda-1} \Gamma\left(\lambda/2\right) \Gamma\left((\lambda+2-n)/2\right)\,. \tag{14}$$

The integral converges for $\mathrm{Re}\,\lambda \geq n$. We transfer the integral in (13) over to D_y via the diffeomorphism $\mathrm{Exp}(= \mathrm{Exp}_y)$. Since

$$dz = dr d\mathbf{w}^r = dr \left(\frac{\sinh r}{r}\right)^{n-1} d\omega^r$$

and since $dr d\omega^r$ equals the volume element dZ on D_y we obtain

$$(I^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_{D_y} (f \circ \mathrm{Exp})(Z) \Big(\frac{\sinh r}{r}\Big)^{\lambda-1} r^{\lambda-n}\, dZ\,,$$

where $r = g(Z,Z)^{1/2}$. This has the form

$$\frac{1}{H_n(\lambda)} \int_{D_y} h(Z,\lambda) r^{\lambda-n}\, dZ\,, \tag{15}$$

where $h(Z,\lambda)$, as well as each of its partial derivatives with respect to the first argument, is holomorphic in λ and h has compact support in the first variable. The methods of Riesz [1949], Ch. III, can be applied to such integrals (15). In particular we find that the function $\lambda \to (I_-^\lambda f)(y)$ which by its definition is holomorphic for $\operatorname{Re}\lambda > n$ admits a holomorphic continuation to the entire λ-plane and that its value at $\lambda = 0$ is $h(0,0) = f(y)$. (In Riesz' treatment $h(Z,\lambda)$ is independent of λ, but his method still applies.) Denoting the holomorphic continuation of (13) by $(I_-^\lambda)f(y)$ we have thus obtained

$$I_-^0 f = f\,. \tag{16}$$

We would now like to differentiate (13) with respect to y. For this we write the integral in the form $\int_F f(z)K(y,z)\,dz$ over a bounded region F which properly contains the intersection of the support of f with the closure of $\mathbf{D}_y$. The kernel $K(y,z)$ is defined as $\sinh^{\lambda-n} r_{yz}$ if $z \in \mathbf{D}_y$, otherwise 0. For $\operatorname{Re}\lambda$ sufficiently large, $K(y,z)$ is twice continuously differentiable in y so we can deduce for such λ that $I_-^\lambda f$ is of class C^2 and that

$$(\Box I_-^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_{\mathbf{D}_y} f(z)\Box_y(\sinh^{\lambda-n} r_{yz})\,dz\,. \tag{17}$$

Moreover, given $m \in \mathbb{Z}^+$ we can find k such that $I_-^\lambda f \in C^m$ for $\operatorname{Re}\lambda > k$ (and all f). Using Lemma 2.10 and the relation

$$\frac{1}{A(r)}\,\frac{dA}{dr} = (n-1)\coth r$$

we find

$$\begin{aligned}\Box_y(\sinh^{\lambda-n} r_{yz}) &= \Box_z(\sinh^{\lambda-n} r_{yz})\\ &= (\lambda-n)(\lambda-1)\sinh^{\lambda-n} r_{yz} + (\lambda-n)(\lambda-2)\sinh^{\lambda-n-2} r_{yz}\,.\end{aligned}$$

We also have

$$H_n(\lambda) = (\lambda-2)(\lambda-n)H_n(\lambda-2)$$

so substituting into (17) we get

$$\Box I_-^\lambda f = (\lambda-n)(\lambda-1)I_-^\lambda f + I_-^{\lambda-2} f\,.$$

Still assuming $\operatorname{Re}\lambda$ large we can use Green's formula to express the integral

$$\int_{\mathbf{D}_y} [f(z)\Box_z(\sinh^{\lambda-n} r_{yz}) - \sinh^{\lambda-n} r_{yz}(\Box f)(z)]\,dz \tag{18}$$

as a surface integral over a part of $\mathbf{C}_y$ (on which $\sinh^{\lambda-n} r_{yz}$ and its first order derivatives vanish) together with an integral over a surface inside $\mathbf{D}_y$

(on which f and its derivatives vanish). Hence the expression (18) vanishes so we have proved the relations

$$\square(I_-^\lambda f) = I_-^\lambda(\square f) \tag{19}$$

$$I_-^\lambda(\square f) = (\lambda-n)(\lambda-1)I_-^\lambda f + I_-^{\lambda-2} f \tag{20}$$

for $\operatorname{Re}\lambda > k$, k being some number (independent of f).

Since both sides of (20) are holomorphic in λ this relation holds for all $\lambda \in \mathbf{C}$. We shall now deduce that for each $\lambda \in \mathbf{C}$, we have $I_-^\lambda f \in \mathcal{E}(X)$ and (19) holds. For this we observe by iterating (20) that for each $p \in \mathbb{Z}^+$

$$I_-^\lambda f = I_-^{\lambda+2p}(Q_p(\square)f)\,, \tag{21}$$

Q_p being a certain p^{th}-degree polynomial. Choosing p arbitrarily large we deduce from the remark following (17) that $I_-^\lambda f \in \mathcal{E}(X)$; secondly (19) implies for $\operatorname{Re}\lambda + 2p > k$ that

$$\square I_-^{\lambda+2p}(Q_p(\square)f) = I_-^{\lambda+2p}(Q_p(\square)\square f)\,.$$

Using (21) again this means that (19) holds for all λ.

Putting $\lambda = 0$ in (20) we get

$$I_-^{-2} = \square f - nf\,. \tag{22}$$

Remark 3.1. In Riesz' paper [1949], p. 190, an analog I^α of the potentials in Ch. V, §5, is defined for any analytic Lorentzian manifold. These potentials I^α are however different from our I_-^λ and satisfy the equation $I^{-2}f = \square f$ in contrast to (22).

We consider next the case

$$X = Q_{+1} = G^+/H = \mathbf{O}_o(2, n-1)/\mathbf{O}_o(1, n-1)$$

and we define for $f \in \mathcal{D}(X)$

$$(I_+^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_{\mathbf{D}_y} f(z) \sin^{\lambda-n}(r_{yz})\, dz\,. \tag{23}$$

Again $H_n(\lambda)$ is given by (14) and dz is the volume element. In order to bypass the difficulties caused by the fact that the function $z \to \sin r_{yz}$ vanishes on $\mathbf{S}_\pi$ we assume that f has support disjoint from $\mathbf{S}_\pi(o)$. Then the support of f is disjoint from $\mathbf{S}_\pi(y)$ for all y in some neighborhood of o in X. We can then prove just as before that

$$(I_+^0 f)(y) = f(y) \tag{24}$$

$$(\square I_+^\lambda f)(y) = (I_+^\lambda \square f)(y) \tag{25}$$

$$(I_+^\lambda \square f)(y) = -(\lambda-n)(\lambda-1)(I_+^\lambda f)(y) + (I_+^{\lambda-2} f)(y) \tag{26}$$

for all $\lambda \in \mathbf{C}$. In particular

(27) $$I_+^{-2} f = \square f + nf\,.$$

Finally we consider the flat case

$$X = \mathbf{R}^n = G^0/H = \mathbf{R}^n \cdot \mathbf{O}_o(1,n-1)/\mathbf{O}_o(1,n-1)$$

and define

$$(I_o^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_{\mathbf{D}_y} f(z) r_{yz}^{\lambda-n}\, dz\,.$$

These are the potentials defined by Riesz in [1949], p. 31, who proved

(28) $$I_o^0 f = f,\ \square I_o^\lambda f = I_o^\lambda \square f = I_o^{\lambda-2} f\,.$$

§4 Determination of a Function from its Integral over Lorentzian Spheres

In a Riemannian manifold a function is determined in terms of its spherical mean values by the simple relation $f = \lim_{r\to 0} M^r f$. We shall now solve the analogous problem for an even-dimensional isotropic Lorentzian manifold and express a function f in terms of its orbital integrals $M^r f$. Since the spheres $\mathbf{S}_r(y)$ do not shrink to a point as $r \to 0$ the formula (cf. Theorem 4.1) below is quite different.

For the solution of the problem we use the geometric description of the wave operator $\square$ developed in §2, particularly its commutation with the orbital integral M^r, and combine this with the results about the generalized Riesz potentials established in §3.

We consider first the negatively curved space $X = G^-/H$. Let $n = \dim X$ and assume n even. Let $f \in \mathcal{D}(X)$, $y \in X$ and put $F(r) = (M^r f)(y)$. Since the volume element dz on $\mathbf{D}_y$ is given by $dz = dr\, d\mathbf{w}^r$ we obtain from (12) and Lemma 2.9 ,

(29) $$(I_-^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_0^\infty \sinh^{\lambda-1} r F(r)\, dr\,.$$

Let $Y_1, \ldots, Y_n$ be a basis of X_y such that the Lorentzian structure is given by

$$g_y(Y) = y_1^2 - y_2^2 - \cdots - y_n^2\,, \quad Y = \sum_1^n y_i Y_i\,.$$

If $\theta_1, \ldots, \theta_{n-2}$ are geodesic polar coordinates on the unit sphere in $\mathbf{R}^{n-1}$ we put

$$\begin{aligned} y_1 &= -r\cosh\zeta \qquad (0 \le \zeta < \infty, 0 < r < \infty)\\ y_2 &= r\sinh\zeta\cos\theta_1\\ &\vdots\\ y_n &= r\sinh\zeta\sin\theta_1\ldots\sin\theta_{n-2}\,. \end{aligned}$$

Then $(r, \zeta, \theta_1, \ldots, \theta_{n-2})$ are coordinates on the retrograde cone D_y and the volume element on $S_r(y)$ is given by

$$d\omega^r = r^{n-1} \sinh^{n-2} \zeta \, d\zeta \, d\omega^{n-2}$$

where $d\omega^{n-2}$ is the volume element on the unit sphere in $\mathbf{R}^{n-1}$. It follows that

$$d\mathbf{w}^r = \sinh^{n-1} r \sinh^{n-2} \zeta \, d\zeta \, d\omega^{n-2}$$

and therefore

$$F(r) = \iint (f \circ \mathrm{Exp})(r, \zeta, \theta_1, \ldots, \theta_{n-2}) \sinh^{n-2} \zeta d\zeta d\omega^{n-2} , \tag{30}$$

where for simplicity

$$(r, \zeta, \theta_1, \ldots, \theta_{n-2})$$

stands for

$$(-r \cosh \zeta, r \sinh \zeta \cos \theta_1, \ldots, r \sinh \zeta \sin \theta_1 \ldots \sin \theta_{n-2}) .$$

Now select A such that $f \circ \mathrm{Exp}$ vanishes outside the sphere $y_1^2 + \cdots + y_n^2 = A^2$ in X_y. Then, in the integral (30), the range of ζ is contained in the interval $(0, \zeta_o)$ where

$$r^2 \cosh^2 \zeta_o + r^2 \sinh^2 \zeta_o = A^2 .$$

Then

$$r^{n-2} F(r) = \int_{\mathbf{S}^{n-2}} \int_0^{\zeta_o} (f \circ \mathrm{Exp})(r, \zeta, (\theta))(r \sinh \zeta)^{n-2} \, d\zeta \, d\omega^{n-2} .$$

Since

$$|r \sinh \zeta| \leq r e^{\zeta} \leq 2A \text{ for } \zeta \leq \zeta_o$$

this implies

$$|r^{n-2} (M^r f)(y)| \leq C A^{n-2} \sup |f| , \tag{31}$$

where C is a constant independent of r. Also substituting $t = r \sinh \zeta$ in the integral above, the ζ-integral becomes

$$\int_0^k \varphi(t) t^{n-2} (r^2 + t^2)^{-1/2} \, dt ,$$

where $k = [(A^2 - r^2)/2]^{1/2}$ and φ is bounded. Thus if $n > 2$ the limit

$$a = \lim_{r \to 0} \sinh^{n-2} r F(r) \quad n > 2 \tag{32}$$

exist and is $\not\equiv 0$. Similarly, we find for $n = 2$ that the limit

$$b = \lim_{r \to 0} (\sinh r) F'(r) \qquad (n = 2) \tag{33}$$

exists.

