p-DIVISIBLE GROUPS AND FORMAL NEARBY CYCLES

XINYU ZHOU

1. Introduction

We are going to explain Sections 2 and 4 of [10]. The rough plan is as follows:

(1) Introduce formal nearby cycles in the sense of Berkovich
(2) Discuss deformation spaces of divisible formal modules
(3) Use the formal nearby cycles to construct test functions
(4) Show certain test functions are associated

2. Formal Nearby Cycles

In this section, we adopt the following notations.

k	a discretely valued field
$\mathscr{O}_k$	the ring of integers of k
$\tilde{k}$	the residue field of k.
K	an extension field of k
$\mathrm{FSch}_{\mathscr{O}_k}$	the category of formal schemes locally of finite type over $\mathrm{Spf}\,\mathscr{O}_k$

Remark 2.1. We use k for the base field (instead of the apparently better choice K) simply because Berkovich does so in [2]. Let me change everything to K later.

Remark 2.2. In [2], Berkovich considers a wider class of formal schemes, called *special formal schemes*, whose category $\mathscr{S}\mathrm{FSch}_{\mathscr{O}_k}$ contains as a full subcategory. But we will only need to consider the smaller class of formal schemes. So we do not bother ourselves introducing new notions.

Definition 2.3. A morphism $\varphi : \mathcal{Y} \to \mathfrak{X}$ in $\mathrm{FSch}_{\mathscr{O}_k}$ is *étale* if for any ideal of definition $\mathscr{I}$ of $\mathfrak{X}$ the morphism of schemes $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}/\varphi^*\mathscr{I}) \to (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathscr{I})$ is étale.

Example 2.4. Let $\pi : Y \to X$ be an étale morphism of schemes locally of finite type over $\mathscr{O}_k$. Write Z for the special fiber of X. Denote $\hat{X}$ the completion of X along Z and $\hat{Y}$ along $\pi^{-1}(Z)$. Then the induced map $\hat{Y} \to \hat{X}$ is étale.

In [3], Berkovich introduces the notion of quasi-étale morphisms of Berkovich analytic spaces. We cite the main properties of quasi-étale morphisms that we are going to use in the following proposition.

Proposition 2.5.

(1) The assignment $\mathfrak{Y} \mapsto \mathfrak{Y}_s$ induces an equivalence between the category of formal schemes étale over $\mathfrak{X}$ and the category of schemes étale over $\mathfrak{X}_s$.

(2) If $\varphi : \mathfrak{Y} \to \mathfrak{X}$ is étale, then $\varphi_\eta(\mathfrak{Y}_\eta) = \pi^{-1}(\varphi_s(\mathfrak{Y}_s))$

$$\begin{array}{ccc} \mathfrak{Y}_\eta & \xrightarrow{\pi} & \mathfrak{Y}_s \\ {\scriptstyle \varphi_\eta}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_s} \\ \mathfrak{X}_\eta & \xrightarrow[\pi]{} & \mathfrak{X}_s \end{array}$$

In particular, $\varphi_\eta(\mathfrak{Y}_\eta)$ is a closed analytic domain in $\mathfrak{X}_\eta$.

(3) If $\varphi : \mathfrak{Y} \to \mathfrak{X}$ is an étale morphism, then the induced morphism $\varphi_\eta : \mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ of k-analytic spaces is quasi-étale.

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Let's fix a quasi-inverse of the reduction functor $\mathfrak{Y} \mapsto \mathfrak{Y}_s$. The composition of functors

$$\mathfrak{Y}_s \mapsto \mathfrak{Y} \mapsto \mathfrak{Y}_\eta$$

induces a morphism of sites

$$\nu : \mathfrak{X}_{\eta_K,\mathrm{qet}} \to \mathfrak{X}_{s_K,\mathrm{et}}$$

Let μ be the morphism of sites

$$\mu : \mathfrak{X}_{\eta_K,\mathrm{qet}} \to \mathfrak{X}_{\eta_K,\mathrm{et}}$$

We get a left exact functor

$$\theta^K = \nu_*\mu^* : \mathrm{Sh}(\mathfrak{X}_{\eta_K,\mathrm{et}}) \to \mathrm{Sh}(\mathfrak{X}_{\eta_K,\mathrm{qet}}) \to \mathrm{Sh}(\mathfrak{X}_{s_K,\mathrm{et}})$$

Remark 2.6. I think everything done in [2] can be rewritten in terms of adic spaces. But I don't know how to do that.

