Preface

This is a latex version of subtle or important materials I encountered while studying in Peking university. I started this project in the fall of my third undergraduate year (October 2019), noticing that I have a poor memory and consistently forget what I have already learned thus struggle to check details. So it came to me that I can compile all the proofs of theorems I cannot recall that is hard and subtle yet appearing over and over again. But finally it turns out I want to make it as comprehensive as possible. That’s it.

I constantly add stuffs to this note, and I regularly put them online. You can find the newest version at https://math.mit.edu/~hao_peng/skyscraper.pdf.

Constitution: The following are principles of the structure of this note, but the current version is far from it. they serve as ultimate goals.

- Be self-contained.
- Notations should be consistent throughout the whole note.
- Logical order is not necessary, but vicious circles are intolerable.
- Put every proposition in the (sub)section of the most advanced term appeared in the statement of this proposition, or as a corollary.
- Theories should be stated at the most generality. No new proof should be given for the special case, but clarify the deduction from the general case, unless it is needed in the proof of the general case, then state it as a lemma.
- When facing multiple proofs, only the most elegant and essential proof should be recorded.
- References should be traced to the original author and his specific paper.

Tips: This is hardly a readable book, I use it as a dictionary. It only contains materials that I’m interested in and many proofs are still missing. Hopefully I can complete them all as time goes by. The most important reason why I have to latex this note is because I need tons of hyper-references. So I believe the best way to read this book is to use a computer, and it’s vital to know how to go forwards and backwards between hyper-references on your computer. For example, on a Mac, the default shortcuts is \(\text{⌘}+\text{Dave}\), \(\text{⌘}+\text{⩴}\) for Preview and \(\text{⌘}+\text{⌫}, \text{⌘}+\text{→}\) for Foxit Reader.

Acknowledgement: Sincere thanks to Yi Tian (田翊) for answering my questions when I was learning algebraic geometry and p-adic geometry. His help is fundamental.

There is already a great online book StackProject that covers considerably many of the Algebraic Geometry part of this note. I haven’t finish reading it but I reordered the materials that I learned and keep track of it in my own way. I changed many proofs and omitted some easy proofs. The ideal to write this book is inspired by [Sta], maintained by Aise Johan de Jong.
Notations:
- Use $\mathcal{\text{mathcal}}$ for symbols representing a sheaf.
- Use $\mathscr{\text{mathscr}}$ for symbols representing a category.

Copyright issues: It should be made clear that I took proofs from many different places, so it should not be considered anything in this book originated from me. Until I get a full extensive reference of this note, I have few rights to the texts. But I am currently just a busy graduate student, so many references are still missing. However, I truly hope these notes can contribute to my study and help anyone who read it, but it comes with no warranty, please use at your own risk.
And they said, Go to, let us build us a city and a tower, whose top may reach unto heaven; and let us make us a name, lest we be scattered abroad upon the face of the whole earth.

—Genesis 11:1-9
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Chapter I
Algebras

I.1 Linear Algebra

References are [H-K71], [线性代数 谢启鸿], [Rom07] and [Determinant, 高等代数 notes, 安金鹏].

In this section, we study vector spaces $V$ over a field $K$ and linear operators between them, without considering the topology on $K$ or $V$.

1 Basics

Def. (I.1.1.1) (Basis). If $M$ is a vector space over a field $K$, then the sets $S$ that are linearly independent over $R$ has maximal objects by Zorn’s lemma, and such a maximal object must span $M$, called a basis of $M$.

Prop. (I.1.1.2) (Dimension). All basis of a linear $k$-vector space $V$ have the same cardinality, this cardinality is called the dimension $\dim_k(V)$ of $V$. This follows immediately from (I.2.4.4).

Prop. (I.1.1.3) (Canonical way of Writing a Basis). After so many years, I still find it confusing to write a basis and observing change of basis, so I will write it here:

A vector should always be written vertically, and so a basis should be $\underline{e} = (e_1, \ldots, e_n)(\text{horizontal})$, and a vector with basis $\underline{d}$ (vertical) is in fact $\underline{e} \cdot \underline{d}$.

A change of basis should be written $\underline{e}' = ea$, with $a \in GL_n$, and then if an operator has matrix $A$ w.r.t. the basis $\underline{e}$, it then map in the basis $\underline{e}' v = \underline{e}' x = \underline{e} a x \Rightarrow \underline{e}' A x = \underline{e}' a^{-1} A x$, so it has matrix $a^{-1}Aa$ w.r.t the basis $\underline{e}'$.

Prop. (I.1.1.4). Let $F$ is a subfield of $K$ and $U$ is a $K$-vector space with a $F$-subspace $U'$. Then if every finite $F$-linearly independent subset of $U'$ is $K$-linearly independent, then $\dim_F(U') \leq \dim_K(U)$.

Proof: This is very simple, if the converse is true, there is a $F$-basis $u'_j$ of $U'$, then some of $u'_j$ is $K$-linearly dependent, contradiction. $\square$

Prop. (I.1.1.5). $A, B$ are two $n \times n$-matrices, if $1 - AB$ is invertible, then so does $1 - BA$, and 

$$(1 - BA)^{-1} = 1 + B(1 - AB)^{-1} A.$$
Proof: Immediate from (I.2.3.2) or (I.1.9.11).

Cor. (I.1.1.6). $AB$ and $BA$ has the same characteristic polynomials.

Prop. (I.1.1.7). For a ring $R$, there is an isomorphism of rings

$$M_n(R)^{op} \cong M_n(R^{op}).$$

The isomorphism is given by $A \mapsto A^t$.

2 Rank

Prop. (I.1.2.1) (Rank Nullity Theorem).

Prop. (I.1.2.2) (Row Rank equals Column Rank). The row rank of a matrix $A$ is the same as the column rank.

Proof: Let $A$ have $n$ rows, the column rank equals dim $\text{Im } f$, and the row rank is $n - \text{dim Ker } f$, so by the rank-nullity theorem (I.1.2.1) $\text{dim Im } f + \text{dim Ker } f = n$, which is because exact sequence of vector spaces split, the conclusion follows.

Prop. (I.1.2.3) (Sylvester’s Inequality). For $U$ a $m \times n$ matrix and $V$ a $n \times k$ matrix,

$$\text{Rank}(UV) \geq \text{Rank}(U) + \text{Rank}(V) - n$$

Proof: This comes from $\text{dim Ker } fg \leq \text{dim Ker } f + \text{dim Ker } g$, which is because $\text{Ker } fg = g^{-1}(\text{Ker } f)$.

Prop. (I.1.2.4) (Finite Field General Linear Group). Over finite field $\mathbb{F}_p^k$, $|GL_n(\mathbb{F}_p^k)| = (p^n - 1)(p^n - p) \cdots (p^n - p^{n-1})$.

Proof: This is because choose the rows are equivalent to choosing a basis for $V = \oplus_{i=1}^n \mathbb{F}_p^k$, and when choosing $n$-th row, it suffices to avoid an element in the span of the first $n - 1$ rows.

3 Dual space

Prop. (I.1.3.1) (Adjoint Map and Transpose). For a linear map between two spaces of the matrix form $A$, the adjoint map between their dual spaces is of the matrix form the transpose of $A$.

Proof:

Cor. (I.1.3.2). A linear map between f.d. spaces is surjective iff its adjoint map is injective.

Proof: Let $f : V_1 \to V_2$, where $\text{dim } V_1 = m, \text{dim } V_2 = n$, then $f$ is represented by a matrix $A$ and $f^t$ is represented by a matrix $A^t$. Then $f$ is surjective iff row rank of $A = n$, and $f^t$ is surjective iff row rank of $A^t = n$ by rank-nullity theorem (I.1.2.1). Then this is equivalent by (I.1.2.2).

Prop. (I.1.3.3) (Infinite Dual space). If $\text{dim}_K V$ is not finite, then $\text{dim}_K V < \text{dim}_K V^*$. 
Proof: Notice $\text{Hom}(\bigoplus_{i \in I} Ke_i, K) = \prod Ke_i^*$. We prove first that if $|K|$ is at most countable, then $|V| = |I|$. Notice the set $S_n(I)$ of all $n$-element subsets of $I$ is of the same cardinality of $I$ (XII.1.5.3). And the finite sums of $K$ and $e_i$ can be seen as a subset of $S_n(I) \times K^n$, so it has the same cardinality of $I$.

Now we can prove if $|K|$ is at most countable, then $\dim V < \dim V^*$. This is because $V^*$ equals the functions from $V$ to $K$, which is bigger than the functions from $V$ to $\{0, 1\}$, which is the power set of $V$, so having cardinality $2^{|V|}$ which is bigger than $|V|$, by Cantor theorem (XIII.1.2.12).

Now generally, $K$ is not countable, but it has a base field $F$, which is countable, so we consider the $F$-vector space $W = \bigoplus_{i \in I} Fe_i$, then $\dim_F W = \dim_K V$, and $\dim_F W < \dim_F W^*$. If we can show $\dim_F W^* \leq \dim_K V^*$, then we are done.

For this, first consider the natural $F$-linear mapping $W^* \to V^*$, which is clearly an imbedding.

Now we want to use (I.1.4.4), so we check the conditions, for $F$-linearly independent $\varphi_1, \ldots, \varphi_n$, if $\sum c_i \varphi = 0$, $c_i \in K$, then if we can find $w_k \in W$ that $\varphi_i(w_j) = \delta_{ij}$, then this is a contradiction. But this is true, by a simple argument, using the $F$-linearity of $F$. □

4 Rational Form and Jordan Form

Prop. (I.1.4.1) (Elementary and Invariant Factors). A linear operator in $L(V)$ is equivalent to a $K[X]$-module structure on $V$, and two operators are similar iff the module structure are isomorphic.

As $K[X]$ is a PID, the elementary factors, invariant factors, cyclic and elementary decomposition theorems (I.2.4.20) can be applied to the case.

Proof: Cf. [Advanced Linear Algebra P168]. □

Cor. (I.1.4.2) (Complex Structure Form). A matrix $J \in M_n(\mathbb{R})$ that $J^2 + 1 = 0$ is similar to

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^n.$$ 

Proof: By the elementary factor theorem (I.1.4.1), the elementary factors of $J$ are all $x^2 + 1$, thus its Rational form is just

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^n.$$ □

Cor. (I.1.4.3) (Jordan Form).

- For a matrix over an alg. closed field, it is similar to a matrix of blocks $\lambda_i I + N, N x_i = x_i + 1$, called the Jordan form.

- For a real matrix, it is similar to a matrix of blocks of the above form together with

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$
on the diagonal and $I_{2 \times 2}$ on the lower side.

Proof: 1: Over an alg. closed field, the elementary factors are all of the form $(x - c_i)^{m_{ij}}$. Now in the basis $v, (T - c_i)v, \ldots, (T - c_i)^{m_{ij} - 1}v$, the matrix is just the Jordan form.

2: Over $\mathbb{R}$, the elementary factors are all of the form $(x - c_i)^{m_{ij}}$ and $((x - a)^2 + b^2)^{m_{ij}}$. Then complexify it and consider a cyclic vector $v$, for $(T - (a + bi)I)$, let $v_{n+1} = (T - (a + bi)I)v_n$, and let $v_n = X_n + iY_n$, then it can be verified that $T$ is of the Jordan form given in the basis $X_i, Y_i$. □
Def. (I.1.4.4) (Companion matrix). The companion matrix for a monic polynomial \( p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0 \in F[X] \) is the matrix
\[
T_{p(x)} = \begin{bmatrix}
0 & 0 & \cdots & 0 & -a_0 \\
1 & 0 & \cdots & 0 & -a_1 \\
0 & 1 & \cdots & \vdots & \\
\vdots & \vdots & \ddots & 0 & -a_{n-1} \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix}
\]

Prop. (I.1.4.5) (Companion Matrix is Nonderogatory). An operator is cyclic iff it is similar to a companion matrix.

A companion matrix is nonderogatory (I.1.5.2). In fact, the minimal polynomial and the maximal polynomial of the companion matrix of \( p(X) \) are both \( p(X) \).

Proof: The operator of the companion matrix is an operator \( A \) with a basis \( \{v, Av, A^2v, \ldots, A^{n-1}v\} \) and \( A^n v = -\sum_{i<n} a_i A^i v \), which just equivalent to the fact the action of \( A \) is cyclic.

The determinant of \( T_{p(X)} \) equals \( p(X) \) by (I.1.4.14), and for the minimal polynomial, in the basis of \( \{v, Av, A^2v, \ldots, A^{n-1}v\} \), clearly for a polynomial \( f(X) \) of degree \( m < n \), \( f(A)v \neq 0 \), and \( p(A)v = \sum_{i<n} a_i A^i v + A^n v = 0 \). So the minimal polynomial of \( A \) is \( p(X) \). \( \square \)

Prop. (I.1.4.6) (Rational Canonical Form). Every matrix is similar to splint of companion matrices (I.1.4.4), corresponding to its elementary divisors.

Prop. (I.1.4.7) (Invariant Factor Form). Every matrix is similar to splint of companion matrices, corresponding to its invariant factors.

Computing the Invariant Factors

Def. (I.1.4.8) (Elementary Row Operation). An elementary row operation for a matrix \( M \) over an algebra \( A \) is one of the following:
- Multiplying one row of \( M \) by a non-zero scalar in \( A \).
- plus the \( r \)-th row by \( c \)-times the \( s \)-th row, where \( c \) is invariant \( A \).

And an elementary matrix is a matrix obtained by the identity matrix by an elementary row operation.

Two matrix is called row equivalent iff they can be connected by f.m. elementary row operations, and this is equivalent to \( M = PN \), where \( P \) is a product of f.m. elementary matrices, because left multiplication by an elementary matrix is equivalent to an elementary row operation.

Similarly we can define elementary column operations.

Lemma (I.1.4.9). The elementary row operation change the determinant only by an invariant element in \( A \). (Clear).

Prop. (I.1.4.10). Let \( P \) be a matrix with entries in \( F[X] \), then the following are equivalent:
- \( P \) is invertible.
- The determinant is a nonzero scalar.
- \( P \) is row equivalent to identity matrix.
• \( P \) is a product of elementary matrices.

*Proof:* The only hard part is \( 2 \to 3 \): this is because it has determinant in \( F^* \), thus the greatest common divisor of the first column is a scalar, thus we can use row operation to make it \((1, 0, \ldots, 0)^t\), and then we can continue to make it a upper triangular matrix with 1 in the diagonal, and also kill the upper half part. So it is row equivalent to the identity matrix. \( \square \)

**Cor. (I.1.4.11).** Let \( M, N \) be two matrices with entries in \( F[X] \), then they are equivalent iff \( N = PM \) for some \( P \) that has determinant in \( F \).

**Def. (I.1.4.12).** We call \( M, N \) equivalent iff \( M, N \) are connected by a sequence of elementary row operations and column operations. This is equivalent to \( M = PNQ \) for \( P, Q \) invertible matrices in \( M_n(F[X]) \).

**Prop. (I.1.4.13) (Normal Form of Companion Matrix).** For a monic polynomial \( p(X) \), consider its companion matrix \( T_p \), then the matrix \( xI - T_p \in M_n(F[X]) \) is equivalent to \( \text{diag}(p(X), 1, \ldots, 1) \).

*Proof:* Clear, if one reduces the \( x \) in the diagonal from the bottom row to the top row one by one. \( \square \)

**Cor. (I.1.4.14).** For a companion matrix \( A \) of \( p \), \( \det(xI - A) = p \).

*Proof:* This is from(I.1.4.9), and the fact both the side are monic polynomials. \( \square \)

**Def. (I.1.4.15) (Smith Normal Form).** A matrix in \( M_n(F[X]) \) is called a Smith normal form iff it is diagonal and diagonal entries \( f_i \in F[X] \) is monic and satisfies \( f_k \) divides \( f_{k+1} \).

**Prop. (I.1.4.16).** Any matrix \( M \) with entries in \( F[X] \) is equivalent to a unique Smith normal form.

*Proof:* This is immediate from(IX.8.6.5) applied to the PID \( F[X](I.2.3.18) \). \( \square \)

**Cor. (I.1.4.17) (Computing Invariant Factors).** The diagonal entries of the Smith form of the matrix \( xI - M \in M_n(F[X]) \) are just the invariant factors of \( M \).

*Proof:* This is because of the uniqueness of Smith form(I.1.4.16) and the invariant factor(I.1.4.7) and(I.1.4.13). \( \square \)

**Applications**

**Prop. (I.1.4.18).** For any two matrices \( A, B \in M_{n \times n}(K) \), \( (AB)^n \) and \( (BA)^n \) are similar.

*Proof:* It suffices to show that they have the same elementary factors. Notice that for any irreducible polynomial \( p \), if \( p \neq x \), then if \( p^k(AB)v = 0 \), then \( p^k(BA)v = 0 \), and then \( p^k(BA)Av = 0 \). Thus there are maps \( B : N(p^k(AB)) \to N(p^k(BA)) \) and \( A : N(p^k(BA)) \to N(p^k(AB)) \). Now their composition are both injective, thus they have the same dimension.

And for \( p = x \), these two both have nullity as the multiplicity of 0 in the charpoly of \( AB, BA \)(I.1.9.12), thus the same. So they have the same elementary factors, thus similar. \( \square \)

**Prop. (I.1.4.19) (Matrix Similar to Transpose).** Any matrix is similar to its transpose.

*Proof:* This is because the invariant factors can be computed using the greatest common divisors of minors by(I.1.4.16) and(I.1.4.17), and they are clearly invariant under conjugation. \( \square \)
5 Minimal and Characteristic Polynomial

Def. (I.1.5.1) (Minimal Polynomial). The minimal polynomial of a matrix $A$ is the minimal polynomial $p$ that $p(A) = 0$. It is equivalent to the maximal invariant factor of $A$, by (I.1.4.1).

Def. (I.1.5.2) (Non-derogatory Operator). An operator is called nonderogatory iff it has only one invariant factor.

Prop. (I.1.5.3) (Generalized Cayley-Hamilton). The characteristic polynomial of $A$ is the product of the elementary divisors of $A$, thus the characteristic polynomial and minimal polynomial (I.1.5.1) of $A$ have the same set of irreducible factors, but may not with the same multiplicity.

Proof: Because charpoly and minipoly are both invariant under similarity, assume $A$ is in rational form (I.1.4.6), so the result follows from (I.1.4.5).

Prop. (I.1.5.4). The linear functor $X \rightarrow AX - XC$ is an isomorphism iff the minimal polynomial of $A$ and $C$ has no common factor.

Proof: Notice if $AX = XC$, then we have $P(A)X = XP(C)$ for every polynomial $P$, in, particular for the minimal polynomials of $A$ and $C$, thus $P(C)$ is non-invertible and $A,C$ has a characteristic value in common. Conversely, if they have a characteristic value, then we upper triangularize $A$ to see clearly that there is a $X$ that $AX = XC$ ($X$ has only the first row).

6 Diagonalization and Triangulation

Prop. (I.1.6.1). If a linear map has matrix form $T$ in a basis $(X_i)$ and there is another basis $(Y_i)$ that $(Y_i) = (X_i)P$, then it has matrix form $PTP^{-1}$ in the basis $(Y_i)$. In particular, if $T$ can be diagonalized, with eigenvectors $(X_i)$, then $T = (X_i)D(X_i)^{-1}$.

Prop. (I.1.6.2) (Relation with Minimal Polynomial).
- An $n \times n$-matrix $A$ is upper-triangulable over a field $K$ iff its minimal polynomial is a product of linear factors.
- An $n \times n$-matrix $A$ is diagonalizable over a field $K$ iff its minimal polynomial is a product of linear factors with no multiple roots.

Proof: 1: If it is upper-triangulable, its minimal polynomial is a product of factors because its characteristic polynomial does (I.1.5.3). Conversely, we can find an eigenvector for $A$, then we quotient this vector and use induction.

2: If it is diagonalizable, then the minimal polynomial is clearly polynomials. Conversely, its elementary factors are all linear factors, thus its Jordan form is just diagonal (I.1.4.3).

Cor. (I.1.6.3) (Upper Triangulation Alg.Closed Field). If $K$ is alg.closed, then any $n \times n$-matrix $A$ is upper-triangulable over $K$. Similarly, it is lower-triangulable.

Proof: It suffices to find a flag that is stabilized by $A$. And for this, it suffices to find an eigenvector of $A$. This is clear, as the characteristic polynomial of $A$ has a root in $K$.

Prop. (I.1.6.4) (Simultaneously Triangulation). If $A_i$ is a commuting family of upper-triangulable $n \times n$-matrices, then they are simultaneously triangular.
**Prop. (I.1.6.5) (Simultaneously Diagonalizable).** If \( A_i \) is a commuting family of diagonalizable \( n \times n \)-matrices, then they are simultaneously diagonalizable.

**Proof:** We may assume there are f.m. matrices and use induction. Consider the diagonal decomposition \( V_i \) of \( A_1 \), then each \( V_i \) is invariant under \( \mathcal{F} \). Notice then each \( A_i \) is diagonalizable on \( V_i \), thus by induction, \( \mathcal{F} \) is simultaneously diagonalizable on each \( V_i \), then \( \mathcal{F} \) is simultaneously diagonalizable.

For the second, induction on the numbers of matrices. If some matrix is \( cI \), then clear, if some are not \( cI \), then choose its eigenvalue decomposition, we conclude by induction hypothesis. □

**Prop. (I.1.6.6) (Invariance of Field Extension).** Let \( A \in M_n(F) \), and \( E \) be the subfield generated by the entries of \( A \), then the invariant factors of \( A \) are polynomials over \( E \). In particular, two matrix are similar over the smallest field that they are defined.

**Proof:** Clear, because we know how an irreducible polynomial over \( E \) factors through \( \overline{E} \). □

**Prop. (I.1.6.7) (Jordan Decomposition).** Let \( k \) be a perfect field and \( A \in M_n(k) \), then there exists unique matrices \( A_s, A_n \) that \( A_s \) is semisimple, \( A_n \) is nilpotent, \( A = A_s + A_n \), and \( A_s, A_n \) are polynomials of \( A \), in particular, \( A_s \) commutes with \( A_n \).

Moreover, if \( A \) is invertible, then there exists unique matrices \( A_s, A_u \) that \( A_s \) is semisimple, \( A_u \) is unipotent, \( A = A_s \cdot A_u \), and \( A_s, A_u \) are polynomials of \( A \), in particular, \( A_s \) commutes with \( A_u \).

**Proof:** If \( k \) is alg.closed, then use(I.1.6.3). And Lagrange interpolation shows \( A_s, A_n \) are polynomials in \( A \). For the uniqueness, notice that because they are polynomials in \( A \), if there are two sets of Jordan decompositions \( A = A_s + A_n = A'_s + A'_n \), then \( (A_s - A'_s) = (A'_n - A_n) \), but the sum of commuting semisimple/nilpotent matrices is semisimple/nilpotent, so it is semisimple and nilpotent, so it can only be 0, so \( A_s = A'_s \).

In general, because \( k \) is perfect, taking a Galois field extension \( k'/k \) containing all eigenvalues of \( A \), then \( A = A_s + A_n \) where \( A_s, A_n \) has entries in \( k' \). But then taking Galois action and using the uniqueness, \( A_s, A_n \) has entries in \( k \).

Finally, by the Galois action again, \( A_s, A_n \) are polynomials in \( A \).

If \( A \) is invertible, then \( A_s \) is also invertible, seen by base change to alg.closure. Then \( A = A_s(1 + A_s^{-1}A_n) \), where \( 1 + A_s^{-1}A_n \) is unipotent. □

**Prop. (I.1.6.8).** If \( T \) is a diagonalizable operator on a subspace \( V \), then for any invariant subspace \( V' \), \( T|_{V'} \) is also diagonalizable.

**Proof:** Use the eigenvalue decomposition \( V = \oplus \lambda_i V_{\lambda_i} \), then \( V' = \oplus \lambda_i (V_{\lambda_i} \cap V') \), which is because if \( \sum v_i \in V' \), where \( v_i \) are in different eigenspaces, then each \( v_i \in V' \). □
### 7 Bilinear & Hermitian Form

A matrix $M$ defines a bilinear form on $V$ by $(x, y) \mapsto x^t M y$, so we will interchange freely between a matrix and a bilinear form on $V$.

**Lemma (I.1.7.1).** Any eigenvalue of a Hermitian(e.g., real symmetric) matrix $M$ is real.

**Proof:** Consider the bilinear form defined by the matrix $M$, then if $x$ is an eigenvector with eigenvalue $\lambda$, then $\lambda(x, x) = (Hx, x) = (x, Hx) = \overline{\lambda}(x, x)$, so if $\lambda$ is not real, $x = 0$. □

**Prop. (I.1.7.2) (Unitarily Diagonalizable).** A symmetric matrix $A$ is orthogonally diagonalizable. Similarly, a skew-symmetric matrix is orthogonally diagonalizable and an (skew)hermitian matrix is unitarily diagonalizable.

**Proof:** Firstly, we can find an eigenvector of $A$: Only the real case needs proof, and this is because any eigenvalue of $A$ is real(I.1.7.1).

Let $v$ be an eigenvector of $A$ of length 1, then the orthogonal complement of $v$ is preserved by $A$, so we can use induction to find an orthonormal basis consisting of eigenvectors of $A$, then these together with $v$ forms an orthonormal basis consisting of eigenvectors of $A$. □

**Prop. (I.1.7.3) (Normal operator).** More generally, a normal operator over $\mathbb{C}$ is unitarily diagonalizable using resolution of identity (X.5.4.3) because the spectrum are discrete thus the point projection is orthogonal.

**Prop. (I.1.7.4) (Gram-Schmidt).** Any symmetric matrix over fields of characteristic $\neq 2$ is congruent to a diagonal matrix.

**Proof:** Any symmetric matrix defines a bilinear form on $V$. If $B$ is not identically 0, then there is a $x$ that $x^t B x \neq 0$, by polarization identity. Then $W = \{Kx\}$ is non-degenerate, so we have $W \oplus W^\perp = V$ by(IV.7.1.7). And by induction, we are done. □

**Prop. (I.1.7.5) (Antonne-Takagi).** For any complex symmetric matrix $A$, there is unitarily matrix $U$ that $UAU^t$ is a real diagonal matrix with non-negative entries.

**Proof:** Consider $B = A^* A$ is Hermitian and positive-semi-definite, thus there is a unitary matrix $V$ that $V^* B V$ is diagonal with non-negative real entries by(I.1.7.2). Now $C = V^t A V$ is complex symmetric with $C^* C$ real diagonal. If we let $C = X + iY$, then $XY = YX$. So by(I.1.6.5), there is a real orthogonal matrix $W$ that $WXW^t$ and $YWY^t$ are diagonal. Now set $U = WV^t$, which is unitary, $UAU^t$ is complex diagonal. And easily we can modify the diagonal entries to be non-negative. □

**Prop. (I.1.7.6) (Symplectic Form).** Over $\mathbb{R}$, a skew-symmetric matrix are orthogonally congruent to $\text{diag}\{\begin{bmatrix} 0 & a_i \\ -a_i & 0 \end{bmatrix}\}_i$.

**Proof:** Choose a $\alpha, \beta$ that $(\alpha, \beta) \neq 0$. and choose their orthogonal complement. □

**Cor. (I.1.7.7).** For a matrix that $J^2 + 1 = 0$, by (I.1.4.2), there is a unique inner product s.t. $J$ is orthogonal and then it is orthogonally congruent to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}_n$. (Use cyclic decomposition).

so this $J$ is equivalent to a complex structure, homeomorphic to $O(n)/U(\frac{n}{2})$. 
Prop. (I.1.7.8). Given a bilinear form on a field, the relation of orthogonality is symmetric iff it is symmetric or alternating, i.e. \( \langle x, x \rangle = 0 \).

Proof: Let \( w = B(x, z) y - B(x, y) z \), then \( B(x, w) = 0 \), hence we have \( B(w, x) = 0 \), that is

\[
B(x, z) B(y, x) - B(x, y) B(z, x) = 0.
\]

Let \( z = x \), then \( B(x, x) [B(x, y) - B(y, x)] = 0 \).

If some \( B(u, v) \neq B(v, u) \) and \( B(w, w) \neq 0 \), then \( B(u, u) = B(v, v) = B(w, v) = B(v, w), B(w, u) = B(u, w) \), Let \( x = u \) or \( v \) se get \( B(w, v) = B(v, w) = 0 = B(w, u) = B(u, w) \).

Now \( B(u, w + v) \neq B(w + v, u) \), hence \( B(w + v, w + v) = 0 = B(w, w) \), contradiction. \( \square \)

Prop. (I.1.7.9). If \( B \) is a non-degenerate bilinear form on an associative algebra \( V \), choose a basis \( x_i \) of \( V \), then choose a dual basis \( y_i \), then \( \sum x_i \otimes y_i \in T(V) \) is independent of \( x_i \) chosen.

Proof: Let \( V' \) be the dual of the vector space \( V \). There is an isomorphism \( V \otimes V' \cong \text{End}(V) \) given by mapping \((v, f)\) to the operator \( v' \mapsto f(v')v \). The non-degenerate bilinear form induces naturally an isomorphism \( \beta : V \cong V' : v \mapsto (., v) \). Then under the isomorphisms

\[
\text{End}(V) \cong V \otimes V' \cong V \otimes V,
\]

where the second isomorphism is \((\text{id}_V, \beta^{-1})\). The identity map \( \text{id}_V \in \text{End}(V) \) is send to \( \sum x_i \otimes y_i \in T(V) \), so \( \sum x_i \otimes y_i \) is independent of the basis chosen. \( \square \)

For symmetric bilinear forms and more about quadratic forms, see IV.7.

Hermitian Forms

Def. (I.1.7.10) (Hermitian Matrix). A Hermitian matrix is a complex matrix \( A \) that satisfies \( A^t = A \).

Prop. (I.1.7.11) (Hermitian Forms). A complex vector space \( V \) is equivalent to a real vector space with an endomorphism \( J \) that \( J^2 = -1 \), by \( i \) acting by \( J \).

A Hermitian form on \( (V, J) \) is an \( R \)-bilinear mapping \(-, - : V \times V \to \mathbb{C} \) that satisfies

\[
(Ju, v) = i(u, v), \quad (u, v) = (v, u).
\]

If we write \( (u, v) = \varphi(u, v) - i\psi(u, v) \), then

\begin{itemize}
  \item \( \varphi \) is symmetric, \( \varphi(Ju, Jv) = \varphi(u, v) \).
  \item \( \psi \) is alternating, \( \psi(Ju, Jv) = \psi(u, v) \).
  \item \( \psi(u, v) = -\varphi(u, Jv), \varphi(u, v) = \psi(u, Jv) \).
\end{itemize}

Conversely, if \( \varphi \) satisfies this condition, then

\[
(u, v) = \varphi(u, v) + i\varphi(u, Jv).
\]

is a Hermitian form. Also \((-, -)\) is positive or non-singular iff \( \varphi \) is.
8 Tensor Algebras

Def. (I.1.8.1) (Tensor Algebras). The tensor product and tensor algebras of modules are defined in(I.2.4.13) and(I.5.1.16). In this subsection, we focus on tensor algebras of vector spaces.

Def. (I.1.8.2) (Symmetrized Tensors). Let V be a vector field over a field of characteristic 0, for any n, we define a multilinear map

\[ V^n \to T^n(V) : (v_1, \ldots, v_n) \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}. \]

and descends to a linear map \( \sigma : S^n(V) \to T^n(V) \), called the symmetrizer. Elements in \( \text{Im}(\sigma) = \tilde{S}^n(V) \) is called symmetrized tensors.

Then \( \sigma^2 = \sigma \), and \( \text{Ker}(\sigma) = I \cap T^n \), where \( I = \text{Ker}(T(V) \to Sym(V)) \). In particular,

\[ T^n(V) = \tilde{S}^n(V) \oplus (T^n(V) \cap I). \]

Proof: \( \sigma^2 = \sigma \) is easy. So \( V = \text{Im}(\sigma) \oplus \text{Ker}(\sigma) \). Now \( T^n(V) \cap I \subset \text{Ker}(\sigma) \), and because \( \text{Im}(\sigma) \to S^n(V) \) is surjective, \( T^n(V) \cap I = \text{Ker}(\sigma) \). \qed

Def. (I.1.8.3) (Anti-Symmetrized Tensors). Let V be a vector field over a field of characteristic 0, for any n, we define a multilinear map

\[ V^n \to T^n(V) : (v_1, \ldots, v_n) \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma)v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}. \]

and descends to a linear map \( \sigma : \wedge^n(V) \to T^n(V) \), called the anti-symmetrizer. Elements in \( \text{Im}(\tau) = \tilde{\wedge}^n(V) \) is called anti-symmetrized tensors.

Then \( \tau^2 = \tau \), and \( \text{Ker}(\tau) = I \cap T^n \), where \( I = \text{Ker}(T(V) \to \wedge(V)) \). In particular,

\[ T^n(V) = \tilde{\wedge}^n(V) \oplus (T^n(V) \cap I). \]

Proof: \( \sigma^2 = \sigma \) is easy. So \( V = \text{Im}(\sigma) \oplus \text{Ker}(\sigma) \). Now \( T^n(V) \cap I \subset \text{Ker}(\sigma) \), and because \( \text{Im}(\sigma) \to S^n(V) \) is surjective, \( T^n(V) \cap I = \text{Ker}(\sigma) \). \qed

9 Determinant and Trace

Def. (I.1.9.1) (Determinant). For a linear operator \( T \in L(V) \), as \( \dim \wedge^n V^* = 1 \), the determinant \( \det T \) is defined by \( \wedge^n(T^t) = \det T \cdot \text{id}_{\wedge^n V^*} \). That is: \( L(T_{\alpha_1}, \ldots, T_{\alpha_n}) = \det TL(\alpha_1, \ldots, \alpha_n) \).

And the determinant of a matrix is defined by the linear operator it associates in a canonical basis.

Prop. (I.1.9.2) (Properties of Determinants).
1. \( \det(\text{id}_V) = 1 \).
2. \( \det(UV) = \det U \cdot \det V \).
3. \( T \) is invertible iff \( \det T \) is invertible, in which case \( \det(T^{-1}) = (\det T)^{-1} \).
4. If \( \{\alpha_1, \ldots, \alpha_n\} \) is a basis for \( V \), and \( f_i \) is its dual basis, then \( \det T = f_1 \wedge \cdots \wedge f_n(T_{\alpha_1}, \ldots, T_{\alpha_n}) \).

Proof: All these are not hard. \qed

Cor. (I.1.9.3). \( \det(P^{-1}AP) = \det(A) \).
Prop. (I.1.9.4). \( \det T = \det T^t \).

Proof: Use (I.1.9.2), if \( \{ \alpha_1, \ldots, \alpha_n \} \) is a basis for \( V \), and \( f_i \) is its dual basis, then
\[
\det T^t = \alpha_1 \wedge \cdots \wedge \alpha_n (T^t f_1, \ldots, T^t f_n) = f_1 \wedge \cdots \wedge f_n (T \alpha_1, \ldots, T \alpha_n) = \det T \]
\[ \square \]

Prop. (I.1.9.5) (Expansion of Determinants). If \( A_i \) be the \( i \)-th column of \( A \), then
\[
\det A = f_1 \wedge \cdots \wedge f_n (A \varepsilon_1, \ldots, A \varepsilon_n) = f_1 \wedge \cdots \wedge f_n (A_1, \ldots, A_n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) A_{\sigma(i)}.
\]

Prop. (I.1.9.6). For a matrix, the determinant satisfies the following properties:
1. adding a multiple of a column/row to another column/row, the determinant doesn’t change.
2. Multiplying a row or a column with a scalar, then the determinant multiplies with this scalar.
3. Changing two rows or two columns makes the determinant multiply by \(-1\).

Proof: All this follows from 4 of (I.1.9.2). Notice the last one follows from the first two. \[ \square \]

Prop. (I.1.9.7) (Laplacian Expansion Formula). Cf.[Determinant 安金鹏 P15].

Prop. (I.1.9.8). Adjunction matrix, Cf.[Determinant 安金鹏 P16].

Proof: \[ \square \]

Cor. (I.1.9.9). If \( AB = 1 \), then \( BA = 1 \).

Prop. (I.1.9.10) (Cramer’s Rule). Cf.[Determinant 安金鹏 P16].

Prop. (I.1.9.11) (Sylvester’s Determinant Identity). If \( A \) and \( B \) are matrices of sizes \( m \times n \) and \( n \times m \), then
\[
\det(I_m + AB) = \det(I_n + BA)
\]

Proof:
\[
\begin{bmatrix}
1 & A \\
B & 1 \\
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
B & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1 - BA \\
\end{bmatrix}
\begin{bmatrix}
1 & A \\
0 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
1 & A \\
0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
-AB & 1 \\
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
B & 1 \\
\end{bmatrix}
\]

and use (I.1.9.2). \[ \square \]

Cor. (I.1.9.12). Multiplying by \( x \), we see that the characteristic polynomial of \( AB \) and \( BA \) are the same.

Remark (I.1.9.13). There is another proof in case \( m = n \): It suffices to show
\[
\det(I + (A + xI)(B + xI)) = \det(I + (B + xI)(A + xI))
\]
But notice \( A + xI \) and \( B + xI \) are invertible in \( M_{n \times n}(K(X)) \), thus
\[
\det(I + (A + xI)(B + xI)) = \det((A + xI)((A + xI)^{-1} + (B + xI))) = \det(I + (B + xI)(A + xI)).
\]
Prop. (I.1.9.14).

\[
\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A) \det(D - CA^{-1}B).
\]

Proof: As before, consider \( \det \begin{bmatrix} A + xI & B \\ C & D + xI \end{bmatrix} \), then \( A + xI \) is invertible, and this equals

\[
\det \begin{bmatrix} A + xI & B \\ C & D + xI - C(A + xI)^{-1}B \end{bmatrix} = \det(A + xI) \det(D + xI - C(A + xI)^{-1}B).
\]

Letting \( x = 0 \), we get the desired result. □

Prop. (I.1.9.15) (Symplectic Group Determinant). The determinant of a symplectic matrix \( \in \text{Sp}(2n, \mathbb{R}) \) has determinant 1.

Proof: A symplectic matrix preserves the symplectic structure thus the symplectic form \( \omega \), hence preserves \( \omega^n \) which is \( n! \) times the volume form, so it has determinant 1 by definition (I.1.9.1). □

Prop. (I.1.9.16) (Vandermonde Matrix). The \( n \times n \) Vandermonde matrix, with the \( k \)-th row \((1, x_k, \ldots, x_k^n)\), has determinant \( \prod_{i<j}(x_i - x_j) \). So it is invertible when \( x_i \) are pairwisely different.

Proof: Eliminate the first row by adding columns. □

Prop. (I.1.9.17) (Pfaffian). There is a polynomial \( Pf \) called Pfaffian s.t. \( \det M = Pf(M)^2 \) for a skew-symmetric matrix. This is because a skew symmetric is equal to \( A^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^k A \) for \( A \) an orthogonal matrix (I.1.7.2), so it has determinant \( (\det A)^2 \) and \( A \) and depends polynomially on the entries of \( M \).

Cor. (I.1.9.18).

\[
Pf(A^t MA) = \det A \cdot Pf(M).
\]

Because we only need to consider the sign and it is determined by letting \( A = \text{id} \).

Traces

Def. (I.1.9.19) (Trace). For a \( n \times n \)-matrix \( A \), define its trace \( \text{tr}(A) \) to be the minus of the coefficient of \( x^{n-1} \) in \( \det(xI - A) \). It is clear that \( \text{tr}(A) = \sum_{i=1}^{n} a_{ii} \), and by (I.1.9.3) that traces are invariant under conjugacy.

Prop. (I.1.9.20). \( \text{tr}(AB) = \text{tr}(BA) \).

Proof: This is because \( \det(x - AB) = \det(x - BA) \) by Sylvester determinant identity (I.1.9.11). □
10 Spectral Theory

See also 4.

Def. (I.1.10.1) (Cayley-transformation). For a field $k$ of $\text{char} k \neq 2$ and a matrix $P \in M_n(k)$ that has no eigenvalue $-1$, there is a Cayley transformation $A = \frac{1 - P}{1 + P}$. $1 + A$ is invertible, and $P = \frac{1 - A}{1 + A}$.

Then $A$ is skew-symmetric iff $P$ is orthogonal.

Proof: If $Av = -v$, then $v - Pv = -v - Pv$, so $2v = 0$, so $v = 0$. So $A + 1$ is invertible.

If $P$ is orthogonal, then $A^t = \frac{1 - P^t}{1 + P^t} = \frac{1 - P^{-1}}{1 + P^{-1}} = -A$. Conversely, if $A^t = -A$, then $P^t = \frac{1 + A}{1 - A} = P^{-1}$. □

Prop. (I.1.10.2). If $\text{char} k \neq 2$ and $P$ is an orthogonal matrix of odd dimension, then $\det P$ is an eigenvalue of $P$.

Proof: multiplying by $-1$, we can assume $\det P = -1$. Consider the Cayley transformation(I.1.10.1), then

$$\det P = \det(1 - A) \det(1 + A)^{-1} = \det(1 - A)^t \det(1 + A)^{-1} = 1.$$ Contradiction. □

11 Positivity(Inner Spaces)

Def. (I.1.11.1) (Inner Spaces). A real inner space is a f.d. real quadratic space that $B(v, v) > 0$ for $v \neq 0$. It is necessarily non-degenerate.

A complex inner space is a f.d. Hermitian space $V$ that $B(v, v) > 0$ for $v \neq 0$.

Prop. (I.1.11.2). An inner metric on a vector space will induce an inner metric on the dual space, that is, asserting the dual basis of an orthonormal basis to be orthonormal. On an arbitrary basis, the matrix on the dual basis is written as $A^{-1}$. because we can write $A = P^t P$, and the dual basis transformation is like $(P^t)^{-1}$, so the metric matrix is $A^{-1}$.

Prop. (I.1.11.3) (Positivity and Principal Minors). A matrix is symmetric positive iff it is symmetric, and all its upper principle minors has positive determinants.

A symmetric(Hermitian) positive matrix is equivalent to a real(complex) inner space.

Proof: Cf.[Hoffman P328]. □

Prop. (I.1.11.4) (Positive Matrices and Symmetric Matrices). The exponential defines a diffeomorphism from the space of symmetric matrices to positive definite symmetric matrices in $GL(n, \mathbb{R})$.

Proof: By(I.1.7.2) it is clearly surjective. For injectivity, consider $\exp(X) = \exp(Y)$, then at least $X, Y$ are both similar to the same diagonal matrix $\text{diag}(d_1, \ldots, d_n)$, and then we have

$$X = \tau Y \tau^{-1}, \quad \text{diag}(D_1, \ldots, D_n) = \tau \text{diag}(D_1, \ldots, D_n) \tau^{-1}$$

where $D_i = e^{d_i}$. Consequently, $\text{diag}(D_1, \ldots, D_n) = \tau \text{diag}(D_1, \ldots, D_n) \tau^{-1}$, and we can choose $c_j$ that $\sum c_i D_{ij} = d_j$ for any $j$, because the Vandermonde matrix is nonsingular. Hence $\text{diag}(d_1, \ldots, d_n) = \tau \text{diag}(d_1, \ldots, d_n) \tau^{-1}$, and $X = Y$. □
Prop. (I.1.11.5) (Farkas’ Lemma). For a matrix \( A \), and a vector \( b \), exactly one of the following equation has a solution:

\[
\left\{
\begin{array}{l}
AX = b, X \geq 0 \\
Y^t A \leq 0, Y^t b > 0
\end{array}
\right.
\]

Proof: First notice if both have a solution, then \( 0 \geq Y^t AX > 0 \), contradiction. The rest follows form the Hahn-Banach separation theorem. \( \square \)

Cor. (I.1.11.6) (Gordan’s Theorem). exactly one of the following has a solution:

\[
\left\{
\begin{array}{l}
AX > 0 \\
Y^t A = 0, Y \geq 0, Y \neq 0
\end{array}
\right.
\]

Proof: If both have a solution, then \( 0 = Y^t AX > 0 \), contradiction. If the first has no solution, then \( A'x = e, z \geq 0 \), where \( A' = [A, -A, -I] \) has no solution, by Farkas’ lemma, there is a solution of \( Y^t A' \leq 0 \) and \( Y^t b = 0 \). Which shows that \( Y^t A = 0 \) and \( Y \neq 0 \). \( \square \)

Cor. (I.1.11.7). For any subspace in \( \mathbb{R}^m \), either it has an intersection with the open first quadrant, or its orthogonal complement has an intersection with the closed first quadrant minus \( 0 \). (Regard it has the image of a \( AX \)).

Quaternion Algebras

Remark (I.1.11.8). The argument below are largely true for any skew field.

Def. (I.1.11.9) (Quaternion Algebra). The Quaternion algebra \( \mathbb{H} \) is the space \( \mathbb{R}\{1, i, j, k\} \) subjects to the relations

\[ ij = -ji = k, jk = -kj = i, ki = -ik = j, i^2 = j^2 = k^2 = -1. \]

Prop. (I.1.11.10) (Module of Quaternion Algebras). There is an involution on \( \mathbb{H} \) that \( \bar{x} = a + bi + cj + dk = a - bi - cj - dk \). Then we define a the module of \( x \in \mathbb{H} \) as

\[ |x|^2 = x\bar{x} = xx = a^2 + b^2 + c^2 + d^2. \]

Then every non-zero element of \( \mathbb{H} \) is invertible. In particular, it is a skew field.

Prop. (I.1.11.11). The center of \( \mathbb{H} \) is \( \mathbb{R} \).

Proof: If \( x = a + bi + cj + dk \) is in the center, then \( ix = xi, jx = xj, kx = xk \), thus \( a = b = c = d = 0 \). \( \square \)

Prop. (I.1.11.12) (Invertible Quaternion Matrices). Denote \( GL(n, \mathbb{H}) \) the set of invertible matrices in \( M_n(\mathbb{H}) \). If \( A \in M_n(\mathbb{H}) \), \( A \) acts on \( \mathbb{H}^n \), and determines a complex matrix \( A' \) in \( M_{2n}(\mathbb{C}) \).

- If \( A, B \in M_n(\mathbb{H}) \) satisfies \( AB = 1 \), then \( BA = 1 \).
- If \( A \in M = GL(n, \mathbb{H}) \) iff \( A' \in GL(2n, \mathbb{C}) \). In particular if we define the determinant of \( A \in M_n(\mathbb{H}) \) as the determinant of \( A' \), then \( \det(A) \neq 0 \) iff \( A \) is invertible.
Proof: 1: \((AB)' = A'B'\), thus \(M_n(\mathbb{H})\) is a subalgebra of \(M_{2n}(\mathbb{C})\). Thus \(B'A' = 1\) by (I.1.9.9).

2: If \(A'\) is invertible, then \(A\) is a bijection, thus there are vectors \(v_1, \ldots, v_n\) that \(Av_i = e_i\). This means
\[
(v_1, \ldots, v_n)A = 1.
\]

Thus by item 1 \(A\) is invertible. \(\square\)

Def. (I.1.11.13) (Sesquilinear Form on \(\mathbb{H}^n\)). Let \(V = \mathbb{H}^n\), then a sesquilinear form on \(\mathbb{H}^n\) is a bi-additive function \((\cdot, \cdot): V \times V \to \mathbb{H}\) such that
\[
(x\alpha, y\beta) = \overline{\pi(x, y)}\beta, \quad \alpha, \beta \in \mathbb{H}.
\]

Moreover if it is called a Hermitian form iff \((x, y) = \overline{(y, x)}\), and skew-Hermitian iff \((x, y) = -(y, x)\).

Prop. (I.1.11.14).
- Every non-degenerate Hermitian quaternion form is of the form
\[
(x, y) = \mathbb{x}_1 y_1 + \cdots + \mathbb{x}_p y_p - \mathbb{x}_{p+1} y_{p+1} - \cdots - \mathbb{x}_n y_n
\]
in some basis.
- Every non-degenerate skew-Hermitian quaternion form is of the form
\[
(x, y) = \mathbb{x}_1 y_1 + \cdots + \mathbb{x}_n y_n
\]
in some basis.

Proof: Firstly, there are some \(v\) that \((v, v) \neq 0\): If \((v, v) = 0\) for all \(v\), then \((x, y) + (y, x) = 0\) for all \(x, y\). If it is skew-Hermitian, this means \((x, y)\) is real, which is impossible, unless \((x, y) = 0\).

If it is Hermitian, this means \((x, y)\) is imaginary, but \((x, yi), (x, yj), (x, yk)\) are all imaginary, thus \((x, y) = 0\).

Then choose this \(v\), and take the orthogonal complement of \(v\), then by induction we can find \(v_1, \ldots, v_n\) that is mutually orthogonal.

If it is Hermitian, then \((v_i, v_i) \in \mathbb{R}\), thus we can find some \(t_i \in \mathbb{R}\) that \((tv_i, tv_i) = \pm 1\).

If it is skew-Hermitian, then \((v_i, v_i)\) is imaginary, thus by (IX.8.4.3), there are \(u_i \in \mathbb{H}\) that \((u_i v_i, u_i v_i) = \pi_i(v_i, v_i)u_i = j\). \(\square\)

Cor. (I.1.11.15).
- Every non-degenerate Hermitian quaternion form is of the form
\[
(x, y) = B_1(x, y) + jB_2(x, y)
\]
in some basis, where \(B_1(x, y)\) is the usual canonical Hermitian form of signature \((2p, 2q)\), and \(B_2\) is the usual canonical skew-symmetric bilinear form. Also \(B_2(x, y) = B_1(xj, y)\).
- Every non-degenerate skew-Hermitian quaternion form is of the form
\[
(x, y) = B_1(x, y) + jB_2(x, y)
\]
in some basis, where \(B_1(x, y)\) is the usual canonical skew-Hermitian form that \(iB_1\) is Hermitian of signature \((n, -n)\), and \(B_2\) is the usual canonical symmetric bilinear form. Also \(B_2(x, y) = B_1(xj, y)\) in some basis.
I.2 Abstract Algebra

References are [Lan05], [Rom07] and [Finite Groups Issac].

This section differs from sections on commutative algebras because it contains more basic, but maybe non-commutative properties.

1 Magmas

Def. (I.2.1.1) (Unital Operator). A unital binary operator on a set $X$ is a map $\circ : X \times X \to X$ that has a left and right identity element.

Def. (I.2.1.2) (Magma). A Magma is a set $X$ with an operator on it. A unital magma is a magma that the operator is unital. It is called Abelian iff $\circ(x, y) = \circ(y, x)$ for any $x, y \in X$.

Prop. (I.2.1.3) (Eckmann-Hilton argument). If $\circ$ and $\otimes$ are two unital binary operators on a set that commute: $(a \circ b) \circ (c \otimes d) = (a \otimes c) \circ (b \otimes d)$, then they are equal and in fact commutative and associative.

Proof: Firstly the units coincide, because

$$1_\circ = 1_\otimes 1_\circ = (1_\otimes \circ 1_\circ) \circ (1_\circ \otimes 1_\otimes) = (1_\otimes \circ 1_\circ) \otimes (1_\circ \otimes 1_\otimes) = 1_\otimes \otimes 1_\circ = 1_\otimes.$$

Next

$$a \circ b = (1 \otimes a) \circ (b \otimes 1) = (1 \circ b) \otimes (a \circ 1) = b \otimes a = (b \otimes 1) \circ (1 \circ a) = (b \otimes 1) \circ (1 \otimes a) = b \circ a.$$

Thus $\circ$ and $\otimes$ coincide and are commutative. Finally for associativity:

$$(a \otimes b) \otimes c = (a \otimes b) \otimes (1 \otimes c) = (a \otimes 1) \otimes (b \otimes c) = a \otimes (b \otimes c).$$

\qed

Def. (I.2.1.4) (Monoid). A Monoid is an associative unital magma $(X, \circ)$.

2 Polynomials

Prop. (I.2.2.1) (Descartes’s Rule of Sign). Let $p(x) = a_0x^{b_0} + a_1x^{b_1} + \cdots + a_nx^{b_n}$ be a real polynomial with nonzero $a_i$, where $A_0 < B_1 < \cdots < b_n$, then the number of positive roots of $p(x)$ is the number of changing signs of $\{a_n\}$ minus $2k$.

Proof: Lemma: when $a_0a_n > 0$, the number of positive roots are even and when $a_0a_n < 0$, it is odd. This is seen by consider $p(0)$ and $p(\infty)$.

Then we consider the derivative $p'$ and use induction. Denote the number of changing sign by $v(p)$ and the number of positive roots by $z(p)$, then if $z_0a_1 > 0$, then $v(p) = v(p')$ and $z(p) \equiv z(p') \mod 2$. Then we have $z(p) \equiv v(p) \mod 2$ and middle value theorem shows that $z(p') \leq z(p) - 1$, hence by induction and parity argument, we have $v(p) \geq z(p)$.

If $a_0a_1 < 0$, then the same method shows that $v(p) = v(p') + 1 \geq z(p') + 1 \geq z(p')$ and the have the same parity by the lemma. \qed

Prop. (I.2.2.2) (Lagrange Interpolation). if $K$ is a field, $a_i$ are $n+1$ elements of $K$, $b_i$ are $n+1$ elements of $K$, then there is a unique polynomial $f$ of degree no greater than $n$ that $f(a_i) = b_i$. 

Then we prove by induction on $\delta_i$ that $h_i$ vanishes at the common zeros of $f$.

**Proof:** The case of $\delta_i = 1$ is trivial. If $\delta_i \geq 2$, then there are some terms of $h_i$ that are divisible by $\prod_{j \neq i} (x - a_j)$.

It is a polynomial of degree smaller than $n + 1$, and it satisfies the hypothesis. And clearly there is at most one such polynomial, otherwise their difference has $n + 1$ zeros.

**Cor. (I.2.2.3).** If $f(x) = a_n x^n + \ldots + a_0$, then for any $n + 1$ different integers $a_0, \ldots, a_n$, there exists some $|f(a_i)| \geq \frac{n!}{2^n} |a_n|$.

**Proof:** Use Lagrange interpolation and consider the leading coefficient.

**Prop. (I.2.2.4).** If a degree $n$ polynomial $p$ satisfies $p(n) = 2^n$ for $n = 0, 1, \ldots, n$, then $p(n + 1) = 2^{n+1} - 1$.

**Proof:** The polynomial in search is $p(x) = \sum_{k=0}^{n} C_x^k$. □

**Prop. (I.2.2.5) (Newton Identities).** For $n$ indeterminants $x_i$, denote $s_k = \sum x_i^k, \sigma_k$, then there are Newton Identities:

\[
\begin{align*}
& s_k - \sigma_1 s_{k-1} + \ldots + (-1)^n \sigma_n s_{k-n} = 0 \quad k \geq n \\
& s_k - \sigma_1 s_{k-1} + \ldots + (-1)^k k \sigma_k = 0. \quad k \leq n
\end{align*}
\]

**Proof:** The case of $k \geq n$ is simple. Now if $k \leq n$, notice that the equation is true for $n = k$. Then we prove by induction on $n$. If $n$ is already proven, then the term is 0 if one of them is 0, but this implies that this equation is divisible by $\prod_{i=1}^{n+1} x_i$, but is has degree $k \leq n$, so it must by 0. □

**Prop. (I.2.2.6) (Combinatorial Nullstellensatz).** If $F$ is a field and $f \in F[X_1, \ldots, X_n]$ is a polynomial. Let $S_1, \ldots, S_n$ be nonempty finite subsets of $F$ and $g_i = \prod_{s \in S_i} (x_i - s)$, then if $f$ vanishes at the common zeros of $g_i$, then there are polynomials $h_i \in F[X_1, \ldots, X_n]$ that $\deg h_i \leq \deg F - \deg g_i$ and $g = \sum h_i g_i$.

**Proof:** The proof is very simple, just replace terms of $f$ by lower degree terms, using equation of $g_i$, then we get a polynomial that has degree in $x_i$ smaller than $|S_i|$ and vanish on $S_1 \times \ldots \times S_n$, so it must be 0, as easily checked. □

**Cor. (I.2.2.7) (Combinatorial Nullstellensatz).** If $F$ is a field and $f \in F[X_1, \ldots, X_n]$ is a polynomial of degree $n$, if $\prod X_i^{t_i}$ is a highest degree term of $f$ and $S_i$ are arbitrary subsets of $F$ that $|S_i| > t_i$, then there are some $s_i \in S_i$ that $f(s_1, \ldots, s_n) \neq 0$.

**Proof:** May assume $|S_i| = t_i + 1$. By combinatorial Nullstellensatz(I.2.2.6), if no such $s_i$ exist, then there are $h_i$ that $f = \sum h_i \prod_{s \in S_i} (x_i - s)$, but the term $\prod X_i^{t_i}$ needs to appear, so it must by some term of $h_i$ times $x_i^{t_i+1}$, which is a contradiction. □

**Remark (I.2.2.8).** For many combinatorial applications of Combinatorial Nullstellensatz, Cf.[Combinatorial Nullstellensatz].
Irreducibility

Prop. (I.2.2.9). If \( f = a_n x^n + \ldots + a_1 x + p \in \mathbb{Z}[X] \) satisfies \( p \) is a prime and \( \sum |a_i| < p \), then \( f \) is irreducible in \( p \).

Proof: The ideal is that all of its roots has norm bigger than 1, because otherwise \( p = |\sum a_k x^k| \leq \sum |a_k| < p \), contradiction. So if now \( f = gh \), then \( g, h \) all have roots with norm greater than 1, in particular it has constant coefficients norm greater than 1, which is a contradiction because \( p \) is a prime.

\[
\square
\]

Prop. (I.2.2.10). The cyclotomic polynomial \( \Psi_n(x) \) is irreducible over \( \mathbb{Z} \).

Proof: It suffices to show that for any irreducible factor \( f | \Psi_n(x) \), if \( \xi \) is a root of \( f \) and \( (p,n) = 1 \), then \( \xi^p \) is also a root of \( f \). Cf.[Lang].

\[
\square
\]

Diophantine Equations

Prop. (I.2.2.11) (Fermat’s Equation in Function Case). If \( n \geq 2 \), then the only non-trivial solution to the equation in \( \mathbb{C}[t] \) of

\[ X^n + Y^n = Z^n \]

is \( n = 2, X = (a^2 - b^2)/2, Y = ab, Z = (a^2 + b^2)/2. \)

Proof: The \( n = 2 \) case is easy. If \( n > 2 \), we show there are no non-trivial solutions: Differentiate it to get:

\[ X^{n-1}X' + Y^{n-1}Y' = Z^{n-1}Z', \]

And cancelling \( X^{n-1} \), we get

\[ Y^{n-1}(X'Y - Y'X) = Z^{n-1}(X'Z - Z'X). \]

Now \( X'Z - Z'X \neq 0 \) because \( X, Z \) are not linearly equivalent, and \( Y, Z \) is coprime, so \( Y^{n-1} | X'Z - Z'X \). But then if we assume \( \dim Y \geq \dim X \), then \( (n - 1) \dim Y \leq 2 \dim Y - 1 \), which means \( n < 2 \).

\[
\square
\]

Resultant

Def. (I.2.2.12). Over a commutative ring \( R \), the result \( \text{resultant } \text{res}(A, B) \) of two polynomials \( A, B \) of degree \( d, e \) respectively is the determinant of the map \( W_e \times W_d \to W_{d+e} \) that \( (X, Y) \mapsto AX + BY \), where \( W_t \) is the free module of polynomials of degree \( < t \).

Prop. (I.2.2.13). The resultant can be seen as the determinant of the matrix with values the coefficient of \( A \) or \( B \) in different places, multiplying \( X^s \)s and \( Y^s \)s together.

\[ \text{res}(A, B) = AC + BD \text{ for some } C, D. \]

Now if \( R \subset S \) and \( A, B \) has common roots in \( S \), then \( \text{res}(A, B) = 0 \).

Cor. (I.2.2.14). Resultant is stable under Euclidean division, so it can be seen as a suitable division remainder of the two polynomial.

Prop. (I.2.2.15). When \( R \subset L \) a field and \( A, B \) decompose into linear factors in \( L \), let \( t_i \) be roots of \( A \) and \( u_j \) be roots of \( B \), then

\[ \text{res}(A, B) = v_0^d w_0^e \prod_{i=1}^d \prod_{j=0}^e (t_i - u_j) \]
\textbf{Proof:} See the resultant as polynomials of the roots of \(A\) and \(B\), then we proved that if they has the same root, then \(\text{res} = 0\), so it is divisible by \((t_i - u_j)\) for all \(i, j\). Then notice the RHS is homogenous of degree \(d\) in \(u_j\) and homogenous of degree \(e\) in \(t_i\), so does \(\text{res}\). So they are equal. \(\square\)

\textbf{Invariant Theory}

\textbf{Prop. (I.2.2.16) (Elementary Symmetric Polynomial).} For \(n\) indeterminants \(x_i\), define the \textit{elementary polynomials} \(\sigma_k = \sum_{1 \leq i_1 < \ldots < i_k \leq n} x_{i_1}x_{i_2} \ldots x_{i_k}\). Then any symmetric polynomial is a polynomial of the fundamental symmetric polynomials.

\textit{Proof:} Set the first coordinate to 0, then the rest is a polynomial of the fundamental symmetric polynomials by induction, \(f(0, x) = h(\sigma_1, \ldots, \sigma_n)\), then consider \(f - h\), with \(h\) the same expression but \(x_n\) is included, we get \(f\) is a polynomial of elementary symmetric polynomials. \(\square\)

\textbf{Prop. (I.2.2.17).} Any polynomial on the entries of matrixes \(M_n(k)\) that is invariant under conjugation is generated by coefficients of \(\det(\lambda I + X)\) and can also be generated by \(\text{tr}(X^k)\).

\textit{Proof:} We notice that the matrixes having disjoint eigenvalues is dense in \(M_n(k)\), thus the restriction of the polynomial on these matrixes is a symmetric polynomial (I.2.2.16) thus identical to a polynomial described above. Hence they are equal. \(\square\)

\textbf{Prop. (I.2.2.18).} For any polynomial on the entries of matrixes \(M_n(k)\) that \(f(BA) = f(A)\) for \(B \in O(n)\), there is a polynomial \(F\) that \(f(A) = F(A^*A)\). Cf. [Heat Equation and the Index Theorem Atiyah P323].

\textbf{Prop. (I.2.2.19) (Weyl).} Any linear map \(f\) from \((\mathbb{R}^m)^\otimes n\) to \(R\) that is \(O(m)\)-equivariant is a linear combinations of maps of the form:

\[v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto \langle v_{i_1}, v_{i_2}\rangle \langle v_{i_3}, v_{i_4}\rangle \cdots \langle v_{n-1}, v_n\rangle.\]

Where \(i_1, \ldots, i_n\) is a permutation of \(1, 2, \ldots, n\) when \(n\) is even and when \(n\) is odd, \(f\) must be 0.

\textit{Proof:} Cf. [Heat Equation and the Index Theorem]. \(\square\)

\section{Ring Theory}

\textbf{Basics}

\textbf{Def. (I.2.3.1) (Terminologies).}
\begin{itemize}
  \item A unital ring is called a \textbf{simple ring} iff it has no non-trivial two-sided ideals.
  \item A non-zero ring is called a \textbf{domain} iff whenever \(ab = 0\), \(a = 0\) or \(b = 0\).
  \item A ring is called \textbf{reduced} iff it has no non-zero nilpotent element.
  \item A ring \(R\) is called \textbf{Dedekind-finite} iff for any \(a, b \in R\), \(ab = 1 \Rightarrow ba = 1\).
\end{itemize}

\textbf{Prop. (I.2.3.2).} If \(1 - ab\) is left(right) invertible in a ring, then so does \(1 - ba\), and

\[(1 - ba)^{-1} = 1 + b(1 - ab)^{-1}a.\]
Proof: Direct Calculation.

**Prop. (I.2.3.3) (Kaplansky).** If \( a \) has a right inverse by no left inverse in a ring, then \( a \) has infinitely many right inverses.

Proof: If \( ab = 1 \), then in fact \( b + (1 - ba)a^n \) are right inverses for \( a \) for any \( n \geq 0 \), and they are distinct, because if \( b + (1 - ba)a^n = b + (1 - ba)a^m \), then \( (1 - ba)a^n = (1 - ba)a^m \). And by multiplying \( b \) on the right, we get \( (1 - ba)a^{n-m} = 1 - ba \), so \( ((1 - ba)a^{n-m} + b)a = 1 \).

**Bezout Domain**

**Def. (I.2.3.4).** A Bezout domain is an integral domain that any sum of two principal domains is also principal.

**Prop. (I.2.3.5).** The localization of a Bezout domain is Bezout. A local ring is Bezout iff it is a valuation ring(by (I.9.2.8)).

**Prop. (I.2.3.6).** Any finite submodule of a free module over a Bezout domain is finite free.

Proof: Cf.[Sta]0ASU.

**Prop. (I.2.3.7).** Finite locally free \( R \)-module over a Bezout domain is free.

Proof:

**Prop. (I.2.3.8).** A module over a Bezout domain is flat iff it is torsion-free.

Proof:

**UFD**

**Def. (I.2.3.9).** An element \( x \) in a domain \( R \) is called irreducible iff for any \( y, z \in R \) that \( x = yz \), either \( y \) or \( z \) is a unit.

An element \( x \) in a domain \( R \) is called a prime iff \((x)\) is a prime ideal. Every prime is irreducible.

A domain \( R \) is called a UFD iff every non-zero element \( x \in R \) has a factorization into irreducibles, unique up to units.

**Prop. (I.2.3.10).** if \( R \) is Noetherian domain, then each element has a decomposition into irreducible.

Proof: Trace the decomposition inductively, if it doesn’t stop, then it contradicts with Noetherian hypothesis.

**Prop. (I.2.3.11).** An integral domain \( R \) is a UFD iff each element \( x \) factors into irreducibles, and every irreducible element is a prime. Also this is equivalent to every non-zero element factors into prime elements.

Proof: If \( R \) is a UFD, then if \( x \) is irreducible, if \( ab \in (x) \), \( ab = xc \), then \( x \) is one irreducible in the decomposition of \( a \) and \( b \), by UFD, so \( a \in (x) \) or \( b \in (x) \), and \((x)\) is a prime ideal.

Conversely, if there are two decompositions \( \prod a_i = \prod b_j \), then some \( b_j \in (a_i) \) by primeness of \((a_i)\), so \( b_j = a_iu \), so \( u \) must by units, so by induction, this two decompositions are the same.

If every element factors into prime elements, then an irreducible element is a prime because it factors as a product of primes, and notice every prime is irreducible(I.2.3.9).
Prop. (I.2.3.12) (Kaplansky). An integral domain $R$ is UFD iff every non-zero prime ideal contains a non-zero principle prime ideal.

Proof: If $R$ is a UFD, let $p$ be a non-zero prime ideal, choose $a \neq 0 \in p$, and write $a = \pi_1 \ldots \pi_n$ as a product of irreducibles. Then $\pi_i \in p$ for some $i$, so $p$ contains the prime ideal $(\pi_i)$ (I.2.3.11).

Conversely, let $S$ be the set of all finite products of prime ideals of $R$, then $S$ is a multiplicative set of $R$. For any non-zero element $a \in R$, if $(a) \cap S = \emptyset$, then there is a prime ideal $P$ containing $(a)$ and is maximal among those avoiding elements of $S$. Then $P$ contains a non-zero prime $\pi$, contradiction. So $(a) \cap S \neq \emptyset$. Let $b \in R$ that $ab = \pi_1 \ldots \pi_n$, then we show $a \in S$ by induction on $n$.

If $n = 1$, then $a$ is a unit or a prime, so we are done. For general $n$, if $\pi_k | b$ for some $k$, then $b = \pi_k c$, and $ac = \pi_1 \ldots \pi_k-1 \pi_{k+1} \ldots \pi_n$, then $a \in S$ by induction hypothesis. Otherwise $\pi_k | a$ for all $k$, thus $a = \pi_1 \ldots \pi_n c$ and $1 = bc$, thus $c$ is a unit and $a \in S$.

Thus we proved that every non-zero element is a product of prime elements, so $R$ is a UFD by (I.2.3.11).

Prop. (I.2.3.13). A polynomial ring over a UFD is a UFD.

Proof:

Prop. (I.2.3.14). Let $k$ be a field, then the power series $k[[X_1, \ldots, X_n]]$ is a UFD.

Proof: Cf.[Algebra Lang P209].

Prop. (I.2.3.15) (Gauss Lemma).

PID

Def. (I.2.3.16) (Euclidean Domain).

Prop. (I.2.3.17) (Euclidean Domain is PID). An euclidean domain is a PID.

Proof:

Cor. (I.2.3.18). If $K$ is a field, then $K[X]$ is a PID.

Prop. (I.2.3.19) (Chinese Remainder Theorem).

Prop. (I.2.3.20) (PID Structures). In a PID,
- An element $t$ is irreducible iff $(t)$ is maximal.
- A PID is UFD hence Noetherian.
- An element $t$ is irreducible iff it is a prime.

Proof: 1: By (I.2.3.12).
2: By (I.2.3.10).
3: By item 2 and (I.2.3.10).
Skew-Field

Def. (I.2.3.21). A skew field (or division algebra) is a unital ring that every non-zero element is invertible (but may not be commutative).

Prop. (I.2.3.22) (Wedderburn). A finite division algebra $D$ is a field.

Proof: Use the class equation for the invertible elements of $D$, if it is not isomorphic, consider the center $Z(D)$ of $D$, let $|Z(D)| = z$, then it is a field, and any other centralizer can be seen as a vector space over $Z(D)$, let it of dimension $k$, then $z^n - 1 = z - 1 + \sum \frac{z^n-1}{z^{n_i}-1}$. But then let $\Psi_n$ be the cyclotomic character of degree $n$, then $\Psi_n(z)$ divides $z - 1$. But this is not true, as it is bigger. □

Prop. (I.2.3.23). If $D$ is a f.d. division algebra over $R$, then it is isomorphic to $R$, $C$ or $H$.

Proof: Cf. [Advanced Linear Algebra P466]. □

Prop. (I.2.3.24). A skew field $A$ of at most countable dimension over $C$ is isomorphic to $C$.

Proof: Notice it suffices to find an eigenvalue of $\phi$ for each $\phi \in A$, but otherwise $\{(\varphi - a)^{-1}\}$ is an uncountable set of linearly elements of $A$, so some $\sum a_i(\varphi - c_i)^{-1} = 0$, spanning the expression, we see $\prod_k(\varphi - \mu_k) = 0$, so some $\varphi - \mu_k = 0$, contradiction. □

Prop. (I.2.3.25) (Hualuogeng Equation). In a skew field $K$, if $a, b \neq 0$ and $ab \neq 1$, then

$$a - (a^{-1} + (b^{-1} - a)^{-1})^{-1} = aba$$

Proof: suffices to show

$$1 = (1 - ab)a(a^{-1} + (b^{-1} - a)^{-1}) = (1 - ab)(1 + a(b^{-1} - a)^{-1}).$$

But this is equal to

$$(1 - ab)(1 - (b^{-1} - a)(b^{-1} - a)^{-1} + b^{-1}(b^{-1} - a)^{-1}) = (1 - ab)(1 - ab)^{-1} = 1.$$  

□

Others

Prop. (I.2.3.26). Any unital ring $R$ of order $p^2$ is commutative.

Proof: Consider the center of $R$, it is non-trivial because □

Prop. (I.2.3.27). Let $A, A'$ be $k$-algebras and $B, B'$ subalgebras of $A, A'$ with centralizers $C, C'$, then the centralizer of $B \otimes_k B' \subset A \otimes_k A'$ is $C \otimes_k C'$.

Proof: It suffices to show that $C \otimes_k A' \cap A \otimes_k C' = C \otimes_k C'$, which is clear because they are flat over $k$. □
4 Module Theory

Remark (I.2.4.1). We usually use left modules. there are several proposition that is written in favor of right modules, they should be rectified.

Def. (I.2.4.2) (Finite Module). A finite module is a module that has a surjection $R^n \to M$ for some $n \geq 0$.

Prop. (I.2.4.3). Let $R$ be a nonzero commutative ring and $M$ be an $R$-module generated by $n - 1$ elements, then any $R$-module map $R^n \to M$ has a nonzero kernel.

Proof: Choose a surjection $R^{n-1} \to M$, then the map $R^n \to M$ can be extended to a map $R^n \to R^{n-1}$. It suffices to assume $M = R^{n-1}$. This map is represented by a matrix. If some entry $a_{ij} = a$ is not nilpotent, then we can localize $R$ to $R_a$ that $a$ is a unit. We can assume $a_{11} = a$, and apply elementary row and column transformation to make $A = \text{diag}(1, B)$, then we finish by induction. Now if all $a_{ij}$ are nilpotent, then $I = (a_{ij})$ is nilpotent, and if $m$ is the maximal integer that $I^m \neq 0$, then $(I^n)^{\leq m}$ is contained in the kernel of this morphism. \[ □ \]

Cor. (I.2.4.4) (Rank of Free Modules). If $M$ is a free module over a nonzero commutative ring $R$, then any basis of $M$ is of the same cardinality, called the rank of $M$, and any spanning subset of $M$ has greater cardinalities. In particular, $R^n \neq R^m$ as $R$-modules.

Prop. (I.2.4.5) (Fitting Lemma). For an endomorphism $T$ of a $R$ module $M$, if we denote $p$ the minimal integer that $R(T^p) = R(T^{p+1})$ and $q$ the minimal integer that $N(T^q) = N(T^{q+1})$. Then the morphisms are stable afterward. Then if there is a $m, n$ that $R(T^m) \oplus N(T^n) = X$ for a $R$-module endomorphism $T \in \text{End}(M)$, then $p, q < \infty$ and they are equal. Moreover, if we know $p, q < \infty$, then we have $R(T^p) \oplus N(T^q) = M$.

Proof: We notice that

\[
T^i : N(T^{i+j})/N(T^i) \to R(T^i) \cap N(T^j), \quad T^i : M/(R(T^j) + N(T^i)) \to R(T^i)/R(T^{i+j})
\]

are isomorphisms. Thus $R(T^m) \oplus N(T^n) = X$ shows $q \leq m$ and $p \leq n$, thus we have $R(T^p) \oplus N(T^q) = M$, which implies $p \geq q$ and $q \geq p$. thus the result. The rest also follows easily from these isomorphisms. \[ □ \]

Prop. (I.2.4.6) (Nakayama). If $M$ is a finite $A$-module, and $I \subset A$ is an ideal that $IM = M$, then there is a $a \in 1 + I$ that $aM = 0$.

In particular, if $I \subset \text{rad}(A)$, then $a$ is a unit(I.5.9.2), so $M = 0$.

Proof: Because $M = IM$, choose a set of generators $\{x_i\}$ of $M$, then $x_i = \sum a_{ij}x_j$, where $a_{ij} \in I$. Then if the matrix $M = (\delta_{ij} - a_{ij})$, then $Mx_i = 0$. So taking the adjoint matrix, then $\text{det}(M)x_i = 0$. Notice $\text{det}(M)$ is a morphism. But the determinant must be element like $1 + k, k \in I$, so we are done. \[ □ \]

Cor. (I.2.4.7). If $M$ is finite and $M = \text{rad}(A)M + N$, then $M = N$.

Cor. (I.2.4.8). If a finite $R$-module $M$ satisfies $M \otimes_R k(p) = 0$, then there is a $f \notin p$ that $M_f = 0$.

Proof: Because $M_p = 0$, and the support of $M$ is closed(finiteness used). \[ □ \]

Cor. (I.2.4.9). If an endomorphism $\varphi$ of a finite module $M$ over $R$ is surjective, then it is injective.
This endomorphism makes \( M \) a finite module over \( R[X] \) by letting \( X \) acts by \( \varphi \). So the hypothesis shows \( IM = M \) where \( I = (x) \subset R[x] \). Then Nakayama (I.2.4.6) shows there are some \((1 + f(X)X)M = 0\). Thus for any \( m \in M \) that \( X(m) = 0, m = 0 \), so \( \varphi \) is injective. \( \square \)

**Prop. (I.2.4.10).** If \( A \) is an algebra of countable dimension over \( C \) with unit, and \( \alpha \in A \) is not nilpotent, then there exists a simple \( A \)-module \( M \) that \( a|_M \neq 0 \).

**Proof:** First we claim that there is some \( \lambda \neq 0 \in C \) that \( a - \lambda \) is not invertible in \( A \), this is nearly the same as the proof of (I.2.3.24), noticing that \( a \) is not nilpotent. Now we can take \( M = A \langle a - \lambda \rangle \), then \( a1 = \lambda \neq 0 \). \( \square \)

**Prop. (I.2.4.11) (Jordan-Horder).** Cf. (I.3.7.6).

**Tensor Module and Hom Module**

**Def. (I.2.4.12) (Hom Module).** If \( B \) is an \( R - S \)-bimodule and \( C \) is a \( T - S \)-bimodule, then \( \text{Hom}_S(B, C) \) is naturally a \( T - R \)-module given by the action \((tfr)(b) = tf(rb)\).

Dually, if \( B \) is an \( S - R \)-bimodule and \( C \) is a \( S - T \)-bimodule, then \( \text{Hom}_S(B, C) \) is naturally an \( R - T \)-bimodule given by the action \((rft)(b) = f(br)t\).

**Def. (I.2.4.13) (Tensor Product).** Given a ring \( R \), a \( T - R \)-bimodule \( A \) and an \( R - S \)-bimodule \( B \), their tensor product is a \( T - S \)-bimodule defined by universal properties: \( A \times B \to A \otimes_R B \) is a \( T - S \)-bimodule map, and any \( T - S \)-bimodule map \( A \times B \to C \) to a \( T - S \)-bimodule \( C \) factors uniquely through \( A \otimes_R B \to C \).

The tensor product can be constructed as:

There is an adjoint:

\[- \otimes_R B : T \text{Mod}_R \leftrightarrow T \text{Mod}_S : \text{Hom}_S(B, -)\ .

In particular, tensoring commutes with colimits.

Similarly, there is an adjoint:

\[ A \otimes_R - : R \text{Mod}_S \leftrightarrow T \text{Mod}_R : \text{Hom}_T(A, -)\ .

**Proof:** We need to give an isomorphism

\[ \tau : \text{Hom}_S(A \otimes_R B, C) \cong \text{Hom}_R(A, \text{Hom}_S(B, C)).\]

Given \( f \in \text{Hom}_S(A \otimes_R B, C) \), we define

\[ \tau(f) : A \to \text{Hom}_S(B, C) : (\tau(f)a)(b) = f(a \otimes b).\]

and conversely, for \( g \in \text{Hom}_R(A, \text{Hom}_S(B, C)) \),

\[ \tau^{-1}(g)(a \otimes b) = (g(a))(b).\]

The verifications is routine and the isomorphism for left modules is dual. \( \square \)

**Prop. (I.2.4.14) ((Co)Induced Modules).** Given a ring homomorphism \( S \to R \), then \( R \) is a \( S - R \)-bimodule as well as a \( R - S \)-bimodule, then we define:
• \( f^* : R \text{ Mod} \to S \text{ Mod} : f^*M = SM = \text{Hom}_R(R, M) = R \otimes_R M \) (I.2.4.12)(I.2.4.13), the restriction.

• \( f_1 : S \text{ Mod} \to R \text{ Mod} : f_1M = R \otimes_S M \) is the induced module, it is left adjoint to restriction, by (I.2.4.13).

• \( f_* : S \text{ Mod} \to R \text{ Mod} : f_*M = \text{Hom}_S(R, M) \) is the coinduced module, it is right adjoint to restriction, by (I.2.4.13).

Dually for left modules, we define:

• \( f^* : \text{Mod}_R \to \text{Mod}_S : f^*M = MS = \text{Hom}_R(R, M) = M \otimes_R R \) (I.2.4.12)(I.2.4.13), the restriction.

• \( f_1 : \text{Mod}_S \to \text{Mod}_R : f_1M = M \otimes_S R \) is the induced module, it is left adjoint to restriction, by (I.2.4.13).

• \( f_* : \text{Mod}_S \to \text{Mod}_R : f_*M = \text{Hom}_S(R, M) \) is the coinduced module, it is right adjoint to restriction, by (I.2.4.13).

**Torsion-Free Module**

**Def.** (I.2.4.15) (Torsion-Free Module). Let \( R \) be a domain, an \( R \)-module \( M \) is called torsion-free iff there are no elements \( 0 \neq x \in R, 0 \neq f \in M \) that \( xf = 0 \).

**Prop.** (I.2.4.16). If \( 0 \to M_1 \to M_2 \to M_3 \) and \( M_1, M_3 \) are torsion-free, then \( M_2 \) is torsion-free. Torsion-free is a stalk-wise property (I.5.1.55).

**Proof:** Trivial. \( \square \)

**Prop.** (I.2.4.17). Let \( M \) be a finite \( R \)-module, then \( M \) is torsion-free if it is a submodule of a finite free module.

**Proof:** One direction is trivial, for the other, if \( M \) is torsion-free, then \( M \subseteq M \otimes_R K \), and \( M \otimes_R K \) is a finite \( K \)-vector space, with basis \( e_i \). Now let \( x_i \) be a basis of \( M \), let \( x_i = \sum a_{ij}b_{ij}e_j \), then let \( e = \prod_{ij}b_{ij}, \) then \( M \subseteq Re_1/b \oplus \ldots \oplus Re_n/b. \) \( \square \)

**Prop.** (I.2.4.18). If \( M, N \) are \( R \)-modules that \( M \) is torsion-free, then \( \text{Hom}(N, M) \) is torsion free.

**Proof:** Choose a surjection \( \bigoplus_I R \to N \to 0 \), then \( \text{Hom}(N, M) \hookrightarrow \prod_I M \) is torsion free. \( \square \)

**Modules over PID**

**Prop.** (I.2.4.19) (Classification of Modules over PID).

1. Submodule of a free module over a PID is free of smaller rank. Thus a projective module over a PID is free

2. Finite torsion-free module over a PID is free.

3. Finite module over a PID has a primary decomposition \( M = \bigoplus_i R/(q_i) \), where \( (q_i) \) is primary ideals.

So projective \( \iff \) free \( \iff \) torsion-free (when f.g.).
Proof: 1: Choose a well ordering on the basis of $F$, let $F_i$ is the submodule generated by $e_j, j \leq i$. Then $\pi_i(P \cap F_i) \subset R$ is a module of the form $(a_i)$, thus choose $u_i \in P$ that $p_i(u_i) = a_i$. Then $u_i$ is a basis for $P$: they are linearly independent, because for any finite linear combination that are 0, the maximal coordinate are 0. It also spans $P$, because we can choose an element in $P - \{u_i\}$ whose maximal nonzero coordinate $\alpha$ is minimal among them, by well-orderedness. But we can subtract a multiple of $u_\alpha$, thus producing a smaller element, contradiction.

2: If it is finite torsion-free, then it is a submodule of a finite free module (I.2.4.17), so it is free by item1.

3: Follows immediately from (I.5.4.23).

Prop. (I.2.4.20) (Primary Cyclic Decomposition). There is a primary cyclic decomposition theorem for a torsion module $M$ over a PID $R$. Thus the multisets of elementary divisors of $M$ is a complete set of invariants for $M$.

Proof: Cf.[Advanced Linear Algebra P153].

Cor. (I.2.4.21) (Invariant Factor Decomposition). By reordering the cyclic decomposition, we can get the invariant factor decomposition of $M$, there are scalars $d_m|d_{m-1}|\ldots|d_1$ that are called the invariant factors of $M$.

Proof: Cf.[Advanced Linear Algebra P157].

Prop. (I.2.4.22) (Elementary Factor Theorem). Let $F$ be a free module over a PID $R$, and let $M$ be a f.g. submodule $\neq 0$, then there exists a basis $B$ of $F$, elements $e_1, \ldots, e_m$ in this basis, and non-zero elements $a_1, \ldots, a_m \in R$ that

- the elements $a_1e_1, \ldots, a_me_m$ forms a basis of $M$.
- $a_i|a_{i+1}$.

And these $a_i$ are uniquely determined up to units.

Proof: Cf.[Lang, P153].

Transfinite Direct Sum dévissage of modules

Def. (I.2.4.23) (Direct Sum Dévissage of Modules). Let $M$ be a module over a ring $R$, then a direct sum dévissage is a family of submodules $M_\alpha$ indexed by an ordinal $S$ such that

- $M_0 = 0$.
- if $\alpha + 1 \in S$, then $M_\alpha$ is a direct sum of $M_{\alpha+1}$.
- if $\alpha$ is a limit ordinal, then $M_\alpha = \cup_{\beta < \alpha} M_{\beta}$.
- $\cup_{\alpha \in S} M_\alpha = M$.

If moreover, for any $\alpha \in S, \alpha + 1 \in S$, $M_{\alpha+1}/M_\alpha$ is countably generated , then $M_\alpha$ is called a Kaplansky dévissage of $M$.

Prop. (I.2.4.24). Let $M_\alpha$ be a direct sum dévissage of $M$, then $M \cong \bigoplus_{\alpha \in S, \alpha + 1 \in S} M_{\alpha+1}/M_\alpha$.

Proof: Cf. [Sta]058V.

Cor. (I.2.4.25). $M$ is a direct sum of countably generated modules iff $M$ admits a Kaplansky dévissage.
Prop. (I.2.4.26). Suppose $M$ is a direct sum of countably generated modules and $P$ is a direct sum of $M$, then $P$ is also a direct sum of countably generated modules.

Proof: Cf.[Sta]058X. □

Prop. (I.2.4.27) (Direct Summand Criterion of Free Modules). Let $M$ be a countably generated $R$-module that for any direct summand $N$ of $M$ and an element $x \in N$, $x$ is contained in a free direct summand of $N$, then $M$ is free.

Proof: Let $x_1, x_2, \ldots$ be a countable set of generators for $M$, then we can use inductions to find free submodules $F_1, F_2, \ldots$ of $M$ s.t. $\oplus_{i=1}^n$ is a direct sum of $M$ and contains $x_1, \ldots, x_n$, thus $M = \oplus F_i$ is free. □

5 Field Extensions

Prop. (I.2.5.1) (Artin). If $G$ is a monoid and $K$ is a field, any distinct characters of $G$ in $K$ are linearly independent over $K$.

Proof: Consider the minimal length of linear combination that is 0, then we multiply a suitable $z$ in it, then we can can cancel a character, contradicting the minimality. □

Cor. (I.2.5.2). If $\alpha_i$ are different elements in $K$ and there are element $\alpha_i$ that $\sum a_i \alpha_i^v = 0$ for every $v \geq 0$, then $a_i = 0$ for all $n$. (Seen as characters from $\mathbb{Z}_{\geq 0} \rightarrow K$).

Field Extensions

Def. (I.2.5.3). A family $L$ of extensions are called distinguished iff it is closed under base change and $k \subset F \subset E \in L$ iff $k \subset F \in L$ and $F \subset E \in L$.

Prop. (I.2.5.4). The family of finite extensions form a distinguished class.
The family of algebraic extensions form a distinguished class.
The family of f.g. extensions form a distinguished class.

Proof: Finite case is trivial. For the alg. extensions, for $k \subset F \subset E$, for any $\alpha \in E$, $\alpha$ satisfies an polynomial function with f.m coefficients in $F$, the coefficients form a subfield $F_0$ of $F$ which is finite over $k$, so $k \subset F_0 \subset F_0(\alpha)$ is a finite tower, so it is finite, hence algebraic. The base change is easy to check.

For f.g. extensions, it suffice to check composition: □

Prop. (I.2.5.5). For an alg.extension $k \subset E$, any injective field map $E \rightarrow E$ over $k$ is an automorphism. (This is because it induce a permutation of any $\alpha$ with its conjugates in $E$, so it is surjective).

Lemma (I.2.5.6). Let $f \in k[X]$ be a polynomial of degree $\geq 1$, then there is a field $K$ that $f$ has a root in $K$. Hence for any finite set of polynomials, there is a field $K$ that all of them have roots in $K$.

Proof: Cf.[Algebra Lang P231]. □

Lemma (I.2.5.7). For any field $k$, there exists uniquely an alg.closed field $K$ containing $k$. 

Proof: Firstly, we construct a field that every polynomial in $k[X]$ of degree $\geq 1$ has a root. Consider the polynomial ring $k[X]$, where there is a indeterminant $X$ for each $f \in k[X]$ of $\deg \geq 1$. Then the ideal generated by $f(X)$ is not a unit ideal, which can be seen by constructing a finite field extension that $f$ all have a root in it (I.2.5.6).

So if $m$ is a maximal ideal containing all $f(X)$, then the quotient field is a field that all $f$ have a root $(X_f)$. So now if we construct inductively like this, and consider their union, then it is clearly a field and any polynomial of degree $\geq 1$ have a root in it. □

Prop. (I.2.5.8) (Algebraic Closure Exists). For any field $K$. There exists uniquely an alg.closed field $K/k$ that is algebraic over $K$, up to isomorphism over $k$.

Proof: Let $E$ be a field that is alg.closed and contains $k$ by (I.2.5.7). Let $k^a$ be the union of subextensions that are algebraic over $k$. $k^a$ is a field, by (I.2.5.4), and $k^a$ is alg.closed, because if $f(X)$ is a polynomial of degree $\geq 1$ in $k^a[X]$, then it has a root $\alpha \in E$, and $\alpha$ is algebraic over $k^a$, so $\alpha \in k^a$. □

Prop. (I.2.5.9). If $E/F$ is an algebraic field extension, then for any $R$ that $E \subset R \subset F$, $R$ is a field.

Proof: If $\alpha \in R$, then $\alpha$ is algebraic over $E$, so there is a relation $\alpha^n + \ldots + a_0 = 0$, so $\alpha^{-1} = -a_0^{-1}(a_1 + \ldots a_n) \in R$. □

Separable, Normal & Galois Extensions

Def. (I.2.5.10). An extension $K/k$ is called normal extension iff it satisfied the following equivalent conditions:

- Any embedding of $K$ into $k^{alg}$ induce an automorphism on $K$.
- $K$ is the splitting field of a family of polynomials in $k[X]$.
- Every irreducible polynomial in $k[X]$ that has a root in $K$ splits in $K$.

Proof: Cf.[Algebra Lang P237]. □

Prop. (I.2.5.11). Normal extension are stable under base change and forms a lattice, this is immediate from the first definition of (I.2.5.10).

Def. (I.2.5.12). If we define the separable degree $[E : k]_s$ of an extension $E/k$ as the number of embedding into a fixed alg.closure, then it commutes with composition and when $E/k$ is finite, $[E : k]_s \leq [E : k]$.

Def. (I.2.5.13). A finite extension is called separable iff $[E : k]_s = [E : k]$, an algebraic number $\alpha$ over $k$ is called separable iff $k(\alpha)/k$ is separable. A polynomial $f \in k[X]$ is called separable iff it has no multiple roots in $k^{alg}$.

Def. (I.2.5.14) (Separable Extensions). An extension $E/k$ is called separable iff it satisfies the following equivalent conditions:

- every f.g. subfield is separable over $k$, (this is compatible because subfield of a finite separable extension is separable, by the compatibility of separable degree).
- Every element of $E$ is separable.
• It is generated by a family of separable elements.

**Proof:** If $E/k$ is separable and $k \subset k(\alpha) \subset E$, then by (I.2.5.13), $k(\alpha)$ is separable. And if it is generated by a family of separable elements $\{\alpha_i\}$, then any f.g. subfield can are f.g. by elements $\{\alpha_i\}$. Now it is a tower of separable extensions, hence separable by the compatibility of separable degree. □

**Prop. (I.2.5.15).** Separable extensions form a distinguished class.

**Proof:** Cf.[Algebra Lang P241]. □

**Prop. (I.2.5.16) (Primitive element Theorem).** A finite extension $E/k$ is primitive iff there are only f.m. middle fields. And if $E/k$ is separable, this is satisfied.

**Proof:** If $k$ is finite, this is simple. Assume $k$ infinite, for any two elements $\alpha, \beta$, consider $k(\alpha + c_\beta)$, if there is only finitely many middle fields, there exists two that is equal, so $k(\alpha, \beta) = k(\gamma)$. Proceeding inductively, $E$ is primitive.

Conversely, if $k(\alpha) = E$, every middle field corresponds to a divisor of the irreducible polynomial of $\alpha$. This map is injective, because for any $g_F$, degree of $\alpha$ over $F$ is the same over the degree over the coefficient field of $g_F$, so it must be equal to $F$.

If $E/k$ is separable, Let

$$P(X) = \prod_{i \neq j}(\sigma_i\alpha + X\sigma_i\beta - \sigma_j\alpha - X\sigma_j\beta)$$

for different embedding $\sigma_i, \sigma_j$ of $E(\alpha, \beta)$ into $k^{alg}$. Then it is not identically zero, thus there exists $c$ that $\sigma_i(\alpha + c\beta)$ is all distinct, thus generate $K(\alpha, \beta)$. □

**Cor. (I.2.5.17).** If $L/K$ is a finite Galois extension, then there is an isomorphism:

$$L \otimes_K L \cong L \times L \times \ldots \times L : (a, b) \mapsto (ab, a\sigma_1(b), \ldots, a\sigma_n b)$$

where $\sigma_i$ are Galois elements.

**Proof:** Choose a primitive element $x$ and its minimal polynomial $f(x)$, then $L \cong K[X]/(f)$, and $L \otimes_K L \cong L[X]/(f)$, but $f$ decomposes completely in $L[X]$, thus by Chinese remainder theorem(I.2.3.19), the given map is an isomorphism of rings. □

**Prop. (I.2.5.18).** Automorphisms of a field $L$ are linearly independent over $L$.

**Proof:** If there are some $\sum \alpha_i\sigma_i(b) = 0$ for all $b$, then if $\alpha_i \neq 0, \alpha_j \neq 0$, choose a $t$ that $\sigma_i(t) \neq \sigma_j(t)$, then we can subtract and make the non-zero terms one less, so by induction this is false. □

### Inseparable Extensions

**Prop. (I.2.5.19).** Any irreducible polynomial of fields of characteristic 0 is separable and if $\text{char}= p$, then all roots have the same multiplicity and thus $[k(\alpha) : k] = p^n[k(\alpha^p) : k]$ for some $n$.

**Proof:** All roots have the same multiplicity because there are Galois actions. If the multiplicity is not 1, the derivative $f'$ is zero, otherwise $f$ is not irreducible. Then $f(X) = g(X^p)$. We can choose $f(X) = h(X^{p^n})$ with $h$ separable, then $[k(\alpha) : k(\alpha^{p^n})] = p^n$, thus the result. □
Def. (I.2.5.20). The **inseparable degree** $[E : k]_i$ is defined as the quotient $[E : k]/[E : k]_s$. An algebraic element $\alpha$ is called **purely inseparable** over $k$ iff there is a $n$ that $\alpha^{p^n} \in k$.

Def. (I.2.5.21). An extension is called **purely inseparable** if it satisfies the following equivalent conditions:

- $[E : k]_s = 1$.
- Every element $\alpha$ of $E$ is purely inseparable over $k$.
- For every $\alpha \in E$, the irreducible equation of $\alpha$ over $k$ is of type $X^{p^n} - a$.
- It is generated by a family of purely inseparable elements.

**Proof:** Cf.[Algebra Lang P249].

Cor. (I.2.5.22). A field over $\mathbb{F}_p$ is perfect iff there are no purely inseparable extensions of it.

Cor. (I.2.5.23). For any field $k$ of char $p$, there is a unique purely inseparable field extension $k_{perf}/k$ that $k_{perf}$ is perfect, called the **perfect closure** of $k$. It is generated by adding all the $p^n$-th roots to $k$.

Prop. (I.2.5.24). Purely inseparably extensions form a distinguished class.

**Proof:** Cf.[Algebra Lang P250].

Prop. (I.2.5.25). If $E/k$ is algebraic and $E_0$ be the maximal separable extension contained in $E$, then $E/E_0$ is purely inseparable. And if $E/k$ is normal, then $E_0/k$ is normal, too.

**Proof:** By the proof of(I.2.5.19), any $\alpha$ has a $p^n$ that $\alpha^{p^n}$ that $\alpha^{p^n}$ is separable, hence it is purely inseparable over $E_0$ by(I.2.5.21). $E_0/k$ is normal because any $\sigma$ maps $E$ to itself, and $E_0$ to $\sigma(E_0) \subset E$ separable, hence $\sigma(E_0) \subset E_0$.

6 Transcendental extension

Def. (I.2.6.1) (Transcendental Basis). Let $K$ be an extension of a field $k$, a **transcendental base** is an algebraically independent set that any element is algebraic over it.

Given any algebraically independent set $S \subset T$ a set over which $K$ is algebraic, $S$ can be extended to a transcendental base contained in $T$, by Zorn’s lemma. In particular, a transcendental basis exists.

Prop. (I.2.6.2). Any two transcendental basis have the same cardinality, called the **transcendental degree** of $K/k$, denoted by $\text{tr deg}_k(K)$.

**Proof:** If $K/k$ has a finite transcendental basis, then let $X = \{x_1, \ldots, x_m\}$ transcendental base of minimal number, $S = \{w_1, \ldots, w_n\}$ an algebraically independent set. If $n > m$, we proceed by changing one element in $X$ a time using induction and prove that $K$ is algebraic over $\{w_1, \ldots, w_r, x_r+1, \ldots, x_m\}$, contradiction.

Because $w_{r+1}$ is algebraic over $\{w_1, \ldots, w_r, x_{r+1}, \ldots, x_n\}$, we have a minimal polynomial

$$f = \sum g_j(w_{r+1}, w_1, \ldots, w_r, x_{r+1}, \ldots, x_m) x^{d_j}_{r+1}$$

s.t. $f(w_{r+1}, w_1, \ldots, x_m) = 0$ (after possibly renumbering $x_i$, this $x$ must exists because $S$ is itself algebraically independent). So $x_r$ is algebraic over $\{w_1, \ldots, w_{r+1}, x_{r+2}, \ldots, x_m\}$, hence $K$ is independent over it, too.
If \( K/k \) has an infinite basis \( B \), let \( B' \) be another basis, then for any \( \alpha \in B^* \), there is a finite set \( B_\alpha \subset B \) that \( \alpha \) is algebraic over \( k(B_\alpha) \), because algebraic equation involves f.m. generators. Then we define \( B^* = \bigcup_{\alpha \in B'} \), then it has cardinality smaller than \( B' \). But \( B^* = B \), because for any \( \beta \in B \), \( \beta \) is algebraic over \( B' \) which is algebraic over \( k(B^*) \), thus \( \beta \) is algebraic over \( k(B^*) \), thus \( \beta \in B^* \).

Prop. (I.2.6.3). If \( K \) is of finite transcendental degree over \( k \), then \(|K| = |k|\).

Proof: We find a purely transcendental \( L/k \) that \( K/L \) is algebraic, then the element of \( L \) are all polynomials of finite indeterminants of elements of \( k \), so \(|L| = |k|\) by (XII.1.5.4), and similarly \(|K| = |L|\).

Prop. (I.2.6.4). Two alg. closed field of the same transcendental degree over base field is isomorphic.

Proof: We define a bijection of the transcendental basis, and then extend it to an isomorphism of fields, by adjunction of algebraic numbers using Zorn’s lemma.

Prop. (I.2.6.5) (Lüroth). The automorphism group of \( K(X) \) is \( PGL_2(K) \).

Proof: Consider \( \theta = \sigma(x) = \frac{f(x)}{g(x)} \), then \( x \) is algebraic over \( K(\theta) : \theta g(x) - f(x) = 0 \). Now \( x \) is transcendental over \( K \), thus \( \theta \) is transcendental over \( K \) as well. Now the minimal polynomial of \( x \) over \( K(\theta) \) is just \( \theta g(x) - f(x) \), because it is irreducible, as it is linear over \( \theta \). But \( K(x) = K(\theta) \), thus the polynomial must have degree 1, so \( f(x), g(x) \) is of degree 1. Now the rest is clear.

Prop. (I.2.6.6) (Lüroth Theorem). Any subfield of \( L = K(X) \) properly containing \( K \) is of the form \( K(u) \) where \( u \in L \) is transcendental over \( K \).

Proof: Cf.[Fields and Galois Theory, Milne, P114].

7 Galois Theory

Prop. (I.2.7.1) (Artin Algebraic Independence). Let \( K \) be an infinite field and \( \sigma_i \) be distinct elements of a finite group of automorphisms of \( K \), then \( \sigma_i \) are alg. independent over \( K \).

Proof: Cf.[Algebra Lang P311].

Prop. (I.2.7.2) ((Artin)Galois Main Theorem). Let \( G \) be a finite group of automorphisms of \( K \). Then \( K/K^G \) is Galois of Galois group \( G \).

Proof: For every element \( x \), set \( \{\sigma_1x, \ldots, \sigma_rx\} \) be distinct conjugates, then \( f(X) = \prod_{i=1}^r (X-\sigma_ix) \) shows that \( K \) is separable and normal over \( K^G \). And primitive element theorem shows that \(|K : K^G| \leq |G|\), so it must equals \( G \).

Prop. (I.2.7.3) (Infinite Galois Theorem). The middle fields correspond to the closed subgp of \( G(L/K) \).

Proof: The highlight is that \( G(L/L^H) = H \) for a closed subgp \( H \) of \( G(L/K) \). If \( \sigma \) fixes \( L^H \) but is not in \( H \), because for every finite field \( M \), \( H \cdot G(L/M) \) corresponds to \( M/(M \cap L^H) \), so \( \sigma G(L/M) \cap H \neq \emptyset \). So \( \sigma \) is in the closure of \( H \) thus in \( H \).

Prop. (I.2.7.4) (Normal Basis Theorem). For a finite Galois extension, normal basis exists.
Proof: Finite case: The Galois group is cyclic, and the linear independent of characters shows that the minimal polynomial of \( \sigma \) is \( n \)-dimensional thus equals \( X^n - 1 \). Regard \( L \) as a \( K[X] \) module thus by (I.2.4.19) is a direct sum of modules of the form \( K[X]/(f(x)), f(x)|X^n - 1 \) and the minimal polynomial for the action of \( X \) is \( X^n - 1 \). So it must be isomorphic to \( K(X)/(X^n - 1) \).

Infinite Case: Let 

\[
f([X_\sigma]) = \det(t_{\sigma_\iota, \sigma_j}), \quad t_{\sigma_\iota, \tau} = X_{\sigma^{-1}\tau}
\]

We see \( f \neq 0 \) by substituting \( 1 \) for \( X_{id} \) and \( 0 \) otherwise. So it won’t vanish for all \( x \) if we substitute \( X_\sigma = \sigma(x) \) because \( [\sigma(x)] \) is pairwisely different. Thus there exists \( w \) s.t.

\[
det(\sigma^{-1}\tau(w)) \neq 0.
\]

Now if

\[
\sum a_\sigma\tau(w) = 0, \quad a_\sigma \in K,
\]

act by \( \sigma \) for all \( \sigma \), we get \( [\sigma^{-1}\tau(w)]|a_\sigma] = 0 \), thus \( [a_\sigma] = 0 \). \( \square \)

Prop. (I.2.7.5) (Kummer Theory). Let \( K \) be a field containing the \( n \)-th roots of unity, a Kummer extension \( L/K \) of order \( n \) is one that the Galois group is Abelian and of exponent \( n \). There exists an inclusion preserving isomorphism between the lattice of Kummer extensions \( L \) of \( K \) and the lattice of subgroups of \( L \) containing \( K^n \):

\[
L \mapsto \Delta = (L^\times)^n \cap K^\times, \quad \Delta \mapsto K(\sqrt[n]{\Delta}).
\]

And \( \Delta/(K^\times)^n \) is isomorphic to \( \text{Hom}_{cont}(G_{L|K}, \mathbb{Q}/\mathbb{Z}) \).

Proof: Notice the composite of two Kummer extension is an extension, so we consider the maximal Kummer extension \( L \), then \( K^* \subset (L^*)^n \), because otherwise we can add a \( \sqrt[n]{a} \), this is another Kummer extension.

We use the exact sequence \( 1 \to \mu_n \to L^* \xrightarrow{n} (L^*)^n \to 0 \), then the profinite cohomology exact sequence says

\[
1 \to K^* \to (L^*)^n \cap K^* \xrightarrow{\delta} H^1_{cts}(G, \mu_n) \to H^1_{cts}(G, L^*) = 1
\]

And \( G \) acts trivially on \( \mu_n \subset K^* \), then \( H^1_{cts}(G, \mu_n) = \text{Hom}_{cts}(G_{L|K}, \mathbb{Q}/\mathbb{Z}) \). \( \delta \) maps \( a \mapsto \chi_a(\sigma) = \sigma(\sqrt[n]{a})/\sqrt[n]{a} \in \mu_n \).

Thus if we let \( L \) be the maximal Kummer extension, then by Galois theory, Kummer extensions of \( K \) corresponds to closed subgroups of \( G \), subgroups of \( \text{Hom}_{cts}(G_{L|K}, \mathbb{Q}/\mathbb{Z}) \) correspond to subgroups of \( K^*/(K^*)^n \). These two correspond by the intersection of the kernel of all them, because they correspond for finite subgroup and open subgroup. And closed subgroups are intersection of open subgroups, and any open subgroup containing it must contain an open subgroup of chosen form, by compactness. Thus they correspond. \( \square \)

Prop. (I.2.7.6). \( \text{Gal}(F_{p^n}/F_p) = \mathbb{Z}/n\mathbb{Z} \), and is generated by the Frobenius.

Brauer Groups

Prop. (I.2.7.7). The Brauer group \( \text{Br}(K) \) is defined as the profinite cohomology \( H^2(G(K^*/K), K^*_1) \). For a Galois extension \( L/K \), \( \text{Br}(L/K) \) is defined as \( H^2(G(L/K), L^*) \). Then by (IV.3.2.2) we have

\[
\text{colim} \text{Br}(L/K) = \text{Br}(K).
\]
Cf. [Neukirch Cohomology of Number Fields Chap 6.3].

**Prop. (I.2.7.8).** The **Brauer group** $\text{Br}(K)$ is defined as the profinite cohomology $H^2(G(K_{s}/K), K^*_s)$. For a Galois extension $L/K$, $\text{Br}(L/K)$ is defined as $H^2(G(L/K), L^*)$. Then by (IV.3.2.2) we have

$$\text{colim} \text{Br}(L/K) = \text{Br}(K).$$

And by Hochschild-Serre spectral sequence and by Hilbert’s multiplicative theorem: $H_1(H, K^*_s) = 0$, we have the low term:

$$0 \to \text{Br}(L/K) \xrightarrow{\text{inf}} \text{Br}(K) \xrightarrow{\text{res}} \text{Br}(L) \xrightarrow{G} H_3(G(L/K), L^*) \to H_3(K, K^*_s).$$

So $\text{Br}(L/K)$ is the kernel of $\text{Br}(K) \to \text{Br}(L)$.

**Cor. (I.2.7.9).** $\text{Br}(\mathbb{F}_q) = 0$ for finite fields, because the finite Galois extension are cyclic and unramified.

In fact, the Brauer group has semisimple algebraic interpretations. Cf. [Milne].

8 **Ordered Rings**

**Def. (I.2.8.1) (Ordered Rings).** An **ordered ring** is a ring $R$ together with a subset $P \subset R$ that $R$ is a disjoint union $P \coprod \{0\} \coprod (-P)$, and if $x, y \in P$, then $x + y, xy \in P$. Elements in $P$ are called **positive elements**.

An **ordered field** is an ordered ring that is also a field. An **orderable ring/field** is a ring/field that can be given an ordered structure.

**Prop. (I.2.8.2).** An orderable field has char0, because $0 \notin P \cup (-P)$.

A square > 0 in an ordered field (trivial).

**Def. (I.2.8.3) (Convex Subgroup).** Let $\Gamma$ be an ordered Abelian group (I.2.8.1), then a **convex subgroup** of $\Lambda$ is a subgroup $\Delta$ that if $a < b < c$ and $a, c \in \Delta$, then $b \in \Delta$. Notice this is in fact equivalent to if $0 < c \in \Delta$, then $0 < b < c$ are also in $\Delta$.

**Prop. (I.2.8.4) (Height).** The set of all convex subgroups of $\Gamma$ is well-ordered, and its ordinal is called the **height** of $\Gamma$.

**Proof:** If $\Delta_1, \Delta_2$ don’t contains each other, let $a \in \Delta_1 \setminus \Delta_2$ and $b \in \Delta_2 \setminus \Delta_1$, then changing $\pm a, \pm b$, we may assume $0 < a < b$, so $a \in \Delta_2$, contradiction. \hfill \Box

**Prop. (I.2.8.5) (Height 1 Case).** Let $\Gamma$ be an ordered Abelian group, then the following are equivalent:

1. ht($\Gamma$) = 1.
2. for all $a, b \in \Gamma$ that $a > 0$ and $b \geq 0$, there is an integer $n$ that $b \leq na$.
3. there exists an injection from $\Gamma$ to $\mathbb{R}$.

**Proof:** 3 $\to$ 1 is easy.

1 $\to$ 2: Consider the convex subgroup generated by $a$, then it is $\Delta$ by height condition, so $b$ must by in it, i.e. $b \leq na$ for some $n$. 
2 → 3: Choose an \( a > 0 \), let the injection \( \varphi \) given by \( \varphi(b) = \sup\{\frac{n}{b} | na \leq kb \} \) for \( b > 0 \) and extends to negative elements.

It is easily verified that \( \varphi(c) + \varphi(b) \leq \varphi(c + b) \), and if \( \varphi(c) + \varphi(b) < \varphi(c + b) \), choose a rational approximation of them, and multiply to get integers, then if \( k(c + b) \leq \varphi(c + b)ka > \varphi(b)a + \varphi(c)a + a \), then either \( kc \geq \varphi(c)ka + a \) or \( kb \geq \varphi(b)ka + a \), contradiction.

So this map is truly a morphism of ordered Abelian groups, and it is injective because if \( b > 0 \), then by 2, there must be an \( n \) that \( a \leq nb \), so \( \varphi(b) \geq 1/n \). \( \square \)

9 Real Fields

**Def. (I.2.9.1) (Real Fields).** A field \( K \) is called real if \(-1\) is not a sum of squares in \( K \). A field \( K \) is called real closed iff it is real, and any alg.extension that is real must be itself. An ordered field is clearly a real field by(I.2.8.2), the converse is in fact true, by(I.2.9.11). In particular, a real field is of characteristic 0.

**Prop. (I.2.9.2).** If \( K \) is real, \( a \in K \), \( a \) and \(-a\) cannot both be sum of squares. If \(-a\) is not a sum of squares in \( K \), then \( K(\sqrt{a}) \) is real. Hence either \( K(\sqrt{a}) \) or \( K(\sqrt{-a}) \) is real.

*Proof:* Suppose \( K(\sqrt{a}) \) is not real. If \( a \) is a square, then \( K(\sqrt{a}) = K \) is real. So \( a \) is not a square,

\[ -1 = \sum (b_i + c_i \sqrt{a})^2 = \sum (b_i^2 + ac_i^2 + 2b_i c_i \sqrt{a}) \]

Since \( \sqrt{a} \notin K \), \( -1 = \sum (b_i^2 + ac_i^2) \), so

\[ -a = \frac{1 + \sum b_i^2}{\sum c_i^2} \]

This implies that \(-a\) is a sum of squares. \( \square \)

**Prop. (I.2.9.3).** If the minimal polynomial \( f \) of an \( \alpha \) algebraic over a real field \( K \) is of odd degree, then \( K(\alpha) \) is real.

*Proof:* If \( K(\alpha) \) is not real, then \( -1 = \sum g_i(X)^2 + h(X)f(X) \), where \( g_i \) has degree smaller than \( n \). This can happen if \( h(X) \) has degree odd and \( \leq n - 2 \). Then if \( \beta \) is a root of \( h \), then \( K(\beta) \) is also not real. So the proof is finished if we use induction. \( \square \)

**Def. (I.2.9.4) (Real Closure Exists).** For any real field \( K \), there exists a real closure \( K^a \) of \( K \). That is, it is real closed and algebraic over \( K \).

*Proof:* This is an easy consequence of Zorn’s lemma. \( \square \)

**Cor. (I.2.9.5) (Real Closed Fields Unique Ordering).** There exists a unique ordering on a real closed field \( R \). The elements \( > 0 \) are just the squares in \( R \). Now every real closed field is assumed to have this ordering tacitly. In particular, any real closed field has char0, so does any real field.

*Proof:* The set of finite sum of squares in \( R \) is closed under addition and multiplication, and all of them are squares, by(I.2.9.2) and maximality of \( R \). Also by(I.2.9.2) either \( a \) is a square or \(-a\) is a square, but not simultaneously. So it is truly an order on \( R \). \( \square \)

**Prop. (I.2.9.6) (Fundamental Theorem of Algebra).** For a field \( R \), \( R \) is real closed iff \( R \neq R[i] \) and \( \overline{R} = R(i) \).
Proof: One direction is trivial, the other follows from the lemma below (I.2.9.7), it satisfies the condition by (I.2.9.2) and maximality.

Lemma (I.2.9.7) (Equivalent Definition of Real Closed Fields). If \( R \) is a real field that: for all \( a \in R \), \( \sqrt{a} \in R \) or \( -\sqrt{a} \in R \), and any polynomial of odd degree has a root in \( R \), then \( K = R(i) \) is alg.closed.

Proof: For any order of \( R \), the first condition in fact says that any \( a > 0 \) in \( R \) is a square. Now \( \frac{a + \sqrt{a^2 + b^2}}{2} \) is non-negative, so there is a \( c^2 = \frac{a + \sqrt{a^2 + b^2}}{2} \), that is \( (c + \frac{b}{2c})^2 = a + bi \), so \( K \) has all squares.

As \( R \) is of char0(I.2.9.5)(I.2.8.2), so it suffices to show any Galois extension \( L/K \) is trivial. Let \( G = G(L/R) \), and \( H \) be its 2-Sylow subgroup, then \( G = H \) by condition. Now if \( G_1 = G(L/K) \), then \( G_1 \) is nontrivial, because otherwise there is a subgroup of index 2, then its fixed field is a square extension of \( K \), which is impossible by what we have proved. So \( G = G_1 \), that is \( L = K \). □

Cor. (I.2.9.8). \( \mathbb{C} = \mathbb{R}[i] \) is alg.closed.

Prop. (I.2.9.9) (Intermediate Property). An ordered field is real closed iff it has the intermediate property.

Proof: If \( R \) is real closed, as \( R[i] \) is alg.closed(I.2.9.6), \( f \) can be decomposed into factors of degree 1 or 2. For a factor \( X^2 + \alpha X + \beta \), \( 4\beta > \alpha^2 \), otherwise it has a root hence not irreducible. So the change of sign is because of a linear factor, the rest is easy.

Conversely, if it has the intermediate property, then for \( a > 0 \), consider \( p(X) = X^2 - \alpha \), then \( p(0) < 0, p(a + 1) > 0 \), so \( p \) has a root, that is, \( a \) is a square. For a polynomial of odd degree, for \( M \) large enough, \( f(M) > 0, f(-M) < 0 \), so \( f \) has a root. So by(I.2.9.7) \( R \) is real closed. □

Prop. (I.2.9.10) (Artin-Schreier). If \( K \) is separably closed and \( F \) is a subfield of finite index in \( K \), then \( F = K \) or \( F \) is real closed and \( K = F(i) \).

Proof: Cf.[Lan05]P299. □

Real Fields and Order

Prop. (I.2.9.11) (Real Field and Order). If \( R \) is a real field, then it is orderable, in fact, if \( -a \) is not a sum of squares in \( F \), then there is an ordering that \( a > 0 \). So a real field is equivalent to an orderable field.

Proof: By(I.2.9.2), \( F(\sqrt{a}) \) is real, so it has a real closure(I.2.9.4) and has the induced order(I.2.9.5), and \( a > 0 \) because it is a square(I.2.8.2). □

Prop. (I.2.9.12) (Existence and Uniqueness of Real Closure). For any ordered field \( F \), there is a unique real closure \( R \) of \( F \) that every positive element of \( F \) is a square in \( R \), thus the ordering is compatible.

Proof: The existence is by adding all the square roots of elements> 0 to \( F \), the resulting field is real closed because of(I.2.9.2) and the fact a union of real fields(I.2.9.2) is real.

The uniqueness: because an ordered field is of char0(I.2.8.2), so the primitive element theorem(I.2.5.16) applies that each finite subextension of \( R_0 \) is of the form \( F(\alpha) \), where \( \alpha \) is a root of a irreducible separable polynomial \( f \). Then the roots of \( f \) are different so can be ordered \( \alpha_1 < \ldots < \alpha_n \).
Similarly, $f$ has the same number of different roots in $R_1 \beta_1 < \ldots < \beta_n$ by (I.2.9.14), so there is a map $h : \alpha_i \to \beta_i$, and it is the unique map that a ordered map from $F(\alpha)$ to $R_1$ extending $\text{id}$ on $F$ can be. It is this uniqueness that makes us able to use Zorn’s lemma to show that there is a maximal ordered map, must be a map from $R_0$ to $R_1$, which is an isomorphism, by primitive element theorem again. \qed

**Prop. (I.2.9.13) (Sturm’s Algorithm).** Cf. [Model Theory Marker P327].

**Cor. (I.2.9.14).** If $F$ is an ordered field and $R_0, R_1$ be two real closure of $F$ that is compatible with the ordering, then any irreducible polynomial has the same number of roots in $R_0$ and $R_1$.

*Proof:* Cf. [Model Theory Marker P328]. \qed

**Prop. (I.2.9.15) (Hilbert’s 17th Problem).** If $f$ is a positive semidefinite rational function over a real closed field $F$, then $f$ is a sum of squares of rational functions.

*Proof:* Let $f(X_1, \ldots, X_n)$ be a positive semidefinite rational function, if $f$ is not a sum of squares of rational functions, then by (I.2.9.11), there is an ordering on $F(X)$ that $f < 0$. Let $R$ be a real closure of $F(X)$, then $R \models \exists \bar{\pi} F(\bar{\pi}) < 0$, as $F(X) < 0$. But $R \subseteq C$ is complete, by (XII.2.7.6), thus $F \models \exists \bar{\pi} F(\bar{\pi}) < 0$ also, contradiction. \qed
I.3 Group Theory

Main References are [Finite Groups, Issac] and [代数学引论，丁石孙].

1 Basics

Def. (I.3.1.1) (Normal Subgroups). A normal subgroup of a group $G$ is a subgroup $N$ that if $x \in G$, then $x^{-1}N x = N$.

Def. (I.3.1.2) (Simple Group). A simple group is a group that has no normal subgroups.

Def. (I.3.1.3) (Finitely Generated Group). A group $G$ is called finitely generated if there is a finite subset $S$ that the only subgroup containing $S$ is $G$ itself.

Prop. (I.3.1.4) (Fundamental Isomorphisms). For a normal subgroups $U, H$ of $G$,

- $G/H U \cong (G/U)/(H/H \cap U)$.
- $U H/U \cong U/U \cap H$.

Proof:

Cor. (I.3.1.5). if $H_1, H_2$ are subgroups of a group $G$ that has finite indexes, then $H_1 \cap H_2$ also has finite index in $G$.

Proof: By fundamental isomorphism(I.3.1.4), $H_1/H_1 \cap H_2 \cong H_1 H_2/H_2 \subset G/H_2$, so $H_1 \cap H_2$ has finite index in $H_1$, so by transitivity of indexes, $H_1 \cap H_2$ has finite index in $G$.

Def. (I.3.1.6) (Index of Subgroup). The index of a group $H$ in a group $G$ is defined to be the number of the left coset $G/H$, if it is finite. Now if $H$ has finite index in $G$, then $|G/H| = |H\backslash G|$.

Proof: Because for any system of representative $a_i$ for the left coset $G/H$, $a_i^{-1}$ is a representative for the right coset $H\backslash G$, and vise versa.

Prop. (I.3.1.7). If a finite group $G$ has an automorphism $\alpha$ that $\alpha^2 = id$ and $\alpha$ has no fixed point other than $e$, then $G$ is an Abelian group of odd order.

Proof: $G$ is clearly of odd order. Consider the map $g \mapsto \alpha(g)g^{-1}$, then it is injective, hence it is also surjective, and consider $\alpha(\alpha(g)g^{-1}) = g\alpha(g)^{-1} = (\alpha(g)g^{-1})^{-1}$, thus $\alpha(h) = h^{-1}$ for all $h \in G$, thus clearly $G$ is Abelian.

Prop. (I.3.1.8). If $H$ is a subgroup of a finite group $G$, then $G \neq \cup g^{-1}Hg$.

Proof: There are at most $|G/H|$ different summands in the right hand side, so it doesn’t have enough elements.

Prop. (I.3.1.9). If $G$ is a f.g. group(I.3.1.3) and $H$ is a group of finite index in $G$, then $H$ is f.g.

Proof: Suppose $G$ has generators $g_i$, we may add their inverses to it, and let $Ht_1, \ldots, Ht_m$ are all the right cosets with $t_1 = 1$, then there are $h_{ij}$ that $t_ig_j = h_{ij}t_{k_{ij}}$, then we claim $H$ is generated by $h_{ij}$.

For this, consider any $h = \prod g_i$, then $g_1 = h_{1i_1}, t_{k_{i_1}}$, and we can do this from left to right. Now $h = \prod h_{i,j}, t_{o}$, then $t_o$ must by 1, and we are done.
2 Abelian Groups

Remark (I.3.2.1). An Abelian group is the same as a module over \( \mathbb{Z} \). Thus the theory of commutative algebra applies in this case.

Prop. (I.3.2.2) (Classifying F.g. Abelian Groups). As \( \mathbb{Z} \) is PID, the classifying theorem follows immediately from (I.2.4.19).

Prop. (I.3.2.3) (Abelianization). There is a functor from the category of Abelian semi-groups to the category of Abelian groups that is left adjoint to the forgetful functor, called the Abelianization.

Proof: Define \( A' \) to be the quotient

\[
\bigoplus_{(a,b) \in A^2} \mathbb{Z} \to \bigoplus_{a \in A} \mathbb{Z} \to A' \to 0,
\]

where \( 1_{(a,b)} \) is mapped to \( 1_a + 1_b - 1_{a+b} \). Then it can be seen this satisfies the universal property and it is functorial in \( A \).

Prop. (I.3.2.4). Filtered Colimits are exact in the category of Abelian groups, by (I.5.1.20).

3 Automorphism Group

Prop. (I.3.3.1). Any group of order \( > 2 \) have at least 2 automorphisms.

Proof: Assume the contrary, consider its inner automorphism, then it is Abelian, and then multiplying by \( p \) for \( p \) large prime is not identity, Then \( |G||p - 1 \) for such \( G \). Now it is clear \( |G| = 2 \), because otherwise we can choose \( p \equiv 2 \mod |G| \).

Def. (I.3.3.2). A group is called complete if all automorphisms of \( G \) are inner.

Prop. (I.3.3.3). \( S_n \) is the automorphism group of \( A_n \) for \( n = 5 \) or \( n \geq 7 \).

Proof:

Prop. (I.3.3.4). If \( G \) is a non-Abelian simple group, then \( \text{Aut}(G) \) is a complete group.

Prop. (I.3.3.5). \( S_n \) are complete groups except for \( S_6 \).

Proof: \( S_n \) is the automorphism group of \( A_n \) for \( n = 5 \) or \( n \geq 7 \) by (I.3.3.3), thus it is complete by (I.3.3.4).

Prop. (I.3.3.6) (Wielandt). If \( G \) is a finite group with trivial center, then the sequence

\[
G < \text{Aut}(G) < \text{Aut}(\text{Aut}(G)) < \ldots
\]

must terminate in finite steps.

Proof:
4 Free Groups and Presentations

Prop. (I.3.4.1) (Nielsen-Schreier). A subgroup $H$ of a free group $G$ is a free group. Moreover, a subgroup $H$ of index $m$ in a free group $G$ on $n$ generators is a free group on $1 + m(n-1)$ generators.