Consider now the case $n > 2$. We can rewrite (29) in the form

$$(I_-^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_0^A \sinh^{n-2} r F(r) \sinh^{\lambda-n+1} r \, dr \,,$$

where $F(A) = 0$. We now evaluate both sides for $\lambda = n-2$. Since $H_n(\lambda)$ has a simple pole for $\lambda = n-2$ the integral has at most a simple pole there and the residue is

$$\lim_{\lambda \to n-2} (\lambda - n+2) \int_0^A \sinh^{n-2} r F(r) \sinh^{\lambda-n+1} r \, dr \,.$$

Here we can take λ real and greater than $n-2$. This is convenient since by (32) the integral is then absolutely convergent and we do not have to think of it as an implicitly given holomorphic extension. We split the integral in two parts

$$(\lambda-n+2) \int_0^A (\sinh^{n-2} r F(r) - a) \sinh^{\lambda-n+1} r \, dr$$
$$+ a(\lambda - n + 2) \int_0^A \sinh^{\lambda-n+1} r \, dr \,.$$

For the last term we use the relation

$$\lim_{\mu \to 0+} \mu \int_0^A \sinh^{\mu-1} r \, dr = \lim_{\mu \to 0+} \mu \int_0^{\sinh A} t^{\mu-1}(1+t^2)^{-1/2} \, dt = 1$$

by (38) in Chapter V. For the first term we can for each $\epsilon > 0$ find a $\delta > 0$ such that

$$|\sinh^{n-2} r F(r) - a| < \epsilon \quad \text{for } 0 < r < \delta \,.$$

If $N = \max |\sinh^{n-2} r F(r)|$ we have for $n-2 < \lambda < n-1$ the estimate

$$\left|(\lambda-n+2) \int_\delta^A (\sinh^{n-2} r F(r) - a) \sinh^{\lambda-n+1} r \, dr\right|$$
$$\leq (\lambda-n+2)(N + |a|)(A-\delta)(\sinh \delta)^{\lambda-n+1} \,;$$
$$\left|(\lambda+n-2) \int_0^\delta (\sinh^{n-2} r F(r) - a) \sinh^{\lambda-n+1} r \, dr\right|$$
$$\leq \epsilon(\lambda - n + 2) \int_0^\delta r^{\lambda-n+1} \, dr = \epsilon \delta^{\lambda-n+2} \,.$$

Taking $\lambda - (n-2)$ small enough the right hand side of each of these inequalities is $< 2\epsilon$. We have therefore proved

$$\lim_{\lambda \to n-2} (\lambda - n + 2) \int_0^\infty \sinh^{\lambda-1} r F(r) \, dr = \lim_{r \to 0} \sinh^{n-2} r \, F(r) \,.$$

Taking into account the formula for $H_n(\lambda)$ we have proved for the integral (29):

$$I_-^{n-2} f = (4\pi)^{(2-n)/2} \frac{1}{\Gamma((n-2)/2)} \lim_{r\to o} \sinh^{n-2} r \; M^r f \,. \tag{34}$$

On the other hand, using formula (20) recursively we obtain for $u \in \mathcal{D}(X)$

$$I_-^{n-2}(Q(\square)u) = u$$

where

$$Q(\square) = (\square + (n-3)2)(\square + (n-5)4)\cdots(\square + 1(n-2))\,.$$

We combine this with (34) and use the commutativity $\square M^r = M^r \square$. This gives

$$u = (4\pi)^{(2-n)/2} \frac{1}{\Gamma((n-2)/2)} \lim_{r\to 0} \sinh^{n-2} r \, Q(\square) M^r u \,. \tag{35}$$

Here we can for simplicity replace $\sinh r$ by r.

For the case $n = 2$ we have by (29)

$$(I_-^2 f)(y) = \frac{1}{H_2(2)} \int_0^\infty \sinh r F(r)\, dr \,. \tag{36}$$

This integral which in effect only goes from 0 to A is absolutely convergent because our estimate (31) shows (for $n = 2$) that $rF(r)$ is bounded near $r = 0$. But using (20), Lemma 2.10, Theorem 2.11 and Cor. 2.14, we obtain for $u \in \mathcal{D}(X)$

$$\begin{aligned} u &= I_-^2 \square u = \tfrac{1}{2} \int_0^\infty \sinh r M^r \square u \, dr \\ &= \tfrac{1}{2} \int_0^\infty \sinh r \square M^r u \, dr = \tfrac{1}{2} \int_0^\infty \sinh r \left(\frac{d^2}{dr^2} + \coth r \frac{d}{dr} \right) M^r u \, dr \\ &= \tfrac{1}{2} \int_0^\infty \frac{d}{dr} \left(\sinh r \frac{d}{dr} M^r u \right) dr = -\tfrac{1}{2} \lim_{r\to 0} \sinh r \frac{d(M^r u)}{dr} \,. \end{aligned}$$

This is the substitute for (35) in the case $n = 2$.

The spaces G^+/H and G^o/H can be treated in the same manner. We have thus proved the following principal result of this chapter.

Theorem 4.1. *Let X be one of the isotropic Lorentzian manifolds G^-/H, G^o/H, G^+/H. Let κ denote the curvature of X ($\kappa = -1, 0, +1$) and assume $n = \dim X$ to be even, $n = 2m$. Put*

$$Q(\square) = (\square - \kappa(n-3)2)(\square - \kappa(n-5)4)\cdots(\square - \kappa 1(n-2))\,.$$

Then if $u \in \mathcal{D}(X)$

$$\begin{aligned} u &= c \lim_{r\to 0} r^{n-2} Q(\Box)(M^r u)\,, \qquad (n \neq 2) \\ u &= \tfrac{1}{2} \lim_{r\to 0} r \frac{d}{dr}(M^r u) \qquad (n = 2)\,. \end{aligned}$$

Here $c^{-1} = (4\pi)^{m-1}(m-2)!$ *and* $\Box$ *is the Laplace-Beltrami operator on* X*.*

§5 Orbital Integrals and Huygens' Principle

We shall now write out the limit in (35) and thereby derive a statement concerning Huygens' principle for $\Box$. As $r \to 0$, $\mathbf{S}_r(o)$ has as limit the boundary $C_R = \partial \mathbf{D}_o - \{o\}$ which is still an H-orbit. The limit

(37) $$\lim_{r\to 0} r^{n-2}(M_r v)(o) \qquad v \in C_c(X - o)$$

is by (31)–(32) a positive H-invariant functional with support in the H-orbit C_R, which is closed in $X - o$. Thus the limit (37) only depends on the restriction $v|C_R$. Hence it is "the" H-invariant measure on C_R and we denote it by μ. Thus

(38) $$\lim_{r\to 0} r^{n-2}\,(M_r v)(o) = \int_{C_R} v(z)\, d\mu(z)\,.$$

To extend this to $u \in \mathcal{D}(X)$, let $A > 0$ be arbitrary and let φ be a "smoothed out" characteristic function of $\operatorname{Exp} B_A$. Then if

$$u_1 = u\varphi\,,\; u_2 = u(1-\varphi)$$

we have

$$\begin{aligned} &\left| r^{n-2}(M^r u)(o) - \int_{C_R} u(z)\, d\mu(z) \right| \\ &\quad \le \left| r^{n-2}(M^r u_1)(o) - \int_{C_R} u_1(z)\, d\mu(z) \right| + \left| r^{n-2}(M^r u_2)(o) - \int_{C_R} u_2(z)\, d\mu(z) \right|\,. \end{aligned}$$

By (31) the first term on the right is $O(A)$ uniformly in r and by (38) the second tends to 0 as $r \to 0$. Since A is arbitrary (38) holds for $u \in \mathcal{D}(X)$.

Corollary 5.1. *Let* $n = 2m\,(m > 1)$ *and* δ *the delta distribution at* o*. Then*

(39) $$\delta = c\, Q(\Box)\mu\,,$$

where $c^{-1} = (4\pi)^{m-1}(m-2)!$.

In fact, by (35), (38) and Theorem 2.11

$$u = c \lim_{r \to 0} r^{n-2}(M^r Q(\Box)u)(o) = c \int_{C_R} (Q(\Box)u)(z)\, d\mu(z)$$

and this is (39).

Remark 5.2. Formula (39) shows that each factor

$$\Box_k = \Box - \kappa(n-k)(k-1) \quad k = 3, 5, \ldots, n-1 \tag{40}$$

in $Q(\Box)$ has fundamental solution supported on the *retrograde conical surface* $\overline{C}_R$. This is known to be the equivalent to the validity of Huygens' principle for the Cauchy problem for the equation $\Box_k u = 0$ (see Gűnther [1991] and [1988], Ch. IV, Cor. 1.13). For a recent survey on Huygens' principle see Berest [1998].

Bibliographical Notes

§1. The construction of the constant curvature spaces (Theorems 1.3 and 1.5) was given by the author ([1959], [1961]). The proof of Lemma 1.4 on the connectivity is adapted from Boerner [1955]. For more information on isotropic manifolds (there is more than one definition) see Tits [1955], p. 183 and Wolf [1967].

§§2-4. This material is based on Ch. IV in Helgason [1959]. Corollary 5.1 with a different proof and the subsequent remark were shown to me by Schlichtkrull. See Schimming and Schlichtkrull [1994] (in particular Lemma 6.2) where it is also shown that the constants $c_k = -\kappa(n-k)(k-1)$ in (40) are *the only ones* for which $\Box + c_k$ satisfies Huygens' principle. Here it is of interest to recall that in the flat Lorentzian case $\mathbf{R}^{2m}$, $\Box + c$ satisfies Huygens' principle only for $c = 0$. Theorem 4.1 was extended to pseudo-Riemannian manifolds of constant curvature by Orloff [1985], [1987]. For recent representative work on orbital integrals see e.g. Bouaziz [1995], Flicker [1996], Harinck [1998], Renard [1997].

CHAPTER V

FOURIER TRANSFORMS AND DISTRIBUTIONS. A RAPID COURSE

§1 The Topology of the Spaces $\mathcal{D}(\mathbf{R}^n)$, $\mathcal{E}(\mathbf{R}^n)$ and $\mathcal{S}(\mathbf{R}^n)$

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Let $\mathbf{R}^n = \{x = (x_1, \dots, x_n) : x_i \in \mathbf{R}\}$ and let ∂_i denote $\partial/\partial x_i$. If $(\alpha_1, \dots, \alpha_n)$ is an n-tuple of integers $\alpha_i \geq 0$ we put $\alpha! = \alpha_1! \cdots \alpha_n!$,

$$D^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}\,,\ x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}\,,\ |\alpha| = \alpha_1 + \cdots + \alpha_n\,.$$

For a complex number c, $\operatorname{Re} c$ and $\operatorname{Im} c$ denote respectively, the real part and the imaginary part of c. For a given compact set $K \subset \mathbf{R}^n$ let

$$\mathcal{D}_K = \mathcal{D}_K(\mathbf{R}^n) = \{f \in \mathcal{D}(\mathbf{R}^n) : \operatorname{supp}(f) \subset K\}\,,$$

where supp stands for support. The space $\mathcal{D}_K$ is topologized by the semi-norms

$$\|f\|_{K,m} = \sum_{|\alpha| \leq m} \sup_{x \in K} |(D^\alpha f)(x))|, \quad m \in \mathbb{Z}^+\,. \tag{1}$$

The topology of $\mathcal{D} = \mathcal{D}(\mathbf{R}^n)$ is defined as the largest locally convex topology for which all the embedding maps $\mathcal{D}_K \to \mathcal{D}$ are continuous. This is the so-called *inductive limit* topology. More explicitly, this topology is characterized as follows:

A *convex* set $C \subset \mathcal{D}$ is a neighborhood of 0 in $\mathcal{D}$ if and only if for each compact set $K \subset \mathbf{R}^n$, $C \cap \mathcal{D}_K$ is a neighborhood of 0 in $\mathcal{D}_K$.

A fundamental system of neighborhoods in $\mathcal{D}$ can be characterized by the following theorem. If B_R denotes the ball $|x| < R$ in $\mathbf{R}^n$ then

$$\mathcal{D} = \cup_{j=0}^{\infty} \mathcal{D}_{\overline{B}_j}\,. \tag{2}$$

Theorem 1.1. *Given two monotone sequences*

$$\begin{aligned} \{\epsilon\} &= \epsilon_0, \epsilon_1, \epsilon_2, \dots, \qquad \epsilon_i \to 0 \\ \{N\} &= N_0, N_1, N_2, \dots, \quad N_i \to \infty \quad N_i \in \mathbb{Z}^+ \end{aligned}$$

let $V(\{\epsilon\}, \{N\})$ denote the set of functions $\varphi \in \mathcal{D}$ satisfying for each j the conditions

$$|(D^\alpha \varphi)(x)| \leq \epsilon_j \quad \textit{for } |\alpha| \leq N_j, \quad x \notin B_j\,. \tag{3}$$

Then the sets $V(\{\epsilon\}, \{N\})$ form a fundamental system of neighborhoods of 0 in $\mathcal{D}$.