Proposition 2.7 ([2] Proposition 2.2)**.** *Let $\mathcal{F}$ be an etale sheaf on $\mathfrak{X}_{\eta_K}$.*

(1) If $\mathfrak{Y}_s$ is etale over $\mathfrak{X}_{s_K}$, then $\theta^K(\mathcal{F})(\mathfrak{Y}_s) = F(\mathfrak{Y}_s)$.

(2) If $\mathcal{F}$ is an abelian sheaf, then $\mathrm{R}^q\theta^K(F)$ *is the sheafification of the presheaf*

$$\mathcal{Y}_s \mapsto \mathrm{H}^q(\mathcal{Y}_\eta, \mathcal{F})$$

(3) If $\mathcal{F}$ is a soft abelian sheaf, then the sheaf $\theta^K(\mathcal{F})$ is flasque.

Denote by θ_K the functor

$$\theta_K : \mathrm{Sh}(\mathfrak{X}_{\eta,\mathrm{et}}) \to \mathrm{Sh}(\mathfrak{X}_{s_K,\mathrm{et}})$$
$$\mathcal{F} \mapsto \theta^K(\mathcal{F}_K)$$

where $\mathcal{F}_K$ is the pullback of $\mathcal{F}$ to $\mathfrak{X}_{\eta_K}$ If $\mathcal{F}$ is soft, then $\mathcal{F}_K$ is also soft. So there is a canonical isomorphism

$$\mathrm{R}^q\theta_K(\mathcal{F}) \xrightarrow{\sim} \mathrm{R}^q\theta^K(\mathcal{F}_K), \qquad \forall q \geq 0$$

We define the nearby cycle functor as

$$\theta = \theta_k : \mathrm{Sh}(\mathfrak{X}_{\eta,\mathrm{et}}) \to \mathrm{Sh}(\mathfrak{X}_{s,\mathrm{et}})$$

and the functor Ψ_η, also called the *nearby cycle functor*, as

$$\Psi_\eta : \theta_{K^s} : \mathrm{Sh}(\mathfrak{X}_{\eta,\mathrm{et}}) \to \mathrm{Sh}(\mathfrak{X}_{s,\mathrm{et}})$$

As for the nearby cycle functors for schemes, the formal nearby cycle sheaves can be compared to the cohomology of the generic fiber in "nice cases".

Theorem 2.8 ([2] Corollary 2.5)**.** *Let $\mathfrak{X} \in \mathrm{FSch}_{\mathscr{O}_k}$ Assume all irreducible components of $\mathfrak{X}_s$ are proper. Then there is a canonical isomorphism*

$$\mathrm{R}\Gamma_c(\mathfrak{X}_s, \mathrm{R}\Phi\,\mathcal{F}^\bullet) \xrightarrow{\sim} \mathrm{R}\Gamma_c(\mathfrak{X}_\eta, \mathcal{F}^\bullet)$$

If the formal scheme $\mathfrak{X}$ is the formal completion of a scheme X, then the formal nearby cycle sheaves can be compared to the corresponding nearby cycle sheaves on X. Let X be a scheme locally of finite type over $S = \mathrm{Spec}\ \mathscr{O}_k$, and $Y \subseteq X_s$ a closed subscheme. Denote by $\hat{X}_{/Y}$ the formal completion of X along Y, and by $\hat{X}$ the formal completion of X along the special fiber X_s. For an étale sheaf $\mathcal{F}$ on $\mathfrak{X}_\eta$, write $\hat{\mathcal{F}}$ (resp. $\hat{\mathcal{F}}_{/Y}$) for the its pullback to $\hat{X}$ (resp. $\hat{X}_{/Y}$).