Proof: A free group is the fundamental group of a graph $X$ which is a wedge sum of circles, and there is a covering $X_H \to X$ the $\pi_1(X_H) = H$ by (IX.6.1.18). And if $H$ is of index $m$ in a free group $G$, $X_H \to X$ has $m$ sheets by (IX.4.2.5). Now (IX.4.2.5) shows $H = \pi_1(X_H)$ is a free group, and the final assertion follows by comparing two ways of counting Euler number $\chi$. □

List of Presentations of Important Groups

Prop. (I.3.4.2). Let $F$ be a field, then $SL(2,F)$ has a representation as:

$$\langle t(y), n(z), w_1 \rangle, \quad y \in F^*, z \in F$$

quotient the relations:

$$t(y_1)t(y_2) = t(y_1y_2), \quad n(z_1)n(z_2) = n(z_1 + z_2), \quad t(y)n(z)t(y)^{-1} = n(y^2z), \quad w_1t(y)w_1 = t(-y^{-1}).$$

$$w_1n(z)w_1 = t(-z^{-1})n(-z)w_1n(-z^{-1}), \quad z \neq 0,$$

And the isomorphism is given by $\Phi$:

$$\begin{align*}
t(y) &\mapsto \begin{bmatrix} y & 0 \\ 0 & y^{-1} \end{bmatrix}, \\
n(z) &\mapsto \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}, \\
w_1 &\mapsto \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.
\end{align*}$$

Proof: The map vanishes on the relations is direct calculation. An inverse of $\Phi$ is constructed by

$$\Psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{cases}
\begin{bmatrix}
\frac{n(a/c)t(-c^{-1})n(d/c)}{t(a)n(b/a)}
\end{bmatrix} & c \neq 0 \\
\begin{bmatrix}
\frac{n}{c}
\end{bmatrix} & c = 0
\end{cases}$$

The verification of the inverse is verified by direct calculation. □

Prop. (I.3.4.3). $SL(2,\mathbb{Z})$ is generated by $S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ with relations

$$S^4 = 1, \quad (ST)^3 = S^2.$$ 

Proof: Serre, Trees, P81. □

Cor. (I.3.4.4) (Modular Group). $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm 1\}$ is generated by $S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ with relations

$$S^2 = 1, \quad (ST)^3 = 1.$$ 

Prop. (I.3.4.5). The Braid group $B_n$, defined by $B_n = \pi_1(\mathbb{C}^n \setminus \bigoplus \{z_i = z_j\}/S^n)$ has a presentation by $b_i, i = 1, \ldots, n - 1$ and relations
• if \(|i - j| \geq 2\), then \(b_ib_j = b_jb_i\).
• \(b_ib_{i+1}b_i = b_{i+1}b_ib_{i+1}\).

Proof: □

Cor. (I.3.4.6) (Pure braids). Due to the covering map \(\mathbb{C}^n - \bigoplus \{z_i = z_j\} \to \mathbb{C}^n - \bigoplus \{z_i = z_j\}/S^n\) with fiber \(S^n\), there is a map \(B_n \to S_n\) which is easily seen to be surjective and with kernel \(P_n = \pi_1(\mathbb{C}^n - \bigoplus \{z_i = z_j\})\), called the group of pure braids.

Prop. (I.3.4.7). There is a group homomorphism \(B_3 \to PSL_2(\mathbb{Z})\) that maps \(a = \sigma_1\sigma_2\sigma_1\) to \(S\) and \(b = \sigma_1\sigma_2\) to \(T\). The kernel of this map is the group generated by \(c = a^2 = b^3\).

5 Sylow Theory

Prop. (I.3.5.1) (Class Equation). For a finite group \(G\), if \(G_x = C((x))\), then
\[|G| = |C(G)| + \sum |G|/|G_x|\]
where the summation is over non-trivial conjugate classes of \(G\).

Proof: Consider the left action of \(G\) on itself, and calculate elements. □

Cor. (I.3.5.2) (p-Group has non-trivial Center). if \(G\) is a \(p\)-group, then \(G\) has a non-trivial center.

Cor. (I.3.5.3). If \(p|G|\), then \(G\) has an element of order \(p\).

Proof: Follows from Sylow theory and any \(p\)-group has a non-trivial center. □

Lemma (I.3.5.4). For any \(p\)-group \(G\) acting on a finite set \(X\), \(|X| \equiv X^G \mod p\).(trivial).

Prop. (I.3.5.5) (Sylow Theorem). For a finite group of order \(|G| = p^km\).
• There is a Sylow \(p\)-group.
• For a Sylow \(p\)-subgroup, any \(p\)-subgroup is contained in a conjugate of \(P\). In particular, any two Sylow \(p\)-subgroups are conjugate.
• the number of Sylow \(p\)-groups \(n_p\) satisfies: \(n_p|m, n_p \equiv 1 \mod p\).

Proof: 1: Use induction, let \(Z = C(G)\), if \(p|Z|\), then \(Z\) contains a cyclic group of order \(p\). Choose a \(p\)-Sylow subgroup of \(G/C\), then its inverse image in \(G\) is a \(p\)-Sylow subgroup. If \(Z\) is prime to \(p\), consider the conjugate action of \(G\) on \(G - Z\), then some conjugacy class has order prime to \(p\), by(I.3.5.4), then the stabilizer \(H\) of this class satisfies \([G : H]\) is prime to \(p\). Thus \(H\) contains a \(p\)-Sylow subgroup by induction.

2: If \(Q\) is a \(p\)-subgroup, then \(Q\) acts on \(G/P\) by left translation, so it has a fixed element by(I.3.5.4), \(QxP = xP\) for some \(x\), thus \(Q \subset xPx^{-1}\).

3: \(n_p|m\) by considering the conjugate action of \(P\) on the set of conjugates of \(P\), then as in the proof of item2, \(P\) is the only fixed element, so \(n_p \equiv 1 \mod p\) by(I.3.5.4). □

Lemma (I.3.5.6). If \(G\) has a sylow subgroup \(H\) that \(|G/H|\) is not divisible by \(|G|\), then \(G\) is not simple.
Proof: Consider the conjugate action of $G$ on the conjugacy classes of $H$, then it is a group homomorphism of $G$ into a subgroup of $S_{G/H}$, but the hypothesis shows that it is not injective, thus the kernel is non-trivial normal. □

Prop. (I.3.5.7) (Frattini Argument). If $G$ is a finite subgroup, $N$ is normal in $G$ and $P$ is a Sylow subgroup of $N$, then $NN_G(P) = G$.

Proof: For any element $g \in G$, consider $g^{-1}Pg \subset N$ is a Sylow subgroup of $N$, thus by Sylow theorem (I.3.5.5), there is a $n \in N$ that $g^{-1}Pg = n^{-1}Pn$, thus $gn^{-1} \in N_G(P)$, thus $g \in NN_G(P)$. □

6 Split Extension

Prop. (I.3.6.1) (Cyclic Central Extension Split). If there is an exact sequence $0 \to Z \to G \to C \to 0$ where $Z \subset C(G)$ and $C$ is cyclic, then $G$ is Abelian.

Proof: This is because we can choose an inverse image of a generator of $C$. □

Prop. (I.3.6.2) (Schur-Zassenhaus). An exact sequence of finite groups $0 \to A \to E \to G \to 0$ must split when $|A|$ and $|G|$ are relatively prime.

Proof: □

Prop. (I.3.6.3). Let $\alpha, \beta : G \to \text{Aut}(H)$ be two actions of $G$ on $H$, then theirs semiproduct sequences

$$1 \to H \to G \ltimes H \to G \to 1$$

are isomorphic iff $\alpha, \beta$ are equivalent modulo Inn($H$).

7 Subnormality

Def. (I.3.7.1) (Normal Series). A normal series of a group $G$ is a descending chain of groups:

$$G = G_0 > G_1 > \ldots > G_r = \{e\}$$

that $G_{k+1}$ is normal in $G_k$. It is called a composite series iff each $G_k/G_{k+1}$ is simple.

Def. (I.3.7.2) (Central Series). A central series of a group $G$ is an ascending chain of groups:

$$\{e\} = Z_0 < G_1 < \ldots < G_r = G$$

that $Z_{k+1}/Z_k$ is in the center of $G/Z_k$.

Prop. (I.3.7.3). A group is
- solvable iff it has a normal series that $G_i/G_{i+1}$ is Abelian.
- defined to be supersolvable iff it has a normal series that $G_i/G_{i+1}$ is cyclic.
- nilpotent iff it has an upper central series.

Proof: □

Lemma (I.3.7.4) ($p$-Group is Nilpotent). Any $p$-group is nilpotent.
Proof: Using induction by (I.3.5.2), we see it has a central series, thus nilpotent (I.3.7.3).

Prop. (I.3.7.5) (Nilpotent Finite Groups). If $G$ is a finite group, then the following are equivalent:

- $G$ is nilpotent.
- $N_G(H) > H$ for every proper subgroup $H < G$.
- Every maximal subgroup of $G$ is normal.
- Every Sylow subgroup of $G$ is normal.
- $G$ is a direct product of its non-trivial Sylow subgroups.

Proof: $1 \to 2$: Choose a central series $Z_n$, let $Z_n \subset H$ and $Z_{n+1} \not\subset H$, then $[Z_{n+1}, H] \subset [Z_{n+1}, G] \subset Z_n \subset H$, thus $Z_{n+1} \subset N_G(H)$.

$2 \to 3, 4$ → $5$: trivial.

$3 \to 4$: For any $p$-Sylow subgroup $G$, if $N_G(P)$ is proper subgroup, then it is contained in some maximal subgroup $M$, and $M$ is normal, thus by Frattini argument (I.3.5.7), $G = N_G(P)M = M$, contradiction.

$5 \to 1$: By lemma (I.3.7.4).

Prop. (I.3.7.6) (Jordan-Horder). For a finite group $G$, any two of its composite series has the same length, then the quotient groups $G_k \triangleleft G_{k+1}$ are in bijection with each other as sets.

Proof: Cf. [代数学引论] P89.

Prop. (I.3.7.7) (Minimal Normal Subgroup). The minimal normal subgroup $N$ of a finite group $G$ is a direct product of simple groups $L^n$.

Proof: Let $N_1$ be a maximal normal subgroup of $N$, then $N/N_1$ is simple, and let $N_i$ be the conjugates of $N_1$ in $G$, then they are all maximal normal subgroup of $N$. the simple groups $N/N_i$ are mutually isomorphic, and $\cap N_i = 1$ by the minimality of $N$.

Now we use induction to prove $N/N_1 \cap \ldots \cap N_i$ is isomorphic to a product of $N/N_1$, which will finish the proof.

Now assume $N_1 \cap \ldots \cap N_{i-1} \not\subset N_i$, then $(N_1 \cap \ldots \cap N_{i-1})N_i = N$, and notice $N/N_1 \cap \ldots \cap N_i \cong N_1 \cap \ldots \cap N_{i-1}/N_1 \cap \ldots \cap N_i \times N_i/N_1 \cap \ldots \cap N_i \cong N/N_i \times N/N_1 \cap \ldots \cap N_{i-1}$.

Prop. (I.3.7.8). If a finite group $|G| = \prod p_i$, where $p_i$ are different primes that $\prod p_i$ and $\prod (p_i - 1)$ are coprime, then $G$ is cyclic.

Proof: We prove all the Sylow groups are normal. Choose the maximal Sylow group $A_n$, then it is normal by Sylow theorem, and other Sylow groups act by conjugation is trivial (consider the center (I.3.5.2)), then the center of the quotient, and so on), hence $A_n$ is in the center. Now consider the quotient, by induction it is cyclic, hence this is a central extension of a cyclic group, hence $G$ is Abelian (I.3.6.1), so cyclic.

Prop. (I.3.7.9). If $G$ is a finite group and $p$ is the minimal prime number of $|G|$, then all subgroups $N$ of $G$ of index $p$ is normal.
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**Proof:** Consider the left action of $G$ on $G/H$, then the kernel is $\cap a^{-1}Ha$, which is the maximal normal subgroup contained in $H$. Now this is group homomorphism of $G$ into $S_p$, thus it has kernel at lest $|G|/p$, so the kernel equals $H$, showing $H$ is normal. □

**Prop. (I.3.7.10) (Burnside’s Theorem).** If $p, q$ are primes, then any finite groups of order $p^aq^b$ is solvable.

**Proof:** Cf.[Serre Linear representations of finite groups, P65]. □

**Prop. (I.3.7.11) (Thompson).** A finite group is not solvable iff there exist non-trivial elements $x, y, z$ of coprime orders $a, b, c$ that $xy = z$.

**Prop. (I.3.7.12) (Feit-Thompson).** All finite groups that has odd order is solvable.

**Proof:** □

8 Commutators

**Def. (I.3.8.1) (Notation).**
- $[a, b] = a^{-1}b^{-1}ab$.
- $x^y = y^{-1}xy$.

**Prop. (I.3.8.2) (Commutator relations).**

**Def. (I.3.8.3).** A metabelian group is a group $G$ that $G'$ is Abelian.

**Prop. (I.3.8.4).** If $G = AB$ where $A, B$ are Abelian, then $[G, G] = [A, B]$ and $G$ is metabelian.

**Proof:** The first one is easy to verify, the second because if we let $b^{a_1} = a_2b_2$, $a_1 = b_3a_3$, then $[a, b]^{a_1b_1} = [a, b]^{a_1}b_1 = [a, b_1]^{b_1} = [a_1, b_2] = [a_3, b_2]$ and similarly, $[a, b]^{b_1a_1} = [a_3, b_2]$, so we have $[a, b]$ commutes with $[b_1^{-1}, a_1^{-1}]$, which shows $[A, B]$ is Abelian. □

**Prop. (I.3.8.5).** If $G$ is a metabelian finite group, then the transfer of $Ver : G \to G'$ is trivial map.

9 Transfer

10 Permutation Groups

**Lemma (I.3.10.1).** If $n \geq 3$, then any proper normal subgroup of $A_n$ has index divisible by 3.

**Proof:** Otherwise consider $n = |G/H|$, then every $p$-power is in $H$. But then an element $c$ of order 3 is in $H$, because $c = c^{3k+1} = (c^{-1})^{3k+2}$ for any $k$. But $A_n$ is generated by 3-Cycles. □

**Lemma (I.3.10.2).** $A_5$ is simple.

**Proof:** By(I.3.10.1), any proper normal subgroup $H$ has order dividing 20. $H$ cannot contain a 5-cycle, because a 5-cycle has □

**Prop. (I.3.10.3).** $A_n$ is simple for $n \geq 5$.

**Proof:** Cf.[代数学引论 P66]. □
11 Classification of Small Groups

Prop. (I.3.11.1) (Classification).
1. A group \(G\) of prime order \(p\) or order \(p^2\) is Abelian.
2. A group \(G\) of order \(p^a q^b\) that it has \(q^b\) \(p\)-Sylow subgroups, then its \(q\)-Sylow subgroup is normal thus it is not simple.
3. A non-Abelian group \(G\) of order 6 is isomorphic to \(S_3\).
4. Any non-Abelian group of order 8 is isomorphic to \(D_4\) or quadratic numbers \(Q\).
5. A group of order smaller than 60 is solvable.
6. Any simple group \(G\) of order 60 is isomorphic to \(A_5\).
7. A group of order 148 is not simple, by (I.3.5.6) applied to the \(37\)-Sylow subgroup.
8. A group of order 150 is not simple, by (I.3.5.6) applied to the \(5\)-Sylow subgroup.

Proof:
1. Because \(G\) has non-trivial center \(Z\) by (I.3.5.2), if \(Z = G\), then it is Abelian, otherwise the \(|Z| = p\), and the quotient \(G/Z\) is cyclic, thus \(G\) is Abelian by (I.3.6.1).
2. Calculating elements.
3. Consider its normal 3-Sylow group, then the quotient is cyclic thus \(G\) is semi-product which must by \(S_3\) when non-Abelian.
4.
5.
6. Consider \(G\) has 6 \(5\)-Sylow groups, thus there are 24 elements of order 5.
\(G\) has 4 or 10 3-Sylow subgroups, if it have 4 3-Sylow subgroups, then the normalizer contains a 5-Sylow subgroup, so we have a subgroup of order 15, which must by \(\mathbb{Z}/15\), so it contains a normal 5-Sylow subgroup, which shows there are at most \(60/15 = 4\) \(5\)-Sylow subgroups, contradiction.
So we have 10 3-Sylow subgroups, which shows there are at most 15 elements of order 2 or 4. So we have 3 or 5 2-Sylow subgroups. If it is 3, then we can do the same as that for 3-Sylow to construct a 20-order group and reach contradiction.
So now it have 5 2-Sylow subgroups, and then we consider the conjugate action on Sylow subgroups, which is transitive, so it has trivial kernel, and \(G \hookrightarrow S_5\). Now \(G = [G, G] \subset [S_5, S_5] = A_5\).

Prop. (I.3.11.2). There is a group that is group that \(a^3 = 1\) for any \(a \in G\), but is not Abelian. It is the uni-upper-triangular matrices in \(M_3(\mathbb{F}_3)\).

12 Profinite Groups

Basic references are [Neukirch Cohomology of Number Fields], [Serre Galois Cohomology] [Profinite Groups Zalesskii] and [Shatz Profinite Groups, Arithmetic and Geometry].

Def. (I.3.12.1). A profinite group is defined as an inverse limit of finite discrete groups.

Proof: $U$ contains a precompact nbhd of $e$, then by (IX.1.12.24), $U$ contains an open subgroup $V$, so by (IX.1.12.6), there is a nbhd $V'$ of $e$ that $xV'x^{-1} \subset V$ for all $x \in G$, this says $\cap x^{-1}Vx$ is open, so it is an open normal subgroup.

Prop. (I.3.12.3) (Profinite Compact and Totally Disconnected). A profinite group is the same thing as a totally disconnected, compact Hausdorff topological group. In particular, $G \cong \varprojlim G/N$ for all open normal subgroups of $G$.

Proof: One way is because $\varprojlim G_i$ is a closed subgroup of $\prod G_i$ which by Tychonoff’s theorem is compact.

Conversely, by (I.3.12.2), $G$ has a basis of $e$ consisting of normal open subgroups, and by (IX.1.12.23), the intersection of open normal subgroups is $\{e\}$. For any open normal subgroup $N$ of $G$, $G/N$ is compact discrete hence finite, the map $G \to \varprojlim G/N$ is continuous and has dense image, but $G$ is compact and the right is Hausdorff, so the image is closed, hence it is surjective. It is injective because $\cap N = \{e\}$. Hence $G \cong \varprojlim G/N$.

Cor. (I.3.12.4). A closed subgroup of a profinite group is profinite, and a quotient group is profinite.

A direct product of profinite groups are profinite, and so the inverse limit profinite groups are profinite, as it is a closed subgroup of a direct product.

Proof: The closed subgroup is totally disconnected by (IX.1.12.22).

To show the quotient group is totally disconnected, by (IX.1.12.23), it suffice to prove $H$ is intersection of compact open nbhds in $G/H$. If $x \notin H$, then there is an open subgroup $U$ disjoint from $xH$ by (IX.1.12.7), so it is closed hence compact. So $UH$ is a compact nbhd of $H$ in $G/H$ that doesn’t contains $xH$, hence the result.

Cor. (I.3.12.5). A closed subgroup of a profinite group is a intersection of open normal subgroups of $G$ containing it, as $G/H$ is profinite and as in the proof of (I.3.12.3), $H$ is the intersection of open normal subgroups of $G/H$.

Prop. (I.3.12.6). A subgroup is a profinite group is

Prop. (I.3.12.7). The category of profinite Abelian groups is Pontryagin dual to the category of torsion abelian group. (not that hard to verify).

Pro-$p$-Groups

Def. (I.3.12.8). To consider indexes of closed subgroups of a profinite group, the notion of surnatural numbers are needed. A surnatural number is a formal product $\prod_p p^{n_p}, n_p \in \mathbb{N} \cup \{0, \infty\}$.

For a closed subgroup $H$ of a profinite group $G$, $[G : H]$ is defined to be the least common multiple of $[G/U : H/H \cap U]$ where $U$ goes over all open normal subgroups of $G$. This also equals the least common multiple of $[G : V]$ for $V$ open containing $H$ (because for any such $V$, there is an open normal subgroup $U$ that $HU \subset V$ (IX.1.12.6)).


**Proof:** \([G/U : K/K \cap U] = [G/U : H/H \cap U][H/H \cap U : K/K \cap U][G : H][H : K]\), giving one way of inequality. For the converse, Cf.[Etale Cohomology Fulei P150]. The quotient case is trivial.

If \([G : H]\) is finite, then For the final assertion, notice for a open subgroup \(V\), \(G - V\) is compact, so \(\cap H_i \subset V\) iff \(\cap H_i \subset V\) for some \(i\).

**Def.** (I.3.12.10). A profinite group \(G\) is called a **pro-p-group** iff \([G : 1]\) is a power of \(p\). This is equivalent to \(G\) is an inverse limit of finite \(p\)-groups \((G = \varprojlim G/N)\).

Given a profinite group, a closed subgroup \(H\) is called **Sylow \(p\)-subgroup** of \(G\) if \(H\) is pro-\(p\) and \([G : H]\) is prime to \(p\).

**Prop.** (I.3.12.11). Any pro-\(p\) subgroup \(H\) of \(G\) is contained in a Sylow \(p\)-subgroup of \(G\), and any two Sylow \(p\)-subgroups are conjugate. And a surjective morphism of profinite groups maps a pro-\(p\) group to a pro-\(p\) group.

**Proof:** For any open normal subgroup \(U\) of \(G\), let \(I_U\) be the sets of all Sylow groups of \(G/U\) containing \(H/H \cap U\), then the map \(G/VG/U\) maps \(I_V\) to \(I_U\), and \(I_U\) is finite nonempty by Sylow theory. So the inverse limit of \(I_U\) is nonempty, and let \((P_U)\) be such an element, and \(P = \varprojlim_U P_U\), then \(P\) is a pro-\(p\) subgroup of \(G\), and \([G : P]\) equals the least common multiple of \([G/U : P_U]\), which is prime to \(p\), so it is a Sylow \(p\)-group. Similarly, for two Sylow-\(p\) subgroup, we consider \(A_U\) the set of all \(x \in G/U\) that \(x^{-1}(PU/U)x = P'U/U\), then there is an inverse element \(x\), and \(x^{-1}Px = P'\).

If \(G' = G/N\), then \([G/N : PN/N] = [G : PN][G : P]\) is prime to \(p\), and \([PN/N : 1] = [P : P \cap N][P : 1]\) is a power of \(p\), so \(PN/N\) is Sylow-\(p\) in \(G'\).

**Prop.** (I.3.12.12). For a pro-\(p\) group \(G\), any nonzero simple \(p\)-torsion \(G\)-module is isomorphic to \(\mathbb{Z}/p\mathbb{Z}\) with trivial \(G\)-action.

**Proof:** The action of \(G\) on \(A\) factors through a finite quotient group which is a \(p\)-group, by??, \(A^G \neq 0\), so \(A = A^G\), then \(A\) must be \(\mathbb{Z}/p\mathbb{Z}\). □
I.4 More on (Non-Commutative) Algebras

Main references are [Noncommutative Rings T.Y.Lam], [Algebra, Lang Chap17] and [Sta] Chap11.

Def. (I.4.0.1) (Notation). In this section, a ring is an unital ring and may not be commutative. An ideal is a two-sided ideal. A simple ring is a ring that has no non-trivial ideals.

1 Semisimplicity

Def. (I.4.1.1). An \( R \)-module \( E \) is called simple iff it has no submodules other than 0 and \( E \). It is called faithful iff there is no nonzero element \( a \in R \) that \( ax = 0 \) for any \( x \in E \).

It is called non-degenerate if \( RE = E \).

Prop. (I.4.1.2) (Shur’s lemma). For a simple module \( E \) over an algebra \( R \), \( \text{End}_R(E) \) is a division ring, this is because the kernel and image are all 0 or \( E \).

Cor. (I.4.1.3) (Uniqueness of Decomposition of Modules). If an \( R \)-module \( E \) can be written as a finite direct sum of simple \( R \)-modules in multiple ways, then the multiplicity of the irreducible modules appearing in it is uniquely determined.

Proof: Cf.[Lang, Algebra, P643] \( \square \)

Def. (I.4.1.4) (Semisimple Module). A module \( E \) is called semisimple iff it satisfies the following equivalent conditions:

- It is a sum of simple modules.
- It is a direct sum of simple modules.
- Any submodule \( F \) of \( E \) has a complement in \( E \).

Proof: 3 \( \rightarrow \) 2: By Zorn’s lemma, it suffices to show any non-zero semisimple module contains a simple submodule: Take a \( m \neq 0 \in M \), then we may assume \( M = Rm \), and by Zorn’s lemma there is a maximal submodule \( N \) that \( m \notin N \), and let \( N \oplus N' = M \), then we show \( N' \) is simple. because any submodule \( N'' \) satisfies \( m \in N \oplus N'' \) thus \( N'' = N' \).

2 \( \rightarrow \) 1 is immediate, it suffices to show 1 \( \rightarrow \) 3: for any submodule \( N \subset M \), consider all the simple modules that intersect \( N \) trivially, denote their sum by \( V \), I claim \( N \oplus V = M \), otherwise, let \( S \) be a simple submodule that contained in \( N + V \), then \( S \cap (N + V) = 0 \), so \( N \cap (S + V) = 0 \), contradicting the maximality. \( \square \)

Cor. (I.4.1.5). Any submodule and quotient module of a semisimple module is semisimple.

Proof: The quotient is clearly a sum of simple modules, and for a submodule, its submodule has a complement. \( \square \)

Density Theorem

Def. (I.4.1.6). If \( E \) be a semisimple \( R \)-module, let \( R' = \text{End}_R(E) \), then \( E \) is also a \( R' \)-module, where the action is given by \( (\varphi, x) \mapsto \varphi(x) \). Then any element of \( R \) defines an element of \( \text{End}_{R'}(E) \) by left multiplication. If we called \( \text{End}_{R'}(E) \) the bicommutant of \( E \) over \( R \), then
Lemma (I.4.1.7). For a module $E$ over $R$, let $\text{End}_{R'}(E)$ be the bicommutant. If $f \in \text{End}_{R'}(E)$ and $x \in E$, then there is an element $\alpha \in R$ that $\alpha x = f(x)$.

Proof: Since $E$ is semisimple, write $E = Rx \oplus F$, and let $\pi$ be the projection unto $Rx$, then $\pi \in \text{End}_R(E)$, and $f(x) = f(\pi(x)) = \pi f(x) \in Rx$. □

Prop. (I.4.1.8) (Jacobson Density Theorem). Let $E$ be semisimple over $R$ and let $R' = \text{End}_R(E)$. If $f \in \text{End}_{R'}(E)$ and $x_1, \ldots, x_n \in E$, then there is an element $\alpha \in R$ that $\alpha x_i = f(x_i)$ for all $i$.

In particular, if $E$ is f.g. over $R'$, the natural map $R \to \text{End}_{R'}(E)$ is surjective.

Proof: Consider $f^n \in \text{End}_R(E^n)$, then it is just a $n \times n$ matrix with entries in $R' = \text{End}_R(E)$. Consider the lemma (I.4.1.7) shows there is an $\alpha \in R$ that

$$(\alpha x_1, \ldots, \alpha x_n) = (f(x_1), \ldots, f(x_n))$$

which is the result. □

Cor. (I.4.1.9) (Wedderburn Theorem). Let $R$ is an algebra and $E$ is a faithful simple $R$-module. Let $D = \text{End}_R(E)$. If $E$ is of f.g. over $D$, then $R = \text{End}_D(E)$.

Proof: The density theorem (I.4.1.8) and the fact $E$ is f.g. over $D$ shows that $R \to \text{End}_k(E)$ is surjective, and also injective because it is faithful, thus $R = \text{End}_k(E)$. □

Cor. (I.4.1.10) (Burnside’s Theorem). Let $E$ is an $R$-module that is f.d over an alg.closed field $k$ and $R$ is a subalgebra of $\text{End}_k(E)$. If $E$ is simple as an $R$-module, then $R = \text{End}_k(E)$.

Proof: It follows from Shur’s lemma (I.4.1.2) and (I.2.3.24) $R' = \text{End}_R(E)$ is just $k$, so we can use Wedderburn’s theorem (I.4.1.9). □

Cor. (I.4.1.11) (F.d. Simple Module over Commutative Algebra). If $R$ is commutative and $E$ is a simple $R$-module of f.d. over an alg.closed field $k$, then $E$ is of 1-dimensional.

Proof: The theorem shows the image of $R$ in $\text{End}_k(E)$ is all of $\text{End}_k(E)$, but Shur’s lemma (I.4.1.2) and (I.2.3.24) that the image of $R$ consists of scalars, thus $\text{dim}_k E = 1$. □

Prop. (I.4.1.12) (Projection Operators). Let $k$ be a field and $R$ is a $k$-algebra. If $V_1, \ldots, V_n$ are pairwise non-isomorphic simple $R$-modules of f.d. over $k$, then there exists elements $e_i$ in $R$ that acts as identity on $V_i$ and 0 on other $V_j$.

Proof: This is an immediate consequence of Jacobson density theorem applied to the projection operator on $\oplus_i V_i$. □

Prop. (I.4.1.13) (Characters Determine Finite Representations (Bourbaki)). Let $R$ be an algebra over a field $k$ of char0, $E_1, E_2$ be two f.d. semisimple $R$-module over $k$, then if the character $\chi_1 = \chi_2$, then the $R$-modules $E_1, E_2$ are isomorphic.

Proof: $E, F$ are isomorphic to direct sums of simple $R$-modules, so it suffices to show the multiplicities $m, n$ of any simple module $V$ is the same. We can find an element $e$ that is identity on $E$ and 0 on other simple modules $V_i$ by (I.4.1.12), thus the trace of $e$ on $E, F$ are $m \text{dim}_k(V), n \text{dim}_k(V)$ respectively, thus $m = n$ because $k$ is of char0. □
\textbf{Def. (I.4.1.14) (Matrix Coefficients).} For an algebra $R$ and an $R$-module $M$ over a field $k$, then a \textit{matrix coefficient} of $M$ is a function $c : R \to k$ of the form $c(r) = L(rx)$, where $L$ is a linear functional on $M$ and $x \in M$.

\textbf{Cor. (I.4.1.15).} If $R$ is an algebra over a field $k$ and two simple $R$-modules that is f.d. over $k$ have a nonzero matrix coefficient in common, then they are isomorphic.

\textit{Proof:} Let $c(r) = L_1(rx_1) = L_2(rx_2)$. If they are non-isomorphic, then we can find a $e \in R$ that is identity on $M_1$ and 0 on $M_2$ by (I.4.1.12). Let $c(u) \neq 0$, then

$$c(ue) = L_1(uex_1) = L_1(ux_1) = c(u) \neq 0, \quad c(ue) = L_2(uex_2) = L_2(0) = 0$$

contradiction. \hfill $\square$

\textbf{Prop. (I.4.1.16) (Simple Module of Tensor Product).} Let $A, B$ be algebras over a field $\Omega$, $R = A \otimes B$, then:

- If $P$ be a simple $R$-module of f.d. over $\Omega$, then there is a simple $A$-module $M$ and a simple $B$-module $N$ that $P$ is isomorphic to a quotient of $M \otimes N$. Also the isomorphism classes of $M, N$ are uniquely determined.

- If $\Omega$ is alg. closed and $M, N$ are simple modules of $A, B$ of f.d. over $\Omega$, then $M \otimes N$ is a simple module over $A \otimes B$.

\textit{Proof:} 1: because $P$ is f.d., it contains a simple $A$-module $M$. Let $N_1 = \text{Hom}_A(M, P)$, then it is a $B$-module as $A, B$ commutes in $A \otimes B$. Then we can define a map $\lambda : M \otimes N_1 \to P$, which is a $A \otimes B$-module morphism. Now $N_1$ is also of f.d., so it contains a simple $B$-module $N$, and $\lambda$ is clearly non-zero on $M \otimes N$, thus its image is all of $P$, as $P$ is simple.

For the uniqueness, let $d = \dim N$, then $P$ is isomorphic to $k \leq d$ copies of $M$ as an $A$-module, so the isomorphism class of $M$ is determined, so does that of $N$.

2: Consider the map of $A, B$ to $\text{End}_\Omega(M), \text{End}_\Omega(N)$ that are surjective by (I.4.1.10). Then it suffices to show that $M \otimes N$ are simple over $\text{End}_\Omega(M) \otimes \text{End}_\Omega(N)$, but this is just $\text{End}_\Omega(M \otimes N)$, over which $M \otimes N$ is clearly simple. \hfill $\square$

\textbf{Semisimple Rings and Simple Rings}

\textbf{Def. (I.4.1.17) (Semisimple Ring).} A ring is called \textit{(left)semisimple} iff it is a \textit{(left)semisimple} module over itself, it is called \textit{simple} iff it is simple module over itself. In (I.4.1.22), we will see a semisimple ring is semisimple in both sides.

\textbf{Prop. (I.4.1.18).} If $R$ is a semisimple ring, then any $R$-module is semisimple.

\textit{Proof:} Any $R$-module is a quotient of a free $R$-module, thus semisimple, by (I.4.1.5). \hfill $\square$

\textbf{Prop. (I.4.1.19).} $R$ is left-semisimple iff all left $R$-modules are projective.

\textit{Proof:} If $R$ is left-semisimple, for any $R$-module $P$ and an exact sequence

$$0 \to N \to M \to P \to 0$$

by (I.4.1.18), there is a complement of $N$ in $M$, thus this sequence splits and $P$ is projective. The converse is also clear that any submodule of any $R$-module has a complement. \hfill $\square$
Lemma (I.4.1.20). Let $D$ be a division ring and $R = M_n(D)$, then
- $R$ is simple, left semisimple and left Noetherian.
- $R$ has a unique left simple module $V$, and $R$ acts faithfully on $V$, with $R \cong nV$.
- $\text{End}_R(V) \cong D$.

Prop. (I.4.1.21) (Wedderburn-Artin). Any left semisimple ring is of the form $R \cong M_{n_1}(D_1) \times \ldots \times M_{n_r}(D_r)$, and $D_k$ are uniquely determined division rings. There are exactly $r$ different left simple $R$-modules and $D_i$ are uniquely determined.

Proof: Consider the decomposition

$$R \cong n_1V_1 \oplus \ldots \oplus n_rV_r$$

where $V_i$ are simple left $R$-modules. Then we can use Schur’s lemma (I.4.1.2) and (I.4.1.20) to calculate the endomorphism ring, so

$$R \cong \text{End}(n_1V) \times \ldots \times \text{End}(n_rV_r) \cong M_{n_1}(D_1) \times \ldots \times M_{n_r}(D_r).$$

For the uniqueness, we use (I.4.1.20), which shows $D_i$ and $V_i$ both can be recovered. □

Cor. (I.4.1.22). A ring is left semisimple iff it is right semisimple. (I.1.1.7 is used).

Cor. (I.4.1.23). A semisimple ring is left and right Artinian.

Cor. (I.4.1.24). A semisimple commutative ring is a finite direct product of fields.

Def. (I.4.1.25) (Semisimple Categories). Let $A$ be a $k$-linear Abelian category that $\text{End}(X)$ are all f.d. over $k$, then $A$ is semisimple iff $\text{End}(X)$ is a semisimple $k$-algebra for any $X$.

Proof: If $A$ is semisimple, then every object $X$ is a direct sum of simple objects, so $\text{End}(X)$ is semisimple.

Conversely, if $\text{End}(X)$ is semisimple thus a product of matrix algebras over division algebras, so $X$ can be indecomposable only if $\text{End}(X)$ is a division algebra. Now if $f : M \to N$ is a morphism of indecomposable objects, if there is a map $g : N \to M$ that $g \circ f \neq 0$, then $g \circ f$ is an automorphism of $M$, and $(g \circ f)^{-1} \circ g$ is a right inverse to $f$, so $N$ is a direct sum of $M$, and thus $f$ is an isomorphism because $M$ is indecomposable.

Now it suffices to show any indecomposable object is simple. If $M$ is an indecomposable object properly contained in another indecomposable object, then

$$\begin{bmatrix} 0 & 0 \\ \text{Hom}(M, N) & 0 \end{bmatrix} \subset \begin{bmatrix} \text{End}(M) & \text{Hom}(N, N) \\ \text{Hom}(M, N) & \text{End}(N) \end{bmatrix} = \text{End}(M \oplus N)$$

is a two sided nilpotent nonzero ideal, contradicting the fact $\text{End}(M \oplus N)$ is semisimple. □

Prop. (I.4.1.26) (Semisimplicity and Base Change). Let $A$ be a $k$-algebra. If $A \otimes_k K$ is semisimple for some field extension $K/k$, then $A$ is semisimple. Conversely, if $A$ is semisimple, then $A \otimes_k K$ is semisimple for every separable field extension $K/k$.

Prop. (I.4.1.27) (Characters Determine F.D. Representations). If $R$ is a semisimple ring over a field $k$ of characteristic 0, then its f.d. representation is determined by its characters, by (I.4.1.13).

Prop. (I.4.1.28) (Simple Artinian Ring). Let $R$ be a simple ring. It is called a simple Artinian ring iff it satisfies the following equivalent conditions:

- $R$ is left semisimple.
- $R$ is left Artinian.
- $R$ has a minimal left ideal.
- $R \cong M_n(D)$ for some division ring $D$.

Proof: The equivalence of 1, 4 is by Wedderburn theorem (I.4.1.21) and (I.4.1.20). 2 $\implies$ 3 is equivalent, and for $1 \implies 2$, $R$ is a finite direct sum of minimal ideals because the existence of 1, so it is Artinian.

For 3 $\implies 1$, consider all the ideals in $R$ that is isomorphic to the minimal ideal $\mathfrak{a}$, then it is also a right ideal of $R$, so equals $R$, hence $R$ is semisimple. □

Cor. (I.4.1.29). And finite simple $k$-algebra $A$ is of the form $M_n(D)$ where $D$ is a finite division ring over $k$. This is because $A$ is clearly left Artinian.

Prop. (I.4.1.30) (Double Centralizer Property). Let $R$ be a simple ring and $\mathfrak{a}$ a nonzero left ideal. Let $D = \text{End}_R(\mathfrak{a})$, then the natural map $f : R \to \text{End}(\mathfrak{a}_D)$ which

Proof: since $R$ is simple, $f$ is injective. To show it is surjective, let $E = \text{End}(\mathfrak{a}_D)$, then for any $a, r \in \mathfrak{a}$ and $h \in E$, we have $h(ra) = h(r)a$, thus

$$(h \cdot f(r))a = h(ra) = h(r)a = f(h(r))a$$

which shows $f(R)$ is a left ideal in $E$. And because $\mathfrak{a} R = R$, we have $f(R) = f(\mathfrak{a})f(R)$, then

$$Ef(R) = Ef(\mathfrak{a})f(R) \subset f(\mathfrak{a})f(R) = f(R).$$

which shows $f(R)$ is a left ideal in $E$, and it contains 1, so $f(R) = E$. □

Semisimple Algebras with Involution

Def. (I.4.1.31) (Involution). Let $*$ be an involution on a semisimple algebra $B$ over a field $k$. It is called an involution of first kind if it fixes elements in the center of $B$. It is called an involution of second kind otherwise.

Prop. (I.4.1.32) (Decomposition). Let $(B, *)$ be a f.d. semisimple $k$-algebra with an involution, if $k$ is alg.closed and has characteristic 0, then it decomposes as products of pairs of the following types:

- $(A) : M_n(k) \times M_n(k), (a,b)^* = (b^t, a^t)$.
- $(C) : M_n(k), b^* = b^t$.
- $(BD) : M_{2n}(k), b^* = Jb^tJ^{-1}$, where $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$. 
Proof: Let $B = B_1 \times \ldots \times B_r$ be the decomposition into products of simple $k$-algebras, where $B_i^*$ are the minimal two-sided ideals of $B$. Applying $\star$, $B = B_1^* \times \ldots \times B_r^*$ so $B_i^*$ is a permutation of $B_i$, so $B$ is a a product of algebras either simple or product of two simple algebras that $\star$ interchanges them.

If $B$ is simple, then $B \cong M_n(k)$ as $k$ is alg.closed, so $b^* = ub^t u^{-1}$ for some $u \in M_n(k)$ by Skolem-Noether[Sta]074Q. Then $b = b^{**} = (u^t u^{-1})^{-1} bu^t u^{-1}$, so $u^t u^{-1}$ is in the center, denote it by $c$, then $u^t = cu$, $u = c^2 u$, so $c = \pm 1$, and $u$ is symmetric or skew-symmetric. Up to a congruence, we see the situation is $(C)$ or $(BD)$.

The other case is also easy. □

2 Jacobson Radical Theory

Def. (I.4.2.1) (Jacobson Radical). For a unital ring $R$, the Jacobson radical is defined to be the intersection of all maximal left ideals in $R$.

A ring $R$ is called Jacobson semisimple if $\text{rad} R = 0$.

A ring $R$ is called semi-local if $R = R \prec \text{rad} R$ is semisimple.

A ring $R$ is called semi-primary if $R$ is semi-local and $\text{rad} R$ is nilpotent.

Prop. (I.4.2.2) (Equivalent Definitions of Radical). For $y \in R$, the following are equivalent:

- $y \in \text{rad} R$.
- $1 - xy$ is left-invertible for any $x \in R$.
- $yM = 0$ for any simple left $R$-module $M$.
- $1 - xyz$ is invertible for any $x, z \in R$.

Proof: 1 → 2: If $1 - xy$ is not left invertible, then it is contained in a maximal left ideal $m$, but $y \in m$, then $1 \in m$, contradiction.

2 → 3: If $ym \neq 0$, then we must have $Rym = M$, so $xym = m$ for some $x \in R$, which shows $(1 - xy)m = 0$, but then $m = 0$.

3 → 1: Consider the simple left $R$-module $R/m$ for any maximal left ideal $m$.

$1 + 2 + 3 \rightarrow 4$: By item3 we know $\text{rad} R$ is an ideal, so $yz \in \text{rad} R$ and there is a $u$ that $u(1 - xyz) = 1$. But then $u = 1 + u(xyz)$ is also left-invertible, so $u$ is invertible and $1 - xyz$ is invertible. □

Cor. (I.4.2.3). $\text{rad} R$ is the largest ideal $\mathfrak{A}$ of $R$ that $1 + \mathfrak{A}$ are all units. In particular, the left radical agrees with the right radical.

Def. (I.4.2.4). A set in a unital ring $R$ is called locally nilpotent iff every element of it is nilpotent.

Lemma (I.4.2.5). If a left or right ideal $\mathfrak{A} \subset R$ is locally nilpotent, then $\mathfrak{A} \subset \text{rad} R$.

Proof: Suppose $y \in \mathfrak{A}$, then $xy \in \mathfrak{A}$ is nilpotent. So $1 - xy$ has an inverse, for any $x$. Then $y \in \text{rad} R$, by(I.4.2.2). □

Prop. (I.4.2.6) (Artinian Radical Nilpotent). In a left Artinian ring, $\text{rad} R$ is the largest nilpotent left ideal, and it is also the largest nilpotent right ideal.

Proof: By the above lemma, it suffices to show $\text{rad} R$ is nilpotent. □
Cor. (I.4.2.7). In a left Artinian ring, any 1-sided locally nilpotent ideal is nilpotent. By left Artinian, there is a $k$ that $(\text{rad } R)^k = (\text{rad } R)^{k+1} = I$. Now if $I \neq 0$, then we can choose a minimal left ideal $\mathfrak{A}$ that $I\mathfrak{A} \neq 0$ by Artinian property. Now there is an $a \in \mathfrak{A}$ that $Ia \neq 0$, so $I(Ia) = Ia \neq 0$, so $Ia = \mathfrak{A}$ and $a = ya$ for some $y \in I$. But $1 - y$ is invertible, so $a = 0$, contradiction.

Prop. (I.4.2.8) (Semisimplicity and Jacobson Semisimplicity). For a unital ring $R$, the following are equivalent:

- $R$ is semisimple.
- $R$ is Jacobson semisimple and left Artinian.
- $R$ is Jacobson semisimple and satisfies DCC on principal left ideals.

Proof: $1 \rightarrow 2$: $R$ is left Artinian by (I.4.1.23), and consider $R = \text{rad } R \oplus \mathfrak{B}$, then $\mathfrak{B}$ is contained in a maximal left ideal $\mathfrak{m}$, which cannot contain $\text{rad } R$, unless $\text{rad } R = 0$.

$3 \rightarrow 1$: $3$ implies that ay left ideal $\mathfrak{A}$ contains a minimal left ideal $I$ (the minimal principal one), and every minimal left ideal $I$ is a direct summand of $rR$ (by choosing the maximal left ideal $\mathfrak{m}$ not containing $I$, because $I \oplus \mathfrak{m} = R$).

Then we can deduce 1: If $R$ is not semisimple, then take a minimal left ideal $\mathfrak{B}_1$, then $R = \mathfrak{B}_1 \oplus \mathfrak{A}_1$, and $\mathfrak{A}_1 \neq 0$ otherwise $R$ is semisimple, and we can choose a minimal left ideal $\mathfrak{B}_2 \subset \mathfrak{A}_1$, then $\mathfrak{A}_1 = \mathfrak{B}_2 \oplus \mathfrak{A}_2$. Continuing this way, we get a chain of left ideals $\mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \ldots$, and they are both principal because they are direct summands of $rR$, contradicting item $3$. ∎

Prop. (I.4.2.9) (Hokins-Levitzki Theorem). Let $R$ be a semi-primary ring (I.4.2.1), then for any $R$-module $M$, the following are equivalent:

- $M$ is Noetherian.
- $M$ is Artinian.
- $M$ has a composition series.

In particular, a ring is left Artinian iff it is left Noetherian and semi-primary.

Proof: It suffices to prove if $M$ is Noetherian or Artinian, $M$ has a composition series. Denote $J = \text{rad } R$, then $J^n$ for some $n > 0$. We consider

$$0 \subset J^{n-1}M \subset \ldots \subset JM \subset M,$$

and the quotient $J^{k-1}M/J^kM$ is Artinian or Noetherian over $\overline{R} = R/\text{rad } R$ which is semisimple, so it is a direct sum of simple modules, and the sum is finite, so there is a composition series.

The last assertion: A left Artinian ring is semi-primary, by (I.4.2.6) and (I.4.2.8). So the assertion follows from the equivalence of item $1$ and $2$. ∎

Lemma (I.4.2.10). Let $x \in \text{rad } R$ where $R$ is a $k$-algebra, then $x$ is algebraic over $k$ iff $x$ is nilpotent.

Proof: One direction is trivial. For the other, if $x^r + a_1x^{r+1} + \ldots + a_nx^{r+n} = 0$, then because $1 + a_1x^1 + \ldots + a_nx^n$ is invertible, we have $x^r = 0$. ∎

Prop. (I.4.2.11) (Amitsur). Suppose $k$ is a field and $R$ is a $k$-algebra that $\dim_k R < |k|$, then $\text{rad } R$ is the largest locally nilpotent ideal of $R$. 


Proof: If $|k| < \infty$, then $R$ is Artinian, so $\text{rad} R$ is nilpotent by (I.4.2.6), and it is the largest by (I.4.2.6) again. Suppose now $k$ is infinite. By the lemma above, it suffices to show every $r \in \text{rad} R$ is algebraic over $k$. Notice that $a - r$ is invertible for $a \in k^*$, and $\{(a - r)^{-1}\}$ cannot be $k$-linearly independent because $\dim_k R < |k|$, so there is a dependence relation

$$\sum_{i=1}^{n} b_i(a_i - r)^{-1} = 0.$$ 

Hence $r$ is algebraic over $k$. \hfill \Box

Prop. (I.4.2.12) (Amitsur). Let $R$ be a ring and $S = R[T]$. Let $J = \text{rad} S$ and $N = R \cap J$, then $N$ is a locally nilpotent ideal in $R$, and $J = N[T]$. In particular, if $R$ is Jacobson semisimple, then $S$ is also Jacobson semisimple (I.4.2.5).

Proof: Cf. [Lam, P71]. \hfill \Box

Lemma (I.4.2.13). Let $R$ be a $k$-algebra and $K/k$ is a separable algebraic field extension, then if $R$ is Jacobson semisimple, so is $R \otimes_k K$.

Proof: Cf. [Lam, P76]. \hfill \Box

Prop. (I.4.2.14) (Jacobson Radical Under Base Change of Fields). Let $R$ be a $k$-algebra and $K/k$ a separable algebraic extension, then $\text{rad}(R \otimes_k K) = (\text{rad} R) \otimes_k K$.

Proof: Cf. [Lam, P76]. \hfill \Box

3 Brauer Group of Fields

Main reference for this subsection is [Sta]Chap 11.

Remark (I.4.3.1). In this subsection we study f.d. simple algebras over a field $k$.

Def. (I.4.3.2) (Central Algebras). A central $k$-algebra is an algebra $A$ that the center of $A$ is the image of $k \rightarrow A$.

Lemma (I.4.3.3). Let $A$ be a $k$-algebra, and $D$ is a central $k$-algebra which is a skew field, then any two-sided ideal of $A \otimes_k D$ is of the form $J \otimes_k D$ where $J$ is a two-sided ideal of $A$.

Prop. (I.4.3.4). if $A, A'$ are two simple $k$-algebras that $A$ is finite central over $k$, then $A \otimes_k A'$ is simple.

Proof: Cf. [Sta] 074F. \hfill \Box

Cor. (I.4.3.5). The tensor product of two finite central simple $k$-algebras are finite central simple.

Proof: Combine the proposition with (I.2.3.27). \hfill \Box

Cor. (I.4.3.6). If $A$ is a finite central simple algebra over $k$, then for a field extension $k'/kk$, $A \otimes_k k'$ is a finite central simple algebra over $k'$.

Proof: Combine (I.2.3.27) with (I.4.3.4). \hfill \Box

Prop. (I.4.3.7). If $A$ is a central simple $k$-algebra, then $A \otimes_k A^\text{op} \cong \text{Mat}(n \times n, k)$, where $n = \dim_k A$. 

Prop. (I.4.3.8). Let \( A \) be a central simple \( k \)-algebra, then \( \dim_k A \) is a square.

Proof: This is true because \( A \otimes_k \overline{k} \) is a matrix algebra by (I.4.3.11).

Prop. (I.4.3.9) (Skolem-Noether). Let \( A \) be a finite central simple \( k \)-algebra and \( B \) is a simple \( k \)-algebra that \( f, g : B \to A \) are two \( k \)-algebra homomorphisms. Then there exists an invertible element \( x \in A \) that \( f = x^{-1}gx \).

Proof: Choose a simple \( A \)-module \( M \), then \( L = \text{End}_A(M) \) is a skew field, and \( M \) has two \( B \otimes_k L^{op} \) structures by \( f \) and \( g \). The \( k \)-algebra \( B \otimes_k L^{op} \) is simple by (I.4.3.4), and also \( B \) is finite simple because there is a \( k \)-homomorphism \( B \to A \), so \( B \otimes_k L^{op} \) is finite simple, thus the two \( B \otimes_k L^{op} \)-structures on \( M \) are isomorphic, which means there is a \( \varphi : M \to M \) intertwining these two structures. But \( \varphi \) commutes with \( L \), meaning that \( \varphi \) is justing multiplying by some \( x \in A \), so \( x \) is what we want.

Cor. (I.4.3.10). Let \( A \) be a finite simple \( k \)-algebra, then any automorphism of \( A \) is inner.

Proof: Because the center of \( A \) is a finite field extension \( k' \) of \( k \) that \( A \) is central simple over \( k' \), thus the Skolem-Noether theorem applies.

\[ Br(k) \]

Def. (I.4.3.11) (Brauer Group). Let \( k \) be a field, then we can define two finite central simple \( k \)-algebras to be similar iff there is some \( n, m > 0 \) that \( M_n(A) \cong M_m(B) \).

If \( A, B \) are finite central simple algebras, then \( A \otimes_k B \) is also central simple by (I.4.3.5), and if \( A \) is similar to \( A' \) then \( A \otimes_k B \) is similar to \( A' \otimes_k B \) because tensoring commutes with taking matrices. Then we get an Abelian group of the similarity classes of central simple algebras over \( k \), called the Brauer group \( Br(k) \) of \( k \).

If \( k \to k' \) is a field extension, then base change gives a functor \( Br(k) \to Br(k') \) by (I.4.3.6), so \( Br \) is a functor from fields to groups. Also for an alg.closed field \( k \), \( Br(k) = 0 \), by (I.4.1.29) and the fact any finite division algebra over \( k \) is a field (I.2.3.24).

4 Idempotented Algebras

Lemma (I.4.4.1). Let \( R \) be a ring and \( e, f \) be idempotents of \( R \) that \( ef = fe = e \), then \( f = e + e' \), where \( e' \) is an idempotent and \( M[f] = M[e] \oplus M[e'] \).

Moreover, if \( R \) is an algebra over an alg.closed field \( k \) and \( M[e] \) is a simple \( R[e] \)-module of f.d. over \( k \), then \( \dim \text{Hom}_{R[e]}(M[e], M[f]) = 1 \).

Proof: Let \( e' = f - e \), then it is easily verified to be an idempotent, and \( ee' = e'e = 0 \), thus \( M[f] = M[e] \oplus M[e'] \) is clear.

For the second, because \( R[e] \) acts by \( 0 \) in \( R[e'] \), \( \text{Hom}_{R[e]}(M[e], M[f]) = \text{Hom}_{R[e]}(M[e], M[e]) \) has dimension 1, by Shur’s lemma (I.4.1.2) and (I.2.3.24).
Def. (I.4.4.2) (Idempotented Algebra). An idempotented algebra \( \mathcal{H} \) is an algebra over a field \( k \) together with a set \( \mathcal{E} \) of idempotents that if \( e_1, e_2 \in \mathcal{E} \), then there exists \( e_0 \in \mathcal{E} \) that
\[
e_0 e_1 = e_1 e_0 = e_1, e_0 e_2 = e_2 e_0 = e_2,
\]
and also for any \( \varphi \in \mathcal{H} \), there exists \( e \in \mathcal{E} \) that \( e \varphi = \varphi e = \varphi \).

We can define a partial order on \( \mathcal{E} \): \( e < f \) iff \( ef = fe = f \), then this order is cofinal.

If \( e \) is an idempotent, denote \( \mathcal{H}[e] = e \mathcal{H} e \), which is a subring of \( \mathcal{H} \) with unit \( e \). Also if \( M \) is an \( \mathcal{H} \)-module, we denote \( M[e] \) the \( R[e] \)-module \( eM \).

Def. (I.4.4.3) (Smooth Representations of Idempotented Algebras). An \( \mathcal{H} \)-module \( M \) is called smooth iff \( M = \bigcup_{e \in \mathcal{E}} M[e] \), and it is called admissible iff it is smooth and each \( M[e] \) is of f.d. over \( k \).

Prop. (I.4.4.4) (Simpleness Checked on Idempotents). Let \( M \) be a non-zero module over an idempotented algebra \( (\mathcal{H}, \mathcal{E}) \) over \( C \), \( \mathcal{E}^0 \) be a cofinal subset of \( \mathcal{E} \), then \( M \) is a simple \( \mathcal{H} \)-module iff \( M[e] \) is simple \( \mathcal{H}[e] \)-modules for any \( e \in \mathcal{E}^0 \).

Proof: If all \( V[e_K] \) are simple over \( \mathcal{H}[e_K] \), because \( V = \bigcup_{e \in \mathcal{E}^0} V[e] \), if \( W \) is a proper subspace of \( V \), \( W = \bigcup_{e \in \mathcal{E}^0} W[e] \), so there is a \( W[e_1] \neq 0, W[e_2] \neq V[e_2] \), and we can find an idempotent \( e < e_1, e < e_2 \), then \( V[e] \) is a proper subspace of \( V[e] \), contradiction.

Conversely, If \( W_0 \subset V[e_K] \) is a proper non-zero \( \mathcal{H}[e_K] \)-submodule, we will show that \( \pi(\mathcal{H})W_0 \cap V[e_K] = W_0 \), which will contradicts the irreducibility of \( V[e_K] \). Notice if \( v \in V[e_K] \) then \( v = e_Kv \). If \( \sum \varphi_i w_i \in V[e_K] \) where \( w_i \in W_0 \), then \( \sum \varphi_i w_i = \sum (e_K \varphi_i)(e_K w_i) = \sum (e_K \varphi_i e_K) w_i \in W_0 \).

Prop. (I.4.4.5) (Isomorphism Checked on Idempotents). Let \( V_1, V_2 \) be two simple admissible modules over an idempotented algebra \( (\mathcal{H}, \mathcal{E}) \), let \( \mathcal{E}^0 \) be a cofinal subset of \( \mathcal{E} \), then \( V_1 \cong V_2 \) iff \( V_1[e] \cong V_2[e] \) as \( \mathcal{H}[e] \)-modules for every \( e \in \mathcal{E}^0 \).

Proof: Fix \( e_{K_0} \) an idempotent that \( V_1[e_{K_0}] \neq 0 \), then we can fix an isomorphism \( \sigma_0 : V_1[e_{K_0}] \cong V_2[e_{K_0}] \), which is unique up to scalar multiple by(I.4.4.4) and Schur’s lemma(I.4.1.2) as \( V[e_{K_0}] \) is of f.d..

Observe for \( e_{K_1} \leq e_{K_0}, \sigma_0 \) naturally extends to a \( \mathcal{H}[e_{K_1}] \)-module isomorphism \( V_1[e_{K_1}] \rightarrow V_2[e_{K_1}] \); there exists an isomorphism \( \sigma_{K_1} \) by hypothesis, and \( e_{K_0} \in \mathcal{H}_{K_1} \), thus \( \sigma_{K_1} \) restricted to \( V[e_{K_0}] \) induces an isomorphism of \( V[e_{K_0}] \), thus it is a multiple of \( \sigma_0 \) by the uniqueness above, so we can assume they are equal on \( V_1[e_{K_0}] \) and extends \( \sigma_0 \).

Now because \( \mathcal{E}^0 \) is cofinal in \( \mathcal{E} \), we can extend \( \sigma_0 \) to a well-defined map on \( V_1 \). And this is an intertwining operator, because for any \( v \in V_1, \varphi \in \mathcal{H} \), we can find a small idempotent \( e_K \) that \( v \in V_1[e_K], \varphi v \in V_2[e_K] \), and \( e_K \varphi = \varphi e_K = \varphi \), then we consider \( (\varphi)v = (e_K \varphi e_K)v \), which shows \( \varphi \) is intertwining. \( \square \)

Prop. (I.4.4.6). If \( \mathcal{H} \) is an idempotented algebra, then \( \mathcal{M}(\mathcal{H}) \) has enough projectives.

Proof: For any idempotent \( e \in \mathcal{H} \), consider the module \( \mathcal{H}e \), then it is projective, because \( \text{Hom}_\mathcal{M}(\mathcal{H}e, X) = eX \), thus it is clearly exact. Now any \( m \in V \) has an idempotent \( e \in \mathcal{H} \) that \( ev = v \) (use definition). Thus by taking the direct sum, we are done. \( \square \)
Spherical Idempotents

Def. (I.4.4.7) (Spherical Idempotents). Let \((\mathcal{H}, \mathcal{E})\) be an idempotented algebra over a field \(\Omega\). Then an idempotent \(e \in \mathcal{E}\) is called spherical idempotent if there exists an anti-involution \(\iota: \mathcal{H} \to \mathcal{H}\) that \(\iota(x) = x\) for any \(x \in \mathcal{H}[e^0]\). Notice this implies \(\mathcal{H}[e^0]\) is commutative, as \(xy = \iota(xy) = yx\).

Def. (I.4.4.8) (Contragradient Module). If \(M\) be a smooth \(\mathcal{H}\)-module, then we can define a contragradient \(\hat{M}\) of \(M\), which is the smooth \(\mathcal{H}\)-module consisting of smooth vectors in \(M^*\), where the action is defined as \(\((\varphi)\lambda)(m) = \lambda(\iota(\varphi)m)\). Notice if \(M\) is admissible, then so does \(\hat{M}\), because an element in \(\hat{M}[e]\) is determined by its restriction on \(M[\iota(e)]\), so \(\hat{M}[e]\) is of f.d. if \(M[\iota(e)]\) does.

Def. (I.4.4.9) (Spherical Vectors). Let \(e^0\) be a spherical idempotent of the idempotented algebra \((\mathcal{H}, \mathcal{E})\) and \(\iota\) is the corresponding involution. If \(M\) is an admissible \(\mathcal{H}\)-module, then \(M\) is called spherical iff \(M[e^0] \neq 0\), and elements in \(M[e^0]\) are called spherical vectors.

Then if \(M\) is simple and spherical, then \(M\) has at most one spherical vector up to scalar, and \(\hat{M}\) is also spherical.

Proof: In fact, \(M[e^0]\) is a simple \(\mathcal{H}[e^0]\)-module (I.4.4.4), and is of f.d., and \(\mathcal{H}[e^0]\) is commutative (I.4.4.7), so (I.4.1.11) shows it is of dimension 1. So if \(m^0 \neq 0 \in M[e^0]\), then we can define a \(\hat{m}^0 \neq 0 \in \hat{M}[e^0]\) as:

\[
e^0 m = \hat{m}^0(m) \cdot m^0.
\]

Prop. (I.4.4.10) (Isomorphism Checked on Spherical Idempotents). Let \((\mathcal{H}, \mathcal{E})\) be an idempotented algebra over a field \(\Omega\) and \(e^0\) is a spherical idempotent. If \(M, N\) are simple admissible spherical \(\mathcal{H}\)-modules that \(M[e^0] \cong N[e^0]\) over \(\mathcal{H}[e^0]\), then \(M \cong N\) as \(\mathcal{H}\)-modules.

Proof: Let \(m^0, n^0\) be non-zero elements in \(M[e^0], N[e^0]\), and let \(\hat{m}^0, \hat{n}^0\) be the corresponding elements in \(\hat{M}[e^0], \hat{N}[e^0]\), respectively.

Firstly we prove: \((\varphi m^0, \hat{m}^0) = (\varphi n^0, \hat{n}^0), \varphi \in \mathcal{H}\). But this is equal to \(((e^0\varphi^0)m^0, \hat{m}^0) = ((e^0\varphi^0)n^0, \hat{n}^0), M[e^0] \cong N[e^0]\).

Next notice each side defines a matrix coefficient, thus for any \(e_K < e^0\), when restricted to \(\varphi \in \mathcal{H}[e_K], M[e_K], N[e_K]\) are of f.d., so they are isomorphic, by (I.4.1.15). Thus (I.4.4.5) shows \(M \cong N\) as \(\mathcal{H}\)-modules. 

Restricted Tensor Product

Def. (I.4.4.11) (Restricted Tensor Product). Give an infinite number of vector spaces \(V_v\) indexed by a set \(\Sigma\) and elements \(x_v^0 \in V_v\) be given for a.e. \(v\), we can define the restricted tensor product \(\prod' V_v\) as the direct limit

\[
\lim_{\mathcal{S} \subset \Sigma \text{ finite}} \bigotimes_{v \in \mathcal{S}} V_v
\]

and think of it as the vector spaces spanned by all symbols \(\bigotimes_v x_v\) where \(x_v = x_v^0\) for a.e. \(v\).

Notice if \(V_v\) are idempotented algebras and \(x_v^0 \in V_v\) are idempotents for a.e. \(v\), then \(\prod' V_v\) also has a natural idempotented algebra structure.
Def. (I.4.4.12) (Tensor Product Module of Idempotented Algebras). Given a set of idempotented algebras \((\mathcal{H}_v, \mathcal{E}_v)\) and specify an idempotent \(e_v^0\) for a.e. \(v\). Let \(M_v\) be \(\mathcal{H}_v\)-modules over \(k\) for all \(v\), and assume \(M_v[e_v^0]\) be of dimension 1 a.e. \(v\), and specify a.e. a non-zero element \(m_v^0 \in M_v[e_v^0]\).

Then we can define tensor product (I.4.4.11) \(\prod' \mathcal{H}_v\) which is an idempotented algebra and \(\prod' M_v\). Then we can define an **restricted tensor module** structure of \(M_v\) over \(\prod' \mathcal{H}_v\) by the action

\[(\otimes_v \varphi_v)x_v = \otimes_v (\varphi_v)x_v.\]

Lemma (I.4.4.13) (Simple Module of Tensor Product). Let \((\mathcal{H}_1, \mathcal{E}_1), (\mathcal{H}_2, \mathcal{E}_2)\) be idempotented algebras over an alg.closed field \(\Omega\), and let \((\mathcal{H}, \mathcal{E})\) be the tensor product. If \(M_1, M_2\) are simple admissible modules over \(\mathcal{H}_1, \mathcal{H}_2\), respectively, then \(M_1 \otimes M_2\) are simple admissible modules over \(\mathcal{H}_1 \otimes \mathcal{H}_2\), and any simple admissible module over \(\mathcal{H}\) comes uniquely from a pair \((M_1, M_2)\) like this.

**Proof:** If \(M_1, M_2\) are simple admissible over \(\mathcal{H}_1, \mathcal{H}_2\) respectively, then if \(e_1 \in \mathcal{E}_1, e_2 \in \mathcal{E}_2\), then \((M_1 \otimes M_2)[e_1 \otimes e_2] = M_1[e_1] \otimes M_2[e_2]\) is simple and of f.d. by (I.4.4.4) and (I.4.1.16), so \(M_1 \otimes M_2\) is simple by (I.4.4.4).

Now if \(M\) is simple admissible over \(\mathcal{H}_1 \otimes \mathcal{H}_2\), then we find an \(e_1^0 \otimes e_2^0 \in \mathcal{E}\) that \(M[e_1^0 \otimes e_2^0] \neq 0\). Let \(\mathcal{E}_i^0 = \{e_i \in \mathcal{E}_i | e_i < e_i^0\}\), then \(\mathcal{E}_i^0\) is cofinal in \(\mathcal{E}_i\). Then for any \(e_i \in \mathcal{E}_i^0\), \(M[e_1 \otimes e_2]\) is non-zero thus simple, thus it is of the form

\[M[e_1 \otimes e_2] = M_1(e_1, e_2) \otimes M_2(e_1, e_2)\]

where \(M_i(e_1, e_2)\) are simple \(\mathcal{H}[e_i]\)-modules by (I.4.1.16).

Now we show that \(M_2(e_1, e_2)\) depends only on \(e_1\): it suffices to show \(M_2(e_1, e_2) = M_2(f_1, e_2)\) for \(f_1 < e_1\). For this, notice that for any idempotents \(g_1, g_2\),

\[M[g_1 \otimes g_2] = M_1(g_1, g_2) \otimes M_2(g_1, g_2)\]

is a finite direct sum of simple modules \(M_2(g_1, g_2)\) as an \(\mathcal{H}_2[g_2]\)-module. Notice that by (I.4.4.1),

\[M[e_1 \otimes e_2] = M[f_1 \otimes e_2] \oplus M[e' \otimes e_2]\]

for \(e' = f_1 - e_1\), so by (I.4.1.3), \(M_2(f_1, e_2) = M_2(e_1, e_2)\). Similarly we know \(M_1(e_1, e_2)\) only depends on \(e_1\).

Next we have:

\[\dim_k \text{Hom}_{\mathcal{H}_1[e_1]}(M_1[e_1], M_1[f_1]) \geq 1, \quad f_1 \leq e_1.\]

and similar for \(\mathcal{H}_2\). For this, it suffices to prove that \(\dim_k \text{Hom}_{\mathcal{H}_1[e_1]}(M[e_1 \otimes e_2], M[f_1 \otimes e_2]) \geq 1\), by what we just said, but this is by the decomposition

\[M[e_1 \otimes e_2] = M[f_1 \otimes e_2] \oplus M[e' \otimes e_2]\]

above. But

\[\dim_k \text{Hom}_{\mathcal{H}_1[e_1] \otimes \mathcal{H}_2[e_2]}(M_1(e_1) \otimes M_2(e_2), M_1(f_1) \otimes M_2(f_2)) = 1, \quad f_1 \leq e_1, f_2 \leq e_2\]

by (I.4.4.1), so the \(\geq\) above should change to \(=\).

Now we know the homomorphism is of dimension 1, we want to choose a family of maps that is compatible for \(g_1 \leq f_1 \leq e_1 \leq e_1^0\). For this, we can choose the maps \(\lambda(e_1, e_1^0)\) arbitrarily, then
choose \(\lambda(e_1, f_1)\) to be be compatible with \(\lambda(e_1, e_1^0)\) and \(\lambda(f_1, e_1^0)\), then these are compatible choices. Hence we can define a direct limit

\[
M_1 = \lim_{\lambda(e_1)} M(e_1).
\]

It is easy to see \(M_1(e_1) \to M_1\) are all injective, and \(M_1(e_1) = M[e_1]\). Similarly, we can define an \(M_2\) that \(M_2[e_2] = M_2(e_2)\).

Finally, we have

\[
M[e_1 \otimes e_2] = M_1[e_1] \otimes M_2[e_2] = (M_1 \otimes M_2)[e_1 \otimes e_2]
\]

for all \(e_1 \in \mathcal{E}_1, e_2 \in \mathcal{E}_2^0\), thus \(M \cong M_1 \otimes M_2\) by (I.4.4.5).

As for the uniqueness of \(M_1, M_2\), notice the decomposition of \(M[e_1 \otimes e_2]\) is unique by (I.4.1.16), so \(M_i[e_i]\) is uniquely determined, thus by (I.4.4.5) \(M_i\) are determined. \(\Box\)

Prop. (I.4.4.14) (Flath Theorem). let \((\mathcal{H}_v, \mathcal{E}_v)\) be an indexed family of idempotented \(k\)-algebras, and for a.e. \(v\) let \(e_v^0 \in \mathcal{E}_v\) be a spherical idempotent. Let \((\mathcal{H}, \mathcal{E})\) be the restricted tensor product of \(\mathcal{H}_v\) w.r.t. \(e_v^0\), which is an idempotented algebra. For each \(v \in \Omega\), there is a simple \(\mathcal{H}_v\)-module \(M_v\), and for a.e. \(v\) we specify a non-zero spherical vector. Let \(\otimes_v M_v\) be the tensor product module, then it is a simple admissible \(\mathcal{H}\)-module, and every simple admissible module is of this type, with \(M_v\) uniquely determined.

Proof: Firstly the tensor product is simple and admissible: For any idempotent \(e = \otimes e_v \in \mathcal{E}\), there is a finite set \(S\) that if \(v \notin S\), then \(e_v = e_v^0\), hence for these \(v\) \(\dim M_v[e_v] = 1\). Then

\[
M[e] = \otimes_{v \in S} M_v[e_v].
\]

which is simple of f.d. by (I.4.4.13), so \(M\) is simple admissible by (I.4.4.4).

Conversely, for any simple \(\mathcal{H}\)-module \(M\), we need to show it is a tensor product. If there are only f.m. indices, then this follows from (I.4.4.13).

Suppose first that \(e_v^0\) is defined and spherical for all \(v\), and \(e = \otimes_v e_v^0\), and \(M[e] \neq 0\). Then \(\dim M[e] = 1\) (I.4.4.9). Let \(m\) be a spherical vector, then there is a ring homomorphism \(\gamma : \mathcal{H}[e] \to k\) defined by \(hm = \gamma(h)m\). Because \(\mathcal{H}[e] = \otimes_v \mathcal{H}_v[e_v^0]\), we have

\[
\gamma(\otimes_v h_v) = \prod_v \gamma_v(h_v).
\]

Now if we decompose \(\mathcal{H} = \mathcal{H}_v \otimes \mathcal{H}_v\), then by (I.4.4.13), there exists simple admissible module \(M_v\) over \(\mathcal{H}_v, M'\) over \(\mathcal{H}_v\) respectively, that \(M = M_v \otimes M'\), thus \(M[e] = M_v[e_v^0] \otimes M'_v[e'_v]\). Now consider \(M_v\) for all \(v\), and \(N = \otimes_v M_v\) w.r.t. \(m_v\), then it is simple admissible \(\mathcal{H}\)-module, and we have \(N[e] = \otimes'_v M_v[e_v^0] \cong M[e]\) as \(\mathcal{H}[e]\)-modules of dimension 1, which is because they are both simple and have the same character

\[
\gamma(\otimes_v h_v) = \prod_v \gamma_v(h_v).
\]

Hence \(M \cong N\) by (I.4.4.10).

Now the general case follows from the two situations above: choose \(e \in \mathcal{E}\) that \(M[e] \neq 0\), then let \(S\) be large that for \(v \notin S\), \(e_v = e_v^0\). Then we decompose \(\mathcal{H}\) as \(\mathcal{H} = \otimes_{v \in S} \mathcal{H}_v \otimes (\otimes_{v \notin S} H_v)\), then

\[
M = \bigotimes_{v \in S} M_v \otimes M' = \bigotimes_{v \in S} M_v \otimes \bigotimes_{v \notin S} M_v = \otimes_v M_v.
\]

\(\Box\)
Hecke Algebras

Prop. (I.4.4.15) (Hecke Algebras of Lie Groups).

- If $K$ is a compact Lie group, then the Hecke algebra $\mathcal{H}_K$ is defined to be the ring of smooth functions on $K$ that is $K$-finite under both left and right translations, where the algebra is given by convolution. By Peter-Weyl theorem, these functions are dense in $C(K)$ and $L^2(K)$, and it is an idempotented algebra over $\mathbb{C}$? Cf.[Bump, P309].

- If $G$ is a real reductive group and $K$ is a maximal compact subgroup, then the Hecke algebra $\mathcal{H}_G$ is defined to be $\mathcal{H}_G = \mathcal{H}_K \otimes_{U(\mathfrak{t}\mathbb{C})} U(\mathfrak{g}\mathbb{C})$, where the right action of $U(\mathfrak{t}\mathbb{C})$ on $\mathcal{H}_K$ is given by

$$f \ast X = \rho(\iota(X)) f,$$

? Cf.[Bump, P312].

Proof: □

Prop. (I.4.4.16) (Equivalence of Representations Lie Group Case).

- For a compact Lie group $K$, the category $\mathcal{M}(K)$ of smooth representations of $\mathcal{H}_K$ is equivalent to the category of unitary representations of $K$.

- For a real reductive group $G$ and a maximal compact subgroup $K$, the category of (admissible) $(\mathfrak{g}, K)$-modules is equivalent to the category of smooth(admissible) modules of $\mathcal{H}_G$.

Proof: Cf.[Cohomological Induction and Unitary Representations, P75]. □

5 Quaternion Algebras

Main references are [Quaternion Algebras].
I.5 Commutative Algebra I

Basic References are \[\text{[Commutative Algebra Atiyah]}\] and \[\text{[Commutative Ring Theory Matsumura], [Sta]Chap10], [Commutative Algebra with a View Towards Algebraic Geometry], [Weibel Homological Algebra Ch4].\]

Remark (I.5.0.1) (Convention). All rings in this section is assumed to be commutative and unital.

1 Basics

Prop. (I.5.1.1) (Quotient). Given an ideal \(I\) of \(R\), there is a quotient ring \(R/\langle I\rangle\) that is an algebra over \(R\) with the universal properties: any map of rings \(R \to R'\) that vanishes on \(I\) factors through \(R/\langle I\rangle\).

Prop. (I.5.1.2). The quotient map induces an order-preserving bijection of ideals of \(A/\langle I\rangle\) and ideals of \(A\) containing \(I\).

Prop. (I.5.1.3) (Prime Avoidance). Let \(I\) be an ideal in a ring, and \(P_i\) are prime ideals that \(I \not\subseteq P_i\) for any \(i\), then \(I \not\subseteq \bigcup P_i\).

Proof: Use induction on the number of primes \(n\). For \(n = 1\) this is trivial. For \(n > 2\), let \(z_i \in I \setminus \bigcup_{j \neq i} P_j\). Now consider \(z = z_1 \cdot z_{n-1} + z_n\). If \(z \in P_i\) for some \(i < n\), then \(z_n \in P_i\), contradiction. If \(z \in P_n\), then some \(z_i, i < n\) is in \(P_n\) because \(P_n\) is a prime ideal, contradiction. □

Prop. (I.5.1.4) (Existence of a Maximal Ideal). Any non-zero commutative ring has a maximal ideal.

Proof: Use Zorn’s lemma, the union of a chain of ideals is an ideal. □

Cor. (I.5.1.5). Any non-trivial ideal is contained in a maximal ideal.

Proof: If \(I \subset A\) is a non-trivial ideal, then \(A/I\) is a non-zero ring, thus \(A/I\) has a maximal ideal, which corresponds to a maximal ideal of \(A\) containing \(I\) (I.5.1.2). □

Prop. (I.5.1.6). If \(r(I)\) and \(r(J)\) are coprime, then \(I, J\) are coprime.

Proof: As \(a + b = 1\), and \(a^m \in I, b^n \in J\), \(1 = (a + b)^{m+n} \in I + J\). □

Def. (I.5.1.7) (Local Ring). A local ring is a commutative ring \(R\) that has only one maximal ideal. Equivalently, there is a prime ideal \(\mathfrak{m}\) that any element in \(R \setminus \mathfrak{m}\) is invertible.

Proof: If \(\mathfrak{m}\) is the maximal ideal, then for any \(x \in R\), if \((x)\) is not all of \(R\), then \(x\) is contained in a maximal ideal by (I.5.1.4), which can only be \(\mathfrak{m}\). Conversely, if there is a prime ideal \(\mathfrak{m}\) that any element in \(R \setminus \mathfrak{m}\) is invertible, then clearly every non-trivial ideal is included in \(\mathfrak{m}\). □

Prop. (I.5.1.8). Any quotient of a local ring is also a local ring.

Def. (I.5.1.9) (Local Ring Map). A map between two local rings are called local ring map if it maps non-invertible elements to non-invertible elements, equivalently, \(f^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R\).

Def. (I.5.1.10) (Ideal of Definition). In a Noetherian local ring \((R, \mathfrak{m})\), an ideal \(I \subset R\) is called an ideal of definition if \(\sqrt{I} = \mathfrak{m}\).
Prop. (I.5.1.11) (Ideals of Products and Filters). If $F_i, i \in I$ is a collection of fields, then the prime ideals in the ring $\prod F_i$ is in bijection with the ultrafilters on $I$, where the ultrafilter $\mathcal{F}$ corresponds to the ideal $p_{\mathcal{F}} = \{(a_i) \mid \text{the set of coordinates that } a_i = 0 \text{ is in } \mathcal{F}\}$. And in the same way, ideals of $\prod F_i$ corresponds to the filters on $I$.

Proof: Clearly $p_{\mathcal{F}}$ is an ideal, and if $\mathcal{F}$ is an ultrafilter, let $Z(a)$ be the coordinates that $a$ is zero on, and notice $Z(ab) = Z(a) \cup Z(b)$, then $ab \in p$ iff $a \in p$ or $b \in p$, by (XII.1.8.7), so it is a prime ideal.

Conversely, notice that any two $a, b$ with $Z(a) = Z(b)$ differs by a unit, so $\mathcal{F}_p = \{Z(a) \mid z \in p\}$ is easily checked to be a filter. And if $p$ is a prime, then for any $A \subset I$, let $a, b \in \prod F_i$ be that $Z(a) = A, Z(b) = I - A$, then $ab = 0 \in p$, so $a \in p$ or $b \in p$.

□

Def. (I.5.1.12) (Torsion-Free Modules). Let $S \subset A$ be a set in a commutative ring, then an $A$-module $M$ is called $S$-torsion-free if for any $\{x \in M \mid Sx = 0\} = 0$.

Prop. (I.5.1.13) (Maximal Torsion-Free Quotient). Let $S \subset A$ be a set, the functor from the category of torsion-free $A$-modules to the category of $A$-modules has a left adjoint, called the maximal $S$-torsion-free quotient.

Proof: A quotient of $M$ is determined by the kernel. It suffices to prove if $M/N_1, M/N_2$ are both $S$-torsion-free, then $M/N_1 \cap N_2$ is also $S$-torsion-free: This is easy.

□

Prop. (I.5.1.14). Let $A$ be a ring and $B$ a finite $A$-algebra. if $A \to B$ is epimorphism in the category of rings, then $A \to B$ is surjective.

Proof: Notice that $h^B \to h^A$ is injective iff $h^B \times_{h^A} h^B \cong h^B$, or equivalently, $B \times_A B \to B$ is an isomorphism. Now we can localize $A$ at maximal ideals, thus we can assume $A$ is local, with maximal ideal $m$ and residue field $k$. And then use Nakayama lemma, it suffices to show that $k \to B/m = C$ is surjective. But $C \times_k C \cong C$, so $\dim_k C = 1$ or $0$, which means $k \to C$ is surjective.

□

Tensor Product, Limits and Colimits

Remark (I.5.1.15) (Tensor Product). Tensor product is defined in (I.2.4.13). Notice that in case the rings are all commutative, there are no need to distinguish between left and right modules.

Def. (I.5.1.16) (Tensor Algebras). For a module $M$ over a commutative ring $R$, we define the

- **tensor algebra** operator from $\text{Mod}_R$ to graded algebras over $R$ that is left adjoint to the forgetful functor. It can be defined as follows:

$$T(M) = \bigoplus_{n \geq 0} \otimes^n M$$

as the module, and the algebra structure determined by the canonical map $\otimes^m M \times \otimes^n M \to \otimes^{m+n} M$.

- **exterior product** $\wedge^k M$ as the module with the universal property that $\text{Hom}_B(\wedge^k M, N)$ is the set of all morphisms $M^n \to N$ that vanishes on all elements that have two equal coordinates.

- **exterior algebra** operator $\wedge$ from $\text{Mod}_R$ to the category of strict graded commutative algebras over $R$ that is left adjoint to the forgetful functor. It can be defined as follows: $\wedge(M) = T(M)/(x \otimes x)$, where $x \in M$, or equivalently $\wedge(M) = \bigoplus_{k \geq 0} \wedge^k(M)$. 

• **symmetric algebra** operator $S$ from $\text{Mod}_R$ to $\text{CRing}_R$ that is left adjoint to the forgetful functor. It can be defined as follows: $\wedge(M) = T(M)/(x \otimes y - y \otimes x)$ where $x, y \in M$.

**Cor. (I.5.1.17).** The construction of $T(M), \wedge M$ and $\text{Sym}(M)$ commutes with all colimits, because they are all left adjoints.

**Prop. (I.5.1.18) (Tensor Product and Quotient).** Let $R$ be a commutative ring and $I, J$ be ideals of $R$, then $R/I \otimes_R R/J \cong R/(I + J)$.

*Proof:* This follows from the universal property of quotient and tensoring. □

**Prop. (I.5.1.19).** There is a pullback square

$$
\begin{array}{ccc}
R/I \cap J & \longrightarrow & R/I \\
\downarrow & & \downarrow \\
R/J & \longrightarrow & R/(I + J)
\end{array}
$$

*Proof:* The pullback is just the elements in $R/I \times R/J$ that map to the same element in $R/(I + J)$. If $(x + I, y + J)$ maps to the same element $z + I + J$, then $x = y + i + j$, so $x - i = y + j$, and the pullback lies in the image of $\Delta : R \to R/I \times R/J$. Now the kernel is just $I \cap J$. □

**Prop. (I.5.1.20) (Filtered Colimits of Modules are Exact).** Let $\mathcal{I}$ be an index category that each connected components of $\mathcal{I}$ is filtered, then taking colimits over $\mathcal{I}$ is exact in the category $\text{Mod}_R$ of modules over a ring $R$.

*Proof:* It is clearly right exact. To check left exactness, Cf.[Sta]04B0. □

**Prop. (I.5.1.21).** Let $k$ be a field and $A, B$ be $k$-algebras and let $b \subset A \otimes_k B$ be an ideal. Then among the ideals $a \subset A$ that $b \subset a \otimes_k B$, there exists a smallest one.

*Proof:* Choose a $k$-basis of $B$, then the smallest ideal $a$ is just the ideal generated by all the $A$-coefficients of elements of $b$. □

**Localization**

**Def. (I.5.1.22) (Localization as Filtered Colimit).** Let $A$ be a commutative ring and $S$ be a multiplicatively closed subset of $A$ containing 1 and not containing 0, the localization $S^{-1}A$ is defined to be a ring over $A$ that any ring map $A \to B$ that maps elements of $S$ to units factors through $S^{-1}A$.

$S^{-1}A$ can be constructed as

$$
\lim_{\substack{\text{inclusions} \ S \subset S' \subset \cdots \subset S'' \subset \cdots}} A/S''
$$

where the ordering is defined to be $s < t$ if $t = sr$ for some $r \in S$, and if $t = sr$, there is a map from $A_s$ to $A_t$ defined by multiplying by $r$. This is easily seen to be a filtered colimit. There is easily seen to be the localization.

**Prop. (I.5.1.23) (Localization is exact).** $S^{-1}$ is an exact functor from $\text{Mod}_R$ to $\text{Mod}_R$. Because it is a filtered colimit(11.2.31)\(1.5.1.20\).

**Cor. (I.5.1.24).** $(R/I)\hat{p} \cong R_p / IR_p$, in particular, $k(R/P) \cong R_p / PR_p$. 

Def. (I.5.1.25) (Total Ring of Fractions). For any ring $R$, the set of non-zerodivisors in $R$ is a multiplicatively closed set $S$, and the localization $\text{Frac}(R) = S^{-1}R$ is called the total ring of fractions of $R$.

Prop. (I.5.1.26) (Localization Along an Ideal). Let $I$ be an ideal of $A$, then the localization of $A$ along $I$ is the ring $\tilde{A}$ that $\text{Spec} \tilde{A}$ is the localization of $\text{Spec} A$ along $V(I)$ as defined in (IX.1.15.19) (because $\text{Spec} A$ is spectral, by (IX.1.15.13)).

Equivalently, it can be defined as $\tilde{A} = S^{-1}A$, where $S = A \setminus \left( \bigcup_{p \in V(I)} p \right)$. (It can be checked that $S$ is multiplicatively closed).

Notice then $I \subset \text{rad} \tilde{A}$.

For $f \in A$, we call the localization of $A$ along $f$ the $f$-localization of $A$. In fact, it is the universal $A$-algebra that $f \in \text{rad} \tilde{A}$.

Prop. (I.5.1.27). Let $R$ be a commutative ring, $f_1, \ldots, f_n \in R$, and $M$ an $R$-module, then $M \to \bigoplus_i M_{f_i}$ is injective iff $M \to \bigoplus_i M : m \mapsto (mf_1, \ldots, mf_n)$ is injective.

Proof: Cf. [Sta]0565. □

Prop. (I.5.1.28). For any ring $A$, $A \to \prod A_m$ is injective, where $m$ are maximal ideals of $A$.

For a domain $A$, $A = \bigcap A_m$ inside the fraction field of $A$, where $m$ are maximal ideals of $A$.

Proof: if $g \in \text{Frac}(A)$ is in the RHS, then $I = \{ x \in A | xg \in A \}$ is an ideal of $A$ not contained in any maximal ideal, thus $I = 1$ (I.5.1.5), and thus $g \in A$. □

Lemma (I.5.1.29). Let $R$ be a ring and $p$ be a prime, then there exists an $f \in R, f \not\in p$ that $R_f \subset R_p$, if any of the following holds:

- $R$ is a domain.
- $R$ is Noetherian.
- $R$ is reduced and has f.m. irreducible components.

Proof: Cf. [Sta]0BX1. □

Def. (I.5.1.30) (Identifying Local Rings). A ring map $A \to B$ is said to identify local rings if for every prime $q \subset B$, the map $A_{\varphi^{-1}(q)} \to B_q$ is an isomorphism.

Prop. (I.5.1.31). The property of identifying local rings is stable under base change and composition. (This is immediate from (I.5.1.32)).

Prop. (I.5.1.32) (Tensor Product and Localization). For a ring map $R \to S$, let $q \subset \text{Spec} S, p = q \cap R$, then $(M \otimes_R S)_q = M_p \otimes_R S_q$ for any $R$-module $M$.

Proof: $(M \otimes_R S)_q = M \otimes_R S_q = M \otimes_R R_p \otimes_R S_q = M_p \otimes_R S_q$. □

Noetherian

Def. (I.5.1.33) (Noetherian Module). Let $R$ be a commutative ring and $M$ an $R$-module, then $M$ is called Noetherian iff every ascending chain of submodules stabilizes.

$R$ is called a Noetherian ring iff $R$ is Noetherian over itself.

Prop. (I.5.1.34). Quotient ring, f.g. module, f.g. algebra, localization and power series of a Noetherian ring $A$ are Noetherian, hence graded algebra of a $A$ by an ideal $I$ is Noetherian. Product of Noetherian rings are Noetherian.
Proof: Only need to prove $A[X]$ and $A[[X]]$, localization and others are quotients of these. For an ascending chain of ideal $I_j$ of $A[X]$, we consider the coefficients ideal $I_{i,j}$ of $X^i$ of $I_j$, then there are only f.m. different $I_{i,j}$‘s, so we have $I_j$ stabilize as well.

Similarly for $A[[X]]$, we prove any ideal $I$ is f.g. Consider the lowest terms coefficient ideal at degree $i$, then it is ascending and stabilize, then a set of generators as a whole generate $I$. □

Remark (I.5.1.35). The subring of a Noetherian ring is NOT necessarily Noetherian, by the example of $k[X_1, \ldots, X_n, \ldots] \subset k(X_1, \ldots, X_n, \ldots)$.

Prop. (I.5.1.36). When $A$ is Noetherian and is quipped with $I$-adic topology, then $I$ is f.g., and there is surjective ring map $A[[X]] \to A^*$ the completion, mapping to the generators of $I$, hence the completion is Noetherian. (It is surjective can be seen by the Cauchy sequence construction of completion).

Prop. (I.5.1.37). If $R \to R'$ is ring map of f.t., then if $R \to S$ and $S$ is Noetherian, then $S \otimes_R R'$ is Noetherian, because $S \times_R R'$ is of f.t. over $S$, and use (I.5.1.34).

Cor. (I.5.1.38). If $S$ is a Noetherian $k$-algebra over a field $k$, then for any f.g. field extension $K/k$, $S \otimes_k K$ is Noetherian. (Because there is a f.g. algebra $B$ over $k$ that $K$ is the localization of $B$, and use (I.5.1.34)).

Prop. (I.5.1.39). If $R$ is Noetherian and $M$ is a f.g. $R$-module, then there is a filtration $\{M_i\}$ of $M$ that the quotients are all isomorphic to $R_{p_i}$ where $p_i$ are primes.

Proof: $M$ is generated by $x_i$, so $(x_1) \cong R/I_1$, and so we modulo $x_i$, then the result follows by induction. So we may assume $M = R/I$. We use Noetherian condition to choose a maximal element $J$ that is a counterexample, then $J$ is not a prime, so there are $a, b \notin J$ that $ab \in J$. Then we have a filtration $0 \subset aR/(J \cap aR) \subset R/J$. Notice $R/(J + bR) \to aR/(J \cap aR) \to 0$, and the second quotient is $R/(J + aR)$, so they all can be factorized. □

Prop. (I.5.1.40). A Noetherian ring has only f.m. minimal prime ideals.

Proof: This is a consequence of (V.4.1.19)(V.4.1.20) and (IX.1.14.4).

Prop. (I.5.1.41) (Cohen). If every prime ideals of a commutative unital ring $R$ is f.g., then $R$ is Noetherian.

Proof: Suppose $P$ is not Noetherian. Firstly the set of non-finitely generated ideals has the chain property: if $I_i$ is a chain of non-f.g. ideals of $R$, then $I = \cup_{i \in \Phi} I_i$ is non-f.g., otherwise there are $(f_i) = I$, but $f_i \in I_h$ for some $h$, thus $I_h$ is f.g.. Next we use the Zorn’s lemma to find a maximal non-f.g. ideal $I$, and show that $I$ is a prime ideal:

$I \neq R$ because $R = (1)$, so if $a, b \in R - I$ that $ab \in I$, then $I + (a)$ and $I + (b)$ is f.g. by $p_i + r_i a$, by maximality, and let $K = (P : a)$, then $I \subset I + (b) \subset K$, thus $K$ is f.g., so does $aK$.

Now I claim $I = (p_1) + \ldots + (p_n) + aK$: one direction is clear, and if $r \in I \subset I + (a)$, then $r = \sum c_i (p_i + r_i a)$, thus $(\sum c_i r_i) a = r - \sum c_i p_i \in I$, thus $\sum c_i r_i \in K$, thus $r = \sum c_i r_i a \in (p_1) + \ldots + (p_n) + aK$.

So now $I$ is f.g., contradiction, which shows $I$ is a prime, but this contradicts the hypothesis. □

Prop. (I.5.1.42) (Modules over Noetherian Ring is Noetherian). Let $R$ be a Noetherian ring, then any submodule of a finite module $M$ over $R$ is finite. Thus any module over $R$ is a Noetherian module (I.5.1.33). In particular, any module over $R$ is of f.p.
Proof: it suffices to prove the first assertion: we use induction on the minimal number of generators of $M$: if it is generated by 1 element, then $M \cong R/I$ for some ideal $I$, thus $N \subset M$ is isomorphic to some $J/I$, so it is finite because $J$ is finite. If the minimal number of generators of $M$ is greater than 1, then there exists an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

where $M', M''$ has fewer number of generators. Now there is also an exact sequence

$$0 \to N \cap M' \to N \to \overline{N} \to 0,$$

and the minimal number of generators of $N$ is smaller than the sum of that of $M'$ and $M''$, thus it is also finite.

□

Prop. (I.5.1.43) (Artin-Tate). Let $R$ be a Noetherian ring and $S$ a f.g. $R$-algebra. If $T \subset S$ is an $R$-subalgebra that $S$ is a finite module over $T$, then $T$ is f.g. over $R$.

Proof: Cf.[Sta]00IS. □

Length

Def. (I.5.1.44) (Length). The length of a $R$-module $M$ is the supremum of lengths of chains of submodules of $M$. It is checked to be an additive function on $R$-modules.

Prop. (I.5.1.45). If $\text{length}_R(M) < \infty$, then any maximal chain of submodules has the same length.

Proof: Let $l(M)$ be the minimal length of a maximal chain, then if $M \subset N$, then firstly $l(M) < l(N)$, because a maximal chain of $M$ restricts to a maximal chain of $N$, and if the length is the same, then each term is in $M$, so $N \subset N$, contradiction. Now any chain has length $l(M)$, because if there is a chain $M_i$, then $l(M_0) < l(M_1) < \ldots < l(M)$. □

Prop. (I.5.1.46). If $R$ is a ring with maximal ideal $m$, $M$ is an $R$-module that $mM = 0$, then $\text{length}_R(M) = \dim_{R/m}(M)$.

Proof: Cf.[Sta]00IY. □

Prop. (I.5.1.47) (Order of Vanishing). If $R$ is a semi-local Noetherian domain of dimension 1 and $a, b$ are not zero-divisors, then $f(a) = \text{length}(R/(a))$ satisfies $f(a) + f(b) = f(ab) < \infty$. So this $R \subset K$ and has fraction field $K$, then $f$ extends to an additive function on $K^*$, denoted by $\text{ord}_R(f)$.

Proof: It is finite by[Sta]00PF. It is additive because length is additive(I.5.1.44) and $0 \to R/(a) \to R(ab) \to R(b) \to 0$. □

Artinian Ring

Def. (I.5.1.48) (Artinian Rings). A ring $A$ is called Artinian if any descending chain of ideals of $A$ stabilizes.

Lemma (I.5.1.49). Let $A$ be an Artinian ring, then $A$ has f.m. maximal ideals.
Proof: Consider \( m_1 \supset m_1 \cap m_2 \supset \ldots \supset \), then it is a descending chain, by Chinese remainder theorem. So it has f.m. maximal ideals.

\[ \square \]

Lemma (I.5.1.50). If \( A \) is an Artinian ring, then the Jacobson radical is nilpotent.

Proof: Consider the Jacobson radical \( I, I^n = I^{n+1} \) for some \( n \), let \( J = \text{Ann}(I^n) \), it suffices to show \( J = A \). If not, choose a minimal \( J' \) that contains \( J \) but not \( J \) (exists by Artinian property), then \( J' = J + Ax \), and \( IJ' \subset J \) by Nakayama, so \( xI^{n+1} \subset JI^n = 0 \), so \( x \in J \), contradiction.

\[ \square \]

Prop. (I.5.1.51) (Characterization of Artinian Rings). The following are equivalent:

1. \( A \) is Artinian.
2. \( A \) is Noetherian of dimension 0.
3. \( \text{length}_A A < \infty \).
4. \( A \) is a finite product of local Artinian rings.
5. \( A \) is Noetherian, Jacobson (I.5.9.3), and has f.m. maximal ideals.

In particular, a \( k \)-algebra that is finite as a \( k \)-module where \( k \) is a field, is an Artinian ring.

Proof: 1 \( \iff \) 3: if \( \text{length}_A A < \infty \), then \( A \) is clearly Artinian. Conversely, if \( A \) is Artinian, then by (I.5.1.49)(I.5.1.50)(I.5.9.6) \( A \) a finite product of its localization of maximal ideals, so we may assume \( A \) is local with maximal ideal \( m \). Then \( m^n = 0 \) for some \( n \) by (I.5.1.50), and \( m^i/m^{i+1} \) has length the same as their dimension as a \( A/m \) vector space, by (I.5.1.46), which is finite because \( A \) is Artinian, so \( \text{length}_A A < \infty \).

1 + 3 \( \rightarrow \) 5: \( A \) has f.m. maximal ideals by (I.5.1.49). It is Jacobson by (I.5.1.50). \( \text{length}_A A < \infty \) clearly implies \( A \) is Noetherian.

5 \( \rightarrow \) 2: By (I.5.9.6).

2 \( \rightarrow \) 5: all prime ideals are maximal, so \( \text{Spec} A \) is discrete, so \( A \) has f.m. maximal ideals, and it is clearly Jacobson.

5 \( \rightarrow \) 3: By (I.5.9.6), \( R \) is a product of its local rings, and the local rings are all Noetherian and Jacobson (I.5.9.5), so by Nakayama, they have finite lengths. So also \( R \) has finite length.

5 \( \rightarrow \) 4: By lemma (I.5.9.6) below, \( A \) is a product of its localization, and its localizations also satisfies 5, and by 5 \( \rightarrow \) 3 \( \rightarrow \) 1, they both have descending conditions.

4 \( \rightarrow \) 5: An Artinian ring is Noetherian and Jacobson by 1 + 3 \( \rightarrow \) 5, then so does their product.

\[ \square \]

Cor. (I.5.1.52). A reduced local Artinian ring is a field. In particular, a reduced Artinian ring is a product of fields.

Proof: An Artinian local ring \( A \) is Jacobson so the maximal ideal \( m = 0 \) as \( A \) is reduced.

\[ \square \]

Prop. (I.5.1.53). For an Artinian local ring \( A \), the following are equivalent:

1. \( A \) is a PID.
2. the maximal ideal \( m \) is principal.
3. \( \dim_k(m/m^2) \leq 1 \).

Proof: It suffices to prove 3 \( \rightarrow \) 1: If \( m = m^2 \), then \( m = 0 \) by Nakayama, so \( A \) is a filed. If \( \dim_k(m/m^2) = 1 \), then \( m \) is principle by Nakayama. And \( m \) is nilpotent by (I.5.1.51), so for any ideal \( a \) there is a minimal \( n \) that \( a \subset m^n \). Now choose \( y \in a - m^{n+1} \), then \( y = ux^n \), and \( u \notin (x) \), so \( u \) is a unit, thus \( x^n \in a \), meaning \( a = m^n \) hence principal.

\[ \square \]
Local Properties

Def. (I.5.1.54) (Local properties). A property $P$ of rings or modules over a ring is called local property iff $X$ has $P$ iff $X_f$ all has $P$ for a covering $(f_1, \ldots, f_n) = 1$.

A property of morphisms of rings is called local on the target iff $R \to S$ has $P$ iff $R_{f_i} \to S_{f_i}$ has $P$ for a covering $(f_1, \ldots, f_n) = 1$ in $R$.

Prop. (I.5.1.55) (Stalkwise Properties). For a commutative ring $R$, a property $P$ is called stalkwise if $A$ satisfies $P$ iff all $A_m$ satisfies $P$ where $m$ are maximal ideals, and iff all $A_p$ satisfies $P$ where $p$ are prime ideals of $A$.

1. Trivial is stalkwise for modules over $R$. Hence so does injectivity and surjectivity because localization is exact.
2. Torsion-free is stalkwise for modules over $R$ integral.
3. Flatness for modules over $R$.
4. Flatness for rings over $R$ on the source.
5. Formal unramifiedness for rings over $R$, both on the target and source.
6. (universally)catenary is stalkwise.
7. reducedness is stalkwise.
8. normal is stalkwise.
9. regular is stalkwise.

Proof:
1. It suffice to prove an element is trivial on every localization then it is 0. For this, consider the annihilator $\text{Ann}(x)$, it is not contained in any maximal ideal so it contains 1.
2. if $xf = 0$ but $f \neq 0$, then $x \in \text{Ann}(f) \neq (1)$, so $\text{Ann}(f) \subset m$ maximal, so $f$ is torsion in $M_m$ over $R_m$. Conversely, if $f$ is torsion in $R_m$, then it is clearly torsion over $R$.
3. We use the definition(I.7.1.2). Notice $(IM)_p = I_p M_q$ and every ideal of $R_p$ is of the form $I_p$. Then use the fact injective is stalkwise(I.5.1.55).
4. We use the definition(I.7.1.2). Notice $(I \otimes_R S)_q = I_p \otimes_{R_p} S_q$ for all primes $q$ of $S$ and $p = q \cap R$. And every ideal of $R_p$ is of the form $I_p$. Then use the fact injective is stalkwise(I.5.1.55)
5. Because formally unramified is equivalent to $\Omega_{R/S} = 0$(I.7.6.1), so we get the result by functorial properties of $\Omega_{S/R}$(I.7.3.6) and triviality is stalkwise(I.5.1.55).
6. For any two prime ideals $p \subset q$, we can choose a maximal ideal containing them.
7. use(I.5.1.28).
8. By definition.
9. By definition.

Remark (I.5.1.56). Our main technique of proving local properties are using affine communication theorem(V.4.1.2).

Prop. (I.5.1.57) (Local Properties). For a fixed ring $R,$
1. Every property that is stalkwise is a local property.\textsuperscript{(I.5.1.55)} The properties listed below should not be stalkwise.

2. Every property that satisfies faithfully flat descent is a local property. The properties listed below should not satisfy faithfully flat descent.

3. Noetherian.

4. F.t. ring maps on the source.

5. F.p. ring maps on the source.

\textbf{Proof:}

1. 

2. 

3. If $A$ is local, then $A_{f_i}$ are local by\textsuperscript{(I.5.1.34)}. Conversely, if $A_{f_i}$ are all Noetherian and \[
I_1 \subset I_2 \subset \ldots
\] is an ascending chain of ideals of $A$, consider $A \rightarrow \prod A_{f_i}$ faithfully flat, thus $I_1 \otimes_A (\prod A_{f_i}) \subset I_2 \otimes_A (\prod A_{f_i}) \subset \ldots$ is an ascending chain of $\prod A_{f_i}$. Now $\prod A_{f_i}$ are Noetherian by\textsuperscript{(I.5.1.34)}, so this chain stabilizes. But this ring map is faithfully flat, so the original chain must also stabilize.

4. Let $(g_1, \ldots, g_n) = 1$, choose $\sum h_i g_i = 1$, and let $x_{ij} - y_{ij} n_{ij}$ generates $S_{g_i}$. Now let $S'$ be the sub-$R$-algebra of $S$ generated by $y_{ij}, g_i, h_j$. Then $(S')_{f_i} \rightarrow S_{f_i}$ is surjective for any $i$, so $S' \rightarrow S$ is also surjective, by\textsuperscript{(I.5.1.55)}. Then $S' = S$, and $S$ is f.g. over $R$.

5. Cf.[Sta]00EP.

\[\square\]

2 Rings and Categories

\textbf{Quotient by Equivalence Relations}

\textbf{Prop. (I.5.2.1) (Quotients by Equivalence Relations).} Let $u_0, u_1 : A_0 \rightarrow A_1$ be an equivalence in the dual category of $\text{Alg}_{R_0}$,\textsuperscript{(I.1.1.17)}. If $u_0$ is locally free of constant rank $r$, then a quotient $u : A \rightarrow A_0$ exists, and $u$ is locally free of constant rank $r$.

\textbf{Proof:} Cf.[Mil17b]P592.

\[\square\]

\textbf{Morita Equivalence}

Basic References are [Morita Equivalence] and [Fuller Rings and Categories of Modules].

\textbf{Def. (I.5.2.2).} Two ring $R, S$ are called \textbf{Morita equivalent} if the category of mod-$R$ is equivalent to the category of mod-$S$.

\textbf{Prop. (I.5.2.3).} For an Abelian category $\mathcal{A}$ satisfying AB3 (i.e arbitrary sum exists), An object $P$ of $\mathcal{A}$ is a \textbf{progenerator} if the functor $h' : X \mapsto \text{Hom}_\mathcal{A}(P, X)$ is exact and and strict: $h'(X) = 0$ implies $X = 0$. Then $h'$ determines an equivalence from $\mathcal{A}$ to mod-$R$, where $R = \text{Hom}_\mathcal{A}(P, P)$.

Similarly, if $\mathcal{A}$ is an Abelian Noetherian category and $P$ is a progenerator, then $R$ is Noetherian and $\mathcal{A}$ is equivalent to the category of finitely generated $R$-categories.
Proof: Essentially surjective: construct using direct limit and cokernel.

Notice that \( h'(X) \cong h'(X') \to X \cong X' \) by strictness and A4 axiom. So let \( X = \text{Coker}(P^{\oplus I}, P^{\oplus J}) \),

\[
\text{Hom}(h'(X), h'(Y)) = \text{Hom}(\text{Coker}(h'(P^{\oplus J}), h'(P^{\oplus I}), h'(Y))
= \text{Ker}(\text{Hom}(h'(P^{\oplus J}), h'(Y)) \to \text{Hom}(h'(P^{\oplus I}), h'(Y)))
= \text{Ker}(h'(Y^{\Pi I}) \to h'(Y^{\Pi J}))
= \text{Hom}(X, Y)
\]

\( \square \)

**Prop. (I.5.2.4).** In the case when \( A \) is the category \( \text{mod-} R \), \( P \) is a generator \( \iff \ h' : X \mapsto \text{Hom}_R(P, X) \) is faithful \( \iff \) every \( M \) is a quotient of direct sums of \( P \). And a **progenerator** is a f.g. projective generator.

**Prop. (I.5.2.5).** Let \( P \) be a \((A, B)\)-bimodule, iff \( P \) is a progenerator as a right \( B \) module, then it is a progenerator as a left \( A \) module.

**Prop. (I.5.2.6).** Let \( P \) be a progenrator as a

**Prop. (I.5.2.7) (Morita).** The following are equivalent:

- categories \( A\)-mod and \( B\)-mod are equivalent.
- categories \( \text{mod-} A \) and \( \text{mod-} B \) are equivalent.
- There exist a finitely generated progenerator \( P \) of \( \text{mod-} A \) that \( B \cong \text{End}_A P \).

**Proof:** \( \to \) \( 3: A \) is a progenerator in \( \text{mod-} A \), thus when \( A \sim B, F : \text{mod-} A \to \text{mod-} B, A \cong \text{End}_A A = \text{End}_B F(A), \) and \( F(A) \) is a left \( A \) module as well as a progenerator of \( B \). Thus there is a \((A, B)\)-bimodule \( P \) that \( A \cong \text{End}_B P \), and a \((B, A)\)-bimodule \( Q \) that \( B \cong \text{End}_A Q \). \( \square \)

**Prop. (I.5.2.8).** There can be defined another Morita invariance that \( R \sim S \) iff there are \((R, S)\)-bimodule \( P \) and \((S, R)\)-bimodule \( Q \) that \( P \otimes S Q \cong R \) as a \((R, R)\)-bimodule and \( Q \otimes_R P \cong S \) as a \((S, S)\)-bimodule. This will immediately generate equivalence between \( R\)-mod and \( S\)-mod as well as equivalence between \( \text{mod-} R \) and \( \text{mod-} S \) by tensoring. And \( P \) and \( Q \) are projective modules respectively, because equivalence is a kind of adjoint.

**Prop. (I.5.2.9).** Let \( D \) be a division ring over \( k \) of finite degree and \( A = M_r(D) \). Let \( S = D^r \) with \( A \) acting by left multiplication and \( D \) acting by right multiplication, then \( S \) is a simple \( A\)-module, and every \( A\)-module is a direct sum of copies of \( S \). This means that \( S \otimes_D - \) induces an equivalence from \( \text{Mod}_D \) to \( \text{Mod}_A \).

**Prop. (I.5.2.10) (Properties Preserved under Morita Invariance).** Cf.[Rings and Categories of Modules P54].

3 Spectra

**Lemma (I.5.3.1) (Affine Chevalley).** The Spec map of a f.p. ring map maps constructible sets to constructible sets.
Proof: Cf.[Sta]00FE. □

Prop. (I.5.3.2). For \( R \subset S \), all the minimal primes of \( R \) are in the image of the Spec map of a minimal prime of \( S \).

Proof: Localize w.r.t. to the minimal prime \( p \), then it is a local ring with only one prime. And \( S_p \) is nonzero because localization is exact, so it has a maximal ideal \( q \). Now we choose a minimal prime of \( S \) contained in \( q \), then it is also mapped to \( p \). □

Lemma (I.5.3.3). For a Noetherian local ring \((A, m)\), \( \text{Spec } A - m \) is affine iff \( \dim A \leq 1 \).

Proof: if \( \dim A = 0 \), this is true, if \( \dim A = 1 \), let \( f \in m \) not in any other minimal primes of \( A \), then \( \text{Spec } A - m = \text{Spec } A_f \).

Conversely, Cf.[[Sta]0BCR]. □

Idempotents

Prop. (I.5.3.4) (Clopen Subsets). The clopen subsets of \( \text{Spec } A \) corresponds to idempotents in \( A \).

Proof: This is all equivalent to the fact that there exists \( e + f = 1, ef = 0 \):

If \( A = U \coprod V \), then both \( U, V \) are closed hence qc, so \( \text{Spec } A = \cup \text{Spec } (f_i) \coprod \cup \text{Spec } (g_j) \), then \( f_j g_j \) is nilpotent by (I.5.9.2). Denote \( I = (f_i), J = (g_j) \), then \((IJ)^N = 0 \) and \( I + J = A \), there are \( 1 = x + y, x \in I^N, y \in J^N \).

For uniqueness, if \( e_1 \neq e_2 \), then \( 0 \neq e_1 - e_2 = e_1(e_2 + f_2) - e_2(e_1 + f_1) = e_1f_2 - e_2f_1 \), so may assume \( e_1f_2 \neq 0 \), and it is not nilpotent, so there is a \( e_1f_2 \subset p \), which is a contradiction. □

Cor. (I.5.3.5). A local ring has no non-trivial idempotents, and then an idempotent is defined by the maximal ideals that it vanishes.

Cor. (I.5.3.6). If \( I \) is an ideal of \( R \) that \( I = I^2 \), and \( I \) is f.g., then \( V(I) \) is open and closed in \( \text{Spec } R \), and \( V(I) = R_e \) for some idempotent \( e \).

Proof: By Nakayama, there is a \( a f = 1 - e \) with \( e \in I \) that \( fI = 0 \). So \( e - e^2 = 0 \) and \( f^2 = f \).

\( V(I) = D(f) = D(e) \). □

Lemma (I.5.3.7). If \( I \) is a locally nilpotent ideal, then \( R \to R/I \) induces a bijection on idempotents.

Proof: Because \( R \to R/I \) induces a homeomorphism on the spectra, and clopen subsets of the spectrum corresponds to the idempotents(I.5.3.4). □

Lemma (I.5.3.8). Let \( R \) be a ring and \( T \subset \text{Spec } R \) is a set. Then the following are equivalent:

- \( T \) is closed and is a union of connected components of \( \text{Spec } R \).
- \( T \) is an intersection of clopen subsets.
- \( T = V(I) \) where \( I \) is generated by idempotents.

Proof: 1 and 2 are equivalent by (IX.1.15.3), and 2 \( \to \) 3 \( \to \) 1 are easy. □

Prop. (I.5.3.9). Let \( R \) be a ring, then any connected component of \( \text{Spec } R \) is of the form \( V(I) \), where \( I \) is an ideal generated by idempotents that any idempotent of \( R \) maps to either 0 or 1 in \( R/I \).

Proof: By (IX.1.15.2) and (IX.1.15.13), a connected component of \( \text{Spec } R \) is an intersection of clopen subsets, so it is of the form \( V(I) \) where \( I \) is generated by idempotents. The last assertion is equivalent to \( V(I) \) being connected. □
Going-up and down

Def. (I.5.3.10) (Going-up and Going Down). Going-up and down for topological spaces is defined in (IX.1.14.6). A ring map $R \rightarrow S$ is said to satisfy the going-up property iff its Spec map does, equivalently, for any prime ideal $q \subset S$ and prime ideal $p \subset q \cap R$, there exists a prime ideal $q' \subset q$ that $q' \cap R = p$.

It is said to satisfy the going-down property iff its Spec map does, equivalently, for any prime ideal $q \subset S$ and prime ideal $q \cap R \subset p$, there exists a prime ideal $q' \subset q$ that $q' \cap R = p$.

Prop. (I.5.3.11). Going-up and Going-down are stable under composition, trivially.

Prop. (I.5.3.12) (Integral Map satisfies Going-Up). Integral ring map satisfies going-up (I.5.5.5). Flat ring map satisfies going-down (I.7.1.19).

Lemma (I.5.3.13). If the image of the Spec map of a ring map is closed under specialization, then this image is closed.

Proof: Let it be $R \rightarrow S$, let $I$ be the kernel, then the image is contained in $V(I)$, so we may replace $R$ be $R/I$, then $R \subset S$. Now we show the image contains all the minimal primes of $R$: for a minimal prime $p$, $A_p \subset B_p$, thus $B_p$ is not-empty, and thus has a maximal ideal, whose intersection with $A_p$ can only be $p$ by hypothesis, thus $p$ is in the image of Spec. Then the image is all of Spec $R$ by hypothesis, thus closed. □

Cor. (I.5.3.14) (Going-up and Spec Closed). Going-up is equivalent to Spec map closed.

Proof: If going-up holds, then Spec map is closed by (I.5.3.13). Conversely, a closed map satisfies going-up, by (IX.1.14.7). □

Prop. (I.5.3.15). If $R \rightarrow S$ is a ring map that satisfies going-up, and $P \subset S$ is a maximal ideal, then $P \cap R$ is also a maximal ideal.

Prop. (I.5.3.16) (Krull). If $A \subset B$ is an integral extension of integral domains, and $A$ is normal, then going-down holds.

Proof: Let $L_1, K$ be the fraction fields of $B, A$ resp., and let $L$ be the normal extension of $K$ contained in $L_1, C$ the integral closure of $A$ in $L$. Let $P \in \text{Spec } B$ and $p = P \cap A, p' \subset p$. Take a prime ideal $Q' \in \text{Spec } C$ lying over $p$, and by going-up applied to $A \subset C$ (I.5.3.12), there is a prime ideal $Q_1$ lying over $p$ that $Q' \subset Q_1$. Take $Q \in \text{Spec } C$ lying over $P$, then by (I.6.5.10) there is a $\sigma \in G_{L/K}$ that $\sigma(Q_1) = Q$. Set $P' = \sigma(Q') \cap B$, then $P' \subset P$ is a prime of $B$ lying over $p'$, so going-down holds for $A \subset B$. □

Prop. (I.5.3.17) (Going-down and Spec Open). If Spec map is open, then going-down holds. Proof:

Minimal Primes and Irreducible Components

Prop. (I.5.3.18) (Minimal Primes Exists). Every nonzero ring contains a minimal prime ideal.

Proof: Firstly prime ideal exists, by (I.5.1.4), and we use Zorn’s lemma to find a minimal prime ideal: it suffices to show the intersection of a chain of prime ideals is a prime ideal, this is not hard. □
Prop. (I.5.3.19). If $p$ is a minimal prime of $R$, then $pR_p$ is locally nilpotent. In particular, if $R$ is reduced, then $R_p$ is a field.

Proof: It suffices to show for every element $x \in pR_p$, $D(x) = 0$ (I.5.9.2). If not, then there is a prime in $pR_p$ not containing $x$, but this is impossible because $p$ is the only prime. □

Prop. (I.5.3.20) (Zerodivisors in a Reduced Ring). Let $R$ be a reduced ring, then

- $R \to \prod_p \text{minimal } R_p$ is an embedding into a product of fields.
- $\bigcup_p \text{minimal } p$ is the set of zerodivisors of $p$.

Proof: Cf. [Sta]00EW. □

Prop. (I.5.3.21). If $R$ is a ring with f.m. minimal primes $q_i$ and $\bigcup_i q_i$ is the set of zerodivisors of $R$, then the ring of fractions of $R$ (I.5.1.25) is equal to $\prod_i R_{q_i}$.

Proof: Cf. [Sta]02LX. □

Universal Homeomorphism

Cf. [Sta]10.45 and [Sta]28.44.

Prop. (I.5.3.22). If $\varphi : R \to S$ is a ring map and $p$ is a prime number that satisfies:

- $S$ is generated over $R$ by elements $x$ that there is $n$ that $x^{p^n} \in \varphi(R)$ and $p^n x \in \varphi(x)$.
- $\ker(\varphi)$ is locally nilpotent.

then $\Spec S \to \Spec R$ is a homeomorphism, and any base change of $\varphi$ satisfies the above conditions, so it is a universal homeomorphism.

In particular, this applies to any base change of a field extension $k'/k$ that is purely inseparable, because it is f.f. hence injective.

Proof: Cf. [Sta]0BRA. □

4 Support and Associated Primes

Def. (I.5.4.1) (Support of a Module). The support $\text{Supp}(M)$ of a module $M$ is the set of all $p$ that $M_p \neq 0$. When $M$ is f.g., $\text{Supp}(M) = V(\text{Ann}(M))$.

Prop. (I.5.4.2). The support of a nonzero module is not empty, because triviality is stalkwise by (I.5.1.55).

Prop. (I.5.4.3). If $0 \to N \to M \to Q \to 0$, then we have $\text{Supp}(M) = \text{Supp}(N) \cup \text{Supp}(Q)$, this is because localization is exact.

Prop. (I.5.4.4). For $E, F$ f.g over a ring $A$, $\text{Supp}(E \otimes F) = \text{Supp}(E) \cap \text{Supp}(F)$, this is because on a local ring $A_p$, $E \neq 0, F \neq 0 \to E \times F \neq 0$, which can be seen by passing to the residue field and use Nakayama.

Prop. (I.5.4.5). Let $A$ be Noetherian and $I$ be an ideal, then $I^n M = 0$ for some $n$ iff $\text{Supp}(M) \subset V(I)$.

Proof: If $I^n M = 0$, then if $I \not\subseteq P$, then $M_P = 0$. Conversely, we have a filtration of $M$, and by (I.5.4.3) we have all the $P_i$ include $I$, so $I^n$ annihilate $M$. □
Associated Primes of a Module

**Def. (I.5.4.6) (Associated Primes of a Module).** The associated primes $\text{Ass}(M)$ of a $A$-module $M$ is the set of primes $p = \Ann(m)$ where $m \in M$. $I$ is called **unmixed** if primes in $\text{Ass}(A/I)$ don’t contain each other, and of the same height.

**Prop. (I.5.4.7) (Associated Primes and Exact Sequence).** Note that $P \in \text{Ass}(M)$ iff $M$ contains a submodule isomorphic to $A/P$. So for an exact sequence $0 \to M_1 \to M \to M_2$, $\text{Ass}(M) \subset \text{Ass}(M_1) \cup \text{Ass}(M_2)$ and $\text{Ass}(M_1) \subset \text{Ass}(M)$. Hence for a f.g module over a Noetherian ring, $\text{Ass}(M)$ is finite by(I.5.1.39).

**Prop. (I.5.4.8) (Associated Primes and Support).** $\text{Ass}(M) \subset \text{Supp}M$, and when $R$ is Noetherian, their minimal elements are the same.

In particular, $\text{Ass}(M)$ is not empty by(I.5.4.2), and $\text{Ass}(A/I)$ contains all the minimal primes over $I$.

**Proof:** If $p = \Ann(m)$, then $m$ is nonzero in $M_p$, so $M_p$ is nonzero, i.e. $p \in \text{Supp}(M)$.

For the second assertion, we first prove for $M$ finite, and then write any module as sum of finite submodules, and use the fact Supp and ass are all unions of those of the submodules. Cf.[Sta]02CE. □

**Prop. (I.5.4.9) (Associated Primes and Zero-divisors).** When $R$ is Noetherian and $M$ a $R$-module, the union of the associated primes of $M$ is the set of zero-divisors in $M$.

**Proof:** Elements in associated points are zero-divisors obviously, and conversely, if $xm = 0$, then $x \in \Ann(m)$ and $\Ann(m)$ has an associated point $q$ by(I.5.4.8). Now $x$ must be in $q$ and $q$ is also an associated point of $M$ by(I.5.4.8).

□

**Cor. (I.5.4.10).** Use the prime avoidance(I.5.1.3), we can prove if $R$ is Noetherian and $M$ is a finite $R$-module, then $I \subset p$ for some $p \in \text{Ass}(M)$ iff $I$ consists of zero-divisors.

**Prop. (I.5.4.11) (Associated Primes and Maps).** For a ring map $\phi: R \to S$ and a $S$-module $M$, then $\text{Spec}(\phi)(\text{Ass}_S(M)) \subset \text{Ass}_R(M)$, and equal if $S$ is Noetherian.

**Proof:** We prove it is equal. If $p = \Ann_R(m)$, then we let $I = \Ann_S(m)$, then $R/p \subset S/I \subset M$, so by(I.5.4.11), there is a minimal prime of $S$ over $I$ that are mapped to $p$, now this prime is in $\text{Ass}(S/I)$ by(I.5.4.8) and also in $\text{Ass}_S(M)$ by(I.5.4.7). □

**Prop. (I.5.4.12) (Associated Primes and Localization).** Let $A$ is Noetherian and $\phi : \text{Spec}S^{-1}A \to \text{Spec}A$, if $M$ is a $A$-module, then

$$\text{Ass}_A(S^{-1}M) = \phi(\text{Ass}_{S^{-1}A}(S^{-1}M)) = \text{Ass}_A(M) \cap \{p | p \cap S = \emptyset\}.$$ 

**Proof:** The first equality is by(I.5.4.11). For the second, if $\Ann_A(x) = p$ and $p \cap S = \emptyset$, then $\Ann_{S^{-1}A}(x/1) = S^{-1}(p)$. Conversely, if $\Ann_{S^{-1}A}(x/s) = S^{-1}p$, then $p \cap S = \emptyset$, and $\Ann_A(x) \subset S^{-1}p \cap A = p$. □

**Prop. (I.5.4.13).** If $R$ is Noetherian and $M$ is a $R$-module, then $M \to \prod_{p \subset \text{Ass}(M)} M_p$ is injective.

**Proof:** Notice $(x) \subset M$, if $(x)$ is nonzero, then there is an associated prime $p$ of $(x)$(I.5.4.8), then it is an associated prime of $M$, and then $(x)_p \subset M_p$ is not zero, contradiction. □
Def. (I.5.4.14). An associated point that is not minimal among them is called a **embedded point**. An embedded point correspond to a nilpotent element, because $px = 0$ is contained in every minimal element but $p$ is not, so $x$ is contained in every minimal prime ideal.

Cor. (I.5.4.15) (Reduced Ring No Embedded Primes). A reduced ring has no embedded primes, because it has no nilpotent elements. Hence all its associated primes are just the minimal primes.