Proof. It is obvious that each $V(\{\epsilon\},\{N\})$ intersects each $\mathcal{D}_K$ in a neighborhood of 0 in $\mathcal{D}_K$. Conversely, let W be a *convex* subset of $\mathcal{D}$ intersecting each $\mathcal{D}_K$ in a neighborhood of 0. For each $j \in \mathbb{Z}^+$, $\exists N_j \in \mathbb{Z}^+$ and $\eta_j > 0$ such that each $\varphi \in \mathcal{D}$ satisfying

$$|D^\alpha \varphi(x)| \leq \eta_j \text{ for } |\alpha| \leq N_j \quad \operatorname{supp}(\varphi) \subset \overline{B}_{j+2}$$

belongs to W. Fix a sequence (β_j) with

$$\beta_j \in \mathcal{D}, \beta_j \geq 0\,,\ \Sigma\beta_j = 1\,,\ \operatorname{supp}(\beta_j) \subset \overline{B}_{j+2} - B_j$$

and write for $\varphi \in \mathcal{D}$,

$$\varphi = \sum_j \frac{1}{2^{j+1}}(2^{j+1}\beta_j\varphi)\,.$$

Then by the convexity of W, $\varphi \in W$ if each function $2^{j+1}\beta_j\varphi$ belongs to W. However, $D^\alpha(\beta_j\varphi)$ is a finite linear combination of derivatives $D^\beta\beta_j$ and $D^\gamma\varphi$, $(|\beta|, |\gamma| \leq |\alpha|)$. Since (β_j) is fixed and only values of φ in $\overline{B}_{j+2} - B_j$ enter, $\exists$ constant k_j such that the condition

$$|(D^\alpha\varphi)(x)| \leq \epsilon_j \text{ for } |x| \geq j \text{ and } |\alpha| \leq N_j$$

implies

$$|2^{j+1}D^\alpha(\beta_j\varphi)(x)| \leq k_j\epsilon_j \quad \text{for } |\alpha| \leq N_j, \quad \text{all } x\,.$$

Choosing the sequence $\{\epsilon\}$ such that $k_j\epsilon_j \leq \eta_j$ for all j we deduce for each j

$$\varphi \in V(\{\epsilon\},\{N\}) \Rightarrow 2^{j+1}\beta_j\varphi \in W\,,$$

whence $\varphi \in W$.

The space $\mathcal{E} = \mathcal{E}(\mathbf{R}^n)$ is topologized by the seminorms (1) for the varying K. Thus the sets

$$V_{j,k,\ell} = \{\varphi \in \mathcal{E}(\mathbf{R}^n) : \|\varphi\|_{\overline{B}_j,k} < 1/\ell \quad j,k,\ell \in \mathbb{Z}^+$$

form a fundamental system of neighborhoods of 0 in $\mathcal{E}(\mathbf{R}^n)$. This system being countable the topology of $\mathcal{E}(\mathbf{R}^n)$ is defined by sequences: A point $\varphi \in \mathcal{E}(\mathbf{R}^n)$ belongs to the closure of a subset $A \subset \mathcal{E}(\mathbf{R}^n)$ if and only if φ is the limit of a sequence in A. It is important to realize that this fails for the topology of $\mathcal{D}(\mathbf{R}^n)$ since the family of sets $V(\{\epsilon\},\{N\})$ is uncountable.

The space $\mathcal{S} = \mathcal{S}(\mathbf{R}^n)$ of rapidly decreasing functions on $\mathbf{R}^n$ is topologized by the seminorms (6), Ch. I. We can restrict the P in (6), Ch. I to polynomials with rational coefficients.

In contrast to the space $\mathcal{D}$ the spaces $\mathcal{D}_K$, $\mathcal{E}$ and $\mathcal{S}$ are Fréchet spaces, that is their topologies are given by a countable family of seminorms.

The spaces $\mathcal{D}_K(M)$, $\mathcal{D}(M)$ and $\mathcal{E}(M)$ can be topologized similarly if M is a manifold.

§2 Distributions

A *distribution* by definition is a member of the dual space $\mathcal{D}'(\mathbf{R}^n)$ of $\mathcal{D}(\mathbf{R}^n)$. By the definition of the topology of $\mathcal{D}$, $T \in \mathcal{D}'$ if and only if the restriction $T|\mathcal{D}_K$ is continuous for each compact set $K \subset \mathbf{R}^n$. Each locally integrable function F on $\mathbf{R}^n$ gives rise to a distribution $\varphi \to \int \varphi(x)F(x)\,dx$. A measure on $\mathbf{R}^n$ is also a distribution.

The *derivative* $\partial_i T$ of a distribution T is by definition the distribution $\varphi \to -T(\partial_i\varphi)$. If $F \in C^1(\mathbf{R}^n)$ then the distributions $T_{\partial_i F}$ and $\partial_i(T_F)$ coincide (integration by parts).

A *tempered distribution* by definition is a member of the dual space $\mathcal{S}'(\mathbf{R}^n)$. Since the imbedding $\mathcal{D} \to \mathcal{S}$ is continuous the restriction of a $T \in \mathcal{S}'$ to $\mathcal{D}$ is a distribution; since $\mathcal{D}$ is dense in $\mathcal{S}$ two tempered distributions coincide if they coincide on $\mathcal{D}$. In this sense we have $\mathcal{S}' \subset \mathcal{D}'$.

Since distributions generalize measures it is sometimes convenient to write

$$T(\varphi) = \int \varphi(x)\,dT(x)$$

for the value of a distribution on the function φ. A distribution T is said to be 0 on an open set $U \subset \mathbf{R}^n$ if $T(\varphi) = 0$ for each $\varphi \in \mathcal{D}$ with support contained in U. Let U be the union of all open sets $U_\alpha \subset \mathbf{R}^n$ on which T is 0. Then $T = 0$ on U. In fact, if $f \in \mathcal{D}(U)$, supp(f) can be covered by finitely many U_α, say $U_1, \ldots, U_r$. Then $U_1, \ldots, U_r, \mathbf{R}^n - \operatorname{supp}(f)$ is a covering of $\mathbf{R}^n$. If $1 = \sum_1^{r+1} \varphi_i$ is a corresponding partition of unity we have $f = \sum_1^r \varphi_i f$ so $T(f) = 0$. The complement $\mathbf{R}^n - U$ is called the *support of* T, denoted supp(T).

A distribution T of compact support extends to a unique element of $\mathcal{E}'(\mathbf{R}^n)$ by putting

$$T(\varphi) = T(\varphi\varphi_0)\,, \quad \varphi \in \mathcal{E}(\mathbf{R}^n)$$

if φ_0 is any function in $\mathcal{D}$ which is identically 1 on a neighborhood of supp(T). Since $\mathcal{D}$ is dense in $\mathcal{E}$, this extension is unique. On the other hand let $\tau \in \mathcal{E}'(\mathbf{R}^n)$, T its restriction to $\mathcal{D}$. Then supp(T) is compact. Otherwise we could for each j find $\varphi_j \in \mathcal{E}$ such that $\varphi_j \equiv 0$ on $\overline{B}_j$ but $T(\varphi_j) = 1$. Then $\varphi_j \to 0$ in $\mathcal{E}$, yet $\tau(\varphi_j) = 1$ which is a contradiction.

This identifies $\mathcal{E}'(\mathbf{R}^n)$ with the space of distributions of compact support and we have the following canonical inclusions:

$$\begin{array}{ccc} \mathcal{D}(\mathbf{R}^n) & \subset \mathcal{S}(\mathbf{R}^n) & \subset \mathcal{E}(\mathbf{R}^n) \\ \cap & \cap & \cap \\ \mathcal{E}'(\mathbf{R}^n) & \subset \mathcal{S}'(\mathbf{R}^n) & \subset \mathcal{D}'(\mathbf{R}^n)\,. \end{array}$$

If S and T are two distributions, at least one of compact support, their *convolution* is the distribution $S * T$ defined by

$$\varphi \to \int \varphi(x+y)\, dS(x)\, dT(y), \quad \varphi \in \mathcal{D}(\mathbf{R}^n)\,. \tag{4}$$

If $f \in \mathcal{D}$ the distribution $T_f * T$ has the form T_g where

$$g(x) = \int f(x-y)\, dT(y)$$

so we write for simplicity $g = f * T$. Note that $g(x) = 0$ unless $x - y \in \operatorname{supp}(f)$ for some $y \in \operatorname{supp}(T)$. Thus $\operatorname{supp}(g) \subset \operatorname{supp}(f) + \operatorname{supp} T$. More generally,

$$\operatorname{supp}(S * T) \subset \operatorname{supp}(S) + \operatorname{supp} T$$

as one sees from the special case $S = T_g$ by approximating S by functions $S * \varphi_\epsilon$ with $\operatorname{supp}(\varphi_\epsilon) \subset B_\epsilon(0)$.

The convolution can be defined for more general S and T, for example if $S \in \mathcal{S}$, $T \in \mathcal{S}'$ then $S * T \in \mathcal{S}'$. Also $S \in \mathcal{E}'$, $T \in \mathcal{S}'$ implies $S * T \in \mathcal{S}'$.

§3 The Fourier Transform

For $f \in L^1(\mathbf{R}^n)$ the *Fourier transform* is defined by

$$\widetilde{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-i\langle x, \xi \rangle}\, dx, \quad \xi \in \mathbf{R}^n\,. \tag{5}$$

If f has compact support we can take $\xi \in \mathbf{C}^n$. For $f \in \mathcal{S}(\mathbf{R}^n)$ one proves quickly

$$i^{|\alpha|+|\beta|} \xi^\beta (D^\alpha \widetilde{f})(\xi) = \int_{\mathbf{R}^n} D^\beta (x^\alpha f(x)) e^{-i\langle x, \xi \rangle}\, dx \tag{6}$$

and this implies easily the following result.

Theorem 3.1. *The Fourier transform is a linear homeomorphism of $\mathcal{S}$ onto $\mathcal{S}$.*

The function $\psi(x) = e^{-x^2/2}$ on $\mathbf{R}$ satisfies $\psi'(x) + x\psi = 0$. It follows from (6) that $\widetilde{\psi}$ satisfies the same differential equation and thus is a constant multiple of $e^{-\xi^2/2}$. Since $\widetilde{\psi}(0) = \int e^{-\frac{x^2}{2}}\, dx = (2\pi)^{1/2}$ we deduce $\widetilde{\psi}(\xi) = (2\pi)^{1/2} e^{-\xi^2/2}$. More generally, if $\psi(x) = e^{-|x|^2/2}$, $(x \in \mathbf{R}^n)$ then by product integration

$$\widetilde{\psi}(\xi) = (2\pi)^{n/2} e^{-|\xi|^2/2}\,. \tag{7}$$

Theorem 3.2. *The Fourier transform has the following properties.*

(i) $f(x) = (2\pi)^{-n} \int \widetilde{f}(\xi) e^{i\langle x,\xi\rangle}\, d\xi$ *for* $f \in \mathcal{S}$.

(ii) $f \to \widetilde{f}$ *extends to a bijection of* $L^2(\mathbf{R}^n)$ *onto itself and*

$$\int_{\mathbf{R}^n} |f(x)|^2 = (2\pi)^{-n} \int_{\mathbf{R}^n} |\widetilde{f}(\xi)|^2\, d\xi\, .$$

(iii) $(f_1 * f_2)^{\sim} = \widetilde{f}_1 \widetilde{f}_2$ *for* $f_1, f_2 \in \mathcal{S}$.

(iv) $(f_1 f_2)^{\sim} = (2\pi)^{-n} \widetilde{f}_1 * \widetilde{f}_2$ *for* $f_1, f_2 \in \mathcal{S}$.