Theorem 2.9 (Comparison Theorem, [2] Theorem 3.1)**.** *Let $\mathcal{F}$ be an étale abelian constructible sheaf on X_η with torsion order prime to* $\mathrm{char}\ \tilde{k}$*. Then for any $q \geq 0$, there are canonical isomorphisms*

$$(\mathrm{R}^q\theta\,\mathcal{F})\,|_Y \xrightarrow{\sim} \mathrm{R}^q\theta(\hat{\mathcal{F}}_{/Y})$$

and

$$(\mathrm{R}^q\Psi\,\mathcal{F})\,|_Y \xrightarrow{\sim} \mathrm{R}^q\Psi(\hat{\mathcal{F}}_{/Y})$$

3. ϖ-DIVISIBLE $\mathscr{O}$-MODULES

In this section, we adopt the following notations.

$F\vert\mathbb{Q}_p$	a finite extension	$\breve{F}$	maximal unramified extension
$\mathscr{O}$	ring of integers of F	$\breve{\mathscr{O}}$	ring of integers of $\breve{F}$
ϖ	a uniformizer of F	σ	arithmetic Frobenius of F_r over F
κ	residue field of F	σ_0	arithmetic Frobenius of $W(\kappa_r)$ over $\mathbb{Z}_p$
$F_r\vert F$	unramified extension of degree r	Frob	fixed geometric Frobenius in W_F
$\mathscr{O}_r$	ring of integers of F_r		
κ_r	residue field of F_r		

We first recall the definition of ϖ-divisible $\mathscr{O}$-module.

Definition 3.1. Let S be an $\mathscr{O}$-scheme on which p is locally nilpotent. A ϖ-divisible $\mathscr{O}$-module H over S is a p-divisible group together with an action $\iota : \mathscr{O} \to \mathrm{End}(H)$ such that the two induced actions of $\mathscr{O}$ on the Lie algebra of H agree.

Example 3.2. Recall the Lubin-Tate formal group LT_F is a formal $\mathscr{O}$-module over $\mathscr{O}$. So its base change $\mathrm{LT}_{F,S}$ to any $\mathscr{O}$-scheme on which p is locally nilpotent is a ϖ-divisible $\mathscr{O}$-module over S. But we can also add étale parts to LT_F. For example, $\mathrm{LT}_F \times (F/\mathscr{O})^j$ base-changed to S is again ϖ-divisible $\mathscr{O}$-module over S.

Now we consider the Diedonné theory (i.e. classification) of ϖ-divisible $\mathscr{O}$-modules. The following is essentially a verbatim reproduction of paragraphs in [10]. Let H be a ϖ-divisible $\mathscr{O}$-module over a perfect field k of characteristic p, which is given the structure of a $\mathscr{O}$-algebra, via a map $\kappa \to k$. Then the usual Dieudonné module (M_0, F_0, V_0) of H carries an action of

$$\mathscr{O} \otimes W(k) = \prod_{\kappa\to k} W_{\mathscr{O}}(k),$$

where $W_{\mathscr{O}}(k)$ is the completion of the unramified extension of $\mathscr{O}$ with residue field k. Let M be the component of M_0 corresponding to the given map $\kappa \to k$, which is a free $W_{\mathscr{O}}(k)$-module. Assume that $\kappa \cong \mathbb{F}_{p^j}$ for some j. Then M carries a σ-semilinear action of F_0^j, which we denote by F in this context. One can check that M also admits a σ^{-1}-semilinear operator V satisfying

$$FV = VF = \varpi.$$

The structure (M, F, V) is functorial in H and is called the relative Dieudonné module of H. It is an easy exercise to see that all of Dieudonné theory goes through in this context.
In particular, to any $\beta \in \mathrm{GL}_n(\mathscr{O}_r)\mathrm{diag}(\varpi, 1, \ldots, 1)\,\mathrm{GL}_n(\mathscr{O}_r)$, one can associate a one-dimensional ϖ-divisible $\mathscr{O}$-module $\overline{H}_\beta$ of height n over κ_r, by taking $F = \beta\sigma$. Conversely, any one-dimensional ϖ-divisible $\mathscr{O}$-module of height n over κ_r is associated to a unique $\mathrm{GL}_n(\mathscr{O}_r)$-$\sigma$-conjugacy class of such β.