**Primary Decomposition**

Def. (I.5.4.16). For $R$ Noetherian, a $R$-module $M$ is called **coprimary** iff it has only one associated primes. A submodule $N$ of $M$ is called $p$-**primary** iff $\text{Ass}(M/N) = \{p\}$. A ring is called $p$-**primary** iff $(0)$ is $p$-primary.

Notice coprimary is equivalent to the following: if $a \in A$ is a zero divisor for $M$, then for each $x \in M$, there is a $n$ that $a^n x = 0$, i.e. **locally nilpotent**. And for ideals in a Noetherian ring, this is equivalent to $r(I)$ is a prime.

**Proof**: If $M$ is $p$-primary, if $x \in M$ is nonzero, then $\text{Ass}(Rx) = \{p\}$, so $p$ is the unique minimal element of $\text{Supp}(Rx) = V(\text{Ann}(x))$ by(I.5.4.8). So $p$ is the radical of $\text{Ann}(x)$, i.e. $a^n x = 0$ for some $n(I.5.9.2)$.

Conversely, we know the ideal $p$ of locally nilpotent elements equals the union of the associated primes(I.5.4.9), so if $q \in \text{Ass}M = \text{Ann}(x)$, then by definition, $p \subseteq q$. So $p = q$, and thus $\text{Ass}M = \{p\}$. \hfill $\square$

**Lemma** (I.5.4.17). A primary ring has no nontrivial idempotent element, because $e$ and $1 - e$ will all belong to the same minimal ideal $p$.

**Lemma** (I.5.4.18). The intersection of $p$-primary submodules are $p$-primary. (Because there is a injection $M/Q_1 \cap Q_2 \rightarrow M/Q_1 \oplus M/Q_2$).

**Lemma** (I.5.4.19) (Associated Prime and Primary Decomposition). If $N = \cap Q_i$ is an irredundant primary decomposition and if $Q_i$ belongs to $p_i$, then we have $\text{Ass}(M/N) = \{p_1, \ldots, p_r\}$.

**Proof**: There is a injection $M/N \rightarrow M/Q_1 \oplus \ldots M/Q_r$ which shows $\text{Ass}(M/N) \subset \{p_1, \ldots, p_r\}$. And for the inverse, notice $Q_2 \cap \ldots \cap Q_r/N$ is a submodule of $M/Q_1$, which shows $\text{Ass}(Q_2 \cap \ldots \cap Q_r/N) = \{p_1\}$ by(I.5.4.8). \hfill $\square$

**Prop.** (I.5.4.20). If $N$ is $p$-primary submodule of a $R$-module $M$, and $p'$ is a prime ideal, then

- $N_{p'} = M_{p'}$ if $p \not\subseteq p'$.
- $N = M \cap N_{p'}$ if $p \subseteq p'$.

**Proof**: $M_{p'}/N_{p'} = (M/N)_{p'}$, and $\text{Ass}((M/N)_{p'}) = \text{Ass}(M/N) \cap \text{primes contained in } p' = \emptyset$ by(I.5.4.12). So $M_{p'} = M_{p'}$ by(I.5.4.8).

For the second, notice it suffices to show $M/N \rightarrow M_{p'}/N_{p'}$ is injective. But this is because $A - p'$ contains no nonzero-divisor, by(I.5.4.9). \hfill $\square$

**Cor.** (I.5.4.21) (Second Uniqueness of Primary Decomposition). For an irredundant primary decomposition $N = \cap Q_i$, if $Q_1$ corresponds to $p_1$ and $p_1$ is minimal in $\text{Ass}(M/N)$, then $Q_1 = M \cap N_{p_1}$. In particular, the minimal prime part of a irredundant primary decomposition is uniquely determined.
Proof: By the above proposition, there are elements $u_i$ of $Q_i$, $i \neq 1$ that are mapped to units in $M_{p_i}$, so $Q_1 \cdot u_2 u_3 \ldots u_e$ is mapped onto the image of $Q_1 \to M_{p_1}$. Then $Q_1 = M \cap (Q_1)_{p_1} = M \cap N_{p_1}$. \hfill \Box

**Prop. (I.5.4.22).** If $R$ is Noetherian and $M$ is a $R$-module, there are $p$-primary submodules $Q(p)$ for each $p \in \text{Ass}M$ that $(0) = \bigcap_{p \in \text{Ass}M} Q(p)$.

**Proof:** For a $p \in \text{Ass}M$, we seek $Q(p)$ to be the maximal submodule $N$ that $p \notin \text{Ass}N$. This has a maximal ideal because of Zorn and the fact $\text{Ass}(\bigcup N_\lambda) = \bigcup \text{Ass}(N_\lambda)$. Then We have $\text{Ass}(M/Q(p)) = \{p\}$, otherwise there is another $p'$, then there is a $Q'/Q(p) \cong A/p'$. Now $Q'$ is bigger than $Q(p)$. Finally, $(0) = \bigcap_{p \in \text{Ass}M} Q(p)$ because it has no associated primes. \hfill \Box

**Cor. (I.5.4.23) (Primary Decomposition).** If $M$ is f.g. over a Noetherian ring $R$, then any submodule has a primary decomposition. (Notice $M$ has only f.m. associated primes).

**Def. (I.5.4.24) (Symbolic Power).** For a prime ideal in a Noetherian ring, The $n$-th symbolic power $p^{(n)}$ of $p$ is defined to be the $p$-primary component of $p^n$, who has only one minimal prime(hence one associated prime). The symbolic power is giving by $p^n A_p \cap A$ by(I.5.4.21).

5 Integral Extensions

**Def. (I.5.5.1) (Totally Integrally Closed).** For two rings $A \to B$, $f \in B$ is called almost integral(or totally integral when almost mathematics is performed:) over $A$ if $f^N$ lies in a f.g. $A$-module of $B$. It is clear that the elements of totally integral elements of $B$ is a subring. And $A$ is called totally integrally closed in $B$ iff any $f \in B$ totally integral over $A$ is in $A$.

**Prop. (I.5.5.2).** For a ring map $\varphi : A \to B$, an element $x$ is integral over $A$ iff $x$ is contained in a finite $A$-module in $B$. In particular, the elements of $B$ that are integral over $A$ is a ring containing $\varphi(A)$.

**Proof:** If $x$ is integral, then $\varphi(A)[x]$ is finite. If $\varphi(A)[x]$ is finite, then there is a set of generators of polynomials in $x$. Then for $m$ large, $x^m = \sum a_i f_i(x)$, so $x$ is integral over $A$. \hfill \Box

**Prop. (I.5.5.3) (Integral Extension of Field).** For $A \subset B$, if $B$ is integral over $A$, then $A$ is a field iff $B$ is a field.

**Proof:** If $A$ is a field, $y^{-1} = -a_{n-1}^{-1}(y^{n-1} + \ldots + a_{n-1}) \in B$. If $B$ is a field, $x^{-1} = -(b_1 + b_2x + \ldots + b_m x^{m-1}) \in A$. \hfill \Box

**Cor. (I.5.5.4).** If $B$ is integral over $A$, then a prime $p$ of $B$ is maximal iff $p \cap A$ is maximal.

**Proof:** Look at the integral extension $A/(p \cap A) \to B/p$. \hfill \Box

**Prop. (I.5.5.5) (Going-Up).** Let $A \to B$ integral. Then:

1. There is no inclusion relation between prime ideals of $B$ lying over a fixed prime ideal of $A$.
2. If $A \subset B$, then thee Spec map is surjective.
3. The going-up holds. In particular, the Spec map of an integral ring map is closed, by(I.5.3.14).

**Proof:**

1. If $p \cap A = p' \cap A = q$, Localize at $q$, then $p, p'$ are both maximal ideals of $B_q$ by(I.5.5.4), they cannot contain each other.
2. For any prime $p$ of $A$, since $A_p \subset B_p$, $B_p \neq 0$, so it has a maximal ideal (I.5.1.4), and use (I.5.5.4).

3. For any prime ideal $q$ of $B$ and $p = q \cap A$, replace $A \to B$ by $A/p \subset B/q$, then we can use 2. \hfill \Box

6 Graded Rings

Cf. [Matsumura Ch11].

**Def. (I.5.6.1) (Graded Rings).** A graded ring is a ring $A = \bigoplus_{n \in \mathbb{Z}} A_n$ that $A_m A_n \subset A_{m+n}$. A graded module over a graded ring $A$ is a module $M = \bigoplus_{n \in \mathbb{Z}} M_n$ that $A_m M_n \subset M_{m+n}$.

Notice that often we mean $\mathbb{Z}_{\geq 0}$-graded rings when we say graded rings. For a $\mathbb{Z}_{\geq 0}$-graded ring $A$, the subset $A_+ = \bigoplus_{n=1}^{\infty} A_n$ is an ideal of $A$, called the *irrelevant ideal*.

**Lemma (I.5.6.2).** Let $A = \bigoplus_{0}^{\infty} A_n$ be a graded ring, then a set of homogenous elements $f_i \in A_+$ generate $A$ as an algebra over $A_0$ iff they generate $A_+$ as an ideal of $A$.

**Proof:** If $f_i$ generate $A$ as algebra over $A_0$, then every element of $A_+$ is a polynomial in $f_i$ with constant coefficients in $A_0$, thus $f_i$ generates $A_+$ as an ideal. Conversely, if $f_i$ generate $A_+$ as an ideal, then for any homogenous element $f$ we can use induction on the degree of $f$ to show that $f$ is a polynomial in $f_i$. \hfill \Box

**Prop. (I.5.6.3) (Noetherian Graded Rings).** A graded ring $A = \bigoplus_{n=0}^{\infty} A_n$ is Noetherian iff $A_0$ is Noetherian and $A_+$ is f.g. as an ideal of $A$.

**Proof:** If $A$ is Noetherian, then clearly $A_+$ is f.g. and $A_0 = A/A_+$ is Noetherian. Now if $A_+$ is f.g. as an ideal of $A$, then it is generated by f.m. homogenous elements $f_i$, so we see $f_i$ generates $A$ as an algebra over $A_0$, which means $A$ is a quotient of a polynomial ring over $A_0$, thus Noetherian by (I.5.1.34). \hfill \Box

**Prop. (I.5.6.4).** Let $A$ be a graded ring that is f.g. over $A_0$ and $M$ be a f.g. graded $A$-module, then each $M_n$ is a finite $S_0$-module.

**Def. (I.5.6.5) (Homogenous Ideals).** Let $A_*$ be a graded ring, then a homogenous ideal $I_*$ of $A_*$ is a ring that is generated by homogenous elements.

**Prop. (I.5.6.6) (Equivalent definition of Homogenous Ideals).** Let $S_*$ be a graded ring, then

- an ideal $I$ of $S_*$ is homogenous iff it contains the degree $n$ part of each of its element for any $n$.
- The set of homogenous ideals of $S_*$ is stable under sum, product, intersection and radical.
- A non-trivial homogenous ideal $I$ of $S_*$ is a prime ideal iff for any homogenous elements $a, b$, if $ab \in I$, then $a \in I$ or $b \in I$.

**Proof:** 1. If $I$ contains the degree $n$ part of each of its element for any $n$, then clearly it is generated by homogenous elements. Conversely, if it is generated by homogenous elements, then
any element \( f = \sum a_i f_i \), where \( f_i \) is homogenous. Then we can see \([f]_n = \sum [a_i]_{n-deg} f_i\) is also in \( I\).

2: Use 1 and the definition. For radicals, we show \( \sqrt{I} \) contains all its homogenous parts: if \( f \in \sqrt{I} \), then \( f^n \in I \) for some \( n \), then we see that the minimal degree part \([f]_m \) of \( f \) also satisfies \([f]_m^n \in I\), because \( I \) contains the homogenous parts of each of its elements. Then we can use induction to show that all the homogenous parts of \( f \) is in \( \sqrt{I} \).

3: One direction is trivial, for the other, if \( a = \sum a_i, b = \sum b_i \) satisfies \( ab \in I \), and \( a \notin I, b \notin I \), and \( a_i, b_j \) homogenous. Let \( i_0, j_0 \) be the minimal numbers that \( a_{i_0} \notin I, b_{j_0} \notin I \), then \( a_{i_0} b_{j_0} \) is not in \( I \), contradicting the fact \( I \) is homogenous. \( \square \)

**Def. (I.5.6.7).** Let \( A \) be a ring and \( a \) be an ideal of \( A \), an \( a \)-filtration of \( M \) is a descending sequence of submodules \( M = M_0 \supset M_1 \supset \ldots \) that \( aM_n \subset M_{n+1} \). It is called a stable filtration iff there is an \( N \) that \( aM_n = M_{n+1} \) for \( n \geq N \).

For an ideal \( a \subset A \), there can be associated a graded ring \( A^* = \oplus a^n \), and an \( a \)-filtration \( M \) can be associated a graded module over \( A^* : M^* = \oplus M_n \). When \( A \) is Noetherian, then so is \( A^* \), because it is a quotient of a polynomial ring over \( A \)(I.5.1.34).

**Lemma (I.5.6.8).** If \( A \) is a Noetherian ring and \( M \) is a f.g. \( A \)-module that has an \( a \)-filtration \( M_n \), then \( M^* \) is f.g. over \( A^* \) iff \( M_n \) is a stable filtration.

**Proof:** As every \( M_n \) is finite over \( A \), if it is stable, then \( M^* \) is generated over \( A^* \) by all the generator of \( M_n, n \leq N \), so it is f.g.. Conversely, if it is f.g., then it is clear that \( M_n \) is a stable filtration. \( \square \)

**Prop. (I.5.6.9) (Artin-Rees).** For \( A \) Noetherian and \( I \) an ideal, let \( N \subset M \) be finite \( A \)-modules, then if \( M_n \) is a stable filtration of \( M \), then \( M_n \cap N \) is a stable filtration of \( N \).

In particular, let \( M_n = I^n \), then \( I^n \cap N = I^{n-r}(I^r M \cap N) \), hence the \( I \)-adic topology on \( M \) induce the \( I \)-adic topology on \( N \).

**Proof:** This is immediate from the lemma above, as \( N^* \) is an \( A^* \)-submodule of \( M^* \), and \( A^* \) is Noetherian(I.5.6.7). \( \square \)

**Cor. (I.5.6.10) (Intersection Theorem).** Notation as above, let \( N = \cap I^n M \), then the \( I \)-adic topology on \( N \) is trivial, by Artin-Rees, thus \( IN = N \). So Nakayama tells us there is an element \( a \in 1 + I \) that \( aN = 0 \). Thus if \( I \subset \text{rad}(A) \) or \( A \) is an integral domain, \( N = 0 \). This can be used to use induction to prove some theorem.

In particular, for any prime ideal \( p \) containing \( I \), use the above on \( R_p \) shows \( N_p = 0 \). But also \( N \) is f.g., so there exists an element \( g \notin p \) that \( N_g = 0 \).

**Cor. (I.5.6.11) (Krull).** For \( A \) Noetherian, if \( I \subset \text{rad}(A) \) or \( A \) is a domain, then \( \cap \infty I^n = 0 \).

**Prop. (I.5.6.12).** Notice for any ring \( A \) and a non-zero-divisor \( f \), if \( I = \cap_n f^n A \), then \( fI = I \), needless of the Noetherian property.

**Proof:** If \( x \in I \), \( x = fy \), because \( x \in f^n A \), \( fy = f^nt \) for some \( t \), so \( y = f^{n-1}t \), so \( f \in I \). Thus \( I = fI \). \( \square \)

**Def. (I.5.6.13) (Hilbert-Serre).** Let \( A \) be a Noetherian graded ring with \( A_0 \) Artinian that \( A_+ \) is generated by \( A_1 \). For a f.g. graded \( A \)-module \( M = \oplus M_n \), we have \( l(M_n) \) is a numerical polynomial of \( n \)(XIV.1.2.12) for \( n \) sufficiently large, called the **Hilbert Polynomial**. Its degree is the dimension of \( \text{Supp} M \subset \text{Proj}(A) \).
Proof: We prove by induction on the minimal number of generators of $A_1$ (it is finite by?? If it is 0, then $M_n = 0$ for $n$ large and the result holds. Now choose $x \in S_1$ as one of the minimal set of generators, then the induction hypothesis applies to $S/(x)$.

Firstly, if $x$ acts nilpotently on $M$, then we do induction on the minimal number $r$ that $x^r M = 0$. If $r = 1$, then $M$ is a module over $S/(x)$ and the assertion holds. If $r > 1$, then we can find an exact sequence $0 \to M' \to M \to M'' \to 0$ that $M', M''$ has smaller $r$, then we have the desired result, because $l$ is additive.

Next, if $x$ doesn’t act nilpotently on $M$, let $M' \subset M$ is the largest submodule that $x$ acts nilpotently, then there is an exact sequence $0 \to M' \to M \to M/M' \to 0$. So we can assume multiplication by $x$ is injective on $M$.

Let $\overline{M} = M/xM$, then for any $d$, there are exact sequences

$$0 \to M_d \xrightarrow{x} M_{d+1} \to \overline{M}_{d+1},$$

so $l(M_{d+1}) = l(M_d) = l(\overline{M}_{d+1})$. Then we finish by (XIV.1.2.13).

Cor. (I.5.6.14). Let $k$ be a field, $I \subset k[X_1, \ldots, X_n]$ be a non-zero graded ideal, and $M = k[X_1, \ldots, X_n]/I$, then the numerical polynomial $n \mapsto \dim_k(M_n)$ has degree $< d - 1$.

Proof: The numerical polynomial associated to $k[X_1, \ldots, X_n]$ is $n \mapsto \binom{n-1+d}{d-1}$, and for any non-zero homogenous element $f \in I$ of degree $e$, $f \cdot k[X_1, \ldots, X_n]_{d-e} \subset I_d$, thus $\dim_k(M_n) < \binom{n-1+d}{d-1} - \binom{n-e-1+d}{d-1}$, which means the numerical polynomial has degree $< d - 1$.

Prop. (I.5.6.15) (Hilbert Polynomial and Dimension). For a Noetherian local ring $A$, the Hilbert polynomial of a f.g. module $M$ w.r.t $\mathfrak{m}$ has degree $\dim M$. And $\dim M$ is the smallest integer $r$ s.t. there exists $x_1, \ldots, x_r$ that $l(M/x_1 M + \ldots, x_r M) < \infty$.

Proof: Cf. [Mat P76].

Prop. (I.5.6.16). For a local ring map of two power series, it is an isomorphism iff its Jacobian is invertible.

Proof: 

7 Completions

This subsubsection should be combined with the derived completion.

Prop. (I.5.7.1). Let the topology on an $A$-module be defined by countable filtration of submodules, then iff $M$ is complete, then $M/N$ is complete in the quotient topology.

Proof: Write $x_{i+1} - x_i = y_i + z_i$ with $y_n \in M_n$ and $z_n \in N$, then the image of the limit of $\sum y_i$ is the limit of $x_i$.

Def. (I.5.7.2) (Completeness). Let $I$ be an ideal of $R$, the $I$-adic completion of a $R$-module is a functor $\varphi : M \mapsto \hat{M} = \lim M/I^n M$. An $R$-module is called $I$-adically complete if the natural map $M \to \hat{M}$ is an isomorphism.

This is compatible with the general notion of completion of a topological Abelian groups (I.9.1.5).

Prop. (I.5.7.3). Let $R$ be a ring and $I \subset R$ be an ideal, $\varphi : M \to N$ be a map of $R$-modules. Then
• If $M/IM \to N/IN$ is surjective, then $\widehat{M} \to \widehat{N}$ is surjective. In particular, this holds for $M \to N$ surjective.
• If $0 \to K \to M \to N \to 0$ is exact and $N$ is flat, then $0 \to \widehat{K} \to \widehat{M} \to \widehat{N} \to 0$ is exact.
• $M \otimes_R \widehat{R} \to \widehat{M}$ is surjective for any finite $R$-module $M$.

**Proof:** Cf.[Sta]0315.

**Prop. (I.5.7.4).** Let $I$ be a f.g. ideal of $A$ and $M$ an $A$-algebra, then $\widehat{M}$ is $I$-adically complete and $I^n\widehat{M} = \text{Ker}(\widehat{M} \to M/I^nM) = (I^nM)^\wedge$.

**Proof:** Because $I$ is f.g., so does $I^n$. If $I^n = (f_1, \ldots, f_r)$. Applying(I.5.7.3) to $(f_1, \ldots, f_r): M^r \to I^nM$ shows

$$\widehat{M}^r \to (I^nM)^\wedge = \lim_{m\geq n} I^nM/I^mM = \text{Ker}(\widehat{M} \to M/I^nM)$$

but the image is clearly $I^n\widehat{M}$, so $\widehat{M}/I^n\widehat{M} \cong M/I^nM$. Taking inverse limit yields $(M^\wedge)^\wedge = M^\wedge$. □

**Cor. (I.5.7.5) (Completion is Complete).** Let $I$ be a f.g. ideal of $A$ and $(M_n)$ an inverse system of $A$-modules that $I^nM_n = 0$, then $M = \lim_n M_n$ is $I$-adically complete.

**Proof:** We have maps $M \to M/I^n \to M_n$, taking limit, we get $M \to \widehat{M} \to M$, so $M$ is a direct summand of $\widehat{M}$. Since $\widehat{M}$ is $I$-adically complete by(I.5.7.4), so does $M$. □

**Prop. (I.5.7.6).** If $I$ is a f.g. ideal of $A$ and $(M_n)$ is an inverse system of $A$-modules that $M_n = M_{n+1}/I^n M_{n+1}$, then $M = \lim_n M_n$ is $I$-adically complete and $M/I^nM = M_n$.

**Proof:** $\widehat{M}$ is $I$-adically complete by(I.5.7.5), and $M \to M_n$ are all surjective because the transition maps are surjective. Consider the inverse system $N_n = \text{Ker}(M \to M_n)$. Since $M_n = M_{n+1}/I^nM_{n+1}$, the map $N_{n+1} + I^nM \to N_n$ is surjective, and thus $N_{n+1}/(N_{n+1} \cap I^{n+1}M) \to N_n/(N_n \cap I^nM)$ is surjective.

Taking the inverse limit of the exact sequences

$$0 \to N_n/(N_n \cap I^nM) \to M/I^nM \to M_n \to 0,$$

we get an exact sequence

$$0 \to \lim_n N_n/(N_n \cap I^nM) \to \widehat{M} \to M.$$

As $M$ is $I$-adically complete, $\widehat{M} = M$, thus $\lim_n N_n/(N_n \cap I^nM) = 0$, thus $N_n/(N_n \cap I^nM) = 0$ for any $n$ as the transition maps are surjective. Then $M/I^nM = M_n$, as desired. □

**Cor. (I.5.7.7) (Spectrum Map of Completions).** Spec $R^\wedge \to$ Spec $R$ has image Spec $R/I \subset$ Spec $\widehat{R}$. This follows from(I.5.7.18) and $R/I \cong R^\wedge/I$.

**Prop. (I.5.7.8).** If $I$ is an ideal of $R$ and $0 \to M \to N \to Q \to 0$ is an exact sequence that $Q$ is annihilated by a power of $I$, then completion produces an exact sequence

$$0 \to \widehat{M} \to \widehat{N} \to Q \to 0$$

**Proof:** If $I^nQ = 0$, then $Q/I^nQ = Q$ cor $n \geq c$, and $I^nM \subset M \cap I^nN \subset I^{n-c}M$ because of this. Then $\widehat{M} = \lim M/(M \cap I^nN)$ by(II.1.1.28), and we apply(I.10.3.2) to the inverse system of exact sequences

$$0 \to M/(M \cap I^nN) \to N/I^nN \to Q \to 0$$

to conclude. □
Cor. (I.5.7.9). If \( A \) is a ring with a nonzero-divisor \( t \) and there is an exact sequence \( 0 \rightarrow M \rightarrow N \rightarrow Q \rightarrow 0 \) of \( A \)-modules that \( IQ = 0 \), then \( M \) is \( t \)-adically complete iff \( N \) is \( t \)-adically complete.

Proof: Use snake lemma. \( \square \)

Cor. (I.5.7.10). Take \( I = (f) \) and \( M = N = R \), then we see that if \( t \) is a nonzero-divisor in \( R \) then \( t \) is a nonzero-divisor in \( R \).

Prop. (I.5.7.11). The completion of a submodule \( N \subset M \) is the closure of \( \varphi(N) \) (By direct construction). The completion of \( M/N \) is \( M^*/N^* \) because it is right exact.

Cor. (I.5.7.12). If \( N \) is open in \( M \) then \( M/N \cong M^*/N^* \) because \( M/N \) is discrete hence complete.

Prop. (I.5.7.13). When \( A \) is Noetherian and \( M \) is finite \( A \)-module, then the natural map \( M \otimes_A A^* \rightarrow M^* \) is an isomorphism (use \( M \) is finite presentation and tensor & completion is right exact), and five lemma.

Cor. (I.5.7.14). When \( A \) is Noetherian, \( A \rightarrow A^\wedge \) is flat (because flatness is check for finite module).

And when \( A \) is complete Hausdorff, any finite module \( M \) is complete Hausdorff and hence any its submodule is complete thus closed in it. Hence the the completion of a submodule \( N \subset M \) is \( \varphi(N)A^* \) in \( M^* = MA^* \). In fact this implies complete Hausdorff adic-ring is Zariski.

Remark (I.5.7.15). WARNING: If \( A \) is not Noetherian, in general \( A \rightarrow A^\wedge \) is not flat, Cf.[Sta]0AL8.

Lemma (I.5.7.16). Let \( A \) be a ring and \( I = (f_1, \ldots, f_r) \) be a f.g. ideal. If \( \lim M/f^n_i M \) is surjective for each \( i \), then \( M \rightarrow \lim M/I^n M \) is also surjective.

Proof: Note that \( \lim M/I^n M = \lim M/(f^n_1, \ldots, f^n_r)M \), as \( I^n \subset (f^n_1, \ldots, f^n_r) \subset I^n \), and elements in \( \lim M/(f^n_1, \ldots, f^n_r)M \) can be written as an infinite sum \( \xi = \sum_n \sum_i f^n_i x_{n,i} \). There is an element \( x_i \) mapping to \( \sum_n f^n_i x_{n,i} \) for any \( i \), thus \( \sum_i x_i \) maps to \( \xi \). \( \square \)

Lemma (I.5.7.17). Let \( A \) be a ring and \( I \subset J \) be ideals, if \( M \) is \( J \)-adically complete and \( I \) is f.g., then \( M \) is \( I \)-adically complete.

Proof: It is clearly \( I \)-adically Hausdorff, and for completeness, by(I.5.7.16) it suffices to show for \( I = (f) \): Let \( x_n \in M \) with \( x_n - x_{n+1} \in f^n M \), then \( \{x_n\} \) is \( J \)-adically Cauchy, thus there is an element \( x \) that \( x - x_n \in J^n \), and we can replace \( x_n \) by \( x_n - x \) to assume \( x_n \in J^n \). Now we prove \( x_n \in (f^n) \): assume \( x_n - x_{n+1} = f^n z_n \), then

\[
x_n = f^n( z_n + f z_{n+1} + \ldots )
\]

This equation is true because it is \( J \)-adically Cauchy. \( \square \)

Properties of Complete Rings

Prop. (I.5.7.18). If \( A \) is \( I \)-adically complete, then \( I \subset \text{rad } A \).

Prop. (I.5.7.19). Let \( A \) be a ring with a non-zero-divisor \( t \), then any limit of \( t \)-adically complete algebras is \( t \)-adically complete.
Proof: Check the definition directly.

Prop. (I.5.7.20) (Zariski Rings). A noetherian $I$-adic ring is called Zariski ring if it satisfies the following equivalent conditions:

- Every finite module is Hausdorff in the $I$-adic topology.
- Every submodule in a finite module is closed in the $I$-adic topology.
- Every ideal is closed.
- $I \subset \text{rad} A$.
- $A^\wedge/A$ is f.f.

Hence every complete Hausdorff ring is Zariski.

Proof: 1 → 2: apply it to the submodule $M/N$.
3 → 4: If $I \not\subset m$, then $I^n + m = A$, thus $\overline{M} = A$, contradiction.
4 → 1: by intersection theorem (I.5.6.10).
4 → 5: for any maximal ideal $m$, $I \subset m$ so it is open, thus $A^*/mA^* = A/m \neq 0$ by (I.5.7.12) thus f.f. by (I.7.1.12).
5 → 1: by (I.7.1.13), for any $m$ maximal, there is a maximal ideal $m'$ lying over $m$, so $IA^* \subset m^*$ by (I.5.7.14), thus $I \subset m$, hence $I \subset \text{rad} A$.

Cor. (I.5.7.21). In a Zariski ring $A$, maximal ideals are open, thus $A/m \cong A^*/mA^*$ by (I.5.7.12), thus Spec $A^* \to \text{Spec} A$ is bijection on closed pt.

Prop. (I.5.7.22) (Cohen Structure Theorem). If $A$ is a complete local ring containing a field $k$ that the residue field is separably generated over $k$, then there is a field $K$ containing $k$ that is a Cohen ring, i.e. complete local ring with a prime number as a uniformizer, that has the same residue field as $A$.

Proof: 

Lemma (I.5.7.23) (Complete Interchanging Lemma). If $R$ is a commutative ring, $x, y \in R$, if $x$ is not a zero-divisor in $R$ and $R$ is $x$-adically complete, and $y$ is not a zero-divisor in $R/x$ and $R/x$ is $y$-adically complete, then the same is true with $x, y$ interchanged.

Proof: 

8 Dimension

Def. (I.5.8.1) (Dimensions and Heights). For a $A$-module $M$, $\dim(M)$ is defined as $\dim(A/\text{Ann}(M))$. The height of an ideal $I$ in $A$ is defined as the infimum of heights of the prime ideals over $I$.

The dimension of a ring $R$ is defined to be the supremum of heights of prime ideals of $R$.

Prop. (I.5.8.2). For any ring $A$ $\dim A = \sup \dim A_p$.

Def. (I.5.8.3) (Catenary Rings). A ring is called catenary if for any pair of primes $p \subset q$, any maximal chain of primes $p = p_0 \subset p_1 \subset \ldots \subset p_e = q$ has the same length. A noetherian ring is called universally catenary if all its f.g. algebra is catenary.
Prop. (I.5.8.4). Any quotient ring and localization of a (universally) catenary ring is catenary. Any quotient of a Noetherian universally catenary ring is universally catenary. Catenary and universally catenary are stalkwise properties (I.5.1.55).

Prop. (I.5.8.5). A Noetherian CM ring is universally catenary.

Proof: Cf.[Sta]00NM.

Prop. (I.5.8.6). Dedekind domain, e.g. field is universally catenary, so f.g. domain over fields is catenary.

Def. (I.5.8.7) (Hilbert Polynomials). Let $(A,\mathfrak{m})$ be a Noetherian local ring, $I$ an ideal of definition, then

**Dimension of Noetherian Local Rings**

Prop. (I.5.8.8). For a Noetherian local ring $R$, the following three numbers are equal:
- $\dim R$.
- $d(R)$.
- the minimal number of elements needed to generate an ideal of definition of $R$ (I.5.1.10).

Proof: Cf.[Sta]00KQ.

Cor. (I.5.8.9). The dimension of a Noetherian local ring is finite. Thus the codimension in any Noetherian scheme is finite.

Prop. (I.5.8.10). If $A$ is a Noetherian local ring with maximal ideal $\mathfrak{m}$, then $\dim A \leq \dim k \frac{\mathfrak{m}}{\mathfrak{m}^2}$.

Proof: Cf.[Matsumura P78].

**Dimension and Ring Extensions**

Prop. (I.5.8.11) (Dimension and Going-Up(Down)). If $A \to B$ is a ring map that Spec map is surjective and $A \to B$ satisfies either going-up or going-down, then $\dim B \geq \dim A$.

Proof: The hypothesis implies any chain of primes in $A$ can lifted to a chain of primes in $B$.

Prop. (I.5.8.12) (Dimensions and Noetherian Ring Extensions). Let $A \to B$ be a map between Noetherian rings, $P$ a prime ideal of $B$, $p = P \cap A$, then:
- $\text{ht}(P) \leq \text{ht}(p) + \text{ht}(P/pB)$ in other words $\dim(B_P) \leq \dim(A_p) + \dim(B_P/pB_P)$.
- equality holds if going-down holds. For example, if it is flat (I.5.3.12).

Proof: 1: Localize at $p$ and $P$, we may assume $A, B$ are local rings with maximal ideals $\mathfrak{p}$ and $P$, then use the characterization (I.5.8.8) of dimension, if $x_1, \ldots, x_d$ generate an ideal of definition of $A_\mathfrak{p}$ and $y_1, \ldots, y_e$ generate an ideal of definition of $B_P/pB_P$, then $x_1, \ldots, x_d, y_1, \ldots, y_e$ generate an ideal of definition of $B_P$.

2: If going down holds, for any chain of primes in $B_P$ containing $pB_P$, we can lift the chain of primes in $A_\mathfrak{p}$ to a chain of primes in $B_P$ to get a longer chain, thus we get the other direction of inequality.
Prop. (I.5.8.13) (Dimension of Integral Extensions). Let \( A \to B \) be an integral ring map, then:
1. Spec maps closed points to closed points, and \( \dim(A) \geq \dim(B) \), which equality if \( A \subset B \).
2. If \( A, B \) is Noetherian, \( \text{ht}(P) \leq \text{ht}(P \cap A) \)
3. If \( A, B \) is Noetherian and going down holds, then \( \text{ht}(J) = \text{ht}(J \cap A) \) for any ideal \( J \subset B \).

Proof: 1: By (I.5.5.5), there is no inclusion relation between prime over a fixed prime, so \( \dim(B) \leq \dim(A) \). On the other hand, if \( A \subset B \), then Spec map is surjective going-up holds (I.5.5.5), so \( \dim(B) \geq \dim(A) \) (I.5.8.11).
2: Follows from (I.5.8.12) since \( \text{ht}(P/(P \cap A)B) = 0 \) by (I.5.5.5).
3: In this case, \( \text{ht}(P) = \text{ht}(P \cap A) \) holds by (I.5.8.12)(2), then use the surjectiveness of Spec for the integral extension \( A/J \cap A \subset B/J \) (I.5.5.5).

Cor. (I.5.8.14). If \( A \to B \) is integral and faithfully flat, then \( \dim A = \dim B \).

Proof: This follows from (I.5.8.13) and (I.7.1.23).

Prop. (I.5.8.15) (Dimension and Completion). For a local ring \( A \), \( \dim A = \dim \hat{A} \).

Proof:

Noetherian Normalization

Prop. (I.5.8.16). For a Noetherian ring \( A \), \( \dim A[X] = \dim A + 1 \).

Proof: Let \( p \) be a prime ideal of \( A \) and let \( q \) be a prime ideal of \( A[X] \) maximal among primes lying over \( p \), then \( \text{ht}(q/pA[X]) = 1 \). In fact, by localizing, we can assume \( p \) is a maximal ideal, then \( A[x]/pA[x] \) is a polynomial ring over a field thus a PID and \( \text{ht}(q/pA[X]) = 1 \). Thus \( \text{ht}(q) = \text{ht}(p) + 1 \) by (I.5.8.12). Now we are done, because Spec \( A[X] \to \text{Spec} A \) is surjective.

Prop. (I.5.8.17) (Krull’s Height Theorem). In a Noetherian domain \( R \), the height of an ideal generated by \( n \) elements is at most \( n \).

Proof: Let \( p \) be a minimal ideal containing \( (f_1, \ldots, f_n) \), then it suffices to show \( \dim(R_p) \leq n \) In this case, \( (f_1, \ldots, f_n) \) is an ideal of definition of \( R_p \), thus we can use (I.5.8.8).

Lemma (I.5.8.18). Let \( A = k[X_1, \ldots, X_n] \) be a polynomial ring over a field \( k \), and \( I \) is an ideal of \( A \) of height \( r \), then we can choose \( Y_1, \ldots, Y_n \in A \) that \( A \) is integral over \( k[Y_1, \ldots, Y_n] \) and \( I \cap k[Y_1, \ldots, Y_n] = (Y_1, \ldots, Y_r) \).

Proof: We use induction on \( r \). \( r = 0 \) is easy. For \( r = 1 \), let \( f(X) \) be any non-zero polynomial in \( I \), then we can assign suitable integral weights \( d_1 = 1, d_2, \ldots, d_n \) to \( X_i \) that monomials of \( f \) have different weights. Put \( Y_1 = X_1 - X_1^{d_1} \) for \( i \geq 2 \), and

\[
Y_1 = f(X) = f(X_1, Y_2 + X_2^{d_2}, \ldots, Y_n + X_n^{d_n}) = a_1 X_1^{N} + g(X_1, Y_2, \ldots, Y_n)
\]

where \( g \) has degree in \( X_1 \) lower that \( N \). Then \( X_1 \) is integral over \( k[Y_1, \ldots, Y_n] \), and hence \( X_i = Y_1 + X_i^{d_i} \) is also integral over \( k[Y] \).

Now \( (Y_1) \) is a prime ideal in \( k[Y] \) of height 1 and \( (Y_1) \in I \cap k[Y] \). Also notice \( I \cap k[Y] \) has height 1 by (I.5.8.13)(Because going-down holds by (I.5.3.16)), so \( (Y_1) = I \cap k[Y] \).
For $r \geq 2$, let $J \subset I$ be an ideal with height $r - 1$, let $J$ be an ideal of $k[X]$ contained in $I$ that $ht(J) = r - 1$. (This is possible by choosing $f_i$ out of all minimal primes containing $(f_1, \ldots, f_{r-1})$ and use Krull’s Height Theorem.) By induction hypothesis, there exists $Z_1, \ldots, Z_n$ that $k[X]$ is integral over $k[Z]$, and $J \cap k[Z] = (Z_1, \ldots, Z_{r-1})$. Now $ht(I \cap k[Z]) = r$ by the same argument above, thus there exist $f \in I \cap k[Z] \setminus (Z_1, \ldots, Z_{r-1})$, and do the same for $r = 1$ again, we can find the desired $Y_i$ that $Y_k = Z_k$ for $k \leq r - 1$.

**Prop. (I.5.8.19) (Noetherian Normalization Theorem).** If $A$ is a f.g. algebra over a field, then there are $r$ alg. independent elements $y_i$ that $A$ is integral over $k[y_i]$.

**Proof:** Let $A = k[X_1, \ldots, X_n]/I$ and height $(I) = n - r$, then by the lemma we can choose $Y_1, \ldots, Y_n$ that $k[X_1, \ldots, X_n]$ is integral over $k[Y_1, \ldots, Y_n]$ (so $Y_1, \ldots, Y_n$ are algebraically independent) and $I \cap k[Y_1, \ldots, Y_n] = (Y_{r+1}, \ldots, Y_n)$. Now we can just choose $y_i = Y_i$ for $i \leq r$.

**Cor. (I.5.8.20) (Dimension and Transcendental Degree).** If $A$ is a f.g. integral ring over a field $k$, then $\dim A = \text{tr} \cdot \deg_k A$.

**Proof:** This is because integral extensions of integral Noetherian rings have the same dimensions (I.5.8.13) and their fraction fields have the same transcendental degrees.

**Cor. (I.5.8.21).** Let $A, B$ be f.g. algebras over a field $k$, then $\dim(A \otimes_k B) = \dim A + \dim B$.

**Cor. (I.5.8.22) (Dimension of Field Base Change).** Let $K/k$ be a field extension and $S$ a f.g. algebra over $k$, then $\dim S = \dim S \otimes_k K$.

**Proof:** By Noetherian normalization, there exists a finite injective map $k[d_1, \ldots, d_n] \to S$ where $n = \dim S$. Then there exists a finite injective map $K[d_1, \ldots, d_n] \to S_K$, so $\dim S \otimes_k K = n$, by(I.5.8.13) and(I.5.8.16).

**Prop. (I.5.8.23) (Codimensions and Field Base Change).** Let $K/k$ be a field extension and $S$ a f.g. $k$-algebra. Let $q$ be a prime of $S$ and $q_K$ be a prime of $S_K$ lying over $q$, then

$$\dim(S_K \otimes_k k(q))_{q_K} = \dim(S_K)_{q_K} - \dim S_q = \text{tr} \cdot \deg_k k(q) - \text{tr} \cdot \deg_K k(q_K).$$

Moreover, for any $q$, we can choose $q_K$ so that this number is 0.

**Proof:** Cf.[Sta]0CWE.

**Local Dimension over Fields**

**Prop. (I.5.8.24) (Local Dimension).** Let $S$ be an algebra f.g. over a field $k$, $X = \text{Spec} \ S$ and $x \in X$, then the following three numbers are equal:

- the local dimension (IX.1.14.23) $\dim_x(X)$.
- $\max \dim(Z)$ where $Z$ runs through irreducible components of $X$ passing through $x$.
- $\min \dim(S_m)$, where $m$ are maximal ideals containing $p_x$.

**Proof:** Cf.[Sta]00OT.

**Prop. (I.5.8.25).** Let $k$ be a field and $S$ a f.g. $k$-algebra, $X = \text{Spec} \ S$, $x \in X$, then

$$\dim_x(X) = \dim S_p + \text{tr} \cdot \deg_k k(p).$$
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Proof: Cf.[Sta]00P1.

Cor. (I.5.8.26). Let $S' \to S$ be a surjection of f.g. algebras over a field $k$, $\mathfrak{p}$ a prime ideal of $S$ and $\mathfrak{p}'$ its inverse image in $S'$, corresponding to $x, x'$ in $X = \text{Spec } S, X' = \text{Spec } S'$ resp., then

$$\dim_{x'}(X') - \dim_x(X) = \text{ht}(\mathfrak{p}') - \text{ht}(\mathfrak{p}).$$

Proof: This is immediate from (I.5.8.25).

Def. (I.5.8.27) (Relative Dimension). Let $R \to S$ be a ring map of f.t., and $\mathfrak{q} \subset S$ be a prime over $\mathfrak{p} \subset R$, then we define the relative dimension of $S/R$ at $\mathfrak{q}$ to be $\dim_{\mathfrak{q}}(\text{Spec } S)_\mathfrak{p}$. The supremum of all these numbers over $\mathfrak{q} \subset \text{Spec } S$ is called the relative dimension of $S/R$, denoted by $\dim(S/R)$.

Prop. (I.5.8.28) (Local Dimension and Field Extension). Let $K/k$ be a field extension, $S$ be a f.g. $k$-algebra, and $X = \text{Spec } S$. Now if $x_K$ is a element of $X_K$ lying over $x \in X$, then $\dim_x(X) = \dim_{x_K}(X_K)$.

Proof: Cf.[Sta]00P4.

Prop. (I.5.8.29) (Semicontinuity of Dimensions). Let $f : R \to S$ be a ring map of f.t., then the map $\mathfrak{q} \mapsto \dim_{\mathfrak{q}}(S/R)$ is a upper-semicontinuous function on $\text{Spec } (S)$.

Moreover, if $f$ is of f.p., then the set $\{\mathfrak{q} \mid \dim_{\mathfrak{q}}(S/R) \leq n\}$ is quasi-compact open in $\text{Spec } (S)$.

Proof: Cf.[Sta]00QH, 00QJ.

9 Jacobson Radical and Nilradical

Nilradical

Def. (I.5.9.1) (Nilradical). The nilradical of a commutative ring $R$ is defined to be the ideal of elements that is nilpotent.

Prop. (I.5.9.2). The nilradical of a ring (I.5.9.1) is the intersection of all prime ideals.

Proof: Every nilpotent element is contained in every prime, and if $a$ is not nilpotent, then the localization $A_a$ is nonzero, hence there is a maximal ideal, i.e. there is a prime of $A$ not containing $a$.

Jacobson Ring

Def. (I.5.9.3) (Jacobson Ring). A commutative ring is called Jacobson if every prime ideal is an intersection of maximal ideals. In particular, the Jacobson radical equals the nilradical. This is equivalent to every radical ideal is an intersection of maximal primes.

Prop. (I.5.9.4). $R$ is Jacobson iff $\text{Spec } R$ is Jacobson space (IX.1.14.19). In particular, the closed pts are dense in any closed subsets (Hilbert’s Nullstellensatz satisfied).

Proof: We need to show that a locally closed subset contains a closed pt, we assume this set is of the form $V(I) \cap D(f)$, $I$ is radical, then $f \notin I$, then by the condition, there is a $I \subset \mathfrak{m}$ that $f \notin \mathfrak{m}$, thus the result.

Conversely, for a radical ideal, let $J = \bigcap_{\mathfrak{I} \subset \mathfrak{m}} \mathfrak{m}$, then $J$ is radical and $V(J)$ is the closure of $V(I) \cap X_0$, $V(I) = V(J)$, and because they are both radical, $I = J$. 


Cor. (I.5.9.5). Being Jacobson is a local property, and quotient of Jacobson ring is Jacobson, and maximal ideals of $R_f$ are maximal in $R$. (Immediate from (I.5.9.4)(IX.1.14.20) and (IX.1.14.21)).

Prop. (I.5.9.6). If a Jacobson ring $A$ has f.m. maximal ideals, then it is the product of its localizations at maximal primes and $\dim A = 0$.

Proof: Any prime ideal $\mathfrak{p}$ is a finite intersection of maximal ideals, so it equals one of them, so $\dim A = 0$. Now $A/I = \bigoplus A/m_i$ by Chinese remainder theorem, so $\text{Spec } A/I$ is discrete with $n$ pts, so by (I.5.3.7) there are $n$ idempotents $e_i$ that $e_i \equiv \delta_{ij} \mod m_j$, $\sum e_i = 1$. Thus $R = \prod Re_i$. And $Re_i$ is just the localization at a maximal prime. □

Lemma (I.5.9.7). If $R$ is a Jacobson domain and $R \subset K$ where $K$ is a field, and $K$ is f.g. over $R$, then $R$ is a field and $K\triangleleft R$ is a finite field extension.

Proof: By induction, it suffices to consider the monogenic case $A = R[a]$. So $a$ is algebraic over quotient field of $R$ because $A$ is a field. Let $\sum r_i t^i$ be a polynomial satisfied by $a$, and let $\mathfrak{m}$ be a maximal ideal of $R$ that $r_n \not\in R$(exists because $\text{rad } R = 0$). Then Nakayama says $\mathfrak{m}A \not\subseteq A$. Then $\mathfrak{m} = 0$ because $A$ is a field, hence $R$ is a field. □

Lemma (I.5.9.8). Let $R \subset A$ be commutative domains s.t. $A$ is f.g. over $R$, then $\text{rad } A = 0$ if $\text{rad } R = 0$.

Proof: By induction, it suffices to consider the case $A = R[a]$. If $a$ is transcendental over quotient field of $R$, then we finish by (I.4.2.12). Now assume $a$ is algebraic over quotient field of $R$, let $\sum r_i t^i, \sum s_i t^i$ be the polynomials satisfied by $a, b$ of minimal degrees, then $s_0 = -\sum_{i=1}^m s_i b_i \neq 0 \in \text{rad } A$, and $r_n s_0 \neq 0$.

From the fact $\text{rad } R = 0$, we can find a maximal ideal $\mathfrak{m}$ that $r_n s_0 \not\in \mathfrak{m}$. Then Nakayama says

$$\mathfrak{m} \cdot S^{-1} A \not\subseteq S^{-1} A.$$  

In particular, $\mathfrak{A} \not\subseteq A$. Choose a maximal ideal of $A$ containing $\mathfrak{m}A$, then it cannot contain $s_0$, contradicting $s_0 \in \text{rad } A$. □

Prop. (I.5.9.9) (Generalized Nullstellensatz). If $R$ is Jacobson and $S$ is a finitely generated $R$-algebra, then:

- $S$ is Jacobson.
- The maximal ideal of $S$ intersect with $R$ a maximal ideal, and the quotient ring extension is finite, (in particular algebraic).

In particular, a f.g. algebra over a ring of dimension $0$, (e.g. Artinian ring or field) is Jacobson.

Proof: To show $S$ is Jacobson, consider for any prime $\mathfrak{p} \subset A$, $A/\mathfrak{p}$ is a f.g. domain over $R/\mathfrak{p} \cap R$. Because $R$ is Jacobson, $\text{rad}(R/\mathfrak{p} \cap R) = 0$, so $\text{rad}(A/\mathfrak{p}) = 0$, by (I.5.9.8). And this shows $A$ is Jacobson.

If $\mathfrak{m}$ is maximal in $S$, then $R/\mathfrak{m} \cap R \to S/\mathfrak{m}$ satisfies the condition of (I.5.9.7), by (I.5.9.5), so the first two assertions are proved. □
Zariski Pairs

Def. (I.5.9.10). A pair $(A, I)$ is called a Zariski pair iff $I$ is contained in the Jacobson radical of $A$.

Prop. (I.5.9.11). If $(A, I)$ is a Zariski pair, then the map $A \rightarrow A/I$ induces a bijection between the idempotents.

Proof: idempotents are determined by the maximal ideals that it vanishes(I.5.3.5), and $A \rightarrow A/I$ induces a bijection on the maximal ideals.

10 Dedekind Domain

Def. (I.5.10.1) (Dedekind Domain). A Dedekind domain is an integrally closed Noetherian domain of dimension 1. A UFD is a Dedekind domain by (I.6.5.6).

Prop. (I.5.10.2) (Equivalent Definitions of Dedekind Domain). For a domain $R$, the following are equivalent:

1. $R$ is a Dedekind domain.
2. $R$ is Noetherian and each $R_m$ is DVR for maximal ideal $m$.
3. each ideal of $R$ can be written as a product of prime ideals uniquely.

Proof: 1 $\iff$ 2 as normal is a stalkwise and (I.6.5.15).

3 $\Rightarrow$ 2: If 3 is true, then $p \neq p^2$ for each prime $p$, so choose $x \in p - p^2$, then for each $y \in p$, $(x, y) = \prod p_i$, then exactly one $p_i$ (may assume $p_1$) is contained in $p$, so $(x, y)R_p = p_1R_p$. Now in fact $(x)R_p = p_1R_p$, because $(x, y^2)R_p$ is also a prime, so $y = ax + by^2$ in $R_p$, $(1 - by)y = ax \in (x)R_p$ is a prime, so $y \in (x)R_p$. This is for all $y \in p$, so $(x)R_p = p_1R_p$.

Now if $(x) = p_1 \cdots p_r$, then $p_1R_{p_r} = pR_{p_r}$, so $p_1 = p$, and $p$ is f.g. by the lemma (I.5.10.3) below. $p$ is arbitrary, so $R$ is Noetherian and $R_m$ is DVR for $m$ maximal, by (I.6.5.15).

1 $\Rightarrow$ 3: if 1 is true, then any ideal is a unique intersection of primary ideals, and primary ideals are their radical are different, so they are coprime (I.5.16), so this is in fact a unique decomposition into products of primary ideals. And any primary ideal is a power of its radical, because this is the case after localization.

Lemma (I.5.10.3). $I, J$ be ideals in a ring $A$ and $IJ = (f)$ where $f$ is a non-zero-divisor, then $I, J$ are f.g. and finitely locally free of rank 1 as $A$-modules.

Proof: The second assertion implies the first, by (I.6.1.7). $f = \sum x_iy_i$, and $x_iy_i = a_if$, so $\sum a_i = 1$ as $f$ is non-zero-divisor. Now we show $I_{a_i}$ as $J_{a_i}$ is free of rank 1. Now after localization, $f = xy$, so $x, y$ are non zero divisors. Now if $x' \in I$, then $x'y = af = axy$ for some $a$, so $x' = ax$.

Prop. (I.5.10.4) (Extension of Dedekind Domain). If $A$ is a Noetherian domain of dimension 1 with fraction field $K$ and $L/K$ is a finite field extension, then the integral closure $B$ of $A$ in $L$ is a Dedekind domain, and Spec $B \rightarrow$ Spec $A$ is surjective, and have finite fibers and induces finite residue field extension.

Proof: Cf. [Sta]09IG.

Cor. (I.5.10.5). The integral closure of a Dedekind domain in a finite extension fields of its quotient fields is again a Dedekind domain.

In particular, the ring of integers in a number field is a Dedekind domain.
Cor. (I.5.10.6). The ring of integers in an algebraic number field \( K \) is a Dedekind domain.

Prop. (I.5.10.7) (Flat Module over a Dedekind Domain). If \( A \) is a Dedekind domain, then an \( A \)-module is flat iff it is torsion-free.

Proof: Because flatness and torsion-freeness is stalkwise (I.5.1.55), so it suffices to prove for its localization, which is DVR (I.5.10.2), so the result follows from (I.7.1.9).

Fractional Ideals

Def. (I.5.10.8). For \( A \) an integral domain and \( K \) its quotient field, then an \( A \)-module \( M \) in \( K \) is called a fractional ideal if \( xM \subset A \) for some \( x \neq 0 \).

Every f.g. submodule in \( K \) is a fractional ideal, and if \( A \) is Noetherian, then the converse is true, because it is of the form \( x^{-1}a \).

Prop. (I.5.10.9). An \( A \)-submodule \( M \) of \( K \) is called an invertible ideal if there is a submodule \( N \) that \( MN = A \). It follows that \( M, N \) are f.g., because there are \( \sum x_iy_i = 1 \), so \( M \) is generated by \( x_i \) and \( N \) is generated by \( y_i \).

Prop. (I.5.10.10). Invertibility is a stalkwise property.

Proof: Notice \( (A : M)_p = (A_p : M_p) \), and \( M \) is invertible iff \( M(A : M) = A \). Then use the fact isomorphism is stalkwise (I.5.1.55).

Prop. (I.5.10.11). A local domain is a DVR iff every non-zero fractional ideal of \( A \) is invertible.

Proof: If is a DVR, let \( m = (x) \), for any fractional ideal \( M \), let \( yM \subset A = (x^r) \), then \( M = (x^{r-s}) \), where \( v(y) = s \). Conversely, if every non-zero fractional ideal of \( A \) is invertible, then they are all f.g. (I.5.10.9), so \( A \) is Noetherian. Now it suffices to prove that every ideal of \( A \) is a power of \( m \), by (I.6.5.15). If this is not true, choose a maximal element \( a \) in the set of ideals that is not a power of \( m \) (by Noetherian), then \( m^{-1}a \subset m^{-1}m = A \), and \( m^{-1}a \supset a \), but it is not \( a \), so \( m^{-1}a = m^k \) for some \( k \), so \( a = m^{k+1} \), contradiction.

Cor. (I.5.10.12) (Dedekind Domain Fractional Ideals are Invertible). An integral domain is a Dedekind domain iff every non-zero fractional ideal of \( A \) is invertible.

Proof: Immediate from the proposition and (I.5.10.2) (I.5.10.10).

Def. (I.5.10.13) (Class Group). Let \( \mathcal{O} \) be a Dedekind domain, we denote \( \text{Cl}(\mathcal{O}) \) the Abelian group of fractional ideals of \( \mathcal{O} \) modulo principal fractional ideals, called the class group of \( \mathcal{O} \).

Prop. (I.5.10.14) (Class Group of UFD). The class group of a UFD is trivial.

Proof: Cf. [Sta]0BCH.
I.6  Commutative Algebra II

1  Projective

References are [Projective Modules] and [Sta].

Def. (I.6.1.1) (Projective Modules). A module $P$ over a ring $R$ is called projective iff $\text{Hom}(P, -)$ is exact, or equivalent, for any surjective map of modules $F \to Q \to 0$, $\text{Hom}(P, F) \to \text{Hom}(P, Q)$ is surjective.

Prop. (I.6.1.2). Localization and tensor product preserves projective because they are left adjoints.

And when tensoring f.f. map, then the converse is also true (I.7.2.1).

Prop. (I.6.1.3). A module over a ring is projective iff it is a direct summand of a free module, in particular, it is flat. Moreover, there is a free module $Q$ that $P \oplus Q = F$ free.

Proof: For the second assertion, we can choose an arbitrary $Q$ that $P \oplus Q$ free, and see $\bigoplus_{i \in \mathbb{N}} (P \oplus Q)$ is free. □

Lemma (I.6.1.4). A projective module is a direct sum of countably generated projective modules.

Proof: This follows from (I.2.4.26). □

Prop. (I.6.1.5) (Projective over Local Ring). A projective module $P$ over a local ring $R$ or a PID is free.

Proof: Local ring case: By (I.6.1.4) and (I.2.4.27), it suffices to show that any element $x$ of $P$ is contained in a free direct summand of $P$. Because $P$ is projective, it is a direct summand of a free module $F$, $F = P \oplus Q$. Let $B$ be a basis of $F$ that the number of basis element in the expression of $x$ is minimal. Let $x = \sum a_i e_i$. Then no $a_i$ is contained in the ideal generated by other $a_j$, otherwise we can choose another basis to show this is minimal. Let $e_i = y_i + c_i$ be decompositions into $P$ and $Q$ components, and write $y_i = \sum a_{ij} e_i + t_i$, where $t_i$ is combination of elements in $B$ other than $e_i$. Now it suffices to show $\det(a_{ij})$ is invertible, because in this way $\{y_i\} \cup (B \setminus \{x_1\})$ is a basis of $F$ and $x = \sum a_i y_i$ because $x \in P$. And $N = \text{span}\{y_i\}$ is a summand of $P$ because $N \subset P$ and both $N, P$ are summands of $F$.

To show $\det(a_{ij})$ is invertible, notice that by plugging $y_i = \sum b_{ij} e_j + t_i$ into $\sum a_{ij} e_i = \sum a_i y_i$ shows $a_j = \sum a_i a_{ij}$, thus by the argument before, $a_{ij}$ are non-invertible for $i \neq j$ and $1 - a_{ii}$ is non-invertible, so $a_{ii}$ is invertible. Because $R$ is a local ring, we can easily see $\det(a_{ij})$ is invertible. PID case: directly from (I.2.4.19). □

Prop. (I.6.1.6). If $R$ is a ring and $I$ is nilpotent ideal and $\mathcal{P}$ is a projective $R/I$-module, then there exists a projective $R$-module $P$ that $P/IP \cong \mathcal{P}$.

Proof: Cf.[Sta]P07LV. □

Finite Projective Modules

Prop. (I.6.1.7) (Finite Projective Modules). Let $M$ be a $R$-module, the following are equivalent:

1. $M$ is finite projective.
2. $M$ is f.p. and flat.
3. $M$ is f.p. and all its localizations at (maximal)primes are free.
4. $M$ is finite locally free.
5. $M$ is finite and locally free.
6. $M$ is finite and all its localizations at primes are free and the function $p \rightarrow \dim_{k(p)} M \otimes_R k(p)$ is a locally constant function on $\text{Spec } R$.

Proof:

1 $\rightarrow$ 2: $M \otimes K = R^m$ for some $K$ and $m$, so $K$ is finite and $M = R^m/K$ is f.p. And $M$ is flat because it is a summand of $R^n$ (I.7.1.4).

2 $\rightarrow$ 4: For any prime $p$, choose a basis for the $k(p)$-vector space $M \otimes k(p)$, then by Nakayama, their inverse image generate $M$ for some $g \in p(I.2.4.8)$, and the kernel $K$ of this generation is finite because $M_g$ is f.p. And $K \otimes k(p) = 0$ by the flatness of $M_g$. Then by Nakayama again there is a $g' \in p$ that $M_{gg'} = 0$ (I.2.4.8).

4 $\rightarrow$ 3: Because f.p. is local (I.5.1.57).

3 $\rightarrow$ 2: Because flatness is trivial.

4 $\rightarrow$ 5: Because finite is local (I.5.1.57).

5 $\rightarrow$ 4, 4 $\rightarrow$ 6: Trivial.

6 $\rightarrow$ 4: Cf.[Sta]00NX.

2 + 3 + 4 + 5 + 6 $\rightarrow$ 1: Cf.[Sta]00NX.

Consider the stalk, it is all free by (I.6.1.2) and (I.6.1.5), thus by (V.3.1.20), it is locally free. □

Cor. (I.6.1.8) (Partially Stalkwise). If $P$ is fo f.p., then finite projectiveness is a stalkwise property for $P$.

Cor. (I.6.1.9) (Projective and Flat). A finite module over a Noetherian ring is projective iff it is flat.

Cor. (I.6.1.10). If $M$ is finite projective, then the canonical map $\text{Hom}(M, N) \otimes L \rightarrow \text{Hom}(M, N \otimes L)$ is an isomorphism.

Proof: By proposition above $M$ is f.p. and finite locally free, so by (I.6.7.7) and tensor commutes with localization, we can check locally, where $M$ is finite free so the isomorphism is obvious. □

Def. (I.6.1.11) (Characteristic Polynomials for Finite Projective Modules). Let $M$ be a finite projective module over a ring $A$, then we can define a characteristic polynomial in $A[X]$ for any map of $A$-modules $M \rightarrow M$: if $M$ is free, then this map is defined as usual. In general, we can find an open covering $\text{Spec } A_{f_i}$ of $\text{Spec } A$ that $M_{f_i}$ is free over $A_{f_i}$. Thus we can define the characteristic polynomial locally and glue them together to get a characteristic polynomial in $A[X]$.

In particular, we can also define trace and norm of a $A$-module map $M \rightarrow M$. And when $B$ is a locally free $A$-algebra, then there are trace and norm maps $\text{tr} : B \rightarrow A$ and $\text{Nm} : B \rightarrow A$.

Prop. (I.6.1.12). Let $R$ be a ring and $I$ be a locally nilpotent ideal. If $\overline{P}$ is a finite projective module over $R/I$, then there exists a finite projective $R$-module $P$ that $P/I P \cong \overline{P}$.

Proof: Cf.[Sta]0D47. □

Prop. (I.6.1.13). Let $M$ be a $R$ module and $I$ a nilpotent ideal of $R$. If $M/IM$ is a projective $R/I$-module and $M$ is flat over $R$, then $M$ is a projective $R$-module.

Proof: Cf.[Sta]05CG. □
Duality of Projective Modules

Prop. (I.6.1.14) (Basis Criterion of Projectiveness). An $A$-module $P$ is projective iff there are elements $x_i$ in $P$ and $f_i$ in $P^*$ that for any $x$, $f_i(x) = 0$ a.e. $i$, and $\sum f_i(x_i) = x$. Moreover, $P$ is finite projective iff there are f.m. of them.

Proof: If $P$ is projective, as a summand of a free module, then we can choose the coordinates of the inclusion map as $f_i$, and choose the image of the quotient map of the coordinate as $x_i$. The converse is verbatim. □

Cor. (I.6.1.15) (Finite Projective Duality). If $P$ is projective, then $P \rightarrow P^{**}$ is injective, and if $P$ is finite projective, then it is an isomorphism.

Proof: If $f(x) = 0$ for all $f \in P^*$, then the proposition says $x = 0$. And if $P$ is finite projective, it can be seen $x_i, f_i$ forms a "basis" of $P^*$ (finiteness used), so $f_i$ generate $P^*$, and similarly $x_i$ generate $P^*$, so $P \rightarrow P^{**}$ is surjective. □

Cor. (I.6.1.16). If $P$ is projective over $R$, then $P^* \neq 0$.

Cor. (I.6.1.17). In the meanwhile of the proof, we already get: if $P$ is finite projective, then $P^*$ is finite projective, by (I.6.1.14).

Cor. (I.6.1.18). If $P$ is finite projective, the the map $P \otimes M \rightarrow \text{Hom}(P^*, M)$ is an isomorphism.

Proof: In (I.6.1.10), let $N = R$ and let $M = P^*$, then use the fact $P \cong P^{**}$. □

Prop. (I.6.1.19). Any finite projective module over $K[X_1, \ldots, X_k]$ is free. (Highly nontrivial).

Proof: □

Prop. (I.6.1.20). $\prod \mathbb{N} \mathbb{Z}$ is not free thus not projective over $\mathbb{Z}$ (I.6.1.5). And

$$\text{Hom}(\prod \mathbb{Z}, \mathbb{Z}) = \bigoplus \mathbb{Z}.$$ 


2 Injective

Prop. (I.6.2.1) (Baer’s Criterion). A right $R$-module $I$ is injective iff for every right ideal $J$ of $R$, every map $J \rightarrow I$ can be extended to a map $R \rightarrow I$. (Directly from (I.11.5.6)).

Cor. (I.6.2.2). A module over a PID is injective iff it is divisible.

Cor. (I.6.2.3). $A$ is injective iff $\text{Ext}^1(R/I, A) = 0$ for every ideal $I$ of $R$.

Cor. (I.6.2.4) (Rank). Let $M$ a finite projective $R$-module, then $M$ is said to have rank $n$ if $M/mM$ is of rank $n$ over the field $R/m$ for any arbitrary maximal ideal $m$ of $R$.

Prop. (I.6.2.5). The category of $R$-mod has enough injectives by (I.11.2.31), and it has enough projectives trivially.
Prop. (I.6.2.6). If $I$ is an injective $A$-module, then for any ideal $\alpha$ of $A$, $\Gamma_\alpha(I) = \{m|\alpha^n m = 0\}$ for some $n$ is injective.

Proof: Use Baer criterion, for any ideal $b$ of $A$, it is f.g. so there is a $n$ that $\phi(\alpha^n b) = 0$, and Artin-Rees tells us that $\phi(\alpha^N \cap b) = 0$ for some $N$. So we have an extension of $\phi$ over $b/b \cap \alpha^N$ to $A/\alpha^N \rightarrow I$, and this obviously factor through $\Gamma_\alpha(I)$, so it is done. □

Prop. (I.6.2.7). For an injective module $A$-module $I$, $I \rightarrow I$ if is surjective.

Proof: we have the sheaf of modules $\tilde{I}$ is flabby (V.6.5.7), thus the map to the stalk is surjective. □

Pontryagin Duality

Basic references are [Weibel Homological Algebra].

Def. (I.6.2.8). The Pontryagin dual $M^\vee$ of a left $R$-module $M$ is the right $R$-module $\text{Hom}_{ab}(M, \mathbb{Q}/\mathbb{Z})$, where $(fr)(b) = f(rb)$.

It is easily verified that if $A \neq 0$, then $A^\vee \neq 0$, and $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, thus the Pontryagin dual is faithfully exact.

Prop. (I.6.2.9). $M$ is flat $R$-module iff $M^\vee$ is an injective right $R$-module (Because $\text{Hom}(\cdot, \mathbb{Q}/\mathbb{Z})$ is exact).

3 Homological Dimension

Def. (I.6.3.1). For a $R$-mod $A$, the projective dimension $\text{pd}(A)$ is the minimal length of a projective resolution of $A$. The injective dimension $\text{id}(A)$ is the minimal length of a injective resolution of $A$. The flat dimension $\text{fd}(A)$ is the minimal length of a flat resolution of $A$.

Prop. (I.6.3.2). If $R$ is Noetherian, then $\text{fd}(A) = \text{pd}(A)$ for every f.g. module $A$.

Proof: Use(I.6.3.3), we see that if we choose a syzygy and look at the $n$-th term, then it is f.p and flat, so we have it is projective by(I.7.1.10). □

Lemma (I.6.3.3) (pd). If $\text{Ext}^{d+1}(A, B) = 0$ for every $B$, then for every resolution

$$0 \rightarrow M \rightarrow P_{d-1} \rightarrow \ldots, P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$$

where $P_k$ is projective, then $M$ is projective. Hence we have $\text{pd}(A) \leq d$. (Use dimension shifting, the following two are the same).

Lemma (I.6.3.4) (id). If $\text{Ext}^{d+1}(A, B) = 0$ for every $A$, then for every resolution

$$0 \rightarrow B \rightarrow P_0 \rightarrow \ldots, P_{n-1} \rightarrow M \rightarrow 0$$

where $P_k$ is injectives, then $M$ is injective. Hence we have $\text{id}(B) \leq d$

Lemma (I.6.3.5) (fd). If $\text{Tor}_{d+1}(A, B) = 0$ for every $B$, then for every resolution

$$0 \rightarrow M \rightarrow F_{d-1} \rightarrow \ldots, F_1 \rightarrow F_0 \rightarrow A \rightarrow 0$$

where $F_k$ is flat, then $M$ is flat. Hence we have $\text{fd}(A) \leq d$
Prop. (I.6.3.6) (Global Dimension Theorem). The following are the same for any ring $R$ and called the left global dimension of $R$:
1. $\sup\{id(B)\}$
2. $\sup\{pd(A)\}$
3. $\sup\{pd(R/I)\}$
4. $\sup\{d : \text{Ext}^d_R(A,B) \neq 0\}$ for some module $A,B$.

Proof: This follows from (I.6.3.3), (I.6.3.4) and (I.6.2.3). □

Prop. (I.6.3.7). A $Z$ has global dimension 1 because injective is equivalent to divisible, and this shows that a quotient of an injective is injective.

Prop. (I.6.3.8) (Tor Dimension Theorem). The following are the same for any ring $R$ and called the Tor dimension of $R$:
1. $\sup\{fd(A)\}$ for $A$ a left module.
2. $\sup\{fd(B)\}$ for $B$ a right module.
3. $\sup\{pd(R/I)\}$ for $I$ a left ideal.
4. $\sup\{pd(R/J)\}$ for $J$ a right ideal.
5. $\sup\{d : \text{Tor}^d_R(A,B) \neq 0\}$ for some module $A,B$.

Proof: This follows from (I.6.3.5) applied to $R$ and $R^{op}$ and also (I.7.1.2). □

Prop. (I.6.3.9) (Change of Rings). Let $S \rightarrow R$ be a ring map, let $A$ be a $R$-mod, then we have $pd_S(A) \leq pd_R(A) + pd_S(R)$.

Proof: Use the Cartan-Eilenberg resolution and the total complex has length $pd_R(A) + pd_S(R)$. □

4 Depth

Regular sequences

Def. (I.6.4.1) (Regular Sequences). If $R$ is a commutative ring and $M$ is an $R$-module, then a sequence $(f_1, \ldots, f_n)$ of elements of $R$ is called a $M$-regular sequence if $f_k$ is a nonzero-divisor of $M/(f_1, \ldots, f_{k-1})$ and $M/(f_1, \ldots, f_n) \neq 0$. If $M = R$, then it is simply called a regular sequence.

Prop. (I.6.4.2). If $R$ is a local ring and $(f_1, \ldots, f_n)$ is a $M$-regular sequence, then a permutation of this sequence is also an $M$-regular sequence.

Proof: By transposition of adjacent indices, we can assume $n = 2$. Now $(x,y)$ is an $M$-regular sequence iff $M \otimes^L_R R/x$ is discrete and $M/x \otimes^L_R R/y$ is discrete, by (I.10.1.3). □

Prop. (I.6.4.3). If $(f_1, \ldots, f_n)$ is a $M$-regular sequence, then $(f_1^{e_1}, \ldots, f_n^{e_n})$ is a regular sequence for any integers $e_i \geq 1$.

Proof: Cf.[Sta]07DV?. □
Depth

Prop. (I.6.4.4) (Rees). For a f.g. module $M$ and $IM \neq M$,

$$\text{depth}_I(M) = \min \{ i | \text{Ext}^i_A(A/I, M) \neq 0 \} = \min \{ i | \text{Ext}^i_A(N, M) \neq 0 \}$$

where $\text{depth}_I(M)$ is the length of the maximal $M$-regular sequence in $I$, $N$ is a finite $A$-module with $\text{Supp}(N) \subset V(I)$.

Proof: If no elements of $I$ are $M$-regular, then $i \subset \cup \text{Ass}(M)$ thus in one of them, so $\text{Hom}_{A_p}(k, M_p) \neq 0$, and we have $N_p/PN_p = N \otimes_A k_p$ nonzero by Nakayama, thus $\text{Hom}_k(N \otimes_A k_p, k_p) \neq 0$, thus $\text{Hom}_{A_p}(N_p, M_p) = (\text{Hom}_A(N, M))_p \neq 0$, so $\text{Ext}^0_A(N, M) \neq 0$. Other dimensions follow by induction, consider the cokernel of $M \xrightarrow{a_1} M$.

Conversely, use induction, then we have an injection $\text{Ext}^i_A(N, M) \xrightarrow{a_1} \text{Ext}^i_A(N, M)$ for $i < n$. And the condition shows that $I \subset \sqrt{\text{Ann}(M)}$, so $a_1^N = 0$, thus the result. \hfill \square

Cor. (I.6.4.5). Two maximal regular sequence in a f.g. module have the same length.

Cor. (I.6.4.6). For a module $M$ over a Noetherian ring $A$, we know $\Gamma_I(M) = \{ m | I^n m = 0 \text{ for some } n \}$, and $H^n_I$ is its right derived functor, then we have $\text{depth}_I(M) \geq n \iff H^n_I(M) = 0$ for $i < n$. (Because derived functor commutes with colimits, consider $N = A/I^k$).

Lemma (I.6.4.7) (Ischebeck). For a Noetherian local ring $A$, if $M, N$ are finite modules, then we have $\text{Ext}^i_A(N, M) = 0$ for $i < \text{depth}(M) - \text{dim} N$.

Proof: \hfill \square

Prop. (I.6.4.8). Let $A$ be a local ring and $M$ is finite $A$-module, then $\text{depth}(M) \leq \dim A/P \leq \dim M$ for every $P \in \text{Ass}(M)$. (Because $\text{Hom}(A/P, M) \neq 0$.)

Proof: \hfill \square

Prop. (I.6.4.9) (Auslander-Buchsbaum Formula). For a local ring $R$, if $M$ is a finitely generated $R$-mod, if $pd(M) < \infty$, then we have $\text{depth}(R) = \text{depth}(M) + \text{pd}(M)$.

Proof: Cf.[Weibel P109]. \hfill \square

Cohen-Macaulay

Def. (I.6.4.10) (Cohen-Macaulay Rings). For a Noetherian local, a f.g. $A$-module $M$ is called Cohen-Macaulay if $\text{depth}(M) = \dim M$. In view of(I.6.4.8), this is equivalence to $\text{depth}(M) = \dim A/P$ for all $P \in \text{Ass}(M)$.

A localization of a C.M local ring is C.M, so we call a ring Cohen-Macaulay if all its localization at primes are C.M.

Prop. (I.6.4.11) (Gorenstein Ring). A ring $R$ is called Gorenstein iff $id_R R < \infty$. A Gorenstein local ring is C.M. In this case, $\text{depth}(R) = id_R R = \dim R$, and $\text{Ext}^0_R(R/m, R) \neq 0 \iff q = \dim R$.

Proof: Cf.[Weibel P107]. \hfill \square

Prop. (I.6.4.12). A ring is C.M. iff for all ideals, the associated primes of $A/I$ all have the same height as $I$, i.e. unmixed.
Proof:

Prop. (I.6.4.13). If a local ring is C.M. and \( I = (x_1, \ldots, x_r) \) is a regular sequence, then there is an isomorphism \( (A/I)[t_1, \ldots, t_r] \to \text{gr}_I A = \oplus I^n/I^{n+1} \). In particular, \( I/I^2 \) is a free \( A/I \) module.

Proof:

Prop. (I.6.4.14). Let \( A \) be a Noetherian local ring and \( M \) a f.g. module, if a set of elements \( (x_1, \ldots, x_r) \) forms a regular sequence for \( M \), then \( \dim M/(x_1, \ldots, x_r) = \dim M - r \). The converse is also true when \( A \) is C.M. If this is the case, then \( A/(x_1, \ldots, x_r) \) is also C.M.

Proof: By (I.5.6.15), we have \(< \), for the converse, \( \text{Supp}(M/fM) = \text{Supp}(M) \cap \text{Supp}(A/fA) = \text{Supp}(M) \cap V(f) \), and when \( f \) is \( M \)-regular, \( V(f) \) doesn’t contain any \( \text{Ass}(M) \) thus no minimal elements of \( \text{Supp}(M) \), so \( \dim(M/fM) < \dim M \), thus we have \( > \).

When \( A \) is C.M.: ?

Prop. (I.6.4.15). Let \( R \to S \) be a local homomorphism of Noetherian local rings, if \( R \) is C.M. and \( S \) is finite flat over \( R \) or \( S \) is flat over \( R \) and \( \dim S \leq \dim R \), then \( S \) is C.M., and \( \dim R = \dim S \).

Proof: Cf. [Sta]00R5.

5 Normal Ring & Regular Local Ring

Normal Ring

Def. (I.6.5.1) (Normal Rings). A normal domain is a domain and is integrally closed in its fraction field.

A domain is normal iff all its localizations are normal, so we can define a normal ring to be a ring that all its stalks are normal local rings. In particular, a normal ring is reduced.

Proof: The localization of a normal domain is normal, and the converse follows from \( A = \cap A_m(I.5.1.28) \).

Prop. (I.6.5.2). A normal ring \( R \) is integrally closed in its ring of fractions.

Proof: Let \( x \in Q(R) \) be integral over \( R \), and \( I = \{ f \in R \mid fx \in R \} \), then for any prime \( p \) of \( R \), \( R \to R_p \) is injective, so \( R_p \subset Q(R) \otimes_R R_p \), and \( x \otimes 1 \) is integral over \( R_p \), thus \( x \otimes 1 \subset R_p \), which means \( x \otimes 1 = 1 \otimes a/f \) for some \( a, f \in R, f \notin p \). This means \( f'(fx - a) = 0 \in Q(R) \) for some \( f' \notin p \), so \( f f' \in I \), and thus \( I \) is not contained in any prime ideal, so \( I = 1 \) and \( x \in R \).

Prop. (I.6.5.3). Let \( R \) be a reduced ring with f.m. minimal prime ideals, then the following are equivalent:

- \( R \) is a normal ring.
- \( R \) is integrally closed in its ring of fractions.
- \( R \) is a finite product of normal domains.

In particular, a Noetherian normal domain is a finite product of normal domains (I.5.1.40).
Proof:

\(3 \to 1\) is trivial, \(1 \to 2\) is by (I.6.5.2), for \(2 \to 3\): let \(p_i\) be the minimal prime ideals of \(R\), then \(\text{Frac}(R) = \prod_{i=1}^{n} Q_{p_i}\) (I.5.3.21), with each \(Q_{p_i}\) field because \(R\) is reduced. Denote the idempotents of \(\text{Frac}(R)\) by \(e_i\). Then \(e_i\) is integral thus in \(R\). These idempotents make \(R\) into a product of domains, which are just \(R/p_i\), because the kernel of the map \(R \to R_{p_i}\) is \(p_i\). Now \(R\) is integrally closed in \(\text{Frac}(R)\) implies each \(R/p_i\) is integrally closed in \(R_{p_i}\), thus \(R\) is a finite product of normal domains.

\[\square\]

Def. (I.6.5.4) (Normalization). The normalization of an integral domain is the alg.closure of it in its quotient field. It commutes with localization.

Def. (I.6.5.5). A domain is called completely normal iff all almost normal elements are in \(A\), i.e. \(\{u \mid 3a, au^n \in A \forall n\} \subseteq A\). For Noetherian ring, completely normal is equivalent to normal.

Proof: Cf.[Matsumura P124].

Prop. (I.6.5.6) (UFD is Normal). A UFD is a normal domain.

Proof:

If \(x\) is integral over \(R\), check the indices of each prime ideal of \(x\) is non-negative.

Prop. (I.6.5.7). \(A\) is a normal domain, then so does \(A[X]\). If \(A\) is Noetherian normal domain, then so does \(A[[X]]\).

Proof: Cf.[Sta]030A, 0BI0.

Prop. (I.6.5.8). Direct limits of normal rings are normal.

Proof:

Let \(p\) be an ideal of \(R = \varinjlim R_i\), \(p = p_i \cap R_i\), then \(R_p = \varprojlim (R_i)_{p_i}\), so it suffices to prove for normal domains, the rest is easy.

Prop. (I.6.5.9). Principal ideals in a Noetherian normal domain is unmixed and \(A = \bigcap_{ht \ p=1} A_p\).

Proof: Cf.[Matsumura P124].

Prop. (I.6.5.10) (Krull). If \(A\) is a normal domain with fraction field \(K\), \(L\) is a normal extension field of \(K\) and \(B\) is the integral closure of \(A\) in \(L\) that \(A \subseteq B\) is integral, then any two prime ideals of \(B\) lying over a prime in \(A\) are conjugate by an action of \(G_{L/K}\).

Proof:

Firstly if \(G_{L/K}\) is finite, and \(P, P'\) be primes of \(B\) that \(P \cap A = P' \cap A\). Let \(P_i = \sigma_i(P)\). If \(P' \neq P_i\) for any \(i\), then \(P' \nsubseteq P_i\) for any \(i\) by (I.5.5.5), so there is some \(x \in P'\) not in any of \(P_i\).

Now let \(y = (\prod \sigma_i(x))^q\) where \(q = 1\) if \(\text{char}(K) = 0\) and \(q = p^r\) for \(r\) large if \(\text{char}(K) = p\), then \(y \in K\) in \(A\) because \(A\) is normal, and it is not in \(P\) by hypothesis. But \(y \in P' \cap A = P \cap A\), contradiction.

For the infinite field extension case, let \(K'\) be the fixed field of \(G_{L/K}\) so that \(K'/K\) is purely inseparable, then there is clearly exactly one prime of \(K'\) over any prime of \(A\). So we can assume \(L/K\) is Galois, then use profinite group technique to find a \(\sigma \in G_{L/K}\) that \(\sigma(P) = P'\).

\[\square\]

Prop. (I.6.5.11) (Hironaka). Let \(A\) be a local Noetherian domain that is a localization of an algebra of f.t. over a field \(k\). Let \(t \in A\) that

- \(t\) generates the maximal ideal of \(A_p\).
- \(A/p\) is normal.
Then $p = tA$ and $A$ is normal.

Proof: Cf.[Hartshorne P264]. \hfill $\Box$

Prop. (I.6.5.12) (Quadratic Normal Ring). If $K$ is a field of char $\neq 2$ and $f \in K[X_1, \ldots, X_n]$ is a polynomial with no square factors, then $K[X_1, \ldots, X_n, Z]/(Z^2 - f)$ is integrally closed.

Proof: It is an integral domain because $K[X_1, \ldots, X_n, Z]$ is UFD thus it suffices to show $Z^2 - f$ is irreducible, and this is easy. Now we see the quotient field is just $K(X_1, \ldots, X_n)[Z]/(Z^2 - f)$. It is Galois extension of degree 2, for any element $g + hZ$, its minimal polynomial is $X^2 - 2gX + (g - hf)$, thus it is integral over $K[X_1, \ldots, X_n, Z]$ iff $g, h$ are both polynomials(char $\neq 2$ and $f$ has no square factors used). \hfill $\Box$

Regular Ring

Def. (I.6.5.13) (Regular Rings). A Noetherian local ring $A$ with residue field $k$ and maximal ideal $m$ is called regular iff it rank$_k m/m^2 = \dim A$. This is equivalent to gr $A \cong k[X_1, \ldots, X_d]$ by (I.5.6.15).

Localization of a regular local ring at primes are regular local. Hence we can call a Noetherian ring regular iff all its localization at primes are regular local.

Proof: Cf.[Matsumura P139]. \hfill $\Box$

Prop. (I.6.5.14). If $A$ is regular, then $A[X_1, \ldots, X_n]$ is regular, and $A[[X_1, \ldots, X_n]]$ is regular.

Proof: Cf.[Matsumura P176]. \hfill $\Box$

Prop. (I.6.5.15). A regular local ring of dimension 1 is a DVR.

Prop. (I.6.5.16) (Auslander-Buchsbaum). A regular local ring is UFD. In particular it is a normal domain. Thus a regular ring is normal and thus reduced (I.6.5.1).

Proof: Cf.[Matsumura P142],[Weibel P106]. \hfill $\Box$

Prop. (I.6.5.17). A regular local ring is Gorenstein hence C.M.

Proof: \hfill $\Box$

Prop. (I.6.5.18). If a quotient of a Noetherian local ring by a non-zerodivisor is regular, then it is itself regular.

Prop. (I.6.5.19) (Serre). A Noetherian local ring $A$ is regular iff the global dimension of $A$ is finite.

Proof: Cf.[Mat P139]. \hfill $\Box$

Prop. (I.6.5.20). For $A$ a regular local ring and $M$ a f.g. $A$-module,

$$pd(M) + \text{depth } M = \dim A.$$ 

Cf.[Hartshorne P237].

Cor. (I.6.5.21). For a f.g. module $M$ over a regular local ring $A$, $pd(M) \leq n$ iff Ext$^i(M, A) = 0$ for all $i > n$. 

Proof: This is because we can use dimension shifting to show \( \text{Ext}^i(M, N) = 0 \) for all \( N \) f.g., then (I.6.3.3) says that \( \text{pd}(M) \leq n \).

Prop. (I.6.5.22). Let \( R \to S \) be a flat local homomorphism of Noetherian local rings that \( R \) is regular, \( S/\mathfrak{m}_S \) is regular, then \( S \) is also regular.

Proof: Cf.[Sta]031E.

Serre Conditions \( R_k \) & \( S_k \)

Def. (I.6.5.23). A ring is called \( R_k \) iff for all prime \( p \) of height \( \leq k \), \( A_p \) is regular.

A ring is called \( S_k \) iff depth\( (A_p) \geq \min(k, \text{ht}(p)) \) for all prime \( p \).

A module \( M \) is called \( S_k \) iff depth\( (M_p) \geq \min(k, \dim \text{Supp} M_p) \) for all prime \( p \).

Prop. (I.6.5.24).

- \( M \) is \( S_1 \) iff \( M \) has no associated embedded primes. Cf.[Sta]031Q.
- A Noetherian ring is reduced iff it is \( R_0 \) and \( S_1 \). Cf.[Sta]031R.
- (Serre Criterion) A Noetherian ring is normal iff it is \( R_1 \) and \( S_2 \). Cf.[Sta]031S.
- A ring is C.M. iff it is \( S_N \).

Proof: □

Cor. (I.6.5.25) (Regular and Normal). A regular ring is normal, and normal ring is regular in codimension 1.

Proof: By(I.6.5.24), it suffices to prove that a regular ring satisfies \( R_1 \) and \( S_2 \). A regular ring is C.M.(I.6.5.17) so it is \( S_2 \) by(I.6.5.24), it is \( R_1 \) by(I.6.5.13) □

Cor. (I.6.5.26) (Normal and Regular Dimension 1). A Noetherian local ring of dim 1 is normal iff it is regular. i.e. integral domain and integrally closed iff maximal ideal is principal.

6 Geometric Properties

Def. (I.6.6.1).

- A \( k \)-algebra \( S \) is called geometrically reduced/integral/connected... over a field \( k \) iff for any field extension \( k'/k \), \( S_{k'} \) is reduced/integral/connected... .
- A Noetherian \( k \)-algebra \( S \) is called geometrically regular iff for any f.g. field extension \( K/k \), \( S_K \) is regular (Notice \( A \otimes_k k' \) is Noetherian(I.5.1.38), so this makes sense).

Prop. (I.6.6.2) (Geometrically reduced). If \( S \) is a \( k \)-algebra, the following are equivalent.

1. \( S \) is geometrically reduced.
2. \( S \otimes_k \overline{k} \) is reduced.
3. \( S \otimes_k k^{\text{per}} \) is reduced.
4. \( S \otimes_k k' \) is reduced for any finite purely inseparable field extension \( k'/k \).
5. \( S \otimes_k k^{1/p} \) is reduced.
6. residue fields of \( S \) at maximal points are reduced.
7. $S \otimes_k R$ is reduced for every reduced $k$-algebra $R$.

\textit{Proof:} \hspace{1em} 1 \to 7: \hspace{1em} We can assume $R$ is f.g., thus $R$ is contained in a finite product of fields $\text{Cf.} \ [\text{Sta}]030V$, and then we can assume $R$ is a product of fields, and we are done. 

\hspace{1em} 1 \to 2 \to 3 \to 4 \text{ is clear. } \hspace{1em} 3 \to 5 \text{ is clear.} 

\hspace{1em} 4 \to 1: \hspace{1em} \text{For any field extension } K/k, \text{ we can assume WLOG } K/k \text{ is f.g., thus } \text{?} 

\hspace{1em} 5 \to 1: \hspace{1em} ? 6: \hspace{1em} \text{Cf.} \ [\text{Sta}]030V \text{ and } \text{[Gortz 135].} \square

\textbf{Prop. (I.6.6.3) (Geometrically Irreducible).} \hspace{1em} \text{Let } S \text{ be a } k\text{-algebra, the following are equivalent:}

\hspace{1em} 1. \hspace{1em} S \text{ is geometrically irreducible.}

\hspace{1em} 2. \hspace{1em} \text{For any finite separable field extension } k'/k, \text{ the spectrum of } S_{k'} \text{ is irreducible.}

\hspace{1em} 3. \hspace{1em} \text{The spectrum of } S_{\kappa} \text{ is irreducible.}

\hspace{1em} 4. \hspace{1em} \text{The spectrum of } S_K \text{ is irreducible.}

\textit{Proof:} \hspace{1em} \text{Cf.} \ [\text{Sta}]037K. \square

\textbf{Prop. (I.6.6.4).} \hspace{1em} \text{Let } S \text{ be a geometrically irreducible } k\text{-algebra and } R \text{ Is a } k\text{-algebra, then the map}

$$\text{Spec}(R \otimes_k S) \to \text{Spec } R$$

\text{induces a bijection on irreducible components.}

\textit{Proof:} \hspace{1em} \text{Cf.} \ [\text{Sta}]037O. \square

\textbf{Prop. (I.6.6.5) (Geometrically Integral).} \hspace{1em} \text{Let } S \text{ be a } k\text{-algebra, the following are equivalent:}

\hspace{1em} 1. \hspace{1em} S \text{ is geometrically integral.}

\hspace{1em} 2. \hspace{1em} \text{For any finite separable field extension } k'/k, \text{ } S_{k'} \text{ is an integral domain.}

\hspace{1em} 3. \hspace{1em} S_K \text{ is an integral domain.}

\hspace{1em} 4. \hspace{1em} S \otimes_k R \text{ is an integral domain for any integral domain } R \text{ over } k.

\textit{Proof:} \hspace{1em} \text{This follows from(1.6.6.3)(1.6.6.2) and(1.6.6.4).} \square

\textbf{Prop. (I.6.6.6).} \hspace{1em} \text{It suffices to check geometrically regular for } k'/k \text{ finite purely inseparable.}

\textit{Proof:} \hspace{1em} \text{Cf.} \ [\text{Sta}]0381. \square

\section{7 Finitely Presented}

\textbf{Finite Presented Module}

\textbf{Def. (I.6.7.1) (Finitely Presented Modules).} \hspace{1em} \text{A \textit{finitely presented module} is a module of the form } R^m/R^n.

\text{Finite presentation is stable under base change because tensoring is right exact.}

\textbf{Prop. (I.6.7.2).} \hspace{1em} \text{Given an exact sequence of } R\text{-modules } 0 \to M_1 \to M_2 \to M_3 \to 0,

\hspace{1em} \text{• \hspace{1em} If } M_1, M_3 \text{ are f.p., then so does } M_2.

\hspace{1em} \text{• \hspace{1em} If } M_3 \text{ is f.p. and } M_2 \text{ is f.g., then } M_1 \text{ is f.g.}

\hspace{1em} \text{• \hspace{1em} If } M_2 \text{ is f.p. and } M_1 \text{ is f.g., then } M_3 \text{ is f.p.}
This can be seen by considering all finite submodules and f.m relations between them.

Proof: 1: e can find an commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & R^m \rightarrow R^{m+n} \rightarrow R^m \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0
\end{array}
\]

and use the snake lemma to see the kernel is f.g..

2: Use the diagram

\[
\begin{array}{ccc}
R^m & \rightarrow & R^n \rightarrow M \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0
\end{array}
\]

\[\alpha\]

cokernel of \(\alpha\) are all finite, so \(M_1\) is finite.

3: Choose a presentation \(R^m \rightarrow R^n \rightarrow M_2\) and a surjection \(f : R^k \rightarrow M_1\), then we can lift \(f\) to \(R^k \rightarrow R^n\), and then \(M_3\) can be written as a quotient \(R^{m+k} \rightarrow R^n \rightarrow M_3\). \(\square\)

Cor. (I.6.7.3). A direct summand of a f.p. module is f.p..

Prop. (I.6.7.4). If \(R \rightarrow S\) is a f.g. ring map and a \(S\)-module \(M\) is f.p. over \(R\), then it is f.p. over \(S\).

Proof: Let \(S = R[x_1, \ldots, x_n]\), and \(M = R[y_1, \ldots, y_m]/(\sum a_{ij}y_j), 1 \leq i \leq t\), then as \(M\) is a \(S\)-module, we let \(x_iy_j = \sum a_{ijk}y_k\), and forms a quotient \(S^{m+n+t} \rightarrow S^m \rightarrow N \rightarrow 0\), where \(S^{m+n+t}\) corresponds to the relations \(\sum a_{ij}y_j\) and \(x_iy_j - \sum a_{ijk}y_k\). Then there is a surjective \(A\)-module map \(N \rightarrow M\), and we check it is injective: if \(z = \sum b_jy_j\) are mapped to 0, where \(b_j \in S\), then we can transform \(z\) into the shape \(\sum c_jy_j\), where \(c_j \in R\) by relations \(x_iy_j - \sum a_{ijk}y_k\). Thus it is zero by definition. \(\square\)

Prop. (I.6.7.5) (Direct Limits of F.P. Modules). Any module is a direct limit of f.p. modules. This can be seen by considering all finite submodules and f.m relations between them.

Prop. (I.6.7.6) (Characterizing Finite and F.P. Modules). Let \(N\) be an \(R\)-module, then

- \(N\) is finite \(R\)-module iff for any filtered colimits \(M = \varinjlim_i M_i\) of \(R\)-modules, the map \(\varinjlim \text{Hom}(N, M_i) \rightarrow \text{Hom}(N, \varinjlim M_i)\) is injective.

- \(N\) is a f.p. \(R\)-module iff for any filtered colimits \(M = \varinjlim_i M_i\) of \(R\)-modules, the map \(\varinjlim \text{Hom}(N, M_i) \rightarrow \text{Hom}(N, \varinjlim M_i)\) is a bijection.

Proof: 1: If \(N\) is generated by \(x_i\) and a map \(f : N \rightarrow M_j\) maps to \(0 \in \text{Hom}(N, \varinjlim M_i)\), then there is a \(j\) that \(f : M_i \rightarrow M_j \rightarrow 0\). Thus \(f = 0\). Conversely, \(N\) is the sum of its f.g. submodules \(N'\), thus \(N \rightarrow \varinjlim N/N_i = 0\), which implies the identity map \(N \rightarrow N\) vanishes for some \(N/N'\) where \(N'\) is a finite submodule of \(N\), so \(N = N'\) and \(N\) is a finite.

2: If \(M\) is f.p., we can get the assertion by writing \(M\) as a quotient of free modules and use the fact filtered colimit is exact (I.5.1.20). Conversely, write \(N\) as a filtered colimits of f.p. modules (I.6.7.5), then \(\text{id} : M \rightarrow M\) factors through some f.p. module, so it is a direct summand of a f.p. module, thus f.p. by (I.6.7.3). \(\square\)

Cor. (I.6.7.7) (FP and Localization). For \(M\) f.p., \(S^{-1}\text{Hom}_R(M, N) = \text{Hom}_{S^{-1}R}((S^{-1}M, S^{-1}N))\) for any \(R\)-module \(N\). (Use the presentation and Hom is left exact).

Proof: \(\text{Hom}_{S^{-1}R}((S^{-1}M, S^{-1}N)) \cong \text{Hom}_R(M, S^{-1}N)\) by duality, and then we can use (I.6.7.6), as localization is a filtered colimit. \(\square\)
Finitely Presented Ring Map

Def. (I.6.7.8) (Finitely Presented Ring Map). A ring map is called **finite presentation** if it is a quotient of a free algebra by a free algebra.

Prop. (I.6.7.9). Finite presentation is stable under composition (choose a presentation form to see) and base change because tensoring is right exact.

It is local on the source and target by (I.5.1.57).

Prop. (I.6.7.10). For S f.g. over R, then the kernel of any surjective ring map \( R[X_1, \ldots, X_n] \to S \) is f.g.

Proof: Let \( S = R[Y_1, \ldots, Y_m]/(f_1, \ldots, f_k) \), then if \( \alpha(X_i) \cong g_i(Y) \), then \( \alpha : R[X_1, \ldots, X_n] \to R[X_1, \ldots, X_m, Y_1, \ldots, Y_m]/(f_1, \ldots, f_k, X_i - g_i) \). And the \( Y_i \) are in the image, thus we let \( Y_i \) are mapped onto by \( h_j(X) \), then \( \text{Ker} \alpha = (f_i(h_j(X)), X_i - g_i(X)) \).

□

Prop. (I.6.7.11). If \( g \circ f : R \to S' \to S \) is of finite presentation and \( f \) is of finite type, then \( g \) is of finite presentation.

Proof: Let \( S' = R[y_1, \ldots, y_n] \) and \( S = R[X_1, \ldots, X_n]/(f_1, \ldots, f_m) \), then let \( h_i(X) \cong y_i \) in \( S \), then \( S = S'[X_1, \ldots, X_n]/(f_1, \ldots, f_m, h_i - y_i) \).

□

Prop. (I.6.7.12). If \( S \) is f.p. over \( R \) that \( S \) has a presentation \( S = R[X_1, \ldots, X_n]/I \) that \( I/I^2 \) is free over \( S \), then \( S \) has a presentation \( R[X_1, \ldots, X_m]/(f_1, \ldots, f_c) \) that \( (f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2 \) is freely generated by \( f_1, \ldots, f_c \).

Proof: Cf. [Sta]07CF.

□

Prop. (I.6.7.13) (Finite Type Locally of Finite Presentation). If \( R \to S \) is a injective map of f.t. of domains, then there are \( f \neq 0 \in R, g \neq 0 \in S \) that \( R_f \to S_{fg} \) is of f.p.

Proof: Use induction on the number of generators of \( S/R \). If \( S = R[x] \), then \( S = R[X]/q \). If \( q = 0 \), then \( S \) is of f.p.. If \( g = f x^d + a_d-1 x^{d-1} + \ldots + a_0 \) be a polynomial of minimal degree in \( q \), then \( R \to S_f \) is of f.p.

The more generator case can be reduced to the single generator case, because f.p. ring map is stable under composition (I.6.7.9).

Lemma (I.6.7.14) (Filtered Colimits and F.P.). Let \( R \to A \) be a ring map, then the category of f.p. \( R \)-algebras \( A' \) with an \( R \)-algebra map \( A' \to A \) is filtered, and the colimit is just \( A \).

Proof: Cf. [0BUF].

□

8 Japanese & Nagata Rings

Def. (I.6.8.1). Let \( R \) be a domain with quotient field \( K \), then \( R \) is called N-1 iff the integral closure of \( R \) in \( K \) is a finite \( R \)-module.

\( R \) is called N-2 or Japanese iff for any finite field extension \( L/K \), its integral closure in \( L \) is a finite \( R \)-module.

A ring \( R \) is called **universally Japanese** if for any finite type domain \( S/R, S \) is Japanese.

A ring \( R \) is called **Nagata** if it is Noetherian and for any prime \( p, R/p \) is Japanese.

Prop. (I.6.8.2). A f.g. algebra \( A \) over a field is Nagata.

Proof: Cf. [Hartshorne P20].

□

Cor. (I.6.8.3). The normalization of a f.g. integral domain over a field is f.g. over \( A \).
9 Separability

Main references are [Matsumura Ch10], [Weibel Chap P309] and [Sta]10.41, 10.43.

Def. (I.6.9.1) (Separable Algebra). A f.d semisimple algebra \( R \) over a field \( k \) is called separable iff for every field extension \( K/k \), \( R \otimes_k K \) is semisimple.

Def. (I.6.9.2). A field extension \( K/k \) is called separably generated iff it \( K \) is a separable algebraic extension of a purely transcendental field \( L/k \).

A field extension \( K/k \) is called separable iff all f.g. subextensions are separably generated.

An algebra \( A/k \) is called separable iff \( A \otimes_k k' \) is reduce for any \( k'/k \) algebraic.

Prop. (I.6.9.3). If \( k \subset K \) is a f.g. field extension, then there is a finite purely inseparable field extension \( k \subset k' \) that \( k' \subset K \) is separable.

Prop. (I.6.9.4) (Separable and Geo.Reduced). Let \( K/k \) be a field extension, then \( K/k \) is separable iff \( K/k \) is geometrically reduced.

Proof: Cf.[Sta]030W. \( \square \)

Cor. (I.6.9.5). If \( K/k \) is a separable field extension and \( S \) is a reduced \( k \)-algebra, then \( S \otimes_k K \) is reduced.

Proof: Cf.[Sta]030U. \( \square \)

Cor. (I.6.9.6). A separably generated field extension is separable.

10 Henselian Ring

Main References are [Sta]Chap10.148.

Def. (I.6.10.1). A local ring \( (R, m, k) \) is called Henselian iff for every \( f \in R[X] \) and \( a_0 \in k \) that \( \overline{f}(a_0) = 0 \) and \( \overline{f}'(a_0) \neq 0 \), then there is a root \( \alpha \) of \( f \) lifting \( a_0 \). It is called strict Henselian if moreover its residue field is separably closed.

Henselian Pairs

Def. (I.6.10.2). A Henselian pair is a pair \( (A, I) \) that is Zariski and for any \( f, g \) in \( A[T] \) monic and \( \overline{f} = \overline{g}h \in A/I[T] \) that is coprime and monic, there is a factorization \( f = gh \) lifting the decomposition.

Prop. (I.6.10.3). Filtered limits of Henselian pairs is Henselian, this is clear from the definition(I.6.10.2).

Lemma (I.6.10.4). If \( A \) is a ring with ideal \( I \), if \( \overline{f} = \overline{g}h \) be a factorization of a polynomial \( f \in A[T] \) in \( A/I[T] \), then there is an étale ring map \( A \to A' \) that \( A/IA \cong A'/IA' \), and a factorization \( f = g'h' \in A'[T] \) lifting the factorization.

Proof: Cf.[Sta]0ALH? \( \square \)

Prop. (I.6.10.5) (Topological Invariance of Étale Sites). If \( I \) is locally nilpotent, then \( (A, I) \) is Henselian, in particularly \( A_{\text{ét}} \cong (A/I)_{\text{ét}} \)(I.6.10.9).
Proof: First if \( A \to S \) is étale, then \( A/I \to S/IS \) is étale by base change(I.7.7.5) and the map is essentially surjective by(I.7.7.15). And any map \( B/IB \to B'/IB' \) can be lifted to \( B \to B' \) because étale is smooth and use(I.7.5.18). And any map \( B/IB \to B' \) can be lifted to \( B \to B' \) because étale is smooth and use(I.7.5.18). And any map \( B/IB \to B' \) can be lifted to \( B \to B' \) because étale is smooth and use(I.7.5.18).

For then Henselian, \( I \) is clearly contained in the Jacobson radical, and for the decomposition, by(I.6.10.4) there is an étale map \( A \to A' \) that \( A/IA \cong A'/IA' \) that lifts the factorization, but \( A = A' \), by what we have seen above.

\[ \square \]

Cor. (I.6.10.6) (Complete Pair is Henselian). If \( (A, I) \) is a pair that \( A \) is \( I \)-adically complete, then \( (A, I) \) is Henselian.

\[ \text{Proof: } I \text{ is in the Jacobson radical because } 1 + I \text{ consists of units, and by(I.6.10.5) and(I.6.10.4) we can lift the decomposition to } A/I^n \text{ inductively. As } A = \lim A/I^n, \text{ we are done.} \]

Prop. (I.6.10.7) (Equivalent Definitions of Henselian Pair). The following are all equivalent to \( (A, I) \) being Henselian:

- Given any étale ring map \( A \to A' \), then any \( A' \to A/I \) lifts to an \( A \)-algebra map \( A' \to A \).
- For any finite/integral \( A \)-algebra \( B \), the map \( B \to B/IB \) induces a bijection on idempotents.
- \((\text{Gabber})\) \( (A, I) \) is Zariski and every monic polynomial \( f(T) \in A[T] \) of the form \( T^n(T - 1) + a_nT^n + \ldots + a_1T + a_0 \) with \( a_i \in I \) has a root \( \alpha \in 1 + I \).

Moreover, root in item3 is unique.

\[ \text{Proof: } \text{Cf.} [\text{Sta}]09XI]. \]

Cor. (I.6.10.8). if \( (A, I) \) is Henselian and \( A \to B \) is integral, then \( (B, IB) \) is also Henselian.

\[ \text{Prop. (I.6.10.9) (Henselian Lifting). If } (A, I) \text{ is a Henselian pair, then there is a natural equivalence of categories: } A_{\text{et}} \cong (A/I)_{\text{et}}. \]

\[ \text{Proof: } \text{Cf.} [\text{Sta}]09ZL]. \]

\[ \text{Prop. (I.6.10.10). A Zariski pair } (R, I) \text{ is Henselian iff the pair } (Z \oplus I, I) \text{ is Henselian. In particular, the property of being Henselian only depends on the non-unital ring } I. \]

\[ \text{Proof: } \text{Cf. [Almost Ring Theory, 5.1.9].} \]

11 Dualizing Module
I.7 Commutative Algebra III

1 Flatness

Def. (I.7.1.1) (Flatness). A module $M$ over a ring $R$ is called flat if $M \otimes_R -$ is an exact functor. This is compatible with the definition (V.2.2.12).

Prop. (I.7.1.2). Flatness need only be checked for finite modules, and it is equivalent to $\text{Tor}_1(M, A/I) = 0$ for any f.g. ideal $I$ (i.e $I \otimes M \to M$ is injective). This is because of (V.6.3.6) and the fact tensor product commutes with colimit.

Cor. (I.7.1.3). If $0 \to M' \to M \to M'' \to 0$, then $M'$ and $M''$ flat implies $M$ is flat.

Prop. (I.7.1.4). If $M$ is flat then $\text{Tor}_i^A(M, N) = 0$ for all $i > 0$, because we have: if $0 \to M_1 \to M_2 \to M_3 \to 0 M_2, M_3$ flat, then $M_1$ is flat (Use 9 entry sequence and the fact that Tor is symmetric (I.10.1.5)). So $\text{Tor}_{n+1}(M_3, N) = \text{Tor}_n(M_1, N) = 0$ by induction.

And a direct summand of a flat module is flat. Thus we have the class of flat modules is adapted $- \otimes N$ for all $N$ (because free is flat).

Prop. (I.7.1.5) (Flatness and Base Change).

- (Faithfully) Flatness is stable under base change.
- If $R \to S$ is f.f., then $M$ is flat iff its base change is flat.
- Flatness is stable under filtered colimit because filtered colimit commutes with tensoring (I.2.4.13) and is exact (I.5.1.20). In particular, $S^{-1}A$ is flat (I.5.1.22).
- If $R \to S$, and a $S$-module is $R$-flat and $S$-f.f., then $R \to S$ is flat.

Proof: Use definition and tensor trick. □

Prop. (I.7.1.6) (Equational Criterion of Flatness). For a $R$ module $M$, a relation $\sum f_i x_i = 0$ of elements of $M$ are called trivial iff $x_i = \sum a_{ij} y_j$ and $0 = \sum f_i a_{ij}$ for any $j$. Then $M$ is flat iff all relations of elements of $M$ is trivial.

Proof: Cf. [Sta]00HK. □

Prop. (I.7.1.7) (Gororov-Lazard). Any flat $A$-module is isomorphic to a direct limit of free modules of finite type.

Proof: ? □

Prop. (I.7.1.8) (Flat over Local Rings). A finite module $M$ over a local ring $A$ is flat iff it is free. In particular, finite modules over a field are all flat.

Proof: Let $A/\mathfrak{m} = k$, choose a $k$-basis $x_i$ of $M/\mathfrak{m}M$, then they generate $M$ by Nakayama. It suffices to prove that $x_i$ are independent over $R$. For this, use equational criterion of flatness (I.7.1.6), we prove that if $x_i$ is independent over $k$, then they are independent over $A$. Use induction, if $x \neq 0$ in $M/\mathfrak{m}$, if $fx = 0$ for some $f \in A$, then $x = \sum a_{ij} y_j$ that $fa_j = 0$, but then some $a_j$ is a unit, so $f = 0$.

If $\sum f_i x_i = 0$, then by hypothesis, $f_i \in \mathfrak{m}$, and there are $y_j$ that $x_i = \sum a_{ij} y_j$, $\sum f_i a_{ij} = 0$. As $x_n \not\in \mathfrak{m}M$, there is a $a_{nj} \not\in \mathfrak{m}$, so $f_n = \sum (-a_{ij}/a_{nj}) f_i$, then $\sum_i f_i (x_i - a_{ij}/a_{nj} x_n) = 0$, but $x_i - a_{ij}/a_{nj} x_n$ is also independent over $k$, so by induction, $f_i = 0$, also does $f_n$, so we are done. □
Prop. (I.7.1.9) (Flat over Valuation Ring Locally Free). A module over a valuation ring is flat iff it is torsion free, because valuation ring is Bezout (I.9.2.8) and use (1.2.3.8).

Prop. (I.7.1.10) (Finite Flat and Projective). Finitely presented flat module is equivalent to finite projective. (Immediate from (I.6.1.7)).

Prop. (I.7.1.11). If $M$ is a flat $R$-module, then $IM \cap JM = (I \cap J)M$ for ideals of $A$.

Proof: Tensoring the exact sequence $0 \to I \cap J \to I \oplus J \to J \cup J \to 0$ with $M$. □

Prop. (I.7.1.12) (Faithfully Flat). The following are equivalent:

- $M$ is f.f.
- $M$ is flat and for any $N \neq 0$, $N \otimes M \neq 0$.
- $M$ is flat and for any (maximal) prime ideal $m$ of $A$, $k_m \otimes_R M \neq 0$. (When $m$ is maximal, this means $mM \neq M$).

Proof: $3 \to 2$: any nonzero module has a submodule $A/I$, and thus $(A/I) \otimes_A M = M/IM \neq 0$. $2 \to 1$: Let $S$ be a complex, if $S \otimes M$ is exact, then $H^*(S) \otimes M = H^*(S \otimes M)$ by flatness, thus $H^*(S) = 0$, so $S$ is exact.

Cor. (I.7.1.14). Integral flat injective ring extension is f.f., by (I.5.5.5).

Cor. (I.7.1.15). Flat local ring map of local rings is f.f.

Cor. (I.7.1.16). Filtered colimits of f.f. rings over $R$ is f.f.

Proof: If is flat by (I.7.1.5), and for a maximal ideal $m$ of $R$, $S_i/mS_i$ is non-zero, hence there direct limit is non-zero because 1 is contained. So $m$ is in the image, hence it is f.f. by (I.7.1.13). □

Cor. (I.7.1.17) (Filtered Colimit of Flat Ring Maps). If $I$ is filtered and $R_i \to S_i$ are (faithfully)flat ring maps, then $\text{colim}_I R_i \to \text{colim}_I S_i$ is (faithfully)flat.

Proof: For any colim $R_i$-module $M$, $(\text{colim}_I S_i) \otimes_{\text{colim}_I R_i} M = \text{colim}(S_i \otimes_{R_i} M)$, so it is flat, because colim is exact. For the faithfully flatness, for any maximal ideal $m$ of colim $R_i$, let $m_i = m \cap R_i$, then $S_i/m_iS_i \neq 0$, thus the direct limit is also $\neq 0$, so $m$ is in the image, hence it is f.f. by (I.7.1.13). □

Prop. (I.7.1.18) (Flatness is Local). Flatness is stalkwise both on the target and source, thus flatness is local both on the target and the source (I.5.1.55).

Cor. (I.7.1.19) (Going-down). Going-down holds for flat ring map.
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Proof: The ring map $R_p' \to S_{q'}$ is flat by (I.7.1.18), thus it is f.f. by (I.7.1.15). Then (I.7.1.13) says $p \subset p'$ is in the image. □

Prop. (I.7.1.20). If rings $A \subset B \subset C$ and $C/A, C/B$ is flat, then $B/A$ is flat.

Proof: Cf. [GAGA Serre P26]. □

Prop. (I.7.1.21). If $R \to S$ is (faithfully)flat ring map and $M$ is a (faithfully)flat $S$-module, then $M$ is a (faithfully)flat $R$-module. In particular, (faithfully) flatness is stable under composition. Also (faithfully) Flatness is stable under base change.

Prop. (I.7.1.22). If $B$ is flat over $A$, then

$$\text{Tor}^A_i(M, N) \otimes B = \text{Tor}^B_i(M_{(B)}, N_{(B)}), \quad \text{Ext}^A_i(M, N) \otimes B = \text{Ext}^B_i(M_{(B)}, N_{(B)}).$$

Prop. (I.7.1.23) (Faithfully Flat Ring Map is Injective). A f.f. ring map $R \to S$ is universally injective. In particular, tensoring with $R/I$, we get $R \cap IS = I$ for an ideal $I$ of $R$.

Proof: Because $R \to S$ is f.f., we only need to show that $N \otimes_R S \to N \otimes_R S \otimes_R S$ is injective for any $N$, but this is true because it has a left inverse. □

Prop. (I.7.1.24). A f.f. map between valuation rings is equivalent to an injective local homomorphism.

Prop. (I.7.1.25). A flat ring map maps a non-zero-divisor to a non-zero-divisor, because if we consider the principal ideal generated by it, then (I.7.1.2) shows the ideal in $M$ is also injective, so it is not a zero-divisor.

Prop. (I.7.1.26) (Noetherian Completion is Flat). If $A$ is Noetherian and $I$ is an ideal, the the $I$-adic completion $\hat{A}$ is flat over $A$ by (I.5.7.14).

Prop. (I.7.1.27). The Spec map of a ring map $R \to S$ of f.p. that satisfies going-down(e.g. flat), is open.

Proof: $S \to S_f$ satisfies going-down and is of f.p, so we see that $R \to S_f$ satisfies going down. It suffice to prove the image of this map is open. By Chevalley, the image is constructible, and it is stable under specialization. So it is closed by (IX.1.15.8). □

Prop. (I.7.1.28). The Spec map $f$ of a f.f. ring map is submersive.

Proof: For a $T$ that $f^{-1}(T)$ is closed, we see that $f^{-1}(T) \to T$ satisfies going-down because $f$ does, so its complement is closed under specialization, so it is closed by (I.5.3.13). So $T$ is open. □

2 Faithfully Flat Descent

Prop. (I.7.2.1) (Faithfully Flat Descent). List of properties that descent through faithfully flat morphism.

1. Projectiveness for modules over a ring.
2. Finiteness for modules over a ring.
3. F.p. for modules over a ring.
4. Flatness for modules over a ring.
5. Mittag-Leffler for modules over a ring.
6. F.g. for ring maps.
7. F.p. for ring maps.
8. (Formal) Smoothness for ring maps, both on target and source.
10. Reducedness for rings.
12. Regular for rings.
13. Being Noetherian and has property $(R_k)$ for rings.

Proof:
5. Cf. [Sta]05A5.
6. Cf. [Sta]00QP.
7. Cf. [Sta]00QQ.
8. Use criterion (I.7.5.4), we see by flatness that the sequence $I/I^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$ commutes with flat base change, and when it is f.f., then use (I.7.3.6) and descent for projectiveness (I.7.2.1) that $\Omega_{S/R}$ is projective, so it is a split exact sequence. The smooth case follows from definition (I.7.5.14) as f.p. can descend.
9. Because for $S \to S'$ faithfully flat and a chain of ideals $I_k$ in $S$, $I_k S' = I_k \otimes_S S'$, and $I_k S'$ is stable if $S'$ is Noetherian, so also $I_k$ is stable because it is faithfully flat.
10. Trivial as $S \to S'$ is f.f. hence injective (I.7.1.23).
11. Cf. [Sta]033G.
12. Cf. [Sta]07NG.

□

Prop. (I.7.2.2) (fpqc-Poincare Lemma). If a ring map $A \to B$, either has a section $B \to A$, or it is faithfully flat, then the Amitsur complex $s(M)$ for the canonical descent datum (with augmentation):

$$0 \to M \to B \otimes_A M \to B \otimes_A B \otimes_A M \to \ldots$$

with Čech-like maps, is exact.

Proof: In the case $A \to B$ has a section $s$, It suffices to construct a nullhomotopy of the case $M = A$. Then we can just let $h(e_0 \otimes e_1 \otimes \ldots \otimes e_r) = s(e_0) e_1 \otimes \ldots \otimes e_r$. The f.f. case can be reduced to the first case by tensoring $B$ to consider $B \to B \otimes_A B$, because it has a section. □
Cor. (I.7.2.3) (Glueing Functions). Let $R$ be a commutative ring, $M$ a $R$-module, and $(f_1, \ldots, f_n) = (1)$, then there is an exact sequence

$$0 \to M \to \prod_i M_{f_i} \to \prod_{i,j} M_{f_if_j}.$$  

In particular this holds for $M = R$.

Proof: This is just (I.7.2.2) applied to $A \to \prod_i A_{f_i}$, which is faithfully flat.

Formal Glueing of Modules

Main references are [Sta] Chap15.80 and 15.81.

Lemma (I.7.2.4). Let $R \to S$ be a ring map and $I = (f_1, \ldots, f_r) \subset R$ be an ideal, then for any $R$-module $M$ we can define a complex

$$0 \to M \xrightarrow{\alpha} M \otimes_R S \times \prod_i M_{f_i} \xrightarrow{\beta} \prod_i (M \otimes_R S)_{f_i} \times \prod_i M_{f_if_j}$$

where $\alpha(m) = (m \otimes 1, m, \ldots, m)$, $\beta(m', m_1, \ldots, m_t) = (m' - m_1 \otimes 1, m' - m_2 \otimes 1, \ldots, m' - m_t \otimes 1, m_1 - m_2, \ldots, m_{t-1} - m_t)$.

Assume that $R \to S$ is flat and $R/I \to S/IS$ is an isomorphism, then this complex is exact.

Proof: Cf. [Sta] 05EK.

Def. (I.7.2.5) (Category of Gluing Data). Let $R \to S$ be a ring map and $I = (f_1, \ldots, f_r) \subset R$ be an ideal, then we define the category $\text{Glue}(R \to S, f_1, \ldots, f_r)$ of gluing data $\text{Glue}(R \to S, f_1, \ldots, f_1)$ consisting of objects $M = (M', M_i, \alpha_i, \alpha_{ij})$ where $M'$ is a $S$-module, $M_i$ are $R_{f_i}$-modules, $\alpha_i : (M')_{f_i} \to M_i \otimes_R S$ and $\alpha_{ij} : (M_i)_{f_j} \to (M_j)_{f_i}$ are isomorphisms that

- $\alpha_{ij} \circ \alpha_i = (\alpha_j)_{f_i}$.
- $\alpha_{ik} \circ \alpha_{ij} = \alpha_{ik}$.

There is a canonical functor $\text{Can} : \text{Mod}_R \to \text{Glue}(R \to S, f_1, \ldots, f_r)$ and also a morphism $H^0 : \text{Glue}(R \to S, f_1, \ldots, f_r) \to \text{Mod}_R$ where

$$H^0(M) = \text{Ker}(M' \times \prod_i M_i \to \prod_i M'_{f_i} \times \prod_i (M_i)_{f_j}).$$  

$H^0$ is a left inverse of $\text{Can}$, by (I.7.2.4).

Lemma (I.7.2.6). If $R \to S$ is flat, then $\text{Glue}(R \to S, f_1, \ldots, f_r)$ is an Abelian category, and $\text{Can}$ is an exact functor that commutes with arbitrary colimits.

If moreover $(f_1, \ldots, f_r) = R$, then $\text{Can}$ and $H^0$ induces an equivalence of categories.

Proof: The kernels and cokernels can be constructed because $- \otimes_R S$ is exact, and $\text{Can}$ is exact because $R_{f_i}$ and $S$ are flat over $R$, and also tensoring commutes with taking colimits.

For the last assertion, by (I.7.2.5) it suffices to show that $\text{Can}$ is essentially surjective. For this, just use (I.7.2.3) on both $R$ and $S$.

Prop. (I.7.2.7). In the setting of (I.7.2.5), if $R/I \to S/IS$ is an isomorphism, then $\text{Can}$ and $H^0$ induces an equivalence of categories.

Proof: Cf. [Sta] 05ER.
Cor. (I.7.2.8). If $R \to S$ is a flat ring map and $f \in R$ that $R/fR \cong S/fS$ is an isomorphism, then there is a pullback diagram of categories:

$$\begin{array}{ccc}
\text{Mod}_R & \to & \text{Mod}_{R_f} \\
\downarrow & & \downarrow \\
\text{Mod}_S & \to & \text{Mod}_{S_f}
\end{array}$$

and we can also restrict to the category of f.g./f.p/flat/projective(any property satisfying f.f. descent) modules.

Proof: For the last assertion, notice $R \to R^{\wedge} \times \prod R_{f_i}$ is f.f. by (I.5.7.7), then use f.f. descent (I.7.2.1). \( \square \)

Cor. (I.7.2.9). If $R$ is a Noetherian ring, $f \in R$ and $R^{\wedge}$ the $f$-adic completion of $R$, then there is an pullback of categories:

$$\begin{array}{ccc}
\text{Mod}_R & \to & \text{Mod}_{R_f^{\wedge}} \\
\downarrow & & \downarrow \\
\text{Mod}_{R^{\wedge}} & \to & \text{Mod}_{R^{\wedge}_f}
\end{array}$$

and we can also restrict to the category of f.g./f.p/flat/projective(any property satisfying f.f. descent) modules.

Proof: This satisfies the hypothesis of (I.7.2.7) by (I.5.7.14). \( \square \)

Def. (I.7.2.10) (Gluing Pairs). Let $R \to R'$ be a ring map and $f \in R$ that induces an isomorphism $R/f^nR \cong R'/f^nR'$ for any $n > 0$, then $(R \to R', f)$ is called a gluing pair if the sequence

$$0 \to R \to R' \oplus R_f \to R'_f \to 0$$

is exact. The pair $(R, f)$ is called a gluing pair if $(R \to \widehat{R}, f)$ is a gluing pair (This makes sense by (I.5.7.6)).

This is equivalent to $R[f^\infty] \to R'[f^\infty]$ is bijective.

Let $M$ be an $R$-module, then $M$ is called a glueable module for $(R \to R'^{\wedge}, f)$ if the sequence

$$0 \to M \to M_{R'} \oplus M_{R_f} \to M_{R'_f} \to 0$$

is exact.

This is equivalent to $M[f^\infty] \to M_{R'}[f^\infty]$ is a bijection. And when $(R \to R', f)$ is a gluing pair, this is equivalent to $M[f^\infty] \to M_{R'}[f^\infty]$ is injective.

Proof: Cf.[Sta]0BNR, 0BNW. \( \square \)

Prop. (I.7.2.11) (Flatness and Gluing). $(R \to R', f)$ is a gluing pair when $R \to R'^{\wedge}$ is flat. In particular $(R, f)$ is a gluing pair then $R$ is Noetherian or $f$ is a nonzero-divisor (I.5.7.10), Cf.[Sta]0BNT.

If $(R \to R', f)$ is gluing, then $M$ is glueable if $\text{Tor}_1^R(M, R')$ is $f$-power torsion, or equivalently $\text{Tor}_1^R(M, R'_f) = 0$. In particular this is the case when $M$ is flat $R$-module or $f$ is not a zero-divisor. And when $R \to R'$ is flat, any $R$-module $M$ is glueable, in particular this is the case for $(R, f)$ when $R$ is Noetherian. Cf.[Sta]0BNX.
Prop. (I.7.2.12) (Beauville-Laszlo). Let $A$ be a commutative ring and $f \in A$ is a nonzero-divisor, let $\hat{A}$ be the $f$-adic completion, then there is a pullback diagram of categories:

$$
\begin{array}{ccc}
A - \text{Mod} & \longrightarrow & A[{\frac{1}{f}}] - \text{Mod} \\
\downarrow & & \downarrow \\
\hat{A} - \text{Mod} & \longrightarrow & \hat{A}[{\frac{1}{f}}] - \text{Mod}
\end{array}
$$

Proof: Cf.[Sta]Ch15.81. \hfill \Box

3 Kahler Differentials

Def. (I.7.3.1) (Derivations). A derivation over $S$ from an $S$-algebra $R$ to an $S$-module $M$ is a morphism of $S$-modules $\delta: R \rightarrow M$ that $\delta(ab) = a\delta(b) + b\delta(a)$. The set of all derivatives from $R$ to $M$ is denoted by $\text{Der}_S(R,M)$.