Proof. (i) The integral on the right equals

$$\int e^{i\langle x,\xi\rangle} \left(\int f(y) e^{-i\langle y,\xi\rangle}\, dy \right) d\xi$$

but here we cannot exchange the integrations. Instead we consider for $g \in \mathcal{S}$ the integral

$$\int e^{i\langle x,\xi\rangle} g(\xi) \left(\int f(y) e^{-i\langle y,\xi\rangle}\, dy \right) d\xi\, ,$$

which equals the expressions

$$\int \widetilde{f}(\xi) g(\xi) e^{i\langle x,\xi\rangle}\, d\xi = \int f(y) \widetilde{g}(y - x)\, dy = \int f(x+y) \widetilde{g}(y)\, dy\, . \tag{8}$$

Replace $g(\xi)$ by $g(\epsilon\xi)$ whose Fourier transform is $\epsilon^{-n}\widetilde{g}(y/\epsilon)$. Then we obtain

$$\int \widetilde{f}(\xi) g(\epsilon\xi) e^{i\langle x,\xi\rangle}\, d\xi = \int \widetilde{g}(y) f(x + \epsilon y)\, dy\, ,$$

which upon letting $\epsilon \to 0$ gives

$$g(0) \int \widetilde{f}(\xi) e^{i\langle x,\xi\rangle}\, d\xi = f(x) \int \widetilde{g}(y)\, dy\, .$$

Taking $g(\xi)$ as $e^{-|\xi|^2/2}$ and using (7) Part (i) follows. The identity in (ii) follows from (8) (for $x = 0$) and (i). It implies that the image $L^2(\mathbf{R}^n)^{\sim}$ is closed in $L^2(\mathbf{R}^n)$. Since it contains the dense subspace $\mathcal{S}(\mathbf{R}^n)$ (ii) follows. Formula (iii) is an elementary computation and now (iv) follows taking (i) into account.

If $T \in \mathcal{S}'(\mathbf{R}^n)$ its Fourier transform is the linear form $\widetilde{T}$ on $\mathcal{S}(\mathbf{R}^n)$ defined by

$$\widetilde{T}(\varphi) = T(\widetilde{\varphi})\, . \tag{9}$$

Then by Theorem 3.1, $\widetilde{T} \in \mathcal{S}'$. Note that

$$\int \varphi(\xi) \widetilde{f}(\xi)\, d\xi = \int \widetilde{\varphi}(x) f(x)\, dx \tag{10}$$

for all $f \in L^1(\mathbf{R}^n)$, $\varphi \in \mathcal{S}(\mathbf{R}^n)$. Consequently

$$(T_f)^\sim = T_{\widetilde{f}} \quad \text{for } f \in L^1(\mathbf{R}^n) \tag{11}$$

so the definition (9) extends the old one (5). If $S_1, S_2 \in \mathcal{E}'(\mathbf{R}^n)$ then $\widetilde{S}_1$ and $\widetilde{S}_2$ have the form T_{s_1} and T_{s_2} where $s_1, s_2 \in \mathcal{E}(\mathbf{R}^n)$ and in addition $(S_1 * S_2)^\sim = T_{s_1 s_2}$. We express this in the form

$$(S_1 * S_2)^\sim = \widetilde{S}_1 \widetilde{S}_2 \,. \tag{12}$$

This formula holds also in the cases

$$S_1 \in \mathcal{S}(\mathbf{R}^n), \quad S_2 \in \mathcal{S}'(\mathbf{R}^n)\,,$$
$$S_1 \in \mathcal{E}'(\mathbf{R}^n), \quad S_2 \in \mathcal{S}'(\mathbf{R}^n)$$

and $S_1 * S_2 \in \mathcal{S}'(\mathbf{R}^n)$ (cf. Schwartz [1966], p. 268).

The classical Paley-Wiener theorem gave an intrinsic description of $L^2(0, 2\pi)^\sim$. We now prove an extension to a characterization of $\mathcal{D}(\mathbf{R}^n)^\sim$ and $\mathcal{E}'(\mathbf{R}^n)^\sim$.

Theorem 3.3. *(i) A holomorphic function $F(\zeta)$ on $\mathbf{C}^n$ is the Fourier transform of a distribution with support in $\overline{B}_R$ if and only if for some constants C and $N \geq 0$ we have*

$$|F(\zeta)| \leq C(1 + |\zeta|^N) e^{R|\operatorname{Im}\zeta|} \,. \tag{13}$$

(ii) $F(\zeta)$ is the Fourier transform of a function in $\mathcal{D}_{\bar{B}_R}(\mathbf{R}^n)$ if and only if for each $N \in \mathbb{Z}^+$ there exists a constant C_N such that

$$|F(\zeta)| \leq C_N (1 + |\zeta|)^{-N} e^{R|\operatorname{Im}\zeta|} \,. \tag{14}$$

Proof. First we prove that (13) is necessary. Let $T \in \mathcal{E}'$ have support in $\overline{B}_R$ and let $\chi \in \mathcal{D}$ have support in $\overline{B_{R+1}}$ and be identically 1 in a neighborhood of $\overline{B}_R$. Since $\mathcal{E}(\mathbf{R}^n)$ is topologized by the semi-norms (1) for varying K and m we have for some $C_0 \geq 0$ and $N \in \mathbb{Z}^+$

$$|T(\varphi)| = |T(\chi\varphi)| \leq C_0 \sum_{|\alpha| \leq N} \sup_{x \in \overline{B_{R+1}}} |(D^\alpha(\chi\varphi))(x)| \,.$$

Computing $D^\alpha(\chi\varphi)$ we see that for another constant C_1

$$|T(\varphi)| \leq C_1 \sum_{|\alpha| \leq N} \sup_{x \in \mathbf{R}^n} |D^\alpha \varphi(x)|, \quad \varphi \in \mathcal{E}(\mathbf{R}^n). \tag{15}$$

Let $\psi \in \mathcal{E}(\mathbf{R})$ such that $\psi \equiv 1$ on $(-\infty, \frac{1}{2})$, and $\equiv 0$ on $(1, \infty)$. Then if $\zeta \neq 0$ the function

$$\varphi_\zeta(x) = e^{-i\langle x, \zeta \rangle} \psi(|\zeta|(|x| - R))$$

belongs to $\mathcal{D}$ and equals $e^{-i\langle x,\zeta\rangle}$ in a neighborhood of $\overline{B}_R$. Hence

$$|\widetilde{T}(\zeta)| = |T(\varphi_\zeta)| \leq C_1 \sum_{|\alpha|\leq N} \sup |D^\alpha \varphi_\zeta| \,. \tag{16}$$

Now $\operatorname{supp}(\varphi_\zeta) \subset \overline{B}_{R+|\zeta|^{-1}}$ and on this ball

$$|e^{-i\langle x,\zeta\rangle}| \leq e^{|x|\,|\operatorname{Im}\zeta|} \leq e^{(R+|\zeta|^{-1})|\operatorname{Im}\zeta|} \leq e^{R|\operatorname{Im}\zeta|+1} \,.$$

Estimating $D^\alpha\varphi_\zeta$ similarly we see that by (16), $\widetilde{T}(\zeta)$ satisfies (13).

The necessity of (14) is an easy consequence of (6).

Next we prove the sufficiency of (14). Let

$$f(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} F(\xi) e^{i\langle x,\xi\rangle} \, d\xi \,. \tag{17}$$

Because of (14) we can shift the integration in (17) to the complex domain so that for any fixed $\eta \in \mathbf{R}^n$,

$$f(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} F(\xi + i\eta) e^{i\langle x,\xi+i\eta\rangle} \, d\xi \,.$$

We use (14) for $N = n+1$ to estimate this integral and this gives

$$|f(x)| \leq C_N e^{R|\eta| - \langle x,\eta\rangle} (2\pi)^{-n} \int_{\mathbf{R}^n} (1+|\xi|)^{-(n+1)} \, d\xi \,.$$

Taking now $\eta = tx$ and letting $t \to +\infty$ we deduce $f(x) = 0$ for $|x| > R$.

For the sufficiency of (13) we note first that F as a distribution on $\mathbf{R}^n$ is tempered. Thus $F = \widetilde{f}$ for some $f \in \mathcal{S}'(\mathbf{R}^n)$. Convolving f with a $\varphi \in \mathcal{D}_{\overline{B}_\epsilon}$ we see that $f * \varphi$ satisfies estimates (14) with R replaced by $R + \epsilon$. Thus $\operatorname{supp}(f * \varphi_\epsilon) \subset \overline{B_{R+\epsilon}}$. Letting $\epsilon \to 0$ we deduce $\operatorname{supp}(f) \subset \overline{B_R}$, concluding the proof.

We shall now prove a refinement of Theorem 3.3 in that the topology of $\mathcal{D}$ is described in terms of $\widetilde{\mathcal{D}}$. This has important applications to differential equations as we shall see in the next section.

Theorem 3.4. *A convex set $V \subset \mathcal{D}$ is a neighborhood of 0 in $\mathcal{D}$ if and only if there exist positive sequences*

$$M_0, M_1, \ldots, \ \delta_0, \delta_1, \ldots$$

such that V contains all $u \in \mathcal{D}$ satisfying

$$|\widetilde{u}(\zeta)| \leq \sum_{k=0}^{\infty} \delta_k \frac{1}{(1+|\zeta|)^{M_k}} e^{k|\operatorname{Im}\zeta|}, \quad \zeta \in \mathbf{C}^n \,. \tag{18}$$

The proof is an elaboration of that of Theorem 3.3. Instead of the contour shift $\mathbf{R}^n \to \mathbf{R}^n + i\eta$ used there one now shifts $\mathbf{R}^n$ to a contour on which the two factors on the right in (14) are comparable.

Let $W(\{\delta\}, \{M\})$ denote the set of $u \in \mathcal{D}$ satisfying (18). Given k the set

$$W_k = \{u \in \mathcal{D}_{\overline{B}_k} : |\widetilde{u}(\zeta)| \le \delta_k(1+|\zeta|)^{-M_k} e^{k|\mathrm{Im}\,\zeta|}\}$$

is contained in $W(\{\delta\}, \{M\})$. Thus if V is a convex set containing $W(\{\delta\}, \{M\})$ then $V \cap \mathcal{D}_{\overline{B}_k}$ contains W_k which is a neighborhood of 0 in $\mathcal{D}_{\overline{B}_k}$ (because the bounds on $\widetilde{u}$ correspond to the bounds on the $\|u\|_{\overline{B}_k, M_k}$). Thus V is a neighborhood of 0 in $\mathcal{D}$.

Proving the converse amounts to proving that given $V(\{\epsilon\}, \{N\})$ in Theorem 1.1 there exist $\{\delta\}, \{M\}$ such that

$$W(\{\delta\}, \{M\}) \subset V(\{\epsilon\}, \{N\}).$$

For this we shift the contour in (17) to others where the two factors in (14) are comparable. Let

$$\begin{aligned} x &= (x_1, \ldots, x_n), \qquad & x' &= (x_1, \ldots, x_{n-1}) \\ \zeta &= (\zeta_1, \ldots, \zeta_n) & \zeta' &= (\zeta_1, \ldots, \zeta_{n-1}) \\ \zeta &= \xi + i\eta, & \xi, \eta &\in \mathbf{R}^n . \end{aligned}$$

Then

$$\int_{\mathbf{R}^n} \widetilde{u}(\xi) e^{i\langle x, \xi\rangle}\, d\xi = \int_{\mathbf{R}^{n-1}} e^{i\langle x', \xi'\rangle}\, d\xi' \int_{\mathbf{R}} e^{i x_n \xi_n} \widetilde{u}(\xi', \xi_n)\, d\xi_n . \tag{19}$$

In the last integral we shift from $\mathbf{R}$ to the contour in $\mathbf{C}$ given by

$$\gamma_m : \zeta_n = \xi_n + im \log(1 + (|\xi'|^2 + \xi_n^2)^{1/2}) \tag{20}$$

$m \in \mathbb{Z}^+$ being fixed.