Consider the deformation functor

$$\mathrm{Def}_\beta : \mathrm{Nilp}_{\mathscr{O}_r} \to \mathrm{Sets}, R \mapsto \{(G, \iota)\}$$

where G is a ϖ-divisible $\mathscr{O}$-module over R and $\iota : G \otimes_R R/\varpi \xrightarrow{\sim} H \otimes_{\kappa_r} R/\varpi$ is an isomorphism. By [6], or well-known theory, Def_β is pro-presented by a ring R_β with a universal deformation H_β. Similarly, we have the deformation function $\mathrm{Def}_{\beta,m}$ with Drinfeld level-m-structure. It is pro-represented by a ring $R_{\beta,m}$.

Proposition 3.3 ([10] Proposition 2.3)**.**

(1) The ring R_β is a formally smooth complete noetherian local $\mathscr{O}_r$-algebra, abstractly isomorphic to $\mathscr{O}_r[[T_1, \ldots, T_{n-1}]]$.
(2) The covering $R_{\beta,m}/R_\beta$ is a finite Galois covering with Galois group $\mathrm{GL}_n(\mathscr{O}/\varpi^m\mathscr{O})$, étale in the generic fibre.
(3) The ring $R_{\beta,m}$ is regular.

Proof. All parts follow from [6] Proposition 4.3 and Proposition 4.5. Notice that [6] actually considers $\mathscr{O}^{\mathrm{ur}}$-modules. But we can use Galois descent to deduce the statements here. □

Remark 3.4. The orginal proof of the proposition reduces the problem to the Lubin-Tate case by using [10] Proposition 5.1. But I think the statements already follow from Drinfeld's paper [6].

In order to compare formal nearby cycles and schematic nearby cycles, we need algebraizations of the deformation spaces $\mathrm{Spf}\, R_{\beta,m}$

Theorem 3.5 ([10] Theorem 2.4)**.** *Associated to any double coset*

$$\bar{\beta} \in (1 + \varpi^m M_n(\mathscr{O}))\backslash \mathrm{GL}_n(\mathscr{O}_r)\mathrm{diag}(\varpi, 1, \ldots, 1)\,\mathrm{GL}_n(\mathscr{O}_r)/(1 + \varpi^m M_n(\mathscr{O}))$$

there is a flat scheme $\mathrm{Spec}\ \mathcal{R}_{\beta,m}$ of finite type over $\mathscr{O}_r$ with smooth generic fiber equipped with an action of $\mathrm{GL}_n(\mathscr{O}/\varpi^m)$, and a finite scheme $Z \subset \mathrm{Spec}\ \mathcal{R}_{\beta,m} \otimes_{\mathscr{O}_r} \kappa_r$ stable under this action *such that the completion* $\operatorname{Spec} \mathcal{R}_{\beta,m}$ *at* Z *is* $\mathrm{GL}_n(\mathscr{O})$*-equivariantly isomorphic to* $R_{\beta,m}$ *for any* $\beta \in \bar{\beta}$*.*