Def. (I.7.3.2) ($A \ltimes R M$). For any $R$-algebra $A$, there is a functor $A \ltimes R M$ from $\text{Mod}_R$ to $(\text{Alg}_R) \ltimes A$ that $A \ltimes R M = A \otimes M$ with the algebra given by

$$(a,x)(b,y) = (ab,ay+bx).$$

Then there is a bijection of sets

$$\text{Der}_R(X,M) \cong (\text{Alg}_R) \ltimes (X,A \ltimes R M).$$

Def. (I.7.3.3) (Kähler Differential). Let $S \rightarrow R$ a ring map, then the Kähler Differential $\Omega_{R/S}$ is defined as a $R$-module that $\text{Der}_S(R,M) \cong \text{Hom}_S(\Omega_{R/S},M)$. In particular, $\text{Der}_S(R,R)$ is the $R$-dual of $\Omega_{R/S}$.

Prop. (I.7.3.4). One construction is by the free group generated by elements of $R$ module some relations.

It can also be constructed as follows: there are two ring maps $\lambda_i$ from $S$ to $R \otimes_R S$, and one map $\varepsilon$ from $S \otimes_R S$ to $S$. Let $I = \text{Ker} \varepsilon$ as a $R$ module by $\lambda_i$, then $I/I^2 \cong \Omega_{S/R}$ by (I.7.3.8) that

$$0 \rightarrow I/I^2 \rightarrow \Omega_{S \otimes_R S/S \otimes_R S} S \rightarrow \Omega_{S/S} \rightarrow 0.$$

So $I/I^2 \cong \Omega_{S/R \otimes_S (S \otimes_R S)} \otimes_{S \otimes_R S} S \cong \Omega_{S/R}$. And it can be verified that $a \otimes 1 - 1 \otimes a$ corresponds to $da$.

Prop. (I.7.3.5) (Adjointness). The functor $X \rightarrow A \otimes_X \Omega_{X/R}$ is left adjoint to the functor $A \ltimes R$ – defined in(I.7.3.2) as a functor from $(\text{Alg}_R) \ltimes A \rightarrow \text{Mod}_A$.

Proof: Because they are both equivalent to $\text{Der}_R(X,M)$. \hfill \Box

Cor. (I.7.3.6) (Functoriality). From the first construction, we can see directly that for a family of morphisms $R_i \rightarrow S_i$,

$$\Omega_{\text{colim} S_i/\text{colim} R_i} = \text{colim} \Omega_{S_i/R_i}.$$

In particular, we have:

$$T^{-1} \Omega_B/A = \Omega_{T^{-1}B/A}, \quad \Omega_{S^{-1}B/S^{-1}A} = S^{-1} \Omega_B/A.$$
Moreover, we have
\[ \Omega_{S/R} \otimes_R R' = \Omega_{S \otimes_R R'/R}, \quad (S \otimes_R \Omega_{T/R}) \oplus (T \otimes_R \Omega_{S/R}) \cong \Omega_{S \otimes_R T/R} \]
by universal properties.

**Proof:** We prove for the localization: it suffices to show the following two assertions:
1. \( S^{-1}\Omega_{A/B} \cong \Omega_{S^{-1}A/B} \).
2. If \( T \subset B \) is a multiplicatively closed subset that \( i(t) \) are all invertible in \( A \), then \( \Omega_{A/T^{-1}B} \cong \Omega_{A/B} \).

We check the universal properties: For any \( S^{-1}A \)-module \( M \),
\[ \Hom_{S^{-1}A}(S^{-1}\Omega_{A/B}, M) \cong \Hom_{S^{-1}A}(S^{-1}A \otimes_A \Omega_{A/B}, M) \cong \Hom_A(\Omega_{A/B}, M) \cong \Der_B(A, M), \]
\[ \Hom_{S^{-1}A}(\Omega_{S^{-1}A/B}, M) \cong \Der_B(S^{-1}A, M) \]

There is a map \( \Der_B(S^{-1}A, M) \to \Der_B(A, M) \) by restriction, and the converse is given by
\[ d \mapsto d\left(\frac{a}{s}\right) = \frac{sda - ads}{s^2}. \]
This is well-defined as
\[ d\left(\frac{at}{st}\right) = \frac{std(at) - atd(st)}{s^2t^2} = \frac{sda - ads}{s^2}, \]
and it satisfies
\[ d\left(\frac{a_1}{t_1} + \frac{a_2}{t_2}\right) = d\left(\frac{a_1t_2 + a_2t_1}{t_1t_2}\right) = \frac{(a_1d(t_2) + t_2d(a_1) + a_2d(t_1) + t_1d(a_2))t_1t_2 - (a_1t_2 + a_2t_1)(t_1d(t_2) - t_2d(t_1))}{t_1^2t_2^2} = d\left(\frac{a_1}{t_1}\right) + d\left(\frac{a_2}{t_2}\right) \]
\[ d\left(\frac{a_1a_2}{t_1t_2}\right) = \frac{(a_1d(a_2) + a_2d(a_1))t_1t_2 - (a_1a_2)\frac{t_1d(a_2) - t_2d(t_1)}{t_1^2t_2^2}}{t_1t_2} = d\left(\frac{a_1}{t_1}\right) + d\left(\frac{a_2}{t_2}\right) \]
\[ d\left(\frac{a}{t} + \frac{b}{t}\right) \]

Thus this \( d \) is an extension of the derivative to \( S^{-1}A \). Thus we get the desired isomorphism by Yoneda lemma.

**Prop. (I.7.3.7) (Jacobi-Zariski Sequences).** For a sequence of commutative rings: \( A \to B \to C \), there is an exact sequence
\[ C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0 \]
of \( C \)-modules. It has a left inverse and splits iff any derivation over \( A \) from \( B \) to a \( C \)-module can be extended to a derivation over \( A \) from \( C \) to \( M \). This is trivially true when \( B \to C \) has a retraction, and true when \( C/B \) is formally smooth by(I.7.5.6).
Proof: Taking Hom with an arbitrary $C$-module $M$, by universal property, we need to check the exactness of $0 \to \text{Der}_B(C, M) \to \text{Der}_A(C, M) \to \text{Der}_A(B, M)$, which is easy.

Prop. (I.7.3.8) (Second Exact Sequence). (This is a special case of (XI.2.1.5)). If $S' = S/I$, then there is an exact sequence of $R'$-modules:

$$I/I^2 \to \Omega_{S/R} \otimes_S S' \to \Omega_{S'/R} \to 0.$$ 

Where $f \in I$ is mapped to $df \otimes 1$ and it has a left inverse and splits iff $S/I^2 \to S'$ has a right inverse.

And in fact $\Omega_{S/R} \otimes S' \cong \Omega_{(S/I^2)/R} \otimes S'$.

Proof: For a $S/I$-module $M$, we check:

$$0 \to \text{Der}_R(S/I, M) \to \text{Der}_R(S, M) \to \text{Hom}_{S/I}(I/I^2, M)$$

To prove $\Omega_{S/R} \otimes S' \cong \Omega_{(S/I^2)/R} \otimes S'$, we apply Hom for a $S'$-module $M$.

So to prove the left exactness, we may assume $I^2 = 0$. If we have an inverse $\Omega_{S/R} \otimes S \to I$, then it gives a derivation $D : A \to I$ that is identity on $I$, so $a - D(a)$ gives a $R$-ring map $S \to S'$ that is trivial on $I$ (because $I^2 = 0$). Hence it gives a $S/I \to S'$ that is inverse to the projection.

For the converse, if $d : S/I \to S'$ is a right inverse, then $a - d(\bar{a})$ is a derivation $S \to I$, which is identity on $I$, so it gives a inverse map $\Omega_{S/R} \otimes S' \to I$ by universal property.

Cor. (I.7.3.9). If $R \to S$ is of f.p., then $\Omega_{S/R}$ is of f.p. over $S$. If $R \to S$ is of f.t. then $\Omega_{S/R}$ is of f.t. over $S$. (Follows from the second exact sequence (I.7.3.8) and (I.7.3.10)).

Cor. (I.7.3.10) (Examples).

1. $\Omega_{A[X_1, \ldots, X_n]/A} = A[X_1, \ldots, X_n]\{dX_1, \ldots, dX_n\}$.

2. If $S = A[X_i]/\{f_j\}$, then $\Omega_{S/A} = S[dX_i]/\{df_j\}$.

3. $\Omega_{A[X_i]/k} = \Omega_A/k \otimes_A A[X_i] \otimes A[dX_i, dX_n]$.

4. (Standard Étale Algebra) For $A = R[x]/(f)$, where $f'$ has image invertible in $A, \Omega_{A/R} = 0$.

5. The differential for the inclusion $k[y^2, y^3] \to k[y]$ is $k[y]/(2y, 3y^2)\{dy\}$.

Cor. (I.7.3.11). 1: Use the differential operator and universal property.

2: Use item1 and (I.7.3.8).

3: Use item1 and the fact (I.7.3.8) splits because any derivative of $A/k$ can be extended to derivative of $B/k$ by acting on the coefficients.

4: 

5: Use definition.

Cor. (I.7.3.12). If $S/I$ is a field $k$ that embeds in $S$, then $I/I^2 \cong \Omega_{S/k} \otimes_S k$.

Prop. (I.7.3.13). Let $k \subset K \subset L$ be fields, and $L/K$ f.g., then

$$\dim_L \Omega_{L/k} \geq \dim_K \Omega_{K/k} + \text{tr.deg}(L/K).$$

Equality holds if $L/K$ is separably generated, i.e. separable over a transcendental basis. If $K = k$, then the equality hold iff $L/k$ is separably generated. In particular, $L/k$ is separable algebraic extension iff $\Omega_{L/k} = 0$. 
Proof: Take a subfield $K \subset K(t_1, \ldots, t_n) \subset L$ that $L$ is separable algebra over $K(t_1, \ldots, t_n)$. Then it suffices to add one element at a time, so we may assume $L = K(\alpha)$.

1: If $\alpha$ is transcendental over $K$, then $\Omega_{K[\alpha]/t} \cong \Omega_{K/k} \otimes K[\alpha] + K[\alpha]dt$ by (I.7.3.10), and by localization (I.7.3.6) we get $\Omega_{L/K} \cong \Omega_{K/k} \otimes K L \oplus Ldt$, thus rank $\Omega_{L/k} = \text{rank} \Omega_{K/k} + 1$.

2: If $\alpha$ is separable over $K$, then there is a monic polynomial $f \in K[X]$ that $K[\alpha] \cong K[X]/(f)$. Then $f'$ is invertible in $K[\alpha]$, and by (I.7.3.10) $\Omega_{K[\alpha]/k} = \Omega_{K/k} \otimes K L \oplus LdX/(d(f) + f'dX) \cong \Omega_{K/k} \otimes K L$. Thus rank $\Omega_{L/k} = \text{rank} \Omega_{K/k}$.

3: If $K$ has characteristic $p$ and $L = K[X]/(X^p - a)$ and $dK/k(a) = 0$, then $\Omega_{K[\alpha]/k} = \Omega_{K/k} \otimes KL \oplus LdX/(d(a)) \cong \Omega_{K/k} \otimes KL$. Thus rank $\Omega_{L/k} = \text{rank} \Omega_{K/k} + 1$.

4: If $K$ has characteristic $p$ and $L = K[X]/(X^p - a)$ and $dK/k(a) \neq 0$, then $\Omega_{K[\alpha]/k} = \Omega_{K/k} \otimes KL \oplus LdX/(d(a))$ has rank rank $\Omega_{K/k}$.

Thus the assertion is clear.

Prop. (I.7.3.14) (Differential and Regularity). Let $B$ be a Noetherian local ring containing its residue field $k$ and $K$ is perfect, then $\Omega_{B/k}$ is a free $B$-module of rank $B$ iff $B$ is regular.

Proof: One way is by (I.7.3.12). Conversely, if $B$ is regular, then it is integral (I.6.5.16), so $\Omega_{B/k} \otimes B K = \Omega_{K/k}$ (I.7.3.6) is of $K$-dimension $tr \deg K/k = \text{dim} B$, where $K$ is the quotient field of $B$, and $\Omega_{B/k} \otimes K \cong m/m^2$ is of $K$-dimension $\text{dim} B$ once again. These two facts shows that $\Omega_{B/k}$ is free $B$-module of rank $B$ by (I.7.8.1).

Prop. (I.7.3.15). The Kähler differential $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$ for an extension of number fields is cyclic.

Proof: Because it is locally cyclic. (IV.2.1.34)

4 Complete Intersections

Koszul Complex

Prop. (I.7.4.1) (Koszul Complex). The complex of $\mathbb{Z}[X_1, \ldots, X_n]$-modules $K_r$, where

$$K_r = \wedge^{r+1} \mathbb{Z}[dX_1, \ldots, dX_n] \otimes \mathbb{Z}[X_1, \ldots, X_n]$$

and the morphism is given linearly by

$$\iota: dX_{i_0} \wedge dX_{i_1} \wedge \ldots \wedge dX_{i_r} \mapsto \sum_k X_{i_k}dX_{i_0} \wedge \ldots \wedge dX_{i_{k-1}} \wedge dX_{i_{k+1}} \wedge \ldots \wedge dX_{i_r}$$

Then this is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[X_1, \ldots, X_n]$.

Proof: We can use induction. If $r = 1$, then this is clear. For the induction process, if $\iota(dX_1 \wedge \alpha + \beta) = 0$, and $\beta \in K_i$, where $\beta, \alpha$ has no $dX_1$ involved. Notice if $dX_1 \wedge \alpha \in K_0$, then we may assume $\alpha$ has no constant coefficient, because this is impossible.

then $X_1 \alpha - dX_1 \wedge \iota(\alpha) + \iota(\beta) = 0$. Then we see that $\iota(\alpha) = 0$. We can write $\alpha = \alpha_0 + X_1 \alpha_1 + X_1^2 \alpha_2 + \ldots$, where $\alpha_i$ has no $X_1$ involved, then we see that $\iota(\alpha_i) = 0$, then by induction hypothesis $\alpha_i = \iota(\alpha_i')$ (notice $\alpha_0$ has no constant coefficient), and $\iota(\beta) + X_1 \iota(\alpha) = 0$. If $\beta = X_1 \beta_1 + \beta_2$, where $\beta_2$ has no $X_1$ involved, then $\iota(\beta_2) = 0$, so $\beta_2 = \iota(\beta_2')$, and $\iota(\beta_1 + \alpha) = 0$, so $\beta_1 + \alpha' = \iota(\beta_1')$. So

$$dX_1 \wedge \alpha + \beta = dX_1 \wedge \iota(\alpha') + \iota(X_1 \beta_1' + \beta_2') - X_1 \alpha' = \iota(X_1 \beta_1' + \beta_2' - dX_1 \wedge \alpha').$$

And in degree 0, this is clear.
Def. (I.7.4.2) (Koszul Complex). Let $A$ be a ring and $I = (f_1, \ldots, f_n) \subseteq A$ is an ideal, then the Koszul complex for $Kos(A, f_1, \ldots, f_n)$ is an object in $D(A)$ defined by

$$Kos(A, f_1, \ldots, f_n) = A \otimes_{\mathbb{Z}[X_1, \ldots, X_n]} \mathbb{Z};$$

where $A$ is a $\mathbb{Z}[X_1, \ldots, X_n]$-algebra by mapping $X_i \to f_i$. If $M$ is an $A$-module, then we define $Kos(M, f_1, \ldots, f_n) = M \otimes_A Kos(A, f_1, \ldots, f_n)$.

Prop. (I.7.4.3). If $f_1, \ldots, f_n \in R$, then $I = (f_1, \ldots, f_n)$ annihilates $H^*(K(f_1, \ldots, f_n))$. In particular, $Kos(A, f_1, \ldots, f_n)$ is in the image of $D(A/I) \subset D(A)$.

Proof: This is because every $H^i(A \otimes_{\mathbb{Z}[X_1, \ldots, X_n]} \mathbb{Z})$ is an $H^0(A \otimes_{\mathbb{Z}[X_1, \ldots, X_n]} \mathbb{Z}) = A/I$-algebra, because it is a simplicial algebra.

Prop. (I.7.4.4). $K(A, f_1, \ldots, f_n, g_1, \ldots, g_m) = K(A, f_1, \ldots, f_n) \otimes_A K(A, g_1, \ldots, g_m)$. (Easy).

Prop. (I.7.4.5). The cone of the map

$$f_n : K(f_1, \ldots, f_{n-1}) \to K(f_1, \ldots, f_{n-1})$$

is isomorphic to $K(f_1, \ldots, f_n)$.

Proof: This is because $\mathbb{Z}[X] \xrightarrow{X} \mathbb{Z}[X] \to \mathbb{Z}$ is an exact triangle, so $A \xrightarrow{f_n} A \to A \otimes_{\mathbb{Z}[X]} \mathbb{Z}$ is an exact triangle, so $Kos(A, f_1, \ldots, f_{n-1}) \xrightarrow{f_n} K(A, f_1, \ldots, f_{n-1}) \to K(A, f_1, \ldots, f_n)$ is an exact triangle.

Prop. (I.7.4.6). Let $A$ be a ring and $M$ be an $A$-module. Let $f_1, \ldots, f_{r-1}, f, g$ be elements of $A$, then there is a natural distinguished triangle

$$Kos(M, f_1, \ldots, f_{r-1}, f) \to Kos(M, f_1, \ldots, f_{r-1}, f, g) \to Kos(M, f_1, \ldots, g).$$

Proof: We use (I.7.4.5) to consider $Kos(M, f_1, \ldots, f_{r-1}, f)$ as a cone of $f_n : Kos(M, f_1, \ldots, f_{r-1}) \to Kos(M, f_1, \ldots, f_{r-1})$. Then this distinguished triangle is given by (II.3.1.4) applied to the diagram

$$\begin{array}{ccc}
Kos(M, f_1, \ldots, f_{r-1}) & \longrightarrow & Kos(M, f_1, \ldots, f_{r-1}) \\
\downarrow f & & \downarrow fg \\
Kos(M, f_1, \ldots, f_{r-1}) & \longrightarrow & Kos(M, f_1, \ldots, f_{r-1})
\end{array}$$

Prop. (I.7.4.7) (Koszul Complex and Čech Complex). Let $A$ be a commutative ring and $I = (f_1, \ldots, f_n) \subseteq A$, $K^*_n = Kos(A, f_1^n, \ldots, f_n^n) = A \otimes_{\mathbb{Z}[X_1, \ldots, X_n]} \mathbb{Z}$. Then there are natural maps

$$\ldots \to K^*_3 \to K^*_2 \to K^*_1$$

compatible with the inverse system $H^0(K^*_n) = A/(f_1^n, \ldots, f_n^n)$. Then there is a description of $R \colim K^*_n$ (I.10.5.2) as the alternating Čech complex

$$R \to \bigoplus_{i_0} Rf_{i_0} \to \bigoplus_{i_0 < i_1} Rf_{i_0}f_{i_1} \to \ldots \to Rf_{1}f_{2} \ldots f_r$$

where $R$ sits in degree 0.
Proof: Cf. [Sta]0913, which is not hard.

Def. (I.7.4.8) (Koszul-Regular Sequence). Let $A$ be a ring and $(f_1, \ldots, f_n) \in A$ be a sequence, then $f_1, \ldots, f_n$ is called $M$-Koszul-regular iff $Kos(M, f_1, \ldots, f_n) = 0$. It is called Koszul-regular iff $Kos(A, f_1, \ldots, f_n) = 0$.

Prop. (I.7.4.9). If $(f_1, \ldots, f_r)$ is $(M)$-regular and $n_i > 0$, then $(f_1^{n_1}, \ldots, f_r^{n_r})$ is also $(M)$-regular.

Proof: This follows from (I.7.4.6).

Prop. (I.7.4.10) (Regular and Koszul-Regular). A $M$-regular sequence (I.6.4.1) is $M$-Koszul-regular. A regular sequence is Koszul-regular.

Proof: Let $(f_1, \ldots, f_r)$ be a regular sequence. The assertion is clear when $r = 1$. For the induction:

$$
Kos(M, f_1, \ldots, f_n) = Kos(M, f_2, \ldots, f_n) \otimes_{\mathbb{Z}[X]} \mathbb{Z} = Kos(A, f_2, \ldots, f_n) \otimes_A^L M \otimes_{\mathbb{Z}[X]} \mathbb{Z}
$$

$$
= Kos(A/f_1, f_2, \ldots, f_n) \otimes_{A/f_1}^L M/f_1 M = Kos(M/f_1 M, f_2, \ldots, f_n)
$$

so we can use induction.

Complete Intersections

Cf. [Sta] Chap23.8, Chap10.135, 10.136.

Def. (I.7.4.11) (Complete Intersections).

- Let $S$ be a f.g. $k$-algebra, then $S$ is a global complete intersection over $k$ iff $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ that $\dim(S) = n - c$.
- Let $S$ be a f.g. $k$-algebra, then $S$ is a local complete intersection over $k$ iff $S$ is locally a global complete intersection over $k$.
- A ring map $R \to S$ is called relative global intersection if $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ that every non-empty fiber of $\text{Spec } S \to \text{Spec } R$ has dimension $n - c$.

Lemma (I.7.4.12). For a f.g. $k$-algebra, (Local)Complete intersection is stable under localization.

Proof: Cf. [Sta] 00SA.

Prop. (I.7.4.13) (Being Local Complete Intersection is Stalkwise). For a f.g. $k$-algebra $S$, being local complete intersection is a stalkwise property.

Proof: Cf. [Sta] 00SH.

Prop. (I.7.4.14). If $S$ is a f.g. algebra over a field $k$ and it is a local complete intersection, then $S$ is a CM ring.

Proof: Cf. [Sta] 00SB.

Local Complete Intersection Rings

Def. (I.7.4.15) (Complete Intersection Local Rings). Let $A$ be a Noetherian local ring, then $A$ is called a complete intersection if there exists a surjection $R \to A$ from a regular local ring that the kernel is generated by a regular sequence.

Prop. (I.7.4.16). When $A$ is f.g. local ring over a field $k$, then $A$ being a complete intersection iff $A$ is a local complete intersection ring (I.7.4.11).
Syntomic

**Def. (I.7.4.17) (Syntomic Maps).** A ring map is called syntomic iff it is of f.p., flat and the fibers are all local complete intersection rings.

Syntomic property is local on the target, because both f.p., flatness and locally complete intersections do.

5 Smoothness

Formally Smoothness

**Def. (I.7.5.1) (Infinitesimal Thickening).** An infinitesimal thickening of a ring \( A \) is of the form \( A \leftarrow I \) where \( I \) is nilpotent.

**Def. (I.7.5.2).** A ring map \( R \rightarrow S \) is called formally smooth if has right lifting property w.r.t all ring maps \( A \rightarrow A/I \) where \( I^2 = 0 \).

Formal smooth is stable under base change and composition, by universal arguments. A polynomial algebra is formally smooth.

**Prop. (I.7.5.3).** Giving a presentation \( S = P/J \) where \( P \) is formally smooth (e.g. polynomial algebra), \( S \) is formally smooth iff there is a map \( S \rightarrow P/J^2 \) that is right converse to the obvious projection.

*Proof:* One way is from the definition of formally smooth applied to \( P/J^2 \) and \( J \). Conversely, for any \( A \) and \( I \), we notice the map \( P \rightarrow S \rightarrow A/I \) can be lifted to \( P \rightarrow A \), and \( J \) is mapped to \( I \), so \( J^2 \) is mapped to 0, so we have a map \( P/J^2 \rightarrow A \). Then \( S \rightarrow P/J^2 \rightarrow A \) is the lifting. □

**Cor. (I.7.5.4).** If \( P \rightarrow S \) is a presentation of \( S/R \) by polynomial algebra with kernel \( I \), then \( S/R \) is formal smooth iff there is a map \( S \rightarrow P/J^2 \) that is right converse to the obvious projection.

*Proof:* This sequence is split exact iff \( P/J^2 \rightarrow S \) has a right converse, by (I.7.3.8). □

Now we consider the relation of Formal Smoothness and Kahler Differentials.

**Cor. (I.7.5.5) (Equivalence Definition).** \( S/R \) is formally smooth iff \( NL_{S/R} \) is quasi-isomorphic to a projective \( S \)-module at degree 0.

*Proof:* If \( S/R \) is formally smooth, then choose a presentation will suffice by (I.7.5.4). The converse is also true by projectiveness and (I.7.5.4). □

**Cor. (I.7.5.6).** If \( C/B \) is formally smooth, then the Jacobi-Zariski sequence

\[
0 \rightarrow C \otimes_B \Omega_{B/A} \rightarrow \Omega_{C/A} \rightarrow \Omega_{C/B} \rightarrow 0
\]

as in (I.7.3.7) is split exact, by (XI.2.1.5). In particular, any derivation of \( B \) to a \( C \)-module can be extended to a derivation \( C \) to a \( C \)-module.

**Cor. (I.7.5.7).** If \( A \rightarrow B \rightarrow C \) with \( A \rightarrow C \) formally smooth and \( B \rightarrow C \) surjective with kernel \( I \), then there is an split sequence

\[
0 \rightarrow I/I^2 \rightarrow \Omega_{B/A} \otimes_B C \rightarrow \Omega_{C/A} \rightarrow 0
\]

by (XI.2.2.6).
Standard Smooth Algebra

Def. (I.7.5.8). A standard smooth algebra over $R$ is an algebra $S = R[X_1, \ldots, X_n]/(f_1, \ldots, f_c)$, where $c \leq n$ and $J(\frac{f_1}{X_1}, \ldots, \frac{f_c}{X_c})$ is invertible in $S$.

Prop. (I.7.5.9) (Standard Smooth Localization). If $R \to S$ is standard smooth, then $R \to S_g$ is standard smooth, and $R_f \to S_f$ is standard smooth (because stable under base change (I.7.5.10)).

Proof: For localization at $g \in S$, let $h$ be an inverse image of $g$ in $R[X_1, \ldots, X_n]$, then $S_g = R[X_1, \ldots, X_n, X_{n+1}]/(f_1, \ldots, f_c, X_{n+1}h - 1)$, and it is standard smooth.

Prop. (I.7.5.10). Standard smoothness is stable under base change and composition.

Proof: For base change, notice the Jacobi matrix is the base change of the Jacobi matrix, so it is also invertible. For composition, write out the presentation, the determinant is the product of the presentation.

Prop. (I.7.5.11). A standard smooth algebra is a relative global complete intersection.

Proof: Cf. [Sta00T7].

Prop. (I.7.5.12) (Jacobian Criterion of Smoothness). For a f.p. ring $S = R[X_1, \ldots, X_n]/(f_1, \ldots, f_c)$, $S/R$ is standard smooth in a nbhd of $q$ iff the Jacobian matrix has rank $c$ at $q$, i.e. $J(\frac{f_1}{X_1}, \ldots, \frac{f_c}{X_c})$ is not in $q$ for some permutation of $X_1, \ldots, X_n$.

Proof: If $h = J(\frac{f_1}{X_1}, \ldots, \frac{f_c}{X_c}) \notin q$, let $S_h = R[X_1, \ldots, X_n, X_{n+1}]/(f_1, \ldots, f_c, X_{n+1}g - 1)$ is a standard smooth algebra. Conversely, $S_h = R[X_1, \ldots, X_n, X_{n+1}]/(f_1, \ldots, f_c, X_{n+1}g - 1)$ is standard smooth for some $g \notin q$, so $J(\frac{f_1}{X_1}, \ldots, \frac{f_c}{X_c}) \notin q$ by calculation.

Lemma (I.7.5.13) (Kahler Differential of Smooth Algebra is Free). The Kahler differential of a standard smooth algebra $S$ over a field $k$ is free of rank $r = \dim(S)$.

Proof: The naive cotangent complex for $S/R$ is

$$d : (f_1, \ldots, f_c)/(f_1, \ldots, f_c)^2 \to S[dX_1, \ldots, dX_n].$$

By hypothesis and linear algebra it is a split injection, and $\Omega_{S/R} = S[dX_{c+1}, \ldots, dX_n]$, so it is a smooth ring map.

Smoothness

Def. (I.7.5.14) (Smooth Ring Map). A ring map $R \to S$ is called smooth if it satisfies the following equivalent conditions:

- It is of f.p. and the naive cotangent complex $NL_{S/R}$ is quasi-isomorphic to a finite locally free $S$-module placed at degree 0. In other words,

$$0 \to I/I^2 \to \Omega_{P/R} \otimes_P S \to \Omega_{S/R} \to 0$$

is exact and $\Omega_{S/R}$ is finite locally free. By (XI.2.1.1), we only need to prove for a single presentation of $S$.

- It is locally standard smooth.
• It is formally smooth and of f.p.

We say $S$ is smooth at $x$ if it is smooth at a nbhd of $x$.

Proof: 1 → 3: by (I.7.5.5). 3 → 1: By (I.7.5.5), $\Omega_{S/R}$ is f.p. and projective, so it is finite projective.

At this point we already know that the first definition is stable under base change and composition, because f.p. and formally smoothness both do (I.7.5.2)(I.6.7.9).

And also the first definition is local on source because f.p. does (I.5.1.57) and NL commutes with localization (XI.2.1.6) so we can use the local properties of triviality (I.5.1.55) and finite projectiveness (I.6.1.7).

Now it is also local on the source because it is stable under base change and composition and $R \rightarrow R_f$ does by locality on the source.

2 → 1: Now the property are all local on source. It suffices to prove a standard smooth map is smooth. This follows from (I.7.5.13).

1 → 2: We need to prove, assuming the first definition, it is locally standard smooth. For this, Cf. [Sta]00TA.

Cor. (I.7.5.15). Smoothness is stable under composition and base change. Smoothness is local on the source and target (In particular, $R \rightarrow R_f$ is smooth). (Already proved in the proof of (I.7.5.14)).

Cor. (I.7.5.16). A smooth map is syntomic, hence flat.

Proof: Clear from (I.7.5.14).

Prop. (I.7.5.17) (Noetherian Descent). A smooth ring map $R \rightarrow S$ is a base change of smooth ring map over a ring f.g. over $Z$.

Proof: Use the equivalence definition (I.7.5.3), we know that there is a map

$$S = R[X_1, \ldots, X_n]/(f_1, \ldots, f_c) \rightarrow R[X_1, \ldots, X_n]/(f_1, \ldots, f_c)^2,$$

which if we write $\sigma(X_i) = h_i$, then must satisfy

$$f_i(h_1, \ldots, h_n) = \sum a_{ijk} f_j f_k.$$  

Then we consider the subalgebra generated by $f_i, h_i, a_{ijk}$, then by the same reason, they form a smooth algebra over $Z$, and its tensor with $R$ gives out $S$.

Cor. (I.7.5.18) (Strong Lifting Property). For a smooth ring map, the lifting property is true for $A \rightarrow A/I$, where $I$ is locally nilpotent.

Proof: By (I.7.5.17), $R \rightarrow S$ is a base change of a smooth ring map $R_0 \rightarrow S_0$ where $R_0$ is f.g. over $Z$. Now if $S_0$ is generated by $x_1, \ldots x_n$ and $a_1, a_n \in A$ maps to the image of $x_1, \ldots, x_n$ in $A/I$, then consider the subring $A_0$ generated by $R_0$ and $a_i$, and let $I_0 = A_0 \cap I$, then it suffices to prove this case followed by base change. But now $A_0$ is f.g. over $Z$, so it is Noetherian, and then $I$ is nilpotent, thus we have a desired lifting.

Prop. (I.7.5.19) (Stalkwise). If $R \rightarrow S$ is f.p., then it is smooth iff it is $S_q/R_p$ is smooth for every (maximal) prime $q$ of $S$ and $p$ under it.

Proof: Because of f.p., we only need to check triviality of $H_1(NL)$ and finite projectivity of $\Omega_{S/R}$ (fp used). But both triviality and finite projectivity is stalkwise (I.5.1.55). (Notice $R \rightarrow R_p$ is smooth).
Cor. (I.7.5.20) (Fiberwise). For a ring map \( R \to S \) and \( q \) is a prime of \( S \) over \( p \). Then \( S/R \) is smooth at \( q \) iff \( S/R \) is of f.p. and \( S_q/R_p \) is flat and \( S \otimes k(p)/k(p) \) is smooth at \( q \).

Proof: One direction is because smooth is flat, f.p. and stable under base change. Conversely, Cf.[Sta]00TF.

Cor. (I.7.5.21) (Smooth Points and Flat Base Change). If \( R \to S \) is of f.p. and \( R \to R' \) is flat. Then the set of primes in \( S' = S \otimes_R R' \) that has a nbhd that is smooth over \( R' \) is the inverse image of set of primes in \( S \) that has a nbhd that is smooth over \( R \).

Proof: One direction is because smooth is stable under base change. Conversely, the local ring map is f.f., so \( H_1(NL_{S'/R',q}) = H_1((NL_{S/R} \otimes_S S')_q) = H_1(NL_{S/R_p} \otimes_{S_p} S'_q) \). Then the result follows as \( S'_q/S'_p \) is f.f. and triviality and finite projective descents for f.f. map(I.7.2.1).

Prop. (I.7.5.22). If \( A \to A[X_1, \ldots, X_n] \to R \) is smooth, then \( A[X_1, \ldots, X_n] \to R \) is smooth.

Proof: The desired map is firstly of f.p. by(I.6.7.11), and it can be verified to be formally smooth, because \( A[X_1, \ldots, X_n] \) is free.

Smooth over Fields

Prop. (I.7.5.23). A smooth \( k \)-algebra is a local complete intersection.

Proof: Immediate from(I.7.5.16).

Lemma (I.7.5.24). Let \( S \) be f.g. over a alg.closed field \( k \) and \( m \) a maximal ideal, then the following are equivalent:
- \( S_m \) is regular.
- \( \dim_k \Omega_{S/k} \otimes_S k \leq \dim S_m \)
- \( \dim_k \Omega_{S/k} \otimes_S k = \dim S_m \)
- \( S/k \) is smooth at \( m \).

Proof: Cf.[Sta]00TS.

Prop. (I.7.5.25) (Differential Criterion of Smoothness). For a ring \( S \) f.g. over a field \( k \), \( S \) is smooth in a nbhd of \( x \) corresponding to a prime \( q \) iff \( \dim_{k(x)} \Omega_{S/k} \otimes k(x) \leq \dim_x(X) \).

And in this case, equality hold, and \( S_q \) is regular.

Proof: Cf.[Sta]00TT.

If \( S \) is smooth at \( x \), then \( \Omega_{S/R} \) is finite free on a nbhd of \( x \) of rank equals to the dimension(I.7.5.13), so the equation holds.

Conversely, if \( \dim_{k(x)} \Omega_{S/k} \otimes k(x) \leq \dim_x(X) \), then

Lemma (I.7.5.26). Let \( k \) be a field and \( (R, m, \kappa) \) be a Noetherian local ring containing \( k \). If the residue field of \( R \) is a f.g. field extension of \( k \), then the derivation map

\[ m/m^2 \to \Omega_{R/k} \otimes R \kappa \]

is injective.

Proof: Cf.[Sta]00TU.
Prop. (I.7.5.27) (Smooth and Regular). Let $S$ be f.g. over a field $k$, if $k(q)/k$ is separable (e.g. char 0) for $q$ a prime of $S$, then $S$ is smooth at $q$ iff $S_q$ is regular.

**Proof:** Let $R = S_q$ with maximal ideal $m$. By (I.7.5.26) and (I.7.3.8) there is an exact sequence
\[ 0 \to m \to m^2 \to \Omega_{R/k} \otimes_R k(q) \to \Omega_{k(q)/k} \to 0. \]

Since $k(q)/k$ is separable, $\dim_{k(q)} \Omega_{k(q)/k} = \text{tr. deg}(k(q)/k)$. So
\[ \dim_{k(q)}(\Omega_{R/k} \otimes_R k(q)) = \dim_{k(q)} m/m^2 + \text{tr. deg}(k(q)/k) \geq \dim R + \text{tr. deg}(k(q)/k) = \dim_q(S) \]

with identity iff $R$ is regular (The last identity comes from (V.5.2.7)). So we are done by differential criterion of smoothness (I.7.25).

□

Prop. (I.7.5.28) (Smooth over Fields and Geo-Regular). Let $S$ be f.g. over a field $k$, then $S$ is smooth over $k$ iff $S$ is geo.regular (I.6.6.1).

**Proof:** If $S$ is smooth at $x$, then all its base change is smooth at $x$ (I.7.5.15), and the stalk is regular by (I.7.5.25), so it is geometrically regular at $x$.

Conversely, if $X$ is geometrically regular, then for any point $x \in X$, $k(x)$ is f.g. over $k$, so by (I.6.9.3) there is a finite purely inseparable extension $k'/k$ that the compositum $k'k(x)$ is separable over $k'$. Then by (I.5.3.22), $\text{Spec } A \otimes_k k'$ is homeomorphic to $\text{Spec } A$, so there is a unique prime $p'$ of $X_{k'}$ over $X$, and its residue field is $k'k(x)$. So by (I.7.5.27), as $k'k(x)/k'$ is separable, $X_{k'}$ is smooth over $k'$ at $p'$. And f.f. descent for smoothness (I.7.2.1) says $X$ is also smooth over $k$ at $p$.

□

Cor. (I.7.5.29) (Differential and Smoothness). Let $k$ be a field of characteristic 0 and $S$ a f.g. algebra over $k$, and $q$ a prime ideal of $S$, if $\Omega_{S/k,q}$ is free over $S_q$, then $S$ is smooth in a nbhd of $q$.

**Proof:** Cf. [Sta]00TX.

□

Prop. (I.7.5.30) (Generic Smoothness). Let $R \to S$ be an injective ring map of f.t. with $R, S$ domains, then it is smooth at (0) iff the quotient field map is separable.

**Proof:** If $S$ is smooth at 0, then replacing $S$ by $S_g$ for some $g$, we can assume $R \to S$ is smooth. Then $K \to S \otimes_R K$ is also smooth (I.7.5.15), and also for any field extension $K'$ of $K$. Then $S \otimes_R K'$ is regular, by (I.7.5.25), a priori reduced (V.4.1.5). Thus $S \otimes_R K$ is geometrically reduced. Hence also $L$ is geometrically reduced over $K$, thus separable, by (I.6.9.4).

Conversely, by?? we may assume $R \to S$ is of f.p., thus to show it is smooth at (0), it suffices to show $S \otimes_R K$ is smooth at (0), by (I.7.5.20). Then this follows from (I.7.5.27).

□

Smoothing Ring Maps

Prop. (I.7.5.31). A regular homomorphism of Noetherian rings is a filtered colimit of smooth ring maps.

**Proof:** Cf. [Sta]07GC.
6 Unramified

Formally Unramified

Def. (I.7.6.1). A ring map \( R \to S \) is called \textbf{formally unramified} if for every \( R \)-ring \( A \) and an ideal \( I \) of \( A \) that \( I^2 = 0 \), a map \( S \to A/I \) has at most one extension to a map \( S \to A \).

Formally unramified is equivalent to \( \Omega_{S/R} = 0 \). So it is stable under composition by Jacobi-Zariski sequence (I.7.3.7).

Proof: Let \( J = \ker(S \otimes_R S \to S) \), let \( A_{univ} = S \otimes_R S/J^2 \), then \( J/J^2 \cong \Omega_{S/R} \) (I.7.3.4), so we have two natural map from \( S \) to \( A_{univ} \), they differ by the universal differential \( S \to \Omega_{S/R} \). If \( S/R \) is unramified, then \( ds = 0 \) for all \( s \in S \), so \( \Omega_{S/R} = 0 \).

Conversely, if there is a \( A \) and \( A/J \) that there are two liftings \( \tau_1, \tau_2 \), then we let \( A_{univ} \to A \) defined by \( s_1 \otimes s_2 \to \tau_1(s_1)\tau_2(s_2) \), this is well-defined, and because \( A_{univ} \cong S \), this map descends to \( S \), so \( \tau_1(s_1s_2) = \tau_2(s_1s_2) \). \( \square \)

Prop. (I.7.6.2) (Formally Unramified Stalkwise). Formally unramified is stalkwise both on the source and target (I.5.1.55).

Prop. (I.7.6.3). Colimits of formally unramified rings over \( R \) is formally unramified. (Trivial as one renders on the diagram in the definition of formally unramified).

Unramified Map

Def. (I.7.6.4). A ring map is called \textbf{unramified} iff it is formally unramified and f.g..

A ring map is called \textbf{\( G \)-unramified} iff it is formally unramified and of f.p.. In particular, an étale map is \( G \)-unramified.

These two notions are stable under composition and base change. These two notions are local on the source and target. \( R \to R_f \) is \( G \)-unramified (I.7.6.1)(I.5.1.55)

Prop. (I.7.6.5). \( R \to R/I \) is unramified, and if \( I \) is f.g., then it is \( G \)-unramified. (Trivial).

Prop. (I.7.6.6) (Stalkwise and Fiberwise). If \( R \to S \) is of f.t(f.p.), then it is unramified(G-unramified) at a prime \( q \) of \( S \) iff \( \Omega_{S/R}_q = 0 \) iff \( \Omega_{S/R} \otimes_S k(q) = 0 \) iff \( \Omega_{S\otimes k(p)/k(p)} \otimes k(q) = 0 \).

Proof: By Nakayama, two pair of them are equivalent, and if \( \Omega_{S/R}_q = 0 \), then \( \Omega_{S/R,g} = 0 \) for some \( g \notin q \)(because support of finite module is open), so \( R \to S_g \) is \( (G-) \)unramified. And notice in fact \( \Omega_{S/R} \otimes_S k(q) = \Omega_{S\otimes k(p)/k(p)} \otimes k(q) \).

Prop. (I.7.6.7) (Equivalent Definition of Unramifiedness). A f.g. ring map \( R \to S \) is unramified at a prime \( q \) of \( S \) over \( p \) iff \( pS_q = qS_q \) and \( k(q)/k(p) \) is finite separable.

Proof: Suppose \( R \to S_g \) is unramified, then \( S \otimes k(p) \) is unramified over \( k(p) \), hence by(I.7.5.25), it is also smooth, so it is étale, and(I.7.7.9) gives the result.

For the converse, Cf[Sta][02FM]. \( \square \)

Prop. (I.7.6.8). A ring map is unramified iff it is locally a quotient of a standard étale map.

Proof: Cf.[Sta][0395]. \( \square \)
Prop. (I.7.6.9). Any $G$-unramified map is a base change of a $G$-unramified map over a ring $R_0$ f.g. over $\mathbb{Z}$. And similarly any unramified map is a quotient of a base change of a $G$-unramified map over a ring $R_0$ f.g. over $\mathbb{Z}$.

Proof: Let $S = R[X_1, \ldots, X_n]/(g_1, \ldots, g_c)$, then we have $dX_i = \sum a_{ij}dg_j + a_{ijk}g_jdX_k$, so we let $R_0$ be generated by $g_i, a_{ij}, a_{ijk}$, so $S_0 = R_0[X_1, \ldots, X_n]/(g_1, \ldots, g_c)$ is $G$-unramified. $\square$

Prop. (I.7.6.10) (Unramifiedness and Idempotent). If $R \rightarrow S$ is of f.t., then it is unramified iff $S \times_R S \rightarrow (S \otimes_R S)_e$ for some diagonal idempotent $e \in S \otimes_R S$ that $e \operatorname{Ker}(\mu) = 0$, i.e. $S \otimes_R S \cong S \times S'$.

Proof: If it is $G$-unramified, the kernel $I$ satisfies $I/I^2 = 0$, and $I$ is f.g.(by $x_i \otimes 1 - 1 \otimes x_i$) so we can use(I.5.3.6).

Conversely, the existence of the diagonal idempotent $e$ implies that $I = I^2$. $\square$

7 Étale

Formally Étale

Def. (I.7.7.1). A ring map $R \rightarrow S$ is called formally étale iff it is formally smooth and formally unramified.

Prop. (I.7.7.2). Colimits of formally étale rings over $R$ is formally étale. (The lifting are compatible because of uniqueness).

Prop. (I.7.7.3). $R \rightarrow S^{-1}R$ is formally étale.

Proof: It suffice to prove that if $\varphi(s)$ is invertible modulo $I$, then $\varphi(s)$ is invertible, but this is true because $I$ is nilpotent. $\square$

Étale Map

Def. (I.7.7.4). A ring map $R \rightarrow S$ is called étale if it is of f.p. and the naive cotangent complex is exact, i.e. $I/I^2 \cong \Omega_{P/R} \otimes_P S$.

In particular, étale is equivalent to smooth+formally unramified($\Omega_{R/S} = 0$).

Cor. (I.7.7.5) (Properties of Étale).
1. Étale map is stable under base change and composition.
2. Étale map is local on the source and target. In particular, $R \rightarrow R_f$ is étale.
3. If $R \rightarrow S$ is of f.p. and $R \rightarrow R'$ is flat. Then the set of primes in $S' = S \otimes_R R'$ that has a nbhd that is étale over $R'$ is the inverse image of set of primes in $S$ that has a nbhd that is étale over $R$. (The same as(I.7.5.21)).
4. Étale map is syntomic, hence flat.
5. Any Étale map is a base change of an étale map over a ring $R_0$ f.g. over $\mathbb{Z}$. (Cf.[Sta]00U2)).

Cor. (I.7.7.6) (Étale Localness of Differential). For ring morphisms $A \rightarrow R \rightarrow S$, if $R \rightarrow S$ is étale, then $\Omega_{S/A} = \Omega_{R/A} \otimes_R S$.

Proof: This follows from(I.7.7.4) and(I.7.5.6). $\square$
Prop. (I.7.7.7) (Jacobson Criterion). Any étale map is equivalent to a standard smooth ring map \( S = R[X_1, \ldots, X_n]/(f_1, \ldots, f_n) \) that \( J(f_1, \ldots, f_n) \) is invertible in \( S \).

\[ I/I^2 \cong \Omega_{P/R} \otimes_P S, \text{ so } I/I^2 \text{ is free, so by I.6.7.12}, \text{ there is a presentation of } S \text{ that } f_1, \ldots, f_c \text{ freely generate } I/I^2, \text{ then obviously } c = n \text{ and } J(f_1, \ldots, f_n) \text{ is invertible in } S, \text{ i.e. } S \text{ is standard smooth. } \]

\[ \square \]

Cor. (I.7.7.8) (Example of Étale Maps). The ring

\[ S = R[X_1, \ldots, X_n, X_{n+1}]/(f_1, \ldots, f_n, X_{n+1} \det(f_1, \ldots, f_n) - 1) \]

is étale over \( R \).

Prop. (I.7.7.9). If \( R \to S \) is étale at a nbhd of a prime \( q \) of \( S \) over \( p \), then \( pS_q = qS_q \), and \( k(q)/k(p) \) is finite separable.

\[ \text{Proof: } \text{We can replace } S \text{ by } S_q \text{ so } S_q/R \text{ is étale. Then } S \otimes k(p)/k(p) \text{ is étale, that is } S_p/pS_p \text{ is a finite product of finite separable fields, so } S_q/pS_q = (S_p/pS_p)_q = \text{some separable closed field. } \]

\[ \square \]

Lemma (I.7.7.10). If \( R \to S \) is an étale map and \( q \) is a prime of \( S \) over \( p \), then \( S/R \) is étale in a nbhd of \( q \) if

- \( R \to S \) is of f.p.
- \( R_p \to S_q \) is flat.
- \( pS_q = qS_q \).
- \( k(q)/k(p) \) is a finite separable field extension.

\[ \text{Proof: } \text{Cf.[Sta]00U6}. \]

\[ \square \]

Prop. (I.7.7.11) (Equivalent Definition of Étale). A ring map \( R \to S \) is étale iff it is flat, of f.p. and \( \Omega_{S/R} \) vanishes.

\[ \text{Proof: } \text{One direction is by definition, and the converse is by I.7.7.10 and I.7.6.7.} \]

\[ \square \]

Prop. (I.7.7.12). A ring map of f.p. is formally étale iff it is étale. (Because in this case, formally smooth is equivalent to smooth I.7.5.14.)

Prop. (I.7.7.13). If \( S/R \) and \( S'/R \) are étale, then any \( R \)-algebra map \( S \to S' \) is étale.

\[ \text{Proof: } S \to S' \text{ is of f.p. by I.6.7.11, the rest Cf.[Sta]00U7}. \]

\[ \square \]

Prop. (I.7.7.14) (Étale Algebra seen explicitly as Finite Projective Modules). Étale algebras are finite projective, by I.6.1.7. And we can see this clearly as follows: There is an diagonal idempotent as it is unramified I.7.6.10. If \( e = \sum a_i \otimes b_i \), then we can realize \( S \) as a direct command of \( R^n \) through maps

\[ S \overset{\alpha}{\to} R^n \overset{\beta}{\to} S \]

where \( \alpha(f) = (\text{tr}_{S/R}(f a_i)) \), and \( \beta((g_i)) = \sum g_i b_i \).

\[ \text{Proof: } \text{We check that } \beta \circ \alpha = \text{id: Notice first that } \text{tr}_{i_e}(e) = \text{tr}_{S/S}(1) = 1, \text{ following from the decomposition above, so } \sum \text{tr}_{S/R}(a_i)b_i = 1, \text{ thus shows that } \beta \alpha(1) = 1. \]

Now for general \( f \), using the formula \((f \otimes 1)e = (1 \otimes f)e\), we get \( \sum \text{tr}(f a_i)b_i = \sum \text{tr}(a_i)b_i f = f. \)

\[ \square \]
Prop. (I.7.7.15). If $R$ is a ring and $I$ is an ideal, then any étale ring map $R/I \to S$ comes from an étale ring map $R \to S$.

Proof: Use (I.7.7.7), an étale map is of the form $S = R[I[X_1, \ldots, X_n]/(f_1, \ldots, f_n)]$ that $\delta = J(f_1, \ldots, f_n)$ is invertible in $S$, then we take $S = R[X_1, \ldots, X_n, X_{n+1}]/(f_1, \ldots, f_n, X_{n+1} \delta - 1)$, then it is étale by (I.7.7.8) and maps to $S$. □

Standard Étale

Def. (I.7.7.16). A ring map $R \to R' = R[X]/(f)$ is called standard étale iff $f$ is monic and the derivative $f'$ is invertible in $R'$.

Standard étale is stable under base change and principal localization, but not stable under composition.

Prop. (I.7.7.17) (Étale and Standard Étale). A ring map is étale iff it is locally standard étale.

Proof: For a standard étale algebra $R[X]/(f) = R[Y]/(f, gY - 1)$ which is standard smooth and $\Omega_{R'/R} = 0$, so it is étale. To prove if it is locally standard étale then it is étale, Cf.[Sta]00UE. □

Prop. (I.7.7.18). Giving any ring $R$ and a prime $p$, if there is a finite separable extension $L/k(p)$, then there is a standard étale map $R \to R'$ that for some $q'$, $k(q') \cong L$ over $k$.

Proof: $L = k(p)[\alpha]$ by primitive element theorem, so the minimal polynomial of $\alpha$ is separable, and if we change $\alpha$ to $c\alpha$ for some $c \in k(p)$, we can assume $f$ can be lifted to a $f \in R[X]$. Now $f'(\alpha)$ is invertible in $L$, so there is a map from $R[X]/f$ to $L$, whose kernel gives the desired prime $q$. □

Étale over Fields

Prop. (I.7.7.19) (Étale over Fields). An algebra over a field $k$ is étale iff it is a finite product of finite separable extensions of $k$.

Proof: If $k'/k$ is finite separable, then $k' = k(\alpha)$ for some $\alpha$ by primitive element theorem, thus $k' = k[X]/(f)$ that $f'$ is invertible in $k'$, thus it is étale by (I.7.7.7). Conversely, Cf.[Sta]00U3. □

Cor. (I.7.7.20) (Finite Étale over Perfect Fields). Let $k$ be a perfect field. If $R$ is a $k$-algebra that is a finite as a $k$-module, then it is étale over $k$ iff it is reduced.

Proof: If it is étale, then it is reduced, by (I.7.7.19). Conversely, if it is finite and reduced, then it is a reduced Artinian ring (I.5.1.51), so a product of fields over $k$, so étale as $k$ is perfect. □

Prop. (I.7.7.21) (Étale and Unramified over Fields). A f.g. algebra is étale over field $k$ if it is $G$-unramified over it, by (I.7.5.25).

Prop. (I.7.7.22) (Maximal Étale Subalgebra). Let $A$ be an algebra of f.t. over a field $k$, then there is a maximal étale $k$-subalgebra of $A$. Also this subalgebra commutes with arbitrary field base change.
Proof: If $R$ is an étale subalgebra of $A$, then $R_k$ is étale over $k$, thus isomorphic to $(k)^n$ for some $n$. Now $n$ is smaller than the number of connected components of $\text{Spec} \ A_k$, which is finite. So it suffices to show the composite of two étale subalgebras of $A$ is étale. For this, notice $RR'$ is a quotient of $R \otimes_k R'$, which is a finite product of finite separable fields over $k$, thus its quotient is also a finite product of finite separable fields over $k$, which is étale. 

\[ \square \]

8 Local Algebra

Main references are [Local Algebra, Serre].

Prop. (I.7.8.1). If $A$ is a Noetherian local integral domain with residue field $k$ and quotient field $K$, if $M$ is a f.g. $A$-module that $\dim_k M \otimes_A k = \dim_K M \otimes_A K = r$, then $M$ is free of rank $r$.

In other words, if the rank of $M$ at the generic point and closed pt of $B$ are the same, then $M$ is free.

Proof: First $M$ is generated by $r$ elements by Nakayama and the kernel $R$ of $A' \to M$ vanishes when tensoring $K$, thus vanish because it is torsion-free. 

\[ \square \]

Prop. (I.7.8.2). Let $A \to B$ be a local ring map of local rings that
- $B$ is finite as an $A$-module.
- $m_B$ is a f.g. ideal.
- $A/m_A \cong B/m_B$.
- $m_A/m_A^2 \cong m_B/m_B^2$.

Then $A \to B$ is surjective.

Proof: By Nakayama, to show it is surjective, it suffices to show $A/m_A \to B/m_A B$ is surjective, then it suffices to show $m_A \otimes_A B/m_B \to m_B$ is surjective. For this, use Nakayama again on $B$ to reduce to the fact $m_A \otimes_A B/m_B \to m_B \otimes B/m_B$ is surjective, which is satisfied because this is just $m_A/m_A^2 \to m_B/m_B^2$. 

\[ \square \]
1.8  $p$-adic Commutative Algebras

$\mathbb{F}_p$-Algebras

**Def. (I.8.1.1) (relative Frobenius).** Let $S \to R$ be a ring map, then the relative Frobenius $\varphi_{R/S}$ is the map $R \otimes_{S,\text{Frob}} S \to R$ induced by universal property.

**Def. (I.8.1.2) (Perfect Rings).** A ring of characteristic $p$ is called perfect iff the Frobenius $\text{Frob}/\varphi : P \to P$ is an isomorphism. It is called semi-perfect iff Frob is surjective.

**Prop. (I.8.1.3) (Perfection and Coperfection).** If $R$ is of char $p$, we define $R_{\text{perf}} = \lim_{\leftarrow \varphi} R$ and $R^{\text{perf}} = \lim_{\rightarrow \varphi} R$.

The $(\cdot)_{\text{perf}}$ and $(\cdot)^{\text{perf}}$ are respectively the left and right adjoint to the forgetful functor from the category of perfect rings to the category of rings of characteristic $p$.

In particular, the category of perfect rings admits limits and colimits, and it equals the limits and colimits in the category of rings.

**Proof:** First both $R_{\text{perf}}$ and $R^{\text{perf}}$ are perfect: for $R_{\text{perf}}$, every element in $R_{\text{perf}}$ is represented by an element $a_n \in R_n$, and this element is equivalent to $a_n^p \in R_{n+1}$, so its $p$-th root is $a_n \in R_{n+1}$.

For $R^{\text{perf}}$, an element $(\ldots, x_n, \ldots, x_0)$ has $p$-th root $(\ldots, x_{n+1}, \ldots, x_1)$.

Second it is easily checked to be a functor because Frobenius is natural. The universal property is easy. □

**Prop. (I.8.1.4) (Perfection Kills Nilextensions).** If $f : R \to S$ is a map of rings of characteristic $p$ that is surjective with nilpotent kernel, then $R_{\text{perf}} \to S_{\text{perf}}$ and $R^{\text{perf}} \to S^{\text{perf}}$ are both isomorphisms.

**Proof:** $-_{\text{perf}}$ map is clearly surjective, and it is injective because if $a$ maps to 0, then $\text{Frob}^k(a) \in \text{Ker} f$ for some $k$, so it is nilpotent, so $\text{Frob}^{k+n}(a) = 0$.

$-_{\text{perf}}$ is clearly injective, and it is surjective because: suppose $\text{Ker} f^n = 0$, then for a $(s_n) \in S$, let $t_m$ be the inverse image of $s_{mn}$, for each $m$, and let $x = (x_n), x_{mn-k} = \text{Frob}^k t_m$, then $(x) \in R_{\text{perf}}$ and $x$ maps to $s$.

**Prop. (I.8.1.5) (Examples of Tilting).**

- $\mathbb{F}_p[t]_{\text{perf}} = \mathbb{F}_p[t^p], \mathbb{F}_p[t]^{\text{perf}} = \mathbb{F}_p$.
- $(\mathbb{Z}_p)^{\text{perf}} = \mathbb{F}_p$.
- If $R$ is a perfect ring of char $p$ and $f \in R$ is a non-zerodivisor, then $(R/f)^{\text{perf}}$ is the $f$-adic completion of $R$. In particular, $(\mathbb{F}_p[t^{1/p}] / (t))^{\text{perf}} \cong \mathbb{F}_p[t^{1/p}]$.
- $(\mathbb{Z}_p[p^{1/p}])^{\text{perf}} \cong \mathbb{F}_p[p^{1/p}] \cong \mathbb{F}_p[t]^{\text{perf}} \cong (\mathbb{F}_p[t]^{\text{perf}} / (t))^{\text{perf}}$.

**Proof:** The first two are trivial, for the third, notice $R_f = \lim_{\to n} R/f^n = \lim_{\to n} R/f^n$, and there are commutative diagram $R/f \xrightarrow{\varphi} R/f$ and $R/f^{p^n+1} \xrightarrow{\varphi^{k+1}} R/f^{p^k}$, so $\lim_{\to n} R/f^{p^n} \cong (R/f)^{\text{perf}}$. 

The $\varphi$-extension $(R/f)^{\text{perf}}$ is a perfect ring.
For the fourth, only the first equivalence needs proving, the others are consequences of the first three items. Then notice
\[
\left( \mathbb{Z}_p[p^{\infty}] \right)^{\text{perf}} \cong \lim_{\leftarrow} (\mathbb{Z}_p[p^{\infty}]/p^k)_{\text{perf}} \cong (\mathbb{Z}_p[p^{\infty}]/p)_{\text{perf}} \cong (\mathbb{F}_p[t^{\infty}]/t)_{\text{perf}} \cong \mathbb{F}_p[t^{\infty}]
\]
The last isomorphism by item 3.

Def. (I.8.1.6) (Perfectly Finitely Presented). A map of perfect \( \mathbb{F}_p \)-algebras \( B \to A \) are called perfectly finitely presented if \( A = (A_0)_{\text{perf}} \) for some f.p. \( B \)-algebra \( A_0 \).

Prop. (I.8.1.7) (Aberbach-Hochster). Let \( R \) be a perfect \( \mathbb{F}_p \)-algebra, \( f_1, \ldots, f_r \in R \), and consider the ideal \( I = \sqrt{(f_1, \ldots, f_r)} \subset R \). Then \( R/I \) has flat dimension \( \leq r \) as an \( R \)-module.

Proof: We only prove for \( r = 1 \).

We show \( I = \lim_{\leftarrow} f_1/p^{a_n} R \). The map is given by \( (a_n) \mapsto f_1/p^a a_n \). This is injective because if \( f_1/p^a a = 0 \), then \( f_1/p^{a+1} a = 0 \) by perfectness, thus \( a \) is killed by the transition map \( f_1/p^{a-1}/p^{a+1} \).

Prop. (I.8.1.8) (Perfect Algebras are Tor Independent). For any two perfect \( A \)-algebras \( B, C \) where \( A \) is a perfect \( \mathbb{F}_p \)-algebra, \( \text{Tor}_i^A(B, C) = 0 \) for \( i > 0 \).

Proof: \( A \to B \) can be written as a composition of a perfection of a free \( A \)-algebra and a quotient. The perfection of free algebra is flat, thus we can assume \( B = A/I \). By a filtered colimit argument again, we can assume \( I = (f_1^{\frac{1}{p^n}}, \ldots, f_r^{\frac{1}{p^n}}) \) is perfectly f.p. By induction, we can assume \( r = 1 \). Now the lemma(I.8.1.7) applied to \( R = C \) shows that \( IC = \lim_{\leftarrow} f_1/p^{a-1}/p^{a+1} C = I \otimes_A^L C \), so \( B \otimes_A^L C = C/IC \) is discrete.

Prop. (I.8.1.9). If \( R \) is a perfect \( \mathbb{F}_p \)-algebra, \( I \) is a radical ideal, and \( J = R[I] \subset R \), then \( J \) and \( I + J \) are both radical, and the square

\[
\begin{array}{ccc}
R & \longrightarrow & R/I \\
\downarrow & & \downarrow \\
R/J & \longrightarrow & R/I + J
\end{array}
\]
is both a fiber pullback and pushout square of commutative rings.

Proof: \( J \) is clearly a radical, and notice that \( I + J \) is the kernel of the map \( R \to R/I \otimes_R R/J = R/I + J \) (I.5.1.18), and the target is a colimit of perfect rings thus perfect, by(I.8.1.3). Thus \( I + J \) is also perfect, thus a radical ideal.

By(I.5.1.19), to show the map is a pullback square, it suffices to show that \( I \cap J = 0 \). If \( x \in I \cap J \), then \( x^2 = 0 \), thus \( x^p = 0 \), so \( x = 0 \).

Tilting

Def. (I.8.1.10) (Tilting). For any ring \( R \), let the tilting of \( R \) is defined to be \( R^\flat = (R/p)_{\text{perf}} \), endowed with the profinite topology.

Prop. (I.8.1.11). If \( R \) is a f.g. algebra over an alg.closed field \( k \) of charp, then \( R^\flat \cong k_{\pi_0}(\text{Spec} R) \).
Proof: It suffice to prove for Spec $R$ connected and reduced, because by (I.8.1.4). We first prove the case Spec $R$ is irreducible, i.e. $R$ is integral:

In this case, choose a closed point $x$, then there is a map $R \to \hat{R}_x$, where $\hat{R}_x$ is the $m_x$-adic completion. By Krull (I.5.6.11) and the fact $R$ is integral, this map is injective, so it suffices to show that $0 = (\hat{R}_x)^{\text{perf}} = \lim(R_x/m^n_x)^{\text{perf}}$. But $(-)^{\text{perf}}$ is a right adjoint so commutes with colimits and $\hat{R}_x = \lim R_x/m^n_x$. But $(R_x/m^n_x)^{\text{perf}} = (R_x/m_x)^{\text{perf}} = k^{\text{perf}} = k$, by (I.8.1.4) again.

If $R$ is not irreducible, ?

Prop. (I.8.1.12). If $R$ is a $p$-adically complete commutative ring, $\pi \in \text{rad}(p)$, then the map $R \to R/p$ induces an homeomorphism of monoids:

$$\lim_{x \to \hat{R}} R \cong \lim_{x \to \hat{R}} R/\pi = \lim_{x \to \hat{R}} R/p(I.8.1.4) = R^\flat$$

Proof: Injectivity: if $(a_n), (b_n) \in \lim_{x \to \hat{R}} R$ satisfies $a_n \equiv b_n \mod \pi$ for all $n$, then applying power lifting (XIV.1.2.6), $a_n \equiv b_n \mod \pi^{n+k}$ for all $k$, so $a_n = b_n$.

Surjectivity: for $(\overline{a_n}) \in R^\flat$, choose arbitrary lifting $a_n$, then $a_{n+k+1}^p \equiv a_{n+k} \mod \pi$ for all $n+k$, so $k \mapsto a_{n+k}^p$ is a Cauchy sequence by power lifting (XIV.1.2.6) again, thus converging to some point $b_n$. then it’s easily checked that $b_{n+1}^p = (\lim a_{n+1+k}^p)^p = \lim a_{n+1+k}^p = b_n$. so $(b_n)$ maps to $(\overline{a_n})$.

For the topology: it is clearly continuous, and for the reverse, if $(a_i), (b_i)$ satisfies that $a_i \equiv b_i \mod \pi$ for $i < k$, then the image in $\lim_{x \to \hat{R}} R$ satisfies $x_i \equiv y_i \mod p^{k-i}$ for $i < k$, thus it is open. □

Cor. (I.8.1.13) (Sharp Map). From this proposition, we get a multiplicative sharp map: $\sharp : R^\flat \to R$, and its image is just the elements that has a compatible system of $p^k$-th roots $x^{1/p^k}$. These elements are also called perfect.

Cor. (I.8.1.14) (Addition in $R^\flat$). From the isomorphism (I.8.1.12) above, we can read what the addition looks like in the presentation $\lim_{x \to \hat{R}} R$: if $(f_n), (g_n)$ are two elements, then their addition is given by $(h_n)$, where $h_n = \lim(f_{n+k} + g_{n+k})^{1/p^k}$.

Cor. (I.8.1.15) (Fontaine’s Map). By (XI.2.4.4), the natural map $R^\flat \to R/\pi$ induces a map $\theta_R : W(R^\flat) \to R$ of rings, called the Fontaine’s functor, which writes as $\sum[a_i]p^i \mapsto \sum a_i^\sharp p^i$. And we denote $A_{inf}(R) = W(R^\flat)$ the Fontaine’s ring of $R$.

Prop. (I.8.1.16). Fontaine’s map $\theta_R$ is surjective iff $R/p$ is semiperfect.

Proof: By Nakayama, $\theta$ is surjective iff it is surjective modulo $o$. Because its reduction mod$p$ is $R^\flat \to R/p$ is surjective as $\varphi : R/p \to R/p$ does. □

Prop. (I.8.1.17) (Tilting as a Valuation Ring). If $R$ is a domain or a valuation ring, then the same is true for $R^\flat$. In the valuation case, the valuation of $R^\flat$ can in fact be chosen to be $| \cdot | \circ \sharp$, so in particular, the rank of $R^\flat$ is no more than the rank of $R$.

Proof: Use the isomorphism $\lim_{x \to \hat{R}} R \cong \lim_{x \to \hat{R}} R/p = R^\flat(I.8.1.12)$.

For the domain case, if $(a_n)(b_n) = 0$, then $a_n b_n = 0$, so $a_0 = 0$ or $b_0 = 0$, so $(a_n) = 0$ or $(b_n) = 0$. Similarly, if $R$ is a valuation ring, then $R^\flat$ is firstly a domain, and it suffices to prove that for any $(a_n), (b_n) \in R^\flat$, the quotient of one by another is in $R^\flat$, by (I.9.2.3). For this, because $R$ is
valuation ring, we may assume $a_0/b_0 \in R$, so $a_n/b_n$ is also in $R$, because their power do, and $R$ is normal(I.9.2.6), thus $(a_n)/(b_n) \in R^\circ$.

For the valuation given, notice in the above proof, $|a_n| \leq |b_n|$ iff $|a_0| \leq |b_0|$, so the valuation are equivalent to $|·|\circ \#$ by(I.9.3.10), so it can be chosen to be so. □

Prop. (I.8.1.18) (Tilting and Completion). If $R = A/I$ with $A/p$ perfect, then $R^\circ$ identifies with the $I$-adic completion of $A/p$.

Proof: $R^\circ = \varprojlim_{\varphi} R/p = \varprojlim_{\varphi} A/(p, I)$. But there are commutative diagrams

$$
\begin{align*}
A/(p, I) & \xrightarrow{\varphi} A/(p, I) \\
\varphi^{n+1} & \downarrow \varphi^n \\
A/(p, p^{n+1}) & \longrightarrow A/(p, p^n)
\end{align*}
$$

where the vertical arrows are isomorphisms because $A/p$ is perfect. So the conclusion follows. □

2 $p$-Local Rings

Def. (I.8.2.1) ($p$-Local Rings). A commutative ring is called $p$-local if $p \in \text{rad } A$.

Prop. (I.8.2.2). Let $A$ be a $p$-adically complete ring, then $A$ is $p$-local, by(I.5.7.18).

Prop. (I.8.2.3). If $A$ is $p$-adically complete and has bounded $p$-torsions, the $p$-completion of a smooth $A$-algebra is $p$-completely smooth.

Proof: This follows from(I.10.7.4) and(I.10.6.16). □

Def. (I.8.2.4) ($p$-Normal Rings). A $p$-torsion-free ring $R$ is called $p$-normal if $R$ is $p$-root closed in $R[\frac{1}{p}]$.

3 Witt Theory and $\delta$-Rings

Complete discrete Valuation Ring

Main references are [Local Fields, Serre], [Integral $p$-adic Hodge, BMS].

Structure of complete discrete valuation ring will be studied in this subsubsection.

Prop. (I.8.3.1) ($0$-Type). Let $A$ be a local ring with maximal ideal $m$. Now if $A/m$ is field of characteristic 0, then $A$ contains a field mapped isomorphically onto $k$.

Proof: $\mathbb{Z} \rightarrow A \rightarrow k$ is injective, so $\mathbb{Z}$ is units in $A$, thus $\mathbb{Q} \in A$, hence $A$ contains a field. Now we show using Zorn’s lemma that the maximal field $S$ of $A$ is mapped onto $k$.

First $k$ is algebraic over $\mathbb{S}$, this is because if there is a transcendent element $\overline{a}$, then the inverse image $a$ is transcendental over $S$ and $S[a] \cap a_1 = 0$, so $S(a) \subset A$. Now for any $a$ that is algebraic over $\mathbb{S}$, the minipoly has no multiple roots, so it has a lifting by Hensel’s lemma, so we are done. This is Cohen’s lemma in [Matsumura P206]. □

Def. (I.8.3.2) (Strict $p$-Ring). A $p$-ring $A$ is a ring which is complete Hausdorff in the topology defined by the decreasing chain of ideals $a_1 \supset a_2 \supset \cdots$ such that $a_n a_n \subset a_{m+n}$ that $k = A/a_1$ is a perfect ring of characteristic $p$.

It is called a strict $p$-ring if moreover $a_n = p^n A$ and $p$ is not a zero-divisor of $A$.
Prop. (I.8.3.3) (Teichmüller Lift). For a $p$-ring, there exists a unique system of representatives $k \rightarrow A$ that $f(\lambda^p) = f(\lambda)^p$, called the Teichmüller lift.

For this representative, it is also multiplicative, and if $A$ has char $p$, then it is also additive. And an element is in the image of $f$ iff it is a $p^n$-th power for any $n$.

Proof: For any $\lambda \in k$, the $\lambda p^n$ is unique in $k$, and if we consider $U_n$ the set of all $x^{p^n}$ where $x$ is a lift of $\lambda p^n$, then $U_n$ is a descending set. Moreover, the diameter converges to 0, because $a \equiv b \mod a_i$ implies $a^{p^n} \equiv b^{p^n} \mod a_{i+1}$ as $p \in a_i$. So it converges to a unique point $f(\lambda)$ in $A$. And we see that any other $f'$ maps $\lambda$ to a $p^n$-th root hence in $U_n$ for any $n$, hence it map be equal to $f(\lambda)$. The rest is easy.

Cor. (I.8.3.4) (Equal Characteristic case). If $A$ is a complete discrete valuation ring with residue field $k$. If $k$ and $A$ have the same characteristic and $k$ is perfect, then $A \cong k[[T]]$.

Cor. (I.8.3.5). When $A$ is a strict $p$-ring, elements of $A$ can be written uniquely as $\sum f(\alpha_i)p^n$.

Def. (I.8.3.6) ($(0,p)$-type case). When $A$ is a complete DVR with residue field $k$ and quotient field $K$. If char$K = 0$ and char$k = p$, then $p$ goes to zero in $k$, so $e = v(p) \geq 1$, called the absolute ramification index of $A$. It is called absolutely unramified iff $e = 1$.

Remark (I.8.3.7) (Canonical Strict $p$-Ring). The canonical strict $p$-ring is the ring $\hat{S} = \hat{Z}[X^p]$ where $X$ is all $p^n$ roots.

Now we consider the $* = \cdot$ in $\hat{S}$. Then there are elements $Q^n_i \in \hat{Z}[X^p, Y^p]$ that $x * y = \sum f(Q^n_i)p^n$ where $f$ is the Teichmüller lift.

Prop. (I.8.3.8) (Universal Law of $p$-Rings). For any $p$-ring $A$ with residue ring $k$, the calculation in $A$ is dominated by $Q^n_i$ defined in (I.8.3.7), i.e.

\[
\left( \sum f(\alpha_i)p^n \right)^* \left( \sum f(\beta_i)p^n \right) = \sum f(\gamma_i)p^n
\]

where $\gamma_i = Q^n_i(\alpha_0, \alpha_1, \ldots, \beta_0, \beta_1, \ldots)$.

Proof: There is a map $\theta$ from $\hat{S} = \hat{Z}[X^p, Y^p]$ to $A$ induced by $f(\alpha_i), f(\beta_i)$ as they all has $p^n$-th roots. Then notice $\theta$ induce a $\theta$ on residue ring and these two $\theta$ commutes with Teichmüller lift, as seen by the definition of the latter. Then the theorem follows immediately.

Cor. (I.8.3.9). For two $p$-ring $A, A'$ that $A$ is strict, then any map $\varphi$ of their residue ring induces a unique ring homomorphism $A \rightarrow A'$. In particular, two strict $p$-ring with the same residue ring is canonically isomorphic.

Proof: We have already seen that ring homomorphism commutes with Teichmüller lift. Now we define

\[
g(a) = \sum g(f(\alpha_i))p^n = \sum f(\varphi(\alpha_i))p^n
\]

and this is the unique choice. It is a ring homomorphism by universal law of (I.8.3.8).

Prop. (I.8.3.10) (Witt Vectors of Perfect Rings). For any perfect ring $k$ of char $p$, there exists uniquely a strict $p$-ring $W(k)$ that has residue ring $k$, called the ring of Witt vectors with coefficients in $k$. $W$ is a faithful functor from perfect rings to strict $p$-rings by (I.8.3.9).
Proof: For a canonical ring $\mathbb{F}_p[X_\alpha^{p^{-n}}]$, $\hat{Z}[X_\alpha^{p^{-n}}]$ is a strict $p$-ring. Now arbitrary perfect $p$-ring is a quotient of $\mathbb{F}_p[X_\alpha^{p^{-n}}]$, so we can construct its strict $p$-ring $W(k)$ as the quotient of $\hat{Z}[X_\alpha^{p^{-n}}]$. Uniqueness is by(I.8.3.9).

Notice it is nothing mysterious, it is just the set of all formal sum $\sum f(x_i)p^i$ under the operation defined in(I.8.3.7). See also(I.8.3.15).

\section*{Witt Vectors}

Def. (I.8.3.11) (Commutative Coalgebra $\Delta$). Let $P \subset \mathbb{N}$ be a subset stable under taking divisors, define $\Delta_P = Z[\theta_n | n \in P]$ to be a polynomial ring where $\theta_n$ are free variables, then it has a commutative cogroup structure, given by

$$+ : \Delta \rightarrow \Delta \otimes \Delta : \theta^n \mapsto \theta_n \otimes 1 + 1 \otimes \theta_n$$

Let $\Delta_P^n$ be the sub commutative coalgebra $Z[\theta_m | m \in P, m \leq n]$. Also, when $P = \{1, p, p^2, \ldots\}$, denote $\Delta_P = \Delta_p$.

Def. (I.8.3.12) (Witt Polynomials). For $n \in \mathbb{N}$, the $n$-th Witt vectors are defined to be

$$W_n = \sum_{d \mid n} dX_d^{n/d}.$$

and we can consider $Z[W_n(\theta_k) | n \in P]$ as a sub coalgebra of $\Delta_n$.

Def. (I.8.3.13) (Witt Vectors). For a subset $P \subset \mathbb{N}$ stable under taking divisors, we define the set $W_P(A) = \text{Hom}(\Delta_P, A)$ which has a natural commutative ring structure. and if $P = \{1, p, p^2, \ldots\}$, we write $W_p(A) = W_P(A)$. Then the Witt vectors $W_n$ for $n \in P$ are functions on $W_P(A)$. For $x \in W_P(A)$, $w_n(x)$ are called the ghost components of $x$, and $x_n$ are called the Witt components.

So there is a map $Z[W_n(\theta_0, \ldots, \theta_n)] \rightarrow$ inducing an morphism of rings: $W(A) \rightarrow \prod_{\mathbb{Z}} A$ that maps

$$(f(\theta_n)) \mapsto (f(\varphi^n))$$

Where the right hand side is the usual addition and multiplication, the left side is the usual coordinate of Witt vector, and $f(\theta_n)$ is called ghost component. This is an embedding if $A$ is $p$-torsionfree, and an isomorphism iff $\frac{1}{p} \in A$, because $\theta_n$ can be presented by $\varphi^n$.

Lemma (I.8.3.14) (Formula for $p$-Rings). For $* = +$ or $\times$, there are integral polynomials $S_*(X_i, Y_i)$ that

$$\left(\sum f(\alpha_i)p^i\right) * \left(\sum f(\beta_i)p^i\right) = \sum f(\gamma_i)p^i$$

where $\gamma_i = Q_i^i(\alpha_0, \alpha_1, \ldots, \beta_0, \beta_1, \ldots)$. And for $+$, when reduced to $\mathbb{F}_p$, $Q_i^+$ are polynomials in $X_i^{p^{-n}}, Y_i^{p^{-n}}$ for $i \leq n$ and homogenous of degree 1. And

$$Q_i^+ = (X_i + Y_i) + (X_i^{p^{-1}} + Y_i^{p^{-1}})R_{n,n-1} + \ldots + (X_i^{p^{-n}} + Y_i^{p^{-n}})R_{n,0}.$$

Proof: We solve $S_n$ by induction. Notice for any lift $\hat{S}_i$ of $S_i$,

$$f(S_i) \equiv \hat{S}_i(X_1^{1/p^{n-i}}, Y_1^{1/p^{n-i}})^{p^{n-i}} \mod p^{n-i+1}$$

so we mod $p^{n+1}$ to solve $S_n$:

$$S_n \equiv 1/p^n\left(X_0 + Y_0 + \ldots + p^nX_n + p^nY_n - \hat{S}_0(X_1^{1/p^n}, Y_1^{1/p^n})p^n - \ldots - p^{n-1}\hat{S}_{n-1}(X_1^{1/p^n}, Y_1^{1/p^n})p^n\right)$$

The rest follows by induction.
Prop. (I.8.3.15). Notice in Serre book, he presented the Witt vectors in $(f(\theta_n))$ coordinates. In this coordinate, if $k$ is a perfect ring and we let

$$T(a_i) = \sum f(a_i)p^{i}p,$$

then $T$ is a ring isomorphism from $W(k)$ to the strict $p$-ring with residue ring $k$.

Proof: We need to prove this is a ring homomorphism. That on $W(A)$ is to make $\varphi$ a ring homomorphism, and that on the right is usual. It suffice to prove for the canonical strict $p$-ring, as seen by the universal law(I.8.3.8).

For this, we let $(\sum X_{i}^{p^{i}}p) = (\sum Y_{i}^{p^{i}}p)^{i}$, and $W_n(a_i)\equiv W_n(b_i) = W_n(\varphi_i)$, where $\varphi_i \in F_{p}[X_i, Y_i]$ and $\varphi_i \in Z[X_i, Y_i]$, they both exist, the latter because of(I.8.4.7).

Then we mod $p^{n+1}$, and let $X_i = X_i^{p^n}, Y_i = Y_i^{p^n}$, so

$$W_n(\varphi_i) = W_n(X_i) \equiv W_n(Y_i) ≡ \sum_{i \leq n} f(\psi_i(X_i^{p^n}, Y_i^{p^n})) p^{i} \equiv W_n(\psi_i) \mod p^{n+1}.$$

Now induction, $\varphi_i \equiv \psi_i \mod p$, then $p^n\varphi_i \equiv p^n\psi_i \mod p^{n+1}$ so this is true for $n$, too. □

Cor. (I.8.3.16). For example, $W(F_p)$ is the unramified extension of $\mathbb{Z}_p$ of degree $n$. And $W(F_p)$ is the completion of the maximal unramified extension of $W(F)$.

Def. (I.8.3.17) (Witt Vectors over Valued Rings). If a perfect ring $R$ itself has a complete valuation $v$, then we can endow $W(R)$ with a finer topology: we let $w_k(x) = \inf_{i \leq k} v(x_i)$, where $x = \sum p^i f(x_i)$. Now $w_k(x + y) \geq \inf(w_k(x), w_k(y))$ by(I.8.3.14). The weak topology of $W(R)$ is defined by the semi-valuations $w_k$.

Prop. (I.8.3.18). If $a, b \in O_R + p^{n+1}W(R)$, then

$$p^n v(a - b) \geq w_n(a - b) \geq \inf_{k \leq n} p^{-k} v(a_n - b_n - b_{n-k}).$$

So we see that a sequence is Cauchy in $W(R)$ if each coordinate is Cauchy in $R$, so $W(R)$ is complete in the weak topology.

Proof: Firstly the last proposition follows from the first because we can always multiply by a $f(a)$ to make the first $n$ coordinate in $O_R$.

The first is nearly an immediate consequence of(I.8.3.14). □

Prop. (I.8.3.19). $O_{\mathcal{E}} = W(K^{\frac{1}{p}})$ is a complete ring with maximal ideal $pO_{\mathcal{E}}$. And $O_{\mathcal{E}}[\frac{1}{p}] = \mathcal{E}$ is complete ring of character $p$. And the same construction of $K^{\frac{1}{p}}$ yields the completion of maximal unramified extension of $O_{\mathcal{E}}$, and the Galois group is the same as $G_{K}$.

Prop. (I.8.3.20) (van Der Kallen’s Theorem). If $A \rightarrow B$ is an étale morphism, then $W_r(A) \rightarrow W_r(B)$ is also étale. Moreover, if $A \rightarrow A'$ is any ring map with $B' = B \otimes_A A'$, then the natural map

$$W_r(A') \otimes_{W_r(A)} W_r(B) \rightarrow W_r(B')$$

is also an isomorphism.

Proof: Cf.[Integral p-adic Hodge, BMS]. □
Truncated Witt Vectors

**Def. (I.8.3.21)** \((W_2)\). The construction of \(W_2\) is as follows: \(W_2(A) = A \times A\) with the addition and multiplication given by

\[
(x_0, x_1) + (y_0, y_1) = (x_0 + y_0, x_1 + y_1 + \frac{x_0^p + y_0^p - (x_0 + y_0)^p}{p}), \quad (x_0, x_1)(y_0, y_1) = (x_0y_0, x_0^py_1 + y_0^px_1 + px_1y_1)
\]

There are two natural morphism of rings \(\varepsilon_1, \varepsilon_2 : W_2(A) \to A:\)

\[
\varepsilon_1((x_0, x_1)) = x_0, \quad \varepsilon_2((x_0, x_1)) = x_0^p + px_1.
\]

**p-Typical Witt Vectors**

**Def. (I.8.3.22).**

4 \(\delta\)-Rings

**Def. (I.8.4.1) (\(\delta\)-Ring).** A \(\delta\)-ring structure on \(R\) characterize the deficit in lifting the Frobenius action on \(R/p\), i.e. \(\varphi(x) = x^p + p\delta(x)\). A \(\delta\)-ring is a pair \((R, \delta)\) where \(R\) is a commutative ring and \(\delta : R \to R\) is a map that \(\delta(0) = \delta(1) = 0\), and satisfies:

\[
\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y), \quad \delta(x + y) = \delta(x) + \delta(y) - \frac{(x + y)^p - x^p - y^p}{p}.
\]

\(\delta\)-rings naturally form a category. And in case \(A\) is \(p\)-torsionfree, a \(\delta\)-structure on \(A\) is the same as a lifting of the Frobenius on \(A/p\).

**Def. (I.8.4.2) (\(\delta\)-Pairs).** The category of \(\delta\)-pairs consists of pairs \((A, I)\) where \(A\) is a \(\delta\)-ring and \(I\) is an ideal of \(A\) that morphisms \(\varphi : (A, I) \to (B, J)\) are \(\delta\)-ring maps \(A \to B\) that \(\varphi(I) \subseteq J\).

**Prop. (I.8.4.3) (\(\delta\)-Rings and \(W_2\).** A \(\delta\)-ring structure on \(A\) is the same as a section of the natural map \(W_2(A) \to A\), and a morphism of \(\delta\)-rings is a commutative diagram of sections.

**Proof:** By the description of \(W_2\) in (I.8.3.21), this is clear, the morphism is given by \(A \to W_2(A) : x \mapsto (x, \delta(x))\).

**Lemma (I.8.4.4) (Initial \(\delta\)-Algebra).** We usually work with \(\delta\)-algebras over \(\mathbb{Z}_{(p)}\). Then there is an initial object in the category of \(\delta\)-rings, given by \(\mathbb{Z}_{(p)}\) with \(\delta(x) = \frac{x - x^p}{p}\).

**Prop. (I.8.4.5).** For a \(\delta\)-ring \(A\), \(\varphi\) commutes with \(\delta\).

**Proof:** We need to check \(\delta(x^p + p\delta(x)) = \delta(x)^p + p\delta(\delta(x))\). This is hard to check, but we can check \(\varphi(\frac{x^p - x^p}{p}) = \varphi(\varphi(x))^p - \varphi(\varphi(x))^p\), so the conclusion is true when \(A\) is \(p\)-torsionfree. But by (I.8.4.9), every \(\delta\)-ring is a quotient of a free thus \(p\)-torsionfree ring, thus the equation is also true for arbitrary \(A\).

**Prop. (I.8.4.6) (Witt Vectors as \(\delta\)-Rings).** Let \(W_p(A)\) be defined as in (I.8.3.13), then there is a natural structure on \(W_p(A)\) as we define now:

Using the formulas in (I.8.4.1), we can write \(\delta^n(xy)\) and \(\delta^n(x + y)\) as functions of \(\delta^i(x), \delta^i(y)\) for \(0 \leq i \leq n\), i.e.

\[
\delta^n(xy) = M_n(x, \delta(x), \delta^2(x), \ldots, \delta^n(x), y, \delta(y), \ldots, \delta^n(y))
\]
Then we use these to define a coalgebra structure on \(\mathbb{Z}[e, \delta, \ldots, \delta^n, \ldots]\), and by (I.8.4.7), there is a coalgebra isomorphism \(\Delta_p \cong \mathbb{Z}[e, \delta, \ldots, \delta^n, \ldots]\), which maps \(\theta_{p^n} \to \Theta_n\).

**Proof:** □

**Lemma (I.8.4.7) (δ-Component).** Let \(\varphi = f^p + p\delta\) be a polynomial in \(e, \delta\), let \(\Theta_n\) be polynomials in \(e, \delta, \delta^2, \ldots, \delta^n\) with integer coefficients that

\[
\varphi^o = \Theta_0^o + p\Theta_1^o + \cdots + p^n\Theta_n = W_p^n(\Theta_0, \ldots, \Theta_n), \forall n
\]

Then \(\mathbb{Z}[\Theta_0, \Theta_1, \ldots, \Theta_n] = \mathbb{Z}[e, \delta, \delta^2, \ldots, \delta^n]\) for any \(n\).

**Proof:** Use equation \(\varphi \circ \varphi^o = \varphi^o \circ \varphi\) and module \(p^n\mathbb{Z}[\Theta_0, \Theta_1, \ldots, \Theta_n]\). □

**Prop. (I.8.4.8) (Limits and Colimits of δ-Rings).** The forgetful functor from the category of \(\delta\)-rings to the category of rings admits a right adjoint given by \(A \mapsto W_p(A)\). It also admits a left adjoint given by free objects (I.8.4.9).

In particular, the category of \(\delta\)-rings admits limits and colimits, and their underlying rings are just the ring-theoretical limit and colimit.

**Proof:** The construction of limits is straightforward. To construct the colimits, use (I.8.4.3), the morphisms \(A_i \to W_2(A_i)\) induces a morphism \(\text{colim} A_i \to \text{colim} W_2(A_i) = W_2(\text{colim} A_i)\), and clearly this map is a section of \(W_2(\text{colim} A_i) \to \text{colim} A_i\), thus given a \(\delta\)-ring structure on \(\text{colim} A_i\). □

**Prop. (I.8.4.9) (Free δ-Rings).** The ring \(\mathbb{Z}\{x_i\}\) is a ring on the free generators \(\{x, \delta(x), \delta^2(x), \ldots\}\) and the Frobenius morphism defined by asserting \(\varphi(\delta^i(x)) = \delta^i(x)^p + p\delta^{i+1}(x)\).

Generally we can define the free \(\delta\)-ring generated by \(\{x_i\}\) over a \(\delta\)-ring \(A\) as the tensor \(A \otimes \mathbb{Z}\{x_i\}\), and it satisfies the universal property.

Then the Frobenius action is f.f.

**Proof:** It is easily verified that for any \(\delta\)-ring \(R\) over \(A\) and a map of sets \(\{x_i\} \to R\), there is a unique morphism \(\mathbb{Z}(\{x_i\}) \to R\) sending \(x\) to \(f\), thus verifying the universal property.

For the f.f., notice it is a colimit of \(\varphi_n : \mathbb{Z}[x, \delta(x), \ldots, \delta^n(x)] \to \mathbb{Z}[x, \delta(x), \ldots, \delta^{n+1}(x)]\), so by (I.7.1.17), it suffices to show \(\varphi_n\) are all f.f.. We decompose this map as \(n\) maps:

\[
\mathbb{Z}[x, \delta(x), \ldots, \delta^n(x)] \cong \mathbb{Z}[x, \delta(x), \ldots, (\delta^i(x))^p, \ldots, \delta^n(x)] \subset \mathbb{Z}[x, \delta(x), \ldots, \delta^n(x)]
\]

which are all f.f., so it is f.f. □

**Cor. (I.8.4.10) (Frobenius is Fpqc locally Surjective).** For a \(\delta\)-ring \(A\) and an element \(x \in A\), there is a faithfully flat morphism of \(\delta\)-rings \(A \to B\) that the image of \(x\) in \(B\) is of the form \(\varphi(y)\) for some \(y \in B\).

**Cor. (I.8.4.11).** Set \(B\) as the pushout of the diagram \(\mathbb{Z}(\{s\}) \leftarrow \mathbb{Z}(\{t\}) \to A\), where the arrows sends \(t\) to \(\varphi(s)\) and \(x\). \(B\) exists by (I.8.4.8) and the underlying ring is the same as the ring pushout, thus \(A \to B\) is faithfully flat by (I.8.4.9).
Extension of $\delta$-Structures

Lemma (I.8.4.12) (Quotients). Let $A$ be a $\delta$-ring and $I$ be an ideal, then if $I$ is stable under $\delta$, then there is a natural $\delta$-structure on $A/I$ compatible with $A$. In general, if $J$ is an ideal of $A$, then there is a universal $\delta$-$A$-algebra $B = A/J$, where $J = \cup_{n \geq 0} \delta^n(I)$. It is the universal $\delta$-$A$-algebra that the image of $I$ is $0$.

Lemma (I.8.4.13) (Localization). Let $A$ be a $\delta$-ring and $S$ be a multiplicative set of $A$ that $\varphi(S) \in S$, then there is a unique $\delta$-structure on $S^{-1}A$, and it satisfies the universal property.

Proof: Firstly if $A$ is $p$-torsionfree, in this case a $\delta$-structure is the same as a lifting of the Frobenius on $A/p$, thus the proposition is clear because $\varphi_{S^{-1}A} : S^{-1}A \rightarrow S^{-1}A$ is uniquely determined.

Generally, we choose a free $\delta$-ring $F$ and a surjection $\alpha : F \rightarrow A$, then $T = \alpha^{-1}S$ is multiplicative, and $T^{-1}F$ admits a unique $\delta$-structure. But now $S^{-1}A = T^{-1}F \otimes_F A$, so there is a $\delta$-structure on $S^{-1}A$ as the colimit, so compatible with that of $S^{-1}A$. Then it’s also the unique one (because if there is another one, the colimit properties gives a morphism of $\delta$-rings above $\text{id}_{S^{-1}A}$, which must by identity). □

Lemma (I.8.4.14) ($p$-adic Localization). If $A$ is a $\delta$-ring with $p \in \text{rad}(A)$, then the formula $\varphi(f) = f^p + p\delta(f)$ shows if $f$ is a unit, then $\varphi(f)$ is also a unit (I.4.2.2), so $S^{-1}A = T^{-1}A$, where $T = \{S, \varphi(S), \varphi^2(S), \ldots\}$.

Thus for any $\delta$-ring $A$ and a multiplicative set $S$, the $p$-localization (I.5.1.26) $(S^{-1}A)_{(p)}$ is the same as the $p$-localization of $T^{-1}A$. Then (I.8.4.13) shows that $(S^{-1}A)_{(p)}$ carries a unique $\delta$-structure compatible with that of $A$.

Lemma (I.8.4.15) (Completions). For a $\delta$-ring $A$ and a f.g. ideal $I$, the $I$-adic completion of $A$ has a unique $\delta$-structure compatible with that of $A$.

Proof: Let $A \rightarrow W_2(A)$ corresponds to the $\delta$-structure by (I.8.4.3), then there is a natural map $A \rightarrow W_2(A) \rightarrow W_2(\hat{A})$, then by the universal property of complete, it extends to a map $\hat{A} \rightarrow W_2(\hat{A})$, which is a ring map and is a section, all by universal properties. □

Lemma (I.8.4.16) (Derived Completion). If $A$ is a $\delta$-ring and $I \subset A$ is an ideal containing $p$, then the derived $I$-completion ring $\hat{A}$ of $A$ admits a unique $\delta$-structure extending that of $A$.

Proof: The proof is similar as (I.8.4.15). □

Prop. (I.8.4.17) (Étale Extension). Let $A$ be a $\delta$-ring with a f.g. ideal $I$ containing $p$. Assume $B$ is a derived $I$-complete and $I$-completely étale $A$-algebra, then $B$ admits a unique $\delta$-structure compatible with that of $A$.

In particular, any $\delta$-structure on an algebra $A$ passes uniquely to its derived $I$-completion for any ideal $I \subset A$ containing $p$.

Proof: By Elkik’s algebraization (I.10.7.9), we can write $B$ as derived $I$-completion of some étale $A$-algebra $B'$. Then $W_2(A) \rightarrow W_2(B')$ is étale by van der Kallen’s theorem (I.8.3.20). Then $W_2(B)$ is the derived $I$-completion of $W_2(B')$, with the $A$-algebra structure given by $A \rightarrow W_2(A) \rightarrow W_2(B')$. For the rest, Cf. [Scholze, Prism, 2.18]. □
Distinguished Elements

Def. (I.8.4.18) (Distinguished Elements). In a $\delta$-ring $A$, an element $d$ is called a distinguished element if $\delta(d)$ is a unit in $A$. A distinguished element is preserved by a $\delta$-ring map.

Lemma (I.8.4.19) (Distinguished up to units). If $A$ is a $\delta$-ring, $d$ is distinguished and $u$ is a unit, then $ud$ is also distinguished, if $d, p \in \text{rad}(A)$.

Proof: $\delta(ud) = u^p\delta(d) + d^p\delta(u) + p\delta(u)\delta(d)$ is a unit. \hfill $\square$

Lemma (I.8.4.20) (Irreducibility of Distinguished Elements). Let $A$ be a $\delta$-ring and $d$ be distinguished element in $A$. If $d = fh$ for some $f, h \in A$ that $f, p \in \text{rad}(A)$, then $f$ is also distinguished and $h$ is a unit.

Proof: Notice that $\delta(d) = f^p\delta(h) + h^p\delta(f) + p\delta(f)\delta(h)$, $\delta(d)$ is a unit, $f^p\delta(h) + p\delta(f)\delta(h) \in \text{rad}(A)$, thus $h^p\delta(f)$ is a unit, so we are done. \hfill $\square$

Prop. (I.8.4.21) (Characterization of Distinguished Elements). Fix a $\delta$-ring $A$ and an element $d$ that $d, p \in \text{rad}(A)$, then $d$ is distinguished iff $p \in (d, \varphi(d))$. In particular, distinguished elements is stable under units.

Proof: If $d$ is distinguished, then $\delta(d)$ is a unit, thus $\varphi(d) = d^p + d^p + p\delta(d)$ shows immediately $p \in (d, \varphi(d))$. Conversely, if $p = ad + b\varphi(d)$, we show $\delta(d)$ is invertible. It suffices to show it is invertible modulo $(d, p)$ as $d, p \in \text{rad}(A)$, or equivalently $(p, d, \varphi(d)) = A$. If it is not the case, then we may take a $(p, d, \varphi(d))$-adic completion to assume $p, d, \varphi \in \text{rad}(A)$, thus the equation simplifies to $p(1 - b\delta(d)) = cd$. The left side is distinguished, by(I.8.4.19), and then $d$ is also distinguished, by(I.8.4.20), so truly $(p, d, \varphi(d)) = A$. \hfill $\square$

Prop. (I.8.4.22) (Examples of Distinguished Elements). The element $d$ is distinguished in the following cases:

- (Crystalline cohomology)Take $A = \mathbb{Z}(p)$ and $d = p$, then $\delta(p) = 1 - p^{p-1}$ is a unit.
- (q-de Rham cohomology) Take $A = \mathbb{Z}_p[[q-1]]$ and $d = [p]_q = \sum_{i=0}^{p-1} q^i \in A$, with the $\delta$-structure determined by $\varphi(q) = q^p$.
- (Breuil-Kisin cohomology)Fix a discretely valued field $K/Q_p$ with uniformizer $\pi$, $W$ the maximal unramified subring of $\mathcal{O}_K$. Take $A = W[[u]]$ with $\delta(u) = u^p$, then any generator of the kernel of the map of $A \to \mathcal{O}_K$: $u \mapsto \pi$ is distinguished?.
- ($A_{inj}$-cohomology)Let $A$ be the $(p, q - 1)$-completion of $\mathbb{Z}_p[q^{1/p}]$. Then $A$ is $p$-torsion free and $\varphi(q) = q^p$ gives a $\delta$-structure. Then $d = [p]_q$ defined in item2 is also distinguished. And $\varphi^n(d)$ is distinguished for any $n \in \mathbb{Z}$ by(I.8.4.5).

Proof: 2: It is clear $\varphi$ is continuous and $\delta$ stabilizes $(q - 1)$, and $d$ is distinguished because the image of $\delta(d)$ in $A/(q - 1) \cong \mathbb{Z}_q$ is $\delta(p) = 1 - p^{p-1}$ is a unit, thus it is also a unit in $A$. \hfill $\square$

Perfect $\delta$-Ring

Def. (I.8.4.23) (Perfect $\delta$-Ring). A perfect $\delta$-ring is a $\delta$-ring that $\varphi$ is an isomorphism.

Prop. (I.8.4.24) (Perfections). The inclusion of the category of perfect $\delta$-rings to the category of $\delta$-rings admits left and right adjoints, $A_{perf}$ and $A^perf$ with definition similar to(I.8.1.10).
Proof: We use (I.8.4.3), the map \( A \to W_2(A) \to W_2(A_{\text{perf}}) \) extends uniquely to a map \( A_{\text{perf}} \to W_2(A_{\text{perf}}) \) lifting the \( \delta \)-action of \( A \). Similarly, because \((-)^{\text{perf}}\) is a limit and \( W_2(-) \) is a right adjoint, then \( W_2(-) \) commutes with \((-)^{\text{perf}}\). In particular, there is a natural map \( A^{\text{perf}} \to W_2(A^{\text{perf}}) \). \( \square \)

Lemma (I.8.4.25) (Frobenius Kills \( p \)-Torsion). If \( A \) is a \( \delta \)-ring and \( x \in A \) satisfies \( px = 0 \), then \( \varphi(x) = 0 \). In particular, if \( A \) is perfect, then \( A \) is \( p \)-torsionfree.

Proof: Applying \( \delta \) to \( px = 0 \), we have \( 0 = p^2\delta(x) + x^p\delta(p) + p\delta(x)\delta(p) = p^2\delta(x) + \varphi(x)\delta(p) \). As \( \delta(p) \) is a unit, and \( p\delta(x) = p^{p-1}(\varphi(x) - x^p) = \varphi(p^{p-1}x) - p^{p-1}x^p = 0 \), thus \( \varphi(x) = 0 \). \( \square \)

Prop. (I.8.4.26) (Perfect \( p \)-Complete \( \delta \)-Rings). The following categories are equivalent:

- The category of perfect \( p \)-adically complete \( \delta \)-rings.
- The category of \( p \)-adically complete and \( p \)-torsionfree rings \( A \) with \( A/p \) perfect.
- The perfect \( \mathbb{F}_p \)-algebras.

In particular, every perfect \( p \)-complete \( \delta \)-ring is of the form \( W(k) \) thus has Teichmuller expansions.

Proof: 2 and 3 are equivalent by Witt vector construction, by (XI.2.4.3), noticing that there is a natural lifting on \( W(k) \) lifting the Frobenius of \( k/\mathbb{F}_p \) that induces a \( \delta \)-functor. There is a forgetful functor from 1 to 2, by (I.8.4.25) (notice \( A/p \) is also perfect because if \( \varphi(x) \in (p) \), then \( x \in \varphi^{-1}(p) = (p) \) as \( \varphi(p) = p\varphi(1) / p \) and it is faithful. Now there is an equivalence from 3 \( \to \) 1 \( \to \) 2, thus 1 \( \to \) 2 is essentially surjective thus is an equivalence. \( \square \)

Prop. (I.8.4.27) (Perfect Element has Rank 1). Fix a \( \delta \)-ring \( A \) and an element \( x \in A \), then \( \delta(x^n) \in p^nA \) for any \( n \). In particular, if \( A \) is \( p \)-adically separated and \( y \) is perfect in \( A \), then \( \delta(y) = 0 \).

Proof: By formal calculation, it suffices to show that \( p\delta(x^n) \in p^{n+1}A \), which is equivalent to \( \varphi(x^n) \equiv x^{p^{n+1}} \mod p^{n+1}A \), which is true by (XIV.1.2.6). \( \square \)

Prop. (I.8.4.28) (Distinguished Element in Perfect \( \delta \)-Rings). Let \( A \) be a perfect \( p \)-complete \( \delta \)-ring (or perfect \( \mathbb{F}_p \)-algebra by (I.8.4.26)), and \( d \in A \), then \( d \) is distinguished iff its coefficient of \( p \) in the Teichmuller expansion (I.8.4.26) is a unit.

If \( d \) is distinguished, then it is a nonzero-divisor, and \( A/d[p^\infty] = A/d[p] \).

Proof: Let \( d = \sum_{i \geq 0} [a_i] p^i \), then

\[
\delta(d) = \frac{1}{p} \left( \sum_{i \geq 0} [a_i p] p^i - \sum_{i \geq 0} [a_i] p^i p^i \right) \equiv [a_1 p] \mod pA
\]

thus it is a unit iff \( a_1 \) is a unit, because \( A \) is \( p \)-complete.

Now if \( d \) is distinguished, and \( fd = 0 \). If \( f \neq 0 \), we may assume \( p \nmid f \), because \( A \) is \( p \)-torsionfree and \( p \)-adically complete (I.8.4.26). Now

\[
\varphi(f)\delta(fd) = \varphi(f)(f^p\delta(d) + \delta(f)\varphi(d)) = \varphi(f)f^p\delta(d) = 0,
\]

so \( f^p\varphi(f) = 0 \) and \( f^{2p} \equiv 0 \mod p \). Hence \( f \equiv 0 \mod p \), but then \( p|f \), contradiction.

For the last assertion, it suffices to show that \( A/d[p^2] = A/d[p] \). If \( p^2f = dg \), then \( \varphi(g)\delta(gd) = \varphi(g)(\delta(d)g^p + \varphi(d)\delta(g)) = \varphi(g)\delta(d)g^p + \varphi(d)\delta(g) \in pA \), thus \( \varphi(g)g^p \in pA \), hence \( g^{2p} \in pA \), and hence \( g \in pA \), showing \( pf \in dA \). \( \square \)
I.9 Topological Commutative Algebra

Main references are [Hub93], [Bos15], [B-S19], [Mor19] and [Sch12].

1 Topological Abelian Groups and Rings

Def. (I.9.1.1) (Topological Rings). A topological Abelian group is an Abelian group with a topology structure that the addition and inversion are all continuous.

A topological ring is a ring endowed with a topology structure that the addition, multiplication and inversion are all continuous.

Similarly we can define a topological module over a topological ring.

Def. (I.9.1.2) (Topologically Nilpotent Element). Let $A$ be a topological ring, then $x \in A$ is called topologically nilpotent iff $x^n \to 0$ when $n \to \infty$.

Def. (I.9.1.3) (Bounded Sets). A subset $S$ in a topological ring is called bounded iff for all open nbhd $U$ of 0, there exists an open nbhd $V$ of 0 that $VS \subset U$.

Def. (I.9.1.4) (Strict Morphism). A strict morphism of topological rings is a continuous morphism that the quotient topology and the subspace topology coincides on the image.

Completion of Topological Abelian Groups

Prop. (I.9.1.5) (Completion). There exists a completion functor left adjoint to the forgetful functor from the category of complete topological Abelian groups to the category of topological Abelian groups, given by Cauchy filters.

Proof: □

Prop. (I.9.1.6) (Subgroups and Completion). Let $A$ be a topological Abelian group, then the completion $i : A \to A^\wedge$ induces a bijection between the set of open subgroups of $A$ and the open subgroups of $A^\wedge$, given by $G \mapsto i(G) = G^\wedge$.

Proof: Cf.[Mor19]P74. □

Def. (I.9.1.7) (Restricted Power Series). Let $R$ be a topological ring, then we can define the restricted power series over $R$ to be

$$R\langle X_1, \ldots, X_n \rangle = \{ \sum a_v X^v \in R[[X_1, \ldots, X_n]] | \lim_{v \to \infty} a_v \to 0 \}$$

Adic Rings

Def. (I.9.1.8) (Adic Rings). An adic ring $R$ is a topological ring that the topology coincides with the $a$-adic topology for some ideal $a$ of $R$, and any such $a$ is called a ideal of definition.

Prop. (I.9.1.9) (Topologically Nilpotent Elements). Let $A$ be an adic ring and $x \in A$, then the following are equivalent:

- $x$ is topologically nilpotent.
- There exists an ideal of definition $I$ that the image of $x$ in $A/I$ is nilpotent.
- There exists an ideal of definition $I$ that $x \in I$. 

In particular, the set $A^{00}$ of nilpotent elements is an open radical ideal of $A$, and it is the union of all ideals of definitions in $A$.

**Proof:** $3 \rightarrow 1 \rightarrow 2$ is trivial. $2 \rightarrow 3$: if $x$ is nilpotent in $A/I$, let $J = I + xA$, then $J$ is an open ideal, and $J^n \in I$, so $I$-adic and $J$-adic topologies on $A$ coincide, so $J$ is an ideal of definition.

The rest is easy. \hfill \qed

**Prop. (I.9.1.10)** (**Adic Localization**). If $R$ is an adic ring with an ideal of definition $a$, the restricted power series $R(\xi)$ is complete w.r.t. the $(a)$-adic topology, and in fact

$$R(\xi) \cong \lim_{n} R/a^n[\xi].$$

For $f \in R$, we can define the **adic completion** of $R$ at $f$ to be the

$$R(f^{-1}) = \lim_{n} (R/a^n[1/f]).$$

The the natural map $R(\xi) \rightarrow R(f^{-1})$ induces an isomorphism

$$R(\xi)/(1 - f\xi) \cong R(f^{-1}).$$

**Proof:** There is an isomorphism $R[f^{-1}]/(a^n) \cong (R/a^n)[f^{-1}]$ because localization is flat, so we are done. \hfill \qed

**Remark (I.9.1.11).** There is another canonical morphism $R[f^{-1}] \rightarrow R(f^{-1})$ which exhibits $R(f^{-1})$ as the completion of $R[f^{-1}]$ w.r.t. the ideal $aR[f^{-1}]$.

Then $R(f^{-1})$ is endowed with an $a$-adic topology, and if $a$ is f.g., then $R(f^{-1})$ is complete w.r.t. to the $aR(f^{-1})$-adic topology. Also we see $R(f^{-1})$ doesn’t depend on the choice of ideal of definition of $a$.

**Proof:** Consider the projective system of exact sequences:

$$0 \rightarrow (1 - f\xi)R/a^n[\xi] \rightarrow R/a^n[\xi] \rightarrow R/a^n[f^{-1}] \rightarrow 0.$$

which is a surjective system so by Mittag-Leffler (II.1.1.32), $\lim_{\leftarrow}$ is exact (I.10.3.2), and there is an exact sequence

$$0 \rightarrow \lim_{\leftarrow} (1 - f\xi)R[\xi] \rightarrow \lim_{\leftarrow} R[\xi] \rightarrow \lim_{\leftarrow} R[f^{-1}] \rightarrow 0.$$

Now $\lim_{\leftarrow}(1 - f\xi)R[\xi] \cong (1 - f\xi) \cong (1 - f\xi)R(\xi)$, because $(1 - f\xi)$ is not a zero-divisor in $R/a^n$, and we get the desired isomorphism. \hfill \qed

**Def. (I.9.1.12)** (**Completed Tensor Product**). Let $(A, a), (B, b)$ be two complete adic rings, then we can define a complete tensor product ring $A\hat{\otimes}B$ as

$$A\hat{\otimes}B = \lim_{n}(A/a^n \otimes B/b^n).$$

this is just the $(a \otimes B + A \otimes b)$-adic completion of the tensor product $A \otimes B$, because $(a \otimes B + A \otimes b)^n$ and $(a^n \otimes B + A \otimes b^n)$ are cofinal.
Properties of \( R \)-Algebras

Remark (I.9.1.13) (Setup). For a good theory of Admissible formal schemes, let the base ring \( R \) be an adic ring with an ideal of definition \( I \) that satisfies either one of the following situation:

- \( R \) is an adic valuation ring with a principal ideal of definition.
- \( R \) is Noetherian and has no \( I \)-torsion.

Def. (I.9.1.14) (Topologically of Finite Presentation). Let \( R \) be an adic ring with an ideal of definition \( I \), then an \( R \)-algebra \( A \) is called

- of topologically finite type if it is isomorphic to an \( R \)-algebra \( R(\zeta_1, \ldots, \zeta_n)/a \) that is endowed with the \( I \)-adic topology and \( a \) is an ideal of \( R(\zeta_1, \ldots, \zeta_n) \).
- of topologically finite presentation if moreover the ideal \( a \) is f.g..
- admissible if it is of topologically finite presentation and has no \( I \)-torsion.

Prop. (I.9.1.15) (Raynaud–Gruson). Let \( A \) be an \( R \)-algebra of topologically finite type and \( M \) a finite \( A \)-module that is flat over \( R \). Then \( M \) is an \( A \)-module of finite presentation.

Proof: Cf.[Bos15]P165.

Cor. (I.9.1.16). Let \( A \) be an \( R \)-algebra of topologically finite type, then if \( A \) has no \( I \)-torsion, then \( A \) is of topologically finite presentation, in particular admissible(I.9.1.14).

Proof: Cf.[Bos15]P166.

Prop. (I.9.1.17) (Topologically of Finite Presentation is Local). Let \( A \) be an \( R \)-algebra that is \( I \)-adically complete and separated, \( f_1, \ldots, f_r \) be a set of elements generating the unit ideal, then \( A \) is of topologically finite type/topologically finite presentation/admissible iff each \( A(\{f_i^{-1}\}) \) does.


2 Valuation Rings

Def. (I.9.2.1) (Valuation Ring). In a field \( K \), the valuation ring is the maximum elements in the dominating ordering of local rings, where \( B \) dominates \( A \) iff \( A \subset B \) and \( m_B \cap A = m_A \).

A valuation ring in \( K \) is called absolutely algebraically closed if \( K \) is alg.closed.

Prop. (I.9.2.2). Any local ring \( A \) in a field \( K \) is dominated by a valuation ring with fractional field \( K \).

Proof: Note that the dominating relation satisfies the condition of the Zorn’s lemma, so it suffices to prove that \( A \) is not maximal if its fractional field is not \( K \). Let \( t \notin K_0 = \frac{A}{K} \). If \( t \) is transcendental over \( K_0 \), then \( A[t] \) with the maximal ideal \( (m, t) \) dominate \( A \). If \( t \) is algebraic over \( K_0 \), then there is a \( a \) that \( at \) is integral over \( A \), hence by(I.5.5.5) there is a maximal ideal of \( A(at) \) above \( A \), which proves the lemma.

Prop. (I.9.2.3) (Valuation Ring Criterion). \( A \) is a valuation ring with field of fraction \( K \) iff for any \( x \in K, x \) or \( x^{-1} \) is in \( A \).
Proof: If $A$ is a valuation ring, then for $x \notin A$, we know that $A[x]$ is a local ring, hence there is no prime over $\mathfrak{m}$ otherwise $A[x]_p$ is a bigger local ring, so we see $\mathfrak{m}A[x] = A[x]$, i.e. $1 = \sum t_i x^i$, so $x^{-1}$ is integral over $A$. Now $A[x^{-1}]$ has a $\mathfrak{m}'$ over $\mathfrak{m}$, so $A = A[x^{-1}]_{\mathfrak{m}'}$, which shows $x^{-1} \in A$.

Conversely, if for any $x \in K$, $x$ or $x^{-1}$ is in $A$, we assume $A$ is not $K$, so $A$ is not field by the condition. Then it has a non-zero maximal ideal, but only one, otherwise we can choose $x, y$ that $x/y, y/x \notin A$. And $A$ is maximal because if there is a $A \subset A'$, and a $x \in A'$, then if $x \notin A$, then $x^{-1} \in A$, hence also in $\mathfrak{m}_A$, so it is in $\mathfrak{m}_{A'}$, but now $x^{-1}$ cannot be in $A'$, contradiction.

\[\square\]

Cor. (I.9.2.4). For $K \subset L$ subfield, if $A$ is a valuation ring of $L$, then $A \cap K$ is a valuation ring of $K$. And if $L/K$ is algebraic and $A$ is not a field, then $A \cap K$ is not a field. (This is because the primes of $A$ are all over 0 so cannot contain each other (I.5.5.5) so $A$ is a field).

Cor. (I.9.2.5). The quotient $A/p$ at a prime is a valuation ring, and any localization of valuation ring is a valuation ring, by this criterion.

Prop. (I.9.2.6) (Valuation Ring is Normal). Valuation ring is normal, because for $x$ algebraic over $A$, either $x \in A$, or $x$ is a combination of $x^{-1}$ thus in $A$, by (I.9.2.3).

Cor. (I.9.2.7) (Integral Closure and Valuation Ring). The integral closure of a subring in a field $K$ is the intersection of valuation rings containing $A$.

Proof: Valuation ring is integrally closed, so it suffices to prove if $x$ is not algebraic over $A$, then there is a valuation ring of $A$ not containing $x$. This is because $x \notin B = A[x^{-1}]$ otherwise $x$ is integral over $A$. Now $x^{-1}$ is not a unit in $B$, hence $x \in p \in B$, hence $B_p$ is dominated by some valuation ring $V$, and $x \notin V$ because $x^{-1} \notin \mathfrak{m}_V$.

\[\square\]

Prop. (I.9.2.8) (Bezout Domain and Valuation Ring). A valuation ring is equivalent to a Bezout local domain.

Proof: One way is because the element of minimum valuation generate the ideal. Conversely, for $f, g \in A$, $(f, g) = (h)$, so $f = ah, g = bh$, and $h = cf + dg$, then $ab + cd = 1$, hence $a$ or $b$ is a unit, so $f/g \in A$ or $g/f \in A$. By (I.9.2.3), $A$ is a valuation ring.

\[\square\]

Prop. (I.9.2.9). A valuation ring is Noetherian iff it is discrete valuation iff it is PID.

Proof: Only need to prove Noetherian then $\Gamma = \mathbb{Z}$. we know ideals of $\Gamma$ of the form $\{x|x \geq \gamma\}$, where $\gamma > 0$ has a maximal element, so there is a minimal element bigger than 0, so $\Gamma \cong \mathbb{Z}$.

\[\square\]

Prop. (I.9.2.10). In a fixed field, any inclusion relation of two valuation ring is given by localization.

Proof: Just localize at the image of the maximal ideal $\mathfrak{m}_B \cap A$, then they are valuation rings (I.9.2.5) that dominate each other, thus they are the same by definition (I.9.2.1).

\[\square\]

Def. (I.9.2.11) (Extension of Valuation Rings). An injective local homomorphism of valuation rings is called an extension of valuation rings. By (I.7.1.24), it is equivalent to a f.f. morphism of valuation rings.
Discrete Valuation Rings

Prop. (I.9.2.12) (Local in Dimension 1 Case). For a Noetherian local domain of dimension 1 with maximal ideal \( \mathfrak{m} \) and residue field \( k \), the following are equivalent:

1. \( A \) is a DVR.
2. \( A \) is normal.
3. \( \mathfrak{m} \) is a principal ideal.
4. \( A \) is regular.
5. Every nonzero ideal is a power of \( \mathfrak{m} \).
6. There exists \( x \in A \) that every nonzero ideal is of the form \((x^k)\).

\( \mathbf{Cf.}\ [\mathbf{Sta}]00PD. \)

Proof: 1 \( \rightarrow \) 2: Valuation ring is integrally closed, by(I.9.2.6).
2 \( \rightarrow \) 3: As the radical of any ideal \( a \) is \( \mathfrak{m} \), and \( A \) is Noetherian, so there is an \( n \) that \( \mathfrak{m}^n \subset a \) and \( \mathfrak{m}^{n-1} \nsubseteq a \). Then choose \( b \in a - \mathfrak{m}^{n-1} \), \( x = a/b \in K \), then \( x^{-1} \notin A \), then it is not integral over \( A \).
So \( x^{-1} \mathfrak{m} \nsubseteq \mathfrak{m} \), but \( x^{-1} \mathfrak{m} \subset A \), so it equals \( A \), which means \( \mathfrak{m} = (x) \).
3 \( \rightarrow \) 4: Clear.
4 \( \rightarrow \) 5: For any ideal \( a \), its radical is \( \mathfrak{m} \) and \( A \) is Noetherian, so \( \mathfrak{m}^n \subset a \). Now \( A/\mathfrak{m}^n \) is Artinian by(I.5.1.51), so by(I.5.1.53) \( a \) is a power of \( \mathfrak{m} \).
5 \( \rightarrow \) 6: And \( x \in \mathfrak{m} - \mathfrak{m}^2 \) will do.
6 \( \rightarrow \) 1: Define \( v(a) = k \) if \( (a) = (x^k) \).

\( \square \)

3 Valuations

Def. (I.9.3.1) (Valuations). A valuation on a field \( K \) is surjective map \( v : K \rightarrow \Gamma \) where \( \Gamma \) is an ordered Abelian group(I.2.8.1), called the value group of \( K \).

The rank of a valuation is defined as the height of its value group(I.2.8.4).

Prop. (I.9.3.2) (Valuation Ring and Valuation). Valuation rings(I.9.2.1) \( A \) of a field \( K \) is equivalent to valuations on \( K \).

The equivalence is given by \( K = Q(A), \Gamma = K*/A^* \) and that \( A = v^{-1}\{x \geq 0\} \).

Proof: These are definitely valuation rings, and if \( A \) is a valuation ring by(I.9.2.3), then we set \( \Gamma = K*/A^* \), where \( A^* \) is the invertible elements of \( A \) and \( x \leq y \) iff \( y/x \in (A - \{0\})/A^* \). This is totally ordered by(I.9.2.3).

Cor. (I.9.3.3) (Rank and Dimension). A valuation ring of rank \( n \) has Krull dimension \( n \), because clearly the convex subgroups of \( \Gamma \) is in bijection with ideals of \( A \).

Valuations of Rank 1

In this subsubsection, all valuations is of rank 1.

Remark (I.9.3.4) (Real Valuations). As an ordered Abelian group of height 1 can be embedded into \( \mathbb{R}(I.2.8.5) \), a valuation \( v \) on a field of rank 1 is equivalent to a real valued valuation.

Def. (I.9.3.5) (Multiplicative Valuation). For a real continuous valuation \( v \), we can define a multiplicative valuation \( |\cdot| \) where \( |a| = \exp(-v(a)) \). Then it is multiplicative.
Def. (I.9.3.6) ((Non-)Archimedean Valuations). A valuation is called non-Archimedean iff \(|x + y| \leq \max\{|x|, |y|\}. It is called Archimedean iff it is not non-Archimedean.

Def. (I.9.3.7) (Non-Archimedean field). A non-Archimedean field is a topological field that the valuation is given by a rank1-valuation.

Prop. (I.9.3.8). A valuation is non-Archimedean iff \(|n| |n \in \mathbb{N}\) is bounded.

Proof: If it is non-archimedean, then clearly by induction and \(n = 1 + (n - 1) |n| \leq 1\). Conversely, if \(|n| \leq M\), then consider \(|x + y|^n = |(x + y)^n| \leq \sum |C_n^k| |x^k y^{n-k}| \leq M \max\{|x|, |y|\}\), so letting \(n\) be large, clearly \(|x + y| \leq \max\{|x|, |y|\} \). \(\Box\)

Cor. (I.9.3.9). Any valuation on a field of char\(\neq 0\) is non-Archimedean.

Prop. (I.9.3.10) (Equivalent Valuations). Two valuation on a field is equivalent iff \(|x|_1 < 1 \Rightarrow |x|_2 < 1\) and is equivalent to \(|x|_1 = |x|_2^s\) for some \(s > 0\).

Proof: if two valuation are equivalent, then \(x^n \rightarrow 0\) in \(\tau_1\) iff \(x^n \rightarrow 0\) in \(\tau_2\), so \(|x|_1 < 1 \Rightarrow |x|_2 < 1\). If \(|x|_1 < 1 \Rightarrow |x|_2 < 1\), then let \(y\) be an element that \(|y|_1 > 1\), then any element \(|x| = |y|_\alpha\) for some \(\alpha \in \mathbb{R}\). Let \(\frac{n}{m} \alpha\) converges to \(\alpha\) from above, then \(\frac{|x|^m}{|y|^n}|_1 < 1\), so \(\frac{|x|^m}{|y|^n}|_2 < 1\), so \(|x|_2 \leq |y|_\alpha^s\).

Similarly, \(|x|_2 \geq |y|_\alpha^s\), so \(|x|_2 = |y|_\alpha^s\). So \(|x|_1 = |x|_2^s\) for some \(s > 0\).

If \(|x|_1 = |x|_2^s\) for some \(s > 0\), then these two valuations are clearly equivalent. \(\Box\)

Cor. (I.9.3.11) (Weak Approximation). If \(\cdot |1, \ldots, |_n\) be pairwise inequivalent valuations on \(K\), then for any \(a_1, \ldots, a_n \in K\) and \(\varepsilon > 0\), there is an \(x \in K\) that \(|x - a_i|_i < \varepsilon\).

Proof: As \(\cdot |1, \ldots, |_n\) are inequivalent, there are \(\alpha, \beta\) that \(|\alpha - 1| < 1, |\alpha|_n \geq 1, |\beta|_n < 1, |\beta|_1 \geq 1\) by (I.9.3.10), so let \(y = \beta/\alpha\), then \(|y|_1 > 1, |y|_n < 1\).