We claim that (cf. Fig. V.1)

$$\int_{\mathbf{R}} e^{i x_n \xi_n} \widetilde{u}(\xi', \xi_n)\, d\xi_n = \int_{\gamma_m} e^{i x_n \zeta_n} \widetilde{u}(\xi', \zeta_n)\, d\zeta_n . \tag{21}$$

Since (14) holds for each N, $\widetilde{u}$ decays between ξ_n-axis and γ_m faster than any $|\zeta_n|^{-M}$. Also

$$\left|\frac{d\zeta_n}{d\xi_n}\right| = \left|1 + i\, m\, \frac{1}{1+|\xi|} \cdot \frac{\partial(|\xi|)}{\partial \xi_n}\right| \le 1 + m .$$

Thus (21) follows from Cauchy's theorem in *one* variable. Putting

$$\Gamma_m = \{\zeta \in \mathbf{C}^n : \zeta' \in \mathbf{R}^{n-1}, \zeta_n \in \gamma_m\}$$

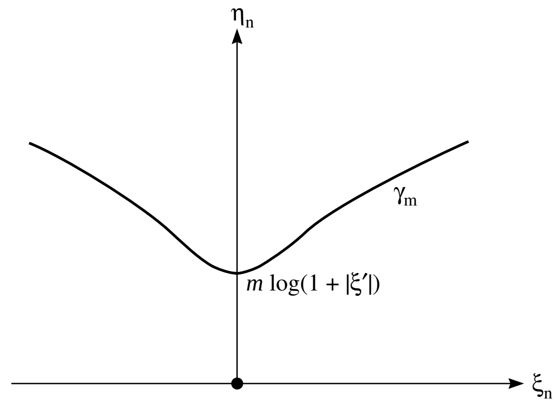

FIGURE V.1.

we thus have with $d\zeta = d\xi_1 \dots d\xi_{n-1}\, d\zeta_n$,

$$u(x) = (2\pi)^{-n} \int_{\Gamma_m} \widetilde{u}(\zeta) e^{i\langle x,\zeta\rangle}\, d\zeta\,. \tag{22}$$

Now suppose the sequences $\{\epsilon\}$, $\{N\}$ and $V(\{\epsilon\},\{N\})$ are given as in Theorem 1.1. We have to construct sequences $\{\delta\}$ $\{M\}$ such that (18) implies (3). By rotational invariance we may assume $x = (0,\dots,0,x_n)$ with $x_n > 0$. For each n-tuple α we have

$$(D^\alpha u)(x) = (2\pi)^{-n} \int_{\Gamma_m} \widetilde{u}(\zeta)(i\zeta)^\alpha e^{i\langle x,\zeta\rangle}\, d\zeta\,.$$

Starting with positive sequences $\{\delta\}$, $\{M\}$ we shall modify them successively such that (18) $\Rightarrow$ (3). Note that for $\zeta \in \Gamma_m$

$$e^{k|\operatorname{Im}\zeta|} \le (1+|\xi|)^{km}$$

$$|\zeta^\alpha| \le |\zeta|^{|\alpha|} \le ([|\xi|^2 + m^2(\log(1+|\xi|))^2]^{1/2})^{|\alpha|}\,.$$

For (3) with $j = 0$ we take $x_n = |x| \ge 0$, $|\alpha| \le N_0$ so

$$|e^{i\langle x,\zeta\rangle}| = e^{-\langle x,\operatorname{Im}\zeta|} \le 1 \quad \text{for } \zeta \in \Gamma_m\,.$$

Thus if u satisfies (18) we have by the above estimates

$$\begin{aligned} &|(D^\alpha u)(x)| \\ &\le \sum_0^\infty \delta_k \int_{\mathbf{R}^n} (1 + [|\xi|^2 + m^2(\log(1+|\xi|))^2]^{1/2})^{N_0 - M_k} (1+|\xi|)^{km}(1+m)\, d\xi\,. \end{aligned} \tag{23}$$

We can choose sequences $\{\delta\}$, $\{M\}$ (all δ_k, $M_k > 0$) such that this expression is $\le \epsilon_0$. This then verifies (3) for $j = 0$. We now fix δ_0 and M_0. Next we want to prove (3) for $j = 1$ by shrinking the terms in $\delta_1, \delta_2, \ldots$ and increasing the terms in $M_1, M_2, \ldots$ (δ_0, M_0 having been fixed).

Now we have $x_n = |x| \geq 1$ so

$$|e^{i\langle x,\zeta\rangle}| = e^{-\langle x, \operatorname{Im}\zeta\rangle} \leq (1+|\xi|)^{-m} \text{ for } \zeta \in \Gamma_m \tag{24}$$

so in the integrals in (23) the factor $(1+|\xi|)^{km}$ is replaced by $(1+|\xi|)^{(k-1)m}$.

In the sum we separate out the term with $k = 0$. Here M_0 has been fixed but now we have the factor $(1+|\xi|)^{-m}$ which assures that this $k = 0$ term is $< \frac{\epsilon_1}{2}$ for a sufficiently large m which we now fix. In the remaining terms in (23) (for $k > 0$) we can now increase $1/\delta_k$ and M_k such that the sum is $< \epsilon_1/2$. Thus (3) holds for $j = 1$ and it will remain valid for $j = 0$. We now fix this choice of δ_1 and M_1.

Now the inductive process is clear. We assume $\delta_0, \delta_1, \ldots, \delta_{j-1}$ and $M_0, M_1, \ldots, M_{j-1}$ having been fixed by this shrinking of the δ_i and enlarging of the M_i.

We wish to prove (3) for this j by increasing $1/\delta_k$, M_k for $k \geq j$. Now we have $x_n = |x| \geq j$ and (24) is replaced by

$$|e^{i\langle x,\zeta\rangle}| = e^{-\langle x, \operatorname{Im}\zeta\rangle} \leq 1 + |\xi|^{-jm}$$

and since $|\alpha| \leq N_j$, (23) is replaced by

$$\begin{aligned}
&|(D^\alpha f)(x)| \\
&\leq \sum_{k=0}^{j-1} \delta_k \int_{\mathbf{R}^n} (1 + [|\xi|^2 + m^2(\log(1+|\xi|))^2]^{1/2})^{N_j - M_k} (1+|\xi|)^{(k-j)m} (1+m)\, d\xi \\
&+ \sum_{k \geq j} \delta_k \int_{\mathbf{R}^n} (1 + [|\xi|^2 + m^2(\log(1+|\xi|))^2]^{1/2})^{N_j - M_k} (1+|\xi|)^{(k-j)m} (1+m)\, d\xi\,.
\end{aligned}$$

In the first sum the M_k have been fixed but the factor $(1+|\xi|)^{(k-j)m}$ decays exponentially. Thus we can fix m such that the first sum is $< \frac{\epsilon_j}{2}$.

In the latter sum the $1/\delta_k$ and the M_k can be increased so that the total sum is $< \frac{\epsilon_j}{2}$. This implies the validity of (3) for this particular j and it remains valid for $0, 1, \ldots j-1$. Now we fix δ_j and M_j.

This completes the induction. With this construction of $\{\delta\}$, $\{M\}$ we have proved that $W(\{\delta\}, \{M\}) \subset V(\{\epsilon\}, \{N\})$. This proves Theorem 3.4.

§4 Differential Operators with Constant Coefficients

The description of the topology of $\mathcal{D}$ in terms of the range $\widetilde{\mathcal{D}}$ given in Theorem 3.4 has important consequences for solvability of differential equations on $\mathbf{R}^n$ with constant coefficients.

Theorem 4.1. *Let $D \neq 0$ be a differential operator on $\mathbf{R}^n$ with constant coefficients. Then the mapping $f \to \mathcal{D}f$ is a homeomorphism of $\mathcal{D}$ onto $D\mathcal{D}$.*

Proof. It is clear from Theorem 3.3 that the mapping $f \to Df$ is injective on $\mathcal{D}$. The continuity is also obvious.

For the continuity of the inverse we need the following simple lemma.

Lemma 4.2. *Let $P \neq 0$ be a polynomial of degree m, F an entire function on $\mathbf{C}^n$ and $G = PF$. Then*

$$|F(\zeta)| \leq C \sup_{|z|\leq 1} |G(z+\zeta)|, \quad \zeta \in \mathbf{C}^n \,,$$

where C is a constant.

Proof. Suppose first $n = 1$ and that $P(z) = \sum_0^m a_k z^k (a_m \neq 0)$. Let $Q(z) = z^m \sum_0^m \overline{a}_k z^{-k}$. Then, by the maximum principle,

$$|a_m F(0)| = |Q(0)F(0) \leq \max_{|z|=1} |Q(z)F(z)| = \max_{|z|=1} |P(z)F(z)| \,. \tag{25}$$

For general n let A be an $n \times n$ complex matrix, mapping the ball $|\zeta| < 1$ in $\mathbf{C}^n$ into itself and such that

$$P(A\zeta) = a\zeta_1^m + \sum_0^{m-1} P_k(\zeta_2, \ldots, \zeta_n)\zeta_1^k, \quad a \neq 0 \,.$$

Let

$$F_1(\zeta) = F(A\zeta), \quad G_1(\zeta) = G(A\zeta), \quad P_1(\zeta) = P(A\zeta) \,.$$

Then

$$G_1(\zeta_1 + z, \zeta_2, \ldots, \zeta_n) = F_1(\zeta_1 + z, \zeta_2, \ldots, \zeta_n) P_1(\zeta + z, \zeta_2, \ldots, \zeta_n)$$

and the polynomial

$$z \to P_1(\zeta_1 + z, \ldots, \zeta_n)$$

has leading coefficient a. Thus by (25)

$$|aF_1(\zeta)| \leq \max_{|z|=1} |G_1(\zeta_1 + z, \zeta_2, \ldots, \zeta_n)| \leq \max_{\substack{z\in\mathbf{C}^n \\ |z|\leq 1}} |G_1(\zeta + z)| \,.$$

Hence by the choice of A

$$|aF(\zeta)| \leq \sup_{\substack{z\in\mathbf{C}^n \\ |z|\leq 1}} |G(\zeta + z)|$$

proving the lemma.

For Theorem 4.1 it remains to prove that if V is a convex neighborhood of 0 in $\mathcal{D}$ then there exists a convex neighborhood W of 0 in $\mathcal{D}$ such that

$$f \in \mathcal{D},\, Df \in W \Rightarrow f \in V\,. \tag{26}$$

We take V as the neighborhood $W(\{\delta\},\{M\})$. We shall show that if $W = W(\{\epsilon\},\{M\})$ (same $\{M\}$) then (26) holds provided the ϵ_j in $\{\epsilon\}$ are small enough. We write $u = Df$ so $\widetilde{u}(\zeta) = P(\zeta)\widetilde{f}(\zeta)$ where P is a polynomial. By Lemma 4.2

$$|\widetilde{f}(\zeta)| \leq C \sup_{|z|\leq 1} |\widetilde{u}(\zeta + z)|\,. \tag{27}$$

But $|z| \leq 1$ implies

$$(1 + |z+\zeta|)^{-M_j} \leq 2^{M_j}(1+|\zeta|)^{-M_j}\,, \quad |\mathrm{Im}\,(z+\zeta)| \leq |\mathrm{Im}\,\zeta| + 1\,,$$

so if $C2^{M_j}e^j\epsilon_j \leq \delta_j$ then (26) holds.

Q.e.d.

Corollary 4.3. *Let $D \neq 0$ be a differential operator on $\mathbf{R}^n$ with constant (complex) coefficients. Then*

$$D\mathcal{D}' = \mathcal{D}'\,. \tag{28}$$

In particular, there exists a distribution T on $\mathbf{R}^n$ such that

$$DT = \delta\,. \tag{29}$$

Definition. A distribution T satisfying (29) is called a *fundamental solution* for D.

To verify (28) let $L \in \mathcal{D}'$ and consider the functional $D^*u \to L(u)$ on $D^*\mathcal{D}$ ($*$ denoting adjoint). Because of Theorem 3.3 this functional is well defined and by Theorem 4.1 it is continuous. By the Hahn-Banach theorem it extends to a distribution $S \in \mathcal{D}'$. Thus $S(D^*u) = Lu$ so $DS = L$, as claimed.

Corollary 4.4. *Given $f \in \mathcal{D}$ there exists a smooth function u on $\mathbf{R}^n$ such that*

$$Du = f\,.$$

In fact, if T is a fundamental solution one can put $u = f * T$.

We conclude this section with the mean value theorem of Ásgeirsson which entered into the range characterization of the X-ray transform in Chapter I. For another application see Theorem 5.9 below

Theorem 4.5. *Let u be a C^2 function on $B_R \times B_R \subset \mathbf{R}^n \times \mathbf{R}^n$ satisfying*

$$L_x u = L_y u\,. \tag{30}$$

Then

$$\int_{|y|=r} u(0,y)\,dw(y) = \int_{|x|=r} u(x,0)\,dw(x) \quad r < R\,. \tag{31}$$

Conversely, if u is of class C^2 near $(0,0) \subset \mathbf{R}^n \times \mathbf{R}^n$ and if (31) holds for all r sufficiently small then

$$(L_x u)(0,0) = (L_y u)(0,0)\,. \tag{32}$$

Remark 4.6. Integrating Taylor's formula it is easy to see that on the space of analytic functions the mean value operator M^r (Ch. I, §2) is a power series in the Laplacian L. (See (44) below for the explicit expansion.) Thus (30) implies (31) for analytic functions.