Proof. Faltings [7] shows the functor $H \mapsto H[\varpi^m]$ from ϖ-divisible $\mathscr{O}$-modules to m-truncated ϖ-divisible $\mathscr{O}$-modules is formally smooth. So $\operatorname{Spf} R_\beta$ is also a versal deformation space of $H_\beta[\varpi^m]$. Now Artin's algebraization theorem shows there is a separated scheme $\mathfrak{R}_{\bar{\beta}}$ of finite type over $\mathscr{O}_r$ together with an m-truncated ϖ-divisible $\mathscr{O}$-module $\mathcal{H}_{\bar{\beta}}$ and a point $x \in \mathfrak{R}_{\bar{\beta}}(k_r)$ such that the completion of $\mathfrak{R}_{\bar{\beta}}$ at x with $\mathcal{H}_{\bar{\beta}}$ restricted to the formal completion is isomorphic to $\operatorname{Spf} R_\beta$ with $H_\beta[\varpi^m]$ for all $\beta \in \bar{\beta}$. Recall the universal deformation ring R_β itself is normal and flat over $\mathscr{O}_r$. So by shrinking the scheme $\mathfrak{R}_{\bar{\beta}}$ and normalizing, we can assume $\mathfrak{R}_{\bar{\beta}} = \operatorname{Spec} \mathcal{R}_{\beta,m}$ is affine, normal, and flat over $\mathscr{O}_r$ with smooth geneic fiber. Now let $\mathcal{R}_{\beta,m}$ be the normalization of $\mathcal{R}_{\beta,m}$ in the covering of the generic fibre parametrizing trivializations of $\mathcal{H}_{\bar{\beta}}$. Let Z be the preimage of x in $\mathcal{R}_{\beta,m}$.

□

Consider the formal nearby cycles

$$\mathrm{R}\Psi_\beta := \varinjlim_m \mathrm{H}^0(\mathrm{R}\Psi_{\operatorname{Spf} R_{\beta,m}}, \overline{\mathbb{Q}}_\ell)$$

and the object

$$[\mathrm{R}\Psi_\beta] := \sum_i (-1)^i \varinjlim_m \mathrm{H}^0(\mathrm{R}^i\Psi_{\operatorname{Spf} R_{\beta,m}}, \overline{\mathbb{Q}}_\ell)$$

in the Grothendieck group of representations of $W_{F_r} \times \mathrm{GL}_n(\mathscr{O})$.

Following the methods of [8], we want to study the representations given by the formal nearby cycle sheaves on $\operatorname{Spf} R_{\beta,m}$. However, in order to have nice representations, we at least need to show $[\mathrm{R}\Psi_\beta]$ has continuous W_{F_r}-action and smooth admissible $\mathrm{GL}_n(\mathscr{O})$-action.

Theorem 3.6 ([10] Theorem 2.5)**.** *The space* $\mathrm{H}^0(\mathrm{R}^i\Psi_{\operatorname{Spf} R_{\beta,m}} \overline{\mathbb{Q}}_\ell)$ *is a finite dimensional continuous representation of* $W_{F_r} \times \mathrm{GL}_n(\mathscr{O}/\varpi^m)$*, which vanishes outside the range of* $0 \le i \le n-1$*.*

Proof. By Theorem 3.5, and the Comparison Theorem 2.9 of formal nearby cycles,

$$\mathrm{R}^i\Psi_{\operatorname{Spf} R_{\beta,m}} \overline{\mathbb{Q}}_\ell = \mathrm{R}^i\Psi_{\operatorname{Spec} \mathcal{R}_{\beta,m}} \overline{\mathbb{Q}}_\ell|_{Z \otimes_{\kappa_r} \bar{k}}$$

Now $\operatorname{Spec} \mathcal{R}_{\beta,m}$ is affine of relative dimension $n-1$ over $\operatorname{Spec} \mathscr{O}$. Thus, $\mathrm{R}^i\Psi_{\operatorname{Spec} \mathcal{R}_{\beta,m}} \overline{\mathbb{Q}}_\ell|_{Z \otimes_{\kappa_r} \bar{k}}$ vanishes if i is not in the range $[0, n-1]$. By a result by Deligne ([5] SGA $4\frac{1}{2}$ Chapitre 7 Theorem 3.2), $\mathrm{R}^i\Psi_{\operatorname{Spec} \mathcal{R}_{\beta,m}} \overline{\mathbb{Q}}_\ell$ is a constructible $\overline{\mathbb{Q}}_\ell$-sheaf on the geometric special fiber $(\operatorname{Spec} \mathcal{R}_{\beta,m})_{\bar{s}}$. Thus, by finiteness results of étale cohomology, the space