Now we prove by induction that there is an \(\alpha\) that \(|\alpha - 1| > 1, |\alpha|_i < 1\) for \(i = 2, \ldots, n\). the case \(n = 2\) is done, for general \(n\), if the \(z\) for \(n - 1\) satisfies \(|z|_n < 1\), then \(z^m y\) will do, for \(m\) large. if \(|z| > 1\), then the sequence \(|t_m|_i = |\frac{z^m}{1+z^m}|_i\) converges to 1 for \(i = 1, n\) and converges to 0 for \(i = 2, \ldots, n - 1\), so \(t^m y\) will do, for \(m\) large. \(\Box\)

Prop. (I.9.3.12) (Gelfand). Any field with an Archimedean valuation \(K\) is a subfield of \(\mathbb{C}\).

Proof: We consider its completion. when it contains \(\mathbb{C}\), this is a corollary of ??, otherwise, we consider \(K \otimes \mathbb{C}\), then it is a finite dimensional module over \(K\) thus also complete. \(\Box\)

Remark (I.9.3.13). Because of this, we usually don’t consider Archimedean valuation fields.

Prop. (I.9.3.14) (Ostrowski). 1. Any non-trivial value on \(\mathbb{Q}\) is equivalent to \(v_p\) or \(|\cdot|\). Thus any complete Archimedean field is isomorphic to \(\mathbb{R}\) or \(\mathbb{C}\) by (I.9.3.12).

2. Any non-trivial valuation on \(\mathbb{F}_q(t)\) is of the form \(|\cdot|_p\) or \(|\cdot|_\infty\), where \(p\) is an irreducible polynomial in \(\mathbb{F}_q[t]\).

Proof: 1: if it is non-Archimedean, then \(|n| \leq 1\), and it is not trivial, so there is a minimal \(p\) that \(|p| < 1\). Then \(p\) is easily seen to be a prime. Then for any \((a, p) = 1, a = dp + r\), so \(|r| = 1\), so \(|a| = 1\).

And if it is Archimedean, then we prove that in \(\mathbb{N}\), \(|m| = m^\lambda\) for some \(\lambda\). Let \(F(n) = |n|\) and \(f = \log_2 F\), then \(f(m + n) \leq \max\{f(m), f(n)\} + 1\), and if \(m = \sum_{i=1}^r d_i n^i\), then \(f(m) \leq r(1 + f(n)) + a_n\), where \(a_n = \sup\{f(k)|k < n\}\). And \(r \leq \log m/\log n\), so

\[
\frac{f(m)}{\log m} \leq \frac{a + f(n)}{\log n} + \frac{b}{\log n}
\]
then letting \( m \to m^k, k \to \infty \), and then let \( n \to n^k, k \to \infty \), we get \( \frac{f(m)}{\log m} \leq \frac{f(n)}{\log n} \) for any \( m, n \).

2: Any valuation on \( \mathbb{F}_q(t) \) is non-Archimedean (I.9.3.9), and \( |n| = 1 \) if \( (n, p) = 1 \), because \( n^{p-1} = 1 \). Similarly, if there is a minimal hence irreducible \( P \) that \( |P| < 1 \), then use induction and \( Q = sP + r \) for some \( s, r \) of degree \( < \deg Q \), so \( |Q| = 1 \) for all \( (Q, P) = 1 \). Otherwise, \( |P| \leq 1 \) for all \( P \), then \( |t| > 1 \), otherwise \( |\cdot| \) is trivial, so it is easy by induction that \( |F(t)| = |t|^{\deg F} \). □

Lemma (I.9.3.15) (Continuity of Roots). For a separable polynomial \( f \) over a valued alg. closed field \( K \), there is a \( \varepsilon \) that every polynomial \( g \) that are closed enough to \( f \), the roots of \( g \) is closed to roots of \( f \) respectively.

Proof: This is easy to see by decomposition as each root of \( f \) is close to a root of \( g \). \( f, g \) have the same degree so the roots correspond to each other. □

Prop. (I.9.3.16) (Fundamental Inequality). If \( (K, v) \) is a valued field and \( L/K \) be a field extension of degree \( n \), if \( w_i \) are the valuations of \( L \) above \( v \), then

\[
\sum e(w_i/v) f(w_i/v) \leq [L : K].
\]

The equality holds when \( v \) is discrete and \( L/K \) is separable.

Proof: Cf.[Clark note Theorem4].

Def. (I.9.3.17). A field \( K \) is called spherically complete iff each descending chain of metric balls has a nonempty intersection.

Microbial Valuations

Prop. (I.9.3.18) (Microbial Valuation). For a valuation ring \( R \subset K \), a \( f \neq 0 \in R \) is called topologically nilpotent iff \( f^n \to 0 \) in the valuation topology of \( A \). The following are equivalent:

- The topology on \( K \) coincides with a rank 1 topology.
- There exists a nonzero topologically nilpotent element in \( K \).
- \( R \) has a prime ideal of height 1.

If this is the case, then the valuation defined by \( A \) is called microbial.

And in this case, for any topological nilpotent element \( \varpi, K = R[\varpi^{-1}] \), and \( \varpi^r \in R \) for some \( r \), and the topology on \( R \) is \( \varpi^r \)-adic. And if \( p \) is a prime ideal of rank 1, then the valuation ring \( R_p \) is of rank 1, and defines the same topology on \( R \).

Proof: 1 \( \to \) 2: if there is a rank 1 valuation \( |\cdot|' \) that defines the same topology as \( R \), then any \( |x|' < 1 \) will be a topological nilpotent element by (I.9.3.19).

2 \( \to \) 3: if \( \varpi \) is a topological nilpotent element, then \( p = \sqrt{(\varpi)} \) is a prime ideal, and it is minimal, because if there is another \( q \subseteq p \), then \( \varpi \notin q \), but \( p \subset (\varpi^n) \) by induction: because \( (\varpi) \notin q, q \subset (\varpi) \), and if \( x \in q \), then \( x = \varpi^n y \), and \( \varpi \notin q \), so \( y \in q \subset (\varpi^n) \), so \( x \in (\varpi^{n+1}) \). Now \( q = 0 \) because \( \varpi \) is topological nilpotent.

3 \( \to \) 1: It suffices to prove that the valuation defined by \( R_p \) is the same as the topology of \( R \). But this is true in general, just notice that the valuation topology of a nontrivial valuation is also defined by \( B(a, \gamma) \).

The final remark is clear as \( x \varpi^n \in R \iff |\varpi^n| \leq |x^{-1}| \). □

Lemma (I.9.3.19). Let \( R \) be a valuation ring, if \( x \in R^* \) is topologically nilpotent, then \( |x| < 1 \), and the converse is also true if \( R \) has rank 1.
Proof: if $|x| \geq 1$, then $x^n \notin B(0,1)$, so it is not topologically nilpotent. And if $R$ has rank 1, $|x| < 1$, then for any $\delta \neq 0$, there is some $n$ that $|\delta^{-1}| < |x^{-n}|(I.2.8.5)$, so $|x^m| < |\delta|$ for $m$ large, thus $x$ is topologically nilpotent.

\[\square\]

Prop. (I.9.3.20) (Constructing Microbial Valuations). If $A$ is a valuation ring and $f \in A$ is a non-zero non-unit, then the $f$-adic Hausdorffization $\overline{A} = A/\cap_n f^nA$ and the completion $A$ are all microbial.

Proof: Easy, Cf.[Bhatt Perfectoid Spaces, P63].

\[\square\]

4 Affinoid Algebras

Tate Algebras

Def. (I.9.4.1) (Tate Algebra). For a complete non-Archimedean field $K$ with residue field $k$, we define the Tate algebra $T_n = K(x_1, \ldots, x_n)$ to be the restricted power series(I.9.1.7) consists of elements $\sum_v a_v x^v$ that $\lim_{|v| \to \infty} |a_v| = 0$. It is endowed with the norm $|f| = \max |a_v|$.

The norm satisfies $|fg| = |f||g|$ and $|f + g| \leq |f| + |g|$.

There is a reduction map from $T_n$ to $k[x_1, \ldots, x_n]$, it is surjective.

Proof: $T_n$ is an algebra because the values of coefficients of $f$ is bounded. $|fg| \leq |f||g|$ is easy, to show $|fg| \geq |f||g|$, we assume $|f| = |g| = 1$, then their reduction in $K[x_1, \ldots, x_n]$ is non-zero, thus $\overline{fg}$ is non-zero, which shows $|fg| \geq 1$.

\[\square\]

Prop. (I.9.4.2) (Maximum Principle). A formal power series $f$ converges in $B^n(K)$ iff it is in $T_n$.

And when it is in $T_n$, $|f(x)|$ attains a maximum in $B^n(K)$.

Proof: If it converges at $(1, \ldots, 1)$, then $\lim_{|v| \to \infty} |a_v| = 0$ by(IV.1.1.20). Conversely, for any point in $B^n(K)$, it can be considered in a finite extension field of $K$, thus complete, hence we can apply(IV.1.1.20) again.

For the second assertion, we assume $|f| = 1$, then consider its reduction to $k[x_1, \ldots, x_n]$, then there is a $\overline{a}$ in the algebra closure of $k$ that $\overline{f}(\overline{a}) \neq 0$. Now $\overline{K}$ can be seen as the residue field of $\overline{K}$. Then the lifting of $\overline{a}$ to a $x \in K$ has valuation 1 and $|f(x)| = 1$.

\[\square\]

Prop. (I.9.4.3). $T_n$ is a Banach algebra(Easy).

Cor. (I.9.4.4). An element $f$ of norm 1 of $T_n$ is invertible in $T_n$ iff its reduction in $k[x_1, \ldots, x_n]$ is a unit. Elements of other norms can be reduced to the case of norm 1.

Proof: One direction is trivial, the other is because $|f-f(0)| < 1$, hence $f = f(0)(1+g)$, this is invertible by power expansion as $T_n$ is complete.

\[\square\]

Def. (I.9.4.5). A restricted power series $g = \sum g_v x^v \in T_n$ with coefficients in $T_{n-1}$ is called $X_n$-distinguished of order $s$ iff $g_s$ is a unit in $T_{n-1}$, $|g_s| = |g|$ and $|g_v| > |g_v|$ for all $v > s$.

Lemma (I.9.4.6). For any f.m. elements $f_i \in T_n$, there is a continuous automorphism of $T_n$ that maps $T_n \to T_n, T_i \to T_i + T_n^s$ that maps $f_i$ to $X_n$-distinguished elements.

Proof: Cf.[Rigid and Formal Geometry P16].

\[\square\]
Prop. (I.9.4.7) (Weierstrass Division). If \( g \in T_n \) is \( X_n \)-distinguished of order \( s \), for any \( f \in T_n \), there is a unique form \( f = qg + r \), where \( q \in T_n \) and \( r \in T_{n-1}[X_n] \) of degree \( r < s \). Moreover, \(|f| = \max\{|q||g|, |r|\} \).

Proof: Cf.[Rigid and Formal Geometry P17]. □

Cor. (I.9.4.8) (Weierstrass Preparation). If \( g \in T_n \) is \( X_n \)-distinguished of order \( s \), then there exists uniquely a \( r \in T_{n-1}[X_n] \) of degree \( s \) and \( g = re \), where \( e \) is a unit in \( T_n \).

Proof: By (I.9.4.7) applied to \( X_n^s = qg + r \) with \(|r| \leq 1 \). Then \( \omega = X_n^s - r \) is \( X_n \) is the desired polynomial, it suffice to show \( g \) is a unit. Let \( g \) be normalized that \(|g| = 1\), then \(|q| = 1\), by reduction to polynomials, we see \( \bar{\omega} = \bar{q}\bar{g} \), and \( \bar{\omega}, \bar{g} \) are both polynomials of degree \( s \), so \( \bar{q} \in k^s \), so \( q \) is a unit by (I.9.4.4).

Uniqueness: if \( g = e\omega \), then \( X_n^s = e^{-1}g + (X_n^s - \omega) \), so uniqueness follows from that of Weierstrass division. □

Prop. (I.9.4.9) (Noether Normalization). For any proper ideal \( \mathfrak{a} \) of \( T_n \), There is a \( d \) and a finite injection \( T_d \to T_n/\mathfrak{a} \).

Proof: We may assume \( \alpha \neq 0 \), thus choose a \( g \in \alpha \neq 0 \), then using (I.9.4.6), we may assume \( g \) is \( X_n \)-distinguished. Now the Weierstrass division theorem (I.9.4.7) says that \( T_{n-1} \to T_n/(g) \) is finite. Hence \( T_{n-1} \to T_n/(g) \to T_n/\mathfrak{a} \) is finite. Now we can use induction to find a \( T_d \to T_{n-1}/T_{n-1} \cap \mathfrak{a} \) finite, thus also \( T_d \to T_n/\mathfrak{a} \) is finite. □

Cor. (I.9.4.10) (Residue Field of Tate Algebra). The residue field of a maximal ideal of \( T_n \) is a finite extension field of \( K \), because \( T_n/\mathfrak{m} \) has dimension 0, thus \( K \to T_n/\mathfrak{m} \) finite injective.

Proof: Because finite injection \( T_d \to T_n/\mathfrak{m} \) shows \( T_n \) is a field (I.5.5.3), thus we must have \( d = 0 \). □

Cor. (I.9.4.11). The map from \( B^n(\overline{K}) \) to the set of maximal ideals of \( T_n \) are surjective.

Proof: Evaluation map defines a map \( T_n \to K(x_1, \ldots, x_n) \) that is surjective, thus the kernel is a maximal ideal. Conversely, for any maximal ideal \( \mathfrak{m} \subset T_n \), \( K' = T_n/\mathfrak{m} \) is finite over \( K \), so we may assume \( K' \subset \overline{K} \).

We show that this map \( \varphi : T_n \to \overline{K} \) is contractive, otherwise there is a \(|\alpha| = 1, |\alpha - \varphi(\alpha)| > 1 \). Consider the minimal polynomial \( p \) of \(|\alpha| \), all the conjugates of \( \alpha \) has the same valuation as \( K \), as \( K \) is Henselian, thus \( p \) has ascending Newton polygon, thus by (I.9.4.4) it is invertible in \( T_n \). But \( \varphi(p(\alpha)) = 0 \), contradiction.

So \(|\varphi(x)| \leq |x| \), then it is continuous, and \(|\varphi(T_1), \ldots, \varphi(T_n)| \subset B^n(K^n) \), so we are done. □

Cor. (I.9.4.12) (Main Theorem). \( T_n \) is Noetherian, UFD, Jacobson of Krull dimension \( n \).

Proof: Noetherian: Use induction, as in the proof of (I.9.4.9), \( T_{n-1} \to T_n/(g) \) is finite for some \( g \subset \mathfrak{a} \), then also \( T_n/\mathfrak{a} \) is finite over \( T_{n-1} \), thus Noetherian as a \( T_{n-1} \) module, thus Noetherian as a ring.

UFD: Cf.[Rigid and Formal Geometry P20].

Jacobson: We need to show that any prime ideal \( \mathfrak{p} \) is an intersection of maximal ideals. The case of \( \mathfrak{p} \) is by (I.9.4.2). For \( \mathfrak{p} \neq 0 \), by Noetherian normalization (I.9.4.9), there is a \( T_d \subset T_n/\mathfrak{p} \) finite. Then use induction and generalized Nullstellensatz (I.5.9.9), \( T_n/\mathfrak{p} \) is Jacobson, thus \( \mathfrak{p} = \text{rad}(T_n/\mathfrak{p}) \).

Dimension \( n \): Cf.[Formal and Rigid Geometry P22]. □
Prop. (I.9.4.13). For an ideal \( \mathfrak{a} \in T_n \), there are \( a_1, \ldots, a_r \) which generate \( \mathfrak{a} \) that \( |a_i| = 1 \), and any elements in \( f \) has a representation of the form \( \sum f_i a_i \) with \( |f_i| \leq |f| \).

The same assertion holds for submodules of \( T_n \).

Proof: Cf. [Formal and Rigid Geometry P27,29].

Cor. (I.9.4.14). Each ideal of \( T_n \) is closed hence complete in \( T_n \). This follows immediately from (I.9.4.12) and (IV.1.2.12).

Cor. (I.9.4.15). For any ideal \( \mathfrak{a} \) of \( T_n \), the distance from an element to \( \mathfrak{a} \) attains minimum.


Affinoid Algebras

Def. (I.9.4.16) (Affinoid Tate Algebra). A normed algebras of the form \( A = T_n/\mathfrak{a} \) are called affinoid (Tate) algebras, so it is Noetherian and Jacobson by (I.9.4.12). An affinoid algebra has a natural semi-norm by \( |f|_{\text{sup}} = \sup |f|_m \) in \( A/\mathfrak{m} \) for a maximal ideal \( \mathfrak{m} \) of \( A \) by (I.9.4.10).

Proof: We need to show the sup is finite, for this, notice \( |f| = |g| \) for some \( g \) in the residue norm (I.9.4.17), so for any maximal ideal \( \mathfrak{m} \) of \( A \), the inverse is a maximal ideal \( \mathfrak{n} \) in \( T_n \) by finiteness, thus \( |f|_m = |g|_n \leq |g|_{\text{sup}} = |g| = |f| \), so \( |f|_{\text{sup}} \leq |f| \).

For the second-last equality, notice on \( T_n \), \( | \cdot |_{\text{sup}} \) and \( | \cdot | \) equal, by (I.9.4.2) and (I.9.4.11).

Def. (I.9.4.17) (Residue Norm). For a Tate algebra \( A = T_n/\mathfrak{a} \), there is a natural residue norm on it. This is a complete \( K \)-algebra norm on \( A \), and \( T_n \to A \) is continuous and open. For any \( f \in A \), the residue norm is attained at an element of \( T_n \).

Any residue norm is bigger than the sup-norm, by the proof of (I.9.4.16).

Proof: It is a \( K \)-algebra norm is easily verified, should notice \( |f| = 0 \) iff \( f = 0 \), because \( \mathfrak{a} \) is closed (I.9.4.14). The last assertion follows from (I.9.4.15).

Remark (I.9.4.18). The sup norm may not even be a norm, if \( \mathfrak{a} \) is not radical, but the fact that sup norm is smaller than any residue norm, together with (I.9.4.22), is enough for use.

Prop. (I.9.4.19). For \( T_d \to A \) a finite injection, assume \( A \) is a torsion-free \( T_d \)-module, then for any \( f \in A \), there is a unique minimal monic polynomial \( P \) of \( f \) over \( T_d \).

In this case, \( |f|_{\text{sup}} = \sup |a_i|^{1/i} \) where \( a_i \) are coefficients of \( P \).

Proof: Because \( A \) is torsion-free, we reduce to the quotient field of \( T_n \), then \( f \) has a minimal monic polynomial, and \( T_n \) is UFD, hence Gauss lemma shows that this polynomial has coefficients in \( T_d \). Hence \( T_n[f] = T_n[X]/(p) \).

For the second, notice first for finite extension the Spec map is surjective, thus we may assume \( A = T_n[f] = T_n[X]/(p) \), and for a maximal ideal \( \mathfrak{m} \) of \( T_n \), let \( T_n/\mathfrak{m} = k \), then \( A/(\mathfrak{m}) = k[X]/(\overline{p}) \), then maximal ideals of \( A/(\mathfrak{m}) \) corresponds to roots \( \alpha_i \) of \( \overline{p} \) in \( k \), so \( \sup_{\mathfrak{m}} |f|_m = \sup |\alpha_i| = \max |a_i|^{1/i} \), so the result follows.

Cor. (I.9.4.20). \( |f|_{\text{sup}} \in \sqrt[n]{|K|} \) for some \( N \) and all \( f \in A \), because the minimal polynomial has coefficients in \( T_n \), and sup norm and Gauss norm coincide on \( T_n \) by the proof of (I.9.4.16).

Cor. (I.9.4.21) (Maximum Principle). \( |f|_{\text{sup}} = |f|_m \) for some maximal ideal \( \mathfrak{m} \).
Proof: Since $A$ is Noetherian, it has f.m. minimal primes, hence $|f|_{\text{sup}} = |f|_{\text{sup}}$ in $A/p_i$ for some minimal prime of $A$. Hence we reduce to the case of $f$ is continuous w.r.t any residue norms. In particular, any $k$-Banach algebra topology on $A$ coincides with the $k$-affinoid topology on $A$, and all residue norms on an affinoid algebra are equivalent.

Moreover, for any morphism of $k$-affinoid algebras $B \to A$, the norm on $A$ can be replaced by an equivalent one that makes $A$ into a normed $B$-algebra.

Proof: Use (IV.1.2.9), it suffices to show the condition holds, for $A/B = \{m^n\}$ where $m$ are maximal ideals of $A$: The residue field is finite by (I.9.4.10), their intersections is $(0)$ because if $f \in \cap m \cap_n m^n$, Krull’s theorem (I.9.5.11) (use localization) says for each maximal ideal $m$ there is a $m \in m$ that $(1 - m)f = 0$, so $Ann(f) = (1)$, so $f = 0$.

For the second assertion, see [non-Archimedean Analysis P229].

Cor. (I.9.4.23). The notion of power-boundedness and topological nilpotence is independent of residue norm chosen.

Cor. (I.9.4.24) (Restricted Power Series). For an affinoid algebra $A$, the restricted power series in $A$:

$$A(X_i) = \left\{ \sum a_v X^v \mid \lim_{|v| \to \infty} a_v = 0 \right\}$$

is an affinoid algebra, this is independent of the residue norm chosen.

Def. (I.9.4.25) (Strongly Noetherian). $A$ is called strongly Noetherian if $A(T_1, \ldots, T_n)$ are Noetherian for any $n \geq 0$.

Lemma (I.9.4.26). The image $A$ is dense in $A(X)/(X - f)(\text{in the residue norm, and thus in all other norms, by (I.9.4.22)(I.9.4.17)})$, this is because a restricted power series can be truncated by a finite part and a part with small norm, and the finite part is in the image of $A$.

Def. (I.9.4.27) (Affinoid Generator). For a morphism of affinoid algebras $A \to A'$, a set of elements $h_i$ in $A'$ is called a set of affinoid generator iff there is a surjection

$$A(X_1, \ldots, X_n) \to A', \quad X_i \mapsto h_i$$

Of course $h_i$ is power-bounded, by the residue norm given.

Lemma (I.9.4.28). If $\pi' : A(X_1, \ldots, X_n) \to A' : X_i \to h_i'$ is a surjective morphism of affinoid algebras that $A(X_1, \ldots, X_n)$ is endowed with the Gauss norm and $A'$ is endowed with the residue norm, then any set of elements $h = (h_1, \ldots, h_n)$ that $|h_i - h'_i| < 1$ is a set of affinoid generators.

Proof: By non-Archimedean property, $|h_i| \leq 1$ thus also $|h'_i| \leq 1$, and Let $\varepsilon = \max\{|h_i - h_i|\} < 1$. The strategy is simple, if for each $g$ in $A'$, we can find a $f$ that $|f| = |g|, |\pi(f) - g| \leq \varepsilon|g|$, then by iteration, there is a $f$ that $\pi(f) = g$. But by (I.9.4.17) and (I.9.4.15), if we choose a $f$ that $\pi'(f) = g$ and $|f| = |g|$, then

$$|\pi(f) - g| = \left| \sum a_v h^v - \sum a_v h'^v \right| = \left| \sum a_v (h^v - h'^v) \right| \leq \varepsilon|f| = \varepsilon|g|.$$
**Def. (I.9.4.29) (Distinguished Element).** For an affinoid algebra $A$ and an element $x \in \Spec A(VIII.4.1.1)$, a element $f \in A(X_1,\ldots, X_n)$ is called $X_n$-**distinguished of order $s$ at $x$ iff it is distinguished in $A/m_x(X_1,\ldots, X_n)$ is distinguished of order $s$ in the sense of(I.9.4.5)(notice $A/m_x$ is a complete valued field by(I.9.4.10)).

**Prop. (I.9.4.30) (Fibered-Pushouts).** When $R, A_1, A_2$ are all affinoid algebras, the amalgamated sum is also an affinoid algebra. In other words, the category of affinoid algebras admits amalgamated sums(fibered pushouts by(IV.1.1.15)).

**Proof:** Cf.[Formal and Rigid Geometry P245].

**Prop. (I.9.4.31).** $T_n \hat{\otimes} T_m \cong T_{m+n}, K' \hat{\otimes} T_{n,K} = T_{n,K'}$.

**Prop. (I.9.4.32).** For affinoid algebras $R, A_1, A_2$ and ideals $a_1 \subset A_1, a_2 \subset A_2$, there is an isomorphism:
\[
(A_1 \hat{\otimes}_R A_2)/\langle a_1, a_2 \rangle \cong (A_1/a_1) \hat{\otimes}_R (A_2/a_2)
\]

**Proof:** Cf.[Rigid and Formal Geometry P248].

**Prop. (I.9.4.33) (Finite Extension of Affinoid Algebras).** If $B$ is an affinoid $K$-algebra and $\varphi : B \rightarrow A$ is a finite ring map, then $A$ can be provided a topology to make it an affinoid $K$-algebra, and $\varphi$ is continuous.

**Proof:** We can associate to $A$ a Banach algebra topology induced from $B^n \rightarrow A \rightarrow 0$ that is continuous. Now it is an affinoid $K$-algebra: we may assume $B = T_n$, then $A = \sum T_n a_i$, and we may assume $|a_i| < 1$ then clearly there is a continuous extension $T_n(X_i) \rightarrow A$ extending this map, so $A$ is affinoid.

**Construction of Affinoid Tate Algebras**

**Def. (I.9.4.34) (Affinoid Localizations).** Let $A$ be an affinoid Tate algebra, then for a finite set of elements $\{f_i, g_j\} \subset A$, we can define the localization
\[
A(f_i, g_j^{-1}) = A(\zeta_1, \ldots, \zeta_i, \xi_1^{-1}, \ldots, \xi_j)/(\zeta_i - f_i, 1 - \xi_j g_j).
\]

5 **Huber Rings**

**Def. (I.9.5.1) (Huber Ring).** A topological ring is called Huber if there exists an open subring $A_0$ that the induced topology on $A_0$ is $I$-adic for some f.g. ideal $I$ of $A$. Such a $A_0$ is called a ring of definition, and $I$ is called the ideal of definition. Morphisms of Huber rings are just a continuous morphisms of topological rings.

**Prop. (I.9.5.2) (Boundedness and Rings of Definition).** A subring $A_0 \subset A$ of a Huber ring is a ring of definition iff it is open and bounded.

**Proof:** Clearly a ring of definition is open and bounded, for the converse, let $(A'_0, I)$ be a couple of definition, and $A_0$ is an open and bounded subset of $A$, then $I^n \subset A_0$ for some $n$, and set $J = I^n A_0$. As $A_0$ is bounded, for any open nbhd $U$ of 0, there exists $m > 0$ that $I^{km} A_0 \subset U$, thus $J^m \subset U$. This shows $J$ is a fundamental system of nbhd of 0, thus $A_0$ is $J$-adic and is a ring of definition.

**Cor. (I.9.5.3).** Let $A$ be a Huber ring, then
• If $A_0, A_1$ are two rings of definition of $A$, then so does $A_0 \cap A_1$ and $A_0A_1$.
• Every open subring $B$ of $A$ is a Huber ring.
• If $B \subset C$ are subrings of $A$ and $B$ is bounded, $C$ is open, then there is a ring of definition $A_0$ that $B \subset A_0 \subset C$.

Proof:  1: By(I.9.5.2).
  2: Let $I^n \subset B$, then $(B \cap A_0, I^n)$ is a couple of definition of $B$.
  3: By2, $C$ is Huber, take a ring of definition $A_0$ of $C$, then $A_0B$ is open and bounded in $C$, thus a ring of definition. \( \Box \)

Lemma (I.9.5.4). If $A$ is a Huber ring and $T \subset A$ is a subset that generates an open ideal of $A$, then for any open nbhd $U$ of $A$, the subgroup $T^nU$ is open.

Proof: Let $(A_0, I)$ be a couple of definition. By assumption the ideal $J$ generated by $T$ is open, thus $J^n$ is also open, and contains some $I^m$. Now we can change $I^m$ to $I$. Now $I$ is f.g., so there is a finite subset $M \subset A$ that $I \subset T^nM$. Notice $M$ is bounded because it is finite, so there is an integer $r$ that $I^rM \subset U$, thus $I^{r+1} \subset T^nU$, and $T^nU$ is open. \( \Box \)

Def. (I.9.5.5) (Tate Huber Ring). A Huber ring is called Tate if there exists an open subring $A_0$ that the induced topology on $A_0$ is $t$-adic for some $t \in A_0$ which becomes a unit in $A$. Such a $t$ is called a pseudo uniformizer.

Prop. (I.9.5.6) (Examples of Tate Huber Rings).
• If $K$ is a complete non-Archimedean field and $R$ is a $K$-Banach algebra, then $R$ is Tate with a ring of definition by $(R_{\leq 1}, t)$, where $t$ is a pseudo-uniformizer of $K$.
• If $A_0$ is any ring and $g \in A_0$ is a nonzero-divisor, and let $A = A_0[g^{-1}]$ equipped with the $g$-adic topology, then it is an Tate Huber ring.

Prop. (I.9.5.7) (Properties of Tate Huber Rings). If a Huber ring $A$ is Tate with a topological nilpotent unit $g$ and $A_0 \subset A$ is any ring of definition, then there exists $n$ large that $g^n \subset A_0$. And in this case, $A_0$ is $g^n$-adic and $A = A_0[(g^n)^{-1}]$.

In this case, a subset $S \subset A$ is bounded iff $S \subset g^{-n}A_0$ for some $n$.

Proof: Because $A_0$ is open in $g$, there is some $n$ that $g^n \subset A_0$, and then with $n$ even larger we can assume $g \in I$, because $g$ is topologically nilpotent, and $gA_0$ is also open in $A_0$, thus it contains $I^m$ for some $m$. So now $g^{mn}A_0 \subset I^m \subset g^nA_0$, which means $A_0$ is $g^n$-adic.

To show $A = A_0[(g^n)^{-1}]$, it suffices to notice $g^{kn}x \to 0$ as $k \to \infty$ for any $x \in A$, so for $k$ large, $g^{kn}x \in A_0$.

The last assertion is easy, as open subsets of $A$ and $g^{kn}A_0$ are cofinal. \( \Box \)

Prop. (I.9.5.8) (Power-Bounded Elements and Topologically Nilpotent Elements). The subset $A^0$ of power-bounded elements in $A$ is a subring, and it is the filtered colimit of all the ring of definition in $A$, thus open. It is also integrally closed in $A$.

The subset $A^{00}$ fo topologically nilpotent elements of $A$ is a radical ideal of $A^0$. But it is in general not an ideal of $A$.

Recall $A$ is called uniform if $A^0$ is bounded(IV.1.1.6), or equivalently $A^0$ is a ring of definition, by(I.9.5.2).
Proof: By (I.9.5.3), every power-bounded element is contained in a ring of definition, and any ring of definition is bounded, so $A^0$ is the union of all rings of definitions of $A$, and this is filtered by (I.9.5.3).

To show $A^0$ is integrally closed, notice by what we already proved, if $a$ is integral over $A^0$, then it is integral over a ring of definition $A_0$, but then $\{a^n\} \subset A_0[a]$ is bounded, so $a \in A^0$.

Showing $A^{00}$ is a radical ideal of $A^0$ is easy and omitted. □

Cor. (I.9.5.9). If a Huber ring $A$ is separated, Tate and uniform, then $A$ is reduced.

Proof: Assume that $A_0$ the set of power-bounded elements is a ring of ideal and $g \in A_0$ is a pseudo-uniformizer. If $x$ is nilpotent, then $g^{-n}$ is nilpotent for any $x$, so power-bounded and $g^{-n}x \in A_0$, which means $x \in g^n A_0$ for any $n$. But $A_0$ is separated, so $x = 0$. □

Cor. (I.9.5.10). Let $A$ be a Huber ring, then an ideal $J$ is open iff $A^{00} \in \sqrt{J}$.

Proof: If $J$ is open, then clearly $A^{00} \subset \sqrt{J}$. Conversely, if $A^{00} \subset \sqrt{J}$ and $(A_0, I)$ is a couple of definition, then $I \subset A^{00}$ by (I.9.1.9), so $I \subset \sqrt{J}$. And $I$ is f.g., so $I^N \subset J$ for some $J$, thus $J$ is open. □

Prop. (I.9.5.11). If $K$ is a complete non-Archimedean field, then any Banach $K$-algebra $R$ is a complete Tate ring, and if $K, R$ are perfectoid, then $R^{00} = K^{00} R^{00}$.

Proof: In the perfectoid case, first $K^{00} R^0 \subset R^{00}$, and for any topological nilpotent $\alpha$, $\alpha^n \subset t R^{00}$ for a pseudo-uniformizer $t$. Thus $R^{00}$ and $K^{00} R^0$ has the same radical, it suffices to show $K^{00} R^0$ is radical, but the quotient $R^0 / K^{00} R^0$ is a perfect $K^0 / K^{00}$-algebra by perfectoidness, thus it must be radical. □

Prop. (I.9.5.12) (Complete Perfect Tate ring is Uniform, André). If $A$ is a complete Tate ring of char p that is perfect, then $A$ is uniform.

Proof: Let $(A_0, t)$ be a ring of definition, let $A_n = A_0^{\frac{1}{t^n}}$, then $A_\infty = \text{colim} A_n = (A_0)_{\text{perf}}$. We check $t^{\frac{1}{t^n}} A^0 \subset A_\infty \subset t^{-1} A_0$, which shows $A^0$ is bounded.

If $f \in A^0$, then $t^m f \subset A \subset A_\infty$, and $A_\infty$ is perfect, so $t^{\frac{m}{t^n}} f \in A_\infty$ for all $n$. Notice the Frobenius is a continuous bijection of Banach spaces, so it is open by Banach theorem (X.3.2.4), so $A^p_0 \supset t^{mp} A_0$, thus $t^m A_1 \subset A_0$, and then $t^{\frac{m}{t^n}} A_{n+1} \subset A_n$. So $t^{\sum_{m/p^n}} A_n \subset A_1$. So $t^c A_\infty \subset A_0$, for $c$ large. □

Huber Pairs

Def. (I.9.5.13) (Huber Pairs). For a Tate ring $A$, a ring of integral elements is an open and integrally closed subring of $A$ contained in $A^0$ (I.9.5.8) (for example $A^0$ itself). A Huber pair is a pair $(A, A^+)$ that $A$ is a Huber ring and $A^+$ is a ring of integral elements. A morphism of Huber pairs should preserve the ring of integers.

A Huber ring is called an Affinoid Tate ring if $A$ is Tate.

Prop. (I.9.5.14). $A^{00} \subset A^+$ as an ideal for any ring of integral elements $A^+$. In particular, $A^+$ contains any pseudo-uniformizer, and the set of rings of integral elements is in bijection with integrally closed subrings of $A/A^{00}$.

Also, $A^+$ is a filtered colimits of rings of definitions.
Proof: \( t \in A^{00} \) is topologically nilpotent hence \( t^n \subset A^+ \) for some \( n \) as it is open, and then \( t \in A^+ \) as it is integrally closed. It is an ideal because it is an ideal of \( A^0(I.9.5.8) \).

For the last assertion, notice that \( A^0 \) is the filtered colimits of rings of definitions (I.9.5.8), and the intersection of a ring of definition with \( A^+ \) is also a ring of definition, because it is open and bounded (I.9.5.2), the result follows. \( \square \)

Def. (I.9.5.15) (Zariski, Henselian, Complete Huber Pairs). A Huber ring \((A, A^+)\) is called
- complete iff \( A \) is complete.
- Henselian iff \((A^+, A^{00})\) is a Henselian pair.
- Zariski iff \((A^+, A^{00})\) is a Zariski pair.

Prop. (I.9.5.16). An affinoid Tate ring \((A, A^+)\) with a ring of definition \((A_0, I)\) that \( A_0 \subset A^+ \) is
- Zariski iff \( I \) is in the Jacobson radical of \( A_0 \).
- Henselian iff the pair \((A_0, I)\) is Henselian.
- Complete then it is Henselian.
- Zariski then it is Zariski.

Proof: 1: We prove that if \( t \in \text{rad}(A_0) \), the for any other \( B_0 \supset A_0 \), \( t \in \text{rad}(B_0) \). If this is true, then as \( A^+ \) is a filtered colimits of rings of definitions (because \( A^0 \) does), it is clear that \( t \) lies in the maximal ideal (check \( 1 + at \) is unit). For this, if \( m \subset B_0 \) is maximal and \( t \notin m \), choose \( n \) that \( t^n B_0 \subset A_0 \), and an element \( b \in B_0 \) that maps to \( t^{-n-1} \) modulo \( m \), then \( a = t^n b \in A_0 \) is mapped to \( t^{-1} \). Thus the composition \( A_0 \to B_0 \to B_0/m \) is surjective: \( b \) is the image of \( a^n(t^n b) \in A \). So \( t \) is not in a maximal ideal of \( A_0 \), contradiction.

Conversely, Cf. [Bhatt Perfectoid Space P57].

2: Cf. [Bhatt Perfectoid Spaces P57].

3: \( A \) is complete then \( A_0 \) is complete, hence \((A_0, I)\) is Henselian by (I.6.10.6), so it is Henselian by item 2. 4: Trivial. \( \square \)

Adic Morphisms

Def. (I.9.5.17) (Adic Morphisms). A morphism of Huber rings \( f : A \to B \) is called an adic morphism if we can choose rings of definitions \( A_0, B_0 \) and an ideal of definition \( I \) of \( A \) that \( f(A_0) \subset B_0 \), and \( f(I)B_0 \) is an ideal of definition of \( B_0 \).

A morphism \((A, A^+) \to (B, B^+)\) of Huber pairs is called adic if \( A \to B \) is.

Prop. (I.9.5.18). Let \( f : A \to B \) be an adic morphism between Huber rings, then:
1. \( f \) is continuous and open.
2. If \( A_0, B_0 \) are rings of definition s.t. \( f(A_0) \subset B_0 \), then for any ideal of definition \( I \subset A_0 \), \( f(I)B_0 \) is an ideal of definition in \( B_0 \).
3. \( f \) maps bounded sets to bounded sets.

Proof: Let \( f(A_0) \subset B_0 \) and \( J = f(I)B_0 \). Then \( I^n \subset f^{-1}(J^n) \), so \( f \) is continuous.

If \( E \) is bounded in \( A \), for any \( n \), let \( m \) be that \( I^m E \subset I^n \), then \( f(E)f(I)^m = f(EI^m) \subset f(I^n) \subset J^n \), thus \( f(E)I^m \subset J^n \), so \( f(E) \) is bounded. \( \square \)
Construction of Huber Rings

Main references are [Mor19].

Prop. (I.9.5.19) (Quotient). Let \((A, A^+)\) be a Huber ring and \(a\) be an ideal of \(A\), then the quotient pair \((A/a, (A/a)^+)\) where \(A/a\) is the integral closure of \(A^+/a\) in \(A/a\).

Prop. (I.9.5.20) (Completion of Huber Rings). Let \(A\) be a Huber ring and \((A_0, I)\) be a couple of definition. Set \(\hat{A} = \lim \downarrow A/I^n\) (as an Abelian group), then:

1. The canonical map \(\hat{A}_0 \to \hat{A}\) is injective and \(\hat{A}_0 \cap A = A_0\).
2. If we put the unique topology on \(\hat{A}\) that \(\hat{A}_0\) is an open subgroup, then \(\hat{A}\) is complete.
3. There is a unique topological ring structure on \(\hat{A}\) that \(A \to \hat{A}\) is continuous.
4. \(\hat{A}\) is Huber with a couple of definition \((\hat{A}_0, I\hat{A}_0)\), and the canonical map \(A \to \hat{A}\) is adic.
5. \(\hat{A}_0 \otimes_{A_0} A \to \hat{A}\) is an isomorphism.

Proof: Cf. [Mor19]P72.

Prop. (I.9.5.21). Let \(A\) be a Huber ring and \(i: A \to \hat{A}\) be the completion, then under the bijection of (I.9.1.6),

- \(\hat{A}^0 = \hat{A}^0_0, A^0 = A^0_0\).
- \(G \subset A\) is a ring of definition iff \(\hat{G} \subset \hat{A}\) is a ring of definition.
- the map \(\text{Cont}(\hat{A}) \to \text{Cont}(A)\) is a bijection.

Proof: Cf [Mor19] P75.

Prop. (I.9.5.22). Let \(A\) be a Huber ring, then under the bijection of (I.9.1.6), an open ring \(A_0\) is a ring of integral elements of \(A\) iff \(A^0_0\) is a ring of integral elements of \(A^\wedge\).

Proof: It is easy to show \(A_0\) is a ring iff \(A^0_0\) is a ring, and by (I.9.5.21), \(A_0 \subset A^0\) iff \(A^0_0 \subset (A^\wedge)^0\).

It suffices to prove that if \(A_0\) is open and integrally closed, then \(A^0_0\) is integrally closed in \(A^\wedge\).

Let \(x \in A^\wedge\) satisfy \(x^d + a_{d-1}x^{d-1} + \ldots + a_0 = 0\), where \(a_i \in A^\wedge_0\), because \(A^\wedge_0\) is open, we can find \(x' \in A\) and \(a_i' \in A_0\) that \((x')^d + a_{d-1}'(x')^{d-1} + \ldots + a_0' \in A_0\), but then \(x' \in A_0\) because \(A_0\) is integrally closed, and thus \(x = (x - x') + x' \in A_0\).

Cor. (I.9.5.23) (Completion of Huber Pairs). The forgetful functor from the category of complete Huber pairs to the category of Huber pairs has a left adjoint called completion, where \((A, A^+)\wedge = (A^\wedge, (A^\wedge)^+)\), where \((A^\wedge)^+\) is the closure of the image of \(A^\wedge\) in \(A^\wedge\).

Prop. (I.9.5.24) (Completion, Henselization, Zariski Localization). There are left adjoint to the forgetful functors from the category of Complete/Henselian/Zariski pairs to the category of pairs, called the Completion/Henselization/Zariski Localization of pairs. And there are natural maps

\[(A, A^+) \to (A, A^+)_{\text{Zar}} \to (A, A^+)_{\text{Hens}} \to (\hat{A}, \hat{A}^+)\]

Proof: 

• If \((A, A^+) \rightarrow (B, B^+)^\) is adic, then pullback along the associated map of topological spaces \(\text{Spa}(B, B^+) \rightarrow \text{Spa}(A, A^+)\) preserves rational subsets.

• Let \((B, B^+) \leftarrow (A, A^+) \rightarrow (C, C^+)\) be a diagram of Huber pairs where both morphisms are adic. Let \(A_0, B_0, C_0\) be rings of definition compatible with the morphisms, and \(I \subset A_0\) be an ideal of definition. Let \(D = B \otimes_A C\) and let \(D_0\) be the image of \(B_0 \times_{A_0} C_0\) in \(D\). Make \(D\) into a Huber ring by declaring \(D_0\) to be a ring of definition with \(ID_0\) as its ideal of definition and \(D^+\) be the integral closure of the image of \(B^+ \otimes_A C^+\) in \(D\). Then \((D, D^+)\) is a Huber pair, and it is the pushout of the diagram in the category of Huber pairs.

\[ \text{Proof:} \quad \text{For } 1, \text{ it suffices to show that if } T \text{ is a finite set of } A \text{ that } TA \text{ is open, then } TB \text{ is open in } B: I \subset TA \text{ for some ideal of definition } I \subset A_0, \text{ in which case } IB_0 \subset B_0 \text{ is also an ideal of definition by (I.9.5.18), thus open, and so } TB \text{ is also open as } IB \subset TB. \]

2 just follows from the definition.

Remark (I.9.5.26) (Non-Adic Morphisms). Pushouts may not exists for non-adic morphisms of Huber rings. For example, \(Z_p \rightarrow \mathbb{Z}_p[[T]]\) is not adic (I.9.5.18), and the diagram

\[ (Z_p[[T]], \mathbb{Z}_p[[T]]) \leftarrow (Z_p, \mathbb{Z}_p) \rightarrow (Q_p, \mathbb{Z}_p) \]

has no pushout in the category of Huber pairs: If there is a pushout \((D, D^+)\), then we will have a morphism

\[ (D, D^+) \rightarrow (Q_p(T, \frac{T^n}{p}), \mathbb{Z}_p(T, \frac{T^n}{p})) \]

for each \(n \geq 1\). But notice that \(T\) is nilpotent in \(D\), and \(1/p \in D < T^n/p \rightarrow 0 \in D\) as \(n \rightarrow \infty\). So \(T^n/p \in D^+\) for some \(n\), but then \(T^n/p \in \mathbb{Z}_p(T, \frac{T^{n+1}}{p})\), which is impossible.

Def. (I.9.5.27) (Topological Polynomial Functions). Let \(A\) be a non-Archimedean topological ring, and \(\{X_i\}_{i \in I}\) be a family of indeterminates, \(\{T_i\}_{i \in I}\) be a family of subsets of \(A\) that satisfies \(T^n_i U\) is open for any \(n > 0, i\) and open nbhd \(U\) of \(A\).

Let \(\mathbb{N}^{(I)}\) be the set of functions \(I \rightarrow \mathbb{N}\) with finite support, then for any \(\nu \in \mathbb{N}^{(I)}\), let \(T^\nu = \prod_{i \in I} T_i^{\nu(i)}\).

For any nbhd \(U\) of \(A\), we set

\[ U_{[X,T]} = \{ \sum_{\nu \in \mathbb{N}^{(I)}} a_\nu X^\nu | a_\nu \in T^\nu U \}, \]

Then there exists a unique topological structure on \(A[X]\) that \(U_{[X,T]}\) form a fundamental system of nbhds of 0, and denote it by \(A[X]_T\). It satisfies:

• the canonical inclusion \(i: A \rightarrow A[X]_T\) is continuous and the set \(\{T_iX_i\}_{i \in I}\) is power-bounded.

• \(i\) satisfies the universal property that any continuous map \(f : A \rightarrow B\) to another non-Archimedean topological ring \(B\) that \(\{f(T_i)X_i\}_{i \in I}\) is power-bounded factors through \(A[X]_T\).

\[ \text{Proof:} \quad \text{Just notice that } (U \cap V)_{[X,T]} \subset U_{[X,T]} \cap V_{[X,T]} \text{ and } U_{[X,T]} \cdot V_{[X,T]} = (UV)_{X,T}, \text{ so they form a topological basis because } A \text{ is topological.} \]

The first properties are easily verified. For the second, the extension \(f' : A[X] \rightarrow B\) exists abstractly, and it suffices to show it is continuous. If we let \(E \subset B\) be the subring generated by \(\{f(T_i)X_i\}_{i \in I}\), then \(E\) is bounded, so for any open subgroup \(H \subset B\), there is some open subgroup \(G \subset B\) that \(EG \subset H\), and thus \(f^{-1}(G)\) is open and contains some nbhd \(U\), then \(U_{[X,T]} \subset (f')^{-1}(G)\), so \(f'\) is continuous.
Def. (I.9.5.28) (Topological Power Series). Let $A, X, T$ as in (I.9.5.27), then the set
\[ A(X)_T = \left\{ \sum_{\nu \in \mathbb{N}(I)} a_\nu X^\nu \in A[[X]] | a_\nu \in T^\nu U \text{ a.e.} \right\}, \]
is a subring of $A[[X]]$, and there is a unique topological structure on $A[[X]]$ that
\[ U_{(X,T)} = \left\{ \sum_{\nu \in \mathbb{N}(I)} a_\nu X^\nu \in A(X)_T | a_\nu \in T^\nu U \right\}, \]
form a fundamental system of nbhds of $A(X)_T$.

Proof: The proof is not hard and similar to that of (I.9.5.27) so omitted. \[ \square \]

Prop. (I.9.5.29). Let $A, X, T$ as in (I.9.5.27), then
\begin{itemize}
    \item $A[X]_T$ is dense in $A(X)_T$ and the topology coincide.
    \item If $A$ is Hausdorff and $T_i$ is bounded for any $i \in I$, then $A[X]_T$ and $A(X)_T$ are all Hausdorff.
    \item If $A$ is complete and $T_i$ is bounded for any $i \in I$, then $A(X)_T$ is complete, so it is the completion of $A[X]_T$.
\end{itemize}

Proof: Only the completeness needs proof, Cf.[?]P80. \[ \square \]

Prop. (I.9.5.30) (Power Series is Huber). Let $A$ be a Huber ring with a couple of definition $(A_0, I)$, and $X = \{X_\lambda\}$ be a finite set of indeterminates, $T_\lambda$ is a family of subsets of $A$ that $T_\lambda A$ is open in $A$, then
\begin{itemize}
    \item $A[X]_T$ is a Huber ring with a couple of definition $((A_0)[X,T], I[X,T])$, and there is a canonical map $A \to A[X]_T$ which is adic.
    \item $A(X)_T$ is Huber with a couple of definition $((A_0)(X,T), I(X,T))$, and there is a canonical map $A \to A(X)_T$ which is adic.
\end{itemize}

Proof: It suffices to prove for any ideal of definition $J$, $J_{[X,T]} = J \cdot (A_0)[X,T]$ and $J_{(X,T)} = J \cdot (A_0)(X,T)$. The first is clear. For the second, use the fact $J$ is f.g. and $\{J^n\}$ is a fundamental basis of nbhds of 0. \[ \square \]

Prop. (I.9.5.31) (Example). Let $A = \mathbb{Z}_p$ and $T = p$, then there is a Huber ring
\[ \mathbb{Z}_p(X)_T = \left\{ \sum_{n \geq 0} a_n X^n \in \mathbb{Z}_p[[X]] | l^{-n} a_n \to 0 \right\} \]
with a ring of definition
\[ (\mathbb{Z}_p)(X)_T = \left\{ \sum_{n \geq 0} a_n X^n \in \mathbb{Z}_p(X)_T | a_n \in p^n \mathbb{Z}_p \right\}. \]

Notice that $\mathbb{Z}_p(X)_T$ is not adic although $\mathbb{Z}_p$ is (because $p$ is nilpotent but $pX$ is not).

Def. (I.9.5.32) (Localizations). Let $A$ be a non-Archimedean topological ring and $\{T_i\}$ is a family of subsets of $A$ satisfying $T_i^n U$ is open for any $n > 0, i$ and open nbhd $U$ of $A$, and $\{s_i\}$ is a family of elements of $A$, which generates a multiplicative subset $R \subset A$.

Then there is a unique non-Archimedean ring structure on $R^{-1}A$, denoted by $A(T_i)$, that the canonical map $\varphi : A \to A(T_i)$ is continuous and the set $\{\frac{\varphi(t)}{\varphi(s_i)}\}$ is power-bounded, and it is the initial map for all maps $A \to B$ satisfying this property.
Proof: Cf. [Mor19]P83.

Cor. (I.9.5.33). Let $J$ be the ideal of $A[X]_T$ generated by \{1 - s_iX_i\}, then $A[X]_T/J$ with the quotient topology satisfies the same universal property as $A(\frac{T}{S})$, so there is a canonical isomorphism

$$A[X]_T/J \cong A(\frac{T}{S}).$$

In particular, $A(\frac{T}{S})$ is a Huber ring, and the canonical map $A \rightarrow A(\frac{T}{S})$ is adic. Explicitly, $B_0$ is the $(A_0)_{X,T}$-subalgebra of $B$ generated by the elements $\frac{T}{s_i}$.

Def. (I.9.5.34). If $A$ is Huber ring, then we denote the completion of $A(\frac{T}{S})$ by $A(\frac{T}{S})$, which is also a Huber ring, and the canonical map $A \rightarrow A(\frac{T}{S})$ is adic, by (I.9.5.30) and (I.9.5.33). It satisfies the natural universal property.

Cor. (I.9.5.35). If $A$ is complete, then we can also regard $A(\frac{T}{S})$ as the quotient of $A(X)_T$ by the closure of the ideal generated by \{1 - s_iX_i\}.

Prop. (I.9.5.36) (Example). Let $A = \mathbb{Z}_p[[T]]$ with the $(p,T)$-adic topology, then

$$A(\frac{p,T}{T}) = \mathbb{Z}_p[[T]][T^{-1}]$$

with a ring of definition $A[\frac{p}{T}]$, and

$$A(\frac{p,T}{p}) = \mathbb{Z}_p[[T]][p^{-1}]$$

with a ring of definition $A[\frac{T}{p}]$.

In $A(\frac{p,T}{T})$, a ring of definition is $A(X)_T/(1 - pX)$, which is isomorphic to

$$A(\frac{T}{p}) = \{ \sum_{n \geq 0} a_n(\frac{T}{p})^n | a_n \in A, a_n \rightarrow 0 \}$$

by (I.9.5.33).

6 Analytic Points and Analytic Huber Pairs

Def. (I.9.6.1) (Analytic Huber Rings). A Huber ring is called analytic if the ideal generated by the topologically nilpotent elements is the unit ideal. Any Tate ring is analytic.

Prop. (I.9.6.2) (Equivalent Definitions of Analytic Rings). For a Huber ring $A$, the following are equivalent:

1. $A$ is analytic.
2. Any ideal of definition in any ring of definition of $A$ generates the unit ideal of $A$.
3. Any open ideal of $A$ is trivial.
4. For any non-trivial ideal $I$ of $A$, the quotient topology on $A/I$ is not discrete.
5. The only discrete $A$-module is the 0-module.
6. The set $\text{Spa}(A, A^+)$ contains no point with induced topology on the residue field trivial.

Prop. (I.9.6.3) (Analytic Open Mapping Theorem). If $A$ is an analytic Huber ring, and $M,N$ are complete Banach $A$-modules, then any continuous surjective map $M \to N$ is open.

Proof: Hub94, L2.4(i). 

Analytic Points

Def. (I.9.6.4) (Analytic Points). Let $A$ be a Huber ring, then a point $x \in \text{Cont}(A)$ is called an analytic point if $p_x$ is not open in $A$. The set of analytic points of $\text{Cont}(A)$ is denoted by $\text{Cont}(A)_{an}$.

If $A$ is Tate, then $\text{Cont}(A)_{an} = \text{Cont}(A)$, because the only open ideal of $A$ is $A$ itself.

Prop. (I.9.6.5) (Characterizations of Analytic Points). Let $A$ be a Huber ring, then for a point $x \in \text{Cont}(A)$, the following is equivalent:

1. $x$ is analytic.
2. $|A^{00}|_{x} \neq 0$.
3. For any ring of definition and ideal of definition $(A_0,I)$ of $A$, $|I|_{x} \neq 0$.

Proof: 1 $\rightarrow$ 2: $p_x$ is non-open, so it cannot contain the open subset $A^{00}$.
2 $\rightarrow$ 3: trivial because any ring of definition contains $A^{00}$ (I.9.5.8). 

Cor. (I.9.6.6). Let $A$ be a Huber ring and $I$ an ideal of definition with a set of generators $(f_1, \ldots, f_n)$, then

$$\text{Cont}(A)_{an} = \bigcup_{i=1}^{n} U(f_1, \ldots, f_n).$$

Prop. (I.9.6.7) (Analytic Valuation Microbial). Let $A$ be a Huber ring and $x \in \text{Cont}(A)_{an}$, then $x$ has rank $\geq 1$, and the valuation $|\cdot|_{x}$ on $k(x)$ is microbial (I.9.3.18).

Proof: If $x$ has rank 1, then $\Gamma_x = 1$, and $p_x = \{a \in A| |a|_{x} < 1\}$ is open.

If $x$ is analytic, then there are some $a \in A^{00}$ that $|a|_{x} \neq 0$ (I.9.3.18), thus the image of $a$ in $k(x)$ is non-zero and topologically nilpotent, thus $k(x)$ is microbial by (I.9.3.18). 

Prop. (I.9.6.8) (Adic Morphism and Analytic Points). Let $\varphi : (A,A^+) \to (B,B^+)$ be a morphism of Huber pairs, then:

- If $x \in \text{Spa}(B,B^+)$ is not analytic, then $\text{Spa}(\varphi)(x)$ is not analytic.
- If $\varphi$ is is adic, then $\text{Spa}(\varphi) : \text{Spa}(B,B^+) \to \text{Spa}(A,A^+)$ carries analytic points to analytic points.
- If $B$ is complete and $\text{Spa}(\varphi) : \text{Spa}(B,B^+) \to \text{Spa}(A,A^+)$ carries analytic points to analytic points, then $\varphi$ is adic.
- If $\varphi$ is adic, then the $\text{Spa}(\varphi)$ maps rational subsets to rational subsets. In particular, $\text{Spa}(\varphi)$ is spectral.

Proof: 1: Trivial.

2: If $f(x)$ is not analytic, then $I \subset \varphi^{-1}(p_x)$, so $f(I) \subset p_x$, which means $p_x$ is not analytic because $f(I)A_0$ is open.

3: Cf.Morel P96?.

4: Only notice that $(f(T))$ is open if $(T)$ is open.
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Cor. (I.9.6.9). If $A$ is an analytic Huber ring, then any continuous morphism $f : A \to B$ is adic, by (VIII.5.4.24).

7 Huber and Banach Rings

Cf. [Ked19] 1.5.

Prop. (I.9.7.1). Let $A$ be a uniform Huber ring, then for $x = \sum_{n=0}^{\infty} x_n T^n \in A\langle T \rangle$ such that the coefficients $x_n$ generate the unit ideal of $A$, then multiplication by $x$ defines a strict inclusion $A\langle T \rangle \to A\langle T \rangle$, i.e. $|xy| \geq |x|$.


8 Perfectoid Fields

Def. (I.9.8.1). The basis setting is a non-Archimedean complete valued field $K$, and denote $K^0 = \{ x \in K | |x| \leq 1 \}$, and $K^{00} = \{ x \in K | |x| < 1 \}$, which are rings because $K$ is non-Archimedean. $k = K^0 / K^{00}$ is the residue field of $K$. A non-zero element of $K^{00}$ is called a pseudo-uniformizer. $K^{00}$ is just the set of topological nilpotent elements of $K$.

Prop. (I.9.8.2). The valuation can in fact be constructed from $K^0$ as $|x| = \sup \{ \frac{1}{n} |x^k \in t^n K^0 \}$ by (I.2.8.5), as it is a rank 1 valuation.

Def. (I.9.8.3) (Perfectoid Field). A perfectoid field is a non-Archimedean complete field with residue field of char $p$ s.t.:

- The value group $|K^*| \subset \mathbb{R}^*$ is not discrete.
- $\mathcal{O}_K/p$ is semi-perfect.

Prop. (I.9.8.4) (Examples of Perfectoid Fields).

- $K = \mathbb{Q}_p(p^{\frac{1}{p^k}})^\wedge$. Its valuation ring $K^0$ is $\mathbb{Z}_p(p^{\frac{1}{p^k}})^\wedge$, and $K^0/p \cong \mathbb{F}_p(t^{\frac{1}{p^k}})/(t)$, which is clearly semi-perfect. And its value group is $\mathbb{Z}[p^{-1}]$.
- $K = \mathbb{C}_p = \hat{\mathbb{Q}_p}$, its value group is $\mathbb{Q}$, and $K$ is alg.closed, so $K^0$ is clearly perfect.
- if $K$ is a non-Archimedean field of char $p$, then $K$ is a perfectoid field iff $K$ is perfect: if $K$ is perfect, then it is clearly perfectoid, and the semi-perfectness of $K^0$ implies its perfectness, so also $K$ is perfect(multiply by a $p$-power of an element in $K^{00}$).
- If $K$ is a perfectoid field and $|p| \leq |\pi| < 1$ is a pseudo-uniformizer, then $K/\pi$ is perfect hence perfectoid.

Prop. (I.9.8.5) (Perfectoid Field and Integral Perfectoid Rings). The ring of integers $\mathcal{O}_K$ for a perfectoid field $K$ is an integral perfectoid ring (VI.5.3.1).

Proof: We assume that $K$ is of char 0, then we check conditions in (VI.5.3.6). It is clear that $\mathcal{O}_K$ is $p$-adically complete, $p$-normal. To find $\varpi^p = pu$, as $|K^*|$ is not discrete, we find $x$ that $x^p$ divides $p$, and then there exists some $y$ that $y^p \equiv x^p/p \mod p$, thus $(xy)^p \equiv 1 \mod p$, thus $\varpi = xy$ satisfies the condition.

Prop. (I.9.8.6). If $K$ is a perfectoid field, then

- $|K^*|$ is $p$-divisible.
\( (K^{00})^2 = K^{00} \), and \( K^{00} \) is flat.

\( K^0 \) is not Noetherian.

**Proof:** 1: First if \(|p| < |x| \leq 1\), we show \(|x| \) is \( p \)-divisible: there is a \( y, z \in K^0 \) that \( y^p = x + pz \), so \(|y|^p = |x|\). Now because \( |K^*| \) is not discrete, so there is a \( |x| \notin |p|^Z \), by rescaling, we may assume \(|p| < |x| \leq 1\), thus \( p = xy \) for some \( y \), and \(|p| < |y| \leq 1\), too. So \(|p|\) is also divisible by \( p \), so it is clear now \(|K^*| \) is divisible by \( p \).

2 follows from (VI.5.3.5)(I.9.8.5).

\( 2 \to 3 \) by Nakayama’s lemma, because otherwise \( K^{00} = 0 \).

**Cor. (I.9.8.7).** The proof of 1 also shows that \(|K^*| \) is generated by \(|x| \) that \(|p| < |x| < 1\).

**Lemma (I.9.8.8).** If \( C^\circ \) is a perfectoid space of char \( p \), then \( 1 + m_{C^\circ} \) is a \( \mathbb{Q}_p \)-algebra.

**Proof:** Both \( \varphi \) and exponentiation of \( \mathbb{Z}_p^* \) is definable, so \( p^n t \cdot (1 + x) = (\varphi^n(1 + x))^k \).

**Tilting**

**Prop. (I.9.8.9).** Fix a pseudo-uniformizer \(|p| \leq |\pi| < 1\), consider the tilting (I.8.1.10) \( K^{0b} \), then by (I.8.1.12), the group together with the topology doesn’t depends on \( \pi \) chosen.

\[
\begin{array}{c}
\text{lim}_{x \to x^p} K^0 \\
\cong \\
\downarrow \\
K^{0b} = \lim_{\varphi} K^0/\pi \to K^0/\pi
\end{array}
\]

**Remark (I.9.8.10).** There are diagrams:

\[
\begin{array}{c}
\text{lim}_{x \to x^p} K^0 \\
\cong \\
\downarrow \\
K^{0b} = \lim_{\varphi} K^0/\pi \to K^0/\pi
\end{array}
\]

**Prop. (I.9.8.11) (Tilting of \( K^0 \)).** Let \( K^{0b} \) be given as in (I.9.8.9), then there is an element \( t \in K^{0b} \) that \(|t^\sharp| = |\pi|\), and \( t \) maps into \((\pi)\) and gives an isomorphism \( K^{0b}/t \cong K^0/\pi \).

Moreover, the \( t \)-adic topology on \( K^{0b} \) is complete, and coincides with the topology of \( K^{0b} \) given as in (I.8.1.10).

**Proof:** There are canonical surjective map \( K^{0b} \to K^0/p \to K^0/\pi \), and by \( p \)-divisibility of the value group (I.9.8.6), there is a \( f \in K^0 \) that \(|f|^p = |\pi|\), so in particular \(|f| > |\pi|\), thus \( f \neq 0 \in K^0/\pi \), and choose a \( g \in K^{0b} \) lifting \( f \) mod \( \pi \), then \( g^\sharp \equiv f \mod \pi \), see diagram (I.9.8.10), so \(|g^\sharp| = |f| \) as \(|f| > |\pi|\). Now let \( t = g^\sharp \), then \(|t^\sharp| = |f|^p = |\pi|\).

Now clearly \( t \) maps into \((\pi)\), and if \( g \) maps to 0 in \( K^0/\pi \), then by the diagram again, \( g^\sharp \in (\pi) \), and \((t^\sharp) = (\pi)\), so \( g^\sharp = at^\sharp \) for some \( a \in K^0 \), so \( t|g \) in \( K^{0b} \), as by (I.8.1.17), \( K^{0b} \) is a valuation ring in the valuation \(|\cdot| \circ t^\sharp\).

For the last assertion, just use the commutative diagrams:

\[
\begin{array}{ccc}
K^{0b}/(p^n) & \rightarrow & K^{0b}/(p^{n-1}) \\
\downarrow & & \downarrow \\
K^0/(\pi) & \varphi & \rightarrow & K^0/(\pi)
\end{array}
\]

the vertical are isomorphisms, and compute their inverse limits.

**Cor. (I.9.8.12) (Tilting of Perfectoid Field).**

- \( K^{0b} \) is a valuation ring of rank 1, with the field of fraction \( K^\flat = K^{0b}[t^{-1}] \) which is a perfectoid.

- Its maximal ideal is \((t^{\frac{1}{n}})\), and of Krull dimension 1.

- The value group and residue field of \( K \) and \( K^\flat \) is canonical isomorphic.
Proof: \( K^{0b} \) has rank no more than \( K^0 \) which is \( 1(I.9.8.3) \), and it is non-trivial because \(|t| = |\pi|\), so the rank is 1, and it is perfect by definition, so \( K \) is perfectoid by \( (I.9.8.4) \).

For the maximal ideal, the maximal ideal of \( K^{0b}/t \) is its nilradical, as it is a valuation ring of rank \( 1(I.2.8.8) \), which is clearly \( \langle t^{1/\pi} \rangle \). For the dimension, by \( (I.9.3.3) \), the Krull dimension equal the rank, which is 1.

For the residue field, use the isomorphism \( (I.9.8.11) \), \( K^{0b}/t = K^0/\pi \) and the second item just proved, and for the value group, the same lemma \( (I.9.8.11) \) gives any \(|p| \leq |\pi| < 1\) are in the value group of \( K^p \), and \(|K^*|\) is generated by these values by \( (I.9.8.7) \).

Prop. \( (I.9.8.13) \) (Tilting Continuous Valuations). If \( K \) is perfectoid, for any continuous valuation on \( K \) of any rank, the function \(|\cdot| = |\cdot| \circ \sharp\) is a continuous valuation on \( K^{0b} \), and all continuous valuation of \( K^{0b} \) comes from this way.

Proof: Clearly \(|-| = |\cdot| \circ \sharp\) is multiplicative and has trivial kernel, and it is non-Archimedean: for \( f = (f_n), g = (g_n) \in K^p \), 
\[
|f| = |(f_n)^{\pi^n}| = |\lim_k (f_{n+k} + g_{n+k})^{\pi^k}| \leq \max\{|f_n|, |g_n|\}^{\pi^n} = \max\{|f_n|, |g_n|\}
\]
so it is non-Archimedean. It is also continuous because \( \sharp\) is continuous.

Conversely, we notice a continuous valuation on a rank 1 valuation field corresponds to valuation rings in the residue field \( K^0/K^{0b} \), so by \( (I.9.8.12) \), we get a bijection on the continuous valuations. □

Prop. \( (I.9.8.14) \) (Almost Purity in Dimension 0). If \( K \) is a perfectoid field, \( L/K \) is a finite field ext, with the natural topology, then:

- \( L \) is perfectoid.
- \([L^p : K^p] = [L : K]\).
- The map \( L \to L^p \) defines an isomorphism \( K_{fet} \cong K_{fet}^p \).

Proof: This is a special case of almost purity theorem \( (I.9.10.1) \).

Lemma \( (I.9.8.15) \) (Kedlaya). If \( K^p \) is alg.closed, then \( K \) is alg.closed.

Proof: Let \( P(X) = X^d + a_{d-1}X^{d-1} + \ldots + a_0 \in K^0[X] \) be a irreducible monic polynomial, then its Newton polygon is a line, and we may assume \(|a_0| = 1\), as \( K^{0b} \) is alg.closed, so \(|K^{0b}| = |K^{0b}|(I.9.8.12) \) is a \( Q = \text{vector space} \).

Next we choose a \( Q(X) \in K^{0b}[X] \) that \( Q(X) \equiv P(X) \mod t \), as \( K^{0b}/t \cong K^0/\pi \). Now we consider \( P(x + y^p) \), then \( P(y^p) \) is divisible by \( \pi \), so its Newton polygon is now of positive slope, so \( c^{-d}P(cx + y^p) \in K^0[X] \) again, where \( c^{-d} = |P(y^p)| \leq |\pi| \). Then notice by iteration this argument, we get a sequence of \( y_n \), and then \( y_1 + c_1 y_2 + c_1 c_2 y_3 + \ldots + c_1 \ldots c_n y_{n+1} \) that converges to a root of \( P(X) \). □

Prop. \( (I.9.8.16) \) (Examples of Tilting).

- If \( K = \mathbb{Q}_p(p^{\pi^{-1}})^{\wedge} \), then \( K^0 = \mathbb{Z}_p[p^{\pi^{-1}}]^{\wedge} \), thus \( K^p = \mathbb{F}_p((\hat{t}))_{\text{per}f}(I.8.1.5) \). And if \( L = K(\sqrt{p}) \), then similarly \( L^0 = \mathbb{Z}_p[p^{\pi^{-1}}] \), and \( L^p = K^p(\sqrt{t}) \).
• If \( K = \mathbb{Q}_p(\mu_{p^n}) \), then \( K^0 = \mathbb{Z}_p[\mu_{p^n}] \), notice there is a map \( \mathbb{Z}_p[\frac{1}{p^n}] \to \mathbb{Z}_p[\mu_{p^n}] \) with kernel \( (1 + \frac{1}{p^2} + \ldots + \frac{1}{p^{n-1}}) \), so

\[
K^0/p = \mathbb{F}_p[\frac{1}{p^n}]/(\frac{1}{p^n} - 1)p^{-1} \cong \mathbb{F}_p[t]/(t^{p-1})
\]

with the substitution \( t = \frac{1}{p^n} - 1 \). Then by(I.9.8.4), \( K^0_\emptyset = \mathbb{F}_p[\frac{1}{p^n}] \), and \( K^\emptyset = \mathbb{F}_p(t)^{\text{perf}} \).

**Remark (I.9.8.17).** Notice that \( K = \mathbb{Q}_p(\mu_{p^n})^\wedge \) and \( K = \mathbb{Q}_p(p^{\frac{1}{p^n}})^\wedge \) have the same tiltings, so the tilting functor is not faithful. This is due to the fact that \( \mathbb{Q}_p \) is not perfectoid. This will not happen over a perfectoid base field, see(I.9.9.13).

9 Perfectoid Algebras

**Def. (I.9.9.1) (Perfectoid Algebra).** For \( K \) a perfectoid field with tilt \( K^\emptyset \), let \( t \in K^\emptyset \) be a pseudo-uniformizer with \( \pi = t^\emptyset \) satisfying \( |p| \leq |\pi| < 1 \), so it has a compatible collection of \( p^n \)-th roots \( (t^{\frac{1}{p^n}})^\emptyset \). Now:

- A **perfectoid algebra** over \( K \) is a uniform Banach \( K \)-algebra \( R^\emptyset/\pi \) that \( R^\emptyset/\pi \) is semi-perfect.

- A **perfectoid algebra** over \( K^0_\emptyset \) is a \( K^0 \)-algebra \( A \) that is \( t \)-adically complete and flat over \( K^0_\emptyset \) (or \( A_\emptyset \) over \( K^0_\emptyset \), by(I.16.3.3)), and \( K^0_\emptyset/\pi \to A/\pi \) is relative perfect, i.e. the Frobenius induces an isomorphism \( A/\pi^{1/p} \cong A/\pi \).

- A **perfectoid algebra** over \( K^0_\emptyset/\pi \) is a \( K^0_\emptyset/\pi \)-algebra \( A \) that is flat over \( K^0_\emptyset/\pi \) (or \( A_\emptyset \) over \( K^0_\emptyset \), by(I.16.3.3)), and the map \( K^0_\emptyset/\pi \to A \) is relatively perfect, i.e. the Frobenius induces an isomorphism \( A/\pi^{1/p} \cong A \).

**Remark (I.9.9.2).** Notice the definition regarding the relative perfectness doesn’t depend on \( \pi \) chosen, by the power lifting theorem(XIV.1.2.6).

**Prop. (I.9.9.3) (Faithfully flatness of Perfectoids).** Nonzero flat \( K^0_\emptyset/\pi \)-algebras are faithfully flat, so does \( t \)-adically complete flat \( K^0_\emptyset \)-algebras. In particular, \( \text{Perf}_{K^0_\emptyset/\pi} \) and \( \text{Perf}_{K^0_\emptyset} \) are all faithfully flat modules.

**Proof:** If \( K^0_\emptyset/\pi \to A \) is not faithfully flat, then there is an ideal \( J \subset K^0_\emptyset/\pi \) that \( K^0_\emptyset/J \neq 0 \) but \( A/J = 0 \). Now this implies \( J \subset I \), so there is a \( \omega \in I - J \) hence \( J \subset (\omega) \). Hence \( A/\omega = 0 \) as well. Now there are exact sequences \( 0 \to K^0_\emptyset /\omega^n \to K^0_\emptyset /\omega^{n+1} \to K^0_\emptyset /\omega \to 0 \), so tensoring with \( A \) and induct, we get \( K^0_\emptyset /\omega^n \otimes A = 0 \), but \( |\omega| < |\pi| \) for some \( n \), so \( A = 0 \).

The other case is similar, now \( A/\omega = 0 \), so use(I.16.3.3), \( A_\emptyset /\omega \subset (A/\omega)_\emptyset = 0 \), but \( A_\emptyset \) is also \( t \)-adically complete, so \( A_\emptyset = 0 \), and \( A = (A_\emptyset)^a = 0 \).

**Prop. (I.9.9.4) (Examples of Perfectoid Algebras).**

- If \( K \) has charp, then a \( K \)-Banach algebra is perfectoid iff it is uniform and perfect. Likewise, a \( \pi \)-adically complete and \( \pi \)-torsion free \( K^0_\emptyset \)-algebra is perfectoid iff it is perfect.

- Let \( A = K^0[\frac{x_1^{p^n}}, \ldots, x_n^{p^n}]^\wedge \), then \( A^\emptyset \in \text{Perf}_{K^0_\emptyset} \), and \( R = A[\pi^{-1}] \in \text{Perf}_K \) in the Banach metric as in(IV.1.2.7).
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**Proof:** 1: a perfectoid algebra of charp is perfect, because by semi-perfectness, \( x = x_1^p + \pi z_1 = x_1^p + \pi x_2^p + \pi^2 z_2 = \ldots \), so \( x = (x_1 + \pi^2 x_2 + \pi^3 x_3 + \ldots)^p \). In fact, uniformity is automatically implied by perfectness, by (I.9.5.12). The case of \( K^{0a} \)-algebra is similar.

2: \( A^a \) is \( K^{0a} \)-flat because \( A_s \) does, because it is a colimit of completions of polynomial algebras over \( I \) and \( I \) is flat over \( K^0 \)(I.9.8.6). and \( R \) is perfectoid by (IV.1.2.7) because \( A \) is totally integrally closed in \( R \), because \( K^0[x_1^{\frac{1}{p}}, \ldots, x_n^{\frac{1}{p}}] \) does (trivially), and use (I.16.2.10).

** Tilting Equivalence **

**Prop. (I.9.9.5) (Tilting Equivalence).** There are canonical isomorphisms of categories:

\[
\text{Perf}_K \cong \text{Perf}_{K^{0a}} \cong \text{Perf}_{K^{0a}/\pi},
\]

where the first map is by \( R \mapsto R^{0a} \) and \( A \to A_s[t^{-1}] \) just as in (IV.1.2.7). The second map is reduction by \( \pi \).

In particular, using tilting (I.9.8.11), there are canonical isomorphisms of categories:

\[
\text{Perf}_K \cong \text{Perf}_{K^{0a}} \cong \text{Perf}_{K^{0a}/\pi} = \text{Perf}_{K_{0a}/t} \cong \text{Perf}_{K_{0a}} \cong \text{Perf}_{K^n}.
\]

Now if \( R \in \text{Perf}_K \) corresponds to \( S \in \text{Perf}_{K^n} \), then we call \( S = R^\flat \) the **tilting** of \( R \) and \( R = S^\flat \) the **untilting** of \( S \).

**Proof:** \([\text{Perf}_K \cong \text{Perf}_{K^{0a}}]\)

Firstly, if \( R \in \text{Perf}_K \), then \( A = R^{0a} \in \text{Perf}_{K^{0a}} \): \( R^0/\pi^\frac{1}{p^2} \to R^0/\pi \) is surjective by definition, for injectivity, if \( x^p/\pi \in R^0 \), then \( x/\pi^\frac{1}{p} \) is also power bounded, thus in \( R^0 \). And by (IV.1.2.7), \( A \) is \( \pi \)-adically complete and \( \pi \)-torsion-free, hence \( R \)-flat by (I.7.1.9).

Next we show if \( A \in \text{Perf}_{K^{0a}} \), then \( A_s \) is \( \pi \)-adically complete, \( t \)-torsion free and \( p \)-root closed in \( A[\pi^{-1}] \), hence is an left inverse to the mapping \( R \to R^{0a} \), by (I.16.3.5). It is complete by (I.16.3.2.3).

For \( p \)-root closedness, by (I.16.3.3), \( A_s/\pi^\frac{k-1}{p} \subset (A/\pi^\frac{k}{p})_s \leftrightarrow (A/\pi)_s \) by Frobenius, and then so does \( A_s/\pi^\frac{k}{p} \to A_s/\pi \). Now if \( x \in A_s[\pi^{-1}] \) satisfies \( x^p \in A_s \), then \( y = \pi^{\frac{k}{p}} x \in A_s \) for some \( k \), and we want to lower \( k \) by 1 inductively, thus showing \( x \in A_s \): As \( y^p \in \pi A_s \), \( y \in \pi^\frac{k}{p} A_s \) by what we have proved, thus \( \pi^{\frac{k-1}{p}} x \subseteq A_s \).

For surjectivity of Frobenius: \( A_s/\pi \to A_s/\pi \), notice first it is almost surjective, because \( (A_s \to A_s/\pi)^a = A \to (A_s/\pi)^a \subset (A/\pi)^a = A/\pi \) is surjective by hypothesis, then by (I.16.2.3), it suffices to show that Frob is surjective on \( A/IA \). For some \( x \in A^s \), choose \( 0 < c < a \), almost surjectivity shows that \( \pi^c x \equiv y^p \) mod \( \pi A_s \), so \( (y/p^c) \equiv (y/p^c)^a \in A_s \), thus \( y \equiv p^c A_s \), thus \( x \equiv (y/p^c)^p \) mod \( \pi^{1-c} A_s \subset IA \), so we are done.

Finally, this is also a right inverse, because we know that \( A_s \cong R^0 \) by (I.16.3.5), thus \( A \cong R^{0a} \) in \( \text{Mod}_R^t \).

**Proof:** \([\text{Perf}_{K^{0a}} \cong \text{Perf}_{K_{0a}/\pi}]\)

Firstly the reduction is a perfectoid \( K^{0a}/\pi \)-algebra: it is flat because flatness is stable under base change, and the rest are trivial. To construct a converse is a problem of deformation theory, we need to lift from \( K^{0a}/\pi \)-algebra via \( K^{0a}/\pi^n \)-algebras to a \( K^{0a} \)-algebra, suppose each lifting is unique up to isomorphism and the lift \( A_n \) is flat over \( K^{0a}/\pi^n \), then we can form their inverse limit, which is flat,
because it is $\pi$-torsion-free: if $\pi(x_n) = 0$, then by $0 \to \pi^n K^{0a}/\pi^{n+1} \to K^{0a}/\pi^{n+1} \xrightarrow{\pi} K^{0a}/\pi^n \to 0$ and the flatness of $A_{\tilde{n}}$, $x_{n+1} \in \pi^n A_{n+1}$, thus $x_n = 0$, and $x = 0$.

Now $0 \neq A \in \text{Perf}_{K^{0a}/\pi}$, then $A$ is faithfully flat by (I.9.9.3), then by (I.16.1.8), $A_\pi$ is faithfully flat, and $(-)_{\pi}$ is preserves all colimits and also Frobenius, so $A_\pi$ is relatively perfect. Then we use the above argument, and (XI.2.4.1) to show that there is a $A \in \mathcal{C}$ which is $\pi$-adically complete and $K^{0a}$-flat, then $A = (A_\pi)^\circ$ is also $p$-adically complete and $K^{0a}$-flat, by (I.16.3.3).

And we check $\tilde{A}/\pi = (A_\pi/\pi)^\circ = (A_\pi)^\circ = A$ as $(-)_{\pi}$ commutes with colimits, and conversely, if $A \in \text{Perf}_{K^{0a}}$, we need to show $A = A/\pi$, notice by hypothesis, $A_\pi$ is faithfully flat $K^{0a}$-algebra that is relatively flat over $K^{0}/\pi$, now it is also complete, because $A_\pi \to A_\pi$ is an injection(because $(-)^\circ$ is exact) and almost isomorphism, so the cokernel is $\pi$-torsion, and $A_\pi$ is complete, so does $A_\pi_{\pi}$ by (I.5.7.9). Now $A_{\pi}/\pi = (A_{\pi}/\pi)$ as $(-)_{\pi}$ commutes with colimits, so $A_\pi$ is just the lift, and $(A_\pi/\pi)^\circ = A_\pi^\circ \cong A$. □

**Cor. (I.9.9.6) (Tilting via Fountain’s Functors).** The tilt $R^0$ is just the Fontaine’s tilting, i.e. $R^0 = R^{0b}[t^{-1}]$, and $R^0 = \lim_{x \to x^{0b}} R$, $R^{0b} = (R^0)^0$.

**Proof:** Consider the diagram

$$
\begin{array}{ccc}
\tilde{K}^{0a}/t^{0c} & \xrightarrow{\varphi} & K^{0a}/\pi \\
\downarrow & & \downarrow \\
K^{0a}/t & \cong & K^0/\pi
\end{array}
$$

Then the upper row is just the unique flat and relative perfect lifting along $K^{0a}/t^{0c} \to K^{0a}/t$. Taking inverse limit, we get the structure map $K^{0a} \to R^{0b}$, so after almostification, this is just the lifting we are looking for, because it is unique. So $(R^{0b})^\circ = (R^{0b})^\circ$, and $R^0 = R^{0b}[t^{-1}]$ unwinding the tilting equivalence.

For $R^0$, notice there is a map

$$R^0 \cong (\lim_{x \to x^{0b}} R^0)[t^{-1}] \to \lim_{x \to x^{0b}} (R^0[\pi^{-1}]) \cong \lim_{x \to x^{0b}} R$$

Now injectivity is clear as $t$ is non-zero-divisor, and if $(f_n) \in \lim_{x \to x^{0b}} R$, then $\pi^c f_n \in R^0$ because $R^0$ is $p$-root closed (I.16.3.5), so $t^c (f_n) \subset R^{0b}$.

For the last assertion, it is true if $R^{0b}$ is totally integrally closed in $R^0$, by (I.16.3.5). For this, if $t^c f_n \subset R^{0b}$, then $\pi^c (f_n) \subset R^0$, thus $f_n \in R^0$. And by $p$-root closedness, $p^n$-th roots of $f^\circ$ are all in $R^0$, so $f = (f_n) \in \lim_{x \to x^{0b}} R$ is in $R^{0b}$. □

**Prop. (I.9.9.7) (Fountain’s Functor $\theta$).** Given a perfectoid field $K$, the kernel of the Fontaine’s map $\theta : A_{inf}(K) \to K^{0}(I.8.1.15)$ is generated by a non-zero-divisor, in fact, if $\text{char} K = 0$, the generator can be chosen to be any element that maps to a generator of $\text{Ker} \tilde{\theta}$ and if $\text{char} K = p$ this diagram is trivial. In particular, the diagram is a pushout.

**Proof:** See the proof of (VI.5.3.6) in the $p$-torsionfree case. □

**Prop. (I.9.9.8) (Untilting via $A_{inf}$).** For any perfect $K^{0b}$-algebra $A$, by deformation theory (in fact Witt theory) there is a unique lifting $W(A)$ lifting it to $A_{inf}(K^{0b})$. And then pushout $W(A) \otimes_{A_{inf}(K^{0b})} K^{0b}$ is just the lifting of $A/\pi$, because the diagram above is pushout. This is in fact the method of [Kedlaya-Liu] used to prove the tilting-equivalence without the use of almost mathematics and deformation theory.
Cor. (I.9.9.9) (Limits and Colimits). Any of the categories in (I.9.9.5) has arbitrary limits and colimits.

Proof: We construct for $\text{Perf}_{K^{0a}}$: The limits is just the limits of topological rings, as the properties of $t$-adically complete, $t$-torsion free and perfect is preserved by limits (I.5.7.19). For the colimit, just use the $t$-adic completion of the left perfection of the colimits in the category of $K^{0a}$-algebras, its $t$-torsion is almost zero because of perfectness, thus it is almost flat (I.16.3.3).

Remark (I.9.9.10). Note also for further reference that in the category $\text{Perf}_{K^{0a}}$, a filtered colimits is just the $\pi$-adically completion of the filtered limits as rings, because perfectness and flatness is preserved (I.7.1.5).

Prop. (I.9.9.11) ((Un)Tilting Preserves Fields). A perfectoid $K$-algebra $R$ is a perfectoid field iff its tilt $R^\flat$ is a perfectoid field.

Proof: It is proven that if $R$ is a perfectoid field, then $R^\flat$ is a perfectoid field. Conversely, $R$ is a perfectoid field if the spectral norm given by $||x|| = \inf\{|t|^{-1} t \in R^*, tx \in R^0\}$ is the Banach valuation of $R$ and $R$ is a field.

For the multiplicativeness of $||-||_R$, notice that $R^\flat$ is a perfectoid field, so its non-Archimedean valuation coincides with the spectral norm of $||-||_{R^\flat}$, and this equals $||-||_R \circ \pi$, because $R^0 = R^{0\flat}$, an element $f \in R^{0\flat}$ iff $f^2 \in R^0$. Now the norm extends that of $K$ and commutes with scalar multiplication, so for any $f, g$, we may assume $f, g \in R^{0 - 0\frac{1}{\pi} R^0}$, now choose $a, b \in R^\flat$ that $a^2 - f, b^2 - g \in \pi R^0$, this can be done because $R^{0\flat} = R^{0\flat} / R^0 / \pi$ is surjective, then $a, b, ab \notin t R^{0\flat}$ because $R^{0\flat} = R^0$. Then clearly $||f||_R = ||a||_{R^\flat}, ||g|| = ||b||_{R^\flat}, ||fg||_R = ||ab||_{R^\flat}$, so it is multiplicative by the multiplicativeness of $R^\flat$.

To show $R$ is a field, consider and $f \in R - \frac{1}{\pi^0} R$, choose $a \in R^\flat$ that $f = a^2 + \pi g$, then as $R^\flat$ is a field, there is a $b$ that $ab = 1$. Now $||\pi||_R < ||\frac{1}{\pi^0}||_R \subset ||f||_R = ||a||_{R^\flat} \leq 1$, so we get $||\pi b^2 g|| < 1$, then

$$f^{-1} = \frac{1}{a^2 + \pi g} = \frac{b^2}{1 + \pi b^2 g} = b^2 (\sum (-\pi b^2 g)^k)$$

can be constructed in $R$. \qed

Perfectoid Affinoid Algebra

Def. (I.9.9.12) (Perfectoid Affinoid $K$-algebras). Fix a perfectoid field $K$ and write $m \subset K^0$ and $m^\flat \subset K^{0\flat}$ for the maximal ideals, and choose a pseudo-uniformizer $t$ that $|p| \leq |t^\flat| < 1$, $\pi = t^\flat$. Then an affinoid $K$-algebra $(R, R^+)$ is just an affinoid Tate ring over $(K, K^0)$. It is called a perfectoid affinoid $K$-algebra iff $R$ is a perfectoid algebra.

Prop. (I.9.9.13) (Tilting Equivalence). The categories of perfectoid affinoid algebras over $K$ and $K^\flat$ are equivalent, where $(R, R^+)$ is identified with $(R^\flat, R^{0\flat})$ iff $R^\flat$ is the tilting of $R$ and

$$\begin{align*}
R^+ / mR^0 & \cong R^{0\flat} / m^\flat R^{0\flat} \\
R^0 / mR^0 & \cong R^{0\flat} / m^\flat R^{0\flat}
\end{align*}$$

Moreover, in this case, $R^+ / \pi$ is semi-perfect, and $R^{0\flat} \cong R^{0\flat}$ as a subring of $R^{0\flat} \cong R^0$. 

Proof: The case \( R^+ = R^0 \) is already known by tilting equivalence(I.9.9.5) and(I.9.9.6).

By(I.9.5.11) and(I.9.5.14), \( mR^0 = R^{00} \subseteq R^+ \subseteq R^0 \), thus \( R^+ \to R^0 \) is an almost isomorphism and \( R^+ \) is determined by its image \( R^+/\pi \), which is integrally closed if \( R^+ \) does, so the identification is clear.

For the semi-perfectness: as \( R^+/mR^0 \) is integrally closed, it is perfect. Now \( R^+ \to R^0 \) is an almost isomorphism, so Frob on \( R^+/\pi \) is almost surjective because it does on \( R^0/\pi \) by definition, and now we know Frob is surjective on \( R^+/\pi \) by(I.16.2.3).

\[
\begin{array}{c}
R^+ \\
\downarrow
\end{array}
\begin{array}{c}
\to R^+ /mR^0 \\
\downarrow
\end{array}
\begin{array}{c}
R^+ /\pi \\
\to R^+ /mR^0 \cong R^+ /m^bR^0 \\
\downarrow
\end{array}
\begin{array}{c}
R^0 \\
\downarrow
\end{array}
\begin{array}{c}
\to R^0 /m^bR^0
\end{array}
\]

is the Cartesian diagram applied the functor \((-)^{\text{perf}}\), which preserves limits(I.8.1.10). (Notice that \( R^+ /mR^0 \cong R^+ /m^bR^0 \) is already perfect).

Cor. (I.9.9.14). Notice that the proof also shows that \( R^+ \to R^0 \) is an almost isomorphism, thus if \( R \) is a perfectoid \( K \)-algebra, then \( R^+ \) is automatically a perfectoid \( K^{0a} \)-algebra by(I.9.9.1).

Cor. (I.9.9.15) (Perfectoid Affinoid Field). A perfectoid affinoid \( K \)-algebra \((R, R^+)\) is called a perfectoid affinoid field iff \( R \) is a perfectoid field and \( R^+ \) is an open valuation ring.

Notice this is equivalent to \( R^+ /mR^0 \) is a valuation ring in \( R^0 /mR^0 \). In particular, combining with(I.9.9.11), affinoid perfectoid fields are preserved under tilting and untilting.

Cor. (I.9.9.16). The tilting equivalence also shows that for any perfectoid affinoid \( K \)-algebra \((R, R^+)\), the tilting induces an equivalence of categories \( \text{Perf}_R \cong \text{Perf}_S \).

Prop. (I.9.9.17) (Filtered Colimits of Perfectoid Affinoid \( K \)-Algebras). The category of perfectoid affinoid \( K \)-algebras has filtered colimits, and it is just the colimits in the category of complete uniform affinoid Tate rings(VIII.5.2.27). In particular, the filtered colimits of \((A_i, A^+_i)\) is \((\colim_i A_i, \colim_i A^+_i)\).

Proof: The colimit is perfectoid because the filtered colimits is exact. \( \square \)

10 Almost Purity Theorem

Prop. (I.9.10.1) (Almost Purity Theorem). For a perfectoid \( K \)-algebra \( R \) and its tilt \( S \),

- Almost purity in characteristic \( p \): (take \((-)_s\) and)Inverting \( t \) gives an equivalence \( S_{0f \text{ ét}}^{0a} \cong S_{f \text{ ét}} \).
- Almost purity in characteristic 0: Inverting \( \pi \) gives an equivalence \( R_{0f \text{ ét}}^{0a} \cong R_{f \text{ ét}} \).
- Tilting and untilting functors induce equivalences \( R_{0f \text{ ét}}^{0a} \cong S_{0f \text{ ét}}^{0a} \). In particular, there are equivalences

\[
S_{f \text{ ét}} \overset{a}{\leftarrow} S_{0f \text{ ét}}^{0a} \overset{b}{\to} (S^{0a}/t)_{af \text{ ét}} \cong (R^{0a}/\pi)_{af \text{ ét}} \overset{c}{\leftarrow} R_{0f \text{ ét}}^{0a} \overset{d}{\to} R_{f \text{ ét}}
\]

Proof: The map \( a \) is already given in(I.16.2.14) by passing the power bounded-elements(equivalently, \( S_s \)) and inverting \( t \). And it is an isomorphism.
The equivalence of $b$ and $c$ follows from [Almost Ring theory, Thm5.3.27].

The functor $d$ is given by $A \to A[[t^{-1}]]$. Firstly, $A$ is a perfectoid $K^{0a}$-algebra. This is because it is almost finite projective thus almost flat, and $R^{0a}/\pi \to A/\pi$ is weakly relative perfect by (I.16.2.15), so does $K^{0a}/\pi \to A/\pi$ because relative perfect is stable under composition. And it is finite projective thus almost direct summand of a finite free module.

So now the tilting equivalence (I.9.9.5) shows that $A[[t^{-1}]] \in \mathcal{R}_{\text{f\'et}}$; it is finite etale because the $A_\ast$ is finite projective by the right adjointness of $(-)_\ast$, and unramified is defined in terms of $A_\ast$. The converse of $d$ is supposed to be the functor that extract from $A_\ast$ from $A[[t^{-1}]]$ the total integral closure $A_{\text{tic}}$ of $R^0$, which is functorial. We already know that $A_\ast$ is totally integrally closed in $A[[t^{-1}]]$ by (I.9.9.5), so $A_{\text{tic}} \subseteq A_\ast$. Conversely, as $A$ is almostly finitely generated over $R^0$, for $f \in A_\ast$, $\pi f^N$ lies in a f.g. $R^0$-submodule of $A_\ast$, so $f^N$ is totally integral over $R^0$, so $A_\ast = A_{\text{tic}}$.

It’s left to show that $d$ is essentially surjective, but this uses perfectoid spaces. For now, we only check that this is true for $R$ being a perfectoid field (of char 0). For this, we show directly that the untilting functor $\sharp : K_{f\text{\'et}}^0 \to K_{f\text{\'et}}$ is essentially surjective. Now $\sharp$ is an equivalence of categories Perf$_K$ $\to$ Perf$_K$, and it preserves degree, at least for field extensions, so it preserves Galois extensions. Now that finite étale algebra over fields are just disjoint of finite separable extensions (I.7.7.19), so it suffices to show that any finite extension of $K$ is contained in some $L^\sharp$.

Consider $M = \widehat{K}$, it is alg.closed of char 0 so clearly a perfectoid field, and by (I.9.8.15) $M^\sharp$ is alg.closed. $M^\sharp$ is just the colimit in the category of uniform Banach $K$-algebras, so its valuation ring is just the completion of the valuation ring of $L^\sharp$ for $L/K^\sharp$ finite Galois. Then if $N = \cup L^\sharp$, then $N$ is dense in $M^\sharp$, and $N/K$ is clearly algebraic and in particular Hensel. So $N \subseteq \overline{N} \subseteq M^\sharp$ is dense, so by Krasner’s lemma (IV.1.1.35), $N = \overline{N}$. Now $N = \cup L^\sharp$ is an alg.closure of $K$, so every finite extension of $K$ is contained in some $L^\sharp$.

The proof of the general case of the essentially surjectivity of $d$ is continued at (VIII.5.7.10). □
I.10 Derived Commutative Algebras

Main references are [Sta].

[Sta]Chap15 contains many beautiful results working on the derived category of rings, and is used heavily in Scholze’s Thesis.

The basic construction of Rtensor and RHom should be redone at the level of ringed sites, Cf.[Sta]Chap21.

Basics

Prop. (I.10.0.1). Let $R$ be a ring and $K_n \in D(R)$, then the product in $D(R)$ of $K_n$ is given by $\prod I_n$, where $I_n$ are $K$-injective resolutions of $K_n$.

Proof: This is immediate from (I.11.5.15).

Def. (I.10.0.2) (Homotopy Fiber Square). A square of Abelian groups is called a homotopy fiber square if it is a homotopy fiber square in the derived category, or equivalently, the kernel of the two rows (or the two columns) are isomorphic.

This notion is identical to the notion of pullback square when the rows or the columns are surjective.

1 Rtensor and Tor

Prop. (I.10.1.1) (Differential Graded Structure). If $K$ is a commutative $A$-algebra object in $D(A)$ in the monoidal structure defined in (V.6.3.3), then $\oplus_{n \geq 0} H^n(K^\bullet)$ carries a natural graded commutative $A$-algebra structure.

Proof: Compare with (XI.1.2.4).

We may replace $K$ by a $K$-flat resolutions $L^1(V.6.5.15)$ that the algebra structure map for $K$ is represented by a morphism $L^\bullet \otimes_A L^\bullet \to L^2$ of complexes where $L^2$ is a complex quasi-isomorphic to $K$, hence the graded $A$-algebra structure is clear by (I.11.3.11), and it is commutative by (I.11.3.10). (How to check the structure is uniquely determined?).

Tor

Def. (I.10.1.2) (Tor). Let $M, N$ be $A$-module, then the torsion group $\text{Tor}_n^A(M, N)$ is defined to be $H^n(M \otimes^L_A N)$, compatible with that of (V.6.3.5).

Def. (I.10.1.3) (Torsion Group). Let $A$ be a commutative ring, $B$ an $A$-algebra and $I$ be an ideal, then the $I$-torsion of $B$ is defined to be $\text{Tor}_i^A(A/I, B)$, denoted by $B[I]$. In case $I = (f)$, it can be checked that $B[f]$ is the set of elements of $B$ that killed by $f$.

Also we denote $B[I^\infty] = \text{colim}_{n \to \infty} B[I^n]$. And $B$ is said to have bounded $f$-torsion iff $B[f^\infty] = B[f^n]$ for some $n$.

Prop. (I.10.1.4). If $A$ is a commutative ring with bounded $I$-torsions and $B$ is a flat $A$-module, then $B$ also has bounded $I$-torsions. (Because tensoring $B$ is exact).

Prop. (I.10.1.5) (Balancing Tor). In the category of rings, $\text{Tor}_n(A, B) = \text{Tor}_n(B, A)$. This can be seen using spectral sequence of the double complex of flat resolutions of $A$ and $B$. Similarly, we have two definitions of $\text{Ext}^i(M, N)$ are compatible.
Proof:

Prop. (I.10.1.6) (Base Change). For a ring extension $R \to S$, using projective resolution and spectral sequence, there is a first quadrant homology spectral sequence:

$$E^2_{pq} = \text{Tor}^R_p(S, \text{Tor}^R_q(A, B)) \Rightarrow \text{Tor}^R_{p+q}(A, B).$$

Similarly, for Ext,

$$E^2_{pq} = \text{Ext}^p_S(A, \text{Ext}^q_R(S, B)) \Rightarrow \text{Ext}^p_R(A, B).$$

Prop. (I.10.1.7) (Universal Coefficient Theorem). Let $P$ be a free $R$-module so $d(P_n)$ are all flat, then $Z(P_n)$ are also flat and

$$0 \to d(P_{n+1}) \to Z_n \to H_n \to 0$$

is a free resolution. We have an exact sequence:

$$0 \to H_n(P) \otimes_R M \to H_n(P \otimes_R M) \to \text{Tor}^R_1(H_{n-1}(P), M).$$

$$0 \to \text{Ext}^R_1(H_{n-1}(P), M) \to H^n(\text{Hom}_R(P, M)) \to \text{Hom}(H_n(P), M) \to 0$$

and these exact sequences non-canonically split because $Z_n$ is a direct summand of $P_n$, thus $Z_n \otimes M$ is a direct summand of $P_n \otimes M$ and a fortiori $Z_n(P_n \otimes M)$. So $H_n(P) \otimes M$ is a direct summand of $H_n(P \otimes_R M)$.

2 RHom and Rtensor

Prop. (I.10.2.1) (Explicit Addition as Extensions). In an Abelian category with enough injectives, the extension $\text{Ext}^1(N, M)$ is equivalent with the Abelian group of extensions with Baer sum as addition.

Proof: We choose a projective resolution $0 \to K \to P \to N \to 0$, so $\text{Hom}(K, M) \to \text{Ext}^1(N, M)$ is surjective, so choose a lifting and the pushout $0 \to M \to L \to N \to 0$ with be the corresponding extension, Now the Baer sum is easy to define and verify. \hfill \square

3 Rlim

Def. (I.10.3.1) (Rlim). Rlim is the derived limit (II.3.1.8) in $D(\text{Ab})$ restricted to the inverse systems consisting of discrete complexes.

Lemma (I.10.3.2). Mittag-Leffler Complex in $\text{Ab}(\mathbb{N})$ is adapted for Rlim.

Proof: Firstly, for any complex $(A_n)$, we can associate to it the complex $(B_n)$ where $B_n = A_n \oplus A_{n-1} \oplus \ldots \oplus A_1$, then $(B_n)$ is a Mittag-Leffler complex and $(A_n) \hookrightarrow (B_n)$. So ML complexes are sufficiently large.

Now for any exact sequence of complexes $0 \to (A_n) \to (B_n) \to (C_n) \to 0$, if $A_n$ is ML, then $\lim B_i \to \lim C_i$ is surjective: for an element $(c_i) \in \lim C_i$, let $E_i = \pi_i^{-1}(c_i) \in B_i$, then $(E_i)$ is an inverse system of nonempty sets, and it suffices to show $(E_i)$ is ML, because then (II.1.1.33) will show there is a element $(e_i) \in \lim E_i \subset \lim B_i$ that maps to $(c_i)$.

For this, Cf.[Sta]0598. \hfill \square
Prop. (I.10.3.3) \(\text{(Rlim)}\).

- If \((A_n)\) is Mittag-Leffler, then \(R^1\lim((A_n)) = 0\).
- \(R\lim((A_n))\) is represented by the complex in degree 0, 1:
  \[
  \prod A_n \to \prod A_n : (x_n) \mapsto (x_n - f_{n+1}(x_{n+1}))
  \]
- for any \((A_n) \in Ab(\mathbb{N})\) we have \(R^p\lim((A)n)) = 0\) for \(p > 1\).

Proof: 1 follows from (I.10.3.2) and (II.3.3.6).

2, 3: We use (II.3.3.6) again. Notice the complex \((B_n)\) where \(B_n = A_n \oplus A_{n-1} \oplus \cdots \oplus A_1\) and the complex \((C_n)\) where \(C_n = A_{n-1} \oplus A_{n-2} \oplus \cdots \oplus A_1\) form an exact sequence of complexes
\[
0 \to (A_n) \to (B_n) \to (C_n) \to 0
\]
where \(B_n \to C_n : (x_i) \mapsto (x_i - f_{i+1}(x_{i+1}))\), and \((B_n), (C_n)\) are both ML, so we are done. □

4 Lifting Complexes

Prop. (I.10.4.1) \(\text{(Lifting Projective Complex Along Thickening)}\). Let \(R\) be a ring and \(I\) be a nilpotent ideal, and \(K \in D(R)\). Now if \(K \otimes_R R/I\) is represented by a bounded above complex of projective \(R/I\)-modules, then there is a complex \(P\) of bounded above complex of projective \(R\)-modules that \(P \cong K \in D(R)\), and \(P \otimes_R R/I \cong E\).

Proof: Cf. [[Sta]09AR]. □

5 Perfect Complexes

Def. (I.10.5.1) \(\text{(Perfect Complexes of Modules)}\). Let \(R\) be a ring, then \(K \in D(R)\) is called perfect if \(K\) is quasi-isomorphic to a bounded complex of finite projective \(R\)-modules. An \(R\)-module \(M\) is called perfect iff \(M[0]\) is perfect.

Prop. (I.10.5.2) \(\text{(Duality of Perfect Complexes)}\). Let \(K\) be a perfect complex of \(D(A)\), then the dual complex \(K^\vee = R\operatorname{Hom}(K, A)\) is also a perfect complex and \((K^\vee)^\vee \cong K\). Also, there is a functorial isomorphism
\[
L \otimes^L_A K^\vee = R\operatorname{Hom}_A(K, L)
\]

Proof: Cf. [[Sta]07VI]. □

Prop. (I.10.5.3). If \(A\) is a ring and \(K_n\) is a system of perfect objects in \(D(A)\), then for any \(E \in D(A)\), there is an isomorphism
\[
R\operatorname{Hom}_A(\underset{\text{ho} \operatorname{colim}}{\text{colim}} K_n, E) \cong R\lim E \otimes^L_A K_n^\vee
\]

Proof: By (I.10.5.2), \(R\lim E \otimes^L_A K_n^\vee = R\lim R\operatorname{Hom}(K_n, E)\) which fits into a distinguished triangle
\[
R\lim R\operatorname{Hom}(K_n, E) \to \prod R\operatorname{Hom}(K_n, E) \to \prod R\operatorname{Hom}(K_n, E)
\]
So it suffices to show that
\[
\prod R\operatorname{Hom}(K_n, E) \cong R\operatorname{Hom}_A(\oplus K_n, E).
\]
This follows from Yoneda lemma and (V.6.3.14)(V.6.3.15). □
Prop. (I.10.5.4) (Perfectness and Thickening). If $R$ is a ring, $I \subset R$ is a nilpotent ideal, and $K \in D(R)$. If $K \otimes_R^L R/I$ is perfect in $D(R/I)$, then $K$ is perfect in $R$. Moreover, if $K \otimes_R^L R/I = 0$, then $K = 0$.

**Proof:** Let $\mathcal{P}^* \cong K \otimes_R^L R/I$ where $P$ is a complex of finite projective $R/I$-modules, then by (I.10.4.1) there is a complex of projective $R$-modules $P$ that $P/IP \cong \mathcal{P}$. Then it follows from Nakayama that $P$ is bounded. \hfill \Box

6 Derived Completeness

Cf. ([Sta] Chap 15.90).

Def. (I.10.6.1). For a ring $A$, $f \in A$ and a complex $K \in D(A)$, we denote by $T(K, f)$ a derived limit of the system

$$
\ldots \rightarrow K \xrightarrow{f} K \xrightarrow{f} K
$$

Prop. (I.10.6.2) (Properties of $T(K, f)$). For a ring $A$, $f \in A$ and $K \in D(A)$, the following are equivalent:

1. $T(K, f) = 0$.
2. $R \text{Hom}_A(A_f, K) = 0$.
3. $\text{Ext}_A^n(A_f, K) = 0$ for all $n$.
4. $\text{Hom}_{D(A)}(E, K) = 0$ for all $E \in D(A_f)$.
5. For any $p \in \mathbb{Z}$, $\text{Hom}_A(A_f, H^p(K)) = 0$ and $\text{Ext}_A^1(A_f, H^p(K)) = 0$.
6. For any $p \in \mathbb{Z}$, $T(H^p(K), f) = 0$.

**Proof:** 2, 3 is clearly equivalent.

4 $\Rightarrow$ 3 is clear, and for 3 $\Rightarrow$ 4: Let $I^*$ be a complex representing $K$, then 3 says $\text{Hom}_A(A_f, I^*)$ is acyclic, and $\text{Hom}_{D(A)}(E, K) = \text{Hom}_{K(A)}(E, I^*) = \text{Hom}_{K(A_f)}(E, \text{Hom}_A(A_f, I^*))$. As $\text{Hom}_A(A_f, I^*)$ is both acyclic and $K$-injective (I.11.6.3), we get it is homotopic to 0 by (I.11.5.12), thus we get 4.

1 $\iff$ 3: There is a free resolution of $A_f$ given by

$$
0 \rightarrow \oplus_n A \rightarrow \oplus_n A \rightarrow A_f \rightarrow 0
$$

where the first map is $(a_n) \mapsto (a_n - fa_{n-1})$, and the second map is $(a_n) \mapsto \sum a_i/f^i$. Applying $\text{Hom}_A(-, I^*)$, we get a distinguished triangle

$$
R \text{Hom}_A(A_f, K) \rightarrow \prod K \rightarrow \prod K.
$$

So this shows $R \text{Hom}_A(A_f, K)$ is just $T(K, f)$, so we get 1 $\iff$ 3.

1 $\iff$ 5 $\iff$ 6: There is a spectral sequence convergence (choose a finite free resolution of $A_f$ then rotate and use (II.3.5.10)):

$$
E_2^{p,q} = \text{Ext}_A^q(A_f, H^p(K)) \Rightarrow \text{Ext}^{p+q}_A(A_f, K)
$$

This spectral sequence degenerates at $E_2$ because $A_f$ has a length 1 resolution by free $A$-modules hence the $E_2$ page has only 2 rows. So there is an exact sequence

$$
0 \rightarrow \text{Ext}_A^1(A_f, H^{p-1}(K)) \rightarrow \text{Ext}_A^1(A_f, K) \rightarrow \text{Hom}_A(A_f, H^p(K)) \rightarrow 0.
$$

Then we are done. \hfill \Box
Lemma (I.10.6.3). Let $A$ be a ring, $K \in D(A)$, then the set $I$ of $f$ that $T(K, f) = 0$ is a radical ideal of $A$.

Proof: If $T(K, f) = 0$ and $g \in A$, then $A_{gf}$ is a $A_f$-module, then

$$\text{Ext}^n_{A}(A_{gf}, K) = \text{Hom}_{D(A)}(A_{gf}[-n], K)(V.6.3.12) = 0$$

by(I.10.6.2) item4. Then $T(K, gf) = 0$ by(I.10.6.2) again. And if $f, g \in I$, there is an exact sequence

$$0 \to A_{f+g} \to A_{f(g+f)} \oplus A_{g(f+g)} \to A_{f(g+f)} \to 0$$

by(I.7.2.3) and a easy check that the last term is surjective. Then from the long exact sequence of Ext, we get $\text{Ext}^n(A_{f+g}, K) = 0$ for any $n$. Finally if $f^n \in I$, then $f \in I$, because $A_f = A_f^n$. □

Def. (I.10.6.4) (Derived Completeness). Let $A$ be a ring, $K \in D(A)$, $I$ is an ideal of $A$, then $K$ is said to be derived complete w.r.t $I$ if $T(K, f) = 0$ for any $f \in A$. Let $D_{\text{comp}}(A, I)$ denote the subcategory consisting of derived $I$-complete objects in $D(A)$.

Let $M$ be an $A$-module, then $M$ is called derived complete w.r.t $I$ if $M[0] \in D(A)$ is derived complete w.r.t $I$.

Prop. (I.10.6.5). A $\aleph_0$-filtered colimit of derived $I$-complete rings is also derived $I$-complete.

Proof: Cf.[Bhatt, Prism, 5.4.3]. □

Prop. (I.10.6.6) (Complete and Derived Complete). Let $A$ be a ring and $I$ be an ideal, $M$ an $A$-module, then

- If $M$ is $I$-adically complete, then $T(M, f) = 0$ for any $f \in I$.
- If $T(M, f) = 0$ for all $f \in I$ and $I$ is f.g., then $M \to \lim M/I^nM$ is surjective.

In particular, if $I$ is f.g., $M$ is $I$-adically complete iff $M$ is derived $I$-adically complete and $\cap I^nM = 0$.

In particular, when $M$ is f.g. over $A$ Noetherian and $I \subset \text{rad}(A)$, derived $I$-complete is equivalent to $I$-complete(I.5.6.10).

Proof: If $M$ is $I$-adically complete, by(I.10.6.2), it suffices to show that $\text{Hom}(A_f, M) = 0$ and $\text{Ext}^1(I, A_f, M) = 0$. But $M = \lim M/I^nM$, and $\text{Hom}(A_f, M/I^nM) = 0$, because $f \in I$. For Ext, use(I.10.2.1), for any extension

$$0 \to M \to E \to A_f \to 0,$$

chose arbitrary $e_n$ that maps to $1/f^n$. Then $\delta_n = fe_{n+1} - e_n \in M$. We consider

$$e'_n = e_n + \delta_n + f\delta_{n+1} + f^2\delta_{n+2} + \ldots,$$

which exist because $M$ is $I$-adically complete. Then $fe'_{n+1} = e'_n$, so this gives a splitting of the extension.

Conversely, if $I = (f_1, \ldots, f_r)$ and $T(M, f_i) = 0$ for any $i$, then by(I.5.7.16), we can assume $I = (f)$. Then consider the extension

$$0 \to M \to E \to A_f \to 0,$$

where $E = (M \oplus \bigoplus A e_n)/(x_n - fe_{n+1} + e_n) \to A_f$ that maps $M$ to 0 and $e_n$ to $1/f^n$. This extension splits by(I.10.6.2) and(I.10.2.1), thus there is an element $x + e_0$ that generate a copy of $A_f$ in $E$.

But then $x + e_0 = x - x_0 + fe_1 = x - x_0 - fx_1 + f^2e_2 - \ldots - f^{n-1}x_{n-1} + f^nE + A_f$ for any $n$. Then $x - x_0 - fx_1 - f^2x_2 - \ldots - f^{n-1}x_{n-1} \in f^nE + A_f$ for any $n$. Then $x - x_0 - fx_1 - f^2x_2 - \ldots - f^{n-1}x_{n-1} \in f^nM$, because $E = M \oplus A_f$. Then we are done. □
Prop. (I.10.6.7). If \( M \in D(A/I) \subset D(A) \), then \( M \) is derived-\( I \)-complete. (This follows from the definition of \( T(M, f) \)).

Prop. (I.10.6.8) (Category of Derived Complete Modules). Let \( I \) be an ideal of \( A \), then the derived \( I \)-complete \( A \)-modules form a weak Serre subcategory of \( \text{Mod}_A \). In particular, \( D_{\text{comp}}(A, I) \) is also a weak Serre subcategory.

Proof: If \( f : M \to N \) is a map of derived \( I \)-complete \( A \)-modules, then we consider the complex \( K = (M \to N) \), then there is an exact sequence \( 0 \to M[1] \to K \to N \to 0 \), so we have \( \text{Ext}^n(A_f, K) = 0 \) for any \( f \in I, n \in \mathbb{Z} \) because \( M, N \) does(I.10.6.2), so \( K \) is derived \( I \)-complete by(I.10.6.2) again. Then we have \( \text{Ker}(f), \text{Coker}(f) \) are derived \( I \)-complete, by(I.10.6.2) again. Extension is also clear.

Lemma (I.10.6.9). If \( R \) is ring, \( I \) is an ideal, and \( K \in D(R) \) that \( K \otimes^L_R R/I = 0 \), then \( K \otimes^L_R M \) for any \( M \in D^b(R) \) with all the cohomology groups \( I \)-power torsions.

Proof: We use the truncation(I.11.3.6), then it suffices to prove for \( M \) discrete. Now \( M = \cup M[n] \), and we have \( K \otimes^L_R M = \text{ho} \text{colim} K \otimes^L_R M[n] \), so we may assume \( I^n M = 0 \) for some \( n \). Consider the \( R \)-algebra \( R' = R/n \oplus M \), where \( M^2 = 0 \), then it suffices to show \( K' = K \otimes^L_R R' = 0 \). Now \( 0 = K \otimes^L_R R/I = K' \otimes^L_R R/I \), so by(I.10.5.4) \( K' = 0 \).

Prop. (I.10.6.10) (Derived Nakayama). the derived tensor product \( - \otimes^L_A A/I \) reflects isomorphism on \( D_{\text{comp}}(A, I) \), i.e. if \( M \otimes^L_A A/I = 0 \), then \( M = 0 \).

Proof: Let \( I = (f_1, \ldots, f_r) \), by(I.10.6.9), \( M \otimes^L_A K_n = 0 \) for any \( K_n = \text{Kos}(A, f_{i_1}^n, \ldots, f_{i_r}^n) \), so \( K = R \lim K \otimes^L_A K_n = 0 \).

Cor. (I.10.6.11). If \( I \) is f.g. and \( M \) is a derived \( I \)-complete \( A \)-module that \( M/IM = 0 \), then \( M = 0 \).

Proof: This should be an immediate corollary.

Let \( I = (f_1, \ldots, f_r) \), if \( M \neq 0 \), let \( i \) be the largest integer that \( M/(f_1, \ldots, f_i)M \neq 0 \), then \( N \) is also derived \( I \)-complete by(I.10.6.8). But \( f_{i+1} : N \to N \) is surjective, so \( T(N, f_{i+1}) \neq 0 \), contradiction.

Prop. (I.10.6.12). If \( A \) is derived \( I \)-complete, then \( (A, I) \) is a Henselian pair.

Proof: Cf.[Sta]0G3H?.

Prop. (I.10.6.13) (Derived \( I \)-Completion). If \( I = (f_1, \ldots, f_n) \) is f.g. in \( A \), the inclusion of categories \( D_{\text{comp}}(A, I) \subset D(A) \) has a left adjoint, which maps \( K \) to

\[ \widehat{K} = R \text{Hom}(A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r}, K) .\]

called the derived \( I \)-completion of \( K \).

Moreover, this construction is identical to \( K \mapsto K^* = R \lim (K \otimes^L_A K_n^\bullet) \), by(I.10.5.3) and(I.7.4.7).

Proof: There is a map \( A \to \prod_{i_0} A_{f_{i_0}} \to \prod_{i_0 < i_1} A_{f_{i_0} f_{i_1}} \to \ldots \to A_{f_1 \ldots f_r} \to A \), which induces a morphism \( K \to \widehat{K} \). Now by(V.6.3.14), \( R \text{Hom}(A_f, \widehat{K}) \) is isomorphic to

\[ R \text{Hom}(A_f \to \prod_{i_0} A_{f f_{i_0}} \to \prod_{i_0 < i_1} A_{f f_{i_0} f_{i_1}} \to \ldots \to A_{f f_1 \ldots f_r}, K) .\]
as $A_f$ is $A$-flat. Now this one is 0 for any $f \in I$, by (I.7.4.7), so $\hat{K}$ is derived $I$-complete.

Conversely, if $\hat{K}$ is derived $I$-complete, then $R \text{Hom}(A_f, K) = 0$ for any $f \in I$, thus $K \to \hat{K}$ is an isomorphism is we inductively use the stupid truncation (I.11.3.6).

□

Cor. (I.10.6.14). If $M$ is an $A$-module, then $H^0(\hat{M})$ is the derived-$I$-completion in the category of modules, by (II.3.2.7).

Cor. (I.10.6.15). (I.10.6.2) show the notion of derived $I$-complete and derived $I$-completion only depends on $\text{rad } I$.

**Principal Ideal Case**

Prop. (I.10.6.16) (Bounded Torsion and Derived Completion). Let $A$ be a commutative ring and $f \in A$. If $M$ is an $A$-module that has bounded $f^\infty$-torsion, then the derived $f$-completion of $M$ as a complex is a module and coincides with the classical $f$-adic completion.

**Proof:** The derived $f$-completion is defined to be

$$\hat{M} = R \lim_n (M \otimes_{A[X]} \mathbb{Z}[X]/(x^n)) = R \lim_n (M \xrightarrow{f^n} M).$$

So by (II.3.1.12), there are exact sequences

$$0 \to R^1 \lim_n M/f^n M \cong H^1(\hat{M}) \to H^0(\hat{M}) \to \lim_n M/f^n \to 0$$

$$H^{-1}(\hat{M}) \cong \lim_n M[f^n]$$

Now the hypothesis implies that $(M[f^n])$ is Mittag-Leffler and $\lim_n M[f^n] = 0$, so we have the desired result. □

Cor. (I.10.6.17). Let $R$ be a perfect $\mathbb{F}_p$-algebra, then the derived $p$-completion and $p$-adic completion of $R$ coincide.

7 Derived Completely Properties

Def. (I.10.7.1) (Derived Completely Properties). Let $A$ be a commutative ring and $I$ is a f.g. ideal, then $M \subset D(A)$ is called $I$-completely (faithfully) flat/smooth/étale/... iff $M \otimes_A^L A/I$ is discrete and is a (faithfully) flat $A/I$-module.

$M$ is said to have finite $I$-completely Tor amplitude if $M \otimes_A^L A/I$ is a bounded complex in $D(A/I)$.

Clearly, any flat/smooth/étale $A$-module $M$ is $I$-completely flat/smooth/étale/... for any $I$. And $M$ has finite $I$-completely amplitude if $M$ has a finite resolution of flat $A$-modules.

Prop. (I.10.7.2) (I-Completely F.F. Descent). If $A \to B \to B'$ are ring maps and $I$ is an ideal of $A$,

1. If $C$ is a $B$-algebra and $B$ is $I$-completely f.f. over $A$, then $C$ is $I$-completely (f.)f. over $A$ iff $C$ is $I$-completely (f.)f. over $B$.

2. If $B \to B'$ is $I$-completely flat, $M$ is a $B$-module, then $M$ is $I$-completely flat over $A$ iff $M \otimes_B^L B'$ is $I$-completely flat over $A$. 


Proof: 1: One direction is easy, for the other, if \( C \) is \( I \)-completely (f.)f. over \( A \), then
\[
C \otimes^L_A A/I = C \otimes^L_B B \otimes^L_A A/I = (C \otimes^L_B B/I) \otimes^L_B (B \otimes^L_A A/I)
\]
is a discrete and (f.)f. \( C/I \)-module iff \((C \otimes^L_B B/I)\) does, as \((B \otimes^L_A A/I)\) is f.f. over \( B/I \).

2: This is because \((B' \otimes^L_B M) \otimes^L_A A/I \cong B' \otimes^L_B (M \otimes^L_A A/I)\).

\[\square\]

Prop. (I.10.7.3). If \( I \) is generated by a Koszul-regular sequence, then any \( A \)-module \( M \) has finite \( I \)-completely Tor amplitude.

Proof: This is because \( A/i \cong A \otimes^L \mathbb{Z}[X_1, \ldots, X_r] \mathbb{Z} \) in this case, so \( M \otimes^L A/I \cong M \otimes^L \mathbb{Z}[X_1, \ldots, X_r] \mathbb{Z} \), which has f.m. homology groups because \( \mathbb{Z} \) has a finite free \( \mathbb{Z}[X_1, \ldots, X_n] \)-resolution (I.7.4.1).

\[\square\]

Prop. (I.10.7.4) \((I\text{-Completely Flatness and Derived Completion})\). Let \( A \) be a commutative ring and \( I \) be f.g., then the derived \( I \)-completion of a \( I \)-completely (faithfully)flat/étale \( M \) is \( I \)-completely (faithfully)flat/étale, both the complex and the module. In fact, \( H^0(\widehat{M}) \otimes^L_A A/I \cong \widehat{M} \otimes^L_A A/I \).

Proof: Because objects in the image of \( D(A/I) \to D(A) \) are all derived \( I \)-complete, by(I.10.6.7), so there is an isomorphism \( M \otimes^L_A A/I \cong \widehat{M} \otimes^L_A A/I \) for any \( M \), because they are both left adjoint to \( \text{Mod}_{A/I} \subset D(A) \), by the definition of derived \( I \)-completion and(V.6.3.17). So \( \widehat{M} \otimes^L_A A/I \cong M \otimes^L_A A/I \), which is a flat \( A/I \)-module.

\[\square\]

Prop. (I.10.7.5) \((\text{Flatness and Completed Derived Tensor})\). The completed derived tensor of a \( I \)-completely flat/étale module is discrete and \( I \)-completely flat/étale.

Proof: If \( M \) is an \( I \)-completely flat \( A \)-module, then \( M \otimes^L_A B \) is \( I \)-complete and \( I \)-completely flat by(I.10.7.4), and \((M \otimes^L_A B) \otimes^L_B B/I = H^0(M \otimes^L_A B) \otimes^L_B B/I = M \otimes^L_A B/I \), because they are both left adjoint of the forgetful functor \( \text{Mod}_{B/I} \subset D(A) \). Then we have \( M \otimes^L_A B \cong H^0(M \otimes^L_A B) \) is discrete, as they are both \( I \)-complete, by(I.10.6.2) and derived Nakayama.

\[\square\]

Prop. (I.10.7.6). \( A \) is derived \( I \)-completely flat iff \( A \) is \( I^n \)-completely flat for any \( n > 0 \). Moreover, in this case, \( A \) is derived \( J \)-completely flat for any ideal \( J \) that \( I \subset \text{rad} J \).