For u of class C^2 we give another proof.

We consider the mean value operator on each factor in the product $\mathbf{R}^n \times \mathbf{R}^n$ and put

$$U(r,s) = (M_1^r M_2^s u)(x,y)$$

where the subscript indicates first and second variable, respectively. If u satisfies (30) then we see from the Darboux equation (Ch. I, Lemma 3.2) that

$$\frac{\partial^2 U}{\partial r^2} + \frac{n-1}{r}\,\frac{\partial U}{\partial r} = \frac{\partial^2 U}{\partial s^2} + \frac{n-1}{s}\,\frac{\partial U}{\partial s}\,.$$

Putting $F(r,s) = U(r,s) - U(s,r)$ we have

$$\frac{\partial^2 F}{\partial r^2} + \frac{n-1}{r}\,\frac{\partial F}{\partial r} - \frac{\partial^2 F}{\partial s^2} - \frac{n-1}{s}\,\frac{\partial F}{\partial s} = 0\,, \tag{33}$$

$$F(r,s) = -F(s,r)\,. \tag{34}$$

After multiplication of (33) by $r^{n-1}\frac{\partial F}{\partial s}$ and some manipulation we get

$$-r^{n-1}\tfrac{\partial}{\partial s}\left[\left(\tfrac{\partial F}{\partial r}\right)^2 + \left(\tfrac{\partial F}{\partial s}\right)^2\right] + 2\tfrac{\partial}{\partial r}\left(r^{n-1}\tfrac{\partial F}{\partial r}\tfrac{\partial F}{\partial s}\right) - 2r^{n-1}\tfrac{n-1}{s}\left(\tfrac{\partial F}{\partial s}\right)^2 = 0\,.$$

Consider the line MN with equation $r + s = \text{const.}$ in the (r,s)-plane and integrate the last expression over the triangle OMN (see Fig. V.2).

Using the divergence theorem (Ch. I, (26)) we then obtain, if $\mathbf{n}$ denotes the outgoing unit normal, $d\ell$ the element of arc length, and $\cdot$ the inner product,

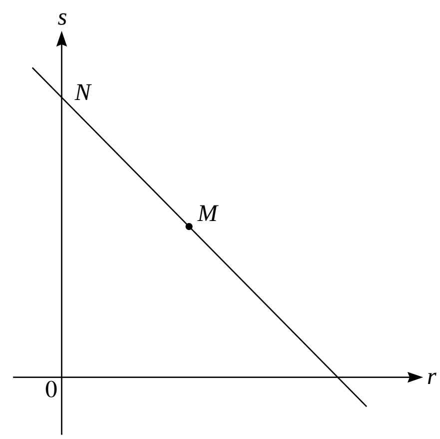

FIGURE V.2.

$$(35)\quad \int_{OMN} \left(2r^{n-1} \frac{\partial F}{\partial r} \frac{\partial F}{\partial s}\, , -r^{n-1} \left[\left(\frac{\partial F}{\partial r} \right)^2 + \left(\frac{\partial F}{\partial s} \right)^2 \right] \right) \cdot \mathbf{n}\, d\ell$$

$$= 2 \iint_{OMN} r^{n-1} \frac{n-1}{s} \left(\frac{\partial F}{\partial s} \right)^2 dr\, ds\,.$$

On $OM : \overline{n} = (2^{-1/2}, -2^{-1/2}), \quad F(r,r) = 0$ so $\frac{\partial F}{\partial r} + \frac{\partial F}{\partial s} = 0$.
On $MN : \overline{n} = (2^{-1/2}, 2^{-1/2})$.

Taking this into account, (35) becomes

$$2^{-\frac{1}{2}} \int_{MN} r^{n-1} \left(\frac{\partial F}{\partial r} - \frac{\partial F}{\partial s} \right)^2 d\ell + 2 \iint_{OMN} r^{n-1} \frac{n-1}{s} \left(\frac{\partial F}{\partial s} \right)^2 dr\, ds = 0\,.$$

This implies F constant so by (34) $F \equiv 0$. In particular, $U(r,0) = U(0,r)$ which is the desired relation (31).

For the converse we observe that the mean value $(M^r f)(0)$ satisfies (by Taylor's formula)

$$(M^r f)(0) = f(0) + c_n r^2 (Lf)(0) + o(r^2)$$

where $c_n \neq 0$ is a constant. Thus

$$r^{-1} \frac{dM^r f(0)}{dr} \to 2c_n (Lf)(0) \ \text{ as } \ r \to 0\,.$$

Thus (31) implies (32) as claimed.

§5 Riesz Potentials

We shall now study some examples of distributions in detail. If $\alpha \in \mathbf{C}$ satisfies $\operatorname{Re}\alpha > -1$ the functional

$$x_+^\alpha : \varphi \to \int_0^\infty x^\alpha \varphi(x)\, dx, \quad \varphi \in \mathcal{S}(\mathbf{R}), \tag{36}$$

is a well-defined tempered distribution. The mapping $\alpha \to x_+^\alpha$ from the half-plane $\operatorname{Re}\alpha > -1$ to the space $\mathcal{S}'(\mathbf{R})$ of tempered distributions is holomorphic (that is $\alpha \to x_+^\alpha(\varphi)$ is holomorphic for each $\varphi \in \mathcal{S}(\mathbf{R})$). Writing

$$x_+^\alpha(\varphi) = \int_0^1 x^\alpha(\varphi(x) - \varphi(0))\, dx + \frac{\varphi(0)}{\alpha+1} + \int_1^\infty x^\alpha \varphi(x)\, dx$$

the function $\alpha \to x_+^\alpha$ is continued to a holomorphic function in the region $\operatorname{Re}\alpha > -2, \alpha \neq -1$. In fact

$$\varphi(x) - \varphi(0) = x \int_0^\infty \varphi'(tx)\, dt,$$

so the first integral above converges for $\operatorname{Re}\alpha > -2$. More generally, $\alpha \to x_+^\alpha$ can be extended to a holomorphic $\mathcal{S}'(\mathbf{R})$-valued mapping in the region

$$\operatorname{Re}\alpha > -n-1, \quad \alpha \neq -1, -2, \ldots, -n,$$

by means of the formula

$$\begin{aligned} x_+^\alpha(\varphi) = & \int_0^1 x^\alpha \left[\varphi(x) - \varphi(0) - x\varphi'(0) - \cdots - \frac{x^{n-1}}{(n-1)!}\varphi^{(n-1)}(0)\right] dx \\ & + \int_1^\infty x^\alpha \varphi(x)\, dx + \sum_{k=1}^n \frac{\varphi^{(k-1)}(0)}{(k-1)!(\alpha+k)}. \end{aligned} \tag{37}$$

In this manner $\alpha \to x_+^\alpha$ is a meromorphic distribution-valued function on $\mathbf{C}$, with simple poles at $\alpha = -1, -2, \ldots$. We note that the residue at $\alpha = -k$ is given by

$$\operatorname*{Res}_{\alpha=-k} x_+^\alpha = \lim_{\alpha \to -k} (\alpha+k) x_+^\alpha = \frac{(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}. \tag{38}$$

Here $\delta^{(h)}$ is the h^{th} derivative of the delta distribution δ. We note that x_+^α is always a tempered distribution.

Next we consider for $\operatorname{Re}\alpha > -n$ the distribution r^α on $\mathbf{R}^n$ given by

$$r^\alpha : \varphi \to \int_{\mathbf{R}^n} \varphi(x)|x|^\alpha\, dx, \quad \varphi \in \mathcal{D}(\mathbf{R}^n).$$

Lemma 5.1. *The mapping $\alpha \to r^\alpha$ extends uniquely to a meromorphic mapping from $\mathbf{C}$ to the space $\mathcal{S}'(\mathbf{R}^n)$ of tempered distributions. The poles are the points*

$$\alpha = -n - 2h \quad (h \in \mathbb{Z}^+)$$

and they are all simple.

Proof. We have for $\operatorname{Re}\alpha > -n$

$$r^\alpha(\varphi) = \Omega_n \int_0^\infty (M^t\varphi)(0) t^{\alpha+n-1}\, dt\,. \tag{39}$$

Next we note (say from (15) in §2) that the mean value function $t \to (M^t\varphi)(0)$ extends to an even $\mathcal{C}^\infty$ function on $\mathbf{R}$, and its odd order derivatives at the origin vanish. Each even order derivative is nonzero if φ is suitably chosen. Since by (39)

$$r^\alpha(\varphi) = \Omega_n t_+^{\alpha+n-1}(M^t\varphi)(0) \tag{40}$$

the first statement of the lemma follows. The possible (simple) poles of r^α are by the remarks about x_+^α given by $\alpha + n - 1 = -1, -2, \ldots$. However if $\alpha + n - 1 = -2, -4, \ldots$, formula (38) shows, since $(M^t\varphi(0))^{(h)} = 0$, ($h$ odd) that $r^\alpha(\varphi)$ is holomorphic at the points $a = -n-1, -n-3, \ldots$.

The remark about the even derivatives of $M^t\varphi$ shows on the other hand, that the points $\alpha = -n-2h$ ($h \in \mathbb{Z}^+$) are genuine poles. We note also from (38) and (40) that

$$\operatorname*{Res}_{\alpha=-n} r^\alpha = \lim_{\alpha\to -n} (\alpha+n) r^\alpha = \Omega_n \delta\,. \tag{41}$$

We recall now that the Fourier transform $T \to \widetilde{T}$ of a tempered distribution T on $\mathbf{R}^n$ is defined by

$$\widetilde{T}(\varphi) = T(\widetilde{\varphi}) \qquad \varphi = \mathcal{S}(\mathbf{R}^n)\,.$$

We shall now calculate the Fourier transforms of these tempered distributions r^α.

Lemma 5.2. *We have the following identity*

$$(r^\alpha)^\sim = 2^{n+\alpha}\pi^{\frac{n}{2}} \frac{\Gamma((n+\alpha)/2)}{\Gamma(-\alpha/2)} r^{-\alpha-n}, \quad -\alpha - n \notin 2\mathbb{Z}^+\,. \tag{42}$$

For $\alpha = 2h$ ($h \in \mathbb{Z}^+$) the singularity on the right is removable and (42) takes the form

$$(r^{2h})^\sim = (2\pi)^n (-L)^h \delta, \quad h \in \mathbb{Z}^+\,. \tag{43}$$

Proof. We use the fact that if $\psi(x) = e^{-|x|^2/2}$ then $\widetilde{\psi}(u) = (2\pi)^{\frac{n}{2}} e^{-|u|^2/2}$ so by the formula $\int f\widetilde{g} = \int \widetilde{f}g$ we obtain for $\varphi \in \mathcal{S}(\mathbf{R}^n), t > 0$,

$$\int \widetilde{\varphi}(x)e^{-t|x|^2/2}\,dx = (2\pi)^{n/2}t^{-n/2}\int \varphi(u)e^{-|u|^2/2t}\,du\,.$$

We multiply this equation by $t^{-1-\alpha/2}$ and integrate with respect to t. On the left we obtain the expression

$$\Gamma(-\alpha/2)2^{-\frac{\alpha}{2}}\int \widetilde{\varphi}(x)|x|^\alpha\,dx\,,$$

using the formula

$$\int_0^\infty e^{-t|x|^2/2}t^{-1-\alpha/2}\,dt = \Gamma(-\tfrac{\alpha}{2})2^{-\frac{\alpha}{2}}|x|^\alpha\,,$$

which follows from the definition

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt\,.$$

On the right we similarly obtain

$$(2\pi)^{\frac{n}{2}}\Gamma((n+\alpha)/2)\,2^{\frac{n+\alpha}{2}}\int \varphi(u)|u|^{-\alpha-n}\,du\,.$$

The interchange of the integrations is valid for α in the strip $-n < \operatorname{Re}\alpha < 0$ so (42) is proved for these α. For the remaining ones it follows by analytic continuation. Finally, (43) is immediate from the definitions and (6).