$$\mathrm{H}^0(\mathrm{R}^i\Psi_{\operatorname{Spec} \mathcal{R}_{\beta,m}} \overline{\mathbb{Q}}_\ell|_{Z \otimes_{\kappa_r} \bar{k}})$$

is a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space. The action of $W_{F_r} \times \mathrm{GL}_n(\mathscr{O}/\varpi^m)$ on

$$\mathrm{H}^0(\mathrm{R}^i\Psi_{\mathrm{Spec}\ \mathcal{R}_{\beta,m}}\overline{\mathbb{Q}}_\ell|_{Z\otimes_{\kappa_r}\bar{k}})$$

comes from the group action on the formal scheme $\mathrm{Spf}\ R_{\beta,m}$. The continuity of the representation is clear. □

For any $\tau \in \mathrm{Frob}^r I_F \subset W_{F_r}$ and $h \in C_c^\infty(\mathrm{GL}_n(\mathscr{O}), \mathbb{Q})$, we define a function $\phi_{\tau,h}$ on $\mathrm{GL}_n(F_r)$ by

$$\phi_{\tau,h}(\beta) = \mathrm{tr}(\tau \times h^\vee|[\mathrm{R}\Psi_\beta])$$

where $h^\vee$ is defined by

$$h^\vee(g) = h({}^tg^{-1})$$

Theorem 3.7 ([10] Theorem 2.6). $\phi_{\tau,h} \in C_c^\infty(\mathrm{GL}_n(F_r), \mathbb{Q})$ *independent of* ℓ.

Proof. By Theorem 3.6, $\phi_{\tau,h}$ is locally constant in β. The independence of ℓ follows from a result by Mieda [9]. □

Finally, we let $f_{\tau,h} \in C_c^\infty(\mathrm{GL}_n(F))$ be associated to $\phi_{\tau,h}$. Recall this means the following:

$$O_\gamma(f) = \begin{cases} \pm TO_{\delta,\sigma}(\phi) & \text{if } \gamma \text{ is conjugate to } N\delta \text{ for some } \delta \\ 0 & \text{else} \end{cases}$$

for all semisimple $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$. Here, the sign is $+$ except if $N\delta$ is a central element, but δ is not σ-conjugate to a central element, when it is $-$. In this situation, we also say ϕ and f have *matching (twisted) orbital integrals.* The existence of $f_{\tau,h}$ follows from [1] Proposition 3.1 and [4] Proposition 7.2.

Remark 3.8. There is also a variant of "being associated", which we may call "being *regularly* associated" or "having *regular* matching (twisted) orbital integrals". This variant appears in [1].

4. Associated Test Functions

We are going to define functions h and ϕ_h in two contexts

(1) D-group H for a semisimple algebra over $\mathbb{Q}_p$
(2) one-dimensional formal $\mathscr{O}$-module of height n

In case (1),

$$\begin{aligned}&h \in C_c^\infty(\mathscr{O}_D^\times) \text{ invariant under conjugation}\\&\phi_h \in C_c^\infty((\mathscr{O}_D \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})^\times),\ \phi_h := h(N\beta)\end{aligned}$$

In case (2),

$$\begin{aligned}&h \in C_c^\infty(\mathcal{D}_r) \text{ which is invariant under } \mathscr{O}_{\mathcal{D}}^\times\text{-conjugation}\\&\phi_h \in C_c^\infty(B_r),\ \phi_h := h(N\beta)\end{aligned}$$

We want to show

Proposition 4.1 ([10] Proposition 4.3, Corollary 4.5 Proposition 4.7, Corollary 4.8)**.** *In both cases, h and ϕ_h are associated. Thus,*

- *in case (1),*
$$\int_{(\mathscr{O}_D \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})^\times} \phi_h(\beta) d\beta = \int_{\mathscr{O}_D^\times} h(\gamma) d\gamma$$
- *in case (2),*
$$\int_{B_r} \phi_h(\beta) d\beta = \int_{\mathcal{D}_r} h(\gamma) d\gamma$$