Proof: The exact sequence
\[
0 \to I/I^2 \to A/I^2 \to A/I \to 0
\]
induces distinguished triangles
\[
(M \otimes^L_A A/I) \otimes^L_A I/I^2 \to M \otimes^L_A I/I^2 \to M \otimes^L_A A/I^2 \to M \otimes^L_A A/I
\]
which shows \( M \otimes^L_A I/I^2 \) is discrete, = \( N \). Then \( N \) is an \( A/I^2 \)-module that \( N \otimes^L_{A/I^2} A/I = M/I \)M is \( A/I \)-flat. Then \( N \) is \( A/I^2 \) flat: for any \( A/I^2 \)-module \( L \), let \( L' = IL \) and \( L'' = L/IL \) which are both \( A/I \)-modules, then there is a distinguished triangle
\[
N \otimes^L_{A/I^2} L' \to N \otimes^L_{A/I^2} L \to N \otimes^L_{A/I^2} L''
\]
and
\[
N \otimes^L_{A/I^2} L' = (N \otimes^L_{A/I^2} A/I) \otimes^L_{A/I} L' = (M/I \)M \)M \( A/I \) \( L'
\]
\[
N \otimes^L_{A/I^2} L'' = (N \otimes^L_{A/I^2} A/I) \otimes^L_{A/I} L'' = (M/I \)M \)M \( A/I \) \( L''
\]
are both discrete, hence so does \( N \otimes^L_{A/I^2} L \), implying it is \( A/I^2 \)-flat.

In a similar fashion, we can show \( M \) is \( I/I^n \)-flat for any \( n \). And if \( I \subset \text{rad} J \), then \( I^n \subset J \), so \( M \otimes^L_A A/J = (M \otimes^L_A A/I^n) \otimes^L_{A/I^n} A/J \) is discrete and \( A/J \)-flat.
**Prop. (I.10.7.7).** Let $A$ be a ring and $I$ be an invertible ideal, then any derived $(p, I)$-complete and $(p, I)$-completely flat $A$-complex $M \in D(A)$ is discrete and $(p, I)$-complete. Moreover, for any $n \geq 0$, we have $M[I^n] = 0$ and $M/I^n M$ has bounded $p^\infty$-torsion.

**Proof:** $M$ is $(I^n, p)$-completely flat by (I.10.7.6), so we fine $M \otimes_A^L A/I^n$ is $p$-completely flat in $D(A/I^n)$. Notice that $A/I^n$ has bounded $p^\infty$-torsion by induction (I invertible used), (I.10.7.11) says $M \otimes_A^L A/I^n$ is discrete in $D(A/I^n)$ and has bounded $p^\infty$-torsion, and is $p$-adically complete. In particular,

\[ M \otimes_A^L A/I^n = (M \otimes_A^L A/I^{n+1}) \otimes_{A/I^{n+1}} A/I^n \]

So if we denote $M \otimes_A^L A/I^n = M_n$, then $M_n = M_{n+1}/I^n M_{n+1}$, as $M$ is derived $I$-complete, we have $M = R\lim(M \otimes_A^L A/I^n)$, so clearly $M$ is discrete. And then we have $M \otimes_A^L A/I^n = M/I^n M$, which means $M[I^n] = 0$, and $M/I^n M$ has bounded $p^\infty$-torsion. Now because $M$ is derived $(p, I)$-complete,

\[ M = R\lim((M \otimes_A I^n \to M) \otimes_{A/I^n} (A/I^n \to A/I^n)) = R\lim(M/I^n p^n \to M/I^n) = R\lim(M/(I^n, p^n)) \]

is $(p, I)$-complete.

**Lemma (I.10.7.8).** Let $C$ be a commutative ring with a f.g. ideal $J$, and $D$ a $C$-algebra that has finite $J$-complete Tor amplitude, then the $J$-completed base change operator $\sim_{C/J}$ commutes with totalization in $D^{\geq 0}(C)$ and $D^{\geq 0}(D)$, i.e. if $M^\bullet$ is a cosimplicial object with in $D^{\geq 0}(C)$ with totalization $M$, then

\[ \text{Tot}(M^\bullet \hat{\otimes}_C^L D) \cong \text{Tot}(M^\bullet \hat{\otimes}_C^L D) \]

via the natural map.

**Proof:** Cf.[Scholze, Prism, 4.22].

**Prop. (I.10.7.9) (Elkik’s Algebrization Theorem).** Let $A$ be a commutative ring and $I$ is a f.g. ideal, then an $A$-algebras is derived $I$-completely étale/smooth iff it is the derived $I$-completion of some étale/smooth $A$-algebra.

**Proof:** if it is the derived $I$-completion of some étale/smooth algebra, then it is derived $I$-completely étale/smooth by (I.10.7.4). Converse, Cf.[A. Arabia, “Relèvements des algèbres lisses et de leurs morphismes”, Commentarii Mathematici Helvetici 76 (2001), 607–639].

**Lemma (I.10.7.10).** Let $A$ be a ring, if $M$ is an $A$-module with bounded $f^\infty$-torsion, i.e. $M[f^\infty] = M[f^c]$ for some $c > 0$, then there are maps

\[ (M \to M) \to M/f^n M, \quad M/f^{n+c} \to (M \to M) \]

in $D(A)$ inducing an equivalence between two pro-objects $\{M \to M\}$ and $\{M/f^n\}$.

**Proof:** The first map is obvious, for the second map, use the following commutative diagram:

\[ \begin{array}{ccc}
M & \xrightarrow{f^n} & M \\
\downarrow \scriptstyle{f^c} & & \downarrow \scriptstyle{f^n} \\
M & \xrightarrow{f^c} & M
\end{array} \]

the upper row is injective thus isomorphic to $M/f^{n+c} M$, then this gives the map. It can be checked that this is an equivalence of pro-objects.
Prop. (I.10.7.11). Let $A$ be a commutative ring that has bounded $f^\infty$-torsion, then for a $M \in D(A)$, the following is equivalent:

- $M$ is derived $f$-complete and $f$-completely flat.
- $M$ is discrete and is represented by a $f$-adically complete module that $M/f^n M$ is $A/f^n$-flat for any $n > 1$ and $M$ has bounded $f^\infty$-torsion.

Furthermore, in this case, $M \otimes_A a[f^\infty] = M[f^\infty]$.

Proof: By (I.10.7.10), $\{A/f^n A\}$ and $\{Kos(A, f^n)\}$ are two equivalence pro-objects in $D(A)$. So if 1 or 2 holds, then $M$ is derived $f$-complete, so

\[ M = R \lim (M \otimes_a L_a Kos(A, f^n)) = R \lim (M \otimes_a L_a A/f^n A). \]

Now if 1 holds, then $M_n = M \otimes_a L_a A/f^n A$ is discrete by (I.10.7.6), and $M_n = M_{n+1}/f^n M_{n+1}$ is surjective:

\[ M \otimes^L AA/f^n A = (M \otimes_a L_a A/f^{n+1} A) \otimes_a L_{A/f^n} A/f^n. \]

So $M = R \lim (M \otimes_a L_a A/f^n A)$ is discrete. Then $M \otimes_a A/f^n = M \otimes_a L_a A/f^n$ is flat over $A/f^n$.

Next we prove $M \otimes_A A[f^\infty] = M[f^\infty]$: There is an exact sequence

\[ 0 \to (A/f^n)[1] \to Kos(A, f^n) \to A/f^n \to 0 \]

Then tensoring $M \otimes_a L_a$ gives a distinguished triangle

\[ (M \otimes_a L_a A[f^n])[1] \to Kos(M, f^n) \to M \otimes_a L_a A/f^n. \]

Notice that

\[ M \otimes_a L_a A[f^n] = (M \otimes_a L_a A/f^n) \otimes_a L_{A/f^n} A[f^n] = (M \otimes_a A/f^n) \otimes_a L_{A/f^n} A[f^n] = M \otimes_a A[f^n] \]

by flatness, so the distinguished triangle shows $M \otimes_a A[f^n] \cong H^{-1}(Kos(M, f^n)) = M[f^n]$.

Conversely, if 2 holds, then there are equivalences of pro-objects

\[ \{M \otimes_a L_a A/f^n\} \cong \{M \otimes_a L_a Kos(A, f^n)\} = \{Kos(M, f^n)\} \cong \{M/f^n M\} \]

by (I.10.7.10) as $M$ has bounded $f^\infty$-torsion. So

\[ \tilde{M} = \lim \{M \otimes_a L_a Kos(A, f^n)\} = \lim \{M/f^n M\} = M[0] \]

so $M$ is derived $p$-complete. And the constant system

\[ \{M \otimes_a L_a A/f\} = \{M \otimes_a L_{A/f^n} A/f^n \otimes_a L_{A/f^n} A/f\} \cong \{M/f^n M \otimes_a L_{A/f^n} A/f\} = \{M/fM\} \]

where we used $M/f^n M$ is $A/f^n$-flat. So $M \otimes_a L_a A/f \cong M/fM$ is flat over $A/f$. \hfill \square
I.11 Homological Algebra

Main references are [Sta]Chap12.

1 Additive Category

Def. (I.11.1.1) (Preadditve Categories). A category $\mathcal{A}$ is called preadditive iff

• $A1$: $\mathcal{A}$ is enriched over the Cartesian category of Abelian groups.

Def. (I.11.1.2) (Zero Element). Let $\mathcal{A}$ be a preadditive category and $x \in \mathcal{A}$, then the following are equivalent:

• $x$ is initial.
• $x$ is final.
• $\text{id}_x = 0 \in \text{Mor}(x,x)$.

Such an element is called a zero element in $\mathcal{A}$, denoted by $0$. If $0$ exists, then a morphism $\alpha : x \to y$ factors through $0$ iff $\alpha = 0$.

Proof: Cf.[Sta]00ZZ.

Cor. (I.11.1.3). An additive functor transforms a zero object to a zero object.

Prop. (I.11.1.4) (Finite Direct Sums). If $\mathcal{A}$ is a preadditive category and $x, y \in \mathcal{A}$. If one of $x \times y$ and $x \coprod y$ exists, then so does the other, and they are isomorphic, called the direct sum of $x, y$.

Proof: Cf.[Sta]0101.

Prop. (I.11.1.5) (Characterizing Direct Sums). If $\mathcal{A}$ is a preadditive category and $x, y, z \in \mathcal{A}$, and there are four morphism that satisfies some identities, then $z$ is the product and sum of $x, y$ in $\mathcal{A}$. Cf.[Sta]0102.

Proof: 

Cor. (I.11.1.6). An additive functor transforms direct sums to direct sums.

Def. (I.11.1.7) (Additive Category). A preadditive category $\mathcal{A}$ is called an additive category iff

• $A2$: There exists an element that is both initial and final, called the zero element.
• $A3$: There exists a canonical finite sums and finite products and they are equal, and the sum induce the Abelian structure of $\text{Hom}(X, Y)$.

Def. (I.11.1.8) (Additive Functors). A functor between preadditive categories $F : \mathcal{C} \to \mathcal{D}$ is called additive if it is a morphism of $\text{Ab}$-enriched categories.

Def. (I.11.1.9) (Kernels, Cokernels, Images and Coimages). Let $\mathcal{A}$ be a preadditive category, then Cf.[Sta]0106.

Def. (I.11.1.10) (Exact Sequence). A diagram $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an exact sequence if $f$ is the kernel of $g$ and $g$ is the cokernel of $f$. 
Karoubian Categories

Def. (I.11.1.11) (Karoubian Categories). A Karoubian category is an additive category that satisfies the following equivalent conditions:

- Every idempotented endomorphism of an object of $\mathcal{C}$ has a kernel.
- Every idempotented endomorphism of an object of $\mathcal{C}$ has a cokernel.
- Every idempotented endomorphism $p : z \to z$ induces a direct sum decomposition $z = x \otimes y$ that $p$ corresponds to the projection $z \to x$(exists by A3).

Proof: Cf.[Sta]09SH. □

Prop. (I.11.1.12). Let $\mathcal{D}$ be a preadditive category,

- if $\mathcal{D}$ has countable products and kernels of morphisms that have a right inverse, then $\mathcal{D}$ is Karoubian.
- Dually, if $\mathcal{D}$ has countable coproducts and cokernels of morphisms that have a left inverse, then $\mathcal{D}$ is Karoubian.

Proof: Given any idempotented morphism $e : X \to X$, $e$ has a kernel iff $W \rightarrowtail \ker\left(\operatorname{Mor}(M,X) \xrightarrow{\Phi} \operatorname{Mor}(M,X)\right)$ is representable. Notice that for any Abelian group $A$,

$$\ker(e : A \to A) = \ker(\Phi : \prod_z A \to \prod_z A)$$

where

$$\Phi(a_1,a_2,\ldots) = (ea_1 + (1 - e)a_2, ea_2 + (1 - e)a_3, \ldots)$$

and it has a right inverse

$$\Psi(a_1,a_2,\ldots) = (a_1, (1 - e)a_1 + ea_2, (1 - e)a_2 + ea_3, \ldots)$$

thus the kernel exists. □

Exact Categories

Def. (I.11.1.13) (Exact Categories). Let $\mathcal{C}$ be an small additive category and $\mathcal{E}$ be a set of short sequences

$$0 \to X \to Y \to Z \to 0$$

in $\mathcal{C}$. If $0 \to X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \to 0$ is in $\mathcal{E}$, then we call $\varphi$ an admissible monomorphism and $\psi$ an admissible epimorphism. then $(\mathcal{C}, \mathcal{E})$ is called an exact category if it satisfies:

- **Ex1**: For any complex $0 \to X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \to 0$, $\varphi$ is the kernel of $\psi$ and $\psi$ is the cokernel of $\varphi$.
- **Ex2**: For any $X,Y \in \mathcal{C}$, $0 \to X \to X \otimes Y \to Y \to 0$ is in $\mathcal{E}$.
- **Ex3**: $\mathcal{E}$ is saturated in the category of short sequences.
- **Ex4**: if $f,g$ are admissible monomorphisms, then so is $gf$.
- **Ex5**: If $f$ is an admissible monomorphism, then any pushout of $f$ exists and is an admissible monomorphism.
• **Ex6**: If \( g \) is an admissible epimorphism, then any pullback of \( g \) exists and is an admissible epimorphism.

**Cor. (I.11.1.14)**. If \( \mathcal{C} \) is an Abelian category and \( \mathcal{E} \) the set of all exact sequences in \( \mathcal{C} \), then \( (\mathcal{C}, \mathcal{E}) \) is an exact category.

**Cor. (I.11.1.15)**. If \( (\mathcal{C}, \mathcal{E}) \) is an exact category, then

- **Ex7**: if \( f : X \to Y \in \mathcal{C} \) is a morphism having a kernel and there is a morphism \( g : Z \to X \) that \( fg \) is an admissible monomorphism, then so is \( f \). Dual argument holds for admissible epimorphisms.

**Proof**: Cf. [Bernhard Keller, Chain complexes and stable categories, P28]. \( \square \)

**Def. (I.11.1.16) (Geometric Exact Categories)**. A geometric exact category consists of an exact category \( (\mathcal{C}, \mathcal{E}) \) and a mapping \( A \) from \( \text{Ob}(\mathcal{C}) \) to \( \mathcal{S}ets \) together with morphisms

- a morphism \( f^* : A(X) \to A(Y) \) for any admissible monomorphism \( f : X \to Y \).
- a morphism \( g_* : A(X) \to A(Z) \) for any admissible epimorphism \( g : X \to Z \).

that satisfies the following axioms:

- **A1**: \( A(0) = \text{pt} \).
- **A2**: If \( i, j \) are admissible monomorphisms, then \( (ji)^* = i^* j^* \).
- **A3**: If \( p, q \) are admissible epimorphisms, then \( (qp)_* = q_* p_* \).
- **A4**: \( \text{id}_{X}^* = (\text{id}_{X})_* = \text{id}_{A(X)} \).
- **A5**: If \( f : X \to Y \) is an isomorphism, then \( f^* f_* = f_* f^* = \text{id} \).
- **A6**: For any Cartesian and Cocartesian diagram

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{p} & & \downarrow{q} \\
Z & \xrightarrow{v} & W
\end{array}
\]

if \( u \) is admissible monomorphism or \( q \) is admissible epimorphism, then \( v^* q_* = p_* u^* \).

- **A7**: If \( X \xrightarrow{u} Y \xrightarrow{v} Z \) is a diagram in \( \mathcal{C} \) that \( u \) is an admissible epimorphism and \( v \) is an admissible monomorphism, and if \( h_X \in A(X), h_Z \in A(Z) \) satisfy \( u_* (h_X) = v^* (h_Z) \), then there exists \( h \in A(X \oplus Z) \) that \( (\text{id}, vu)(h) = h_X \) and \( \pi_2 (h) = h_Z \). (Notice \( (\text{id}, vu) \) is an admissible monomorphism because it is composition of \( X \to X \oplus Z \) with the isomorphism \( (\pi_1, vu \pi_2) : X \oplus Z \to X \oplus Z \). For any \( X \in \mathcal{C} \), an element of \( A(X) \) is called a geometric structure on \( X \). Exact categories can be viewed as geometric exact categories by asserting \( A(X) = \text{pt} \) for all \( X \in \mathcal{C} \).

**Def. (I.11.1.17) (Morphisms compatible with the Geometric Structure)**. Let \( (\mathcal{C}, \mathcal{E}, A) \) be a geometric exact category, if \( (X', h'), (X'', h'') \) are two geometric objects, then a morphism \( f : X' \to X'' \) is said to be a morphism compatible with the geometric structure if there exists a geometric object \( (X, h) \) and an admissible monomorphisms \( u : X' \to X \) and an admissible epimorphism \( v : X \to X'' \) that \( h' = u^* (h) \) and \( h'' = v_* (h) \) and \( f = vu \).

The composition of two morphisms compatible with the geometric structure is also a morphism compatible with the geometric structure. We denote \( \mathcal{C}_A \) the category of geometric objects of \( \mathcal{C} \), and \( \mathcal{E}_A \) the set of diagrams of geometric objects \( 0 \to X' \xrightarrow{u} X \xrightarrow{v} X'' \to 0 \) that the underlying diagram is in \( \mathcal{E} \) and \( u, v \) are compatible with the geometric structures.
Prop. (I.11.1.18) (Hermitian Spaces). The f.d. Hermitian spaces over $\mathbb{C}$ or f.d. normed vector spaces over $\mathbb{R}$ form a geometric exact category.

Proof: Cf.[Harder-Narasimhan Categories, P4].

Prop. (I.11.1.19) (F.D. Ultranormed Banach Spaces). Let $K$ be a complete valued field, then the category of f.d. ultranormed Banach spaces over $K$ (IV.1.2.4) is a geometric exact category.

Proof: It suffices to check axiom A7, but $vu$ has norm $\leq 1$, and for any $\varphi : E \to F$ of norm $\leq 1$, we can endow $E \oplus F$ with the maximum norm, then in the decomposition $E \xrightarrow{(id,\varphi)} E \oplus F \xrightarrow{\pi_2} F$, we have $(id,\varphi)^*(h) = h_E$ and $(\pi_2)_*(h) = h_F$. □

Prop. (I.11.1.20) (Filtrations in an Abelian Category). The filtrations (II.2.2.1) in an Abelian category form a geometric exact category.

Proof: Cf.[Harder-Narasimhan Categories, P5]. □

Def. (I.11.1.21) ($K_0$ Group). Let $\mathcal{A}$ be an exact category, then $K_0$-group $K_0(\mathcal{A})$ is defined to be the quotient Abelian group

$$\bigoplus_{1_0 \to A \to B \to C \to 0} \mathbb{Z} \to \bigoplus_{A \in \mathcal{C}} \mathbb{Z} \to K_0(\mathcal{A}) \to 0,$$

where $1_{0 \to A \to B \to C \to 0}$ is mapped to $1_B - 1_A - 1_C$.

2 Abelian Categories

Def. (I.11.2.1) (Abelian Categories). An Abelian category $\mathcal{A}$ is an additive category that satisfies the follows axiom:

- **A4**: The kernels and cokernels exist, and the coimage equals image.

Remark (I.11.2.2). WARNING: An additive category that epimorphism+monomorphism is isomorphism need not be an Abelian category. Cf.[https://mathoverflow.net/questions/41722/is-every-balanced-pre-abelian-category-abelian] for a counter-example.

Prop. (I.11.2.3). In an Abelian category, the functor $X \mapsto \text{Hom}(X,Y)$ and $X \mapsto \text{Hom}(Y,X)$ is both left exact. (Note that left and right is seen on the image).

Def. (I.11.2.4) (Injectives and Surjectives). A morphism $f$ is an Abelian category $\mathcal{A}$ is called an injection if $\text{Ker} f = 0$. It is called a surjection if $\text{Coker} f = 0$. $f$ is an injection iff it is a monomorphism, it is a surjection iff it is an epimorphism.

Prop. (I.11.2.5). Axiom A3 asserts the good existence of product and sum of objects as we wanted, and it can be used to prove that monomorphism and epimorphism are stable under pushout and pullback.

But this uses A4 strongly, Cf.[MacLane Categories for working mathematicians P203]. (For epimorphism, first prove $0 \to X \times_U Y \to X \times Y \to U \to 0$ is exact when $X \to U$ is epi).

Prop. (I.11.2.6). equalizer and finite product derives finite limit, thus finite limits and finite colimits exists in Abelian categories.
Prop. (I.11.2.7) (Mitchell’s embedding theorem). If \( \mathcal{A} \) is a small Abelian category, then there exists a unital ring \( R \), not necessary commutative and a fully faithful and exact functor \( \mathcal{A} \to R - \text{Mod} \) that preserves kernel and cokernel. WARNING: it may not preserve sum and product, let alone limits and colimits.

Proof: □

Prop. (I.11.2.8). If \( \mathcal{C}, \mathcal{A} \) are categories and \( \mathcal{A} \) is Abelian, then \( \text{Hom}(\mathcal{C}, \mathcal{A}) \) is an Abelian category. In particular, \( \text{Ch}(\mathcal{A}) \) is Abelian.

Localization

Prop. (I.11.2.9) (Localization Category). If \( \mathcal{C} \) is a preadditive category and \( S \) is a left or right localizing system of \( \mathcal{C} \), then there exists a natural additive structure on \( S^{-1}\mathcal{C} \) and a localizing functor \( \mathcal{C} \to S^{-1}\mathcal{C} \) that is additive.

Proof: Cf. [Sta]05QD. □

Lemma (I.11.2.10). If \( \mathcal{C} \) is additive and \( S \) is localizing, let \( X \) be an element of \( \mathcal{C} \), then: \( Q(X) = 0 \) iff there is a morphism \( 0 : X \to Y \) that is an element of \( S \) iff there is a morphism \( 0 : Z \to X \) that is an element of \( S \).

Proof: If such \( 0 : X \to Y \in S \), then it maps to isomorphisms in \( S^{-1}\mathcal{C} \) by (II.1.1.45), so \( Q(X) = 0 \). If \( Q(X) = 0 \), then the morphism \( 0 \to X \) is mapped to an isomorphism, so by (II.1.1.44), there are \( g, h \) that \( fg = hf = 0 \), so \( Z \to 0 \to X \in S \). Dually for the other direction. □

Prop. (I.11.2.11). If \( \mathcal{A} \) is Abelian and \( S \) is localizing, then \( S^{-1}\mathcal{A} \) is an Abelian category and \( \mathcal{A} \to S^{-1}\mathcal{A} \) is exact.

Proof: Cf. [Sta]05QG. □

Serre Subcategory

Def. (I.11.2.12) (Serre Subcategories). A Serre subcategory of an Abelian category is a non-empty full subcategory \( \mathcal{C} \) that if

\[ A \to B \to C \]

is exact and \( A, C \in \text{Ob}(\mathcal{C}) \), then \( B \in \text{Ob}(\mathcal{C}) \).

A weak Serre subcategory of an Abelian category is a non-empty full subcategory \( \mathcal{C} \) that if

\[ A \to B \to C \to D \to E \]

is exact and \( A, B, D, E \in \mathcal{C} \), then \( C \in \mathcal{C} \).

Prop. (I.11.2.13).

- A Serre category is equivalent to a full subcategory \( \mathcal{A} \) that contains 0, all the subobjects and quotient objects of \( \mathcal{A} \), and extensions of objects of \( \mathcal{A} \) are in \( \mathcal{A} \).

- A weak Serre category is equivalent to a full subcategory \( \mathcal{A} \) that contains 0, and all the kernels, cokernels of objects in \( \mathcal{A} \), and all the extensions of objects in \( \mathcal{A} \).

In these cases, \( \mathcal{A} \) is an Abelian category and the functor \( i : \mathcal{A} \to \mathcal{C} \) is exact.
Proof: One direction of these two are trivial, it suffices to prove the converse. For the first, \(0 \rightarrow \text{Im} A \rightarrow B \rightarrow \text{Im} B \rightarrow 0\), so \(B \in \mathcal{C}\). For the second, \(0 \rightarrow \text{Coker}(A \rightarrow B) \rightarrow C \rightarrow \text{Ker}(D \rightarrow E) \rightarrow 0\), so \(C \in \mathcal{C}\). □

Prop. (I.11.2.14) (Quotients by Serre Subcategory). For an exact functor between Abelian categories, the objects that mapped to 0 forms a Serre subcategory. And any Serre subcategory is the kernel of an essentially surjective exact functor, i.e. the quotient Abelian category \(\mathcal{C}/\mathcal{A}\) can be constructed, and satisfies the universal property.

Proof: The full subcategory of \(\text{Ker}(F)\) is clearly a Serre subcategory by checking the definition. Conversely, consider \(S = \text{all the morphisms that has kernel and cokernel in} \mathcal{C}\), first we prove it is a localizing system (II.1.1.42).

The long exact sequence (I.11.4.6) shows that if \(f, g \in S\), then \(gf \in S\). For other verifications, Cf.[Sta]02MS?

Next we construct \(\mathcal{C}/\mathcal{A}\) as \(S^{-1}\mathcal{C}\). Consider which objects are mapped to 0 in \(\mathcal{C}/\mathcal{A}\), use (I.11.2.10) and consider the kernel and cokernel, it is easy to see that \(\text{Ker}(Q) = \mathcal{C}\). If another category \(\mathcal{D}\) and \(F : \mathcal{C} \rightarrow \mathcal{D}\) satisfies \(\mathcal{C}\) is mapped to 0, then it is clear that elements in \(S\) is mapped to isomorphism, so it factors through \(\mathcal{C}/\mathcal{A}\) by universal property (II.1.1.45). □

Prop. (I.11.2.15). For a Serre subcategory \(\mathcal{B}\) of an Abelian category \(\mathcal{A}\), the set of all complexes that has cohomology group in \(\mathcal{B}\) is a strictly full triangulated subcategory of \(D(\mathcal{A})\).

Proof: Cf.[Sta]06UQ. □

Prop. (I.11.2.16) \((K_0 \text{ Group of Serre Subcategory})\). Let \(\mathcal{A}\) be an Abelian category and \(\mathcal{C}\) a Serre subcategory, with \(\mathcal{A}/\mathcal{C} = B(I.11.2.14)\). Then

- The exact functors \(\mathcal{C} \rightarrow \mathcal{A} \rightarrow \mathcal{B}\) induces an exact sequence \(K_0(\mathcal{C}) \rightarrow K_0(\mathcal{A}) \rightarrow K_0(\mathcal{B}) \rightarrow 0\),

- The kernel of \(K_0(\mathcal{C}) \rightarrow K_0(\mathcal{A})\) is generated by elements of the form \([\text{Ker}(\psi)/\text{Im}(\varphi)] - [\text{Ker}(\varphi)/\text{Im}(\psi)]\)

where \(\varphi, \psi : M \rightarrow M\) are pairs of maps that \(\varphi \circ \psi = \psi \circ \varphi = 0\).

Proof: Cf.[Sta]02MX. □

Artinian Abelian Categories

Def. (I.11.2.17) (Artinian Abelian Categories). An Artinian Abelian category is an Abelian category that

- \(\text{Hom}(A, B)\) are all f.d. vector spaces over \(k\).
- Then length of any filtrations \(0 = X_0 \subset X_1 \subset \ldots \subset X_l = X\) for any object \(X\) is bounded. The maximal length is called the length of \(X\).

Prop. (I.11.2.18) (Jordan-Holder). Let \(\mathcal{C}\) be an Artinian category, then any maximal length filtration of an element \(X\) has the same length, and the set of quotients \(X_{k+1}/X_k\) is the same, up to order.

Proof: □
Others

Def. (I.11.2.19) (Essential Morphism). In an Abelian category, an injection \( A \to B \) is called **essential** iff every non-zero subobject of \( B \) intersects \( A \). A surjection is called **essential** iff every proper subobject of \( A \) is not mapped to \( B \).

Def. (I.11.2.20) (Noetherian Abelian Category). In a Grothendieck Abelian category \( \mathcal{A} \), an object \( M \) is called **finitely generated** if for every ascending chain

\[
M_1 \subset M_2 \subset \ldots \subset M
\]

with \( \cup_i M_i = M \), we have \( M_i = M \) for some \( i \).

\( \mathcal{A} \) is called **Noetherian** iff a subobject of a f.g. object is f.g.. \( \mathcal{A} \) is called **Artinian** iff every f.g. object has finite length.

Def. (I.11.2.21) (Compact Objects). An object \( M \) in an additive category \( \mathcal{C} \) with arbitrary direct sum is called **compact** if \( \text{Hom}(M, -) \) commutes with arbitrary direct sums.

Grothendieck Abelian Category

Def. (I.11.2.22) (Axioms for Grothendieck Abelian Category).

- **AB3**: It is an Abelian category and arbitrary direct sums exist. (Thus colimits over small categories exists.)
- **AB5**: Filtered colimits over small categories are exact. This is equivalent to \{ for any family of subobjects \( \{A_i\} \) of \( A \) to \( B \) indexed by inclusion can induce a morphism \( \sum A_i \to B \) (internal sum) \}?
- **GEN**: It has a generator, that is, an object \( U \) s.t. for any proper subobject \( N \subset M \), there is a map \( U \to M \) that doesn’t factor through \( N \).

Def. (I.11.2.23) (Further Grothendieck Axioms).

- **AB4**: Arbitrary direct sums are exact.
- **AB6**: For any index set \( J \) and filtered categories \( I_j, j \in J \) and diagrams indexed over \( I_j \), the natural map

\[
\lim_{i_j \in I_j} \prod_{j \in J} M_{i_j} \to \prod_{j \in J} \lim_{i_j \in I_j} M_{i_j}
\]

is an isomorphism.

- Axioms with an \( * \) means that the dual category satisfies something.

Prop. (I.11.2.24). The presheaf category \( \mathcal{A}^C \) is a Grothendieck Abelian category if \( \mathcal{A} \) is Grothendieck Abelian.

*Proof:* For the presheaf, the only problem is the existence of generator, for that, just construct a family of presheaves and sum them. Take \( Z_X = i_{fX}(U) \), where \( U \) is the generator of \( \mathcal{A} \) and \( f = pt \to \mathcal{A}^C : pt \to U \). Then \( F(X) = \text{Hom}(Z_X, F) \) by adjointness(V.1.2.8). So they are a family of generators.

Prop. (I.11.2.25). Any Grothendieck Abelian category has a functorial injective embedding.
Prop. (I.11.2.26) (Representability on Grothendieck Category). A contravariant functor from a Grothendieck category to $\text{Sets}$ is representable iff it takes colimits to limits.

Proof: Consider $M \oplus M \to M$ with induce a map $F(M) \times F(M) \to F(M)$ thus $F(M)$ is a semigroup, and the inverse of $\text{id}_M$ in $\text{Hom}(M, M)$ maps to a $F(M) \to F(M)$ which is the inverse, Thus in fact $F$ is a left adjoint functor to $\text{Ab}$.

Let $U$ be a generator, $A = \sum_{s \in F(U)} s \in F(A) = \prod_{s \in F(U)} F(U)$. Let $A'$ be the largest objects that $s_{\text{univ}}$ restricis to $0$ in $A'$, let $\sigma_{\text{univ}}$ be in $F(A/A')$ that maps to $s_{\text{univ}}$ in $F(A)$ (because $F$ is left exact). Then we claim $(A/A', \sigma_{\text{univ}})$ represents $F$. Cf.[Sta]07D7. □

Cor. (I.11.2.27). Grothendieck Category satisfies $AB3^*$ (because $F = \prod_i \text{Hom}(-, M_i)$ commutes with colimits).

Examples of Grothendieck Category

Prop. (I.11.2.28). The category of sheaf of $\mathcal{O}_X$-modules on a ringed site is a Grothendieck Abelian category. Moreover, injectives are flabby.

Proof: It is obviously an Abelian category and have filtered colimits as presheaves, which are exact because colimits in the category of Abelian groups are exact, and for a family of generators, take $j_! \mathcal{O}_U$ as the representative for $\Gamma(U, -)$, which is the sheaf associated to the sheaf $\mathcal{Z}_U$ in the proof of(I.11.2.24). Use $j_! \mathcal{O}_U$, we can see injectives are flabby, (because $j_! \mathcal{O}_U \to j_! \mathcal{O}_V$ is a monomorphism for $V \subset U$). □

Cor. (I.11.2.29). The category of Abelian presheaves and Abelian sheaves on a site is a Grothendieck category.

Proof: For the presheaf, Cf(I.11.2.24). For the sheaf, it follows from (I.11.2.28). □

Remark (I.11.2.30). The category of Abelian sheaves doesn’t satisfy $AB4^*$, i.e. not every limit of epimorphisms is epimorphism.

Proof: Consider the constant sheaf $\oplus \mathcal{B}(\mathbb{P}^{1}_{\mathbb{Q}})$ on $[0, 1]$. □

Cor. (I.11.2.31). The category of $R$-modules is a Grothendieck Abelian category.

Prop. (I.11.2.32). The category of $\text{Qco}$ sheaves on a scheme is Grothendieck category, and there is a coherentor left adjoint to the forgetful functor.

Proof: $\text{Qco}$: First by(V.3.1.3), $\text{Qco}$ is an Abelian category, and on affine open set, the colimit is an $\text{Qco}$ sheaf, thus the colimit exists in $\text{Qco}$ and equals the colimit in the category of sheaves, thus filtered colimits is exact because $\mathcal{O}_X$-Mod is Grothendieck(I.11.2.28). The generator exists, Cf.[Sta]077P.

The coherentor exists by the fact that $h_{\text{F}}$ commutes with colimits and by the property of Grothendieck category(I.11.2.26). □
3 Chain Complexes

Def. (I.11.3.1) (Chain Complex). A chain complex over an Abelian category is . The category of complexes over \( \mathcal{A} \) is denoted by \( \text{Comp}(\mathcal{A}) \).

Prop. (I.11.3.2). For any Abelian category, \( \text{Comp}(\mathcal{A}) \) is an Abelian category.

Prop. (I.11.3.3). The natural inclusion \( \mathcal{A} \subset \text{Comp}(\mathcal{A}) \) embeds \( \mathcal{A} \) as a full subcategory of \( D(\mathcal{A}) \), and \( H^0 \) is just the left adjoint.

An object \( K \in D(\mathcal{A}) \) is called discrete if it is in the essential image of this embedding.

Remark (I.11.3.4). Remember the translation operator \( K[n] \) makes the complex lower \( n \) dimensions.

Def. (I.11.3.5) (Truncation of Complexes). Let \( \mathcal{A} \) be an Abelian category and \( A^\bullet \) be a complex, there are several ways to truncate \( A^\bullet \):

- The stupid truncation \( \sigma_{\leq n} A = \ldots \to A_{n-1} \to A_{n} \to 0 \to \ldots \) There is a morphism \( A \to \sigma_{\leq n} A \).
- The stupid truncation \( \sigma_{\geq n} A = \ldots \to 0 \to A_{n} \to A_{n+1} \to \ldots \) There is a morphism \( \sigma_{\geq n} A \to A \).
- The canonical truncation \( \tau_{\leq n} A = \ldots \to A_{n-1} \to \ker(d_n) \to 0 \to \ldots \) There is a natural morphism \( \tau_{\leq n} A \to A \) that induces isomorphism on cohomology groups on degree \( \leq n \).
- The canonical truncation \( \tau_{\geq n} A = \ldots \to 0 \to \coker(d_{n-1}) \to A_{n+1} \to \ldots \) There is a natural morphism \( A \to \tau_{\geq n} A \) that induces isomorphism on cohomology groups on degree \( \geq n \).

Cor. (I.11.3.6). There are exact sequences of complexes

\[
0 \to \tau_{\leq n} A^\bullet \to A^\bullet \to \tau_{\geq n+1} A^\bullet \to 0
\]

\[
0 \to \sigma_{\geq n+1} A^\bullet \to A^\bullet \to \sigma_{\leq n} A^\bullet \to 0
\]

Def. (I.11.3.7) (Cone & Cylinder). The mapping cone of \( f : K^\bullet \to L^\bullet \) is the complex \( C(f)^\bullet \) that:

\[
C(f) = K[1]^\bullet \oplus L^\bullet, \quad d(k^{i+1}, l^i) = (-d_K k^{i+1}, f(k^{i+1}) + d_L l^i)
\]

The mapping cylinder of \( f : K^\bullet \to L^\bullet \) is the complex \( \text{Cyl}(f) \) that:

\[
\text{Cyl}(f) = K^\bullet \oplus K[1]^\bullet \oplus L^\bullet, \quad d(k^i, k^{i+1}, l^i) = (d_K k^i - k^{i+1}, d_R k^{i+1}, f(k^{i+1}) + d_L l^i)
\]

It is a shame I haven’t see clearly the similarity of this with the topological cone and cylinder, should study it further.

Def. (I.11.3.8) (Double Complexes). A double complex over an Abelian category is a complex over \( \text{Comp}(\mathcal{A}) \) (I.11.3.2).

Def. (I.11.3.9) (Totalization). Given a double complex \( K^{p, \bullet} \) over an Abelian category \( \mathcal{A} \), the associated total complexes is defined to be

\[
(\text{Tot}^{\Pi}(K))^n = \prod_{n=p+q} K^{p, q}, \quad d^n = \prod_{n=p+q} (d_1^{p, q} + (-1)^p d_2^{p, q})
\]

\[
(\text{Tot}^{\bigoplus}(K))^n = \prod_{n=p+q} K^{p, q}, \quad d^n = \sum_{n=p+q} (d_1^{p, q} + (-1)^p d_2^{p, q})
\]
Lemma (I.11.3.10) (Koszul Sign Rule). There is an isomorphism of complexes
\[ \sigma : C^\bullet \otimes D^\bullet \rightarrow D^\bullet \times C^\bullet : \sigma(x \otimes y) = (-1)^{mn}y \otimes x \]
where \( x \in C^m \) and \( y \in D^n \).

Proof:
\[
\partial(\sigma(x \otimes y)) = \partial((-1)^{mn}y \otimes x) = (-1)^{mn}(\partial y \otimes x) + (-1)^{mn+n}(y \otimes \partial x) = (-1)^n(\sigma(x \otimes \partial y) + \sigma(\partial x \otimes y)) = \sigma(\partial(x \otimes y)) \]  
\[ \square \]

Prop. (I.11.3.11) (Monoidal Structure on \( \text{Comp}(\mathcal{A}) \)). The functor
\[ \text{Comp}(\mathcal{A}) \times \text{Comp}(\mathcal{A}) \rightarrow \text{Comp}(\mathcal{A}) : C \times D \mapsto \text{Tot}^\oplus(C \otimes D) \]
endows \( \text{Comp}(\mathcal{A}) \) with a commutative monoidal structure.

Proof: Commutativity follows from (I.11.3.10). For the associativity, ?

Unbounded Complexes

Lemma (I.11.3.12) (Left Resolutions of Unbounded Complexes). Let \( \mathcal{A} \) be an Abelian category and \( \mathcal{P} \) be a subset of objects of \( \mathcal{A} \). Assume that every object of \( \mathcal{A} \) is a quotient of an object of \( \mathcal{P} \), then for any complex \( K^\bullet \), there exists a commutative diagram

\[
\begin{array}{ccc}
P_1^\bullet & \rightarrow & P_2^\bullet & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \\
\tau_{\leq 1} K^\bullet & \rightarrow & \tau_{\leq 2} K^\bullet & \rightarrow & \cdots
\end{array}
\]

where the vertical arrows are quasi-isomorphisms, and each \( P_n^\bullet \) is a bounded above complex with terms in \( \mathcal{P} \), and each \( P_n^\bullet \rightarrow P_{n+1}^\bullet \) are termwise-split injections and the cokernel is also a complex with terms in \( \mathcal{P} \).

Proof: Cf. [Sta]06XX.

4 Cohomology of Complexes

Prop. (I.11.4.1) (Distinguished Triangle of \( K^*(\mathcal{A}) \)). For any morphism \( K^\bullet \rightarrow L^\bullet \), there exists a termwise-splitting exact sequence of Complexes commuting in \( K(\mathcal{A}) \).

\[
\begin{array}{ccc}
K^\bullet & \rightarrow & L^\bullet \\
\downarrow & & \downarrow \alpha \\
0 & \rightarrow & K^\bullet \rightarrow \text{Cyl}(f) \rightarrow C(f) \rightarrow 0 \\
\downarrow \beta & & \\
0 & \rightarrow & L^\bullet \rightarrow C(f) \rightarrow K^*[1] \rightarrow 0
\end{array}
\]

where \( \beta \alpha = \text{id} \) and \( \alpha \beta \sim \text{id} \). And \( K^\bullet \rightarrow L^\bullet \rightarrow C(f) \rightarrow K^*[1] \) is called a distinguished triangle. Any exact triple of complexes in \( \text{Kom}(\mathcal{A}) \) is quasi-isomorphic to a distinguished triangle. In fact, we can define the distinguished triangle in \( K(\mathcal{A}) \) as that induced by a split exact sequence, Cf. [Sta]014L.

Notice all this can imitate the similar parallel construction in the category of topological spaces.
**Proof:** Cf.[Gelfand P157]

**Cor. (I.11.4.2).** A distinguished triangle will induce a long exact sequence, for this, just need to verify that the $\delta$-homomorphism coincide with the morphism that $C(f) \to K^*\{1\}$ induces.

**Cor. (I.11.4.3).** A morphism $f : K \to L$ is quasi-iso iff $C(f)$ is acyclic. It is homotopic to 0 iff $f$ can be extended to a morphism $C(f) \to L$.

**Prop. (I.11.4.4) (Five lemma).** In an Abelian category, if there is a diagram

$$
\begin{array}{cccccc}
* & \rightarrow & * & \rightarrow & * & \rightarrow * \\
\downarrow s & & \downarrow g & & \downarrow f & & \downarrow h & & \downarrow i \\
* & \rightarrow & * & \rightarrow & * & \rightarrow * 
\end{array}
$$

Where the rows are exact and $g, h$ are isomorphisms. If $i$ is injective, then $f$ is surjective; if $s$ is surjective, then $f$ is injective.

**Proof:** Rotate the diagram counterclockwise $90^\circ$. Then use the two different filtration both converge(II.3.5.8).

**Prop. (I.11.4.5) (Snake lemma).** In an Abelian category, if there is a diagram

$$
\begin{array}{cccccc}
* & \rightarrow & * & \rightarrow & * & \rightarrow 0 \\
\downarrow f & & \downarrow g & & \downarrow h \\
0 & \rightarrow & * & \rightarrow & * & \rightarrow *
\end{array}
$$

where the rows are exact, then there is a long exact sequence

$$
\text{Ker } f \rightarrow \text{Ker } g \rightarrow \text{Ker } h \rightarrow \text{Coker } f \rightarrow \text{Coker } g \rightarrow \text{Coker } h
$$

And if $i$ is injective, then the first one is injective; if $s$ is surjective, then the last one is surjective.

**Proof:** Rotate the diagram counterclockwise $90^\circ$. Then use the two different filtration both converge(II.3.5.8).

**Cor. (I.11.4.6).** In an Abelian category, if $f : A \to B, g : B \to C$, then there is a long exact sequence:

$$
0 \rightarrow \text{Ker } f \rightarrow \text{Ker } gf \rightarrow \text{Ker } g \rightarrow \text{Coker } f \rightarrow \text{Coker } gf \rightarrow \text{Coker } g \rightarrow 0.
$$

**Proof:** Use snake lemma(as modules), there is a diagrams:

$$
\begin{array}{cccccc}
A & \rightarrow & B & \rightarrow & \text{Coker } f & \rightarrow 0 \\
\downarrow gf & & \downarrow g & & & \\
0 & \rightarrow & C & \rightarrow & C & \rightarrow 0
\end{array}
$$

So by Snake lemma,

$$
\text{Ker } gf \rightarrow \text{Ker } g \rightarrow \text{Coker } f \rightarrow \text{Coker } gf \rightarrow \text{Coker } g \rightarrow 0.
$$

As Abelian category is dual, we can do this dually to get:

$$
0 \rightarrow \text{Ker } f \rightarrow \text{Ker } gf \rightarrow \text{Ker } g \rightarrow \text{Coker } f \rightarrow \text{Coker } gf.
$$

They splint together to get the desired long exact sequence.
Prop. (I.11.4.7). For a $3 \times 3$ diagram of complexes, the connection homomorphism satisfies an anti-commutative diagram:

$$
\begin{array}{ccc}
H^q(Z') & \xrightarrow{\delta} & H^q(X') \\
\downarrow{\delta} & & \downarrow{\delta} \\
H^q(Z) & \xrightarrow{\delta} & H^{q+1}(X)
\end{array}
$$

by (II.3.1.4) as the category $K(A)$ is triangulated.

Prop. (I.11.4.8) (Universal Coefficient Theorem). Should be somewhere in [Weibel].

Def. (I.11.4.9) (Herbrand Quotient). For a complex of $R$-modules cyclic of order 2, we define the additive Herbrand quotient as $\text{length}_R(H^0) - \text{length}_R(H^1)$, when both are definable and the multiplicative Herbrand quotient as $|H^0|/|H^1|$ when they are both finite.

Prop. (I.11.4.10). For an exact sequence $0 \to M \to N \to K \to 0$ of complexes of cyclic order 2, we have $h(N) = h(M) + h(K)$ and $h^*(N) = h^*(M)h^*(K)$ in the sense that if two of them are definable, then so is the third. This is an easy consequence of long exact sequence.

Prop. (I.11.4.11). If each term of this complex has finite length, then $h(M) = 0$. If each term is finite, then $h^*(M) = 0$. This is an consequence of isomorphism theorem. So we have, if a morphism of complexes has kernel and cokernel finite, then it induce an isomorphism on $h$ or $h^*$.

Proof:

5 Injectives & Projectives

Def. (I.11.5.1). An injective object in a Abelian category is a $I$ s.t. $\text{Hom}(-, I)$ is an exact functor, equivalently, maps to $I$ can be extended along injections.

A projective object in a Abelian category is a $I$ s.t. $\text{Hom}(I, -)$ is an exact functor, equivalently, maps to $I$ can be pulled back along surjections.

Prop. (I.11.5.2). Product of injective elements are injective, sums of projective elements are projective.

Prop. (I.11.5.3). In an Abelian category, the direct summand of a projective object is projective. (The summand has definition in an Abelian category).

Prop. (I.11.5.4). If a functor $f$ between Abelian categories is left adjoint to an exact functor, then it preserves injectives. Dually for projectives.

Prop. (I.11.5.5). If $A$ is an Abelian category, the chain complex category $Ch(A)$ is abelian by (I.11.2.8). A chain complex $P$ is projective iff it is a split exact complex of projective objects. The same is true by dual argument for injectives.

Proof:

If $K$ is projective, use the surjection $C(id_K) \to K[1]$, there is a homotopy between $id_K$ and 0. Thus we have $x = dhx + hdx$. And if $dhx = hdx$, then $hdx = 0$, thus $dy = 0$, so $K = dhK \oplus hdK$ and thus $K[n] = B_n \oplus B_{n+1}$. Thus $K$ is a direct product of $0 \to B \to B \to 0$. And this one is projective if $B$ is projective. □
Prop. (I.11.5.6) (Check Injectives). In a Grothendieck Abelian category with generator $U$, an object is injective if it is extendable over subobjects of $U$. (AB5 assures we can extend by Zorn’s lemma. Then use GEN, Cf.[Sta]079G). If it is a family of objects, it suffice to extend over each one of them.

Proof: □

Injective Resolutions

Prop. (I.11.5.7) (Horseshoe Lemma). For a exact sequence $0 \to X_1 \to X \to X_2 \to 0$ and a injective resolution of $X_1$ and $X_2$, there is a injective resolution of $X$ commuting with them. (Choose them one-by-one, in fact, $I_n = I_n^1 \oplus I_n^2$ using the injectivity of $I_n^1$. Snake lemma told us that the cokernel is an exact sequence, use that to define the next one.

Prop. (I.11.5.8). For two lifting of morphisms $X_1 \to Y_1$ and $X_2 \to Y_2$, there is a lifting of the morphism $X \to Y$ compatible with that. Cf.[Weibel P2.4.6].

Prop. (I.11.5.9) (Cartan-Eilenberg Resolution). For a complex $K \in D(A)$, a Cartan-Eilenberg resolution of $K$ consists of a 2-complex $I_ullet$ and a map of complexes $K \to I_0$ that the induced complexes:

\[
\begin{align*}
0 &\to K^i \to I^{i,0} \to I^{i,1} \to \ldots \\
0 &\to B^i(K) \to B^i_x(I^{0,0}) \to B^i_x(I^{0,1}) \to \ldots \\
0 &\to Z^i(K) \to Z^i_x(I^{0,0}) \to Z^i_x(I^{0,1}) \to \ldots \\
0 &\to H^i(K) \to B^i_x(I^{0,0}) \to H^i_x(I^{0,1}) \to \ldots 
\end{align*}
\]

are all injective resolutions, and the exact sequences

\[
\begin{align*}
0 &\to B^i_x(I^{0,j}) \to Z^i_x(I^{0,j}) \to H^i_x(I^{0,j}) \to 0 \\
0 &\to Z^i_x(I^{0,j}) \to I^{0,j} \to Z^i_x(I^{0,j}) \to 0
\end{align*}
\]

split.

Then if $I_B$ is sufficiently large, for any $K$ in $K(B)$ there is a Cartan-Eilenberg resolution.

Proof: Cf.[Gelfand P210],[Weibel P146]. □

Cor. (I.11.5.10). For a CE resolution of a complex $K \in K^+(B)$, the spectral sequence can be applied and shows $K \to \operatorname{Tot}(L)$ is a quasi-isomorphism, i.e. $\operatorname{Tot}(L)$ is a injective resolution of $K$.

Cor. (I.11.5.11) (Functoriality of Cartan-Eilenberg Resolution). If $f: A \to B$ is a chain map and $A \to P, B \to Q$ are Cartan-Eilenberg resolutions, then there is a double complex map $\tilde{f}: P \to Q$ extending $f$. And if $f$ is homotopic to $g$, then $\tilde{f}$ is homotopic to $\tilde{g}$.

In particular, for any two Cartan-Eilenberg resolutions $P, Q$ of $A$ and an additive functor $F$, the chain complex $\operatorname{Tot}^\Pi(F(P))$ and $\operatorname{Tot}^\Pi(F(Q))$ are chain homotopy equivalent.
**K-injective**

**Prop. (I.11.5.12) (K-injective).** For an Abelian category \( \mathcal{A} \), a complex \( I^\bullet \) in \( K(\mathcal{A}) \) is called a \( K \)-injective object iff it satisfies the following equivalent conditions:

- \( \text{Hom}_{K(\mathcal{A})}(S^\bullet, I^\bullet) = 0 \) for any acyclic \( S^\bullet \) in \( K(\mathcal{A}) \).
- \( \text{Hom}_{K(\mathcal{A})}(N^\bullet, I^\bullet) \cong \text{Hom}_{K(\mathcal{A})}(M^\bullet, I^\bullet) \) for quasi-iso \( M^\bullet \to N^\bullet \).
- \( \text{Hom}_{D(\mathcal{A})}(X^\bullet, I^\bullet) \cong \text{Hom}_{D(\mathcal{A})}(X^\bullet, I^\bullet) \) for every \( X^\bullet \). In particular, a quasi-iso between two \( K \)-injective objects is a homotopy equivalence.

Dually, we can define \( K \)-projectives.

**Proof:** 1 \( \to \) 2 is by(I.11.4.2); 2 \( \to \) 3 use(II.3.2.2), for 3 \( \to \) 1, notice an acyclic is quasi-iso to 0. \( \square \)

**Prop. (I.11.5.13) (Injectives and K-Injective).** Objects in \( K^+(\mathcal{I}) \) are all \( K \)-injectives. thus the injective resolution is unique in \( K^+ \). Dually \( K^-(\mathcal{P}) \) are all \( K \)-projectives.

**Proof:** Use the first definition of \( K \)-injectives. Use induction, we construct the first homotopy, because \( I^\bullet \) is bounded below, we see the map \( h^n \) factors through \( \text{Coker} \ d^{n-1} = \text{Im} \ d^n \) because \( S^\bullet \) is acyclic, so by injectivity, it can be extended to \( S^{n+1} \to I^n \). \( \square \)

**Prop. (I.11.5.14).** If a functor \( f \) between Abelian categories is left adjoint to an exact functor, then it preserves \( K \)-injectives (use definition1).

**Prop. (I.11.5.15) (Products of K-Injectives).** Let \( \mathcal{A} \) be an Abelian category, \( I_\ell \) be \( K \)-injective complexes. If the termwise product of \( I_\ell \) exists, then it is also \( K \)-injective, and is the products of \( I_\ell \) in \( D(\mathcal{A}) \).

**Proof:** It is clearly \( K \)-injective by(I.11.5.12) item1 as it is the product in the category \( K(\mathcal{A}) \). Thus we can easily use the (I.11.5.12) item3 to see that \( I^\bullet \) is also the product in the category \( D(\mathcal{A}) \). \( \square \)

**Lemma (I.11.5.16).** [Sta070M]

**Prop. (I.11.5.17) (K-injective Resolution).** If \( \mathcal{A} \) is an Abelian category having enough injectives and exact countable products, then every complex is quasi-isomorphic to a \( K \)-injective complex.

Moreover, if \( \mathcal{A} \) is a Grothendieck category, then the \( K \)-injective resolution can be chosen to be functorial.

**Proof:** By(I.11.5.16), it suffices to show that \( K \to R \lim \tau_{\geq -n} K \) is a quasi-isomorphism for all complex \( K \). But this is clear from the distinguished triangle

\[
R \lim \tau_{\geq -n} K \to \prod \tau_{\geq -n} K \to \prod \tau_{\geq -n} K \to R \lim \tau_{\geq -n} K[1]
\]

and the fact \( H^p(\prod \tau_{\geq -n} K) = \prod_{p \geq -n} H^p(K) \).

For the second assertion, Cf.[Sta079P]. \( \square \)
6 Ring Category Case

Prop. (I.11.6.1). An $R$-module $I$ is injective iff for any injective homomorphism from $I$ to any $R$-module splits.

Proof: The critical point is that we can always embed $I$ to an injective hull $J$ by (I.11.2.25), then $J = I \oplus J'$, so $I$ is clearly injective. □

Prop. (I.11.6.2). If $A$ is Noetherian and $C^\bullet$ is a complex of $A$-modules bounded above that every cohomology group $H^i$ is a finite $A$-module, then there is a complex $L^\bullet$ of finite free $A$-modules, that $g : L^\bullet \to C^\bullet$ is a quasi-isomorphism.

Moreover, if $C^i$ are all flat $A$-modules, then $L^\bullet \otimes_A M \to C^\bullet \otimes_A M$ is quasi-isomorphism for every $M$.

Proof: $C^\bullet$ is bounded above so we choose $L^n = 0$, and use induction to construct $L^n$ that $H^i(L) \to H^i(C)$ is isomorphism for $i > n+1$ and surjection for $i = n+1$. For this, choose a generator $x_1, \ldots, x_r$ of $H^n(C)$ in $Z^n(C)$, and let $y_{r+1}, \ldots, y_s$ be a generator of $g^{-1}(B^{n+1}(C))$(Noetherian used), and let $g(y_i) = dx_i$ for $x_i \in C^n$.

Now let $L^n$ be freely generated by $e_1, \ldots, e_s$ and $de_i = 0$ for $i \leq r$ and $de_i = y_i$ for $i > r$, and let $g : L^n \to C^n$ be $ge_i = x_i$. Then it can be verified to be a quasi-isomorphism.

If $C^i$ are all flat, we check isomorphism for all f.g. modules $M$, because $\otimes$ and cohomology all commutes with direct limits. Use induction, for $n$ large, both are 0, and if we write $0 \to R \to F \to M \to 0$, for $F$ finite free, then there is a commutative diagram of long exact sequences, and for $F$, $H^i$ are obviously isomorphism, so I can use five lemma. □

Prop. (I.11.6.3) (K-Injective under Change of Rings). If $R \to S$ is a ring map, then

1. If $R \to S$ is flat and $I^\bullet$ is a $K$-injective complex of $S$-modules, then $I^\bullet$ is $K$-injective as a complex of $R$-modules.

2. If $R \to S$ is surjective and $I^\bullet$ is a complex of $S$-modules that is $K$-injective as a complex of $R$-modules, then it is $K$-injective as a complex of $S$-modules.

3. If $I^\bullet$ is a $K$-injective complex of $R$-modules, then $\text{Hom}_R(S, I^\bullet)$ is $K$-injective as a complex of $S$-modules.

Proof: 1: This is because $\text{Hom}_{K(R)}(M^\bullet, I^\bullet) = \text{Hom}_{K(S)}(M^\bullet \otimes_R S, I^\bullet)$ and (I.11.5.12), as tensoring $S$ is exact.

2: This is because $\text{Hom}_{K(R)}(N^\bullet, I^\bullet) = \text{Hom}_{K(S)}(N^\bullet, I^\bullet)$ for a complex of $S$-modules $N^\bullet$ and (I.11.5.12).

3: This is because $\text{Hom}_{K(S)}(N^\bullet, \text{Hom}_R(S, I^\bullet)) = \text{Hom}_{K(R)}(N^\bullet, I^\bullet)$, and (I.11.5.12). □
I.12 Lie Algebras

Basic references are [Car05], [Complex Semisimple Lie Algebras, Serre], [Mil13], [Kna96] and [Lie Algebras and Lie Groups Serre]. Notice that [Lie Algebras Humphreys] is too complicated.

In this section, if not otherwise pointed out, $k$ is assumed to be a field of char0 or just $\mathbb{C}$.

1 Basics

Def. (I.12.1.1) (Lie Algebras). A Lie algebra $L$ is an non-associative algebra over a field $k$ with a bilinear Lie bracket operation that satisfies:

$$[x,x] = 0, \quad [x[yz]] = [[xy]z] + [y[xz]]$$ (Jacobi Identities).

It is easily deduced that $[xy] = -[yx]$.

Denote $\text{ad}_x(y) = [xy]$, then $\text{ad}_x$ are all derivatives of $L$.

An element $x \in g$ is called nilpotent or semisimple if $\text{ad}_x$ is nilpotent or semisimple.

Prop. (I.12.1.2) (Associative Algebra). For any associative algebra $A$ over $k$, it can be given naturally a Lie algebra structure by defining $[xy] = xy - yx$. In this way, we get a natural functor $\text{AssAlg}_k \to \text{Lie}_k: A \mapsto [A]$.

Prop. (I.12.1.3) (Derivatives form a Lie Algebra). Given a $k$-algebra $A$, the set of derivatives $\text{Der}_k(A) = \text{Der}_k(A, A)$ is a Lie algebra under the associative bracket.

Proof:

$$[D_1, D_2](ab) = D_1(D_2(a)b + aD_2(b)) - D_2(D_1(a)b + aD_1(b)) = D_1D_2(a)b + D_2(a)D_1(b) + D_1(a)D_2(b) + aD_1D_2(b) - (D_2D_1(a)b + D_1(a)D_2(b) + D_2(a)D_1(b) + aD_2D_1(b)) = [D_1, D_2](a)b + a[D_1, D_2](b)$$

Prop. (I.12.1.4) (Base Change of Fields). Let $a$ be a subalgebra of a Lie algebra $g$ over $k$, and $k'/k$ a field extension, then $a_{k'}$ is a subalgebra of $g_{k'}$. Then

$$N_{g_{k'}}(a_{k'}) = N_g(a)_{k'}.$$  

$$c_{g_{k'}}(a_{k'}) = c_g(a)_{k'}.$$  

Proof: This is because the normalizer and centralizer is defined by linear equations with coefficients in $k$, thus the vector space is defined over $k$.

Prop. (I.12.1.5). Let $D$ be a derivative of a $k$-algebra $A$ that is nilpotent, then $e^D$ is an automorphism of $A$(as an algebra).

Proof: Routine calculation.

Def. (I.12.1.6) (Elementary Automorphisms). Let $g$ be a Lie algebra, a special automorphism is an automorphism of $g$ of the form $e^{ad_g(x)}$, where $x$ is in the nilpotent radical?

The group of elementary automorphisms is the subgroup of the automorphism group of $g$ generated by the automorphisms of the form $e^{ad_g(x)}$ where $ad(x)$ is nilpotent.
Def. (I.12.1.7) (Ideals). A subspace \( a \subseteq \mathfrak{g} \) is called an ideal of \( \mathfrak{g} \) iff \( [\mathfrak{g}, a] \subseteq a \). If \( I \) is an ideal of \( L \), then \( L/I \) can be made into a Lie algebra by defining \( [I + x, I + y] = I + [xy] \).

Def. (I.12.1.8) (Center). The center of a Lie algebra \( \mathfrak{g} \) is the elements \( a \) that \( \text{ad} a = 0 \). It is an ideal.

Def. (I.12.1.9) (Simple Lie Algebras). A Lie algebra \( \mathfrak{g} \) is called simple if it is not 1-dimensional and it has no nontrivial ideal.

Def. (I.12.1.10) (Lie Algebra of Affine Maps). Let \( V \) be a f.d. \( k \)-vector space. If we regard \( V \) as a commutative algebra, then \( \text{Der}_k(V) = \mathfrak{gl}_V \). Then \( V \times \mathfrak{gl}_V \) is a Lie algebra, denoted by \( \mathfrak{af}(V) \).

Let \( V' = V \oplus k \), and let \( \mathfrak{h} = \{ w \in \mathfrak{gl}_{V'}, \|w(V') \subset V \} \), which is a Lie subalgebra of \( \mathfrak{gl}_{V'} \). If we define

\[
\eta : \mathfrak{h} \rightarrow \mathfrak{gl}_V : \eta(w) = w|_V, \quad \zeta : \mathfrak{h} \rightarrow V : \zeta(w) = w(0,1),
\]

then \((\eta, \zeta)\) defines a Lie algebra homomorphism from \( \mathfrak{h} \) to \( \mathfrak{af}(V) \). This map is bijective, with the inverse given by sending \((v, f) \in \mathfrak{af}(V)\) to the morphism

\[
(v', c) \mapsto (f(v') + cv, 0).
\]

Lemma (I.12.1.11). Let \( \lambda, \mu \in k \) and \( x, y, z \in \mathfrak{g} \), then we have

\[
(ad(x) - \lambda - \mu)^m[y, z] = \sum_{i=1}^{m} \binom{m}{i} (ad(x) - \lambda)^i y, (ad(x) - \mu)^{m-i} z.
\]

Proof:

\[
(ad(x) - \lambda - \mu)^m[y, z] = \sum_{p+q+r+s=m} (-1)^{p+s} \frac{m!}{p!q!r!s!} \lambda^p \mu^s [(ad x)^p(y), (ad x)^q(z)]
\]

\[
= \sum_{k+l=m} \frac{m!}{k!l!} [(ad(x) - \lambda)^k y, (ad(x) - \mu)^l z]
\]

\(\square\)

Def. (I.12.1.12) (Killing Form). A bilinear form \( B \) on \( \mathfrak{g} \) is called invariant if \( B([x, y], z) + B(x, [y, z]) = 0 \).

The Killing form on a Lie algebra \( \mathfrak{g} \) of f.d. is the invariant symmetric bilinear form defined by \( B(x, y) = \text{tr}(\text{ad} x \circ \text{ad} y) \).

If \( a \) is an ideal of a Lie algebra \( \mathfrak{g} \), then the Killing form on \( a \) is that of the Killing form on \( a \) as a Lie algebra. This is linear algebra.

Prop. (I.12.1.13). Any invariant symmetric bilinear form on a simple Lie algebra \( \mathfrak{g} \) is a multiple of the Killing form.

Prop. (I.12.1.14). A subalgebra \( \mathfrak{h} \) of a Lie algebra \( \mathfrak{g} \) is commutative if it consists of semisimple elements.

Proof: For an element \( x \in \mathfrak{h} \), we need to show that \( \text{ad}_\mathfrak{h}(x) = 0 \). If it is not, because \( \text{ad}_\mathfrak{h}(x) \) is semisimple by (I.1.6.8), there is a nonzero eigenvector \( y \) at least after a base change, so \( [x, y] = cy, y \neq 0 \in \mathfrak{h} \). So \( \text{ad}(y)(x) = -cy \), and \( \text{ad}(y)^2(x) = 0 \), so \( \text{ad}(y) \) is non-semisimple on the subspace \( \{x, y\} \), which means \( \text{ad}(y) \) is non-semisimple on \( \mathfrak{g} \), by (I.1.6.8) again. \(\square\)
Lemma (I.12.1.15). If $g \subset g_{ln}$ is a Lie subalgebra, and $a \in g$ is a nilpotent matrix, then $\text{ad}(a)$ is also nilpotent.

Proof: This is because $\text{ad}(a) = l(a) - g(a)$, where $l(a)$ is left multiplication and $r(a)$ is right multiplication. The left and right multiplication commutes, so it is clear $\text{ad}(a) 2^n = 0$ if $\text{ad}(a)^n = 0$. □

Nilpotent and Solvable Lie Algebras

Def. (I.12.1.16) (Nilpotent and Solvable Lie Algebras). Let $g$ be a Lie algebra, the lower central series of $g$ is the descending sequence of ideals of $g$ defined inductively by $C^1g = g$ and $C^n g = [g, C^{n-1}g]$.

Let $g$ be a Lie algebra, the derived series of $g$ is the descending sequence of ideals of $g$ defined inductively by $D^1g = g$ and $D^n g = [D^{n-1}g, D^{n-1}g]$.

A Lie algebra is called nilpotent if there is an $n$ that $C^n g = 0$. This is equivalent to $\text{ad}x_1 \text{ad}x_2 \cdots \text{ad}x_n = 0$ for any $n$ element $x_1, \ldots, x_n$. It is called solvable if $D^n = 0$ for some $n$. It is clear that $D^n \subset C^n$, so nilpotent Lie algebra is solvable.

Prop. (I.12.1.17). The lower central series satisfies: $[C^ng, C^mg] \subset C^{m+n} g$.

The operation of taking derived series or lower central series commutes with base change of fields.

Proof: Prove by induction on $n$: $n = 0, 1$ is trivial, and if the assertion is true for $n \geq k$, then for $n = k + 1$, $[C^mg, C^{n+1}g] \subset [g, [C^{n-1}g, C^mg]] + [C^{n-1}g, C^{m+1}g] \subset C^{m+n} g$. □

Cor. (I.12.1.18). Let $g$ be a Lie algebra over a field $k$, and $k'/k$ is a field extension, then $g$ is solvable/nilpotent if $g \otimes_k k'$ is solvable/nilpotent.

Prop. (I.12.1.19). If $g$ is a nilpotent Lie algebra, then for any subalgebra $h \subset g$, $N_g(h) \neq h$.

Proof: Because $g^n = 0$ for some $n$, take $n$ to be the maximal one that $h \nsubseteq g^n$, the $[g^n, h] \subset g^{n+1} \subset h$, so $g^n \subset N_g(h)$, so $N_g(h) \neq h$. □

Prop. (I.12.1.20). Subalgebras, quotient algebras and extension algebras of solvable algebras are solvable.

Proof: Let $h \subset g$, then $D^n(h) \subset D^n(g)$, so if $g$ is solvable, then so is $h$. Also the quotient is clearly solvable. For an extension of Lie algebras

$$0 \to h \to g \to \tau \to 0,$$

if $h, \tau$ are both solvable, let $D^n(h) = 0, D^n(\tau) = 0$, then the image of $D^n(g)$ is $0$ in $\tau$, so $D^n(g) \subset h$, so $D^{m+n}(g) = 0$. □

Cor. (I.12.1.21) (Radical). If $a, b$ are solvable ideals of a Lie algebra $g$, then the ideal $a + b$ is also solvable, because $0 \to a \to a + b \to b / a \cap b$.

Let $r \subset g$ be the sum of all solvable ideals of $g$, called the radical $\text{Rad}(g)$. When $g$ is of f.d., this is the maximal solvable ideal.

Def. (I.12.1.22) (Semisimple Lie Algebra). A Lie algebra $L$ is called semisimple if $\text{Rad} L = 0$, or equivalently $g$ has no solvable ideals or no commutative ideals. Notice $L / \text{Rad}(L)$ is semisimple, by(I.12.1.21).
Prop. (I.12.1.23) (Lie). Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a solvable lie algebra over an alg.closed field $k$ of char0, then $\mathfrak{g}$ is upper triangulable. Equivalently, there exists a vector $v \in V$ which is a common eigenvector for all $X \in \mathfrak{g}$, and moreover equivalently, any irreducible representation of $\mathfrak{g}$ is 1-dimensional.

Proof: Idea is to prove by induction on dimension of $\mathfrak{g}$.

Produce a codimension 1 ideal $\mathfrak{h}$ of $\mathfrak{g}$. Let $\mathfrak{g}$ be generated (as a vector space) by $\mathfrak{h}$ and $Y$. Being a subalgebra of solvable algebra $\mathfrak{g}$, $\mathfrak{h}$ is itself a solvable lie algebra. Apply induction step on $\mathfrak{h}$ and choose $v \in V$ such that $v$ is an eigenvector for all $X \in \mathfrak{h}$.

The idea is to consider set $W$ all common eigenvectors of elements of $\mathfrak{h}$ and produce an eigenvector of $Y$ from this $W$. Let

$$W = \{v \in V | X(v) = \lambda(X)v \ \forall \ X \in \mathfrak{h} \ \text{for a fixed} \ \lambda(X) \in \mathbb{C}\}.$$ 

Suppose $W$ is an invariant subspace of $Y$, we then have restriction map $Y : W \rightarrow W$. As we are in complex vector space (algebraically closed) there exists an eigenvector for $Y$ in $W$ say $w_0$. Thus, $w_0$ is common eigenvector for all elements of $\mathfrak{g}$.

It remains to show that $W$ is an invariant subspace of $Y$ i.e., $Y(w) \in W$ for all $w \in W$ i.e., given $X \in \mathfrak{h}$, we need to have $X(Y(w)) = \lambda(X)Y(w)$.

Let $w \in W$, we have

$$X(Y(w)) = Y(X(w)) + [X,Y](w) = Y(\lambda(X)w) + \lambda([X,Y])w = \lambda(X)Y(w) + \lambda([X,Y])w$$

This is almost the same as what we want but with an extra term $\lambda([X,Y])w$. Suppose we prove $\lambda([X,Y])w = 0$ for all $X \in \mathfrak{h}$ then we are done.

Then considers subspace $U$ spanned by elements $\{w, Y(w), Y^2(w), \cdots \}$ and then says that $U$ is invariant subspace of each element of $\mathfrak{h}$ and (assuming $n$ is the smallest integer such that $Y^{n+1}w$ is in the subsapce generated by $\{w, Y(w), \cdots, Y^n(w)\}$) representation of an element $Z$ of $\mathfrak{h}$ with the basis $\{w, Y(w), \cdots, Y^n(w)\}$ is an upper triangular matrix with $\lambda(Z)$ in the diagonal. So, $tr(Z) = n\lambda(Z)$.

So, $tr([X,Y]) = n\lambda([X,Y])$. As $[X,Y] = XY - YX$, we have $tr([X,Y]) = tr(XY) - tr(YX) = 0$. Thus, $\lambda([X,Y]) = 0$ and we are done.

Cor. (I.12.1.24). If $\mathfrak{g}$ is a solvable algebra over an alg.closed field $k$ of char 0, then all irreducible representations of $\mathfrak{g}$ is of dimension 1.

Cor. (I.12.1.25). $\mathfrak{g}$ is a solvable algebra iff $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Proof: If $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, then clearly $\mathfrak{g}$ is solvable. Conversely, if $\mathfrak{g}$ is solvable, we need to prove $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. For this, we can assume $k$ is alg.closed, and then $ad(\mathfrak{g}) \subset \mathfrak{b}_g$ for some basis, thus $ad([\mathfrak{g}, \mathfrak{g}]) \subset n_g$ is nilpotent, and the kernel of $ad$ is an Abelian subalgebra, so $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Cor. (I.12.1.26). If $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(n,k)$ where $k$ is an alg.closed field of char0, then

$$\mathfrak{g} \text{ is solvable} \iff tr(xy) = 0, \ \forall x \in \mathfrak{g}, y \in [\mathfrak{g}, \mathfrak{g}].$$

Proof: Firstly if $\mathfrak{g}$ is solvable, then by Lie’s theorem, we can assume $\mathfrak{g} \subseteq \mathfrak{b}_V$ the upper-triangular matrices, so $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{n}_V$ is nilpotent, and so $xy \in \mathfrak{n}_V$ is also nilpotent, and $tr(xy) = 0$.

Conversely, if $tr(xy) = 0$ for all $x \in \mathfrak{g}, y \in [\mathfrak{g}, \mathfrak{g}]$, we only need to prove $[\mathfrak{g}, \mathfrak{g}]$ is solvable, so we may change $\mathfrak{g}$ to $[\mathfrak{g}, \mathfrak{g}]$ and assume $tr(xy) = 0$ for all $x, y \in \mathfrak{g}$. 


Now to show $\mathfrak{g}$ is solvable, it suffices to show $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, or by Engel’s theorem (I.12.1.28) all $x \in [\mathfrak{g}, \mathfrak{g}]$ defines a nilpotent endomorphism on $V$. Choose a basis that $x$ is upper-triangular by (I.1.6.3), and let $x_s$ be the semisimple part of $x$, then it suffices to show $x_s = 0$, or equivalently $\text{tr}(\tau x) = 0$. To show this, notice $x \in [\mathfrak{g}, \mathfrak{g}]$, so it suffices to show $\text{tr}(\tau [y, z]) = 0$ for any $y, z$. But this equals $\text{tr}([\tau y, z])$. Finally, this is 0 because of the hypothesis and the fact $\tau$ is a polynomial in $x$ (I.1.6.7) so $\tau [y, z]$ is nilpotent. □

Cor. (I.12.1.27) (Cartan’s Criteria for Solvability). A Lie algebra $\mathfrak{g}$ over a field of characteristic 0 is solvable if $\kappa(x, y)$ for any $x \in \mathfrak{g}, y \in [\mathfrak{g}, \mathfrak{g}]$, where $\kappa$ is the Killing form.

Proof: By (I.12.1.18), it suffices to show for $k$ alg. closed. Because the kernel of the adjoint map is the center of $\mathfrak{g}$, so $\mathfrak{g}$ is solvable iff $\text{ad}(\mathfrak{g})$ is solvable. □

Prop. (I.12.1.28) (Engel). If $(V, \rho)$ is a representation of a Lie algebra $\mathfrak{g}$ that $\rho(x)$ is nilpotent for all $x \in \mathfrak{g}$, then there is a basis that $\rho(\mathfrak{g})$ is contained in $\mathfrak{n}_V$, in particular $\mathfrak{g}$ is nilpotent.

Proof: It suffices to show if a sub-Lie algebra of $\mathfrak{gl}_V$ consists of nilpotent elements, then there is a common 0-eigenvector. Use Induction, choose a maximal subalgebra $K$ of $L$, then notice the normalizer of $K$ in $L$ is strictly containing $K$, because we can let $K$ acts by adjoint on $L/K$, and notice $\text{ad} x = \lambda x - \rho x$ is nilpotent for $x$ a nilpotent matrix, so by induction hypothesis there is an $x \in L$ that $[x, K] \subset K$. But $K$ is maximal, so it must be of codimension 1, and $L = K + Fz$. The 0-eigenvectors for $K$ is a nonzero subspace by induction hypothesis. Now this space is invariant under $z$: for any $h \in K$,

$$h(z(v)) = [h, z](v) + zh(v) = 0.$$ 

So now a 0-eigenvector for $z$ in this space will suffice. □

Cor. (I.12.1.29). If all elements of $L$ are ad-nilpotent (i.e. $\text{ad} x = 0$), then $L$ is nilpotent. Equivalently, elements of $L$ has a common 0-eigenvector.

Proof: Consider $\text{ad} : \mathfrak{g} \to \mathfrak{gl}_\mathfrak{g}$, then the image is nilpotent by Engel’s theorem (I.12.1.28), and the kernel of $\text{ad}$ is the center of $\mathfrak{g}$, so $\mathfrak{g}$ is also nilpotent. □

Cor. (I.12.1.30). Let $\mathfrak{a}$ be an ideal of a Lie algebra $\mathfrak{g}$. Then $\mathfrak{a}$ is nilpotent if $\text{ad}_\mathfrak{g}(a)$ is nilpotent for any $a \in \mathfrak{a}$.

Proof: If $\text{ad}_\mathfrak{g}(a)$ is nilpotent for any $a \in \mathfrak{a}$, then also $\text{ad}_\mathfrak{a}(a)$ is nilpotent, so $\mathfrak{a}$ is nilpotent by Engel’s theorem. Conversely, if $\mathfrak{a}$ is nilpotent, then $\text{ad}_\mathfrak{a}(a)$ is nilpotent for any $a \in \mathfrak{a}$. And $\text{ad}(a)(\mathfrak{g}) \subset \mathfrak{a}$, so $\text{ad}_\mathfrak{g}(a)$ is nilpotent. □

Cor. (I.12.1.31). The sum of two nilpotent ideals of $\mathfrak{g}$ is nilpotent.

Proof: We need to show for any $a \in \mathfrak{a}, b \in \mathfrak{b}$, $\text{ad}_\mathfrak{g}(a + b)$ is nilpotent. For this, we need to factor $\mathfrak{g}$ as Jordan sequence $0 \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \ldots \subset \mathfrak{g}_n = \mathfrak{g}$ over itself via the adjoint representation, then I claim that $\text{ad}(a)(\mathfrak{g}_k) \subset \mathfrak{g}_{k-1}$: Because $V = \mathfrak{g}_k/\mathfrak{g}_{k-1}$ is simple, let $V' \subset V$ consists of vectors $v$ that $a(v) = 0$ for any $a \in \mathfrak{a}$, then it is non-empty by Engel’s theorem, and also it is invariant under action of $\mathfrak{g}$: $a(gv) = [a, g](v) + g(a(v)) = 0$. So it is all of $V$.

Then, we have $\text{ad}(a + b)(\mathfrak{g}_k) \subset \mathfrak{g}_{k-1}$, thus $\text{ad}(a + b)$ is nilpotent. □

Cor. (I.12.1.32) (Maximal Nilpotent Ideal). For any Lie algebra $\mathfrak{g}$, there exists a maximal nilpotent ideal, denoted by $\mathfrak{n}$. 

Def. (I.12.1.33) (Nilpotent Radical). The nilpotent radical \( s = s(\mathfrak{g}) \) of a Lie algebra is the intersection of the kernels of simple representations of \( \mathfrak{g} \).

\( s \) is nilpotent for any f.d. representation of \( \mathfrak{g} \), in particular the adjoint representation of \( \mathfrak{g} \). Thus it is nilpotent, by(I.12.1.30).

Lemma (I.12.1.34). Let \( \mathfrak{g} \subset \mathfrak{gl}_V \) be a subalgebra, and let \( \mathfrak{a} \) a commutative ideal of \( \mathfrak{g} \). If \( V \) is simple as a \( \mathfrak{g} \)-module, then \( [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{a} = 0 \).

Proof: Cf.[Mil13]P58. \( \square \)

Prop. (I.12.1.35). Let \( \mathfrak{g} \) be a Lie algebra, and \( \mathfrak{r} \) its radical, \( s \) it nilpotent radical, then

\[ s = D(\mathfrak{g}) \cap \mathfrak{r} = [\mathfrak{g}, \mathfrak{r}] \]

In particular, \([\mathfrak{g}, \mathfrak{r}]\) is nilpotent(I.12.1.33).

Proof: To show \( D(\mathfrak{g}) \cap \mathfrak{r} \subset \mathfrak{s} \), we need to show that \( \rho(D(\mathfrak{g}) \cap \mathfrak{r}) = 0 \) for any simple representation \( \rho \). Because \( \mathfrak{r} \) is solvable, let \( r \) be the smallest integer that \( \rho(D^{r+1}(\mathfrak{r})) = 0 \), then \( \mathfrak{a} = \rho(D^r(\mathfrak{r})) \) is a commutative ideal of \( \rho(\mathfrak{g}) \). Hence by(I.12.1.34) \( D(\rho(\mathfrak{g})) \cap \mathfrak{a} = 0 \), so \( \rho(D(\mathfrak{g}) \cap D^r(\mathfrak{r})) = 0 \). Now if \( r > 0 \), then \( \rho(D^r(\mathfrak{r})) = 0 \), contradicting the minimality of \( r \), so \( \rho(D(\mathfrak{g}) \cap \mathfrak{r}) = 0 \).

To show \( s \subset [\mathfrak{g}, \mathfrak{r}] \), let \( \mathfrak{q} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{r}] \), and \( f \) the quotient map. Then because the kernel is solvable, \( f(\mathfrak{r}) \) is the radical of \( \mathfrak{q}(I.12.21) \), but it is also contained in the center of \( \mathfrak{q} \), so \( \mathfrak{q} \) is reductive, and thus has a faithful semisimple representation(I.12.4.4), then the kernel of this representation is just \( [\mathfrak{g}, \mathfrak{r}] \), showing that \( s \subset [\mathfrak{g}, \mathfrak{r}] \). \( \square \)

Def. (I.12.1.36) (Levi Subalgebras). Let \( \mathfrak{g} \) be a Lie algebra and \( \mathfrak{r} \) its radical, \( s \) it nilpotent radical, then a Lie subalgebra \( \mathfrak{s} \) is called a Levi subalgebra if \( \mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s} \).

Prop. (I.12.1.37) (Levi-Malcev). Every Lie algebra over a field \( k \) of char0 has a Levi subalgebra, and any two Levi subalgebras of \( \mathfrak{g} \) are conjugate by a special automorphism of \( \mathfrak{g}(I.12.1.6) \).

Proof: If \( \mathfrak{g} \) is reductive, then Levi subalgebras exists uniquely, by(I.12.4.2) and(I.12.4.3).

if \( \mathfrak{r} \) is a minimal ideal of \( \mathfrak{g} \), then \([\mathfrak{g}, \mathfrak{r}] = \mathfrak{r} \), and \([\mathfrak{r}, \mathfrak{r}] = 0 = Z(\mathfrak{g}) \). Consider the adjoint action of \( \mathfrak{g} \) on \( \text{End}_k(\mathfrak{g}) \). Also consider the subspaces \( V, W \) of \( \text{End}_k(\mathfrak{g}) \), where \( V \) is the subspace of maps from \( \mathfrak{g} \) to \( \mathfrak{r} \) that restriction to \( \mathfrak{r} \) is a constant multiple of identity, and \( W \) is the subspaces of \( W \) consisting of maps vanishing on \( \mathfrak{r} \). Both of \( V, W \) are invariant under action of \( \mathfrak{g} \).

Let \( \varphi : \mathfrak{r} \to \mathfrak{g} \) be the adjoint action, which is injective and has image \( P \subset W \). Also \( \varphi \) is invariant under action of \( \mathfrak{g} \)(because \( \mathfrak{r} \) is an ideal).

For \( x \in \mathfrak{r}, y \in \mathfrak{g}, \alpha \in V, (x\alpha)(y) = [x, \alpha(y)] − \alpha([x, y]) = −\lambda(\alpha)[x, y], \) so \( x\alpha = \text{ad}(\lambda(\alpha)x) \), which means elements of \( \mathfrak{r} \) map \( V \) into \( P \). Thus now \( \mathfrak{r} \) acts trivially on \( V/P \), and \( W/P \) is invariant under the action of \( \mathfrak{g}/\mathfrak{r} \), which is a semisimple Lie algebra. Thus by Weyl’s theorem(III.6.1.2), there exists a \( \mathfrak{g} \)-stable line \( L \) that \( V/P \) is spanned by \( L \). But \( \mathfrak{g} \) acts trivially on \( L \) by(III.6.1.1).

Let \( \alpha_0 \) generates \( L \) and normalized that \( \lambda(\alpha) = 1, \) then \( \mathfrak{g}\alpha_0 \subset P \). We consider the map \( \mathfrak{g} \to \mathfrak{g}\alpha_0 \) \( P \xrightarrow{-1} \mathfrak{r} \), whose restriction to \( \mathfrak{r} \) is the identity map, so its kernel is a Levi subgroup for \( \mathfrak{r} \).

Still in this case, let \( \mathfrak{s}' \) be a second Levi subgroup for \( \mathfrak{r} \). For each \( x \in \mathfrak{s}' \), there is a unique \( h(x) \in \mathfrak{r} \) that \( x + h(x) \in \mathfrak{s} \). Hence this \( h \) satisfies \( h([x, y]) = [h(x), y] + [x, h(y)] \). But then by(III.6.1.3), \( h(x) = [a, x] \) for some \( a \in \mathfrak{r} \). Then \( 1 + \text{ad}(a) \) maps \( \mathfrak{s} \) to \( \mathfrak{s}' \), and \( \text{ad}(a)^2 = 0 \) because \( \mathfrak{r} \) is commutative. And \( \mathfrak{r} = [\mathfrak{r}, \mathfrak{g}] \), so \( a \) is in the nilpotent radical of \( \mathfrak{g} \), thus \( \mathfrak{s}, \mathfrak{s}' \) are conjugate by a special automorphism.
For the general case, we can use induction on the dimension of \( g \). After the first two cases, we can assume that \( [g, r] \neq 0 \) and \( r \) contains a proper non-trivial ideal. As \( [g, r] \) is nilpotent (I.12.1.35), its center is non-zero. So we can choose a maximal ideal \( m \) contained in the center of \( [g, r] \), and \( m \neq r \). Now \( g/m \) has radical \( r/m \) because \( m \) is solvable, so we can apply induction to find a Levi subgroup \( h' \) for \( g/m \) in \( g/m \), and let \( h'' \) be its preimage in \( g \). Then \( h'' \) has radical \( m \), and thus by induction there is a Levi subgroup \( h' \) for \( m \) in \( h'' \), and let \( h'' \) be its preimage in \( g \). Similarly any two such Levi subgroups are conjugate by a special automorphism.

This has another cohomological proof in [Etingof, Section48].

2 Semisimple Lie Algebra

Prop. (I.12.2.1) (Cartan-Killing Criteria for Semisimplicity). A f.d. Lie algebra \( g \) is semisimple iff its Killing form (I.12.1.12) is non-degenerate.

Proof: If \( g \) is semisimple, then the adjoint representation is faithful, thus by (I.12.9.6) the Killing form is non-degenerate. Conversely, if the Killing form is non-degenerate and \( a \) is a commutative ideal of \( g \) and \( a \in a, g \in g \), then \( (\text{ad} a \circ \text{ad} g)^2 = 0 \), so \( \text{ad} a \circ \text{ad} g \) is nilpotent and has trace 0. so \( a \in g^\perp \), which is 0 because the Killing form is non-degenerate.

Cor. (I.12.2.2). Let \( a \) be a semisimple ideal of a Lie algebra \( g \), then the orthogonal space \( a' \) w.r.t the Killing form is an ideal and is a complement for \( a \) in \( g \), and \( g \cong a \times a' \).

Proof: \( a^\perp \) is an ideal because the Killing form is invariant. The Killing form of \( a \) is the restriction of the Killing form of \( g \) (I.12.1.12), so \( a \cap a' = 0 \) because \( \kappa_a \) is non-degenerate.

Cor. (I.12.2.3). A Lie algebra is semisimple iff it is isomorphic to a product of simple algebras \( g = a_1 \times \cdots \times a_r \), and these \( a_i \) are all its minimal ideals, (Not only up to isomorphism).

Proof: The Killing form of \( a \cap a^\perp \) is 0, thus it is solvable, by (I.12.1.27), and then 0, so we can continue the decomposition.

For any minimal nonzero ideal \( a \subset g \), then \( [g, a] \) is an ideal contained in \( a \). which is nonzero because \( g \) has trivial center. Then

\[
    a = [g, a] = \bigoplus_i [a, a_i]
\]

so \( a \subset [a, a_i] \subset a_i \), and then \( a = a_i \) by simplicity.

Cor. (I.12.2.4). If \( g \) is semisimple, then \( [g, g] = g \).

Cor. (I.12.2.5). Let \( g \) be a Lie algebra over a field \( k \) and \( k'/k \) is a field extension, then \( g \) is semisimple iff \( g \otimes_k k' \) is semisimple.

Prop. (I.12.2.6) (Examples of Semisimple Lie Algebras).

- The subalgebra \( \mathfrak{sl}(V) \) of all elements of \( \text{End}(V) \) of trace 0 is semisimple.

Prop. (I.12.2.7). If \( L \) is semisimple, then every derivative of \( L \) is inner.

Proof: This is a special case of (III.6.1.3) applied to the adjoint representation.

Prop. (I.12.2.8). If \( g \) is a semisimple algebra of \( \mathfrak{gl}_n(k) \) where \( k \) is a field of char0, then it contains the semisimple and nilpotent parts of each of its elements under the Jordan decomposition (I.1.6.7).
Proof: We may assume $k$ is alg.closed, because the Jordan decomposition is invariant of the field that contains $\mathfrak{g}$, and an element is contained is a vector space can be checked after base change to a larger field. For any subspace $W \subset V$, let $\mathfrak{g}_W = \{ \alpha \in \mathfrak{gl}_V | \alpha(W) \subset W, \text{tr}(\alpha|_W) = 0 \}$, then if $\mathfrak{g}W \subset W$, then $\mathfrak{g} \subset \mathfrak{g}_W$, because every element of $\mathfrak{g}$ is a sum of brackets by (I.12.2.4), thus have zero trace. Now consider

$$ \mathfrak{g}' = n_{\mathfrak{gl}_V} (\mathfrak{g}) \bigcap_{\mathfrak{g}W \subset W} \mathfrak{g}_W. $$

If $x \in \mathfrak{g}'$, then so does $x_s$ and $x_n$, because they are polynomials in $x$ without constant terms, and $\text{ad}(x)_s = \text{ad}(x_s), \text{ad}(x)_n = \text{ad}(x_n)$.

So it suffices to show that $\mathfrak{g} = \mathfrak{g}'$. We claim that $\mathfrak{g}' = \mathfrak{g}$: As $\mathfrak{g}$ is a semisimple ideal of $\mathfrak{g}'$, by (I.12.2.2), we have a decomposition

$$ \mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{g}^\perp. $$

Let $\alpha \in \mathfrak{g}^\perp$ and $W$ a simple $\mathfrak{g}$-module of $V$, then $\alpha$ acts on $W$ as a scalar, which must be $0$ because $\alpha \in \mathfrak{g}_W$ and $k$ has char $0$. As $W$ is a sum of simple $\mathfrak{g}$-modules by Weyl’s theorem (III.6.1.2), we get the desired conclusion.

Prop. (I.12.2.9) (Abstract Jordan Decomposition). A semisimple/nilpotent element $x$ in a Lie algebra $\mathfrak{g}$ is an element that $\rho(x)$ is semisimple/nilpotent for any representation $\rho$ of $\mathfrak{g}$. And $x = x_s + x_n$ is called a Jordan decomposition iff $\rho(x) = \rho(x_s) + \rho(x_n)$ is a Jordan decomposition (I.1.6.7) for any representation $\rho$ of $\mathfrak{g}$.

Every element of a semisimple Lie algebra $\mathfrak{g}$ over a field of characteristic $0$ has a unique Jordan decomposition, and $x = x_s + x_n$ is a Jordan decomposition if $\rho(x) = \rho(x_s) + \rho(x_n)$ is a Jordan decomposition for one faithful representation $\rho$ of $\mathfrak{g}$. In particular, this is holds for the adjoint representation $\text{ad}$.

Proof: Let $x \in \mathfrak{g}$ and $(V, \rho)$ a faithful representation of $\mathfrak{g}$ (for example the adjoint representation), then there is at most one $x = x_s + x_n$ that $\rho(x) = \rho(x_s) + \rho(x_n)$ is the Jordan decomposition, which proves the uniqueness.

Now for any $x \in \mathfrak{g}$, as (I.12.2.8) shows, there do exist these two elements that $\rho(x) = \rho(x_s) + \rho(x_n)$. But then it can be checked directly $\text{ad}(x) = \text{ad}(x_s) + \text{ad}(x_n)$ is the Jordan decomposition of $\text{ad}(x)$ as an endomorphism of $\mathfrak{g}$, by (I.12.1.15). As the adjoint representation is faithful, this shows the Jordan decomposition is independent of the faithful representation chosen.

Now every representation is a subrepresentation of a faithful representation, so we can prove the existence. 

Recovering Semisimple Lie Algebras from a Dynkin Diagram

Main references are [Car05] Chap7.

Simple Lie Algebras

Main references are [Car05] Chap8.

Def. (I.12.2.10) (Lie Algebras of Type $A_l$).

Def. (I.12.2.11) (Lie Algebras of Type $A_1$). $A_1$ is also called $\mathfrak{sl}_2(\mathbb{C})$. It has a basis $f, h, e$ with

$$ [he] = 2e, \quad [hf] = -2f, \quad [ef] = h. $$
In matrix form,
\[ h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \]

It can also be realized by
\[ H = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad R = \frac{1}{2} \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}, \quad L = \frac{1}{2} \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}. \]

These two representation differ by a conjugation by the Cayley transformation \( C = -\frac{1+i}{2} \begin{bmatrix} i & 1 \\ i & -1 \end{bmatrix} \).

It is simple by \( ? \). The subalgebra \( \mathfrak{b} \) generated by \( X \) and \( H \) is solvable, called the canonical Borel subalgebra of \( \mathfrak{sl}_2 \).

**Proof:**

Prop. (I.12.2.12) (Lie Algebras of Type \( B_l \)).

Prop. (I.12.2.13) (Lie Algebras of Type \( C_l \)).

Prop. (I.12.2.14) (Lie Algebras of Type \( D_l \)).

Prop. (I.12.2.15) (\( G_2 \)).

Prop. (I.12.2.16) (\( F_4 \)).

Prop. (I.12.2.17) (\( E_8 \)).

Prop. (I.12.2.18) (Classification of F.D. Complex Simple Lie Algebras).


Prop. (I.12.2.20). \( \mathfrak{sl}_2(k) \) is simple if \( k \) has characteristic \( \not= 2 \).

### 3 Cartan Subalgebras

Lie algebras in this subsection are assumed to be of f.d..

**Def. (I.12.3.1) (Cartan Subalgebras).** A **Cartan subalgebra** of a Lie algebra \( \mathfrak{g} \) is a nilpotent subalgebra \( \mathfrak{h} \) that equals to its own normalizer in \( \mathfrak{g} \).

**Remark (I.12.3.2).** Recall that a proper subalgebra of a nilpotent algebra is never its own normalizer(I.12.1.19), so a Cartan subalgebra is a maximal nilpotent subalgebra, but a maximal nilpotent subalgebra may not be a Cartan subalgebra.

If \( k'/k \) is a field extension, then \( \mathfrak{h} \) is a Cartan subalgebra of \( \mathfrak{g} \) iff \( \mathfrak{h}_{k'} \) is a Cartan subalgebra of \( \mathfrak{g}_{k'} \). This is because being nilpotent and the normalizer is also compatible with base change(I.12.1.4).

**Prop. (I.12.3.3) (Diagonal Cartan Matrices).** Let \( \mathfrak{g} \subset \mathfrak{gl}_V \) be a subalgebra containing a diagonal matrix \( a = \text{diag}(a_1, \ldots, a_n) \) with distinct \( a_i \), and let \( \mathfrak{h} \) be the subspace of all diagonal matrices in \( \mathfrak{g} \), then \( \mathfrak{h} \) is a Cartan subalgebra of \( \mathfrak{g} \).
Proof: Firstly \( h \) is Abelian, and if \( b = \sum b_{ij} e_{ij} \in N_\mathfrak{g}(h) \), then \([a, b] \in h\). But

\[
[a, b] = \sum_{ij} (a_{ii} - a_{jj}) b_{ij} e_{ij}
\]

is in \( h \) iff \( b \) is diagonal, or \( b \in h \), so \( h = N_\mathfrak{g}(h) \) and \( h \) is a Cartan subalgebra. \( \square \)

Def. (I.12.3.4) (Regular Elements). Let \( \mathfrak{g} \) be a f.d. Lie algebra, for any \( x \in \mathfrak{g} \), let \( P_x(T) \) be the characteristic polynomial of \( \text{ad}(x) \):

\[
P_x(T) = \det(T - \text{ad}(x)) = T^m + a_{m-1}(x)T^{m-1} + \ldots + a_0(x).
\]

Then the **rank** of \( \mathfrak{g} \) is the minimal \( n \) that \( n(x) \neq 0 \) for some \( x \in \mathfrak{g} \). A **regular element** is an element \( x \in \mathfrak{g} \) that \( a_n(x) \neq 0 \).

Prop. (I.12.3.5) (Regular Elements and Cartan Subalgebras). For any regular element \( x \in \mathfrak{g} \), the nilspace \( \mathfrak{g}_x^0 \) is a Cartan subalgebra of \( \mathfrak{g} \).

Proof: Let

\[
U_1 = \{ y \in \mathfrak{g}_x^0 \mid \text{ad}_\mathfrak{g}(y)|_{\mathfrak{g}_x^0} \text{ is not nilpotent} \},
\]

\[
U_2 = \{ y \in \mathfrak{g}_x^0 \mid \text{ad}_\mathfrak{g}(y)|_{(\mathfrak{g}/\mathfrak{g}_x^0)} \text{ is invertible} \}.
\]

They are both Zariski open subsets of \( \mathfrak{g}_x^0 \). According to Engel’s theorem (I.12.1.28), to show \( \mathfrak{g} \) is nilpotent, it suffices to show that \( U_1 \) is empty. \( U_2 \) is non-empty because it contains \( x \), so if \( U_1 \) is non-empty, \( U_1 \cap U_2 \) is non-empty and there is a \( y \in U_1 \cap U_2 \). For this \( y \), \( n(y) < \dim \mathfrak{g}_x^0 = n(x) \), contradicting the regularity of \( x \).

It remains to show that \( \mathfrak{g}_x^0 \) is its own normalizer. If \( z \) normalizes \( \mathfrak{g}_x^0 \), then \([z, x] \in \mathfrak{g}_x^0 \), which means \((\text{ad}(x))^n[z, x] = 0\), so \( \text{ad}(x)^{n+1}(z) = 0 \), thus \( z \in \mathfrak{g}_x^0 \). \( \square \)

Cor. (I.12.3.6) (Cartan Subalgebras Exist). Let \( \mathfrak{g} \) be a Lie algebra over an infinity field \( k \) contains some Cartan subalgebra, and when \( k \) is alg.closed, all Cartan subalgebras come from some regular element, by (I.12.3.13).

Proof: Regular elements exist because \( k \) is infinite. \( \square \)

Cor. (I.12.3.7). Every Lie algebra over an infinite field is a sum of Cartan subalgebras.

Proof: This is because the sum of Cartan subalgebras is a vector space thus Zariski closed but it contains all regular elements, which is a Zariski open subset. \( \square \)

Cor. (I.12.3.8). Let \( \mathfrak{a} \) be a subalgebra of a Lie algebra \( \mathfrak{g} \) that \( \text{ad}_\mathfrak{g}(a) \) is semisimple for any \( a \in \mathfrak{a} \), then \( \mathfrak{a} \) is contained in a Cartan subalgebra of \( \mathfrak{g} \).

Proof: Cf.[Mil13]P81.\?

Prop. (I.12.3.9). Let \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{g} \) over an alg.closed field \( k \). Consider the generalized eigenvalue decomposition (I.12.3.12), if \( x \in \mathfrak{g}^\alpha \), then \( \text{ad}(x)(\mathfrak{g}^\beta) \in \mathfrak{g}^{\alpha+\beta} \) (I.12.1.11), and thus \( \text{ad}(x) \) is nilpotent. Let \( E(\mathfrak{h}) \) be the subgroup of the group of elementary automorphisms (I.12.1.6) of \( \mathfrak{g} \) generated by the set of all the automorphisms \( e^{\text{ad}_\mathfrak{g}(x)} \), where \( x \in \mathfrak{g}^\alpha \) for some \( \alpha \in \mathfrak{h}^* \)\( \setminus \{0\} \).

Now let \( \mathfrak{h}, \mathfrak{h}' \) be two Cartan subalgebras of \( \mathfrak{g} \), then there exists \( u \in E(\mathfrak{h}), u' \in E(\mathfrak{h}') \) that \( u(\mathfrak{h}) = u'(\mathfrak{h}') \).
Proof: Number the elements of \( h^* \setminus 0 \) as \( \alpha_1, \ldots, \alpha_n \), and consider the map
\[
f: g^{\alpha_1} \times \ldots \times g^{\alpha_n} \times h \to g: (x_1, \ldots, x_n, h) \mapsto e^{ad(x_1)} \ldots e^{ad(x_n)} h.
\]
Given a \( h_0 \in h \), it can be shown that \( (df)_{|_{(0, \ldots, 0, h_0)}} \). Thus if we choose a regular \( h_0 \in h \), then \( E(h)h_r \) contains a dense open subset of \( g \). Similarly, \( E(h'h)_r' \) contains a dense open subset of \( g \). So their intersection is not empty, i.e. \( u(h) = u'(h') \) for some \( u, u', h, h' \). Now
\[
u(h) = u(g^0_h) = g^0_{u(h)} = g^0_{u'(h')} = u'(g^0_{h'}) = u'(h').
\]
\( \square \)

Cor. (I.12.3.10). All Cartan subalgebras in a Lie algebra have the same dimension, which is the rank of \( g \). (I.12.3.4).

Proof: Because we can take a base change to an alg.closed field, under which a Cartan subalgebra is also a Cartan subalgebra by (I.12.3.2), then they have the same rank. \( \square \)

Cor. (I.12.3.11) (Cartan Subalgebras are Conjugate). Any two Cartan subalgebras of a f.d. Lie algebra over an alg.closed field \( k \) are conjugate by an elementary automorphism (I.12.1.6).

### Cartan Subalgebra of Semisimple Lie Algebras

**Lemma (I.12.3.12) (Decomposition w.r.t. a Cartan Subalgebra).** Let \( h \) be a Cartan subalgebra of a semisimple Lie algebra \( g \), and assume that
\[
g = h \oplus \bigoplus_{\alpha \in h^* \setminus 0} g^\alpha.
\]
This is true, for example, when \( k \) is alg.closed, by (I.12.9.10). (See (I.12.3.21) for when this decomposition is possible).

**Cor. (I.12.3.13).** If \( k \) is alg.closed, the set \( h_r \) of regular elements \( h \) in \( h \) that \( g^0_h = h \) is open and dense in \( h \) in the Zariski topology.

**Proof:** The condition is equivalent to \( \prod_{\alpha \in h^* \setminus 0} \alpha(h) \neq 0 \), which is an open condition. \( \square \)

**Lemma (I.12.3.14).** In the decomposition above, if \( \alpha + \beta \neq 0 \), then \( g^\alpha \) and \( g^\beta \) is orthogonal w.r.t. the Killing form.

**Proof:** \( \text{ad}(x) \text{ad}(y) g^\gamma \subset g^{\alpha + \beta + \gamma} \), so if \( \alpha + \beta \neq 0 \), then \( \text{ad}(x) \text{ad}(y) \) is nilpotent, thus \( \kappa(x, y) = 0 \). \( \square \)

**Prop. (I.12.3.15) (Cartan Subalgebras of Semisimple Lie Algebras).** Let \( h \) be a Cartan subalgebra of a semisimple Lie algebra \( g \), then
- Every element of \( h \) is semisimple. In particular, \( h \) is commutative (I.12.1.14).
• The centralizer of $\mathfrak{h}$ in $\mathfrak{g}$ is $\mathfrak{h}$.
• The restriction of the Killing form to $\mathfrak{h}$ is non-degenerate.

Proof: It suffices to prove this after $k$ is replaced by its alg.closure, so the generalized eigenvalue decomposition (I.12.3.12) holds. We prove 3 first: by (I.12.3.14), $\mathfrak{h}$ is orthogonal to all $[\mathfrak{h}, x]$ for any $x$. But if $x \in \mathfrak{g}^a$, we can see that $[\mathfrak{h}, x] = \mathfrak{g}^a$, so $\mathfrak{h}$ is orthogonal to all $\bigoplus \mathfrak{g}^a$, so $\kappa$ must be non-degenerate on $\mathfrak{h}$.

Because $\mathfrak{g}$ has trivial center, the adjoint representation realizes $\mathfrak{h}$ as a subalgebra of $\mathfrak{gl}_\mathfrak{g}$, and Lie’s theorem (I.12.1.23) shows there is a basis that $\mathfrak{h} \subset \mathfrak{b}_\mathfrak{g}$, hence $\text{ad}([\mathfrak{h}, \mathfrak{h}]) \subset \mathfrak{n}_\mathfrak{g}$, and so $\text{tr}(\mathfrak{h}, [\mathfrak{h}, \mathfrak{h}]) = 0$. As $\kappa$ is non-degenerate on $\mathfrak{h}$, $[\mathfrak{h}, \mathfrak{h}] = 0$, thus $\mathfrak{h}$ is commutative. Now $\mathfrak{h} \subset c_\mathfrak{g}(\mathfrak{h}) \subset N_\mathfrak{g}(\mathfrak{h})$, thus $\mathfrak{h} = c_\mathfrak{g}(\mathfrak{h})$.

If $x \in \mathfrak{h}$, and $x = x_s + x_n$ is the Jordan decomposition (I.12.2.9), then $\text{ad}(x_n)$ are polynomials of $\text{ad}(\mathfrak{g})$ thus lies in $\mathfrak{h}$. Now $\text{ad}(x_n)$ commutes with all $\text{ad}(y)$ for $y \in \mathfrak{h}$, thus $\text{ad}(y)\text{ad}(x_n)$ is nilpotent, thus $\kappa(y, x_n) = 0$. Thus $x_n = 0$ as $\kappa$ is non-degenerate on $\mathfrak{h}$.

Cor. (I.12.3.16). The Cartan subalgebras of a semisimple Lie algebra are the maximal subalgebras consisting of semisimple elements (I.12.2.9).

Proof: A subalgebra consisting of semisimple elements is contained in a Cartan subalgebra, by (I.12.3.8).

Conversely, if $\mathfrak{h} \subset \mathfrak{h}'$ and $\mathfrak{h}$ is a Cartan subalgebra and $\mathfrak{h}'$ consists of semisimple elements, then $\text{ad}(\mathfrak{h})$ is commutative, and thus $\mathfrak{h}' \subset c_\mathfrak{g}(\mathfrak{h})$, so $\mathfrak{h} = \mathfrak{h}'$.

Cor. (I.12.3.17). Every regular element is semisimple, because it is contained in a Cartan subalgebra by (I.12.3.5).

Split Semisimple Lie Algebras

Def. (I.12.3.18) (Split Semisimple Lie Algebras). A split Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$ over a field $k$ is a Cartan subalgebra that all the eigenvalues of the linear maps $\text{ad}(\mathfrak{h})$ lies in $k$ for all $h \in \mathfrak{h}$. A split semisimple Lie algebra is a pair $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is semisimple and $\mathfrak{h}$ is a split Cartan subalgebra.

Remark (I.12.3.19). For example, the diagonal matrices in $\mathfrak{sl}_n$ is a splitting Cartan subalgebra over any field.

$\mathfrak{sl}_2(\mathbb{R})$ has a non-split Cartan subalgebra $\left\{ \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} \mid a \in \mathbb{R} \right\}$.

Prop. (I.12.3.20). Let $\alpha$ be a root of the split semisimple Lie algebra $(\mathfrak{g}, \mathfrak{h})$, then

• The subspace $\mathfrak{g}^\alpha$ and $\mathfrak{h}_\alpha = [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ are both 1-dimensional.
• There is a unique element $h_\alpha \in \mathfrak{h}_\alpha$ such that $\alpha(h_\alpha) = 2$.
• For each nonzero $x_\alpha \in \mathfrak{g}^\alpha$, there is a $y_\alpha \in \mathfrak{g}^{-\alpha}$ such that $[x_\alpha, y_\alpha] = h_\alpha$, $[h_\alpha, x_\alpha] = 2x_\alpha$, $[h_\alpha, y_\alpha] = -2y_\alpha$.

Proof: Define $\mathfrak{h}_\alpha = [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] \subset \mathfrak{h}$. Because the Killing form is non-degenerate on $\mathfrak{h}$, we can define for each $\alpha \in R$ a unique element $h_\alpha \in \mathfrak{h}$ that $\alpha(h) = \kappa(h, h_\alpha)$ for all $h \in \mathfrak{h}$. Then $h_\alpha$ is the subspace spanned by $h_\alpha$: This is because for $x \in \mathfrak{g}^\alpha$, $y \in \mathfrak{g}^{-\alpha}$, $\kappa(h, [x, y]) = \kappa([h, x], y) = \alpha(x)\kappa(x, y)$.
so \([x, y] = \kappa(x, y)h^\alpha\). Combine this with the fact \(\kappa(h^\alpha, h^{-\alpha}) \neq 0\), we get the fact \(h_\alpha = kh^\alpha\) is 1-dimensional.

Next, there is a unique element \(h_\alpha\) that \(\alpha(h_\alpha) = 0\). For this, it suffices to show that \(\alpha\) doesn’t vanish on \(h_\alpha\). Otherwise let \(x \in g^\alpha\) and \(y \in g^{-\alpha}\) that \([x, y] = h\neq 0\), then \([h, x] = \alpha(h)x = 0 = [h, y]\). So \(\{x, y, h\}\) spans a solvable subalgebra \(a\) of \(g\). As \(h \in [a, a]\). By Lie’s theorem, \(\rho(h)\) is nilpotent for any representation \(\rho\) of \(a\). But \(h\) is in the Cartan subalgebra so \(ad_g(h)\) is semisimple(1.I.2.3.15), contradiction.

Because \(x_\alpha \neq 0\), there exists a unique \(y_\alpha \in g^{-\alpha}\) that \([x_\alpha, y_\alpha] = h_\alpha\). Now \([h_\alpha, x_\alpha] = \alpha(h_\alpha)x_\alpha = 2x_\alpha, [h_\alpha, y_\alpha] = -\alpha(h_\alpha)y_\alpha = -2y_\alpha\).

\[\text{Prop. (I.12.3.21) (Root Decompositions). If } h \text{ is a split Cartan subalgebra, then } ad(h) \text{ is a commuting family of semisimple endomorphisms with eigenvalues in } k(1.I.2.3.15). \text{ so the decomposition}\]

\[g = h \oplus \bigoplus_{\alpha \in R} g^\alpha\]

holds as in(I.12.3.12), where \(R = R(g, h)\) is the set of roots of \((g, h)\). And it is in fact an eigenvalue decomposition, not only generalized eigenvalue decomposition.

Then \(R\) is a reduced root system(I.14.1.6) in \(h^\vee\).

\[\text{Proof: Firstly } R \text{ spans } h^\vee: \text{ if } h \in h \text{ lies in the center of all } \alpha \in R, \text{ then } [h, g^\alpha] = 0 \text{ for all } \alpha \in R, \text{ and as } [h, h] = 0, \text{ this means } h \text{ is in the center of } g, \text{ which is trivial, so } h = 0. \text{ So } R \text{ spans } h^\vee. \]

\[\text{Prop. (I.12.3.22) (Criterion of Semisimplicity). Let } g \text{ be a Lie algebra and } h \text{ a commutative Lie subalgebra. If}\]

\[\bullet \text{ there is a decomposition } g = h \oplus \bigoplus_{\alpha \in R} g^\alpha, \]

where \(R \in h^\vee\) is the finite set of \(\alpha \in h^\vee \setminus \{0\}\) that \(g^\alpha \neq 0\). \(\dim g^\alpha = 1\) for all \(\alpha \in R\).

\[\bullet \text{ If } \alpha \in R, \text{ then } -\alpha \in R, \text{ and } [[g^\alpha, g^{-\alpha}], g^\alpha] \neq 0.\]

Then \(g\) is semisimple and \(h\) is a split Cartan subalgebra of \(g\).

\[\text{Proof: Cf.[Mil13]P90.}\]

\[\text{Prop. (I.12.3.23) (Criterion of Simplicity). Let } (g, h) \text{ be a split semisimple Lie algebra. A decomposition } g = g_1 \oplus g_2 \text{ of Lie algebras defines a decomposition } (g, h) = (g_1, h_1) \oplus (g_2, h_2), \text{ and hence a decomposition of the root system } R(g, h)\]

In particular, if the root system \(R(g, h)\) is indecomposable, then \(g\) is simple.

\[\text{Proof: Let } g = h \oplus \bigoplus_{\alpha \in R} g^\alpha, \quad g_1 = h_1 \oplus \bigoplus_{\alpha \in R_1} g_1^\alpha, \quad g_2 = h_2 \oplus \bigoplus_{\alpha \in R_2} g_2^\alpha\]

be the eigenvalue decomposition of \(g, g_1, g_2\) w.r.t. the adjoint action of \(h\), then \(h = h_1 \oplus h_2\), and \(R = R_1 \coprod R_2\).

\[\text{Prop. (I.12.3.24) (Splitting Cartan Subalgebras are Conjugate). The group of elementary automorphisms of } g \text{ acts transitively on the set of pairs } (b, h) \text{ consisting of a Borel subalgebra and a splitting Cartan subalgebra of } g.\]

\[\text{Proof: Cf.[Mil13]P98.}\]
4 Reductive Lie Algebra

Def. (I.12.4.1) (Reductive Lie Algebra). A Lie algebra is called reductive if \( \text{rad}(L) = Z(L) \), or equivalently \( Z(L) \subset \text{rad}(L) \).

Prop. (I.12.4.2). The following conditions on a Lie algebra \( g \) are equivalent:

- \( g \) is reductive.
- The adjoint representation of \( g \) is semisimple.
- \( g \) is a product of a commutative Lie algebra \( c \) and a semisimple Lie algebra \( b \).

Proof: 1 \( \rightarrow \) 2: The adjoint representation factors through the center of \( g \), which is also the radical of \( g \), so it is a representation of \( g/\text{rad}(g) \), which is semisimple (I.12.1.22), so Weyl’s theorem (III.6.1.2) shows the adjoint representation is semisimple.

2 \( \rightarrow \) 3: If the adjoint representation is semisimple, then \( g \) decomposes as a sum of minimal nonzero ideals \( a_i \) of \( g \), and then \( g \) is a product of these \( a_i \). Let \( c \) be the product of the one-dimensional ideals, then \( c \) is in the center thus commutative, and \( b \) the product of the remaining ideals, then \( b \) is semisimple because it has no solvable ideals.

3 \( \rightarrow \) 1 is trivial. \( \square \)

Cor. (I.12.4.3). The decomposition of \( g \) into a product of commutative Lie algebra and a semisimple Lie algebra is unique: in fact \( c \) is the center of \( g \), and \( b = [g,g] \), by (I.12.2.4).

Prop. (I.12.4.4). A Lie algebra \( g \) is reductive iff it has a faithful semisimple representation iff it has a trivial nilpotent radical (I.12.1.33).

Proof: If \( g \) has a faithful semisimple representation, then the nilpotent radical \( s = 0 \), thus by (I.12.1.35), \( r \) is in the center of \( g \), thus \( g \) is reductive.

Conversely, if \( g \) is reductive, then we need to show \( g \) has a faithful semisimple representation: For this, we can take the tensor product of the trivial representation of the commutative part and the adjoint representation of the semisimple part (I.12.2.3).

The last assertion is clear from (I.12.1.35). \( \square \)

Cor. (I.12.4.5) (Trace Form Criterion for Reductiveness). If the trace form \( B_\rho \) (I.12.9.5) is non-degenerate for some representation \((\rho,V)\) of \( g \), then \( g \) is reductive.

Proof: If \( x \in s \), then \( \rho(x) = 0 \), thus \( B_\rho(x,y) = 0 \) for any \( y \), thus \( x = 0 \). So \( s = 0 \), and \( g \) is reductive. \( \square \)

Cor. (I.12.4.6) (Classical Lie Algebras are Reductive). All classical Lie groups over \( \mathbb{R} \) or \( \mathbb{C} \) are reductive.

Proof: Apply (I.12.4.5) their standard representations. \( \square \)

Def. (I.12.4.7) (Borel Subalgebras). Let \( g \) be a split reductive Lie group and \( h \) a Cartan subalgebra with a system of positive roots \( \Pi \), and consider the corresponding triangular decomposition \( g = n_- \oplus h \oplus n_+ \) (I.12.3.21). Denote \( b_+ = h \oplus n_+ \), and call any Lie subalgebra of \( g \) conjugate to \( b_+ \) a borel subalgebra of \( g \). The definition is independent of the choice of the Cartan subalgebra \( h \), by (I.12.3.24).
5 Compact Lie Algebras

Main references are [李群讲义, 项武义] and [Kna96].

Def. (I.12.5.1) (Compact Lie Algebras). A **compact Lie algebra** is Lie algebra that is the Lie algebra of a compact Lie group.

Prop. (I.12.5.2) (Killing Form of Compact Lie Algebra). The Killing form of a compact Lie algebra $\mathfrak{g}$ is negatively semi-definite, with the kernel the center of $\mathfrak{g}$.

Proof: Choose an invariant inner product on $\mathfrak{g}$ w.r.t. the adjoint representation of $G$ by (X.6.4.2). Take derivative w.r.t the equation $(\text{Ad}(g)Y, \text{Ad}(g)Z) = (Y, Z)$, we get by (IX.8.1.12),

$$(\text{ad}(X)Y, Z) + (Y, \text{ad}(X)Z) = 0.$$ 

so $\text{ad}(X)$ is skew-symmetric w.r.t. this inner product, thus the eigenvalues are all purely imaginary. Then $B(x, x) = \text{tr}(\text{ad}(x) \text{ad}(x)) \leq 0$. □

Cor. (I.12.5.3) (Compact Lie Algebra is Reductive). A compact Lie algebra $\mathfrak{g}$ is reductive.

Proof: Because of the invariant inner form, any ideal $\mathfrak{a}$ of $\mathfrak{g}$ has a complement $\mathfrak{a}^\perp$, thus the adjoint representation of $\mathfrak{g}$ is semisimple, thus $\mathfrak{g}$ is reductive (I.12.4.2). □

Cor. (I.12.5.4). For a compact Lie algebra $\mathfrak{g}$, the eigenvalues of any adjoint operator $\text{ad}(x)$ is purely imaginary.

Proof: Because the Killing form is negative definite, thus its negation is an inner product on $\mathfrak{g}$, and $\text{ad}(x)$ acts by skew-Hermitian matrices, thus has purely imaginary eigenvalues. □

Prop. (I.12.5.5). If $\mathfrak{g}$ is reductive and the Killing form is negative definite on $[\mathfrak{g}, \mathfrak{g}]$, then $\mathfrak{g}$ is compact.

Proof: For the commutative part we can take the torus $(S^1)^n$, so it suffices to prove the semisimple case: for this, we consider $\text{Int}(\mathfrak{g})^0 \subset G\mathfrak{l}(\mathfrak{g})$, it is contained in $O(\mathfrak{g})$ with the inner product defined by the negation of the Killing form, thus it is compact. And the Lie algebra of it is $\text{Der}(\mathfrak{g}) = \text{ad}(\mathfrak{g})(I.12.2.7)$. □

6 Singular element

Introduction

Singular element in $\mathfrak{g}$ is a linear space and is defined by some homogenous ideal in $S(\mathfrak{g})$.

The paper [Singular element] of Kostant tells in fact it is defined by some $r$-homogenous functions $M^r$ in $S(\mathfrak{g})$, and further describes the properties of this ideal such as the $G$-module decomposition and as span of determinant minors.

Preliminary

Let complex simple Lie algebra $\mathfrak{g} = \text{Lie} \ G, n = l + 2r$. The non-degenerate Killing form $\mathcal{B} \triangleq (x, y)$ on $\mathfrak{g}$ generate a nonsingular pair on $S(\mathfrak{g})$ and $\wedge(\mathfrak{g})$ by

$$(x_1 \cdots x_k, y_1 \cdots y_k) = \sum_{\sigma \in \Sigma_k} (x_1, y_{\sigma(1)}) \cdots (x_k, y_{\sigma(k)})$$
\[ (x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots y_k) = \sum_{\sigma \in \Sigma_k} sg(\sigma)(x_1, y_{\sigma(1)}) \cdots (x_k, y_{\sigma(k)}) \]

So \( g \leftrightarrow g', S(g) \leftrightarrow S(g') \leftrightarrow \) polynomial functions on \( g \); and \( S(g) \) and \( \wedge(g) \) are \( g \) thus \( G \) modules extending the adjoint representation.

recall that \( \delta \) and \( \partial \) are called \( B \)-dual if \( (\delta x, y) = (x, \partial y) \). Set antiderivation \(-d \) \( B \)-dual to the operator
\[
\partial(x_1 \wedge \cdots \wedge x_p) = \sum_{i<j} (-1)^{i+j+1}[x_i, x_j] \wedge x_1 \wedge \cdots \hat{x}_i \wedge \cdots \hat{x}_j \wedge \cdots \wedge x_p
\]
on \( \wedge(g) \) and antiderivation \( \iota(u) \) \( B \)-dual to the operator \( \epsilon(u)v = u \wedge v \) on \( \wedge(g) \).

Element \( v \) of \( S(g) \) are called invariant iff \( gv = v, \forall g \in G \) and element \( u \) of \( S(g) \) are called harmonic iff \( (u,v) = 0, \forall v \) invariant and no constant term.

Denote by \( J, H \) respectively the graded subspace of invariant and harmonic elements, then:

**Prop. (I.12.6.1) (Separation of Variables (in [Kos63])).** \( S(g) \cong J \otimes H \).

the Ideal of \( \text{Sing } g \)

In the projection \( \tau : T(g) \longrightarrow U(g) \), PBW theorem asserts that \( S(g) \longrightarrow U(g) \) is an isomorphism. Denote:
\[
\Gamma = \tau^{-1}_{S(g)} \circ \tau
\]
\( \Gamma \) is a \( G \)-map (as a consequence of the next prop).

Denote by \( \Gamma_{2r,2} \) the subgroup of permutation that preserves the set of unordered pairs \{1,2),(3,4),...,(2r-1,2r)\} and let \( \Pi_r \) be a left coset representative of \( \Gamma_{2r,2} \) in \( \Gamma_{2r} \) that \( sg(\Pi_r) = 1 \)

In [Amitsur-Levitski],Kostant proved:

**Prop. (I.12.6.2) (in [Kos81]).**
\[
\Gamma(\wedge^{2k}(g)) = R^k \in S^k(g)
\]
\[
\Gamma(x_1 \wedge \cdots \wedge x_k) \longrightarrow \sum_{v \in \Pi_k} [x_{v(1)}, x_{v(2)}] \cdots [x_{v(2k-1)}, x_{v(2k)}]
\]

**Prop. (I.12.6.3) (in [Kos81]).**
\[
M = R^r \in H^r
\]
so it consists of harmonic functions.

Let \( w \in \wedge^2 g \) of rank \( k \) standardized as \( v_1 \wedge v_2 + \cdots + v_{2k-1} \wedge v_{2k} \). Let
\[
\text{Rad}w = \{ y \in g|\iota(y)w = 0 \} = \{ y \in g|(w, \epsilon(y)z) = 0, \forall z \}
\]
then \( w \) of rank \( 2k \iff w^{k} \neq 0 \ & w^{k+1} = 0 \iff \dim \text{Rad}w = n - 2k \).