By the analytic continuation, the right hand sides of (42) and (43) agree for $\alpha = 2h$. Since

$$\operatorname*{Res}_{\alpha=2h}\Gamma(-\alpha/2) = -2(-1)^h/h!$$

and since by (40) and (38),

$$\operatorname*{Res}_{\alpha=2h} r^{-\alpha-n}(\varphi) = -\Omega_n\frac{1}{(2h)!}\left[\left(\frac{d}{dt}\right)^{2h}(M^t\varphi)\right]_{t=0}$$

we deduce the relation

$$\left[\left(\frac{d}{dt}\right)^{2h}(M^t\varphi)\right]_{t=0} = \frac{\Gamma(n/2)}{\Gamma(h+n/2)}\frac{(2h)!}{2^{2h}h!}(L^h\varphi)(0)\,.$$

This gives the expansion

$$M^t = \sum_{h=0}^{\infty}\frac{\Gamma(n/2)}{\Gamma(h+n/2)}\frac{(t/2)^{2h}}{h!}L^h \tag{44}$$

on the space of analytic functions so M^t is a modified Bessel function of $tL^{1/2}$. This formula can also be proved by integration of Taylor's formula (cf. end of §4).

Lemma 5.3. *The action of the Laplacian is given by*

$$
\begin{aligned}
(45)\qquad Lr^{\alpha} &= \alpha(\alpha+n-2)r^{\alpha-2}, \quad (-\alpha-n+2 \notin 2\mathbb{Z}^+) \\
(46)\qquad Lr^{2-n} &= (2-n)\Omega_n\delta \quad (n \neq 2)\,.
\end{aligned}
$$

For $n=2$ this 'Poisson equation' is replaced by

$$
(47)\qquad L(\log r) = 2\pi\delta\,.
$$

Proof. For $\operatorname{Re}\alpha$ sufficiently large (45) is obvious by computation. For the remaining ones it follows by analytic continuation. For (46) we use the Fourier transform and the fact that for a tempered distribution S,

$$
(-LS)^{\sim} = r^2\widetilde{S}\,.
$$

Hence, by (42),

$$
(-Lr^{2-n})^{\sim} = 4\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}-1)} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}(n-2)\widetilde{\delta}\,.
$$

Finally, we prove (47). If $\varphi \in \mathcal{D}(\mathbf{R}^2)$ we have, putting $F(r) = (M^r\varphi)(0)$,

$$
(L(\log r))(\varphi) = \int_{\mathbf{R}^2} \log r (L\varphi)(x)\,dx = \int_0^{\infty} (\log r)2\pi r(M^rL\varphi)(0)\,dr\,.
$$

Using Lemma 3.2 in Chapter I this becomes

$$
\int_0^{\infty} \log r\ 2\pi r(F''(r) + r^{-1}F'(r))\,dr\,,
$$

which by integration by parts reduces to

$$
\Big[\log r(2\pi r)F'(r)\Big]_0^{\infty} - 2\pi\int_0^{\infty} F'(r)\,dr = 2\pi F(0)\,.
$$

This proves (47).

Another method is to write (45) in the form $L(\alpha^{-1}(r^{\alpha}-1)) = \alpha r^{\alpha-2}$. Then (47) follows from (41) by letting $\alpha \to 0$.

We shall now define fractional powers of L, motivated by the formula

$$
(-Lf)^{\sim}(u) = |u|^2\widetilde{f}(u)\,,
$$

so that formally we should like to have a relation

$$
(48)\qquad ((-L)^pf)^{\sim}(u) = |u|^{2p}\widetilde{f}(u)\,.
$$

Since the Fourier transform of a convolution is the product of the Fourier transforms, formula (42) (for $2p = -\alpha - n$) suggests defining

$$
(49)\qquad (-L)^pf = I^{-2p}(f)\,,
$$

where I^γ is the *Riesz potential*

$$(I^\gamma f)(x) = \frac{1}{H_n(\gamma)} \int_{\mathbf{R}^n} f(y)|x-y|^{\gamma-n}\, dy \tag{50}$$

with

$$H_n(\gamma) = 2^\gamma \pi^{\frac{n}{2}} \frac{\Gamma(\frac{\gamma}{2})}{\Gamma(\frac{n-\gamma}{2})}\,. \tag{51}$$

Note that if $-\gamma \in 2\mathbb{Z}^+$ the poles of $\Gamma(\gamma/2)$ cancel against the poles of $r^{\gamma-n}$ because of Lemma 5.1. Thus if $\gamma - n \notin 2\mathbb{Z}^+$ we can write

$$(I^\gamma f)(x) = (f * (H_n(\gamma)^{-1} r^{\gamma-n}))(x), \quad f \in \mathcal{S}(\mathbf{R}^n)\,. \tag{52}$$

By (12) and Lemma 5.2 we then have

$$(I^\gamma f)^\sim(u) = |u|^{-\gamma} \widetilde{f}(u)\,, \qquad \gamma - n \notin 2\mathbb{Z}^+ \tag{53}$$

as tempered distributions. Thus we have the following result.

Lemma 5.4. *If $f \in \mathcal{S}(\mathbf{R}^n)$ then $\gamma \to (I^\gamma f)(x)$ extends to a holomorphic function in the set $\mathbf{C}_n = \{\gamma \in \mathbf{C} : \gamma - n \notin 2\mathbb{Z}^+\}$. Also*

$$I^0 f \;=\; \lim_{\gamma\to 0} I^\gamma f = f\,, \tag{54}$$

$$I^\gamma L f \;=\; L I^\gamma f = -I^{\gamma-2} f\,. \tag{55}$$

We now prove an important property of the Riesz' potentials. Here it should be observed that $I^\gamma f$ is defined for all f for which (50) is absolutely convergent and $\gamma \in \mathbf{C}_n$.

Proposition 5.5. *The following identity holds:*

$$I^\alpha(I^\beta f) = I^{\alpha+\beta} f \text{ for } f \in \mathcal{S}(\mathbf{R}^n), \quad \operatorname{Re}\alpha, \operatorname{Re}\beta > 0, \quad \operatorname{Re}(\alpha+\beta) < n\,,$$

$I^\alpha(I^\beta f)$ being well defined. The relation is also valid if

$$f(x) = 0(|x|^{-p}) \text{ for some } p > \operatorname{Re}\alpha + \operatorname{Re}\beta\,.$$

Proof. We have

$$\begin{aligned} I^\alpha(I^\beta f)(x) &= \frac{1}{H_n(\alpha)} \int |x-z|^{\alpha-n} \left(\frac{1}{H_n(\beta)} \int f(y)|z-y|^{\beta-n}\, dy \right) dz \\ &= \frac{1}{H_n(\alpha) H_n(\beta)} \int f(y) \left(\int |x-z|^{\alpha-n} |z-y|^{\beta-n}\, dz \right) dy\,. \end{aligned}$$

The substitution $v = (x-z)/|x-y|$ reduces the inner integral to the form

$$|x-y|^{\alpha+\beta-n} \int_{\mathbf{R}^n} |v|^{\alpha-n} |w-v|^{\beta-n}\, dv\,, \tag{56}$$

where w is the unit vector $(x-y)/|x-y|$. Using a rotation around the origin we see that the integral in (56) equals the number

$$c_n(\alpha,\beta)=\int_{\mathbf{R}^n}|v|^{\alpha-n}|e_1-v|^{\beta-n}\,dv\,, \tag{57}$$

where $e_1=(1,0,\dots,0)$. The assumptions made on α and β insure that this integral converges. By the Fubini theorem the exchange order of integrations above is permissible and

$$I^\alpha(I^\beta f)=\frac{H_n(\alpha+\beta)}{H_n(\alpha)H_n(\beta)}c_n(\alpha,\beta)I^{\alpha+\beta}f\,. \tag{58}$$

It remains to calculate $c_n(\alpha,\beta)$. For this we use the following lemma which was already used in Chapter I, §2. As there, let $\mathcal{S}^*(\mathbf{R}^n)$ denote the set of functions in $\mathcal{S}(\mathbf{R}^n)$ which are orthogonal to all polynomials.

Lemma 5.6. *Each $I^\alpha(\alpha\in\mathbf{C}_n)$ leaves the space $\mathcal{S}^*(\mathbf{R}^n)$ invariant.*

Proof. We recall that (53) holds in the sense of tempered distributions. Suppose now $f\in\mathcal{S}^*(\mathbf{R}^n)$. We consider the sum in the Taylor formula for $\widetilde{f}$ in $|u|\leq 1$ up to order m with $m>|\alpha|$. Since each derivative of $\widetilde{f}$ vanishes at $u=0$ this sum consists of terms

$$(\beta!)^{-1}u^\beta(D^\beta\widetilde{f})(u^*)\,,\qquad |\beta|=m$$

where $|u^*|\leq 1$. Since $|u^\beta|\leq|u|^m$ this shows that

$$\lim_{u\to 0}|u|^{-\alpha}\widetilde{f}(u)=0\,. \tag{59}$$

Iterating this argument with $\partial_i(|u|^{-\alpha}\widetilde{f}(u))$ etc. we conclude that the limit relation (59) holds for each derivative $D^\beta(|u|^{-\alpha}\widetilde{f}(u))$. Because of (59), relation (53) can be written

$$\int_{\mathbf{R}^n}(I^\alpha f)^\sim(u)g(u)\,du=\int_{\mathbf{R}^n}|u|^{-\alpha}\widetilde{f}(u)g(u)\,du\,,\quad g\in\mathcal{S}\,, \tag{60}$$

so (53) holds as an identity for functions $f\in\mathcal{S}^*(\mathbf{R}^n)$. The remark about $D^\beta(|u|^{-\alpha}\widetilde{f}(u))$ thus implies $(I^\alpha f)^\sim\in\mathcal{S}_0$ so $I^\alpha f\in\mathcal{S}^*$ as claimed.

We can now finish the proof of Prop. 5.5. Taking $f_o\in\mathcal{S}^*$ we can put $f=I^\beta f_o$ in (53) and then

$$\begin{aligned}(I^\alpha(I^\beta f_0))^\sim(u)&=(I^\beta f_0)^\sim(u)|u|^{-\alpha}=\widetilde{f}_0(u)|u|^{-\alpha-\beta}\\&=(I^{\alpha+\beta}f_0)^\sim(u)\,.\end{aligned}$$

This shows that the scalar factor in (58) equals 1 so Prop. 5.5 is proved. In the process we have obtained the evaluation

$$\int_{\mathbf{R}^n}|v|^{\alpha-n}|e_1-v|^{\beta-n}\,dv=\frac{H_n(\alpha)H_n(\beta)}{H_n(\alpha+\beta)}\,.$$

We now prove a variation of Prop. 5.5 needed in the theory of the Radon transform.

Proposition 5.7. *Let $0 < k < n$. Then*

$$I^{-k}(I^k f) = f \qquad f \in \mathcal{E}(\mathbf{R}^n)$$

if $f(x) = 0(|x|^{-N})$ for some $N > n$.

Proof. By Prop. 5.5 we have if $f(y) = 0(|y|^{-N})$

$$I^\alpha(I^k f) = I^{\alpha+k} f \quad \text{for } 0 < \operatorname{Re}\alpha < n-k\,. \tag{61}$$

We shall prove that the function $\varphi = I^k f$ satisfies

$$\sup_x |\varphi(x)|\,|x|^{n-k} < \infty\,. \tag{62}$$

For an $N > n$ we have an estimate $|f(y)| \leq C_N(1+|y|)^{-N}$ where C_N is a constant. We then have

$$\left(\int_{\mathbf{R}^n} f(y)|x-y|^{k-n}\,dy\right) \leq C_N \int_{|x-y|\leq\frac{1}{2}|x|} (1+|y|)^{-N}|x-y|^{k-n}\,dy$$
$$+C_N \int_{|x-y|\geq\frac{1}{2}|x|} (1+|y|)^{-N}|x-y|^{k-n}\,dy\,.$$

In the second integral, $|x-y|^{k-n} \leq (\frac{|x|}{2})^{k-n}$ so since $N > n$ this second integral satisfies (62). In the first integral we have $|y| \geq \frac{|x|}{2}$ so the integral is bounded by

$$\left(1+\frac{|x|}{2}\right)^{-N} \int_{|x-y|\leq\frac{|x|}{2}} |x-y|^{k-n}\,dy = \left(1+\frac{|x|}{2}\right)^{-N} \int_{|z|\leq\frac{|x|}{2}} |z|^{k-n}\,dz$$

which is $0(|x|^{-N}|x|^k)$. Thus (62) holds also for this first integral. This proves (62) provided

$$f(x) = 0(|x|^{-N}) \text{ for some } N > n\,.$$

Next we observe that $I^\alpha(\varphi) = I^{\alpha+k}(f)$ is holomorphic for $0 < \operatorname{Re}\alpha < n-k$. For this note that by (39)

$$\begin{aligned}(I^{\alpha+k}f)(0) &= \frac{1}{H_n(\alpha+k)}\int_{\mathbf{R}^n} f(y)|y|^{\alpha+k-n}\,dy \\ &= \frac{1}{H_n(\alpha+k)}\Omega_n \int_0^\infty (M^t f)(0)t^{\alpha+k-1}\,dt\,.\end{aligned}$$

Since the integrand is bounded by a constant multiple of $t^{-N}t^{\alpha+k-1}$, and since the factor in front of the integral is harmless for $0 < k + \operatorname{Re}\alpha < n$, the holomorphy statement follows.