Let's first explain some of the notations. Let D be a semisimple algebra over $\mathbb{Q}_p$ with a maximal order $\mathscr{O}_D$, and let $\mathcal{D}$ be the central divisor algebra over F with invariant $1/n$ with ring of integers $\mathscr{O}_{\mathcal{D}}$. There is a natural valuation $v : \mathcal{D}^\times \to \mathbb{Z}$, taking $\varpi \in F \subset \mathcal{D}$ to n. Let B_r be the set of basic elements in $\mathrm{GL}_n(\mathscr{O}_r)\mathrm{diag}(\varpi, 1, \ldots, 1)\,\mathrm{GL}_n(\mathscr{O}_r)$, by which we mean those elements that are basic as elements of $\mathrm{GL}_n(\breve{F})$. Also, we let $\mathcal{D}_r$ be the set of elements of $\mathcal{D}^\times$ whose valuation is r.

Definition 4.2. Let S be scheme on which p is locally nilpotent. A D-group over S is an étale p-divisible group H over S together with an action

$$\iota : \mathscr{O}_D^{\mathrm{op}} \to \mathrm{End}(H)$$

such that $H[p]$ is free of rank 1 over $\mathscr{O}_D^{\mathrm{op}}/p$.

The main tool we use is the two parametrizations of ϖ-divisible $\mathscr{O}$-modules over $\mathbb{F}_{p^r}$: Dieudonné and Galois parametrizations.

Dieudonné parametrization. Let H be a ϖ-divisible $\mathscr{O}$-module over $\mathbb{F}_{p^r}$, and let M be the contravariant Dieudonné module of H. Then $M \cong \mathscr{O}_D \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r}$ as an $\mathscr{O}_D^{\mathrm{op}}$-module. The Frobenius endomorphism F on M can be written as $F = \beta^{-1}\sigma_0$ for some $\beta \in (\mathscr{O}_D \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})^\times$. The element β is well-defined up to σ_0-conjugation by elements of $(\mathscr{O}_D \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})^\times$. This establishes a natural bijection between the set of isomorphism classes of H over $\mathbb{F}_{p^r}$ and the set of σ_0-conjugacy classes in $(\mathscr{O}_D \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})^\times$.

Galois parametrization. Over the algebraic closure $\bar{\mathbb{F}}_p$, there is a unique D-group $\tilde{H}$ (up to isomorphism). So parametrizing D-groups H over $\mathbb{F}_{p^r}$ is equivalent to adding a descent datum to $\mathbb{F}_{p^r}$, i.e., an isomorphism

$$\alpha : \mathrm{Frob}^* \tilde{H} \cong \tilde{H}$$

Let

$$F : \bar{\mathbb{F}}_p \to \bar{\mathbb{F}}_p$$

be the p-th power map. Then there is the natural Frobenius isogeny $F : \tilde{H} \to \tilde{H}$ defined for any p-divisible group, which is an isomorphism in the case of étale p-divisible groups. Then giving a descent datum α is equivalent to giving $\gamma = \alpha \circ F^{-r} : \tilde{H} \to \tilde{H}$, which is an $\mathscr{O}_D^{\mathrm{op}}$-linear automorphism. Since $\mathrm{End}_{\mathscr{O}_D^{\mathrm{op}}}(\tilde{H}) = \mathscr{O}_D$, it follows that giving a descent datum is equivalent to giving an element $\gamma \in \mathscr{O}_D^\times$. One easily checks that γ is well-defined up to $\mathscr{O}_D^\times$-conjugation.

The following proposition is clear.

Proposition 4.3. *The two parametrizations define a bijection*

$$\begin{Bmatrix} \sigma_0\text{-conjugacy clasees} \\ \text{in } (\mathscr{O}_D \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})^\times \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{conjugacy clasees in} \\ \mathscr{O}_D^\times \end{Bmatrix}$$
$$\beta \longmapsto N\beta$$

Now we give a proof of Proposition 4.1 in Case (1).