**Lemma (I.12.6.4).**
\[
\iota(y)dx = [y, x]
\]
Thus
\[
\text{Radd}x = g^x, \ \text{Sing}g = \{ x \in g|(dx)^r = 0 \}.
\]
Proof: \((\iota(y) dx, z) = (dx, y \wedge z) = (x, [y, z]) = ([y, x], z)\)

So in order to find the module \(M\), it’s the best to find the dual of
\[
\gamma : S(\mathfrak{g}) \rightarrow \wedge^{\text{even}} \mathfrak{g} : x \mapsto -dx
\]

Luckily:

**Prop. (I.12.6.5) (in [Kos81]).** \(\gamma\) is \(B\) dual to \(\Gamma\) in particular,\]
\[
(\Gamma(\zeta), x) = \left(\frac{(-1)^r}{r!} (\zeta, (dx)^r) \right) (\forall \zeta \in \wedge^2 \mathfrak{g} \text{ and } x \in \mathfrak{g})
\]

So: \(f(x) = 0, \forall f \in M \iff x \in \text{Sing}\mathfrak{g}\).

**Cor. (I.12.6.6).** Let \(a\) be a CSA of \(\mathfrak{g}\), \(\Delta_+(a)\) be the positive roots, then
\[
f|a = C_f \cdot \prod_{\beta \in \Delta_+(a)} \beta \quad (\forall f \in M)
\]

Proof: This is because that an element in a CSA is singular iff it commutes with an element outside this CSA, and taking root decomposition, this is equivalent to terminated by a root, and by counting degree, the cor follows.

By propositions of [[Kos59] a regular nilpotent element \(e\) is uniquely in a nilpotent radical \(n\) of a Borel subalgebra and that \(\mathfrak{g}^e \cap [n, n] = (\text{Sing}\mathfrak{g}) \cap \mathfrak{g}^e\). So there is a linear function \(\xi\) on \(\mathfrak{g}^e\) that \(\text{Ker}\xi = (\text{Sing}\mathfrak{g}) \cap \mathfrak{g}^e\). Thus:

**Cor. (I.12.6.7).** \(f|\mathfrak{g}^e = C_f : \xi^r \quad (\forall f \in M)\).

Proof: By counting degree, the same reason as before. \(\Box\)

Now we think of a natural question: Can singular elements be defined be functions of even lower degree? The answer is NO.

**Prop. (I.12.6.8).** Assume \(0 \neq f\) homogenous vanishes on \(\text{Sing}\mathfrak{g}\), then \(\text{deg } f \geq r\)

Proof: By the last cor, if \(f\) has degree less than \(r\) then \(f\) vanish on any CSA, but semisimple regular element, thus CSAs are Zariski dense in \(\mathfrak{g}\) (this is because semisimple element are defined by a polynomial), so \(f = 0\). \(\Box\)

Thus we have established that \(\text{Sing}\mathfrak{g}\) is an algebraic set defined by a set of harmonic \(r\)-homogenous functions on \(\mathfrak{g}\) and not by functions of degree lower than \(r\).

Next we offer a different formation of \(M\).

\(M\) as minors of determinants

for a \(B\) dual basis \(y_i, w_j\), define a derivation
\[
d_W(f \otimes u) = \sum_i \partial y_i f \otimes c(w_i) u \quad \text{on } S(\mathfrak{g}) \otimes \wedge \mathfrak{g}.
\]

Here \(\sum a_i \partial x_i\) is defined as \(\sum a_i \frac{\partial}{\partial x_i}\) for a standard basis \(x_i\) of \(\mathfrak{g}\). It’s easy to verify that \(d_W\) is well defined and is a \(G\)-map (Take a different basis \(Aw_i\) and \(Bz_i\), then \(AB^t = I\), substitute into the formula of \(d_W\), it doesn’t change).
Chevalley Thm tells us $J$ is a polynomial ring $\mathbb{C}[p_1, \ldots, p_l]$, where $p_i$ are homogenous polynomials of fixed degree $d_i$ and $\sum_{j=1}^r (d_j - 1) = r$. So:

$$d_W p_1(x) \wedge \cdots \wedge d_W p_l(x) = \sum_{1 \leq i_1 < \cdots < i_l \leq n} \phi(y_{i_1}, \ldots, y_{i_l})(x) w_{i_1} \wedge \cdots \wedge w_{i_l}$$

Where $\phi(y_{i_1}, \ldots, y_{i_l}) = \det \partial_{y_i} p_j$ is homogenous of degree $r$. (counting degree).

To see this, notice that $f \otimes u$ acts as a function from $g$ to $\wedge g : f \otimes u(x) = f(x) u$. So:

$$d_W p_j(x) = \sum_{i=1}^n \partial z_i p_j(x) w_i.$$

Prop. (I.12.6.9). For any CSA $h$ of $g$ and a basis $\{v_i\}$ of $h$, $\forall x \in h$,

$$d_W p_1(x) \wedge \cdots \wedge d_W p_l(x) = \kappa \cdot \prod_{\phi \in \Delta^+} \phi(y) v_1 \wedge \cdots \wedge v_l$$

Lemma (I.12.6.10) (in [Kos63]). $\{d_W p_1(x), \ldots, d_W p_l(x)\}$ is linearly independent iff $x \in \text{Regg}$. Proof: Notice that $d_W p_j$ is a $g$-map, $\text{ad} \cdot d_W p_j = d_W p_j([y, x])$ so $d_W p_j(x)$ commutes with $g^x$; so $g^x$. Then the lm tells us when $y$ is regular, $d_W p_j(y)$ forms a basis of $g^y$. Considering in $g^y$, $x$ is regular iff $\prod_{\phi \in \Delta^+} \phi(x) \neq 0$, the prop follows. Next we give an explicit expression for $\gamma_r$.

It can be verified (taking a $z_i$ basis) that $dx = \frac{1}{2} \sum_{i=1}^n w_i \wedge [z_i, x]$. Now $x \in h$,

$$dx = \sum_{\phi \in \Delta^+} \phi(x) e_\phi \wedge f_{-\phi}$$

(Just take the basis $w_i$ and $z_i$ as a standard basis of $g$ consisting of $\{h_i, \ldots, h_l, e_\phi, f_\phi\}$)

So

$$\gamma_r(x^r) = r!(-1)^r \prod_{\phi \in \Delta^+} \phi(x) e_\phi \wedge f_{-\phi}$$

Let $\mu = i^r v_1 \wedge \cdots \wedge v_l \wedge \prod_{\phi \in \Delta^+} e_\phi \wedge f_{-\phi}$ then $(\mu, \mu) = 1$. Denote $v^* = \iota(v) \mu$ for $v \in \wedge g$, then

$$(v_1 \wedge \cdots \wedge v_l)^* = i^r \prod_{\phi \in \Delta^+} e_\phi \wedge f_{-\phi} = C_\alpha \gamma_r(x^r)$$

(notice that $\iota(u)\iota(v) = \iota(u \wedge v)$ and use lm 1)

Prop. (I.12.6.11). $(d_W p_1(x) \wedge \cdots \wedge d_W p_l(x))^* = \kappa_\alpha \gamma_r(x^r) \neq 0$. Proof: For $y \in h$ regular, this follows from previous calculations, and notice both side are $G$-maps, and semisimple regular elements are Zariski open, conclusion follows.

Lemma (I.12.6.12). $(s, t) = (s^*, t^*)$, so that $-*$ is a $B$-isomorphism.

Prop. (I.12.6.13). Let $\{w_1, \ldots, w_{2r}\}$ be linearly independent and $\{u_1, \ldots, u_l\}$ be a basis of $\{w_1, \ldots, w_{2r}\}^\perp$, then

$$\Gamma(w_1 \wedge \cdots \wedge w_{2r}) = \kappa_1 \det \partial_{u_i} p_j \neq 0$$

Thus, $M$ is the span of all the minors $\det \partial_{u_i} p_j$. 
Proof: By the preceding props,
\[
\det \partial u_i p_j = \phi(u_1, \ldots, u_l)(x)
\]
\[
= (d_W p_1(x) \wedge \cdots \wedge d_W p_l(x), u_1 \wedge \cdots \wedge u_l)
\]
\[
= ((d_W p_1(x) \wedge \cdots \wedge d_W p_l(x))^*, (u_1 \wedge \cdots \wedge u_l)^*)
\]
\[
= \kappa_0 \kappa_2 (\phi_r(x^r), w_1 \wedge \cdots \wedge w_{2r})
\]
\[
= \kappa^{-1} \Gamma(w_1 \wedge \cdots \wedge w_{2r})(x).
\]

\[\square\]

**G-module structure of M**

Now we show the $G$-module structure of $M$.

Let $\theta$ be the derivative that $\theta(x)(y) = [x, y]$ on $\mathfrak{g}$. Cas = $\sum_{i=1}^n \theta(z_i) \theta(w_i)$.

It’s in fact just the action of the Casimir element in center of $U(\mathfrak{g})$. Let $m_l$ and $M_l$ be the maximal eigenvalue and eigenspace of Cas.

For a commutative Lie subalgebra $\mathfrak{c}$ of rank $l$, denote by $[\mathfrak{c}]$ the line it defines on $\wedge^l \mathfrak{g}$. The span of these $[\mathfrak{c}]$ is denoted $A_l$. Notice that $[g^p] \subset A_l$ for a regular $g$, and $A_l$ is a $G$-submodule.

**Prop. (I.12.6.14) (in [Kos65]).**

\[A_l = M_l; \quad m_l = l.\]

An ideal in a Borel subalgebra of $\mathfrak{g}$ is necessarily spanned by root vectors and a prop of $[[K-W09]]$ says any ideal of dim $l$ is (denoted by $\mathcal{I}$) in fact abelian.

A prop in $[[Kos65]]$ asserts that for two different ideals $\Phi_1, \Phi_2$, sum of their weight vectors $\langle \Phi \rangle$ is distinct.

So $G[\Phi_l]$ is an irreducible $G$-module $V_\Phi$ with highest weight $\langle \Phi \rangle$ and $V_\Phi$ are inequivalent $G$-modules (because an irreducible representation have only one highest vector).

**Prop. (I.12.6.15) (in [Kos65]).**

\[M_l = \bigoplus_{\Phi \in \mathcal{I}} V_\Phi.\]

Now denote $M_{2r}$ image of $M_l$ under the isomorphism $u \rightarrow u^*$, then

**Prop. (I.12.6.16) (in [K-W09]).** $M_l$ is the span of $G \cdot [\mathfrak{g}^x]$ for $x$ regular.

but by precious prop,
\[ [\mathfrak{g}^x] = C d_W p_1(x) \wedge \ldots \wedge p_l(x). \]

thus $M_{2r}$ is the span of $G \cdot (\gamma_r(x^r))$, $x$ regular.

**Prop. (I.12.6.17) (Final).** $\Gamma|_{M_{2r}} : M_{2r} \rightarrow M$ is an isomorphism and $M \cong M_{2r} \cong M_l = A_l$ as $G$-module.

So $M$ is a multiplicity one module with $|\mathcal{I}|$ irreducible components.

**Proof:** Notice that $\Gamma(\zeta)(x) = (\zeta, \gamma_r(x^r))$ and $M_{2r}$ is the span of $G \cdot (\gamma_r(x^r))$, the first part follows, and the rest is a recapitulation of previous props.  

\[\square\]
7 Real Lie Algebra

Prop. (I.12.7.1) (Passage from Real to Complex). If $\mathfrak{g}_0$ is a Lie algebra over $\mathbb{R}$ and $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ its complexification, then $\mathfrak{g}_0$ is Abelian/nilpotent/solvable/semisimple iff $\mathfrak{g}$ does.

Def. (I.12.7.2). A compact real form is a real subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ s.t. $\mathfrak{g}$ is the complexification of $\mathfrak{l}$ and $\mathfrak{l}$ is the Lie algebra of a compact simply-connected Lie group.

Prop. (I.12.7.3). A real Lie algebra is compact iff there exists an invariant inner product iff the Killing form is negative definite.

Proof: One direction is easy, just use the average method to find a $G$-invariant inner product and then take derivative. For the other direction, the identity shows that a complement of an ideal is an ideal so $\mathfrak{g}$ is decomposed into simple Lie groups and reduce to the case that $\mathfrak{g}$ is simple. The ideal is to show that $\mathfrak{g} \cong \text{ad}(\mathfrak{g})$ is the whole outer derivative group $\partial(\mathfrak{g})$ (the following lm). So $\mathfrak{g}$ equals to the identity component of $\text{Aut}(\mathfrak{g})$ which is a closed subgroup thus closed but it is also a subgroup of the compact group $O(\mathfrak{g})$ thus it is compact.

Lemma (I.12.7.4). If a real semisimple Lie algebra $X$ has an invariant inner product, then every outer derivative is inner. (In fact, this is true by Cartan Criterion for semisimplicity (I.12.2.7)).

Proof: since $\text{ad}(X)$ is skew-symmetric, it’s diagonalizable and its eigenvalue is pure imaginary, so the Killing form of $X$ is negative definite. Now choose the complement $\mathfrak{a}$ of $\text{ad}(X)$ in $\partial(X)$, then $\mathfrak{a} \cap X = 0$. Thus for $D \in \mathfrak{a}$, $\text{ad}(D(g)) = [D, \text{ad}(g)] = 0$ for all $g$ in $X$, so $D = 0$, thus $\text{ad}(X) = \partial(X)$. □

Prop. (I.12.7.5). -

1. The complexification of the Lie algebra of a connected compact Lie group is reductive.

2. A complex Lie algebra is semisimple iff it is isomorphic to the complexification of the Lie algebra of a simply-connected compact Lie group. i.e. every complex semisimple Lie algebra has a compact real form.

Proof: 1: Because a connected compact Lie group is completely reducible so the does the Lie algebra and so does the complexification. So it is reductive by (I.12.4.1)4.

2: Cf.[Varadarajan Lie Groups Lie algebras and Their Representations]. The idea is to find a real form whose corresponding simply-connected group is compact. □

Prop. (I.12.7.6). If $\mathfrak{g}$ is the Lie algebra of a matrix Lie group $G$, then:

1. every Cartan subalgebra comes from a maximal commutative subalgebra of a compact real form and any two Cartan subalgebras are conjugate under the Ad-action of $G$.

2. any two compact real form is conjugate under the Ad-action of $G$.

3. any two maximal commutative subalgebra of a compact real form is conjugate under the Ad-action of the corresponding compact compact subgroup.

Prop. (I.12.7.7). A real Lie algebra is semisimple iff its complexification is semisimple. Cf.[Varadarajan].

Cor. (I.12.7.8). The real Lie algebra of a compact simply-connected group is semisimple.

Note: For the classification of real semisimple Lie algebras, Cf.[李群讲义项武义 §6]
Prop. (I.12.7.9). If a complex representation of a Lie group admits an invariant bilinear form, then it is non-degenerate and unique. In fact, this is equivalent to a $G$-map from $V$ to $V^*$. Thus there is unique invariant inner product in a compact real form by the preceding proposition.

8 Universal Constructions

In this subsection $k$ can be a field of any characteristics.

Def. (I.12.8.1) (Universal Enveloping Algebra). The universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is defined to be

$$U(\mathfrak{g}) =\frac{T(\mathfrak{g})}{J}, \quad T(\mathfrak{g}) = \bigoplus_{n=0}^\infty \mathfrak{g}^\otimes n$$

which is a graded algebra $T(\mathfrak{g})$ quotients the ideal $J = \{x \otimes y - y \otimes x - [xy]\}$. There is a natural linear map $\sigma : \mathfrak{g} \to U(\mathfrak{g})$.

Prop. (I.12.8.2). The universal enveloping algebra $U : \mathfrak{g} \mapsto U(\mathfrak{g})$ defines a functor $\text{LieAlg} \to \text{AssAlg}$ that is left adjoint to the canonical functor $\text{AssAlg}_k \to \text{LieAlg}_k$(I.12.1.2).

Proof: For any associative algebra $A$ and a morphism of Lie algebras $\mathfrak{g} \to [A]$, there is easily seen a morphism $U(\mathfrak{g}) \to A$, and it is unique. \qed

Cor. (I.12.8.3) (Representation as Modules). A representation of $\mathfrak{g}$ (I.12.12.4) is he same as a representation of $U(\mathfrak{g})$.

Prop. (I.12.8.4) (Poincaré-Birkhoff-Witt). Let $\mathfrak{g}$ be a Lie algebra, define a filtration on $U(\mathfrak{g})$ by assigning $F_nU(\mathfrak{g}) = \text{the image of } \bigoplus_{i=0}^n \mathfrak{g}^\otimes i \text{ in } U(\mathfrak{g})$. We have $[F_{n+1}U(\mathfrak{g}), F_iU(\mathfrak{g})] \subset F_{i+j-1}U(\mathfrak{g})$, thus $grU(\mathfrak{g})$ has a graded commutative ring structure. Thus there is an algebra homomorphism $S(\mathfrak{g}) \to grU(\mathfrak{g})$. Then this homomorphism is an isomorphism.

If $\mathfrak{g}$ has a basis $\{x_i\}, i \in I$ and $<$ is an order on $I$, then $U(\mathfrak{g})$ has a basis consisting of elements $\{x_{i_1} \cdots x_{i_k} \}$ where $i_1 < i_2 < \ldots < i_r$.

Proof: We assign any monomial $a_{i_1} \cdots a_{i_n}$ in $\mathfrak{g}$'s a pair $(k, N)$ where $k$ is the number of factors in a monomial and $N$ is the number of inversions (meaning the number of pairs $1 \leq r, s \leq n$ that $i_r > i_s$), pairs $(k, N)$ are lexicographically ordered. Let $T(k, N)$ be the space of $T(\mathfrak{g})$ generated by monomials of index $(k, N)$, $T_k = \bigcup_{N=1}^\infty T(k, N)$, and $U(k, N)$ the space of $T(\mathfrak{g})$ the image of $T(k, N)$, $T_k$ in $U(\mathfrak{g})$. We will use induction on $(k, N)$. Notice for any $k$, $T_k = T(k, N)$ for $N$ large.

To show that the monomials in (*) generates $U(\mathfrak{g})$, if we have a monomial $a_{i_1}a_{i_2} \cdots a_{i_s}a_{i_{s+1}} \cdots a_{i_k}$ that $i_s > i_{s+1}$, then

$$a_{i_1}a_{i_2} \cdots a_{i_s}a_{i_{s+1}} \cdots a_{i_k} = a_{i_1}a_{i_2} \cdots (a_{i_{s+1}}a_{i_s} + [a_{i_s}, a_{i_{s+1}}]) \cdots a_{i_k},$$

which is in $\bigcup_{(k', N') \subseteq (k, N)} U(k', N')$. So we can use induction on $(k, N)$ to show that any element of $U(\mathfrak{g})$ is in $\bigcup_{k=1}^\infty T(k, 0)$.

From now on, we write $a_{i_1}a_{i_2} \cdots a_{i_n}$ as $a_{i_1}a_{i_2} \cdots a_{i_n}$ for simplicity.

To show that the monomials are linearly independent, we first show that there is a linear map

$$\theta : T(\mathfrak{g}) \to R = \mathbb{C}[z_i]_{i \in I}$$
satisfying the following conditions:

\[ \theta(a_{i_1} \ldots a_{i_n}) = z_{i_1} \ldots z_{i_n}, \text{if } i_1 \leq i_2 \ldots \leq i_n, \quad (**) \]

\[
\theta(a_1 \ldots a_k a_{ik+1} \ldots a_{i_n}) \\
= \theta(a_1 \ldots a_{ik+1} a_{ik} \ldots a_{i_n}) + \theta(a_1 \ldots [a_{ik} a_{ik+1}] \ldots a_{i_n}). \quad (***)
\]

We construct this map by construction on \( \cup(k',N') \leq (k,N) \mathcal{U}^{(k',N')} \) and use induction on \((kN)\). For \( k = 0 \), let \( \theta(1) = 1 \). If \( \theta \) is defined for any monomials of index \((k,N)\) that \( k < n \), define \( \theta \) on \( T^n.0 \) by \( \theta(a_1 \ldots a_{i_n}) = z_{i_1} \ldots z_{i_n} \), then it satisfies \((***)\).

And if \( \theta \) is already defined for any \( T^n.k \) that \( k < i \), suppose that the monomial \( a_{i_1} \ldots a_{i_n} \) has index \((n,i)\), then there is a smallest \( k \) that \( a_{i_1} \ldots a_{ik+1} a_{ik} \ldots a_{i_n} \) has index \( i-1 \). Then we define

\[
\theta(a_{i_1} \ldots a_{i_n}) = \theta(a_{i_1} \ldots a_{ik+1} a_{ik} \ldots a_{i_n}) + \theta(a_{i_1} \ldots [a_{ik}, a_{ik+1}] \ldots a_{i_n}).
\]

Now we need to check that this definition satisfies \((***)\):

If there is another \( k' \) that \( i_k > i_{k'+1} \), then \( k < k' \). Suppose first that \( k+1 < k' \), let \( a_{i_k} = a, a_{ik+1} = b, a_{i_{k'+1}} = c, a_{i_{k'+2}} = d \), then

\[
\theta(\ldots ab \ldots cd \ldots) = \theta(\ldots ba \ldots cd \ldots) + \theta(\ldots [a,b] \ldots cd \ldots) \\
= \theta(\ldots ba \ldots dc \ldots) + \theta(\ldots ba \ldots [cd] \ldots) + \theta(\ldots [ab] \ldots cd \ldots) \\
= \theta(\ldots ab \ldots dc \ldots) + \theta(\ldots ab \ldots [cd] \ldots)
\]

where the terms except the first one are all in \( \cup(k',N') \leq (k,N) \mathcal{T}^{(k',N')} \) so the equalities come from induction hypothesis. So it satisfies \((***)\).

Suppose next that \( k' = k+1 \), let \( a_{i_k} = a, a_{ik+1} = b, a_{i_{k+2}} = c \), then

\[
\theta(\ldots abc \ldots) = \theta(\ldots bac \ldots) + \theta(\ldots [ab]c \ldots) \\
= \theta(\ldots bca \ldots) + \theta(\ldots [bc]a \ldots) + \theta(\ldots [bc] \ldots a \ldots) + \theta(\ldots [ab]c \ldots) + \theta(\ldots [ab] \ldots c \ldots) \\
= \theta(\ldots cba \ldots) + \theta(\ldots [bc]a \ldots) + \theta(\ldots [bc] \ldots a \ldots) + \theta(\ldots [ac]b \ldots) + \theta(\ldots [ac] \ldots b \ldots) \\
= \theta(\ldots cab \ldots) + \theta(\ldots [ac]b \ldots) + \theta(\ldots [ac] \ldots b \ldots)
\]

where the terms except the first one are all in \( \cup(k',N') \leq (k,N) \mathcal{T}^{(k',N')} \) so the equalities come from induction hypothesis, and in the third equality we used the Jacobi identity. So it satisfies \((***)\).

Now all elements in \( J \) is a linear combination of elements of the form

\[ a_1 \ldots a_{i_k} \ldots a_{i_n} - a_1 \ldots a_{i_k+1} a_{i_k} \ldots a_{i_n} - a_1 \ldots [a_{i_k} a_{i_k+1}] \ldots a_{i_n}, \]

so the map \( \theta \) factors through \( T(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \) to a map \( \overline{\theta} : \mathcal{U}(\mathfrak{g}) \to R \), and the elements \( a_{i_1} a_{i_2} \ldots a_{i_k}, i_1 \leq i_2 \leq \ldots \leq i_k \) are mapped to \( z_{i_1} \ldots z_{i_n} \), which are linearly independent in \( R \), so the elements \( a_{i_1} a_{i_2} \ldots a_{i_k}, i_1 \leq i_2 \leq \ldots \leq i_k \) are also linearly independent in \( \mathcal{U}(\mathfrak{g}) \).

\[ \square \]

Cor. (I.12.8.5). The map \( \mathfrak{g} \to U(\mathfrak{g}) \) is injective.

Cor. (I.12.8.6). \( U(\mathfrak{g}) \) has no zero-divisors.
The PBW theorem shows this map is injective.

Proof: We can use the identities \( x \otimes y - y \otimes x = [xy] \) to make any element in their right representations under the PBW prop(I.12.8.4), so it is clear that the product of two nonzero elements cannot be 0.

□

Cor. (I.12.8.7) If \( \mathfrak{h} \subset \mathfrak{g} \), then the subalgebra of \( U(\mathfrak{g}) \) generated by \( \mathfrak{h} \) is isomorphic to \( U(\mathfrak{h}) \).

Proof: There is a natural map \( U(\mathfrak{h}) \to U(\mathfrak{g}) \), and the image is just the subgroup generated by \( \mathfrak{h} \). The PBW theorem shows this map is injective.

□

Cor. (I.12.8.8) If \( \mathfrak{g} = \mathfrak{a} \times \mathfrak{b} \), then \( U(\mathfrak{g}) = U(\mathfrak{a}) \otimes U(\mathfrak{b}) \).

Def. (I.12.8.9) (Coproduct of \( U(\mathfrak{g}) \)). Let \( \mathfrak{g} \) be a Lie algebra, there is a coproduct \( \Delta \) on \( T(\mathfrak{g}) \) defined by \( \Delta(\mathfrak{g}) = \mathfrak{g} \otimes 1 + 1 \otimes \mathfrak{g} \). This coproduct descends to a coproduct \( \Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g}) \).

Proof: It suffices to check that \( \Delta(J) \subset J \otimes T(\mathfrak{g}) + T(\mathfrak{g}) \otimes J \), and this is because

\[
\Delta(x \otimes y - y \otimes x - [xy]) = (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y) - (y \otimes 1 + 1 \otimes y)(x \otimes 1 + 1 \otimes x) - ([xy] \otimes 1 + 1 \otimes [xy]) \\
= (x \otimes y - y \otimes x - [xy]) \otimes 1 + 1 \otimes (x \otimes y - y \otimes x - [xy])
\]

□

Prop. (I.12.8.10) (\( \mathfrak{g} \)-Module Structure). Let \( \mathfrak{g} \) be a Lie algebra, then \( T(\mathfrak{g}) \) is a \( \mathfrak{g} \)-module, and this action descends to a \( \mathfrak{g} \)-module structure on \( U(\mathfrak{g}) \).

Proof: It suffices to show that \( \mathfrak{g}J \subset J \) and \( \mathfrak{g}I \subset I \):

\[
y(a \otimes b - b \otimes a - [ab]) = [ya] \otimes b + a \otimes [yb] - [yb] \otimes a - b \otimes [ya] - [yab] \\
= [ya] \otimes b - b \otimes [ya] - [[ya]b] + a \otimes [yb] - [yb] \otimes a - [a[yb]]
\]

□

Prop. (I.12.8.11) (Transpose). There is an anti-automorphism \( u \mapsto u^t \) of \( U(\mathfrak{g}) \) that \( X^t = -X \) for \( X \in \mathfrak{g} \).

Proof: We first extend this \( t \) to an automorphism of \( T(\mathfrak{g}) \), then we compose with the obvious anti-automorphism \( T(\mathfrak{g}) \to T(\mathfrak{g}) \). Then we check that this map descends to \( U(\mathfrak{g}) \): \( (X \otimes Y - Y \otimes X - [X,Y])^t = Y \otimes X - X \otimes Y - [Y,X] \in J \), so \( J^t \in J \).

□

Prop. (I.12.8.12) (Graded Algebra of \( U(\mathfrak{g}) \)). Let \( L \) be a Lie algebra, if we let \( S(L) = T(L)/(x \otimes y - y \otimes x) \) be the universal symmetric algebra of \( L \), then it is a graded algebra. There is a filtered structure on \( U(L) \) given by \( U_i = \{ \text{subalgebra generated by } a_1a_2 \ldots a_j, j \leq i \} \), then the associated graded algebra of \( U(L) \) is isomorphic to \( S(L) \) by PBW theorem.

Cor. (I.12.8.13). If \( W \) is a subspace of \( T^n(L) \) that is sent isomorphically onto \( S^n(L) \), then the image of \( W \) is a complement of \( U_n(L) \) complementary to \( U_{n-1}(L) \).

Cor. (I.12.8.14) (Symmetrization Map). Over a field of characteristic 0, the symmetrization map \( \sigma : S(\mathfrak{g}) \to T(\mathfrak{g}) \to U(\mathfrak{g})(\text{I.1.8.2}) \) is an isomorphism of \( \mathfrak{g} \)-modules that

\[
U^n(\mathfrak{g}) = \sigma(S^n(\mathfrak{g})) \oplus U^{n-1}(\mathfrak{g})
\]
Proof: It is clearly an isomorphism of vector spaces. It suffices to show the map is compatible with \( g \)-actions: Because
\[
\sigma(y_1 \ldots y_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} y_{\sigma(1)} \ldots y_{\sigma(n)},
\]
\[
\sigma(g(y_1 \ldots y_n)) = \sigma([gy_1] \ldots y_n + \ldots + y_1 \ldots [gy_n])
\]
\[
= \frac{1}{n!} \sum_{\sigma \in \Sigma_n} ([gy_{\sigma(1)}] \ldots y_{\sigma(n)} + \ldots + y_{\sigma(1)} \ldots [gy_{\sigma(n)}])
\]
\[
= g(\frac{1}{n!} \sum_{\sigma \in \Sigma_n} y_{\sigma(1)} \ldots y_{\sigma(n)})
\]
\[\square\]

Prop. (I.12.8.15). If \( g \) is a Lie algebra over a field \( k \) of characteristic 0, then the set of primitive elements (I.15.1.3) are just \( g \).

Proof: If \( f \) is primitive, then the leading term \( f_0 \) of \( f \) is also primitive in \( grU(g) \cong S(g) \). Now consider \( S(g) \xrightarrow{\Delta} S(g) \otimes S(g) \xrightarrow{\mu} S(g) \), then if \( f \) is of degree \( n \), then \( 2^n f_0 = 2f_0 \), which means \( n = 1 \). So \( f = c + f_0 \), and \( c = 0 \).

Prop. (I.12.8.16). \( U(g) \) is Noetherian.

Proof:

Free Lie Algebras

Def. (I.12.8.17) (Free Lie Algebra). Let \( X \) be a set, then we define the free Lie algebra \( FL(X) \) to be the intersection of Lie subalgebras in \( [F(X)] \) containing \( \sigma(X) \), where \( F(X) \) is the free algebra generated by \( X \).

Then the free Lie algebra \( FL : X \mapsto FL(X) \) defines a functor \( Set \to LieAlg \) that is left adjoint to the forgetful functor.

Proof: We need to show that for any Lie algebra \( L \) and a map \( \theta : X \to L \), there is a unique \( \varphi \) completing the upper left triangular diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{i} & FL(X) & \xrightarrow{\varphi} & F(X) \\
\downarrow{\theta} & & \downarrow{\beta} & & \\
L & \xrightarrow{\sigma} & U(L)
\end{array}
\]

Notice \( \varphi^{-1}(\sigma(L)) \) is a Lie algebra containing \( X \) thus containing \( FL(X) \), so it induces a \( \varphi \).

And for the uniqueness, if there are two \( \varphi_1, \varphi_2 \), then the element that they coincide is a Lie algebra containing \( X \), thus containing \( FL(X) \), so \( \varphi_1 = \varphi_2 \).

Cor. (I.12.8.18). \( U(FL(X)) \cong FX \) for any set \( X \).

Proof: Because \( U \circ FL \) and \( F \) are both left adjoint to the forgetful functor \( AssAlg \to Set \). \[\square\]
Center of $U(\mathfrak{g})$

Prop. (I.12.8.19) (g Action on $U(\mathfrak{g})$). $\mathfrak{g}$ acts on $T(\mathfrak{g})$ by adjoint (I.12.9.2), and notice
\[
\text{ad}(z)(x \otimes y - y \otimes x - [xy]) = [zx] \otimes y + x \otimes [zy] - [zy] \otimes x - y \otimes [zx] - [z[xy]] \in J,
\]
so the action of $\mathfrak{g}$ descends to an action on $U(\mathfrak{g})$.

In fact, this action is inner: $\text{ad}(g)(z) = gz - zg$ for $z \in U(\mathfrak{g})$. In particular, $Z(U(\mathfrak{g})) = U(\mathfrak{g})^{ad(\mathfrak{g})}$.

Invariant Polynomials

Prop. (I.12.8.20) (Chevalley). The center of the universal enveloping algebra is isomorphic to the polynomial ring over $\mathbb{C}$ of $l$ elements, where $L$ is a semisimple Lie algebra of rank $l$. In particular, the center for $\mathfrak{sl}_2$ is the algebra generated by the Casimir element $1/2h^2 + ef + fe$.

Proof: Because there is a commutative diagram of isomorphisms of algebras:

\[
\begin{array}{ccc}
S(L)^G & \xrightarrow{\alpha} & P(L)^G \\
\downarrow{\eta} & & \downarrow{\phi} \\
S(H)^W & \xrightarrow{\beta} & P(H)^W \\
\end{array}
\]

Where $P$ is the polynomial ring $\cong S(L^*)$, the horizontal is Killing isomorphisms and vertical is the restriction maps. Cf. [Carter prop 13.32].

The twisted Harish-Chandra map gives an isomorphism of algebras $Z(L) \to S(H)^W$ (It just maps $z \in Z(L)$ to its pure $H$ part and transform every indeterminants $h_i$ to $h_i - 1$). e.g. $z = h^2 + 2h + 1 + 4fe \in Z(\mathfrak{sl}_2)$ is mapped to $h^2$ in $S(H)$. And $P(H)^W$ is isomorphic to a polynomial ring in $l$ generators over $\mathbb{C}$.

Def. (I.12.8.21) (Casimir Element). If $L$ is semisimple Lie algebra, by (I.12.2.1) the Killing form is non-degenerate, thus we choose a basis $x_i$ of $L$ and a dual basis $y_i$, then $c = \sum x_iy_i$ is independent of $x_i$ chosen by (I.1.7.9), and is called the Casimir element of $U(L)$.

Prop. (I.12.8.22). The Casimir element lies in the center of $U(L)$.

Proof: \[\square\]

Prop. (I.12.8.23) (Quillen’s lm). If $K$ is an alg.closed field of char 0 that $\mathfrak{g}$ is a f.d. Lie algebra over $K$. If $U = U(\mathfrak{g})$ is its universal enveloping algebra, then for any irreducible $U$-module $M$, $\text{End}_U(M) = K$.

Miscellaneous

Prop. (I.12.8.24) (Grading on $U(\mathfrak{sl}_2(\mathbb{C}))$). Let $H, R, L$ be a basis of $\mathfrak{sl}_2(\mathbb{C})$(I.12.2.11), if we define a grading as $\deg R = 1, \deg H = 0, \deg L = -1$, then this is descends to a grading on $U(\mathfrak{g})$, and the degree 0 part is the ring $\mathcal{R} = \mathbb{C}[\Delta, H]$. Also, there is a decomposition:

\[
U(\mathfrak{g}) = \bigoplus_{i \geq 0} L^i\mathcal{R} \oplus \bigoplus_{i > 0} R^i\mathcal{R}.
\]
9 Representations

Def. (I.12.9.1) (Representations). A representation of a Lie algebra $\mathfrak{g}$ over a vector space $V$ is a Lie homomorphism $\mathfrak{g} \rightarrow \mathfrak{gl}_V$.

Def. (I.12.9.2) (Tensor Product of Representations). Let $(V_1, \pi_1), (V_2, \pi_2)$ be two representations of a Lie algebra $\mathfrak{g}$, then $(V_1 \otimes V_2, \pi_1 \otimes \pi_2)$ is a representation of $\mathfrak{g}$ given by

$$(\pi_1 \otimes \pi_2)(g)(v_1 \otimes v_2) = \pi_1(g)v_1 \otimes v_2 + v_1 \otimes \pi_2(g)v_2.$$ 

Def. (I.12.9.3) (Representation on Tensor Algebras). If $(V, \rho)$ is a representation of $\mathfrak{g}$, then $\mathfrak{g}$ acts on $T(V)$ via (I.12.9.2). Also, the ideals $I, J$ are invariant under action of $\mathfrak{g}$, thus the representation extends to $\text{Sym}(V)$ and $\wedge(V)$. Also it preserves degree, thus it induces representations on $\text{Sym}^k(V)$ and $\wedge^k(V)$.

Proof: 

$$g(a \otimes a) = g(a) \otimes a + a \otimes g(a) = (a + g(a)) \otimes (a + g(a)) - a \otimes a - g(a) \otimes g(a)$$

and

$$g(a \otimes b - b \otimes a) = g(a) \otimes b + a \otimes g(b) - g(b) \otimes a - b \otimes g(a) = g(a) \otimes b - b \otimes g(a) + a \otimes g(b) - g(b) \otimes a$$

Def. (I.12.9.4) (dual representation). If $(\varphi, V)$ is a representation of $\mathfrak{g}$, we define the dual representation $(\varphi^*, V^*)$ as

$$(\varphi^*(v^*), v) = (v^*, \varphi(g)v).$$

Def. (I.12.9.5) (Trace Form). The trace form of a representation $(V, \rho)$ of a Lie algebra $\mathfrak{g}$ is an invariant symmetric form $\beta_\rho$ defined by $(x, y) \mapsto \text{tr}(\rho(x) \circ \rho(y))$.

Proof: It is invariant because

$$\text{tr}(\rho([x, y]) \circ \rho(z)) = \text{tr}(\rho(x)\rho(y)\rho(z)) - \text{tr}(\rho(y)\rho(x)\rho(z)) = \text{tr}(\rho(x)\rho(y)\rho(z)) - \text{tr}(\rho(x)\rho(z)\rho(y)) = \text{tr}(\rho(x)\rho([y, z])).$$

Prop. (I.12.9.6). If $\rho$ is a faithful representation of $\mathfrak{g}$ and $\mathfrak{g}$ is semisimple, then $\beta_\rho$ is non-degenerate.

Proof: The Cartan’s criteria (I.12.1.26) shows $\mathfrak{g}^\perp$ is a solvable sub-Lie algebra, so it must be 0 as $\mathfrak{g}$ is semisimple (I.12.1.22).

Prop. (I.12.9.7). Let $L$ be a simple lie algebra, then any two non-degenerate symmetric invariant bilinear forms on $L$ is proportional. Because any of this form corresponds to a $L$-morphism from $L$ to $L^*$. In particular, when $L \subset \mathfrak{gl}_n$, the usual trace is proportional to the Killing form.

Remark (I.12.9.8) (Rep($\mathfrak{g}$)). Let $\mathfrak{g}$ be a Lie algebra over a field $k$, let Rep($\mathfrak{g}$) denote the category of f.d. representations of $\mathfrak{g}$. 


Prop. (I.12.9.9) (Schur’s Lemma). Let \( g \) be a finite Lie algebra, \( M \) be an irreducible \( g \)-module, then \( \dim M \) is countable. In particular, Shur’s lemma holds by (III.1.1.2).

Proof: It is of countable dimensional because \( \dim U(g) \) is countable.

Prop. (I.12.9.10) (Generalized Eigenspace Decomposition for Nilpotent Lie Algebras). Assume \( k \) is alg.closed, and \( g \) is a nilpotent algebra, and \( (V, \rho) \) is a representation of \( g \), then there is a generalized eigenspace decomposition

\[
V = \bigoplus_{\lambda \in \mathfrak{g}^*} V^\lambda,
\]

where \( V^\lambda \) are the generalized eigenspaces, and they are stable under action of \( g \).

Proof: We use an induction argument:
- If each \( a \in \mathfrak{h} \) has only one eigenvalue, then \( V \) is the generalized eigenspace \( V_\lambda \) for some function \( \lambda \) on \( \mathfrak{h} \). Then it suffices to show that \( \lambda \) is linear. But this is because by Lie’s theorem elements of \( \mathfrak{h} \) has a common eigenvector.
- If for some \( a_0 \), \( \text{ad}(a_0) \) has two eigenvalues. Now \( \mathfrak{h} \) is nilpotent, so \( \mathfrak{h} \subset \mathfrak{h}_0^{a_0} \), and hence \( \pi(h)V_{a_0}^\lambda \subset V_{a_0}^\lambda \) for any \( \lambda \) by (I.12.1.11).
- As \( k \) is alg.closed, \( V \) can be written as a sum of generalized eigenspaces of \( a_0 \), and each generalized eigenspace is a subrepresentation of \( \mathfrak{h} \), thus we can use induction.

Def. (I.12.9.11) (Casimir Operator). Let \( g \) be a semisimple Lie algebra of dimension \( n \), and \( \beta : g \times g \to k \) a non-degenerate invariant bilinear form on \( g \). Let \( e_i \) be a basis of \( g \) and \( e'_i \) be the dual basis under \( \beta \), then \( c = \sum e_i e'_i \in U(g) \) is independent of the basis, and lies in the center of \( U(g) \).

Now the trace form \( \beta_V \) for a faithful representation \( g \to gl_V \) of \( g \) is non-degenerate and invariant (I.12.9.6), then the corresponding elements \( c_\rho \) is called the \textbf{Casimir element} of \( (V, \rho) \), and the action \( c_\rho \) of \( c_\rho \) on \( V \) is called the \textbf{Casimir operator} of \( (V, \rho) \).

The Casimir operator \( c_\rho \) is a \( g \)-module homomorphism, and has trace \( n \).

Proof: The independence of basis is by (I.1.7.9). To show it is in the center of \( U(g) \), Cf.[Mil13]P50.

Casimir operator \( c_\rho \) is a \( g \)-module homomorphism follows from the fact \( c_\rho \) is in the center of \( U(g) \), and its trace is

\[
\text{tr}(c_\rho) = \sum_i \text{tr}(e_i e'_i) = \sum_i (\beta_V(e_i, e'_i)) = n
\]

Def. (I.12.9.12) (Unimodular Lie Algebras). A f.d Lie algebra \( g \) is called \textbf{unimodular} if \( \wedge \text{ad} \) is a trivial representation of \( g \).

Prop. (I.12.9.13).
- If \( g = [g, g] \), then \( g \) is unimodular.
- If \( g \) is nilpotent, then \( g \) is unimodular.
- If \( g_1, g_2 \) is nilpotent, then \( g_1 \oplus g_2 \) is unimodular.
- If \( g \) is reductive, then \( g \) is unimodular.
Lemma (I.12.9.14) (Zassenhaus). Let \( g \) be a Lie algebra and \( g' \) an ideal of \( g \). A representation \( \rho' \) of \( g' \) extends to a representation \( \rho \) of \( g \) that \( \rho'(g') \subseteq n_\rho(g) \) if there exists a Lie subalgebra \( h \) of \( g \) that \( g = g' \oplus h \) and \( [h, g'] \subseteq n_\rho'(g') \). If moreover \( \text{ad}_g(x)|_{g'} \) is nilpotent for all \( x \in h \), then \( \rho \) can be chosen so that \( h \subseteq n_\rho(g) \).


Prop. (I.12.9.15) (Nilpotent Representation Extension). Let \( g \) be a Lie algebra, \( a \) a nilpotent ideal of \( g \), and \( \rho \) a representation of \( a \) that \( \rho(x) \) is nilpotent for all \( x \in a \). Then \( \rho \) extends to a representation \( \rho' \) of \( g \) that \( \rho'(x) \) is nilpotent for all \( x \in n \) the largest nilpotent ideal of \( g \).

Proof:

Thm. (I.12.9.16) (Ado). Let \( g \) be a Lie algebra over a field \( k \) of char 0, then there exists a faithful representation \( \rho \) of \( g \) that \( \rho(n) = 0 \), where \( n \) is the largest nilpotent radical. If \( g \) is of f.d., then this representation can be chosen to be of f.d.. In particular, any finite dimensional Lie algebra can be embedded in some \( gl(n, k) \).

Proof: This is true for any commutative Lie algebras, for example we can use tensor products of \( c \mapsto \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} \). Choose a faithful representation of the center \( c \) of \( g \) that every element is mapped to a nilpotent endomorphism, and then extend it to a representation \( \rho_1 \) of \( g \) by (I.12.9.15). Let \( \rho_2 \) be the adjoint representation of \( g \), and \( \rho = \rho_1 \oplus \rho_2 \). Then \( \text{Ker}(\rho) = \text{Ker}(\rho_1) \cap \text{Ker}(\rho_2) = \text{Ker}(\rho_1) \cap c = 0 \), so this is faithful. And it sends an element in \( n \) to a nilpotent endomorphism because each \( \rho_1 \) and \( \rho_2 \) do.

Remark (I.12.9.17). In fact, this is true for Lie algebras over a field of char \( p \), too, Cf. [JACOBSON, N. 1962. Lie algebras.] Chap 6.3.

10 Lie Algebra Cohomology

For representations of a semisimple Lie algebra, See III.6.

11 Amitsur-Levitski

Prop. (I.12.10.1) (Chevalley-Eilenberg resolution).

Preliminary

Notice that in this paper, Kostant considers reductive lie groups. But in the range of this paper, the abelian part makes no contribution in the alternative part because they commute with all elements. So We well just consider a semisimple Lie algebra in order to get a non-degenerate Killing form.

Prop. (I.12.11.1).

\[
\Gamma(\wedge^{2k}(g)) = R^k \subseteq S^k(g)
\]

\[
\Gamma(x_1 \wedge \cdots \wedge x_k) = \sum_{v \in \Pi_k} [x_{v(1)}, x_{v(2)}] \cdots [x_{v(2k-1)}, x_{v(2k)}]
\]
Proof: The proof is in fact simple, just notice that for every $v \in \Pi_k$ a representative of the subgroup $\Sigma_{2k,2}$ permuting the unordered pairs $\{(1,2), (3,4), \ldots, (2k-1,2k)\}$, the element in $v\Sigma_{2k,2}$ in fact combine in pairs to $[x_{v(2i-1)}, x_{v(2i)}]$ and together the $k!$ permutation of them compose a $[x_{v(1)}, x_{v(2)}] \cdots [x_{v(2k-1)}, x_{v(2k)}]$. Later he finds the dual of $\Gamma$, that is:

**Prop. (I.12.11.2).** $\gamma$ is $\mathcal{B}$ dual to $\Gamma$,

$$(\Gamma(\zeta), y_1 \cdots y_r) = (-1)^r (\zeta, dy_1 \wedge \cdots \wedge dy_r) \quad \forall \zeta \in \wedge^r \mathfrak{g} \text{ and } y_i \in \mathfrak{g}.$$ In particular,

$$\Gamma(\zeta)(x) = \frac{(-1)^r}{r!} (\zeta, (dx)^r) \quad \forall \zeta \in \wedge^r \mathfrak{g} \text{ and } x \in \mathfrak{g}.$$ For the proof just notice that $(-dw, x_i \wedge x_j) = (w_i, [x_i, x_j])$ and

$$(x_1 \otimes \cdots \otimes x_r, y_1 \otimes \cdots \otimes y_r) = \sum_\sigma (x_1, y_{\sigma(1)}) \cdots (x_r, y_{\sigma(r)})$$

So dim of $R^k$ equals the dim of image of $\gamma_r$, that is, spanned by $(dx)^k$ (because they are dual). We say that a representation of $\mathfrak{g}$ satisfies m-fold standard identity if the alternating sum of any $m$ elements of image of $\mathfrak{g}$ is 0. Obviously, this is equivalent to:

$$\tau(R^k(\mathfrak{g})) \subset \text{Ker} \pi_V$$

Now let $o(\mathfrak{g})$ be the maximum rank of $dw, w \in \mathfrak{g}$, then by the discussion of the first paper, when $\mathfrak{g}$ is semisimple, $o(\mathfrak{g}) = r$. So the $2r$-identity is satisfied by any representation of $\mathfrak{g}$.

Furthermore a prop of [Harish-Chandra] assert for any nonzero element $u \in U(\mathfrak{g})$, there is a representation such that $\pi(U) \neq 0$. So this is a sharp bound for general representations.

But one might naturally ask: Can we find the specific bound for a particular representation of a specific $\mathfrak{g}$? The answer is YES.

**Prop. (I.12.11.3).** $\gamma$ vanishes on the ideal $J_+ \cdot S(\mathfrak{g})$.

Proof: The proof comes from the observation $\pi$ is a $G$-map and by Cartan-Koszul theory, invariant elements in $\wedge \mathfrak{g}$ are naturally isomorphic to the cohomology of $\mathfrak{g}$ and $\gamma(w) = -dw$ is clearly exact, Thus $\gamma(w_1w_2 \ldots w_i) = (-1)^i dw_1 \wedge \cdots \wedge dw_i$ is exact too. So $\gamma(w) = 0$. □

**Cor. (I.12.11.4).** $M = R^r \in H^r$, so it consists of harmonic functions.

Proof: $(u, \Gamma(y)) = (\gamma(u), y) = 0 \forall u$ invariant, by Thm; so $R^k =$ Image $\Gamma$ is harmonic. □

### Generalized Amitsur-Levitski

Let $E^k \subset U(\mathfrak{g})$ be spanned be $y^k, y$ nilpotent in $\mathfrak{g}$, and $Z$ the center.

In [Kos97] Kostant proved that the PBW isomorphism $\delta : S(\mathfrak{g}) \to U(\mathfrak{g})$ induces $\delta(J) = Z, \delta(H) = E$.

So $\tau : T(\mathfrak{g}) \to U(\mathfrak{g})$ induces

$$\tau(A^{2k}(\mathfrak{g})) = \delta(R^k(\mathfrak{g})) \subset E^k.$$ Define $\epsilon(\pi)$ the minimum integer $k$ that $\pi(y)^k = 0, \forall y$ nilpotent in $\mathfrak{g}$. Then clearly:
Prop. (I.12.11.5) (Generalized Amitsur-Levitski). $\pi$ satisfies the $2\epsilon(\pi)$-fold standard identity.

Prop. (I.12.11.6). If $\pi$ satisfies the $m$-identity, it satisfies the $m + 1$-identity (by taking a summation on a fixed first element $\sigma(1)$).

Prop. (I.12.11.7).

- Let $\pi$ be the natural representation of $\mathfrak{gl}_n$ on $\mathbb{C}^n$ then $\epsilon(\pi) = n$.
- If $n$ even and $\pi$ the natural representation of skew-symmetric matrices on $\mathbb{C}^n$ then $\epsilon(\pi) = n - 1$. From this one derives the classical Amitsur-Levitski prop that $GL_n(\mathbb{C})$ satisfies the $n$-fold standard identity.

Proof: 1: the abstract Jordan decomposition which assures $x$ is nilpotent in $\mathfrak{gl}_n$ if $\pi(x)$ is nilpotent.

2: comes from the Lacobson-Morosov Thm that any nilpotent element of $\mathfrak{g}$ is contained in a $\mathfrak{sl}_2$-triple. Thus we only need to show that $W$ is reducible considered as this $\mathfrak{sl}_2$-triple-module.

But then an irreducible representation of $\mathfrak{sl}_2$ preserves a non-degenerate bilinear form it must be odd dimensional cause a non-degenerate bilinear form is equivalent to a $\mathfrak{g}$-map from $V$ to $V^*$.

And there can be constructed an anti-symmetric form defined on the $\mathfrak{sl}_2$-representation on $\text{Sym}_{2k}[x, y]$ by (I.12.11.9), so there can’t exist symmetric $\mathfrak{g}$-invariant form. So this representation must be reducible.

□

Cor. (I.12.11.8) (Classical Amitsur-Levitski). By (I.12.11.5),

$$[[x_1, x_2, \ldots, x_n]] = \sum_{\sigma} \text{sgn}(\sigma)x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(k)} = 0 \quad \forall x_i \in \mathfrak{gl}_n$$

called $n$-fold standard identity.

Prop. (I.12.11.9). For a construction of the anti-symmetric form, notice

$$\pi(g)f(x_1, x_2) = f(g_{11}x_1 + g_{12}x_2, g_{21}x_1 + g_{22}x_2).$$

Set

$$v_k = \binom{m}{k} x_1^{m-k} x_2^k, \quad \Omega(v_k, v_{m-k}) = (-1)^k \binom{m}{k} \cdot \Omega(v_k, v_p) = 0, \quad k + p \neq m.$$  

One verifies:

$$\Omega(g \cdot u, g \cdot v) = (\det g)^{m} \Omega(u, v) \quad \forall g \in GL(2, \mathbb{C})$$

So when $m = n - 1$ is odd, this is a symplectic form preserved by $\mathfrak{sl}_2$.

A computable Formula

Finally, Kostant gave a computable formula for determining $\epsilon(\pi)$. Clearly we just need to consider irreducible representation.

Let $\pi_\lambda$ be the irreducible representation of highest weight $\lambda$, then the dual representation $\pi_{\lambda'}$ has highest weight the negative of the lowest weight of $\pi_\lambda$, that is, $-w_\alpha(\gamma)$.

But then $\lambda + \lambda'$ is a sum of simple positive roots, $\lambda + \lambda' = \sum_{i=1}^{n} m_i \alpha_i$. Put $\epsilon(\lambda) = 1 + \sum_{i=1}^{n} m_i$. Then:

Prop. (I.12.11.10). $\epsilon(\pi_{\lambda}) = \epsilon(\lambda)$.

Proof: Just choose a $\mathfrak{sl}_2$-triple $\{e, x, f\}$ with $\alpha(x) = 2$ $\forall$ simple root. Then $\lambda(x)$ and $-\lambda'(x)$ are respectively the maximal and minimal eigenvalues of $\pi(x)$. $(\lambda + \lambda')(x) = 2(\epsilon(\lambda) - 1)$. Thus $f$ has nilpotent degree $\epsilon(\lambda)$. And any nilpotent element action increases the eigenvalue of an eigenvector of $x$ by at lest $2$, the prop follows.
Further Work

(cf. [Pro76]) Another different proof of the Amitsur-Levitski theorem is given by Kostant using
techniques related to trace identities. It turns out that method sheds more light. Later is studied
the polynomial of matrices invariant under the conjugation action.

Artin conjectured that all the invariants is polynomials of the Trace polynomial
\( Tr(A_1 A_2 \cdots A_n) \) (Proved)

And further, the relations among these invariant all turned out to be consequences of the prop
of Hamilton-Cayley. All this is made into the Invariant Theory.

Prop. (I.12.11.11) (Interesting results).
1. If an algebra over a field of characteristic 0 satisfies the identity \( X^n = 0 \), then it satisfies all
   the identities of \( n \times n \) matrices.
2. The space of multilinear identities of degree \( m \) of \( n \times n \) matrices can be described completely
   in terms of Young diagrams.

12 Lie \( p \)-Algebras

Remark (I.12.12.1). In this subsection, let \( k \) be a field of characteristic \( p \).

Def. (I.12.12.2) (Lie \( p \)-Algebras). Let \( x_0, x_1 \) be elements of a Lie algebra over \( k \) of characteristic
\( p \), then for \( 0 < r < p \), let \( s_r(x_0, x_1) \) denote \( \frac{1}{r!} \) times coefficient of \( t^{r-1} \) in the expression of \( \text{ad}^{p-1}_x y(x) \).

Then a Lie \( p \)-algebra is a Lie algebra \( g \) over \( k \) equipped with a map \( x \mapsto x^{[p]} : g \to g \) such that
- \( (cx)^{[p]} = c^p x^{[p]} \) where \( c \in k \).
- \( \text{ad}(x^{[p]}) = (\text{ad}(x))^p \).
- \( (x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{r=1}^{p-1} s_r(x, y) \).

Example (I.12.12.3). If \( L \) is an Abelian Lie algebra, then it can be regarded as a Lie \( p \)-algebra
by assigning \( x^{[p]} = 0 \).

If \( A \) is an associative algebra, then it can be given a Lie \( p \)-algebra structure by assigning \( x^{[p]} = x^p \).

Proof: To show \( A \) is a Lie \( p \)-algebra, we check \( \text{ad}(x)^p(y) = (l_x - r_x)^p(y) = l_x^p - r_x^p(y) = \text{ad}(x^p)(y) \).

For the third formula, notice
\[
\text{ad}(tx + y)^{p-1}(x) = \sum_{i=0}^{p-1} (-i)^i \binom{p-1}{i} (tx + y)^{p-1-i} x(ty + y)^i
\]
Notice that \( (-i)^i \binom{p-1}{i} \equiv 0 \mod p \), so \( s_r(x, y) \) is equivalent to the sum of words of \( x, y \) with \( r \) xs. So
the third formula clearly holds.

Def. (I.12.12.4) (Representations). A representation of a Lie \( p \)-algebra \( g \) over a vector space
\( V \) is a homomorphism \( g \to gl_V \) of Lie \( p \)-algebras.

Def. (I.12.12.5) (Universal Enveloping \( p \)-Algebra). The universal enveloping \( p \)-algebra
of a Lie \( p \)-algebra \( g \) is defined to be
\[
U^{[p]}(g) = T(g)/J^{[p]}, \quad T(g) = \bigoplus_{n=0}^{\infty} g^{\otimes n}
\]
which is a graded algebra \( T(g) \) quotients the ideal \( J = \{ x \otimes y - y \otimes x - [xy], x^{\otimes p} - x^{[p]} \} \).

There is a natural linear map \( \sigma : g \to U^{[p]}(g) \).
Prop. (I.12.12.6). The universal enveloping $p$-algebra $U[p] : \mathfrak{g} \to U[p](\mathfrak{g})$ defines a functor $\text{LieAlg} \to \text{AssAlg}$ that is left adjoint to the canonical functor $\text{AssAlg}_k \to \text{LieAlg}_k$(I.12.12.3).

Proof: For any associative algebra $A$ and a morphism of Lie $p$-algebras $\mathfrak{g} \to [A]$, there is easily seen a morphism $U[p](\mathfrak{g}) \to A$, and it is unique. □

Cor. (I.12.12.7) (Representation as Modules). A representation of $\mathfrak{g}$(I.12.12.4) is the same as a representation of $U[p](\mathfrak{g})$.

Prop. (I.12.12.8). Let $e_i$ be a $k$-vector space basis of $\mathfrak{g}$, then the monomials

\[ \{ e_{i_1}^{n_1} e_{i_2}^{n_2} \ldots e_{i_r}^{n_r} | i_1 < i_2 < \ldots < i_r, 0 < n_{i_k} < p \} \]

for different $r$ form a basis of $U[p](\mathfrak{g})$.

Proof: This is a consequence of PBW theorem(I.12.8.4). □

Cor. (I.12.12.9). If $\mathfrak{g}$ is of finite dimensional over $k$, then so does $U[p](\mathfrak{g})$, and the map $i : \mathfrak{g} \to U[p](\mathfrak{g})$ is injective.
I.13 Divided Power Algebras

Basics

Def. (I.13.0.1) (PD-Structures). Let $I$ be an ideal of a commutative ring $A$, a divided power structure or pd-structure on $I$ is a collection of maps $I_n : I \to A, n \geq 0$ that

- $\gamma_0(x) = 1, \gamma_1(x) = x, \gamma_n(x) \in I \forall n$.
- $\gamma_n(x + y) = \sum \gamma_{n-i}(x) \gamma_i(y)$.
- $\gamma_n(\lambda x) = \lambda^n \gamma_n(x)$, $\lambda \in A, x \in I$.
- $\gamma_m(x) \gamma_n(x) = \binom{m+n}{n} \gamma_{m+n}(x)$.
- $\gamma_n(\gamma_m(x)) = \frac{(mn)!}{(m!)^n} \gamma_{mn}(x)$.

It is a simulation of the divided power $\gamma_n(x) = \frac{x^n}{n!}$ in case $n!$ is definable.

A divided power ring is a triple $(A, I, \gamma)$ where $I$ is an ideal of a commutative ring $A$ and $\gamma$ is a pd-structure on $I$. A morphism of divided power rings is a morphism of pairs $(A, I)$ that preserves pd-structures.

For a pd-structure $(A, I)$, denote $I[^n]$ the ideal generated by $\prod x_i$ where $x_i \in I$ and $\sum n_i \geq n$.

Prop. (I.13.0.2) (Limits and Colimits). The category of divided power rings has all limits and colimits, the limits commute with forget functors but the colimits don’t. However, the colimit always commutes with the functor taking $(A, I)$ to $A \lll I$. This can be seen from the universal property of colimit applied to the pd-structures that $I = 0$.

Proof: The construction of the limit is clear. For the colimits, we use representability criterion (II.1.1.19), Cf.[[[Sta]07GX]].

Prop. (I.13.0.3). Let $A$ be a ring and $I$ an ideal of $A$, then if $\gamma$ is a pd-structure on $I$, then $n! \gamma(x) = x^n$.

Proof: If $\gamma$ is a pd-structure, then we have $n \gamma_n(x) = \gamma_1(x) \gamma_{n-1}(x)$, so we can use induction.

Prop. (I.13.0.4). If $I, J$ are two ideals of $A$ and $\gamma$ a pd-structure on $I$ and $\delta$ a pd-structure on $J$, then

- $\gamma, \delta$ agree on $IJ$.
- $\gamma, \delta$ agree on $I \cap J$, then they extends to a pd-structure on $I + J$.

Proof: 1: for $x \in I, y \in J$, $\gamma_n(xy) = y^n \gamma_n(x) = n! \gamma_n(x) = \delta(xy)$.

2: direct calculation.

Prop. (I.13.0.5) ($p$-Nilpotent and Thickening). Let $p$ be a prime and $(A, I, \gamma)$ a pd-structure. Assume $p$ is nilpotent in $A/I$, then $I$ is locally nilpotent iff $p$ is nilpotent in $A$, equivalently $(A, I, \gamma)$ is a pd-thickening.

Proof: If $p^N = 0 \in A$, then for any $x \in I$, $x^{p^N} = (pN)! \gamma_{pN}(x) = 0$. Then converse is trivial.
Constructing PD-Structures

Prop. (I.13.0.6) \((\mathbb{Z}(p)\text{-Algebras})\). Cf.\([\text{Sta}07\text{GN}]\).

Prop. (I.13.0.7). Let \(A\) be a \(\mathbb{Z}(p)\)-algebra and \(I\) is an ideal, then two pd-structures \(\gamma, \gamma'\) on \(\gamma\) are equal iff \(\gamma_p = \gamma'\). Moreover, given a map \(\delta : I \to I\) that
- \(p\delta(x) = x^p\),
- \(\delta(ax) = a^p\delta(x)\) for any \(a \in A, x \in I\),
- \(\delta(x + y) = \delta(x) + \delta(y) + \sum_{i+j=p, i>0, j>0} \frac{1}{ij!} x^i y^j\).

Then there exists a unique pd-structure on \(I\) that \(\gamma_p = \delta\).

Proof: Just notice that \(\gamma_n(x) = dx\gamma_{n-1}(x)\) for some \(c\) invertible in \(\mathbb{Z}(p)\), and also \(\gamma_{pm}(x) = c\gamma_m(\gamma_p(x))\) for some \(c\) invertible in \(\mathbb{Z}(p)\), thus \(\gamma\) is uniquely determined, and we can also define \(\gamma_n\) inductively in this way, and the verification of axioms in \([\text{Sta}07\text{GS}]\). \(\square\)

Prop. (I.13.0.8). Let \(A\) be a \(\mathbb{Z}\)-torsion-free ring and \(I\) an ideal of \(A\), then:
- \(I\) has at most one pd-structure.
- if \(\gamma_n : I \to I\) are maps, then \(\gamma\) is a pd-structure iff \(n!\gamma_n(x) = x^n\).
- \(I\) has a pd-structure iff there is a set of generators \(\{x_i\}\) of \(I\) that \(x_i^n \in n!I\).

Proof: 1 is clear from(I.13.0.3).
2: because \(A \subset A \otimes \mathbb{Z} \mathbb{Q}\), we can verify in \(A \otimes \mathbb{Z} \mathbb{Q}\), then the verifications are trivial.
3: Use the axioms to extend linearly and additively. \(\square\)

Prop. (I.13.0.9) (DVR). If \(R\) is a DVR in char \(0\) with residue field of char \(p\) and ramification \(p\) and maximal ideal \(m\), then \(m\) has a pd-structure iff \(e \leq p - 1\).

Proof: As \(R\) has char0, it has at most one pd-structure by(I.13.0.8), and we need to show that \(x^n/n!\) is in \(m\) for any \(x \in m\). And using(XIV.1.2.1), we are done. \(\square\)

Extending PD-Structure

Def. (I.13.0.10) (Extending PD-Structure). Let \((A, I, \gamma)\) be a pd-structure and \(B\) is an A-algebra, we say that \(\gamma\) extends to \(B\) if \(A \to B\) extends to a morphism of pd-structures \((A, I, \gamma) \to (B, IB, \gamma')\).

Let \((A, I), (B, J)\) be two pd-structures and \(B\) is an A-algebra, then these two pd-structures are said to be compatible iff the pd-structure on \(A\) extends to \(B\) and the pd-structure on \(J\) and \(IB\) coincides, or equivalently, there is a pd-structure on \(IB + J\) compatible with \(IB\) and \(J\), by(I.13.0.4).

Prop. (I.13.0.11) (Extendability). Let \((A, I)\) is a pd-structure and \(B\) an A-algebra, if any of the following holds:
- \(IB = 0\),
- \(I\) is principal,
- \(B[I] = 0\), (e.g. \(A \to B\) is flat).
then \(\gamma\) extends to \(B\).
Proof: 1 is trivial.

2: if $I = \{x\}$, we define $\gamma_n(bx) = b^n\gamma_n(x)$. This is well defined: if $(b - b')x = 0$, then $(b^n - (b')^n)\gamma_n(x) = 0$ because $\gamma_n(x) \in \{x\}$. Verifications of axioms is routine.

3: The condition shows $I \otimes_A B \cong IB$, thus it suffices to define $\gamma$ on $I \otimes_A B$. For this we define on $I \times B$ and descend: let $\gamma_n((x, b)) = b^n\gamma_n(x)$ and extend by freeness and axioms in (I.13.0.1), then it is easy to show it is bi-additive and $A$-linear, so descend to $I \otimes B$ by (I.2.4.13).

Prop. (I.13.0.12) (PD-Structure and Completions). Let $(A, I, \gamma)$ be a pd-structure that $p$ is nilpotent in $A/I$, then each $\gamma_n$ is continuous in the $p$-adic topology and extends to a pd-structure $\tilde{\gamma}$ on $\hat{I}$.

If moreover $A$ is a $\mathbb{Z}(p)$-algebra, then for $e$ large, $p^e A \in I$ is preserved by $\gamma$ and

$$(\hat{A}, \hat{I}, \tilde{\gamma}) = \mathrm{colim}_e (A/p^eA, I/p^eA, \gamma).$$

Proof: Let $p^i \in I$, then 1 follows from (I.5.7.8). $\gamma_n$ is clearly continuous, and $\gamma_n$ preserves $p^e A$ because

$$\gamma_n(p^e a) = p^n \gamma_n(p^{e-1}a) = \frac{p^n}{n!} \gamma_m(x)^n \in p^e A.$$ 

The limit equation follows from (I.13.0.2).

Prop. (I.13.0.13) (Quotient). Let $(A, I, \gamma)$ be a pd-structure and $a \subset A$ is an ideal, and $I' = I \cap a$, then the following are equivalent:

- $\delta$ extends to $A/a$.
- $I'$ is preserved by $\gamma$.
- There is a set of generator $x_i$ of $I'$ that $\gamma_n(x_i) \in I'$ for all $n, i$.

Proof: 1 $\rightarrow$ 2 $\rightarrow$ 3 is clear. 2 $\rightarrow$ 1: we can just define $\gamma(x + I) = \gamma(x) + I$, which is well defined by axiom 2 of (I.13.0.1). 3 $\rightarrow$ 2 is clear.

Def. (I.13.0.14) (Free PD-Algebra). For a pd-structure $(A, I, \delta)$, the free pd-algebra $A\langle t_1, \ldots, t_n \rangle$ is defined to be the $A$-algebra generated by symbols $t_i^{[n]}$, where $n_i > 0$, modulo the algebraic relations $t_i^{[m]} t_i^{[n]} = (\frac{m+n}{n})^{[m+n]} t_i^{[m+n]}$. Denote by $A\langle t_1, \ldots, t_n \rangle_+$ the ideal generated by $t_i^{[n]}$ where $n_i > 0$.

Then the ideal $J$ generated by $I$ and $A\langle t_1, \ldots, t_n \rangle_+$, where $n_i > 0$ has a unique pd-structure that $\gamma_n(t_i) = t_i^{[n]}$ and $(A, I, \delta) \rightarrow (A\langle t_1, \ldots, t_n \rangle_+, J, \gamma)$ is a morphism of pd-structures.

It has a universal property that $\mathrm{Hom}((A\langle t_1, \ldots, t_n \rangle_+, J, \gamma), (C, K, \varepsilon))$ is the same as $\mathrm{Hom}((A, I, \delta), (C, K, \varepsilon))$ with specified $n$ elements in $K$.

Proof: Because $IA(t_i) \cap A\langle t_1, \ldots, t_n \rangle_+ = IA(t_i) + A\langle t_1, \ldots, t_n \rangle_+$, by (I.13.0.4), it suffices to construct pd-structures on $IA(t_i)$ and $A\langle t_1, \ldots, t_n \rangle_+$. The former is by (I.13.0.11) and for the latter: if $A$ is torsion-free, then we can use (I.13.0.8) because $\gamma_m(x)^n = n! \gamma_m(\gamma_m(x)) = \frac{mn}{ml} \gamma_m(x) \in n! A\langle t_1, \ldots, t_n \rangle_+$. In general, we write $A = R/a$ where $R$ is a torsion-free pd-structure (can choose $\mathbb{Z}(A))$, so there is a pd-structure on $R\langle t_i \rangle$, and $R\langle t_i \rangle/a\langle t_i \rangle = A\langle t_i \rangle$, then we can use (I.13.0.13) to construct a pd-structure on $A\langle t_i \rangle$ compatible with that of $A$.

The verification of universal property is omitted.
PD-Envelopes

Prop. (I.13.0.15) (PD-Envelope). Let \((A, I, \gamma)\) be a pd-structure, then there is a pd-envelope functor \(B, J \rightarrow (D_B(J), J, S)\) from the the category of pairs over \((A, I)\) to the category of pd-structures over \((A, I, \gamma)\) that is left adjoint to the forgetful functor.

In particular, by the universal property of pd-envelope, there is a morphism \((B, J) \rightarrow (D_B(J), J) \rightarrow B/J, 0\) of pairs, so in particular \(D/J \rightarrow B/J\) is surjective.

Proof: We use adjoint functor theorem (II.1.1.24), the forgetful functor preserves limit by (I.13.0.2), and it satisfies the set-theoretical condition: for any pair \((B, J)\) over \((A, I)\) and a morphism \(\psi : (B, J) \rightarrow (C, K)\) over \((A, I)\) where \(\varphi : (A, I, \gamma) \rightarrow (C, K, \delta)\) is a pd-morphism, then we can consider the subring \(C' \subset C\) generated by all \(\varphi(A), \psi(B)\) and \(\delta_m(J), \) and \(K' \subset K \cap C'\) the ideal of \(C'\) generated by \(\varphi(I), \delta_n(\psi(J))\), then \(|C'| < |A| \otimes |B|^{k_0}\) and its type is bounded by a cardinal, so does \((C, I)\).

\(\square\)

Prop. (I.13.0.16) (PD-Envelope of Quotients). Let \((A, I, \gamma)\) be a pd-structure and \(\varphi : B' \rightarrow B\) be a surjection of \(A\)-algebras with kernel \(K\). Let \(IB \subset J \subset B\) an ideal and \(J' = \varphi^{-1}(J)\) and \(D_{B', \gamma}(J') = (D', J', \gamma)\), then \(D_{B, \gamma}(J) = (D'/K', J'/K', \gamma)\) where \(K'\) is the ideal generated by all \(\tau_n(k)\) for \(n \geq 0\) and \(k \in K\).

Proof: There is a pd-structure on \((D'/K', J'/K', \gamma)\) by (I.13.0.13). A map of pais \((B, J) \rightarrow (T, J', \gamma)\) is equivalent to a map of pairs \((B', J') \rightarrow (T', J', \gamma)\) that vanishes on \(K\), or a map of pd-structures \((D', J', \gamma) \rightarrow (T', J', \gamma)\) that vanishes on \(\gamma_n(K)\), thus this is clearly represented by \((D'/K', J'/K', \gamma)\).

\(\square\)

Prop. (I.13.0.17). If \((A, I)\) is a pd-structure and \((B, J)\) is a pair over \((A, I)\), then

\[
D_{B[X_i], \gamma}(J(B[X_i]) + (X_i)) \cong D_{B, \gamma}(J(X_i))
\]

Proof: This follows from the universal property of free pd-structure (I.13.0.14).

\(\square\)

PD-Structures and \(\delta\)-Rings

Lemma (I.13.0.18). If \(A\) is a \(p\)-torsionfree \(\mathbb{Z}_p\)-\(\delta\)-ring, denote \(\gamma_n(z) = \frac{z^n}{p^n}\). If \(z \in A\) satisfies \(\gamma_p(z) \in A\), then \(\gamma_n(z) \in A\) for any \(n\).

Proof: WARNING: this is not an easy consequence of power counting. We first prove for \(n = p^2\): as \(A\) is a \(\delta\)-ring, \(\delta(\frac{z^p}{p}) \in A\)

\[
\delta(\frac{z^p}{p}) = \frac{1}{p} (\varphi(z)^p - \frac{z^p}{p}) = \frac{(z^p + p\delta(z))^p}{p^2} - \frac{z^p}{p^2 + 1} \in A.
\]

The first term is in \(A\) by assumption, thus the second term is also in \(A\), proving the case for \(n = p^2\).

Now for general \(n\), it suffices to prove for \(n = kp\). But it can be checked that \(\gamma_{nk}(z) = u\gamma_k(\gamma_p(z))\) where \(u\) is a unit. Now by what we just proved, we can use induction hypothesis for \(z = \gamma_p(z)\), and conclude that \(\gamma_{nk}(z) \in A\).

\(\square\)

Prop. (I.13.0.19). The ring \(C = \mathbb{Z}_p\{x, \varphi(x)\}\) identifies with the pd-envelope \(D = D_{\mathbb{Z}_p\{x\}}(x) = \mathbb{Z}_p\{x\}\{\gamma_n(x)\}\) where \(\gamma_n(x) = \frac{x^n}{p^n}\). Moreover, it also equals to \(\mathbb{Z}_p[X_1, X_2, \ldots ]/(pX_1 - x^p, pX_2 - x^p, \ldots )\).
Proof: It suffices to show that the smallest $\delta$-ring of $\mathbb{Z}(p) \{x\}[[x]]$ containing $\mathbb{Z}(p) \{x\}$ and $\varphi(x)/p$ is the same as the smallest ring of $\mathbb{Z}(p) \{x\}[[x]]$ containing $\mathbb{Z}(p) \{x\}$ and $\varphi(x)/p$.

$D \subset C$ is immediate from (I.13.0.18). To show $C \subset D$, notice $\varphi^p \in D$, it suffices to show $\varphi$ preserves $D$, or equivalently, $\varphi(y) - y^n \in pD$ for any $y \in D$. Now

$$\varphi(x^n) = (x^p + p\delta(x))^n = \sum_{i=0}^{n} \binom{n}{i}(pi)p^{n-i}\delta(x)^{n-i}$$

The coefficients

$$\binom{n}{i}(pi)!p^{n-i}$$

are all in $p\mathbb{Z}(p)$, thus $\varphi(x^n/p) \in D$. On the other hand,

$$\left(\frac{x^n}{n!}\right)^p = \gamma_p(\gamma_n(x)) = u\gamma_{pn}(x) \cdot p! \in pD$$

where $u$ is a unit in $\mathbb{Z}(p)$ by (I.13.0.1), thus we are done.

Lemma (I.13.0.20). If $A$ is a $p$-torsionfree (equivalently, flat) $\mathbb{Z}(p)$-algebra and $(a, p)$ is a regular sequence in $A$, then $D_A((a)) \cong A \otimes_{\mathbb{Z}(p)} D_{\mathbb{Z}(p)}((x)) = A[X_1, X_2, \ldots]/(pX_1 - a^p, pX_2 - X_1^{p}, \ldots)$.

Proof: By (I.13.0.19),

$$A \otimes_{\mathbb{Z}(p)} D_{\mathbb{Z}(p)}((x)) = A \otimes_{\mathbb{Z}(p)} \{\{\gamma_n(x)\}\} = A[X_1, X_2, \ldots]/(a^p - pX_1, X_1^{p} - pX_2, X_2^{p} - pX_3, \ldots)$$

thus there is a natural map from $A \otimes_{\mathbb{Z}(p)} D_{\mathbb{Z}(p)}((x))$ to $D_A((a))$ given by

$$X_k \mapsto \frac{a^p}{p^{\text{ord}_p(n!)}} = \frac{a^p}{p! + p! + \cdots + p! -1} \in D_A((a))$$

It is surjective, and it is an isomorphism when inverting $p$. Thus the kernel are all $p^\infty$-torsions. Then to show it is an isomorphism, it suffices to show $A[X_1, X_2, \ldots]/(a^p - pX_1, X_1^{p} - pX_2, X_2^{p} - pX_3, \ldots)$ is $p$-torsion-free. It is a filtered colimit, so it suffices to show $A[X_1, X_n]/(a^p - pX_1, X_1^{p} - pX_2, X_2^{p} - pX_3, \ldots)$ is $p$-torsion-free.

For this, it suffices to prove $A' = A[X_1, \ldots, X_k]/(px_1 - a^k, px_2 - x_1^{k}, \ldots, px_k - x_{k-1}^{p})$ is $p$-torsion-free. If we denote $K(R) = K^s(R[X_1, \ldots, X_k], px_1 - a^k, px_2 - x_1^{k}, \ldots, px_k - x_{k-1}^{p})$ for any ring $R$, then we have a distinguished triangle

$$K(A) \xrightarrow{p} K(A) \rightarrow K(Kos(A[X_1, \ldots, X_k], p)) = K(A/p)$$

as $A$ is $p$-torsionfree. Now we can consider the spectral sequence associated to this distinguished triangle. Notice first that $(px_1 - a^k, px_2 - x_1^{k}, \ldots, px_k - x_{k-1}^{p})$ is a regular sequence in $A'/p$, which is clear. So the $E_1$ page looks like

$$\begin{array}{cccc}
A' & \xrightarrow{p} & A' & \rightarrow A'/p \\
\uparrow & & \uparrow & \\
\ast & \rightarrow & \ast & \rightarrow 0 \\
\uparrow & & \uparrow & \\
\ast & \rightarrow & \ast & \rightarrow 0 \\
\uparrow & & \uparrow & \\
\vdots & \rightarrow \ldots & \rightarrow \ldots & \\
\end{array}$$
Prop. (I.13.0.21) (PD-Envelope for Regular Sequence). Let $A$ be a $p$-torsionfree $\delta$-ring and $p, f_1, \ldots, f_r$ define a regular sequence in $A$, then $A\{\frac{\varphi(f_1)}{p}, \ldots, \frac{\varphi(f_r)}{p}\}$ identifies with the pd-envelope $D_A(I)$ of $I = (f_1, \ldots, f_r)$ as a subring of $A[p]$.

Proof: In case $r = 1$, $A\{\frac{\varphi(f_1)}{p}\} = A \otimes_{\mathbb{Z}(p)} \{x, \varphi(x)\} = A \otimes_{\mathbb{Z}(p)} D_{\mathbb{Z}(p)}((x)) \cong D_A((f_1))$ by (I.13.0.20).

The general case follows from this, by considering the tower

$$
(A, (f_1)) \rightarrow (D_A(f_1), (f_2)) = (A\{\frac{\varphi(f_1)}{p}\}, (f_2))
$$

$$
\rightarrow (D_A\{\frac{\varphi(f_1)}{p}\}, (f_3)) = (A\{\frac{\varphi(f_1)}{p}, \frac{\varphi(f_2)}{p}\}, (f_3))
$$

$$
\rightarrow \ldots = A\{\frac{\varphi(f_1)}{p}, \ldots, \frac{\varphi(f_r)}{p}\}
$$

The equations are true because $(p, f_k)$ are regular in $A\{\frac{\varphi(f_1)}{p}, \ldots, \frac{\varphi(f_{k-1})}{p}\}$:

$$
A\{\frac{\varphi(f_1)}{p}, \ldots, \frac{\varphi(f_{k-1})}{p}\}/p = A/p[X_1, X_2, \ldots, X_1, X_{k-1}, X_{k-2}, \ldots] /
$$

$$(f_1^p, f_2^p, \ldots, f_{k-1}^p, X_1^p, X_{12}^p, \ldots, X_{k-1}^p, X_{k-2}^p, \ldots)$$

and $f_k$ is a non-zero divisor in it because it is a non-zero divisor in $A/(p, f_1^p, \ldots, f_{k-1}^p)$, as $(p, f_1^p, \ldots, f_k^p)$ is also a regular sequence by (I.6.4.3). Then a map of pairs of rings $(A, I) \rightarrow (C, J)$ will lift through this tower uniquely by the universal property of pd-envolop.

then we see $A\{\frac{\varphi(f_1)}{p}, \ldots, \frac{\varphi(f_r)}{p}\}$ is just the pd-envolop of $(A, I)$, by the universal property. □

Lemma (I.13.0.22). Let $A$ be an $\mathbb{F}_p$-algebra and $B$ an $A$-algebra. If $(x_1, \ldots, x_n) \in B$ is regular w.r.t. $A$ (XI.1.3.7), then $D_B((x_1, \ldots, x_n))$ is $A$-flat.

Proof: By (I.13.0.19) and base change, it is clear that $D_B(I)$ is a free $B/I^p$-algebra. Thus we need to show that $B/I^p$ is a flat $A$-module. And this is true as the sequence $x_1^p, \ldots, x_n^p$ is also regular w.r.t. $A$ (XI.1.3.8). □

Prop. (I.13.0.23) (Flatness of PD-Envelope). If $A \rightarrow B$ is a map of simplicial rings, $A$ is naturally a simplicial pd-structure with $I_\bullet = pA_\bullet$, and $B$ is $p$-completely flat over $A$ (XI.1.3.5), and $(x_1, \ldots, x_n) \in \pi_0(B)$ is $p$-completely regular w.r.t. $A$ (XI.1.3.7), then the completed pd-envelope $D = D_B((x_1, \ldots, x_n))$ is $p$-completely flat over $A$. 

Diagram:

```
\begin{array}{c}
A'[p] \\
\rightarrow \quad 0 \rightarrow 0 \\
\uparrow \quad \uparrow \quad \uparrow \\
* \rightarrow * \rightarrow 0 \\
\uparrow \quad \uparrow \quad \uparrow \\
* \rightarrow * \rightarrow 0 \\
\uparrow \quad \uparrow \quad \uparrow \\
\ldots \rightarrow \ldots \rightarrow \ldots
\end{array}
```

This spectral sequence converges to 0, so $A'[p] = 0$, we win. □
Proof: By definition (I.13.3.5), to check it is\( p \)-completely flat, it suffices to check
\[
\text{Kos}(A,p) \rightarrow \text{Kos}(A,p) \otimes_A^L D
\]
is flat. and by (I.13.3.4) it suffices to check that
\[
\pi_0(\text{Kos}(A,p)) \rightarrow \pi_0(\text{Kos}(A,p)) \otimes_A^L D
\]
is flat. The formation of pd-envelope commutes with derived base change by universal property, and tensor with \( A/p \)-algebra undoes the completions, so we are reduced to the the case \( A' = \pi_0(\text{Kos}(A,p)) \) and \( B' = A' \otimes_A^L B \) is flat over \( A' \) and to show that
\[
A' \rightarrow D_{B'}((x_1, \ldots, x_n))
\]
is flat. Notice \( B' \) is a discrete flat \( A' \)-algebra by definition (I.13.3.5), and \( (x_1, \ldots, x_n) \) is a sequence in \( B' \) that \( \text{Kos}(B, x_1, \ldots, x_n) \) is a sequence regular w.r.t. \( A' \):
\[
\text{Kos}(B, x_1, \ldots, x_n) = \text{Kos}(B' = A' \otimes_A^L B, x_1, \ldots, x_n) = A' \otimes_A^L \text{Kos}(B, x_1, \ldots, x_n)
\]
\[
= A' \otimes_{A', \text{Kos}(A,p)}^L \text{Kos}(A, p) \otimes_A \text{Kos}(B, x_1, \ldots, x_n)
\]
is flat because \( \text{Kos}(A, p) \rightarrow \text{Kos}(B, p, x_1, \ldots, x_n) \) does by definition (I.13.3.7) and (I.13.3). So we are done by (I.13.0.22).

Cor. (I.13.0.24). If \( A \) is a \( p \)-complete simplicial \( \delta \)-ring and \( B \) is a \( p \)-completely flat simplicial \( \delta \)-\( A \)-algebra, and if \( x_1, \ldots, x_r \in \pi_0(B) \) is \( p \)-completely regular w.r.t. \( A \), then
\[
C_r = B_r\{\frac{x_1}{p}, \ldots, \frac{x_r}{p}\}
\]
is \( p \)-completely flat over \( A \).

\textbf{Proof:} Let \( C'_r = C'_r\{\frac{\varphi(x_1)}{p}, \ldots, \frac{\varphi(x_r)}{p}\} \), then \( C'_r \) is \( p \)-completely flat over \( A \), by (I.13.0.21)\(^?\) why \( B \) is \( p \)-torsionfree and \( (p, x_1, \ldots, x_n) \) is regular sequence) and (I.13.0.23). Now there is a commutative diagram
\[
\begin{array}{cccccc}
A & \rightarrow & A\{x_1, \ldots, x_r\} & \rightarrow & B & \rightarrow & B\{\frac{x_1}{p}, \ldots, \frac{x_r}{p}\} & \rightarrow & C_r \\
\text{id} & & \downarrow \psi & & \downarrow \psi_B & & \downarrow \psi' & & \downarrow \psi_C \\
A & \rightarrow & A\{\varphi(x_1), \ldots, \varphi(x_r)\} & \rightarrow & B' & \rightarrow & B'\{\frac{\varphi(x_1)}{p}, \ldots, \frac{\varphi(x_r)}{p}\} & \rightarrow & C'_r
\end{array}
\]
where \( \psi \) is the relative Frobenius and \( \psi_B, \psi_C \) is the derived base change and completed derived base change. Then \( \psi \) is f.f. as it is the base change of the Frobenius on the free \( \delta \)-ring \( \mathbb{Z}\{x_1, \ldots, x_r\} \) (I.8.4.9), thus so does \( \psi' \). Then \( \psi_C \) is \( p \)-completely flat, because it is the completion (use (I.10.7.2)(I.10.7.4)). Now the conclusion follows, by completely f.f. descent (I.10.7.2).

\section{de Rham Complex}

\textbf{Def. (I.13.1.1) (PD-Differentials).} Let \( B \) be an \( A \)-algebra and \( (B, J, \delta) \) be a pd-structure and \( M \) a \( B \)-module, then a \textbf{pd-A-derivation} from \( B \) to \( M \) is an element \( \theta \) of \text{Der}_A(B, M)\) that \( \theta(\delta_n(x)) = \delta_{n-1}(x) \theta(x) \) for \( n \geq 1 \) and \( x \in J \).
As in (I.7.3.4), there is a **pd-differential** $\Omega_{B/A,\delta}$ that

$$\text{Hom}_B(\Omega_{B/A,\delta}, M)$$

is isomorphic to the set of pd-A-derivations of $B$ to $M$, functorially in $M$.

**Prop. (I.13.1.2) (PD-Differential of PD-Envelope).** Let $(A, I, \gamma)$ be a pd-structure and $(B, J)$ be a pair over $(A, I)$, and let $D_{B/A,\gamma}(J) = (D, J, \tilde{\gamma})$ be the pd-envelope, then $\Omega_{D/A,\tilde{\gamma}} = \Omega_{B/A} \otimes_B D$.

**Proof:** It suffices to show that for any $D$-module $M$, the set of $A$-derivations $B \to M$ is isomorphic to the set of pd-$A$-derivations $D \to M$.

Let $D \otimes M$ be the ring that $M^2 = 0$ and a pd-structure on $\tilde{J} \oplus M$ is given by $\delta_n(x + m) = \delta_n(x) + \delta_{n-1}(x)m$. Then a pd-$A$-derivations $D \to M$ is equivalent to a pd-ring map $(D, \tilde{J}) \to (D \oplus M, \tilde{J} \oplus M)$, and an $A$-derivations $B \to M$ is also equivalent to a map of pairs $(B, J) \to (D \oplus M, \tilde{J} \oplus M)$, thus we are done by the universal property of $D$.

**Prop. (I.13.1.3).** Let $B$ be an $A$-algebra and $(B, J, \delta)$ be a pd-structure, then

- if $(B[X], JB[X], \delta')$ is the $\delta$-structure extended from that of $(B, J, \delta)$ as in (I.13.0.11), then
  $$\Omega_{B[X]/A,\delta'} = \Omega_{B/A,\delta} \otimes_B B[X] \oplus B[X]dx.$$  

- Let $B\langle x \rangle$ be the free pd-algebra over $B$(I.13.0.14), then
  $$\Omega_{B\langle x \rangle/A,\delta'} = \Omega_{B/A,\delta} \otimes_B B\langle x \rangle \oplus B\langle x \rangle dx.$$  

- Let $K \subset J$ be an ideal preserved by $\delta$ and then consider the quotient $(B' = B/K, J = J/K, \tilde{\delta})$, then $\Omega_{B'/A,\tilde{\delta}}$ is quotient of the module $\Omega_{B/A,\delta} \otimes_B B'$ by the $B'$-submodule generated by $dk$ where $k \in K$.

**Proof:** These are all somewhat trivial.

**Prop. (I.13.1.4) (PD-Differential and Completion).** Let $A$ be a $\mathbb{Z}_p$-algebra, $B$ be an $A$-algebra and $(B, J, \delta)$ be a pd-structure and $p$ is nilpotent in $B/J$, then

$$\lim_{e} \Omega_{B_e/A,\tilde{\delta}} = (\Omega_{B/A,\delta})^\wedge = (\Omega_{B/A,\delta})^\wedge.$$  

where $B_e = B/p^e$.

**Proof:** By (I.13.0.12), the terms make sense. Now by (I.13.1.3) and the observation $d(p^e) = 0$, we have $\Omega_{B_e/A,\tilde{\delta}} = \Omega_{B/A,\delta}/p^e = \Omega_{B/A,\delta}/p^e$, thus we are done.

**Def. (I.13.1.5) (PD-de Rham Complex).** Let $\Omega^i_{B/A,\delta} = \wedge^i \Omega_{B/A,\delta}$, then the surjection $\Omega_{B/A} \to \Omega_{B/A,\delta}$ satisfies the condition of (VI.4.1.4), thus there is a **pd-de Rham complex** $\Omega^*_{B/A,\delta}$.

**PD-Poincaré Lemma**

**Lemma (I.13.1.6) (PD-Poincaré Lemma).** Let $A$ be a ring, $P = A(X_i)$ is the free PD-algebra over $A$, then for any $A$-module, the complex

$$0 \to M \to M \otimes_A^e P \to M \otimes_A^e \Omega^1_{P/A,\delta} \to \ldots$$

is exact. And if $D = \hat{P}$ and let $\Omega^n_{D} = \Omega^n_{P/A,\delta}$, then for any $p$-complete $A$-module $M$, the complex

$$0 \to M \to M \hat{\otimes}_A P \to M \hat{\otimes}_A \Omega^1_{P/A,\delta} \to \ldots$$

is exact.
Proof: It suffices to show that \( 0 \to M \to M \otimes_A P \to M \otimes_A \Omega^1_{P/A,\delta} \to \ldots \) is homotopic to 0. For this, notice every element of \( \Omega^n_{P/A,\delta} \) is of the form \( \sum P \prod_{j=0}^n dx_{i_0} dx_{i_1} \ldots dx_{i_n} \), so we can let
\[
f(\omega) = f(\gamma_{n,0} (x_{i_0}) dx_{i_0} \wedge \omega) = \gamma_{n,0+1} (x_{i_0}) \omega
\]
where \( \omega \) doesn't divide \( dx_k \) for \( k < i_0 \). Then it can be checked that \( df + fd = id \), so we are done.

\[\square\]

**Prop. (I.13.1.7).** If \( A \) is a ring and \((B,J,\delta)\) is a pd-structure, and let \( P = B(X_i) \) be the free pd-structure (I.13.0.14). Let \( M \) be a \( B \)-module endowed with an integrable connection \( \nabla : M \to M \otimes_B \Omega^1_{B/A,\delta} \), then the map of de Rham complexes
\[
M \otimes_B \Omega^*_{B/A,\delta} \to M \otimes_B \Omega^*_{P/A,\delta}
\]
is a quasi-isomorphism. And if we denote \( D, D' \) the \( p \)-adic completions of \( B, P \), and \( \Omega_D, \Omega_{D'} \) the \( p \)-adic completion of \( \Omega_{B/A,\delta}, \Omega_{P/A,\delta} \), and \( M \) is a \( p \)-complete \( B \)-module endowed with an integrable connection \( \nabla : M \to M \otimes_D \Omega_D \), then the map of de Rham complexes
\[
M \otimes_D \Omega^*_D \to M \otimes_{D'} \Omega^*_{D'}
\]
is a quasi-isomorphism.

**Proof:** Consider the filtration \( F^* \) on \( \Omega^*_{B/A,\delta} \) given by the stupid truncation \( \sigma \geq i \Omega^*_{B/A,\delta} \), and consider the filtration on \( \Omega^*_{P/A,\delta} \) given by
\[
F^*(\Omega^*_{P/A,\delta}) = F^*(\Omega^*_{B/A,\delta}) \wedge \Omega^*_{P/A,\delta}.
\]

Notice that we have a split exact sequence
\[
0 \to \Omega^1_{B/A,\delta} \otimes_B P \to \Omega^1_{P/A,\delta} \to \Omega^1_{B/P,\delta} \to 0
\]
and \( \Omega^1_{P/B,\delta} \) is free on \( X_i \) over \( B \) (pondering the universal property, this is for the same reason as (I.7.3.7).

Then we see that \( F^i(\Omega^*_{P/A,\delta}) \to \Omega^*_{P/A,\delta} \) is termwise split injection for any \( i \), and the graded is \( \Omega^1_{B/A,\delta} \otimes_B \Omega^*_{P/B,\delta} \). Thus if we let \( F^i(M \otimes_B \Omega^*_{P/A,\delta}) = M \otimes F^i(\Omega^*_{P/A,\delta}) \), then the graded is \( M \otimes_B \Omega^1_{B/A,\delta} \otimes_B \Omega^*_{P/B,\delta} \), which is quasi-isomorphic to \( M \otimes_B \Omega^1_{B/A,\delta} \) by (I.13.1.6). Then the original map is a filtered complexes that induces quasi-isomorphism on gradeds, so it induces a quasi-isomorphism, because it induces an morphism between two convergent spectral sequences, by (II.3.5.6) and (II.3.5.5).
I.14 Reflection Groups and Coxeter Groups

Main references are [Hum90], [Ser87].

1 Root Systems

Let $V$ be a vector space over a field $k$ of char0.

**Def. (I.14.1.1) (Reflections).** A reflection of a vector space $V$ of dimension $n$ is a linear transformation that has $n - 1$ eigenvalues 1 and one eigenvalue $-1$.

**Prop. (I.14.1.2).** Let $\alpha \in V$ and $\alpha^\vee \in V^\vee$ that $(\alpha, \alpha^\vee) = 1$, then

$$s_\alpha : x \mapsto x - 2(x, \alpha^\vee)\alpha$$

is a reflection, and every reflection with vector $\alpha$ is of this form.

**Proof:** Clearly $s_\alpha$ is a reflection, and if $s$ is any reflection, then $\alpha^\vee$ is the composition of the quotient map $V \to V/H$ with the map $V/H \to \mathbb{F}$ sending $\alpha + H$ to 1. \qed

**Lemma (I.14.1.3).** Let $R$ be a spanning set of $V$. Then for any $\alpha \in V$, there exists at most one reflection $s$ with vector $\alpha$ that $s(R) \subset R$.

**Proof:** Let $s, s'$ be two such reflections, then $t = ss'$ is an automorphism that is identity on both $\mathbb{F}\alpha$ and $V/\mathbb{F}\alpha$. So $(t - 1)^2 = 0$ on $V$. So the minimal polynomial of $T$ divides $(T - 1)^2$. Also because $R$ is finite there exists an $m$ that $t^m = 1$ on $R$ thus on $V$, so $t = 1$ as the greatest divisor of $(T - 1)^2$ and $T^m - 1$ is $T - 1$. \qed

**Lemma (I.14.1.4).** Let $V$ be an inner product space, then for any vector $\alpha$, there exists a unique reflection $s_\alpha$ that respects the inner product, which is

$$s_\alpha(x) = x - 2\frac{(x, \alpha)}{(\alpha, \alpha)}\alpha.$$

**Def. (I.14.1.5) (Root System).** A subset $R$ of a vector space $V$ over $\mathbb{F}$ is called a root system if

- $R$ spans $V$ and doesn’t contain 0.
- For each $\alpha \in R$, there exists a unique reflection $s_\alpha = \text{id} - \alpha^\vee \otimes \alpha$ with vector $\alpha$ that $s_\alpha(R) \subset R$.
- For any $\alpha, \beta \in R$, $s_\alpha(\beta) - \beta$ is a multiple of $\alpha$, or equivalently $2\alpha^\vee(\beta) \in \mathbb{Z}$.

Elements of $R$ are called the roots of $R$, and dimension of $V$ is called the dimension of the root system. And the subgroup of $GL(V)$ generated by all $s_\alpha$ is called the Weyl subgroup of $R$. The Weyl subgroup is a finite group, as a subgroup of the group of permutations of $R$.

A root system is called indecomposable iff it cannot be written as a direct sum of two root systems.

**Def. (I.14.1.6) (Reduced Root System).** Let $\alpha, \beta$ be roots that are multiples of each other, then $\beta = c\alpha$ for some $c \in \mathbb{F}$, then $2(\beta, \alpha^\vee) = 2c \in \mathbb{Z}$, also $2c^{-1} \in \mathbb{Z}$, so $c \in \{-1, -1/2, 1/2, 1\}$. A reduced root system is a root system that there are no roots $\alpha, \beta$ that $\alpha = 2\beta$. 
Prop. (I.14.1.7) (Invariant Quadratic Form). Let $R$ be a root system in $V$, then there is a positive bilinear form in $V$ that is invariant under the action of the Weyl group of $R$.

Notice that for such a bilinear form, the reflection must be of the form given in (I.14.1.4), or in other words, $\alpha^\vee(\beta) = (\frac{2\alpha}{(\alpha,\alpha)}, \beta)$.

Proof: This follows entirely because the Weyl group of $R$ is finite, as we can take the average of any positive bilinear form under the action of reflections. □

Def. (I.14.1.8) (Dual Root System). Let $R$ be a root system in $V$, the set of dual vectors $\alpha^\vee$ for $\alpha \in R$ is also a root system in $V^\vee$, called the dual system of $R$. And it has the same Weyl group as $R$.

Proof: Take an invariant quadratic form on $V$ (I.14.1.7), then it gives an isomorphism $V \rightarrow V^\vee$. Then $\alpha^\vee$ corresponds to $\frac{2\alpha}{(\alpha,\alpha)}$ under this isomorphism, and there obviously generates $V$. If $\alpha^\vee \in R^\vee$, we can take the corresponding reflection to be $s_{\alpha^\vee} = 1 - \alpha \otimes \alpha^\vee$. Then

$$s_{\alpha^\vee}(\beta^\vee)(x) = (\beta^\vee - \beta^\vee(\alpha)\alpha^\vee)(x)$$
$$= (x, 2\beta/(\beta, \beta) - \frac{(\alpha, \beta)}{(\beta, \beta)}(x, 2\alpha/(\alpha, \alpha))$$
$$= (x, \frac{2(\beta - \alpha(\alpha, \beta))/(\alpha, \alpha)}{(\beta, \beta)})$$
$$= (x, \frac{2s_{\alpha}(\beta)}{(\beta, \beta)}) = (s_{\alpha}(\beta))^\vee(x).$$

So $s_{\alpha^\vee}(\beta^\vee) = (s_{\alpha}(\beta))^\vee \in R^\vee$. In this way, we see $\alpha^{\vee\vee} = \alpha$, and $s_{\alpha^\vee}(\beta^\vee) - \beta^\vee = \beta^\vee(\alpha)\alpha^\vee$ is an integral multiple of $\alpha^\vee$. Also it can be seen easily from the formula above that the Weyl group is the same as the Weyl group of $R$. □

Prop. (I.14.1.9) (Real Root Systems). Let $R$ be a roots system in a vector space $V$ over a field $F$, and let $V_0$ be the $\mathbb{Q}$-vector space spanned by $R$, then $R$ is a root system in $V_0$ and $V_0 \otimes \mathbb{Q} F \cong V$.

So from now on we focus on a real root system.

Proof: The non-trivial part is that the isomorphism $V_0 \otimes \mathbb{Q} F \cong V$. The natural map $i : V_0 \otimes \mathbb{Q} F \rightarrow V$ is surjective because $R$ generates $V$, and consider its transpose

$$i^* : V^\vee \rightarrow V_0^\vee \otimes \mathbb{Q} F,$$

then we can see that this maps $\alpha^\vee$ to $\alpha^\vee_0$, and $R^\vee_0$ is a root system in $V_0^\vee$ (I.14.1.8), thus $\alpha^\vee_0$ generates $V_0^\vee$, and $i^*$ is surjective, showing $i$ is injective. □

Def. (I.14.1.10) (Weyl Chambers). Let $(V, R)$ be a real root system, then a Weyl chamber is a connected component of $V \setminus \bigcup_{\alpha \in R} H_{\alpha}$, where $H_{\alpha}$ is the fixed hyperplane of vectors fixed by $s_{\alpha}$.

Def. (I.14.1.11) (Base). A subset $S \subset R$ is called a base of $R$ if the every elements of $R$ can be written uniquely as a linear integral combination of elements of $S$ with the same sign.

If $S$ is a base of $R$, then we let $R^+$ denote the set of $R$ that is non-negative integral combinations of elements of $S$, called the positive roots of $R$, and $R^-$ the set of $R$ that is non-positive integral combinations of elements of $S$, called the negative roots of $R$. $R = R^+ \sqcup R^-$. 

Prop. (I.14.1.12) (Base exists). Let \( t \in V^\vee \) be an element that \( t(\alpha) \neq 0 \) for all \( \alpha \in R \). Let \( R_t^+(R_t^-) \) be the set of all \( \alpha \in R \) that \( t(\alpha) > 0(< 0) \), then \( R = R_t^+ \cup R_t^- \). An element of \( R_t^+ \) is called indecomposable if it cannot be written as the sum of two elements in \( R_t^+ \). Let \( S_t \) be the set of indecomposable elements of \( R_t^+ \), then \( S_t \) is a base of \( R \).

In particular, every root system \((R, V)\) contains a base. And if \( S \) is a base and \( t \in V^\vee \) that \( t(S) > 0 \), then \( S = S_t \).

Proof: Cf. [Ser87]P38.

Prop. (I.14.1.13). Let \( S \) be a base of a root system \((R, V)\), then every positive root \( \beta \) can be written as

\[
\beta = \alpha_1 + \ldots + \alpha_k
\]

in such a way that all the partial sums are roots.

Proof: Cf. [Ser87]P40.

Prop. (I.14.1.14). Let \( R \) be a reduced root system and \( S \) a base, then for any \( \alpha \in S \),

\[
s_\alpha(R^+ \setminus \{\alpha\}) = R^+ \setminus \{\alpha\}.
\]

In particular if \( \rho = \frac{1}{2}(\sum_{\alpha \in R^+} \alpha) \), then \( s_\alpha = \rho - \alpha \).

Proof: Cf. [Ser87]P40.

Prop. (I.14.1.15) (Base and Dual System). Let \( R \) be a reduced system and \( S \) a base, then \( S^\vee \) is a base of \( R^\vee \).

Proof: By the isomorphism between \( V \) and \( V^\vee \), it suffices to show the vectors \( \{2\alpha/(\alpha, \alpha), \alpha \in S\} \) is a base for \( R^\vee = \{2\alpha/(\alpha, \alpha), \alpha \in R\} \). But it is clear if \( S \) is a base corresponding to a vector \( t \in V^* \), then \( \{2\alpha/(\alpha, \alpha), \alpha \in S\} \) is the extremal vectors corresponding to \( t \) too, so it is a base by (I.14.1.12).

Prop. (I.14.1.16) (Weyl Group and Bases). Let \( W \) be the Weyl group of a reduced root system \( R \) and \( S \) a base of \( R \), then

1. For any \( t \in V^* \), there exists \( w \in W \) that \( (w(t), \alpha) \geq 0 \) for all \( \alpha \in S \).
2. \( W \) acts transitively on the set of bases of \( R \).
3. For each \( \beta \in R \), there exists some \( w \in W \) that \( w(\beta) \in S \).
4. The group \( W \) is generated by \( s_\alpha \) where \( \alpha \in S \).
5. \( W \) acts simply transitively on the set of bases of \( R \).

Proof: We can in fact prove this for \( W_S \) the group generated by \( s_\alpha \) where \( \alpha \in S \).

1. Let \( \rho \) be defined in (I.14.1.14), and choose an element \( w \in W_S \) that \( w(t)(\rho) \) is maximal, then \( w(t)(\rho) \geq w(t)s_\alpha(\rho) = w(t)(\rho) - w(t)(\alpha) \). So \( w(t), \alpha \geq 0 \).
2. Let \( S' \) be a base and \( t' \in V^\vee \) that \( t'(S') > 0 \). Also by 1 we can find \( w \in W \) that \( w(t)(\alpha) \geq 0 \) for all \( \alpha \in S \). And in fact \( w(t)(\alpha) > 0 \) for all \( \alpha \in S \). So \( S = S_t, S' = S_{w(t)} \). Thus \( w \) sends \( S' \) to \( S \).
3. Finally we prove that \( W_S = W \): because for any \( \beta \in R \), there exists \( w \in W_S \) that \( w(\beta) \in S \), so \( s_\beta = w^{-1}s_{w(\beta)}w \in W_S \).
4. See [Ser87]P70.
Cor. (I.14.1.17). The Weyl group \( W \) acts transitively on the set of Weyl chambers of \( R \).

Proof: ? \qed

Prop. (I.14.1.18) (Highest Root). Let \((R, V)\) be a root system and \( S \) a base. If \( S \) is indecomposable, then there exists a root \( \tilde{\alpha} = \sum_{\alpha \in S} n_{\alpha} \alpha \) that for any other root \( \sum_{\alpha \in S} n_{\alpha} \alpha, n_{\alpha} \geq m_{\alpha} \).

Proof: □

Cartan Matrix and Dynkin Diagrams

Prop. (I.14.1.19) (Angles Between Roots). Let \( \alpha, \beta \) be two non-propositional roots in a root system \( R \), then we can put \( n(\alpha, \beta) = (\alpha, \beta^\vee) = 2(\alpha, \beta) / (\beta, \beta) \). Then \( n(\alpha, \beta)n(\beta, \alpha) = 4 \cos^2 \varphi_{\alpha, \beta} \), where \( \varphi_{\alpha, \beta} \) is the angle between these two vectors. Then because \( 4 \cos^2 \varphi_{\alpha, \beta} \) is an integer, it can only take values \( 0, 1, 2, 3 \). Then there are only 7 possibilities of the angles between \( \alpha, \beta \), and if \( \alpha, \beta \) are not orthogonal, their ration of lengths are determined also by the angle.

Prop. (I.14.1.20). If \( \alpha, \beta \) are not proportional and \( n(\beta, \alpha) > 0 \), then \( \alpha - \beta \) is a root. then(I.14.1.19) shows \( n(\beta, \alpha) = 1 \) or \( n(\alpha, \beta) = 1 \). Now \( \alpha - \beta = s_{\beta}(\alpha) \) or \( -s_{\alpha}(\beta) \) is a root of \( R \).

Def. (I.14.1.21) (Cartan Matrix). Let \( R \) be a root system with a base \( S \). Then the Cartan matrix is \( (n(\alpha, \beta))_{\alpha, \beta \in S} \).

Prop. (I.14.1.22). The Cartan matrix depends only on \((V, R)\) and not on \( S \), and if \( R \) is reduced, it is determined by its Cartan matrix up to isomorphisms.

Proof: This follows from(I.14.1.16). \qed

Prop. (I.14.1.23). Let \( E \) be the group of automorphisms of \( S \) that leaves that Cartan matrix invariant, then it can be identified with the set of automorphisms of \( R \) that leave the base \( S \) invariant. Then the group \( \text{Aut}(R) \) is isomorphic to the semi-product \( E \times W \).

Proof: Let \( W \) is generated by \( s_{\alpha} \) so invariant under \( \text{Aut}(R) \). Now if \( u \in \text{Aut}(R) \), then \( u(S) \) is a base of \( R \), so there exists some \( w \in W \) that \( w(u(S)) = S \), thus \( u \in EW \). Also if \( w \in W \cap E \), then □

Def. (I.14.1.24) (Coxeter graph). A coxeter graph is a finite graph that each pair of distinct vertices are connected by 0, 1, 2 or 3 vertices.

Def. (I.14.1.25) (Coxeter Graph associated to a Root System). Let \((R, V)\) be a root system and \( S \) a base of \( R \), then there is a coxeter graph whose nodes are indexed by the elements of \( S \), and two distinct nodes \( \alpha, \beta \) are connected by \( n(\beta, \alpha)n(\alpha, \beta) = 4 \cos^2 \varphi_{\alpha, \beta} \) edges. This is independent of the choice of \( S \), by(I.14.1.22).

Prop. (I.14.1.26). \( R \) is indecomposable iff the Coxeter graph is connected.

Proof: By the formula in(I.14.1.4), \( R \) is decomposable iff \( R = R_1 \rightarrow R_2 \) where \( R_1, R_2 \) are orthogonal to each other. Then this is equivalent to \( \varphi_{\alpha, \beta} = \pi/2 \) for any \( \alpha \in R_1, \beta \in R_2 \). \qed

Lemma (I.14.1.27) (Listing of Indecomposable Root Systems). The coxeter diagrams arising from indecomposable root systems are exactly the diagram \( A_n(n \geq 1), B_n(n \geq 2), D_n(n \geq 4), E_6, E_7, E_8, F_4, G_2 \).
Proof: ?

The existence is these types are given by (I.14.1.28).

\[ \Box \]

Prop. (I.14.1.28) (List of Irreducible Root Systems). Let \( e_n \) be the standard basis of \( \mathbb{R}^n \) with the standard bilinear form, and let \( L_n \) be the subgroup generated by \( e_n \). Then

- \( A_n \): Let \( V \) be the hyperplane of \( \mathbb{R}^{n+1} \) orthogonal to the vector \( e_1 + \ldots + e_{n+1} \), and \( R \) be the subset of \( L_{n+1} \cap V \) consisting of vectors of length \( \sqrt{2} \). Then \( (R, V) \) is a root system, and the Weyl group is the permutation group of \( e_1, \ldots, e_{n+1} \). For a base \( S \), we can take the set of all vectors \( e_i - e_{i+1}, i = 1, \ldots, n \).

- \( B_n \): Let \( V = \mathbb{R}^n \) and \( R \) be the subset of \( L_n \) consisting of vectors of length 1 or \( \sqrt{2} \). Then \( (R, V) \) is a root system, and the Weyl group is the permutation and sign changes of the vectors \( e_i \). For a base \( S \), we can take the set \( \{e_1 - e_2, \ldots, e_{n-1} - e_n, e_n\} \).

- \( C_n \): Let \( C_n \) be the dual system of \( B_n \) (I.14.1.8), which by the invariant form isomorphic to the set of \( \mathbb{R}^n \) consisting of vectors \( \pm e_i \), \( 2e_i \). It has the same Weyl group as \( B_n \). Also it has a base \( \{e_1 - e_2, \ldots, e_{n-1} - e_n, 2e_n\} \) (I.14.1.15).

- \( D_n \): Let \( V = \mathbb{R}^n \) and \( R \) be the set of all vectors of \( L_n \) of length \( \sqrt{2} \). The Weyl group consists of permutations and sign changes of an even number of the vectors \( e_i \). Also it has a base \( \{e_1 - e_2, \ldots, e_{n-1} - e_n, e_{n+1} + e_n\} \).

- \( G_2 \): Let \( V = \mathbb{R}[\omega] \) and \( R \) be the subset of \( \mathbb{Z}[\omega] \) of norm 1 or 3. The Weyl group is isomorphic to the dihedral group. Also it has a base \( \{1, \omega - 1\} \).

- \( F_4 \): Let \( V = \mathbb{R}^4 \), and let \( R \) be the set of vectors \( \pm e_i, \pm e_i \pm e_j, \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \). It has a base consisting of \( \{e_2 - e_3, e_3 - e_4, e_4, \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\} \).

- \( E_8 \): Let \( V = \mathbb{R}^8 \) and

\[ R = \{\pm e_i \pm e_j, i \neq j\} \cup \left\{ \frac{1}{2} \sum_{i=1}^{8} \pm e_i \right\} \text{ with even number of minus signs} \}

It has a base \( \{e_2 - e_3, \ldots, e_7 - e_8, \frac{1}{2}(e_1 - e_2 - \ldots - e_7 + e_8)\} \).

- \( E_7 \)

- \( E_6 \)

Def. (I.14.1.29) (Dynkin Diagram). The coxeter graph cannot determine the root system up to isomorphism, because it cannot distinguish between \( n(\alpha, \beta), n(\beta, \alpha) \). So there is a Dynkin diagram which is constructed from the coxeter diagram by adding a vector from the longer vector to the shorter vector when \( n(\alpha, \beta)n(\beta, \alpha) = 2 \) or 3.

Prop. (I.14.1.30) (Listing of Dynkin Diagrams). The Dynkin diagrams arising from indecomposable root systems are exactly the diagram \( A_n(n \geq 1), B_n(n \geq 2), C_n(n \geq 3), D_n(n \geq 4), E_6, E_7, E_8, F_4, G_2 \).

Proof: This follows easily from (I.14.1.27).

\[ \Box \]

Prop. (I.14.1.31) (Non-Reduced Root Systems). One can show that for any \( n \geq 1 \), there is exactly one non-reduced root system, which is \( BC_n \), the union of \( B_n \) and \( C_n \) in (I.14.1.28).
Weight Lattices

Def. (I.14.1.32) (Weight Lattice). Let \((R, V)\) be a root system, we can define the root lattice as the \(\mathbb{Z}\)-lattice \(Q(R)\) generated by \(R\), and also weight lattice \(P(R) = \{ x \in V | \alpha^\vee(x) \in \mathbb{Z}, \alpha \in R \} \). The elements of \(P(R)\) are called weights of \(R\). We have \(Q(R) \subset P(R)\) by definition, and \(P(R)/Q(R)\) is finite, because they both generate \(V\).

If \(S\) is a base for \(R\), then \(S^\vee\) is a base for the dual system \(R^\vee\), so the dual basis for \(S\) in \(V^\vee\) is a basis for the lattice \(P(R)\), and its elements are called fundamental weights.
I.15 Hopf Algebras

Main references are [Quantum Groups, Drinfeld], [A Brief Introduction To Quantum Groups Pavel Etingof And Mykola Semenyakin].

1 Hopf Algebras

Coalgebras and Bialgebras

Def. (I.15.1.1) (Coalgebras). Let \( R \) be a commutative ring, a coalgebra is a monoid object in the category \( \text{Mod}^{op}_R \).

Remark (I.15.1.2) (Yoneda Interpretation). We do not need to verify all the relations defining a coalgebra \( C \), whenever we have a functorial monoidal structure on all the set \( \text{Hom}_R(H,T) \), we immediately recover the maps

- (Comultiplication): \( \mu : C \to C \otimes_R C \) as \( i_1 \cdot i_2 \) in \( \text{Hom}_R(C,C \otimes C) \),
- (Counit): \( \varepsilon : C \to R \) as \( 1 \) in \( \text{Hom}_R(C,R) = C^\vee \).

by Yoneda lemma.

\( C^\vee \) is an monoid by definition, and \( C \) is called cocommutative iff \( C^\vee \) is commutative.

Def. (I.15.1.3) (Primitive Elements). Let \( H \) be a coalgebra over \( R \), an element \( x \in H \) is called primitive if \( \mu(x) = 1 \otimes x + x \otimes 1 \). It is called group-like if \( \Delta(x) = x \otimes x \).

Def. (I.15.1.4) (Bialgebras). Let \( R \) be a commutative ring, a bialgebra is a monoid object in the category of coalgebras over \( R \). Equivalently, in the Yoneda interpretation, \( C \) is an \( R \)-algebra that \( \Delta \) is a homomorphism of algebras.

Def. (I.15.1.5) (Hopf Algebra). A Hopf algebra over a commutative algebra \( R \) is a bialgebra \( A \) together with a \( R \)-linear map \( S : A \to A \) that satisfies:

\[
m \circ (\text{id} \otimes S) \circ \mu = m \circ (S \otimes \text{id}) \circ \mu = \eta \circ \varepsilon : A \to A.
\]

If \( S^2 = \text{id}_A \), then \( A \) is called an involutive Hopf algebra.

Prop. (I.15.1.6). In a Hopf algebra, \( S \) is an anti-homomorphism both for the algebra structure and coalgebra structure.

Proof: Cf.https://ncatlab.org/nlab/show/Hopf+algebra. \( \square \)

Example (I.15.1.7) (Group Algebras). Let \( \Gamma \) be a group, the group algebra \( R[\Gamma] \) with the coalgebra structure

\[
\mu : R[\Gamma] \to R[\Gamma] \times R[\Gamma] : g \mapsto g \otimes g, \varepsilon : R[\Gamma] \to R : \sum a_g g \mapsto \sum a_g
\]

and

\[
S : R[\Gamma] \to R[\Gamma] : g \mapsto g
\]

is a Hopf algebra.

Def. (I.15.1.8) (Dual Hoof Algebra). Let \( H \) be a Hopf algebra, matrix coefficients. \( H^0 \).

Prop. (I.15.1.9). \( \text{Rep}(H) = \text{Mod}_{H^0} \).
Co-modules

Def. (I.15.1.10) (Co-Module). Let $A$ be a coalgebra over a field $k$, then a right co-module is a $k$-vector space $V$ together with a $k$-linear map $\rho : V \to V \otimes A$ that satisfies 

$$(\text{id}_V \otimes \mu) \circ \rho = (\rho \otimes \text{id}_A) \circ \rho, \quad (\text{id}_V \otimes \varepsilon) \circ \rho = \text{id}_V.$$ 

The map $\rho$ is called the co-action, and a $k$-subspace $W \subset V$ that $\rho(W) \subset W \otimes A$ is called a sub-comodule of $V$.

Def. (I.15.1.11) (Tensor Product of Co-modules).

Topologists’ Hopf Algebras

Def. (I.15.1.12) (Topologist’s Hopf Algebra). A topologist’s Hopf algebra over a commutative ring $R$ is a unital magma object in the dual category of graded algebras with 0-degree term $R$, given by maps $\Delta : A \to A \otimes A$, and $\varepsilon : A \to R$ the canonical projection map. So in particular, $\Delta$ is of the form 

$$\Delta(\alpha) = 1 \otimes \alpha + \alpha \otimes 1 + \sum_i \alpha'_i \otimes \alpha''_i, |\alpha'_i| > 0, |\alpha''_i| > 0$$

Prop. (I.15.1.13). The tensor product of two topologists’ Hopf algebra is a topologists’ Hopf algebra.

Prop. (I.15.1.14). Let $F$ be a field, then $F[\alpha]/(\alpha^n)$, where $\alpha$ is placed at even dimension or $F$ has characteristic 2, is a topologists’ Hopf algebra iff $F$ has positive characteristic $p$ and $n$ is a power of $p$.

Proof: By definition, it is easy to see that $\alpha$ is primitive, thus 

$$\Delta(\alpha^n) = 0 = \sum_{0<i<n} \binom{n}{i} \alpha^i.$$ 

Which then implies that $n$ is a $p$-power. \hfill \Box

Prop. (I.15.1.15) (Hopf-Borel). Let $A$ be a commutative topologists’ Hopf algebra over a field $F$, and $A$ is of f.d. in each degree, then:

- If $F$ is of characteristic 0, $A$ is isomorphic as an algebra to the tensor product of an exterior product of odd-dimensional generators and a polynomial ring of even-dimensional generators.

- If $F$ is a finite field of characteristic $p$, then $A$ is isomorphic as an algebra to the tensor product of algebras of the following types:

  - $F[\alpha]/(\alpha^n)$, where $\alpha$ is even-dimensional if $p \neq 2$.

  - $\wedge F[\alpha]$ where $\alpha$ is odd-dimensional.

  - $F[\alpha]/(\alpha^p)$, where $\alpha$ is even-dimensional if $p \neq 2$.

Proof: We only prove the 0-characteristic case, Cf.[Hat02]P285. \hfill \Box
2 Commutative Hopf Algebras

Prop. (I.15.2.1) (Commutative Hopf Algebra). Let \( R \) be a commutative ring, a commutative Hopf algebra is equivalent to a cogroup object in \( CAlg_R \). A homomorphism of Hopf algebras is a morphism of algebras that represents a natural transformation of functors from \( CAlg_R \) to \( Grp \).

Proof: The critical point is to look at the definition of Hopf algebra. In this case, \( S \) is a homomorphism of algebras (I.15.1.6), and notice tensor product is just the product in the dual category, and \( m : A \otimes A \to A \) is the diagonal in the dual category, thus a cogroup object is a map \( \Delta : A \to A \otimes A \) together with a map \( inv : A \to A \) that

\[
m \circ (id \otimes inv) \circ \Delta = m \circ (inv \otimes id) \circ \Delta \circ id = id : A \to A,
\]

which is exactly the definition of the Hopf algebra (I.15.1.5).

\[ \square \]

Cor. (I.15.2.2) (Yoneda Interpretation). We do not need to verify all the relations defining a Hopf algebra \( H \), whenever we have a functorial commutative group structure on all the set \( \text{Hom}_R(H,T) \), we immediately recover the maps:

- (Comultiplication): \( \mu : H \to H \otimes_R H \) as \( i_1 \cdot i_2 \) in \( \text{Hom}_R(H,H \otimes H) \),
- (Antipode): \( \eta : H \to R \) as \( inv \) in \( \text{Hom}_R(H,H) \),
- (Counit): \( \varepsilon : H \to R \) as \( 1 \) in \( \text{Hom}_R(H,R) = H^\vee \).

by Yoneda lemma.

For convenience, we denote the structure maps \( \eta_H : R \to H, \mu : H \times_R H \to H, (-)^{-1} : H \to H \).

Prop. (I.15.2.3) (Group Algebras). Let \( \Gamma \) be a commutative group, then the group algebra \( R[\Gamma] \) (I.15.1.7) represents the group functor that maps a commutative \( R \)-algebra \( S \) to the commutative group \( \text{Hom}_{Grp}(\Gamma,S) \).

Cor. (I.15.2.4) (Multiplicative Groups). \( \mathbb{G}_{m,R} = R[t, t^{-1}] \) is a Hopf algebra that represents the group functor of multiplicative groups on \( CAlg_R \).

Cor. (I.15.2.5) (Roots of Unity \( \mu_n,R \)). Let \( \Gamma \cong \mathbb{Z}/n\mathbb{Z} \), then \( R[\Gamma] \cong R[t]/(t^n - 1) \) is a Hopf algebra that represents the group functor of multiplicative groups of \( n \)-th roots of unity on \( CAlg_R \).

Prop. (I.15.2.6) (Additive Groups). \( R[t] \) can be given a Hopf algebra structure that represents the group functor that maps a commutative \( R \)-algebra \( S \) to the additive group \( S^+ \).