We claim now that $I^\alpha(\varphi)(x)$, which as we saw is holomorphic for $0 < \operatorname{Re}\alpha < n-k$, extends to a holomorphic function in the half-plane $\operatorname{Re}\alpha < n-k$. It suffices to prove this for $x = 0$. We decompose $\varphi = \varphi_1 + \varphi_2$ where φ_1 is a smooth function identically 0 in a neighborhood $|x| < \epsilon$ of 0, and $\varphi_2 \in \mathcal{S}(\mathbf{R}^n)$. Since φ_1 satisfies (62) we have for $\operatorname{Re}\alpha < n-k$,

$$\begin{aligned}\left|\int \varphi_1(x)|x|^{\operatorname{Re}\alpha-n}\,dx\right| &\leq C\int_\epsilon^\infty |x|^{k-n}|x|^{\operatorname{Re}\alpha-n}|x|^{n-1}d|x| \\ &= C\int_\epsilon^\infty |x|^{\operatorname{Re}\alpha+k-n-1}d|x| < \infty\end{aligned}$$

so $I^\alpha\varphi_1$ is holomorphic in this half-plane. On the other hand $I^\alpha\varphi_2$ is holomorphic for $\alpha \in \mathbf{C}_n$ which contains this half-plane. Now we can put $\alpha = -k$ in (61). As a result of (39), $f(x) = 0(|x|^{-N})$ implies that $(I^\lambda f)(x)$ is holomorphic near $\lambda = 0$ and $I^0 f = f$. Thus the proposition is proved.

Denoting by C_N the class of continuous functions f on $\mathbf{R}^n$ satisfying $f(x) = 0(|x|^{-N})$ we proved in (62) that if $N > n$, $0 < k < n$, then

$$I^k C_N \subset C_{n-k}\,. \tag{63}$$

More generally, we have the following result.

Proposition 5.8. *If* $N > 0$ *and* $0 < \operatorname{Re}\gamma < N$*, then*

$$I^\gamma C_N \subset C_s$$

where $s = \min(n, N) - \operatorname{Re}\gamma \quad (n \neq N)$.

Proof. Modifying the proof of Prop. 5.7 we divide the integral

$$I = \int (1+|y|)^{-N}|x-y|^{\operatorname{Re}\gamma-n}\,dy$$

into integrals I_1, I_2 and I_3 over the disjoint sets

$$A_1 = \{y : |y-x| \leq \tfrac{1}{2}|x|\}, \qquad A_2 = \{y : |y| < \tfrac{1}{2}|x|\,,$$

and the complement $A_3 = \mathbf{R}^n - A_1 - A_2$. On A_1 we have $|y| \geq \frac{1}{2}|x|$ so

$$I_1 \leq \left(1+\frac{|x|}{2}\right)^{-N}\int_{A_1}|x-y|^{\operatorname{Re}\gamma-n}\,dy = \left(1+\frac{|x|}{2}\right)^{-N}\int_{|z|\leq|x|/2}|z|^{\operatorname{Re}\gamma-n}\,dz$$

so

$$I_1 = 0(|x|^{-N+\operatorname{Re}\gamma})\,. \tag{64}$$

On A_2 we have $|x| + \frac{1}{2}|x| \geq |x-y| \geq \frac{1}{2}|x|$ so

$$|x-y|^{\operatorname{Re}\gamma-n} \leq C|x|^{\operatorname{Re}\gamma-n}, \quad C = \text{const.}.$$

Thus

$$I_2 \leq C|x|^{\mathrm{Re}\,\gamma-n} \int_{A_2} (1+|y|)^{-N} .$$

If $N > n$ then

$$\int_{A_2} (1+|y|)^{-N}\, dy \leq \int_{\mathbf{R}^n} (1+|y|)^{-N}\, dy < \infty .$$

If $N < n$ then

$$\int_{A_2} (1+|y|)^{-N}\, dy \leq C|x|^{n-N} .$$

In either case

$$I_2 = 0(|x|^{\mathrm{Re}\,\gamma-\min(n,N)}) . \tag{65}$$

On A_3 we have $(1+|y|)^{-N} \leq |y|^{-N}$. The substitution $y = |x|u$ gives (with $e = x/|x|$)

$$I_3 \leq |x|^{\mathrm{Re}\,\gamma-N} \int_{|u|\geq\frac{1}{2},|e-u|\geq\frac{1}{2}} |u|^{-N}|e-u|^{\mathrm{Re}\,\gamma-n}\, du = 0(|x|^{\mathrm{Re}\,\gamma-N}) . \tag{66}$$

Combining (64)–(66) we get the result.

We conclude with a consequence of Theorem 4.5 observed in John [1935]. Here the Radon transform maps functions $\mathbf{R}^n$ into functions on a space of $(n+1)$ dimensions and the range is the kernel of a single differential operator. This may have served as a motivation for the range characterization of the X-ray transform in John [1938]. As before we denote by $(M^r f)(x)$ the average of f on $S_r(x)$.

Theorem 5.9. *For f on $\mathbf{R}^n$ put*

$$\widehat{f}(x,r) = (M^r f)(x) .$$

Then

$$\mathcal{E}(\mathbf{R}^n)\widehat{\ } = \{\varphi \in \mathcal{E}(\mathbf{R}^n \times \mathbf{R}^+) : L_x\varphi = \partial_r^2\varphi + \frac{n-1}{r}\partial_r\varphi\} .$$

The inclusion $\subset$ follows from Lemma 3.2, Ch. I. Conversely suppose φ satisfies the Darboux equation. The extension $\Phi(x,y) = \varphi(x,|y|)$ then satisfies $L_x\Phi = L_y\Phi$. Using Theorem 4.5 on the function $(x,y) \to \Phi(x + x_0, y)$ we obtain $\varphi(x_0,r) = (M^r f)(x_0)$ so $\widehat{f} = \varphi$ as claimed.

Bibliographical Notes

§1-2 contain an exposition of the basics of distribution theory following Schwartz [1966]. The range theorems (3.1–3.3) are also from there but we have used the proofs from Hörmander [1963]. Theorem 3.4 describing the topology of $\mathcal{D}$ in terms of $\widetilde{\mathcal{D}}$ is from Hörmander [1983], Vol. II, Ch. XV. The idea of a proof of this nature involving a contour like Γ_m appears already in Ehrenpreis [1956] although not correctly carried out in details. In the proof we specialize Hörmander's convex set K to a ball; it simplifies the proof a bit and requires Cauchy's theorem only in a single variable. The consequence, Theorem 4.1, and its proof were shown to me by Hörmander in 1972. The theorem appears in Ehrenpreis [1956].

Theorem 4.5, with the proof in the text, is from Ásgeirsson [1937]. Another proof, with a refinement in odd dimension, is given in Hörmander [1983], Vol. I. A generalization to Riemannian homogeneous spaces is given by the author in [1959]. The theorem is used in the theory of the X-ray transform in Chapter I.

§5 contains an elementary treatment of the results about Riesz potentials used in the book. The examples x_+^λ are discussed in detail in Gelfand-Shilov [1959]. The potentials I^λ appear there and in Riesz [1949] and Schwartz [1966]. In the proof of Proposition 5.7 we have used a suggestion by R. Seeley and the refinement in Proposition 5.8 was shown to me by Schlichtkrull. A thorough study of the composition formula (Prop. 5.5) was carried out by Ortner [1980] and a treatment of Riesz potentials on L^p-spaces (Hardy-Littlewood-Sobolev inequality) is given in Hörmander [1983], Vol. I, §4.

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Notational Conventions

Algebra As usual, $\mathbf{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively, and $\mathbb{Z}$ the ring of integers. Let

$$\mathbf{R}^+ = \{t \in \mathbf{R} : t \geq 0\}, \quad \mathbb{Z}^+ = \mathbb{Z} \cap \mathbf{R}^+ .$$

If $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha$ denotes the real part of α, $\operatorname{Im}\alpha$ its imaginary part, $|\alpha|$ its modulus.

If G is a group, $A \subset G$ a subset and $g \in G$ an element, we put

$$A^g = \{gag^{-1} : a \in A\}, g^A = \{aga^{-1} : a \in A\}.$$

The group of real matrices leaving invariant the quadratic form

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

is denoted by $\mathbf{O}(p, q)$. We put $\mathbf{O}(n) = \mathbf{O}(o, n) = \mathbf{O}(n, o)$, and write $\mathbf{U}(n)$ for the group of $n \times n$ unitary matrices. The group of isometries of Euclidean n-space $\mathbf{R}^n$ is denoted by $M(n)$.

Geometry The $(n-1)$-dimensional unit sphere in $\mathbf{R}^n$ is denoted by $\mathbf{S}^{n-1}$, Ω_n denotes its area. The n-dimensional manifold of hyperplanes in $\mathbf{R}^n$ is denoted by $\mathbf{P}^n$. If $0 < d < n$ the manifold of d-dimensional planes in $\mathbf{R}^n$ is denoted by $G(d, n)$; we put $G_{d,n} = \{\sigma \in G(d, n) : o \in \sigma\}$. In a metric space, $B_r(x)$ denotes the open ball with center x and radius r; $S_r(x)$ denotes the corresponding sphere. For $\mathbf{P}^n$ we use the notation $\beta_A(0)$ for the set of hyperplanes $\xi \subset \mathbf{R}^n$ of distance $< A$ from 0, σ_A for the set of hyperplanes of distance $= A$. The hyperbolic n-space is denoted by $\mathbf{H}^n$ and the n-sphere by $\mathbf{S}^n$.

Analysis If X is a topological space, $C(X)$ (resp. $C_c(X)$) denotes the sphere of complex-valued continuous functions (resp. of compact support). If X is a manifold, we denote:

$$\begin{aligned}
C^m(X) &= \left\{ \begin{array}{l} \text{complex-valued } m\text{-times continuously} \\ \text{differentiable functions on } X \end{array} \right\} \\
\mathcal{C}^\infty(X) &= \mathcal{E}(X) = \cap_{m>0} C^m(X) . \\
\mathcal{C}_c^\infty(X) &= \mathcal{D}(X) = C_c(X) \cap \mathcal{C}^\infty(X) . \\
\mathcal{D}'(X) &= \{\text{distributions on } X\} . \\
\mathcal{E}'(X) &= \{\text{distributions on } X \text{ of compact support}\} . \\
\mathcal{D}_A(X) &= \{f \in \mathcal{D}(X) : \text{ support } f \subset A\} . \\
\mathcal{S}(\mathbf{R}^n) &= \{\text{rapidly decreasing functions on } \mathbf{R}^n\} . \\
\mathcal{S}'(\mathbf{R}^n) &= \{\text{tempered distributions on } \mathbf{R}^n\} .
\end{aligned}$$

The subspaces $\mathcal{D}_H$, $\mathcal{S}_H$, $\mathcal{S}^*$, $\mathcal{S}_o$ of $\mathcal{S}$ are defined pages in Ch. I, §§1–2.

While the functions considered are usually assumed to be complex-valued, we occasionally use the notation above for spaces of real-valued functions.

The Radon transform and its dual are denoted by $f \to \widehat{f}$, $\varphi \to \check{\varphi}$, the Fourier transform by $f \to \widetilde{f}$ and the Hilbert transform by $\mathcal{H}$.

I^α, I_-^λ, I_o^λ and I_+^λ denote Riesz potentials and their generalizations. M^r the mean value operator and orbital integral, L the Laplacian on $\mathbf{R}^n$ and the Laplace-Beltrami operator on a pseudo-Riemannian manifold. The operators $\Box$ and Λ operate on certain function spaces on $\mathbf{P}^n$; $\Box$ is also used for the Laplace-Beltrami operator on a Lorentzian manifold, and Λ is also used for other differential operators.

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