Proof of Proposition 4.1: Case (1). Since every conjugacy class in $\mathscr{O}_D^\times$ is matched with a σ_0-conjugacy class in $(\mathscr{O}_D \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})^\times$, we only need to show for any $\beta \in (\mathscr{O}_D \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})^\times$,

$$TO_{\beta\sigma_0}(\phi_h) = O_{N\beta}(h)$$

Let H be the D-group over $\mathbb{F}_{p^r}$ associated to β. Consider the set X of D-groups H' over $\mathbb{F}_{p^r}$ together with an $\mathscr{O}_D^{\mathrm{op}}$-linear quasi-isogeny $\alpha : H' \to H$. On this set X, we have an action of $\Gamma = (\mathrm{End}(H) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^\times$ by composition.

It is easy to see that for any $x \in X$, the stabilizer $\Gamma_x \subset \Gamma$ is a maximal compact subgroup. This shows that all Γ_x have the same volume. We may normalize the Haar measure by requiring that these subgroups have volume 1.

We can define a Γ-invariant function $\tilde{h}$ on X by requiring $\tilde{h}(H', \alpha) = h(\gamma(H))$, where

$\gamma(H')$ is the element in $\mathscr{O}_D^\times$ associated to H' as in the Galois parametrization. We want to show the following equality

$$TO_{\beta\sigma_0}(\phi_h) = O_{N\beta}(h) = \sum_{x \in X/\Gamma} \tilde{h}(x).$$

Now on the one hand, Dieudonné theory gives an isomorphism between Γ and the twisted centralizer of g, i.e.,

$$\Gamma = \{g \in (D \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r})^\times \mid g^{-1}\beta g^\sigma = \beta\} =: G_{g\sigma}$$

and an identification of X with

$$\{g \in (D \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r})^\times \mid g^{-1}\beta g^\sigma \in \mathscr{O}_D^\times\}/(\mathscr{O}_D \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})^\times$$

by sending g to the D-group H' given by the lattice $gM \subset M \otimes M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and the corresponding quasi-isogeny $\alpha : H' \to H$. Also $\tilde{h}(H', \alpha) = \phi_h(g^{-1}\beta g^\sigma)$ under this correspondence.

We may regard the function $\phi_h \in C_c^\infty((\mathscr{O}_D \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})^\times)$ as a function on $(D \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r})^\times$ by setting $\phi_h(g)$ if $g \notin (\mathscr{O}_D \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})^\times$. Write $W = \{g \in (D \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r})^\times \mid g^{-1}\beta g^\sigma \in \mathscr{O}_D^\times\}$ and $W = X \times (\mathscr{O}_D \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})^\times$. Then the twisted orbital integral $TO_{\beta\sigma_0}(\phi_h)$ is

$$\begin{aligned}
TO_{\beta\sigma_0}(\phi_h) &= \int_{G_{g\sigma} \backslash (D \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^r})^\times} \phi_h(g^{-1}\beta g^\sigma) dg \\
&= \int_{(X \times (\mathscr{O}_D \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})^\times)/\Gamma} \phi_h(g^{-1}\beta g^\sigma) dg \\
&= \int_{X/\Gamma} \int_{(\mathscr{O}_D \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})^\times/\Gamma} \tilde{h}(x) d\alpha dx \\
&= \int_{X/\Gamma} \tilde{h}(x) dx \\
&= \sum_{x \in X/\Gamma} \tilde{h}(x)
\end{aligned}$$

where the second last equality is given by the normalization of the Haar measure.

The other equality, $O_{N\beta}(h) = \sum_{x \in X/\Gamma} \tilde{h}(x)$ is given by applying similar arguments to the Galois parametrization.

Now since ϕ_h and h are associated, the equality

$$\int_{(\mathscr{O}_D \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r})^\times} \phi_h(\beta) d\beta = \int_{\mathscr{O}_D^\times} h(\gamma) d\gamma$$

follows from the Weyl integration formula. □

Remark 4.4. There are still some problems with the calculation of the twisted orbital integral. They will be fixed later.

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