Def. (I.15.2.7) (\( G_{(a,b),R} \)). Given elements \( a, b \in R \) that \( ab = 2 \), for any commutative \( R \)-algebra \( S \), the group \( G_{(a,b),R}(S) \) of elements \( x \) of \( S \) that \( x^2 + ax = 0 \) is a group under the mapping \( (m,n) \mapsto m + n + bmn \). Notice the inverse of \( m \) is \( m \) itself. Then \( R[t]/(t^2 + at) \) can be given a Hopf algebra that represents the functor \( S \mapsto G_{(a,b),R}(S) \).

Def. (I.15.2.8) (\( V_a \)). Let \( V \) be a vector space over \( k \), then \( \text{Sym}(V^\vee) \) can be given a group structure representation the functor \( V_a : R \mapsto R \otimes_k V \cong \text{Hom}_k(V^\vee,R) \).

Def. (I.15.2.9) (Locally Constant Functions). Let \( \Gamma \) be a group, then \( \Gamma_R = \prod_{\gamma \in \Gamma} R \) represents the group functor that maps an \( R \)-algebra \( T \) to the group of locally constant functions on \( \text{Spec} R \) with value in \( \Gamma \).
**Remark (I.15.2.10).** To illustrate the philosophy of (V.9.1.2), we figure out the Hopf structure of the local constant functions: a map $\prod_{\gamma \in \Gamma} R \to T$ is equivalent to a set of idempotents $e_{\gamma}$ of $T$ that $\sum e_{\gamma} = 1$. This is equivalent to a locally constant function on $\text{Spec} T$ that takes value $\gamma$ on $V(e_{\gamma})$. Then the product takes values $\gamma \delta$ on $V(e_{\gamma} \otimes e_{\delta}) \subset \text{Spec} T \otimes T$, or equivalently takes values $\gamma$ on $V(\sum_{gg'=\gamma} e_g \otimes e_{g'})$, so $\Delta(e_{\gamma}) = \sum_{gg'=\gamma} e_g \otimes e_{g'}$.

**Lemma (I.15.2.11) (Modulo Ker(\varepsilon)).** For a Hopf algebra $A$ over $R$, the comultiplication and counit are determined by $\ker \varepsilon$:

- $R \oplus \ker \varepsilon \to A : (a,b) \mapsto a + b$ is an isomorphism of $R$-modules.
- $\mu(a) \equiv -\varepsilon(a) + a \otimes 1 + 1 \otimes a \mod \ker \varepsilon \otimes_R \ker \varepsilon$.
- $\iota(a) \equiv -a \mod (\ker \varepsilon)^2$ for $a \in \ker \varepsilon$.

**Proof:**
1: this is because the counit $0 \to \ker \varepsilon \to A \to R \to 0$ has an inverse by the $R$-algebra map $R \to S$.
2: item 1 allows us to write

$$A \otimes_R A = R \oplus (\ker \varepsilon \otimes_R R) \oplus (R \otimes_R \ker \varepsilon) \oplus (\ker \varepsilon \otimes \ker \varepsilon)$$

so for $a \in A$,

$$\mu(a) = b + c \otimes 1 + 1 \otimes d + z$$

where $b \in R, c, d \in \ker \varepsilon, z \in \ker \varepsilon \otimes_R \ker \varepsilon$. Then $a = (\varepsilon \otimes \text{id}_A)(b + c \otimes 1 + 1 \otimes d + z) = b + d$, and also $a = b + c$. Applying $\varepsilon$ shows $b = \varepsilon(a)$, and thus

$$\mu(a) = \varepsilon(a) + (a - \varepsilon(a)) \otimes 1 + 1 \otimes (a - \varepsilon(a)) + z = -\varepsilon(a) + a \otimes 1 + 1 \otimes a + z.$$  

3: Let $\iota(a) = b + c$ where $b \in R, c \in \ker \varepsilon$, then

$$\varepsilon(a) = (\text{multi})(\iota \otimes \text{id})(-\varepsilon(a) + a \otimes 1 + 1 \otimes a + z) = -\varepsilon(a) + \iota(a) + a + (\text{multi})(\iota \otimes \text{id})(z)$$

so for $a \in \ker \varepsilon$, $\iota(a) \equiv -a \mod (\ker \varepsilon)^2$, as $(\text{multi})(\iota \otimes \text{id})(z) \in (\ker \varepsilon)^2$, because $\iota$ commutes with $\varepsilon$.

**Def. (I.15.2.12) (Hopf Ideal and Quotient Hopf Algebra).** Let $A$ be a Hopf algebra, then a quotient Hopf algebra is a quotient $A/I$ that has a Hopf algebra structure compatible with that of $A$. In another words, there are commutative diagrams

$$\begin{array}{ccc}
A & \xrightarrow{\mu} & A \otimes_R A \\
\downarrow & & \downarrow \\
A/I & \xrightarrow{\pi} & A/I \otimes_R A/I
\end{array}$$

$$\begin{array}{ccc}
A & \xrightarrow{\varepsilon} & R \\
\downarrow & & \downarrow \\
A/I & \xrightarrow{\varepsilon} & A/I
\end{array}$$

$$\begin{array}{ccc}
A & \xrightarrow{\iota} & A \\
\downarrow & & \downarrow \\
A/I & \xrightarrow{\varepsilon} & A/I
\end{array}$$

In particular, quotient Hopf algebra corresponds to ideals of $A$ that

$$\mu(I) \subset \ker A \otimes_R A \to A/I \otimes_R A/I,$$

$\varepsilon(I) = 0, \quad \iota(I) \subset I,$

called Hopf ideals of $A$.

**Remark (I.15.2.13) (Examples).** If $\Gamma'$ is a subgroup of a commutative group $\Gamma$, then $R[\Gamma']$ has a quotient Hopf algebra $R[\Gamma/\Gamma']$. In particular, $R[t]/(t^n - 1)$ is a quotient Hopf algebra of $R[t, t^{-1}]$. 

Prop. (I.15.2.14) (Hopf Ideals of Additive Groups). Let $f = \sum_{i=0}^{d} a_i t^i \in R[t]$ be a monic polynomial $\not= t$, then $(f)$ is a Hopf ideal of $R[t]$ iff $R$ is of char $p > 0$ and the derivative $f' = 0$.

Proof: We check conditions in (I.15.2.12), the first says $\sum a_i (t \otimes 1 + 1 \otimes t)^i$ vanishes in $R[t]/(f) \otimes_R R[t]/(f)$. But $f$ is monic, so this is equivalent to $a_i (i^j) = 0$ for any $0 < j < i \leq d$. But we have $gcd_{0 < j < i} (i^j) = p$ if $i = p^r$ for some $r \geq 1$ and 1 otherwise, so $a_d = 0$ unless $d$ is a power of $p$, and $p = 0 \in R$ because $a_d = 1$. In particular, $f = \sum_{i=0}^{k} b_i t^{p^i}$, and it automatically satisfies the other two conditions.

Cor. (I.15.2.15) ($\alpha_{p^r,R}$). Let $R$ be a commutative ring of char $p > 0$, then the quotient Hopf algebra $R[t]/(t^{p^r})$ corresponding to the Hopf ideal $(t^{p^r})$ is denoted by $\alpha_{p^r,R}$.

**WARNING:** $\alpha_{p^r,R}$ is isomorphic to $\mu_{p^r,R}$ as $R$-algebras, but they are not isomorphic as Hopf algebras.

Prop. (I.15.2.16) (Irreducibility of Hopf Algebras). Let $k$ be a field, then the following Hopf algebra contains no proper Hopf ideals:

- $G_{a,k}$ if char $R = 0$.
- $\alpha_{p,k}$ if char $R = p > 0$.

Proof: Any Hopf ideal fo $k[t]$ is principal, thus this proposition follows from (I.15.2.14).

Def. (I.15.2.17) (Cokernel Hopf Algebra). Let $A \to B$ be a monomorphism of Hopf Algebras over $R$, then the cokernel Hopf algebra is the quotient kernel of $B$ defined by $B \to B \otimes_A R$, which represents the functor of kernels of $\text{Hom}_R(B,T) \to \text{Hom}_R(A,T)$.

Lemma (I.15.2.18). A Hopf algebra over a field $k$ is direct limit of Hopf algebra of f.t. over $k$.

Def. (I.15.2.19) (Group-Like Elements). Let $A$ be a Hopf algebra, then a group-like element $a \in A$ is an invertible element that satisfies $\mu(a) = a \otimes a \in A \otimes A$.

If $a$ is a group-like element in a Hopf algebra $A$, then $a = (\varepsilon, \text{id})\mu(a) = \varepsilon(a)a$, so $\varepsilon(a) = 1$.

Prop. (I.15.2.20) (Group-Like Elements are Linearly Independent). Let $A$ be a Hopf algebra over a field $k$, then the set of group-like elements are linearly independent over $k$.

Proof: If $e = \sum a_i e_i$ that $e, e_i$ are all group-like elements, then

$$\mu(e) = e \otimes e = \sum c_i c_j e_i \otimes e_j = \sum c_i \mu(e_i) = \sum c_i e_i \otimes e_i$$

so $c_i^2 = c_i$ and $c_i c_j = 0$. Now also notice $1 = \varepsilon(e) = \sum c_i \varepsilon(e_i) = \sum c_i (\text{I.15.2.19})$, contradiction.

Cor. (I.15.2.21). The set of group-like elements in the Hopf algebra $k[\Gamma]$(I.15.2.3) are just the set $\Gamma \subseteq k[\Gamma]$.

Prop. (I.15.2.22) (Cartier Theorem). A Hopf algebra $A$ over a field $k$ of characteristic 0 is reduced.

Proof: We can base change to the algebraic closure $\overline{k}$ of $k$ and assume $k$ is alg. closed. Because reducedness is stalkwise and Hilbert’s Nullstellensatz, it suffices to show $A_s$ is reduced at each $s \in G(k)$. The translation by $g \in G(k)$ acts transitively on $G(k)$, so it suffices to show it vanishes at the kernel of the counit map $\text{Ker}(\varepsilon)$. Cf. [Jakob, Stix].

We may assume by taking direct limits that $A$ is f.g. over $k$. Let $m$ be the kernel of the counit $\varepsilon : A \to k$, then $m/m^2$ is a f.g. $k$-vector space with a basis lifting to $x_1, \ldots, x_r \in m$.

We prove that $\text{gr}_m(A)$ is a polynomial ring in $x_i$, Cf. [Shatz].
Prop. (I.15.2.23) (Faithfully Flatness). If \( A \subset B \) are f.g. Hopf algebras over a field \( k \), then \( B \) is f.f. over \( A \).

\[ \text{Proof:} \quad \text{Cf.}[\text{Milne, P73}]. \]

Cor. (I.15.2.24). If \( A \subset B \) are Hopf algebras that \( B \) is an integral domain, and let \( K, L \) be their fraction field, then \( B \cap L = A \). In particular, if \( K = L \), then \( A = B \).

\[ \text{Proof:} \quad \text{Since} \ A \to B \text{ is f.f. by(I.15.2.23), } cB \cap A = cA \text{ for any } c \in A \text{ by(I.15.2.23). Thus if } a/c \in B \text{ for } a,c \in A, \text{ then } a \in cB \cap A = cA, \text{ so } a/c \in A. \]

3 Quiver Hecke Algebra

\[ \text{Cf.}[\text{Bru13}]. \]
I.16 Almost Ring Theory

References are [Almost Ring Theory Gabber/Ramero] and [Almost Ring Theory Foundations Gabber/Ramero].

Def. (I.16.0.1). The setup of most mathematics is an flat ideal \( I \subset R \), that \( I^2 = I \). This implies that \( I \otimes I \cong I^2 = I \).

Denote \( i : R \to R/I \). Then there is a map \( i_* : M \mapsto M_R \), which has a left adjoint \( i^* : N \mapsto N \otimes_R R/I \), and a right adjoint \( i^! : N \mapsto \text{Hom}(R/I, N) \).

1 Homological Theory

Almost Modules

- If \( K \) is a perfectoid field, \( R = K^0, I = K^{00} \), then \( I \) is flat over \( R \), because if \( \pi \) is a pseudo-uniformizer of \( K(I.9.8.9) \), then \( I = (\pi^{1/\pi^n}) \), which is a colimit of free modules thus flat, and \( I^2 = I \) clearly.
- Let \( R \) be a ring and \( f \) is an arbitrary element with compatible \( p^n \)-th roots, let \( I = (f^{1/\pi}) \), then \( I^2 = I \). To show \( I \) is flat, consider:

\[
M_0 \xrightarrow{f - \frac{1}{p}} M_1 \xrightarrow{f - \frac{1}{p}} M_2 \to \cdots \to M_n \xrightarrow{f^{1/p} - \frac{1}{p^{n+1}}} M_{n+1} \to \cdots
\]

where \( M_i \cong R \), and \( M = \text{colim} M_i \), then \( M \) is flat, and there is a map \( M \to I : 1 \in M_n \to f^{1/\pi^n} \), then this map is surjective, and it is injective: if \( \alpha \) maps to 0, then \( \alpha f^{1/\pi^n} = 0 \), so \( \alpha^{p^n} f = 0 \) for all \( m \geq n \), and by perfectness of \( R \), \( \alpha f^{1/\pi^n} = 0 \), so in particular, \( \alpha = 0 \in M_{n+1} \).

Prop. (I.16.1.2) (The Category of Almost \( R \)-Modules in disguise). Let \( \mathcal{A} \subset \text{Mod}_R \) be the category of all \( R \)-modules \( M \) that the action \( I \otimes M \to M \) is an isomorphism(By \( I \otimes I = I \) this is equivalent to \( M = I \otimes N \) for some \( N \)) then:

- The inclusion \( j_I : \mathcal{A} \to \text{Mod}_R \) is exact, i.e. the cokernel, kernel of objects in \( \mathcal{A} \) are also in \( \mathcal{A} \).
- \( j_I \) has a right adjoint \( j^* : M \mapsto I \otimes M \), and the unit map \( N \to j^* j_I N \) is an isomorphism on \( \mathcal{A} \).
- \( j^* \) has its right adjoint \( j_* (M) = \text{Hom}(I, M) \), and the counit \( j^* j_* M \to M \) is an isomorphism on \( \mathcal{A} \).

Proof: 1: an easy consequence of five-lemma.
2: We need to show for \( N \in \mathcal{A} \), \( \text{Hom}(N, I \otimes M) \cong \text{Hom}(N, M) \). Notice there is a distinguished triangle

\[
I \otimes M \to M \to M \otimes R/I,
\]

as \( - \otimes_R^{L} M \) is a derived functor and \( I \) is flat. So it suffices to show

\[
R \text{Hom}_R(N, M \otimes_R^{L} R/I) = 0 = R \text{Hom}_R(N \otimes_R^{L} R/I, M \otimes_R^{L} R/I).
\]

And in fact \( N \otimes_R^{L} R/I = 0 \), because \( N \otimes_R^{L} R/I = N \otimes_R^{L} I \otimes_R^{L} R/I \), and \( I \otimes_R^{L} R/I = I \otimes R R/I = I/I^2 = 0 \) by flatness and hypothesis.

\( N \cong j^* j_I N \) is an easy consequence of \( I \otimes I = I \).
3: The adjointness is just Tor-Hom-adjunction, and for the isomorphism $I \otimes \text{Hom}(I, M) \cong M$, as $I$ is flat, it suffices to prove the stronger result that $I \otimes^L_R \text{RHom}(I, M) = M[0]$. As there is an exact triangle

$$R\text{Hom}(R/I, M) \to M \to R\text{Hom}(I, M),$$

so it suffices to show $I \otimes^L_R R\text{Hom}(R/I, M) = 0$, because $I \otimes^L_R M = M$. But this is because $I \otimes^L_R R\text{Hom}(R/I, M) = I \otimes^L_R R/I \otimes^L_R R\text{Hom}(R/I, M)$, and $I \otimes^L_R R/I = 0$ as before. □

**Prop. (I.16.1.3) (Category of Almost $R$-modules).**
- The image of the functor $i_* : \text{Mod}_{R/I} \to \text{Mod}_R$ is a Serre subcategory of $\text{Mod}_R$, so the quotient $\text{Mod}_R^a = \text{Mod}_R/\text{Mod}_{R/I}$ exists by (I.11.2.14),
- The quotient $q : \text{Mod}_R \to \text{Mod}_R^a$ admits fully faithful left and right adjoints. In particular, $q$ preserves all limits and colimits.
- The image of $i$ is a 'tensor ideal' of $\text{Mod}_R$, so the quotient $\text{Mod}_R^a$ inherits a natural symmetric monoidal $\otimes$-product structure.
- There is a functor $\text{alHom} : (\text{Mod}_R^a)^{\text{op}} \times \text{Mod}_R^a \to \text{Mod}_R^a : (X, Y) \to \text{alHom}(X, Y)$ that $\text{alHom}(X, -)$ is right adjoint to $- \otimes X$:

$$\text{Hom}(Z \otimes X, Y) \cong \text{Hom}(Z, \text{alHom}(X, Y)).$$

**Proof:** 1: the image of $i_*$ is just the category of modules killed by $I$, if $M$ is killed by $I$, then subobjects and quotients of $M$ is killed by $I$, and if $M$ is an extension of two elements killed by $I$, then $IM = I^2M = 0$.

2: In fact we show that the category $A$ in (I.16.1.2) and the functor $j^*$ is just equivalent to $\text{Mod}_R^a$:
First: $j^*(\text{Mod}_{R/I}) = 0$, because $I \otimes M = I \otimes R/I \otimes R/I M = I/I^2 \otimes R/I M = 0$ as $I = I^2$, and $j^*$ is exact because $I$ is flat.
And for any $R$-module $M$, consider $I \otimes M \to M$, it has kernels and cokernels, then tensoring $I$, it becomes $I \otimes M \to I \otimes M$ (I.16.0.1). as $I$ is flat, the kernel and cokernels are killed by $I$, so for any functor $q$ to another category that kills $\text{Mod}_{R/I}$, $q(M) = q(I \otimes M) = qj_ij^*(M)$, so $q$ factors through $M$, uniquely, as $j^*$ is surjective.
Now the left/right adjoints exist by (I.16.1.2).
3: if $IM = 0$, then $IM \otimes N = 0$, so the tensor products pass to the quotient, and $j^*$ is a map between symmetric monoidal categories.
4: $\text{alHom}$ is defined by $\text{alHom}(j^*M, j^*N) = j^*(\text{Hom}(M, N)) = \text{Hom}(M, N)^a$. This is well defined, because if $IM = 0$ or $IN = 1$, then $I\text{Hom}(M, N) = 0$. □

**Cor. (I.16.1.4).**
- $i^*j_i = 0$.
- $i^*j_i = 0$.
- $j^*i_* = 0$, and the kernel of $j^*$ is just $i_*(\text{Mod}_{R/I})$.

**Proof:** 1: $R/I \otimes I \otimes M = 0$, because $I \otimes R/I = 0$.
2: $\text{Hom}(R/I, \text{Hom}(I, M)) = 0$, because $I \otimes R/I = 0$.
3: This is by 2 of the proposition (I.16.1.3). □
Remark (I.16.1.5). The construction above can be summarized as the following diagram:

\[
\begin{array}{ccc}
\text{Mod}_{R/I} & \xrightarrow{i^*} & \text{Mod}_R \\
\text{Mod} & \xleftarrow{j_*} & \text{Mod}^a_R
\end{array}
\]

with four adjoint pairs and three vanishing. This should be seen as an analogy of the case of topology: \(X\) is a space and \(i: U \to X\) is open in \(X\), and \(j: Z \to X\) is closed, \(Z = X - U\), then the defined sheaf operations are the same as written above.

However, one should not consider \(\text{Mod}^a_R\) as the sheaf of modules on the open subscheme \(\text{Spec} R_f\) for some pseudo uniformizer, because the map \(M \to M \otimes R_f\) factors through \(\text{Mod}^a_R\) as it vanishes on \(\text{Mod}_{R/I}\), but it is not \(\text{Mod}^a_{R/I}\). For example, if \(k\) is a perfect field, and consider \(R = k[[t^{1/\infty}]]\), then the module \(M = R\langle t \rangle\) is also killed by \(\otimes R_f\), but it is not killed by \(I\).

Then one may consider it is the category of \(\text{Qco}\) sheaves on \(D(I)\), but first this is not an affine scheme, and second this is false, anyway. And we should imagine an non-existent open subscheme \(U\) bigger than \(U\), as it contains any affine opens of \(U\).

Remark (I.16.1.6). Notice \(j^*\) is both left exact and right exact, so it preserves both arbitrary limits and colimits, so almostification nearly loses anything. In particular, the category \(\text{Mod}^a_R\) has all colimits and limits.

Def. (I.16.1.7) (Almost Commutative Algebras). As \(\text{Mod}^a_R\) has a symmetric monoidal structure, it is possible to define the category of almost commutative algebras as the category of commutative unitary monoids in \(\text{Mod}^a_R\), denoted by \(\text{CAlg}(\text{Mod}^a_R)\). Notice that its unit object is \(I = R^a\).

There is an obvious map

\[(-)^a: \text{Alg}(\text{Mod}_R) \to \text{Alg}(\text{Mod}^a_R),\]

and yet another functor

\[(-)_*: \text{Alg}(\text{Mod}^a_R) \to \text{Alg}(\text{Mod}_R),\]

because \(M \to M_*\) is lax symmetric monoidal, i.e. there are natural maps \(M_* \otimes N_* \to (M \otimes N)_*\). This is a right adjoint of \((-)^a\), as \(j^*\) and \(j_*\) is adjoint.

Finally there is a functor

\[(-)!!: \text{Alg}(\text{Mod}^a_R) \to \text{Alg}(\text{Mod}_R),\]

whose construction is a little complicated, first notice the functor \((-)_1\) preserves multiplication but it has no units, so in order to give it a unit, consider the module pushout: \((A_1 \oplus V)/I\), which has a natural multiplicative structure that can be made into a \(R\)-module, and \((-)!!\) is left adjoint to \((-)^a\), Cf.[Almost Ring Theory P22].


Proof: Cf.[Almost Ring theory P52] \(\Box\)

Def. (I.16.1.9). For an almost commutative algebra \(A\), a left module is an almost module \(M \subset \text{Mod}^a_R\) that has a left action \(A \otimes M \to M\) that has natural commutative diagrams as one expects. And for any \(R\)-algebra \(A\), there are natural maps \(\text{Mod}_A \to \text{Mod}^a_A\).
Almost Homological Algebra

2 Almost Commutative Algebra

Def. (I.16.2.1) (Almost Notations). Given a $R$-module $M$, an element $f \in M$ is called almost zero if $I \cdot f = 0$, and $M$ is called almost zero if all $f \in M$ is almost zero.

Denote

$$M^a = j^a M \in \text{Mod}_R^a, \quad M_* = j_* M^a = \text{Hom}(I, M), \quad M_I = j_I M^a = I \otimes M.$$ 

Then there are morphisms $M_I \to M \to M_*$, which becomes isomorphisms after almostification.

Prop. (I.16.2.2). If $I = (f^{1/p^\infty})$, then $M_* = \{ x \in M[f^{-1}] | f^{1/p^\infty} x \in A \}$ for all $n$.

Prop. (I.16.2.3). If $M \to N$ is almost surjective maps of $K^0$-algebras that $M/I \to N/I$ is surjective, then $M \to N$ is surjective.

Proof: As $I$ is flat over $K^0$, if $M \to N \to Q \to 0$ is the cokernel, tensoring $A/I$, as $M/I \to N/I$ is surjective, $Q/IQ = 0$, but $Q$ is almost zero thus $IQ = 0$, so $Q = 0$. \hfill \Box

Def. (I.16.2.4) (Almost Properties). Something is called almost XXX if it is XXX when passed to the category of almost $R$-modules. For example,

- elements of $M_*$ are called almost elements of $M$.
- $M$ is called almost flat iff $M^a \otimes -$ is exact on $\text{Mod}_R^a$, which is equivalent to $\text{Tor}^R_{\geq 0}(M, N)$ is almost zero for all $N$.
- $M$ is called almost projective iff $\text{alHom}(M,-)$ is exact on $\text{Mod}_R^a$, which is equivalent to $\text{Ext}^R_{\geq 0}(M, N)$ is almost zero for all $N$.

Notice this is not equivalent to projective in $\text{Mod}_R^a$, because $R$ is almost projective, but $\text{Hom}_R(R^a, M^a) = \text{Hom}(I, M)$ is not exact as $I$ is not projective: $\text{Ext}^1_R(R^a, R^a) = \text{Ext}^1_R(I, R) = \text{Ext}^2(k, R)$, which is not 0 if $R$ is the valuation ring of a non-spherically complete perfectoid field $K$, like $\hat{\mathbb{Q}}_p$?

- $M$ is called almost finitely generated/almost finitely presented if for any $\varepsilon \in I$, there is a f.g./f.p. $M_\varepsilon \to M$ with $N_\varepsilon$ generators that the kernel and cokernel are killed by $\varepsilon$. It is called uniformly almost finitely generated iff $N_\varepsilon$ is independent of $\varepsilon$.

Notice this definition doesn’t depends on $M$ chosen?
- If $S$ is of charp, it is called almost perfect iff $S_*$ is perfect.

Prop. (I.16.2.5) (Enough Almost Injectives). The category $\text{mod}_R^a$ has enough injectives. In fact $j^a, j_*$ both preserves injectives, because they has exact left adjoints, so $I$ is injective $R$-module iff $I^a$ is injective $R^a$-module, and $J$ is injective $R^a$-module iff $J_*$ is injective $R$-module. So to construct an injective resolution in $R^a$, pass to $R$-modules using either $(-)_*$ or $(-)_!$; and find an injective resolution, then almostificat it.

Prop. (I.16.2.6) (Derived Functors of $(-)_*$). Notice that $\text{Hom}_{R^a}(M^a, N^a) = \text{Hom}_R(I \otimes M, N)$ by adjointness, so using (I.16.2.5),

$$\text{Ext}^k_{R^a}(M^a, N^a) = \text{Ext}^k_R(M_*, N) = \text{Ext}^k_R(M, R \text{Hom}(I, N)),$$

then as $M_* = \text{Hom}(I, M)$, the derived functor of $(-)_*$ is just $\text{Ext}^k_R(I, M) = \text{Ext}^k_{R^a}(R^a, M^a)$.

Notice that $\text{Ext}^k_{R^a}(M, N)$ are all almost zero, as $j^a j_* = \text{id}$, and use trivial Grothendieck spectral sequence.
Prop. (I.16.2.7) (Example) A Quadratic Extension of a Perfectoid Field. If \( K = Q_p[p^{1/p}] \) and \( L = K(\sqrt{p}) \) with \( p \neq 2 \), then \( L^0 \) is a uniformly almost f.p. projective \( K^0 \)-module.

Proof: It suffices to find for each \( n \) a \( K^0 \)-module \( R_n \) of rank 2 that \( R_n \rightarrow L^0 \) is injective with cokernel annihilated by \( p^{1/p^n} \). For this, consider \( R_n = K^0 \oplus K^0 p^{1/p^n} \), then \( L^0 = \text{colim}_n R_n \).

Notice that the cokernel of \( R_n \rightarrow R_{n+1} \) is killed by \( p^{1/p^n} \), because

\[
p^{1/p^n} \cdot p^{1/p^{n+1}} = p^{(n+1)/2} \cdot p^{1/p^n} \subset R_n.
\]

So by killing one by one, the cokernel of \( R_n \rightarrow \text{colim}_n R_n \) is killed by any \( p \)-power with power larger than \( \sum \frac{1}{p^n} \), in particular by \( p^{1/p^n} \). So \( \text{colim}_n R_n \) is an extension of \( R_0 \) by a cokernel killed by \( p \), so it is also \( p \)-adically complete, and \( L^0 = \text{colim}_n R_n \). Now Consider \( 0 \rightarrow R_n \rightarrow \text{colim}_n R_n \rightarrow \text{Coker} \), then \( \text{Ext}^n(\text{colim}_n R_n, N) = \text{Ext}^n(\text{Coker}, N) \) is killed by \( p^{1/p^n} \) for all \( n \), so it is killed by \( I \), thus \( \text{colim}_n R_n \) is almost projective.

\( \square \)

Completions and Closures

Prop. (I.16.2.8) (prc and Completion). If \( A \) is a ring with a non-zero-divisor \( f \) that \( A \subset A[f^{-1}] \) is \( p \)-root closed (prc), then:

- \( A \subset \hat{A}[f^{-1}] \) is \( p \)-root closed.
- If \( f \) admits a compatible \( p \)-power roots, then \( A_s \subset A_s[f^{-1}] \) is \( p \)-root closed (where almost mathematics is performed w.r.t. \( f^{1/p^n} \)).

Proof: We first replace \( A \) with its maximal separated quotient \( A/(\cap_n f^n A = I) \): \( f \) is still non-zero-divisor, because if \( fg \in I \), then \( fg \in f^n A \) for all \( n \), so \( g \in f^{n-1} A \) as \( f \) is non-zero-divisor. And it is \( p \)-root closed, because if \( a^p \in A/I[f^{-1}] \), then \( a^p = b + f^{-c}d \) for \( c \) integer and \( d \in I \). Notice \( I = fI = f^{-c}f \) by (I.5.6.12), so \( f^{-c}d \in I \) as well, so \( a \in A \).

Now \( A \) is \( f \)-separated, in particular, \( A \hookrightarrow \hat{A} \).

1: If \( g \in \hat{A}[f^{-1}] \) and \( g^p \in \hat{A} \), then \( f^Ng \in \hat{A} \) for some \( N \) and choose a \( m \geq N(p-1) \), then by the density, \( g = g_0 + f^mg_1 \) for some \( g_0 \in A[f^{-1}], g_1 \in \hat{A} \). Notice \( f^Ng_0 \in \hat{A} \), now

\[
g^p = g_0^p + pg_0^{p-1}f^mg_1 + \ldots + (f^m g_1)^p,
\]

By definition of \( m \), all terms except \( g_0^p \) are in \( \hat{A} \), so \( g_0^p \in A \), so \( g_0 \in A \), and \( g \in \hat{A} \).

2: Use the convention (I.16.2.2), if \( g \in A_s[f^{-1}] \) that \( g^p \in A_s \), then \( f^{1/p}g^p \in A \) for all \( n \), so \( (f^{1/p^{n+1}}g)^p \in A \), thus \( f^{1/p^{n+1}}g \in A \), hence \( g \in A_s \). \( \square \)

Prop. (I.16.2.9) (ic and Completion). Let \( A \) be a ring with a non-zero-divisor \( f \), if \( A \subset A[f^{-1}] \) is integrally closed, then:

- \( A \subset \hat{A}[f^{-1}] \) is integrally closed.
- If \( f \) admits a compatible \( p \)-power roots, then \( A_s \subset A_s[f^{-1}] \) is integrally closed (where almost mathematics is performed w.r.t. \( f^{1/p^n} \)).

Proof: We first replace \( A \) with its maximal separated quotient \( A/(\cap_n f^n A = I) \): \( f \) is still non-zero-divisor and \( I \) is \( f \)-divisible as in the proof of (I.16.2.8). And it is integrally closed, because if \( g \)
satisfies a monic polynomial \( h(X) \in A/I[f^{-1}][X] \), then choose a lifting, \( h(g) \in I[f^{-1}] = I \subset A \), so \( g \) is integral over \( A \) thus \( g \in A \), and \( g \in A/I \). Now \( A \) is \( f \)-separated and \( A \hookrightarrow \hat{A} \).

1: If \( g \in \hat{A}[f^{-1}] \) satisfies a polynomial \( H \in \hat{A}[X] \), then \( g = f^{-c}h \) for \( h \in \hat{A} \), and then \( h \) satisfies a polynomial \( H(f^c x) \), and choose an approximation of coefficients of \( H(x) \) and \( h_0 \) of \( h \mod f^{cn} \), then it is clear that \( H(f^c h_0) \in f^{cn}A \cap A = f^{cn}A \), so when dividing back, \( g_0 = f^{-c}h_0 \) is integral over \( A \) thus \( g_0 \in A \), thus \( h_0 \in f^c A \), and \( h \equiv h_0 \mod f^{cn} \), thus \( h \in f^c A \), and \( g \in A \).

2: Use the convention (I.16.2.2), if \( g \in A_+[f^{-1}] \) is integral over \( A_+ \), then there are polynomial \( H \) that \( H(g) = 0 \), now if \( \varepsilon = f \frac{1}{p^k} \) consider another polynomial \( H(x/\varepsilon) \), then its coefficients are all in \( A \), thus \( \varepsilon g \) is integral over \( A \) thus \( \varepsilon g \in A \), and then \( g \in A_+ \). \( \square \)

**Prop. (I.16.2.10) (Tic and Completion).** Let \( A \) be a ring with a non-zero-divisor \( f \) that admits a compatible system of \( p \)-power roots \( f^{\frac{1}{p^k}} \) for all \( n > 0 \), and \( A \) is totally integrally closed (tic) in \( A[f^{-1}] \), then \( \hat{A} \) is totally integrally closed in \( \hat{A}[f^{-1}] \) and \( A = A^* \).

**Proof:** 1: Notice totally integrally closed is \( p \)-root closed, so \( \hat{A} \subset \hat{A}[f^{-1}] \) is \( p \)-root closed. Now if \( f^k g^n \subset \hat{A} \) for some \( k \), then by prc, \( f^{\frac{n}{p^k}} g \in \hat{A} \) for all \( n \), thus \( g \) in an almost zero element in \( \hat{A}[f^{-1}]/A \cong A[f^{-1}]/A \), and then \( g \) is totally integrally closed over \( A \), because for any \( n \), let \( n < p^k \), then \( f^{\frac{n}{p^k}} g \in A \), thus \( f^{\frac{n}{p^k}} g \in A \), and \( f g^n \in A \).

2: Because \( f^{\frac{1}{p^k}} A_+ \subset A \) by convention (I.16.2.2), clearly \( A_+ \) is totally integrally closed in \( A \), thus \( A_+ \subset A \). \( \square \)

**Almost Étale Map**

**Def. (I.16.2.11).** A map \( A \to B \) of \( R^2 \)-algebras is called **almost étale** iff:

- \( B \) is almost f.p. projective over \( A \).

- (Unramifiedness(I.7.6.10)) There exists a diagonal idempotent \( e \in (B \otimes_A B)_+ \), i.e. \( e^2 = e \) and \( \mu_*(e) = 1 \), and \( \text{Ker}(\mu)_+ \cdot e = 0 \), where \( \mu : B \otimes_A B \to B \) is the multiplication map.

**Prop. (I.16.2.12) (Example of Almost Étale Maps).** Let \( K = Q_p[p^{\frac{1}{p^k}}] \) and \( L = K(\sqrt[p]{p}) \) with \( p \neq 2 \), then \( L^0/K^0 \) is uniformly almost f.p projective \( K^0 \)-module, by(I.16.2.7). We show it is finite étale: flatness is clear, as \( L^0/K^0 \) is torsion-free and \( K^0 \) is a valuation ring and use(I.16.3.3).

For unramifiedness, notice that

\[
L \otimes_K L \cong L \times L : (a,b) \mapsto (ab,a\sigma(b)).
\]

by(I.2.5.17), the diagonal idempotent \( e \) is given by

\[
e = \frac{1}{2p^{\frac{1}{2p^k}} \otimes 1}(1 \otimes p^{\frac{1}{2p^k}} + p^{\frac{1}{2p^k}} \otimes 1)
\]

for any \( n \geq 0 \), then we see \( p^{\frac{1}{p^k}} e \in L^0 \otimes_{K^0} KL^0 \) for all \( n \), thus \( e \in (L^0 \otimes_{K^0} L^0)_+ \).

**Lemma (I.16.2.13) (Lemma for Almost Purity in Characteristic \( p \)).** If \( \eta : R \to S \) is an integral map of perfect rings. If \( \eta[t^{-1}] \) is finite étale for some \( t \in R \), then \( \eta \) is almost finite étale w.r.t the ideal \( I = (t^{\frac{1}{p^k}}) \).
Proof: Firstly, we may assume $R, S$ are both $t$-torsion-free, because the $t$-torsion part $R[t^\infty]$ and $S[t^\infty]$ is almost zero: if $t^c \alpha = 0$, then $t^c \alpha^{p^n} = 0$, so $t^{cp^n} \alpha = 0$. So we reduce to $R/R[t^\infty] \to S/S[t^\infty]$, which doesn’t change anything.

Now we reduce to the case that $R, S$ are integrally closed in $R[t^{-1}]$ and $S[t^{-1}]$: it suffices to show that $R_{\text{int}} \subset R_s$, thus they are almost isomorphic. For this, an element $f \in R_{\text{int}}$ satisfies $f^N t^k \in R$ for some $k$, so by perfectness, $ft^\frac{1}{p^n} \in R$ for all $n$, so $f \in R_s$.

Now check unramified: let $e \in (S \otimes_R S)[t^{-1}]$ be a diagonal idempotent, then $te \in S \otimes_R S$ for some $c$, now $e^2 = e$, so easily $e \in (S \otimes_R S)_s$.

Now check almost finite projective: for $m > 0$, represent $t^\frac{1}{p^n} e = \sum a_i \otimes b_i \in S \otimes_R S$, then use the map $S \xrightarrow{\alpha} R^n \xrightarrow{\beta} S$ as in(1.7.7.14), then $\beta \alpha = t^\frac{1}{p^n}$ on $S$, as $S$ is $t$-torsion free, $R^n \to S$ is injective with $t^\frac{1}{p^n}$-torsion cokernel, for any $m$. So $S$ is almost finite projective. □

**Prop. (I.16.2.14) (Almost Purity in Characteristic $p$).** If $R$ is a perfect ring of char$p$, then using the almost mathematics w.r.t. $I = (t^\frac{1}{p^n})$, $S \to S_s[t^{-1}]$ gives an isomorphism of categories: $R_{af\acute{e}t} \cong R[t^{-1}]_{af\acute{e}t}$.

**Proof:** As in the proof of(I.16.2.13), we may assume $R$ is $t$-torsion-free. Notice that any integral extension of $R[t^{-1}]$ comes from an integral extension of $R$(choose the integral closure), so the lemma above(I.16.2.13) tells us the functor is essentially surjective.

Now we construct an inverse functor, $S_s[t^{-1}]$ maps to $T^a$, where $T$ is the integral closure of $R$ in $S_s[t^{-1}]$. By lemma(I.16.2.15) below, $S$ is almost perfect. So $S_s$ is $t$-torsion-free, as if $t^c f = 0$, then $t^\frac{1}{p^n} f = 0$ for all $n$, so $f = 0 \in (S_s)_s = S_s$. So now $S_s \subset S_s[t^{-1}]$. Clearly $T$ is also perfect and $t$-torsion-free. So $R \to T$ is an integral extension that is identified with $R[t^{-1}] \to S_s[t^{-1}]$ after inversion of $t$.

To show that $T^a = S$, it suffices to show $T_s = S_s$. for $f \in T$, $f^N$ spans a finite module of $T[t^{-1}] = S_s[t^{-1}]$, so $t^c f^N \subset S_s$, then by perfectness, $f \in (S_s)_s = S_s$, so $T_s \subset S_s$. Conversely, if $g \in S_s$, then $t^c g^N$ lies in a f.g. $R$-module of $S_s[t^{-1}]$, by almost f.g.. So $t^c g^N \subset T$, and then by perfectness $g \in T_s$. □

**Lemma (I.16.2.15).** Almost finite étale map of rings of char$p$ is almost relatively perfect.

**Proof:** Cf.[Bhatt notes on Perfectoid Spaces P28]. □

3 Almost Mathematics on Perfectoid Fields

**Prop. (I.16.3.1) (Almost Elements).** If $K$ is a perfectoid field, $R = K^0$ and $I = K^{00}$, $M$ is an $R$-module, then

- If $M$ is torsion-free, then $M_s = \{m \in M \otimes_{K^0} K|Im \in M\} = \{m \in M \otimes_{K^0} K|t^\frac{1}{p^n} m \in M\}$, by(I.9.8.12) and(I.16.2.2).
- $I_s = R_s = R$. More generally, for an ideal $J \subset R$, let $c = \sup\{|x||x \in J\}$, then $J_s = \{a \in K|a| \leq c\}$.

**Prop. (I.16.3.2).** Let $K$ be a perfectoid field with a pseudo-uniformizer $\pi$. If $\alpha : M \to N$ is an almost surjective map of $K^0$-algebras that $M$ is $\pi$-adically separated and $N$ is $\pi$-torsion-free that $\alpha \mod \pi$ is an almost isomorphism, then $\alpha$ is an almost isomorphism.
Prop. (I.16.3.3) (Almostification and Completeness). Let $K$ be a perfectoid field with a pseudo uniformizer $t$ and $R = K^0, I = K^{00}$, let $M \in \text{Mod}_t^R$, then:

- $M$ is almost flat iff $M_\ast$ is $R$-flat iff $M_!$ is $R$-flat.
- Assume $M$ is almost flat, then $M$ is $t$-adically complete iff $M_\ast$ does.
- Assume $M$ is almost flat, then for each $f \in K^0$, $fM_\ast \cong (fM)_\ast$, and $M_\ast/fM_\ast \subset (M/fM)_\ast$.

And for any $\varepsilon \in I$, the image of $(M/f\varepsilon M)_\ast$ and $M_\ast/fM_\ast$ in $(M/fM)_\ast$ are identical.

Proof: 1: $R$ is a valuation ring, so $M_\ast$ is $R$-flat iff $M_\ast[t]$ is flat by (I.7.1.9), as $t$ is a pseudo uniformizer. As $(-)_\ast$ is left exact, $M_\ast[t] = (M[t])_\ast$, so if $M$ is almost flat, then $M[t] = 0$ as $t$ is nonzero-divisor, so $M_\ast$ is $R$-flat. The converse is true as $M = (M_\ast)^a$, and the tensor is compatible.

For $(-)_\ast$, this follows from the observation that $(-)_!$ and $(-)^a$ are both exact and commute with tensor products, and notice $M_! \otimes N = (M \otimes N)^a$.

2: As $(-)^a$ commutes with all limits and colimits, if $M_\ast$ is $t$-adically complete then so does $M = (M_\ast)^a$. Conversely, if $M$ is $R$-flat and $t$-adically complete, then $M_!, M_\ast$ are also $R$-flat, and consider the commutative diagram:

\[
\begin{array}{ccc}
M_! & \xrightarrow{a} & \lim (M/t^n M)_! = \lim M_!/t^n M_! = \hat{M}_! \\
& \downarrow{d} & \downarrow{b} \\
M_\ast & \xrightarrow{c} & \lim (M/t^n M)_\ast
\end{array}
\]

then $d$ is almost isomorphism by (I.16.1.2) and so does $b$ because $(-)^a$ commutes with all limits, and $c$ is an isomorphism as $(-)_\ast$ commutes with limits and $M$ is $t$-adically complete. So $a$ is also almost isomorphism.

Now notice $M_!$ is flat hence $t$-torsion-free, so the kernel of $a,d$ must be 0, with almost zero cokernels. Now (I.5.7.9) shows first $M_!$ is complete and next $M_\ast$ is complete.

3: Notice $(fM_\ast)^a = fM$ as $(-)^a$ is exact, so

\[
(fM)_\ast = \text{Hom}(I, fM_\ast) = \{y \in M_\ast[t^{-1}] | Iy \subset fM_\ast\} = f\{y \in M_\ast[t^{-1}] | Iy \subset M_\ast\} = fM_\ast
\]

and $M_\ast/fM_\ast \subset (M/fM)_\ast$ follows from the left exactness of $(-)_\ast$.

For the last assertion, consider the commutative diagram:

\[
\begin{array}{c}
0 \longrightarrow M \xrightarrow{f} M \longrightarrow M/fM \longrightarrow 0 \\
0 \longrightarrow M/\varepsilon M \xrightarrow{f} M/f\varepsilon M \longrightarrow M/fM \longrightarrow 0
\end{array}
\]

and apply $(-)_\ast = \text{Hom}_{R^0}(R^a, -)$ and use (I.16.2.6), then

\[
\begin{array}{c}
0 \longrightarrow M_\ast/fM_\ast \xrightarrow{a} (M/fM)_\ast \longrightarrow \text{Ext}^1_{R^0}(R^a, M)[f] \longrightarrow 0 \\
0 \longrightarrow (M/f\varepsilon M)_\ast \xrightarrow{b} (M/fM)_\ast \longrightarrow \text{Ext}^1_{R^0}(R^a, M/\varepsilon M)
\end{array}
\]
To show $a, b$ has the same image, it suffices to show that $c$ is injective. For this, it suffices to show
$\text{Ext}^1_R(R^a, M) \to \text{Ext}^1_R(R^a, M/\varepsilon M)$ is injective. Consider the exact sequence $0 \to M \xrightarrow{\varepsilon} M \to M/\varepsilon M \to 0$, it suffices to show that $\varepsilon \text{Ext}^1_R(R^a, M) = 0$, and this is obvious as $\varepsilon \in I(I.16.2.6)$. □

**Prop. (I.16.3.4) (General Completeness and Almostification).** More generally, if $J = (f_1, \ldots, f_r) \subset R$ is a f.g. ideal, then an $R^a$-module $M$ is $J$-adically complete iff $M_*$ does.

**Proof:** Cf.[Perfectoid Spaces Bhatt P32]. □

**Banach Space**

**Prop. (I.16.3.5) (Uniform Banach $K$-Algebra).** If $K$ is a non-Archipedian perfectoid(perfect) field with a pseudo uniformizer $t$, then the following categories are equivalent:

- The category of uniform Banach $K$-algebras.
- The category $D_{tic}$ of $t$-adically complete and $t$-torsionfree $K^0$-algebras $A$ with $A$ totally integrally closed in $A[t^{-1}]$ (I.5.5.1).
- The category $D_{ic}$ of $t$-adically complete and $t$-torsionfree $K^0$-algebras that $A$ is integrally closed in $A[t^{-1}]$ and $A = A_*$.
- The category $D_{prc}$ of $t$-adically complete and $t$-torsionfree $K^0$-algebras that $A$ is $p$-root closed in $A[t^{-1}]$ and $A = A_*$.  

**Proof:** The last three are equivalent, because if $A \in D_{tic}$, then $A = A_*$ by (I.16.2.10), as $K$ is perfect by (I.9.8.4). So $D_{tic} \subset D_{ic} \subset D_{prc}$, so it suffices to show that $D_{prc} \subset D_{tic}$. Now for any $f$ that $f^N \subset t^{-k} A$, then $t^k f^p$ $\subset A$, and $A$ is $p$-root closed, so $t^{k^p} f \subset A$ for all $n$, so $f \in A_*$ (I.16.2.3), but $A_*$ = $A$.

The equivalence of 1, 2 is general, by (IV.1.2.7). □

**Prop. (I.16.3.6).** If $K$ is a perfectoid field, then the category of uniform Banach spaces has all colimits and limits.

**Proof:** Cf.[Bhatt P38]. □
Chapter II

Categories

II.1 Categories

Main references are [?], [Coend Calculus, Fosco Loregian].

1 Basics

Def. (II.1.1.1) (Category). A category is

A morphism of categories consists of

Def. (II.1.1.2) (Filtered Category). A filtered category is a category \( I \) that:

- It is nonempty.
- for any \( a, b \in I \), there is some \( c \in I \) with morphisms \( a \to c, b \to c \)
- for any two morphisms \( a, b : x \to y \), there is a morphism \( c : y \to z \) that \( c \circ a = c \circ b \).

Def. (II.1.1.3). A morphism of category \( C \to D \) is called:

- full/faithful if for any \( A, B \in C \), \( f : \text{Hom}(X,Y) \to \text{Hom}(f(X),f(Y)) \) is surjective/injective.
- essentially surjective if any object of \( D \) is equivalent to some \( f(C) \).
- an equivalence if there is a functor \( g : D \to C \) that \( f \circ g \cong \text{Id}_D \) and \( g \circ f \cong \text{Id}_C \).

Prop. (II.1.1.4) (Category Equivalence). A Functor \( \mathcal{C} \to \mathcal{D} \) is an equivalence if and only if it’s fully faithful and essentially surjective.

Proof: There exist an object \( G(X) \in \mathcal{C} \) and an isomorphism \( \xi_X : FG(X) \to X \) for every \( X \in \mathcal{D} \). Because \( F \) is fully faithful, there exists a unique morphism \( G(f) : G(X) \to G(Y) \) such that \( F(G(f)) = \xi_Y^{-1} \circ f \circ \xi_X \) for every morphism \( f : X \to Y \) in \( \mathcal{D} \). Thus we obtain a functor \( G : \mathcal{C} \to \mathcal{D} \) as well as a natural isomorphism \( \xi : F \circ G \to \text{Id}_\mathcal{D} \). Moreover, the isomorphism \( \xi_{F(Z)} : FGF(Z) \to F(Z) \) decides an isomorphism \( \eta_Z : GF(Z) \to Z \) for every \( Z \in \mathcal{C} \). This yields a natural isomorphism \( \eta : G \circ F \to \text{Id}_\mathcal{C} \).

Def. (II.1.1.5) (Setoid). A setoid is the same as a category that has only the identity morphisms.

Def. (II.1.1.6) (Groupoid). A groupoid is a category that all morphisms are isomorphisms.

Prop. (II.1.1.7). An equivalence relation is just a groupoid that \( |\text{Mor}_C(x,y)| \leq 1 \) for any \( x, y \in \mathcal{C} \).
Prop. (II.1.1.8). A category is equivalent to a setoid iff it is an equivalence relation.

Proof: □

Def. (II.1.1.9). A subcategory of a category $\mathcal{B}$ is a category $\mathcal{A}$ that all the objects and morphisms of $\mathcal{A}$ are subsets of that of $\mathcal{B}$. $\mathcal{A}$ is called a full subcategory if $\text{Mor}_\mathcal{A}(x,y) = \text{Mor}_\mathcal{B}(x,y)$ for $x,y \in \mathcal{A}$. It is called a strictly full subcategory if it is full and any objects in $\mathcal{B}$ that is isomorphic to an object in $\mathcal{A}$ is in $\mathcal{A}$.

Def. (II.1.1.10) (Comma Topology). For a category $\mathcal{C}$ and an object $S$, the comma category $\mathcal{C}/S$ is defined to be the category of arrows $T \to S$ with the arrows being compatible arrows over $S$.

Def. (II.1.1.11) (Morphism Category). For a category $\mathcal{C}$, the category of arrows $\text{Mor}(\mathcal{C})$ is a category whose objects are arrows in $\mathcal{C}$ and a morphism $f \to g$ is a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{h} & B \\
\downarrow{f} & & \downarrow{g} \\
A' & \xleftarrow{k} & B'
\end{array}
\]

Def. (II.1.1.12) (Twisted Arrow Category). For a category $\mathcal{C}$, the category of twisted arrows $\text{TW}(\mathcal{C})$ is a category whose objects are arrows in $\mathcal{C}$ and a morphism $f \to g$ is a diagram

\[
\begin{array}{ccc}
A & \xleftarrow{h} & B \\
\downarrow{f} & & \downarrow{g} \\
A' & \xrightarrow{k} & B'
\end{array}
\]

Def. (II.1.1.13) (Final Object). An object $Z$ of a category is called final if $|\text{Hom}(X,Z)| = 1$ for any object $X$. It is called weakly final if $\text{Hom}(X,Z) \neq \emptyset$ for every object $X$.

Dually an object is called (weakly)initial if it is (weakly)initial as an object of $\mathcal{C}^{\text{op}}$.

Def. (II.1.1.14) (Epimorphisms and Monomorphisms). An epimorphism in a category is a morphism $X \to Y$ that the map $\text{Hom}(Y,Z) \to \text{Hom}(X,Z)$ induced by composition is injective. Dually, an monomorphism is an epimorphism in the dual category.

Def. (II.1.1.15) (Projective and Injective). A projective object $X$ in a category is an object that for any epimorphism (II.1.1.14) $Y \to Z$, $\text{Hom}(X,Y) \to \text{Hom}(X,Z)$ is surjective. Dually, an injective object $X$ is a projective object in the dual category.

Def. (II.1.1.16) (Retract). A morphism $f$ in a category $\mathcal{C}$ is called a retract of $g$ if there are morphisms $F,G : f \to g$ in $\text{Mor}(\mathcal{C})$ (II.1.1.11) that $G \circ F = \text{id}_f$.

Def. (II.1.1.17) (Equivalence relation). Two morphisms of functors $u_0,u_1 : F_1 \to F_0$ on a category is called an equivalence relation iff for any object $X$, the map

\[
F_1(X) \xrightarrow{(u_0,u_1)} F_0(X) \times F_0(X)
\]

is a bijection of $F_0(X)$ onto the graph of an equivalence relation on $F_1(X)$.

Two morphism between objects $X_1 \to X_0$ is called an equivalence relation iff their induced map $h_{X_1} \to h_{X_0}$ is an equivalence relation.
Def. (II.1.1.18) (Quotient Functor). Given an equivalence relation on functors (II.1.1.17) \( u_0, u_1 : F_1 \to F_0 \), a morphism of functors \( u : F_0 \to F \) is called the quotient functor if
- \( u \circ u_0 = u \circ u_1 \).
- The morphism \( (u_0, u_1) : F_1 \to F_0 \times_F F_0 \) is an isomorphism.
- For any functor \( T \), the map
  \[ \text{Hom}(F, T) \to \text{Hom}(F_0, T) \Rightarrow \text{Hom}(F_1, T) \]
  is exact.

Similarly, for an equivalence relation \( u_1, u_2 : X_0 \to X_1 \), a morphism \( X \to X_0 \) is called a quotient iff \( h^{X_0} \to h^X \) is a quotient functor.

Representable Functors

Prop. (II.1.1.19) (Representability Criterion). Let \( C \) be a complete category, \( F : C \to \text{Sets} \) be a functor. Assume that \( F \) commutes with limits, and the category \( I \) of pairs \((x, f)\) where \( x \in C, f \in F(x) \) has a cofinal family of objects indexed by a set \( I \), then \( F \) is representable, i.e. there is an object \( x \) that \( F(y) = \text{Mor}_C(x, y) \), functorial in \( y \).

Proof: Because \( C \) has small limits, let \( I' \) be the full subcategory of \( I \) generated by \((x_i, f_i)\), set \( x = \lim_{(x_i, f_i) \in I'} x_i \). As \( F \) commutes with limits, \( F(x) = \lim_{(x_i, f_i) \in I} F(x_i) \). Hence there is a universal element \( f \in F(x) \) that maps to \( f_i \) under \( F(x \to x_i) \). \( f \) induces a natural transformation \( \xi : \text{Mor}_C(x, -) \to F(-) \).

The assumption shows \( \xi \) is surjective. Now let \( x' \to x \) be the equalizer of all maps \( \varphi : x \to x \) that \( F(\varphi)f = f \), then there is a \( f' \in F(x') \) mapping to \( f \). then the transformation \( \xi' \) defined by \( f' \) is also surjective. Now we also want to show it is injective: if \( a, b \in \text{Mor}_C(x', y) \) maps to the same element, then we consider the equalizer \( e' : x'' \to x' \) of \( a, b \), then the assumption and the fact \( \xi \) commutes with equalizer shows there is a \( f'' \in F(x'') \) mapping to \( f' \).

By universality consider a morphism \( \psi : x \to x'' \) that \( F(\psi)f = f'' \), then \( e \circ e' \circ \psi \) is a morphism \( x \to x \) that fixes \( f \), thus by construction \( ee'\psi e = e \), so \( e'\psi e = id \), because \( e \) is a monomorphism. Then \( e' \) is an epimorphism, thus \( a = b \). □

Adjunctions

Def. (II.1.1.20). Two functors \( f : C \to D, g : D \to C \) are called left/right adjoint iff there is natural isomorphism of functors:
\[ C^{op} \times D \to \text{Set} : \text{Hom}(fX, Y) \cong \text{Hom}(X, gY) \]

Prop. (II.1.1.21) (Units and Counits). If \( f, g \) are adjoints, then there are natural maps \( u : X \to gfx, \) and \( v : fgy \to Y \), called unit/counit maps. They satisfies \( fX \to gfxX \to fX \) is id, and \( gY \to gfxgY \to fY \) is id.

Conversely, if there are natural morphisms \( u, v \) satisfying these two identities, then \( f, g \) are adjoint, by
\[ \text{Hom}(fX, Y) \to \text{Hom}(gfx, gY) \to \text{Hom}(X, gY) \to \text{Hom}(fx, fgy) \to \text{Hom}(fX, Y). \]

Proof: □
Prop. (II.1.1.22) (Units and Equivalences). Let \((F, G)\) be a pair of adjoint functors, then the following are equivalent:

- \(F, G\) are both fully faithful.
- the unit and counit are both natural isomorphisms.
- \(F, G\) defines an equivalence of categories.

Proof: □

Prop. (II.1.1.23) (Adjoints Preserves (Co)Limits). A right adjoint functor is left exact and it preserves injectives if its left adjoint is exact.

A left adjoint functor is right exact and it preserves projectives if its right adjoint is exact.

Prop. (II.1.1.24) (Adjoint Functor Theorem). Let \(G : \mathcal{C} \to \mathcal{D}\) be a functor, assume \(\mathcal{C}\) is complete and \(G\) commutes with them. Assume for every \(y \in \mathcal{D}\), the category of pairs \((x, f)\) where \(x \in \mathcal{C}\) and \(f \in \text{Mor}_\mathcal{D}(y, G(x))\) has a cofinal family of objects indexed by a set \(I\), then \(G\) has a left adjoint.

Similarly the dual statement holds.

Proof: The assumption shows that for any \(y \in \mathcal{D}\), the functor \(x \mapsto \text{Mor}_\mathcal{D}(y, G(x))\) satisfies the condition of (II.1.1.19), thus it is representable by an object denoted by \(F(y)\). By Yoneda lemma, \(F\) underlies a functor, and this functor is a left adjoint of \(G\). □

Prop. (II.1.1.25) (Examples of Adjoint Functors).
- The valuation at \(k\)-th coordinate is left adjoint to the functor \(k^*(A)(i) = \prod_{\text{Hom}_{i,k}} A\) and is exact. So \(k^*\) preserves injectives.
- The sheaf \(\Gamma\) functor is right adjoint to the constant sheaf functor over arbitrary site.
- The inclusion functor is right adjoint to the sheification functor over arbitrary site.
- The forgetful functor is right adjoint to the sheification functor, and sheification is exact, so it preserves injectives.
- The stalk functor is left adjoint to the skyscraper sheaf operator.

Limits and Colimits

Def. (II.1.1.26) ((Co)Complete Category). A category is called (co)complete if it has all small (co)limits.

Def. (II.1.1.27) (Inverse System). An inverse system in \(\mathcal{C}\) is a diagram \(Z^\text{op} \to \mathcal{C}\), where \(Z_+\) is the category of non-negative integers with a unique morphism \(n \to m\) iff \(n \leq m\).

Prop. (II.1.1.28) (Equivalent Inverse System). Two inverse systems \(\{A_n\}, \{B_n\}\) are called equivalent if there are two non-decreasing unbounded maps \(\alpha, \beta : Z_+ \to Z_+\) and maps \(\alpha : A_n \to B_{\alpha(n)}, \beta : B_n \to A_{\beta(n)}\) that is compatible with the transition maps and for any \(n\), there is a \(m\) large that \(A_m \to A_{\beta\alpha(m)} \to A_n\) coincide with the transition map \(A_m \to A_n\), and similar for \(B_n\). And similarly for colimits.

The limit for two inverse systems are the same, and similarly for colimits.

Prop. (II.1.1.29). Any presheaf on a small category is a colimit of representable sheaves \(h_X\).
(Consider all \(h_X \to \mathcal{F}\) and take colimit, prove it is isomorphism).
II.1. CATEGORIES

Prop. (II.1.1.30) (Filtered Colimit of Sets).

Prop. (II.1.1.31) (Cofiltered Limits of Sets). A cofiltered limit of nonempty sets is nonempty.

Proof: Cf. [Sta]086J □

Def. (II.1.1.32) (Mittag-Leffler). If \((A_i, \varphi_{ji})\) is a directed inverse system of sets over \(I\), then it is said to satisfy the **Mittag-Leffler condition** if \(\varphi_{ji}(A_j) \in A_i\) stabilizes. This is clearly true if \(\varphi_{ji}\) is surjective for any \(i, j\).

Prop. (II.1.1.33). If \((A_n)\) where \(n \in \mathbb{Z}\) is a Mittag-Leffler inverse system of nonempty sets, then \(\lim A_i\) is also nonempty.

Proof: Let \(A'_j = \bigcap_{j \geq i} \varphi_{ji} A_j\), then \((A'_j)\) is a filtered system that the transition maps are all surjective, and clearly \(\lim A_j = \lim A'_j\), so it is nonempty by (II.1.1.31). □

Prop. (II.1.1.34) (Products and Equalizers Implies Limits). If a category admits arbitrary products and equalizers, then it admits all limits. Dually a category that admits coproducts and coequalizers admits all colimits.

Proof: The limits over a category \(\mathcal{C}\) is an equalizer of products over the category of arrows in \(\mathcal{C}\). □

Fiber Product

Prop. (II.1.1.35). For a category \(\mathcal{C}\), the following are equivalent:

- It has arbitrary limits.
- it has arbitrary products and equalizer.
- it has arbitrary products and fibered products.

Proof: \(1 \to 2, 1 \to 3\) is trivial. \(3 \to 2\): The equalizer for \(f, g : X \to Y\) can be constructed as the base change of \(Y \to Y \otimes Y\) along \((f, g) : X \to Y \times Y\). \(2 \to 1\): for any diagram \(F : I \to \mathcal{C}\), the fibered pullback can be constructed as the equalizer of two morphisms:

\[
\begin{align*}
  s, t : \prod_{i \in \text{Ob}(I)} F(i) &\to \prod_{f : j \to k \in \text{Mor}(I)} F(k) \\
\end{align*}
\]

where \(\pi_{(f:j\to k)}s = \pi_k\), and \(\pi_{(f:j\to k)}t = (Ff)\pi_j\). □

Prop. (II.1.1.36) (Diagonal Base Change). The diagonal commutes with base change:

\[
\begin{array}{ccc}
  X \times_Y Z & \xrightarrow{\Delta} & (X \times_Y Z) \times_Z (X \times_Y Z) \\
  \downarrow & & \downarrow \\
  X & \xrightarrow{\Delta} & X \times_Y X
\end{array}
\]

Proof: □

Prop. (II.1.1.37). \((X \times_F Y) \times_S (Z \times_F W) = (X \times_S Z) \times_{E \times_S F} (Y \times_S W)\).

Proof: □
Prop. (II.1.1.38). For \( f : X \to T \) and \( g : Y \to T \) and \( h : T \to S \), \( X \times_T Y = T \times_{T \times S} (X \times_S Y) \).
In particular, \( X \times_T Y \to X \times_S Y \) is a base change of \( T \to T \times S \).

Proof: For any object \( U \),
\[
\text{Hom}(U, X \times_T Y) = \{ s : U \to X, t : U \to Y | f \circ s = g \circ t \},
\]
\[
\text{Hom}(U, T \times_{T \times S} (X \times_S Y)) = \{ r : U \to T, s : U \to X, t : U \to Y | h \circ f \circ s = h \circ g \circ t, r = g \circ t, r = f \circ s \},
\]
so they are functorially isomorphic. Then by Yoneda lemma we have the desired isomorphism. □

Prop. (II.1.1.39). The diagonal map \( X \to X \times_Y X \) is an isomorphism iff \( X \to Y \) is monomorphism. (Because this is equivalent to \( \text{pr}_1 = \text{pr}_2 \)).

Def. (II.1.1.40) (Mapping graph). Let \( f : X \to Y \) be a morphism in a category with fiber products and finite products, then the mapping graph \( \Gamma_f \) of \( f \) is defined to be the pullback
\[
\begin{array}{ccc}
\Gamma_f & \to & X \times Y \\
\downarrow & & \downarrow \\
Y & \to & Y \times Y
\end{array}
\]

it can be seen that \( \Gamma_f \) is isomorphic to \( X \).

Localization

Def. (II.1.1.41) (Localizing Categories). Let \( C \) be a category and \( S \) be a class of morphisms, then a functor \( F : C \to D \) is called the localizing category of \( C \) w.r.t \( S \) if it maps morphisms in \( S \) to isomorphisms, and any other functor with this property factors uniquely through \( F \).

Def. (II.1.1.42) (Localizing System). A class of morphisms \( S \) in a category is called left (resp. right) localizing if:
- \( S \) is closed under composition and has all the identities.
- for every \( s \in S \) and \( f \) with the same source, there is a \( t \in S \) and \( g, s.t. f \circ t = s \circ g \) (resp. \( t \circ f = g \circ s \)).
- the existence of a \( t \in S \) s.t. \( ft = gt \) implies (resp. is implied by) the existence of a \( s \in S \) s.t. \( sf = sg \).

It is called localizing if it is both left localizing and right localizing.

Def. (II.1.1.43) (Saturation).

Prop. (II.1.1.44). If \( S \) is localizing in a category \( C \), then the morphisms in \( C \) that is mapped to an isomorphism in \( S^{-1} C \) is exactly the saturation of \( S \).

Proof: Cf. [Sta]05Q9. □

Prop. (II.1.1.45) (Localization Category). If \( C \) is a category and \( S \) is a left localizing class, then the rule \( X \to X, (f : X \to Y) \mapsto \text{id}^{-1} \) \( f \) is a functor \( Q : C \to S^{-1} C \) that represents \( S^{-1} C \) as a localizing category of \( C \) w.r.t \( S \). And \( Q \) preserves finite colimits.

Group Objects

Def. (II.1.1.46) (Group Object). In a category $C$ with finite products and a final object $e$, a(n) (Abelian) group object is an object $G$ that $h_G$ is a functor from $C$ to $Grp(Ab)$. And a homomorphism of group objects is a natural transformation as a functor from $C$ to $Grp$.

This is in fact equivalent to a morphism $m_G : G \times G \to G$ and $i_G : G \to G$, $e_G : e \to G$ that satisfy the desired commuting diagrams.

Def. (II.1.1.47) (Group Action). A (left) action of a group object $G$ on an object $X$ is a map of presheaves $h_G \times h_X \to h_X$ that for any $U$, $h_G(U) \times h_X(U) \to h_X(U)$ is a group action. This is equivalent to a morphism $\mu : G \times X \to X$ that satisfies the desired commuting diagrams.

Prop. (II.1.1.48). Let $\mu : G \times X \to X$ be an action of a group object $G$ on an object $X$, there is a commutative diagram

$$
\begin{array}{ccc}
G \times X & \xrightarrow{(g,x) \mapsto (g, gx)} & G \times X \\
\downarrow{(g,x) \mapsto gx} & & \downarrow{\pi_2} \\
X & \xrightarrow{id} & X
\end{array}
$$

and the horizontal maps are isomorphisms.

Cor. (II.1.1.49). Considering the action of $G$ on itself, we get

$$
\begin{array}{ccc}
G \times G & \xrightarrow{(g,h) \mapsto (g, gh)} & G \times G \\
m & & \pi_2 \\
\downarrow & & \downarrow \\
G & \xrightarrow{id} & G
\end{array}
$$

and the horizontal maps are isomorphisms.

Prop. (II.1.1.50). A unital magma object $G$ in the category of groups is an Abelian group.

Proof: A unital magma object structure endows $G$ with maps $(\tilde{m}, \tilde{e})$, and $\tilde{m}(ab, cd) = \tilde{m}(a, c)\tilde{m}(b, d)$, because $\tilde{m}$ is a morphism of groups. So we can use Eckmann-Hilton argument(I.2.1.3) to show the category multiplication is the same as the group multiplication. So the commutativity of $\tilde{m}$ with the inverse $i$ implies that it is Abelian. □

Prop. (II.1.1.51). A right-lax monoidal functor(II.1.4.9) between Cartesian monoidal structures maps a unital magma object to a unital magma object.

Def. (II.1.1.52) (Categorical Quotient). Let $C$ be a category with finite products, $G$ be a group object in $C$, $G \times X \to X$ is a left action(II.1.1.47), then a morphism $X \to Y$ is called the categorical quotient of $X$ iff $Y$ is the coequalizer of $G \times X \xrightarrow{pr_2} X$. It is called the universal categorical quotient of $X$ iff its product with $S$ is the categorical quotient for each element $S \in C$, in the category $C/S$. 
2 Ends and Coends

Def. (II.1.2.1) (Dinatural Transformation). Given categories $\mathcal{C}, \mathcal{D}$ and functor $P, Q : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D}$, then a dinatural transformation $\alpha : P \to Q$ is a family of arrows $\alpha_C : P(C, C) \to Q(C, C)$ where $C \in \mathcal{C}$ that for any arrow $C \to C' \in \mathcal{C}$, there is a commutative diagram

\[
\begin{array}{ccc}
P(C', C) & \to & P(C, C) \\
\downarrow & & \downarrow \\
Q(C', C') & \to & Q(C, C')
\end{array}
\]

Def. (II.1.2.2) (Wedge and Cowedge). Let $P : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor, then a wedge for $P$ is a binatural functor $\Delta_D \to P$, where $D$ is an object of $\mathcal{C}$. Similarly a cowedge is a binatural functor $P \to \Delta_D$.

Def. (II.1.2.3) (End and Coend). For a functor $P : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathcal{D}$, the wedges and cowedges of $P$ form categories, and we define end of $P$ is just a terminal wedge, denoted by $\int_C F(C, C)$, and the coend of $P$ is the initial cowedge, denoted by $\int_C F(C, C)$.

Ends and coends are functorial w.r.t natural transformations.

Prop. (II.1.2.4) (Ends as Colimits). There is a morphism

\[ F \mapsto F : \text{Fun}(\mathcal{C}^{\text{op}} \times \mathcal{C}, \mathcal{D}) \to \text{Fun}(\text{Tw}(\mathcal{C}), \mathcal{D})(II\text{\textsuperscript{1.1.1.2})}\]

where we maps $f : C \to C'$ to $F(C, C')$. Then it can be checked that

\[
\int_C F(C, C) = \lim_{\text{Tw}(\mathcal{C})} F, \quad \int_C F(C, C) = \colim_{\text{Tw}(\mathcal{C})} F.
\]

Cor. (II.1.2.5). A functor that preserves (co)limits preserves (co)ends.

Cor. (II.1.2.6). For an object $D \in \mathcal{D}$, we have isomorphisms:

\[
\text{Hom}(\int_C F(C, C), D) \cong \int_C \text{Hom}(F(C, C), D),
\]

\[
\text{Hom}(D, \int_C F(C, C)) \cong \int_C \text{Hom}(D, F(C, C)).
\]

Prop. (II.1.2.7) (Fubini). Cf.[Coend Calculus, P20].

Prop. (II.1.2.8) (Natural Transformation as Ends). Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors, then the set of natural transformations is an coend.

\[
\text{Map}(F, G) \cong \int_C \text{Hom}_\mathcal{D}(FC, GC)
\]

Kan Extension

Cf.[All Concepts are Kan Extensions].
Def. (II.1.2.9) (Kan Extensions). Given functors \( F : C \to \mathcal{E} \) and \( K : C \to \mathcal{D} \), a left Kan extension of \( F \) along \( K \) is a functor \( \text{Lan}_K F : \mathcal{D} \to \mathcal{E} \) together with a natural transformation \( \eta : F \to \text{Lan}_K F \circ K \) that for any other pair \((G : \mathcal{D} \to \mathcal{E}, \gamma : F \to G \circ K)\), \( \gamma \) factors uniquely through \( \eta \).

Dually, a right Kan extension of \( F \) over \( K \) is equivalent to a left Kan extension of \( F^{op} \) over \( K^{op} \).

Prop. (II.1.2.10) ((Co)Limits as Kan Extensions). If \( \mathcal{D} \) is the final category 1, then the left Kan extension is just the colimit of the diagram defined by \( F \), and the right Kan extension is just the limit of the diagram defined by \( F \).

Prop. (II.1.2.11) (Yoneda Lemma). \( h_X : Y \mapsto \text{Hom}(Y, X) \) is a presheaf, and \( \text{Hom}(h_X, \mathcal{F}) \cong \mathcal{F}(X) \) for any presheaf \( \mathcal{F} \).

So \( X \to h_X \) is a fully faithful embedding from \( C \) to \( \hat{\mathcal{C}} = \text{Func}(\mathcal{C}^{op}, \text{Set}) \). In particular, if a \( X \to Y \) induces isomorphism \( \text{Hom}(W, X) \to \text{Hom}(W, Y) \) for every \( W \), then \( X \cong Y \).

So we can regard \( C \) as a fully faithful subcategory of \( \hat{\mathcal{C}} \).

Proof: The map \( \text{Hom}(h_X, \mathcal{F}) \to \mathcal{F}(X) \) maps a \( u \) to \( u(X)(\text{id}_X) \). And the inverse map is defined to be \( x \in \mathcal{F}(X) \mapsto (s \in \text{Hom}(Y, X) \mapsto s^\circ(x) \in \mathcal{F}(Y)) \in \text{Hom}(h_X, \mathcal{F}) \).

Cor. (II.1.2.12). A universal object for a presheaf \( \mathcal{F} \) is a pair \( (X, \zeta) \) that \( \zeta \in \mathcal{F}(X) \) with the property that for any \( U \) and a \( \sigma \in \mathcal{F}(U) \), there is a unique arrow \( U \to X \) that \( \mathcal{F}(\zeta) = \sigma \).

In fact, a universal object is equivalent to an isomorphism \( h_X \cong \mathcal{F} \), and this makes it easy to check whether a presheaf \( \mathcal{F} \) is representable.

Prop. (II.1.2.13) (Presheaves as Colimits of Presentable Presheaves). A presheaf of sets in \( C \), i.e. \( C^{op} \to \text{Set} \) is a colimit of presentable presheaves of \( C \). More precisely, there is an isomorphism
\[
\mathcal{F} \cong \lim_{h_X \to \mathcal{F}} h_X.
\]

From this we see that any morphism \( \hat{\mathcal{C}} \to D \) is determined by its restriction on \( C \).

Proof: For any presheaf \( \mathcal{G} \), there is a morphism \( \text{Hom}(\mathcal{F}, \mathcal{G}) \to \text{Hom}(\lim_{h_X \to \mathcal{F}} h_X, \mathcal{G}) \), i.e. a set of sections \( f_s \in \mathcal{G}(X) \) for every \( h_X \to \mathcal{F} \), that if \( t \circ u = s \), then \( u^\circ(f_t) = f_s \). Conversely, by Yoneda lemma, this just says that there is a morphism of presheaves \( \mathcal{F} \to \mathcal{G} : F(X) \to G(X) : s \mapsto f_s \).

Cor. (II.1.2.14) (Kan Extension). For a cocomplete category \( \mathcal{D} \), there is a natural bijection between functor \( \hat{\mathcal{C}} \to \mathcal{D} \) that commutes with colimits and functors \( \mathcal{C} \to \mathcal{D} \) by Yoneda embedding.

Proof: For this, we only have to notice the functor \( \mathcal{D} \to \hat{\mathcal{C}} : D \to \text{Hom}(FX, D) \) is right adjoint to \( F : \hat{\mathcal{C}} \to \mathcal{D} \) when \( F \) is defined by colimit as in (II.1.2.13).

Cor. (II.1.2.15). Any contravariant functor \( F : \hat{\mathcal{C}} \to \text{Set} \) that take colimits to limits, \( F \) is representable. (Just use \( G \) in the last proof, \( F \) is representable by \( G(\text{pt}) \)).

Def. (II.1.2.16) (I-Free Diagram). For a cocomplete category \( \mathcal{C} \) and a small category \( I \), let \( I^\delta \) be the subcategory of \( I \) that has the same objects but only the identity morphisms, then an \( I \)-diagram in \( \mathcal{C} \) is called \( I \)-free iff it is a left Kan extension of some diagram \( I^\delta \to \mathcal{C} \).
3 2-Categories

Remark (II.1.3.1) (2-Categories). 2-categories are defined as in (II.7.2.25).

Def. (II.1.3.2) ((2,1)-Categories). A (2,1)-category is a 2-category that all the 2-morphisms (corresponding to a 2-simplex) are isomorphisms.

Def. (II.1.3.3) (Final Objects). A final object in a (2,1)-category is an object $x$ that for any object $y$ there is a morphism $y \to x$, and any two morphisms $y \to x$ are isomorphic by a unique 2-morphism.

Lemma (II.1.3.4) (2-Commutative diagrams). Let $C$ be a 2-category, and $g : y \to z, f : x \to z$ are arrows in $C$, then the diagrams in $C$:

\[
\begin{array}{ccc}
w & \xrightarrow{a} & x \\
\downarrow b & & \downarrow f \\
y & \xrightarrow{g} & z
\end{array}
\]

gether with a 2-morphism from $gb$ to $fa$, naturally form a 2-category. A diagrams with invertible 2-morphisms are called 2-commutative diagram in $C$.

Def. (II.1.3.5) (2-Fibered Products). Let $C$ be a 2-category, and $g : y \to z, f : x \to z$ are arrows in $C$, a 2-fibered product of $f, g$ is a final object in the category of 2-commutative diagrams as defined in (II.1.3.4), and it is denoted by $x \times_z y$.

4 Monoidal Categories

Main references are [Tensor Categories, Etingof], [Lur09].

Def. (II.1.4.1) (Monoidal Category). A monoidal category is a category $C$ with a functor $\otimes : C \times C \to C$ and a unit object $1$ together with isomorphisms

\[ \eta_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C), \quad \alpha_A : A \otimes 1 \to A, \quad \beta_A : 1 \otimes A \to A \]

that satisfies the following commutative diagrams: Cf. [HTT, P791]. (Pentagon diagram)

Def. (II.1.4.2) (Dual and Opposed Category). Let $C$ be a monoidal category, then the opposed category $C^{\text{op}}$ is the same category as $C$ with the tensoring switched, which is also a monoidal category.

The dual category $C^{\text{op}}$ with the same tensoring is also a monoidal category.

Def. (II.1.4.3). A strict monoidal category is a monoidal category that the isomorphisms above are all identities.

Prop. (II.1.4.4) (Mac Lane Coherence). In a monoidal category, any two morphisms between two bracketings of a product $X_1 \otimes \ldots X_n$ constructed using the isomorphisms $\eta_{A,B,C}$ are equal. Also if some $X_i$ are 1, then we can also use $\alpha_A$ and $\beta_A$.

Prop. (II.1.4.5) (Cartesian Monoidal Category). For a category that has finite products, the product makes $C$ a monoidal category, called the Cartesian monoidal structure.

The Cartesian monoidal structure of the category of sets the default monoidal structure.
Prop. (II.1.4.6). For any category $\mathcal{C}$, the category $[\mathcal{C}, \mathcal{C}]$ of endofunctors has a natural monoidal structure.

Def. (II.1.4.7) (Closedness). A monoidal category $(\mathcal{C}, \otimes)$ is called left-closed if for every $A \in \mathcal{C}$, the functor $N \mapsto A \otimes N$ has a right adjoint $Y \mapsto A^Y$ (or denoted by $\text{Hom}(A, Y)$ when $\mathcal{C}$ is symmetric). Dually it is called right-closed if the dual structure $(\mathcal{C}, \otimes^{op})$ is left-closed. It is called closed if it is both left-closed and right-closed.

Prop. (II.1.4.8). If $\mathcal{C}$ is closed, then
\[
\lim_{\longrightarrow} A_i B \cong \lim_{\longrightarrow} (A_i B) \quad A(\lim_{\longrightarrow} B_i) \cong \lim_{\longleftarrow} (A B_i)
\]

Proof: Because $\mathcal{C}$ is closed, left and right tensor $A \otimes -$ and $- \otimes A$ are both left adjoints thus commutes with colimits. Now $B \mapsto A B$ is a right adjoint, thus it commutes with limits. And for any $\mathcal{C} \in \mathcal{C}$,
\[
\text{Hom}(\mathcal{C}, \lim_{\longrightarrow} A_i B) \cong \text{Hom}(\mathcal{C} \otimes (\lim_{\longrightarrow} A_i), B) \cong \text{Hom}(\lim_{\longrightarrow} C \otimes A_i, B)
\]
\[
\cong \lim_{\longleftarrow} \text{Hom}(C \otimes A_i, B) \cong \lim_{\longleftarrow} \text{Hom}(\mathcal{C}, A_i B) \cong \text{Hom}(\mathcal{C}, \lim_{\longleftarrow} (A_i B))
\]
so $\lim_{\longrightarrow} A_i B \cong \lim_{\longleftarrow} (A_i B)$ by Yoneda lemma. \[\square\]

Def. (II.1.4.9) (Monoidal Functor). Let $(\mathcal{C}, \otimes), (\mathcal{D}, \otimes)$ be monoidal categories, then a right-lax monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor $G$ together with morphisms $\gamma_{A,B} : G(A) \otimes G(B) \to G(A \otimes B)$ for any $A, B \in \mathcal{C}$ and a morphism $e : 1_D \to G(1_C)$ that satisfies the following compatibility conditions: (Hexagon diagram).

Moreover, it is called a monoidal functor if $\gamma_{A,B}$ and $e$ are all isomorphisms.

Def. (II.1.4.10) (Equivalence of Monoidal Categories). An equivalence of monoidal categories is a map that is an equivalence of categories as well as a monoidal functor.

Prop. (II.1.4.11) (Mac Lane Strictness). Any monoidal category is equivalent to a strict monoidal category.

Prop. (II.1.4.12) (Examples). For a monoidal category $(\mathcal{C}, \otimes)$, the functor $X \mapsto \text{Hom}(1, X)$ is a right-lax monoidal functor from $\mathcal{C} \to \text{Sets} (\Pi.1.4.5)$.

The morphism $\pi_0 : A \mapsto \pi_0(A)$ is a monoidal functor from the category of topological spaces $\text{Top}$ to the category $\text{Sets}$ because it commutes with products.

Proof: \[\square\]

Symmetric Monoidal Categories

Def. (II.1.4.13) (Symmetric Monoidal Category). A symmetric monoidal category is a monoidal category $(\mathcal{C}, \otimes)$ together with a natural transformation $\psi$ between $\otimes$ and $\otimes \circ \iota : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that $\psi^2 = \text{id}$ and the following commutative hexagon diagram is commutative:

A symmetric monoidal functor between symmetric monoidal categories are required to commutes with the braiding.
Rigid Monoidal Categories

Def. (II.1.4.14) (Duals). Let \( \mathcal{C} \) be a monoidal category and \( V \in \mathcal{C} \), a left dual of \( V \) is an element \( V^* \) together with maps

\[
ev_V : V^* \otimes V \to 1, \quad coev_V : 1 \to V \otimes V^*
\]

that satisfies

\[
V \to V \otimes V^* \otimes V, \quad V^* \to V^* \otimes V \otimes V^* \to V^*
\]

are identities.

Similarly we can define right dual. if \( Y \) is a left/right dual to \( X \), then \( X \) is right/left dual to \( Y \).

Def. (II.1.4.15) (Dual Morphisms). Let \( f : X \to Y \) be a morphism and \( X^*, Y^* \) the left duals of \( X \) and \( Y \), then there is a natural left dual map

\[
Y^* \to Y^* \otimes X \otimes X^* \to Y^* \otimes Y \otimes X^* \to X^*.
\]

Prop. (II.1.4.16). Let \( F : \mathcal{C} \to \mathcal{D} \) be a monoidal functor of monoidal categories, \( X \in \mathcal{C} \) is an object with left dual \( X^* \). Then \( F(X^*) \) is a left dual of \( F(X) \) with evaluation and coevaluation maps

\[
ev_{F(X)} : F(X^*) \otimes F(X) \to F(X^* \otimes X) \to F(1_C) \to 1_D,
\]

\[
coev_{F(X)} : 1_D \to F(1_C) \to F(X \otimes X^*) \to F(X) \otimes F(X^*).
\]

and similarly for right duals.

Prop. (II.1.4.17) (Adjointness). Let \( \mathcal{C} \) be a monoidal category and \( V \in \mathcal{C} \) with left dual \( V^* \), then there are natural adjunction maps

\[
\text{Hom}(U \otimes V, W) \cong \text{Hom}(U, W \otimes V^*), \quad \text{Hom}(V^* \otimes U, W) \cong \text{Hom}(U, V \otimes W).
\]

Cor. (II.1.4.18). In particular, we can use Yoneda lemma to show the left/right adjoints are unique if they exist.

Def. (II.1.4.19) (Rigid Tensor Categories). Let \( \mathcal{C} \) be a monoidal category, an element \( X \in \mathcal{C} \) is called rigid if it has left and right duals. A rigid monoidal category is a monoidal category that every object is rigid. In particular, by (II.1.4.17), a rigid monoidal category is closed.

Prop. (II.1.4.20). If \( \mathcal{C} \) is a rigid tensor category, then the functor

\[\text{Equivalence of } \mathcal{C}^{\text{op}} \text{ and } \mathcal{C}^{\text{opp}.}\]

Cor. (II.1.4.21). Any natural transformation of monoidal functors between rigid monoidal categories is an isomorphism.

Proof: Cf.[Milne, Tannakian Categories, P13].

Prop. (II.1.4.22) (Trace and Rank). Let \( \mathcal{C} \) be a rigid symmetric monoidal category, we can define a trace morphism

\[
\text{tr}_X : \text{End}(X) \to \text{End}(1) : f \mapsto 1 \to X \otimes X^* \xrightarrow{f \otimes \text{id}} X \otimes X^* \to X^* \otimes X \to 1
\]

And the dimension of \( X \) is defined to be \( \text{tr}_X(\text{id}_X) \in \text{End}(1) \).
Invertible Objects and Grothendieck Categories

Def. (II.1.4.23) (Invertible Objects). An invertible object in a monoidal category is a rigid object $L$ that the evaluation maps and coevaluation maps are all isomorphisms.?

Tensor Categories

Def. (II.1.4.24) (Tensor Categories). A tensor category over a field $k$ is an Artinian Abelian category over $k$ together with a rigid monoidal structure that is bilinear and satisfies $\text{End}(1) = k$.

A tensor functor is a functor between tensor categories that is additive and monoidal.

Prop. (II.1.4.25) ($\text{End}(1)$). If $C$ is a rigid additive tensor category, then $\text{End}(1)$ is a ring that acts on objects $X \in C$ via $X \cong 1 \otimes X$. The action of $\text{End}(1)$ commutes with $\text{End}(X)$, in particular, $\text{End}(1)$ is commutative and $C$ is $\text{End}(1)$-linear.

Prop. (II.1.4.26) (Abelian Rigid Tensor Category Exact). If $C$ is an rigid Abelian tensor category, then $\otimes$ commutes with inverse limits and direct limits in each variable.

Proof: It commutes with direct limits because it is left adjoint to the Hom functor, and it commutes with inverse limits by considering the opposite category (II.1.4.20). \hfill \square

Prop. (II.1.4.27). $\dim(X \otimes Y) = \dim X \cdot \dim Y$. If there is an exact sequence $0 \to X \to Z \to Y \to 0$, then $\dim(Z) = \dim(X) + \dim(Y)$.

Prop. (II.1.4.28) (Decompositions). Let $(C, \otimes)$ be a rigid Abelian tensor category and if $U$ is a subobject of $1$, then $1 = U \oplus U^\perp$ where $U^\perp = \text{Ker}(1 \to U^\vee)$. Consequently, $1$ is a simple object iff $\text{End}(1)$ is a field. And any rigid tensor category can be decomposed as rigid Abelian tensor categories $C_I$ with $\text{End}(1_I)$ being fields.

Proof: Let $V = \text{Coker}(U \to 1)$, by tensoring $0 \to U \to 1 \to V \to 0$ with $U \hookrightarrow 1$, we get exact sequences

$$
\begin{array}{cccccc}
0 & \longrightarrow & U & \longrightarrow & 1 & \longrightarrow & V & \longrightarrow & 0 \\
\uparrow & & \uparrow 0 & & \uparrow & & \uparrow & \\
0 & \longrightarrow & U \otimes U & \longrightarrow & U & \longrightarrow & V \otimes U & \longrightarrow & 0 \\
\end{array}
$$

thus $V \otimes U = 0$ and $U \otimes U = U$ via $1 \otimes 1 \cong 1$.

For the rest, Cf. [Tannakian Categories, Milne, P14]. \hfill \square

Def. (II.1.4.29) (Associative Algebra in a Symmetric Tensor Category).

Prop. (II.1.4.30) (Fiber Functors). If a symmetric tensor category admits an additive monoidal functor from $C \to \text{Vect}_k$, then it is unique up to isomorphisms. Such a functor is called a fiber functor.

Def. (II.1.4.31) (Tannakian Categories). A Tannakian category is a symmetric tensor category $C$ over $k$ which admits a fiber functor $F : C \to \text{Vect}_k$.

Prop. (II.1.4.32). An Tannakian category is equivalent to the symmetric tensor category $\text{Rep}(G)$ for an affine group scheme $G$, and $G$ is uniquely defined up to inner automorphisms.
5 Enriched Category

Def. (II.1.5.1) (Enriched Category). Given a monoidal category \((\mathcal{C}, \otimes)\), an \textit{enriched category} \(\mathcal{D}\) over \(\mathcal{C}\) consists of the following data:

- a collection of objects.
- For objects \(X, Y \in \mathcal{D}\), a mapping object \(\text{Map}_\mathcal{D}(X, Y) \subset \mathcal{C}\).
- For objects \(X, Y, Z \in \mathcal{D}\), a composite map \(\text{Map}(X, Y) \otimes \text{Map}(Y, Z) \to \text{Map}(X, Z)\) that is associative.
- For every \(X \in \mathcal{D}\), a morphism \(1 \mapsto \text{Map}(X, X)\) that satisfies the commutative diagrams of the identity morphism.

Def. (II.1.5.2) (Category of Enriched Categories). We can naturally define morphisms and natural transformations of categories enriched over a monoidal category \(\mathcal{C}\).

Prop. (II.1.5.3) (Transfer of Enriched Structure). Given a right-lax monoidal functor between monoidal categories \(F : \mathcal{C} \to \mathcal{C}'\) and a category \(\mathcal{D}\) enriched over \(\mathcal{C}\), we may obtain a category \(F(\mathcal{D})\) enriched over \(\mathcal{C}'\) by asserting \(\text{Map}_{F(\mathcal{D})}(X, Y) = F(\text{Map}(X, Y))\). It is an enriched category just by the definition of right-lax monoidal functors.

Prop. (II.1.5.4) (Underlying Category). For a category enriched \(\mathcal{D}\) over \(\mathcal{C}\), by (II.1.4.12), we can transfer the structure via \(\mathcal{C} \to \text{Sets} : X \mapsto \text{Hom}(1, X)\), and the resulting category is called the underlying category of \(\mathcal{D}\).

Prop. (II.1.5.5).

- A category enriched in the Cartesian monoidal structure of the category of sets is just a usual category.
- A right-closed monoidal category is enriched over itself if we define \(\text{Map}(X, Y) = Y^X\). (Check).

Def. (II.1.5.6) (Tensored Category). Let \(\mathcal{C}\) be a right-closed monoidal category and \(\mathcal{D}\) a category enriched over \(\mathcal{C}\), then \(\mathcal{D}\) is called \textit{tensored over} \(\mathcal{C}\) if for any \(C \in \mathcal{C}\) and \(X \in \mathcal{D}\), there is an isomorphism of functors

\[
\eta : \text{Map}_\mathcal{D}(X, -)^C \cong \text{Map}_\mathcal{D}(X \otimes C, -).
\]

for some element \(X \otimes C \in \mathcal{D}\).

In particular, this implies \(\text{Hom}_\mathcal{C}(C, \text{Map}(X, Y)) \cong \text{Hom}_\mathcal{D}(X \otimes C, Y)\), thus \(X \otimes C\) is determined up to a unique isomorphism, and the map \((X, C) \mapsto X \otimes C\) defines a functor \(\mathcal{D} \times \mathcal{C} \to \mathcal{D}\) that there are natural morphisms

\[
X \otimes (C \otimes D) \cong (X \otimes C) \otimes D.
\]

Dually, if \(\mathcal{C}\) is left-closed and there is an object \(C^X\) that there is an isomorphism of functors

\[
C \text{Map}(-, X) \cong \text{Map}(-, C^X)
\]

then \(\mathcal{D}\) is called \textit{cotensored} over \(\mathcal{C}\).

Proof:

\[
\square
\]

Prop. (II.1.5.7). If \(\mathcal{C}\) is a right-closed monoidal category, then it is naturally tensored over itself, as defined in (II.1.5.5).
6 Others

**Def. (II.1.6.1) (Compact Objects).** Fix a regular cardinal $\kappa$. Let $C$ be a category that admits small colimits, and $J$ be a $\kappa$-filtered poset and a diagram $\{Y_\alpha\}$ indexed by $J$, then for $X \in C$, there is a natural map

$$\lim \rightarrow \text{Hom}(X, Y_j) \to \text{Hom}(X, \lim \rightarrow Y_j).$$

$X$ is called $\kappa$-compact if this is an isomorphism for any $\kappa$-filtered diagram $J$. $X$ is called small if it is $\kappa$-compact for some small (XII.1.11.12) regular cardinal $\kappa$.

**Def. (II.1.6.2) (Presentable Category).** Fix a regular cardinal $\kappa$, a regular category is a category $C$ that satisfies:

- $C$ admits all small colimits.
- $C$ is generated by a small set $S$ of objects of $C$ under small colimits.
- Any object of $C$ is small.
- For $X, Y \in C$, $\text{Hom}(X, Y)$ is small.

**Remark (II.1.6.3).** HTT Chap5.5. has a $\infty$-generalization of this notion.

**Lifting Property and Small Object Argument**

**Def. (II.1.6.4) (lifting Properties).** Let $C$ be a category and $p : A \to B, q : X \to Y$ be morphisms, then $p$ is said to have left lifting property w.r.t $q$ and $q$ is said to have right lifting property w.r.t $p$ if given any diagram $\begin{array}{ccc} A & \to & X \\ \downarrow & & \downarrow \\ B & \to & Y \end{array}$, there is a dotted arrow completing the diagram.

For a set $A$ of morphisms of $C$, let $l(A)$ denote the morphisms that have left lifting property w.r.t $A$ and $r(A)$ the morphisms that have right lifting property w.r.t $A$.

**Def. (II.1.6.5) (Weakly Saturated Class).** Let $C$ be a category with all small colimits, then a class of morphisms of $C$ is called weakly saturated if it satisfies:

- Closed under pushout.
- Closed under transfinite composition: Let $\alpha$ be an ordinal and $\{D_\beta\}_{\beta < \alpha}$ be a system of objects in $C_{C/}$ indexed by $\alpha$. For $\beta < \alpha$, let $D_{<\beta}$ be the colimit of system $\{D_\gamma\}_{\gamma < \beta}$ in $C_{C/}$, then if each $D_{<\beta} \to D_\beta$ is in $S$, then $C \to D_{<\alpha}$ is in $S$.
- Closed under Retraction: In the category of morphisms of $C$, if there is a morphism $F : f \to g, G : g \to f$ that $G \circ F = \text{id}$, and $g \in S$, then $f \in S$.

**Cor. (II.1.6.6).** The second condition implies all isomorphisms are in $S$, and $S$ is closed under composition.

**Prop. (II.1.6.7) (Lifting and Retraction).** For a diagram $\begin{array}{ccc} X & \to & X \\ \downarrow & & \downarrow \\ Z & \to & Y \end{array}$ represents $p$ as a retraction of $u$, as the diagram $\begin{array}{ccc} X & \to & Z & \to & X \\ \downarrow & & \downarrow & \uparrow & \downarrow \\ Y & \to & Y & \to & Y \end{array}$ shows.
Dually a diagram \[ \begin{array}{ccl} & X & \xrightarrow{i} & Y \\ q \downarrow & \searrow & \nearrow \\ Z & \rightarrow & Z \end{array} \]
represents \( q \) as a retraction of \( i \).

Prop. (II.1.6.8) (Small Object Argument). Let \( \mathcal{C} \) be a presentable category and \( A_0 = \{ \varphi_i : C_i \rightarrow D_i \} \) be a small collection of morphisms in \( \mathcal{C} \), then there is a morphism \( T : \mathcal{C}^{[1]} \rightarrow \mathcal{C}^{[2]} \) taking morphisms \( f : X \rightarrow Z \) in \( \mathcal{C} \) to the diagram
\[ \begin{array}{ccc} & Y \\ f' \rightarrow & \searrow & \nearrow \\ X & \xrightarrow{f} & Z \end{array} \]
That \( f' \) belongs to the weakly saturated class generated by \( A_0 \) and \( f'' \in r(A_0) \).
Moreover, if \( \kappa \) is a regular cardinal that each \( C_i, D_i \) is \( \kappa \)-compact, then \( T \) commutes with \( \kappa \)-filtered colimits.

Proof: Cf.[HTT, P788]. \( \square \)

Lemma (II.1.6.9). \( l(A) \) is weakly saturated for any set of morphisms \( A \) (Clear).

Cor. (II.1.6.10) (Generated Weakly Saturated Class). For any presentable category \( \mathcal{C} \) and \( A \) a set of morphisms in \( \mathcal{C} \), \( l(r(A)) \) is the smallest weakly saturated class of morphisms containing \( A \).

Proof: One direction of inclusion is by(II.1.6.9), for the other, if \( f : X \rightarrow Z \in l(r(A)) \), then there is a factorization \( X \xrightarrow{f'} Y \xrightarrow{f''} Z \), where \( f' \in A \) and \( f'' \in r(A) \), thus \( f \in l(f'') \), thus \( f \) is retraction of \( f' \):
\[ \begin{array}{ccc} & Y \\ f' \rightarrow & \searrow & \nearrow \\ X & \rightarrow & X \\ \downarrow & \downarrow & \downarrow \\ Z & \rightarrow & Z \end{array} \]
Thus \( f \in A \). \( \square \)

Trees
Cf.[HTT, Appendix]

7 Fibered Categories

Categories of Categories

Def. (II.1.7.1) (2-Category of Categories over Categories). There is a 2-category of categories over \( \mathcal{C} \), where the 1-morphisms are morphisms of categories over \( \mathcal{C} \) and the 2-morphisms are base-preserving natural transformations.

Two categories over \( \mathcal{C} \) are called equivalent if they are equivalent in this 2-category.

Prop. (II.1.7.2) (2-Fibered Products in the Categories of Categories). 2-fibered products exists in the categories of categories.
II.1. CATEGORIES

Proof: Let \( F : \mathcal{A} \to \mathcal{C}, G : \mathcal{B} \to \mathcal{C} \) be functors, then we can define a category \( \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \) as follows:

- Objects are the triples \((A, B, f)\) where \( A \in \mathcal{A}, B \in \mathcal{B} \) and \( f : F(A) \to G(B) \) is an isomorphism in \( \mathcal{C} \).
- Morphisms from \((A, B, f)\) to \((A', B', f')\) are pairs \((a, b)\) where \( a : A \to A', b : B \to B' \) s.t. the diagram

\[
\begin{array}{ccc}
F(A) & \xrightarrow{f} & G(B) \\
\downarrow^{F(a)} & & \downarrow^{G(b)} \\
F(A') & \xrightarrow{f'} & G(B')
\end{array}
\]

is commutative.

\( \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \) is a category both over \( \mathcal{A} \) and over \( \mathcal{B} \), and it fits into a 2-fiber products diagram

\[
\begin{array}{ccc}
\mathcal{A} \times_{\mathcal{C}} \mathcal{B} & \rightarrow & \mathcal{B} \\
\downarrow & & \downarrow \\
\mathcal{A} & \rightarrow & \mathcal{C}
\end{array}
\]

where the invertible 2-morphism is given by \( \psi_{(A,B,f)} = f : F(A) \to G(B) \).

The verification that this defines a final object in the 2-category of 2-commutative diagrams is in [Sta]02X9. \( \square \)

Cor. (II.1.7.3). The 2-fibered product \( \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \) is a groupoid iff \( \mathcal{A}, \mathcal{B} \) are all groupoids.

Prop. (II.1.7.4). Let \( \mathcal{A} \to \mathcal{C}, \mathcal{B} \to \mathcal{C}, \mathcal{C} \to \mathcal{D} \) be functors between categories, then there is a 2-fiber product diagram

\[
\begin{array}{ccc}
\mathcal{A} \times_{\mathcal{C}} \mathcal{B} & \rightarrow & \mathcal{A} \times_{\mathcal{D}} \mathcal{B} \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{\Delta_{\mathcal{C}/\mathcal{D}}} & \mathcal{C} \times_{\mathcal{D}} \mathcal{C}
\end{array}
\]

Proof: \( \square \)

Prop. (II.1.7.5) (2-Fibered Products of Categories). The (2,1)-category of categories over \( \mathcal{C} \) has 2-fiber products\( (\mathcal{X}, \mathcal{Y}) \). More explicitly, suppose \( F : \mathcal{X} \to \mathcal{S}, Y \to \mathcal{S} \) be morphisms of categories over \( \mathcal{C} \), \( \mathcal{X} \times_{\mathcal{C}} \mathcal{Y} \) is given as follows:

- Objects of \( \mathcal{X} \times_{\mathcal{C}} \mathcal{Y} \) are quadruples \((U, x, y, f)\), where \( U \in \mathcal{C}, x \in \mathcal{X}_U, y \in \mathcal{Y}_U \), and \( f : F(x) \to G(y) \) is an isomorphism in \( \mathcal{S}_U \).
- A morphism \((U, x, y, f) \to (U', x', y', f')\) is given by a pair \((a, b)\), where \( a : x \to x' \) is a morphism in \( \mathcal{X} \) and \( y \to y' \) is a morphism in \( \mathcal{Y} \) that
  - \( a, b \) induce the same morphism \( U \to U' \).
  - the diagram

\[
\begin{array}{ccc}
F(x) & \xrightarrow{f} & G(y) \\
\downarrow^{F(a)} & & \downarrow^{G(b)} \\
F(x') & \xrightarrow{f'} & G(y')
\end{array}
\]

is an isomorphism.
$\mathcal{X} \times_S \mathcal{Y}$ is endowed with morphisms to $\mathcal{X}$ and $\mathcal{Y}$ over $\mathcal{C}$ that the invertible 2-morphism giving the 2-commutativity is $\psi_{(U,x,y,f)} : F(x) \to G(y)$.

The verification of the universal properties are similar to that of (II.1.7.2).

**Cor. (II.1.7.6).** There is an identification of fiber categories:

$$(\mathcal{X} \times_S \mathcal{Y})_U = \mathcal{X}_U \times_S \mathcal{Y}_U.$$

**Fibered Categories**

**Def. (II.1.7.7) ((Co)Cartesian Arrows).** Let $p : \mathcal{F} \to \mathcal{C}$ be a morphism, then a **Cartesian arrow** is an arrow $\varphi : C' \to C$ in $\mathcal{F}$ that for any object $C'' \in \mathcal{F}$ the map

$$\text{Hom}(C'', C') \to \text{Hom}(C'', C) \times_{\text{Hom}(p(C''), p(C))} \text{Hom}(p(C''), p(C'))$$

is a bijection. A **coCartesian arrow** is an arrow that corresponds to a Cartesian diagram in $\mathcal{F}^{\text{op}} \to \mathcal{C}^{\text{op}}$.

**Prop. (II.1.7.8).**

- If $f$ is Cartesian, then $f \circ g$ is Cartesian iff $g$ is Cartesian.
- An arrow in $\mathcal{F}$ whose image is an isomorphism is Cartesian iff it is itself an isomorphism.

**Proof:** Easy.

**Def. (II.1.7.9) (Quasi-Fibrantion).** A functor $F : \mathcal{C} \to \mathcal{D}$ is called a **quasi-fibration** if for any $X \in \mathcal{C}$ and an isomorphism $f : F(X) \cong Y$, there is an isomorphism $\overline{f} : X \to Y$ mapping to $f$.

**Def. (II.1.7.10) (2-Category of Fibered Categories).** A **fibered category** over $\mathcal{C}$ is a category over $\mathcal{C}$ $p : \mathcal{F} \to \mathcal{C}$ that for any $\eta \in \mathcal{F}$ and an arrow $f : U \to p(\eta)$, there is a Cartesian arrow $\xi \to \eta$ in $\mathcal{F}$ lifting $f$. A **cofibered category** over $\mathcal{C}$ is a category $p : \mathcal{F} \to \mathcal{C}$ that the dual category $p^{\text{op}} : \mathcal{F}^{\text{op}} \to \mathcal{C}^{\text{op}}$ is a fibered category.

A morphism of fibered categories over $\mathcal{C}$ is a morphism of categories over $\mathcal{C}$ that maps Cartesian morphisms to Cartesian morphisms. A 2-morphism of fibered categories is the same as a 2-morphism of categories over categories.

**Lemma (II.1.7.11).** Let $F : \mathcal{F} \to \mathcal{G}$ be a morphism of fibered categories over $\mathcal{C}$ that $\mathcal{F}(U) \to \mathcal{G}(U)$ are fully faithful for any $U \in \mathcal{C}$, then $F$ is fully faithful.

**Proof:** To show $F$ is fully faithful, it suffices to show for objects $X, Y$ lying over $U, V$, $F$ induces a bijection of morphisms from $x$ to $y$ lying over a fixed $f : U \to V$. Choose a Cartesian morphism $f^*y \to y$ in $\mathcal{S}_1$ lying over $f$, then this induces a bijection between morphisms from $x$ to $y$ lying over $f$ and $\text{Hom}_{\mathcal{S}_1(U)}(x, f^*y)$. Similarly, because $F$ preserves Cartesian morphisms, we get a bijection between morphisms $F(x) \to F(y)$ lying over $f$ and $\text{Hom}_{\mathcal{S}_1(U)}(F(x), F(f^*y))$. Then the desired bijection follows from the hypothesis.

**Prop. (II.1.7.12) (Equivalence of Fibered Categories).** Let $F : \mathcal{F} \to \mathcal{G}$ be a morphism of fibered categories. Then $F$ is an equivalence iff the restriction $F_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an equivalence of categories for any object $U$ of $\mathcal{C}$.
Proof: One direction is trivial, for the other, the proof is similar to the fact essentially surjective+fully faithful implies equivalence.

Because \( F(U) \) are equivalences, for any object \( \xi \) of \( G \) over \( U \), pick an object \( G\xi \) in \( F(U) \) together with an isomorphism \( \alpha_\xi : \xi \cong F(G\xi) \). And for any morphism \( \xi \to \eta \), by (II.1.7.11), there is a unique arrow \( G\varphi : G\xi \to G\eta \) that \( F(G\varphi) = \alpha_\eta \circ \varphi \circ \alpha_\xi^{-1} \).

Thus clearly there is a 2-isomorphism \( \text{id}_\xi \cong F \circ G \). It remains to construct an 2-isomorphism \( \text{id}_\xi \cong G \circ F \): For any object \( \xi' \) over \( U \), since \( F(U) \) is fully faithful, there is a unique isomorphism \( \beta_\xi : \xi' \cong G \circ F(\xi) \) in \( F(U) \) that \( F\beta_\xi = \alpha_{F\xi'} \). Then this is easily checked to be an 2-isomorphism \( \beta : \text{id}_\xi \cong F \circ G \).

Prop. (II.1.7.13) (2-Fiber Products of Fibered Categories). 2-fiber products exists in the category of fibered categories, and it coincides with that defined in (II.1.7.5).

Proof: it suffices to show for fibered categories \( X, Y \) over \( C \), \( X \times_C Y \) is also fibered over \( C \). Let \((x, y, \varphi)\) be an object of \( X \times_C Y \) mapping to \( U \in C \) and \( f : V \to U \) is a morphism in \( C \), choose Cartesian morphisms \( a : f^*x \to x \) \( b : f^*y \to y \) lying over \( f \), then \( F(a) \) and \( G(b) \) are Cartesian. Since \( \varphi : F(x) \to G(y) \) is an isomorphism, by the property of Cartesian morphisms, there exists a unique isomorphism \( f^*\varphi : F(f^*x) \to G(f^*y) \in S_V \) that \( G(b) \circ f^*\varphi = \varphi \circ F(a) \). In other words, \((F(a), F(b)) : (V, f^*x, f^*y, f^*\varphi) \to (U, x, y, \varphi)\) is a morphism in \( X \times_S Y \).

The verification that this morphism is Cartesian is omitted?.

Lemma (II.1.7.14). Let \( S \to C \) be a fibered category that factors through \( C/U \) where \( U \in C \), then \( S \to C/U \) is also a fibered category.

Proof: Cf.[[Sta]02XR].

Def. (II.1.7.15) (G-Equivariant Object). Let \( G : \mathcal{C}^{op} \to \text{Grp} \) be a group functor and \( p_\mathcal{F} : \mathcal{F} \to C \) a fibered category, and \( X \) an object of \( \mathcal{C} \) with an action of \( G \). A \textbf{G-equivariant object} of \( \mathcal{F}(X) \) is an object \( \rho \) of \( \mathcal{F}(X) \) that there is an action of \( G \circ p_\mathcal{F} \) on \( \rho \) and for any object \( U \) and \( \xi \in \mathcal{F}(U) \), the function \( p_\mathcal{F} : \text{Hom}_\mathcal{F}(\xi, \rho) \to \text{Hom}_C(U, X) \) is \( (G(U)) \)-equivariant.

The category \( \mathcal{F}^G(X) \) of \( G \)-equivariant objects of \( \mathcal{F}(X) \) consisting of \( G \circ p_\mathcal{F} \)-equivariant morphism of \( G \)-equivariant objects.

Prop. (II.1.7.16). Let \( \pi_2 \) is the projection \( G \otimes X \to X \), \( \rho \) be an object of \( \mathcal{F}(X) \), then a \( G \)-equivariant structure on \( \rho \) is the same as a Cartesian arrow \( \beta : \pi_2\rho \to \rho \) that \( p_\mathcal{F}\beta = \alpha \), and satisfies the desired commutative diagram corresponding to \((gh)x = g(h(x))\). And a morphism of \( G \)-equivariant objects just corresponds to a morphism of pairs \((\rho, \beta)\).

Proof: Cf.[Vistoli, P68].

Def. (II.1.7.17) (Presheaf of Arrows). Let \( \mathcal{F} \) be a fibered category over \( C \), and \( \xi, \eta \in \mathcal{F}(S) \), then we can define a quasi-functor \( \text{Hom}_S(\xi, \eta) \to (\mathcal{C}/S) \), where

\[
\text{Hom}_S(\xi, \eta)(U/S) = \{ (\xi_1 \to \xi, \eta_1 \to \eta, \varphi) \}
\]

where \( \xi_1 \to \xi, \eta_1 \to \eta \) are Cartesian arrows over \( U \to S \), and \( \varphi : \xi_1 \to \eta_1 \in \mathcal{F}(U) \). The arrows in \( \text{Hom}_S(\xi, \eta) \) are uniquely defined by the property of Cartesian arrows. Then it is a quasi-functor, by (II.1.7.27).

Then it is equivalence to a presheaf \( \text{Hom}_S(\xi, \eta) \), by (II.1.7.28). Equivalently, this presheaf can be defined by designating a choice of Cartesian arrows.
Prop. (II.1.7.18) (Splitting a Fibered Category). Let $F \to C$ be a fibered category, then there exists a canonically defined split fibered category $\tilde{F} \to C$ with a canonical equivalence of fibered categories $\tilde{F} \to F$ over $C$.

Proof: There is a functor $C^\text{op} \to \text{Cat}: U \mapsto \text{Hom}(h_U, F)$, with corresponds to a split fibered category $\tilde{F}$. There is an obvious morphism $\tilde{F} \to F$, sending an object $\varphi: h_U \to F$ to $\varphi(\text{id}_U) \in F(U)$. And for any $f: U \to V \in C$ and a $\varphi: h_U \to F$, we send $f^*\varphi \to \varphi \in \tilde{F}$ to $\varphi(f: V/U \to U/U) \in \tilde{F}$, Then we get a canonical map of fibered categories $\tilde{F} \to F$ over $C$. It is an equivalence of categories by (II.1.7.12) and (II.1.7.30). □

Categories Fibered in Groupoids, Sets and Equivalence Relations

Def. (II.1.7.19) (Category Fibered in Groupoids). A category (co)fibered in groupoids sets/equivalence relations over $C$ is a category $F$ (co)fibered over $C$ that $F(U)$ is a groupoid/set/equivalence relations of $F$ for any $U \in C$. Also we call a category fibered in equivalence relations over $C$ a quasi-functor.

Prop. (II.1.7.20) (Characterization of Category Fibered in Groupoids). Let $F$ be a category over $C$, then $F$ is fibered over groupoids over $C$ iff

- Every morphism in $F$ is Cartesian.
- Given any $\eta \in F, U \in C$ and a morphism $f: U \to p_F(\eta)$, there is an arrow $\varphi: \xi \to \eta \in F$ mapping to $f$.

And dually for cofibered categories.

Proof: If these two holds, then $F$ is clearly fibered over $C$, and for any arrow $f: \xi \to \eta$ in $F(U)$, there is a morphism $g: \eta \to \xi$ that is right inverse to $f$, then clearly it is the inverse.

Conversely, if $F$ is fibered over $C$, it suffices to check: for any arrow $f$, the image in $C$ can be lifted to a Cartesian diagram in $F$, and differs $f$ by an isomorphism, thus $f$ is also Cartesian by (II.1.7.8). □

Prop. (II.1.7.21). if $A$ is a category fibered in groupoids over $B$ and $B$ is a category fibered in groupoids over $C$, then $A$ is a category fibered in groupoids over $C$.

Proof: Cf.[Sta]09WW. □

Prop. (II.1.7.22) (Associated Category Fibered in Groupoids). Let $F \to C$ be a fibered category, then the associated category fibered in groupoids $F_{\text{cart}}$ is the category obtained by deleting all the non-Cartesian arrows. Then $F_{\text{cart}}$ is a category fibered in groupoids over $C$.

Proof: Firstly $F_{\text{cart}}$ is a category by (II.1.7.8), and it is a category fibered in groupoids by (II.1.7.20). □

Prop. (II.1.7.23) (2-Fibered Products of Categories Fibered in Groupoids). The 2-fibered products of categories fibered in groupoids over $C$ is a category fibered in groupoids over $C$.

Proof: The 2-fibered products exist by (II.1.7.13), and using (II.1.7.6), it is fibered in groupoids because the 2-fibered products of groupoids is a groupoid by (II.1.7.3). □

Prop. (II.1.7.24) (Characterization of Categories Fibered in Sets). Let $F$ be a category over $C$, then $F$ is fibered in sets over $C$ iff for any object $\eta$ of $F$ and an arrow $f: U \to p_F\eta \in C$, there is a unique arrow $\xi \to \eta$ mapping to $f$. 
Proof: Let $\mathcal{F}$ be a category fibered in sets, then pick a Cartesian arrow $\bar{f}$ over $f$, then any other lifting factors through this lifting by the property of Cartesian, then it is identity, because $\mathcal{F}(U)$ is a setoid.

Conversely, if the hypothesis holds, then clearly $\mathcal{F}(U)$ is a set, and the fibered category condition holds, because of the uniqueness. □

Cor. (II.1.7.25) (Presheaves and Categories Fibered in Sets). Let $\mathcal{C}$ be a category, then a category fibered in sets over $\mathcal{C}$ is the same thing as a presheaf on $\mathcal{C}$.

Cor. (II.1.7.26). In particular, for any object $X \in \mathcal{C}$, the presheaf $h_X$ determines a category fibered in sets, which is just the comma category $\mathcal{C}/X \to \mathcal{C}$.

Prop. (II.1.7.27) (Characterization of Quasi-Functors). A category $\mathcal{F}$ over $\mathcal{C}$ is a quasi-functor iff:

- Given any object $\eta \in \mathcal{F}$ and an arrow $f : U \to p_{\mathcal{F}}\eta$, there is a lifting $\xi \to \eta$ mapping to $f$. And given any two such extensions, there is a morphism $\xi' \to \xi$ commuting them.
- Given any two objects $\xi, \eta \in \mathcal{F}$ and an arrow $f : p_{\mathcal{F}}\xi \to p_{\mathcal{F}}\eta$, there is at most one arrow $\bar{f} : \xi \to \eta$ lifting $f$.

Proof:

Prop. (II.1.7.28). A fibered category over $\mathcal{C}$ is a quasi-functor iff it is equivalent to a presheaf.

Proof: If it is equivalent to a functor, then $\mathcal{F}(U)$ is equivalent to a setoid, thus it is an equivalent relation, by (II.1.1.8). Conversely, if $\mathcal{F}$ is a quasi-functor, then it is fibered in groupoids, thus by (II.1.7.20) every morphism is Cartesian, and if we denote $\Phi(U)$ the equivalence classes of $\mathcal{F}(U)$, then a morphism $U \to V$ will induce a morphism $\Phi(V) \to \Phi(V)$ by the property of Cartesian arrows. Then clearly this defines a presheaf that is equivalent to $\mathcal{F}$. □

Representability

Def. (II.1.7.29) (Representable Fibered Category). A fibered category is called representable if it is equivalent to the fibered category $\mathcal{C}/X$ defined in (II.1.7.26).

Prop. (II.1.7.30) (2-Categorical Yoneda Lemma). Let $\mathcal{F}$ be a fibered category over $\mathcal{C}$ and $X \in \mathcal{C}$, there is an equivalence of categories:

$$\text{Hom}_\mathcal{C}((\mathcal{C}/X), \mathcal{F}) \cong \mathcal{F}(X) : \varphi \mapsto \varphi(\text{id}_X)$$

Proof: To show this functor is essentially surjective, choose a choice of pullbacks of $\mathcal{F}$, for any $\xi \in \mathcal{F}(X)$, we define a $\xi : \mathcal{C}/X \to \mathcal{F}$ that maps a $\varphi : U \to X$ to $\varphi^*\xi$, and to any morphism in $\mathcal{C}/X$ an arrow in $\mathcal{F}$ induced by Cartesian property.

To show it is fully faithful, notice a natural transformation of $\varphi, \psi \in \text{Hom}_\mathcal{C}((\mathcal{C}/X), \mathcal{F})$ is determined by their value on $\text{id}_X$, and any map $\varphi(\text{id}_X) \to \psi(\text{id}_X)$ induces a natural transformation, by Cartesian properties. □

Cor. (II.1.7.31) (Characterization of Representabilities). Translating from the 2-Yoneda lemma, we see that a fibered category $\mathcal{F}$ is representable by $X \in \mathcal{C}$ if $\mathcal{F}$ is fibered in groupoids, and there is an object $\xi \in \mathcal{F}(X)$ that for any object $\rho \in \mathcal{F}$, there is a unique arrow $\rho \to \xi$. 

Def. (II.1.7.32) (Representable 1-Morphisms). Let $\mathcal{C}$ be a category and $F : \mathcal{X} \to \mathcal{Y}$ be a morphism of categories fibered over $\mathcal{C}$, then $F$ is called representable if for any $U \in \mathcal{C}$ and a morphism $C/U \to \mathcal{Y}$, the fibered category $(C/U) \times_{\mathcal{Y}} \mathcal{X} \to C/U$ is representable (II.1.7.29) (Notice it is a fibered category by (II.1.7.13) and (II.1.7.14)).

Prop. (II.1.7.33) (Diagonal and Representability). Let $S$ be a category fibered in groupoids over $\mathcal{C}$. Assume $\mathcal{C}$ has fibered product, then the following are equivalent:

- $\Delta_S : S \to S \times_{\mathcal{C}} S$ is representable.
- For every $U \in \mathcal{C}$, any $G : C/U \to S$ is representable.

Proof: Cf. [Sta]02YA.

The key to this proposition is the fibered product diagram (II.1.7.13) (II.1.1.36)

\[
\begin{array}{ccc}
X \times_S Y & \longrightarrow & F \\
\downarrow & & \downarrow \Delta \\
X \times_S Y & \xrightarrow{f \times g} & F \times_S F
\end{array}
\]

which still holds in the 2-commutative sense.

So if $\Delta_F$ is schematic, then $X \times_F Y$ is a scheme, so $X \to F$ is schematic, for any scheme $X$. Conversely, consider the fibered products

\[
\begin{array}{ccc}
\mathcal{F} \times_{\mathcal{F} \times S} \mathcal{F} X & \longrightarrow & \mathcal{F} \\
\downarrow & & \downarrow \Delta \\
X & \xrightarrow{\Delta} & X \times_S X \xrightarrow{f \times g} \mathcal{F} \times_S \mathcal{F}
\end{array}
\]

induced by $h = f \times g : X \to \mathcal{F} \times_S \mathcal{F}$. So in order to prove $\Delta_{\mathcal{F}}$ is schematic, it suffices to prove $\mathcal{F} \times_{\mathcal{F} \times S} \mathcal{F} X$ is a scheme, and for this, it suffices to prove $X \times_{\mathcal{F}} X$ is a scheme. But $X \times_{\mathcal{F}} X \to X$ is a pullback of $X \to \mathcal{F}$, so it is a scheme. \qed
II.2 Categorical Logic

Main references are [Categorical Logic Notes, Jacob Lurie], [Coend Calculus, Fosco Loregian], [Harder-Narasimhan Filtrations, Huayi Chen], [Harder-Narasimhan Theory, Jonathan Pottharst], [Coend Calculus].

1 Monads and Categories

Def. (II.2.1.1) (Monad). Let \( C \) be a category, a monad on \( C \) is an endofunctor \( C \to C \) together with two natural morphisms:
- (multiplication) \( \mu : T \circ T \to T \).
- (unit) \( \text{id}_C \to T \).
that satisfies associativity and unit diagrams.

Def. (II.2.1.2) (Algebras over Monads). An algebra over a monad \( T \) is an object \( X \) together with a morphism \( \alpha : TX \to X \) that satisfies the diagrams for an algebra.

2 Filtrations of a Category

Def. (II.2.2.1) (Filtrations in a Category). Let \( C \) be a category with an initial object, a filtration of an object \( X \) in \( C \) is a family \( F = (X_t)_{t \in \mathbb{R}} \) of subobjects of \( X \) indexed by \( \mathbb{R} \) that satisfies:
- (decreasing property) if \( s \leq t \), then \( X_s \to X \) factors through \( X_t \).
- (separation property) for sufficiently small \( t \), \( X_t \) is the initial object.
- (exhaustiveness) for sufficiently large \( s \), \( X_s = X \).
- (right locally constant property) for any \( t \in \mathbb{R} \), there exists \( \delta > 0 \) that for any \( s \in [t, t + \delta) \), the morphism \( X_t \to X_s \) are isomorphisms.
- (finite jump) the jump set of \( F \) is finite.
Naturally, we define morphisms of filtrations.

Def. (II.2.2.2) (Pullback and Pushforward Filtrations). Suppose fibered products exist in \( C \), for any morphism \( f : X \to Y \) and a filtration \( G = (Y_t) \to Y \), then the family \( f^*G = (Y_t \otimes_Y X) \to X \) is a filtration on \( X \), called the pullback filtration.
If \( f : X \to Y \) and \( F : (X_t) \to X \) is a filtration on \( X \), then if there is a filtration \( f_*F \) on \( Y \) and a morphism of filtrations \( F \to f_*F \) compatible with \( f \), then \( f_*F \) is called the pushforward filtrations.

3 Harder-Narasimhan Formalism

Main references are [Jon20].

Def. (II.2.3.1) (Harder-Narasimhan Formalism). A Harder-Narasimhan formalism consists of
- An exact category \( C(I.11.1.13) \).
- A function \( \text{deg} : \text{Ob}(C) \to \mathbb{Z} \) that is additive w.r.t short exact sequences.
• An exact faithful \textbf{generic fiber functor} to an Abelian category $F : \mathcal{C} \to \mathcal{A}$ that induces for each object $F : \mathcal{X} \in \mathcal{C}$ a bijection

\[ \{ \text{strict objects of } \mathcal{X} \} \cong \{ \text{subobjects of } F(\mathcal{X}) \} \]

where a \textbf{strict subobject} is an object that can be prolonged to an exact sequence.

• An additive function $\text{rank} : \mathcal{A} \to \mathbb{N}$ on $\mathcal{A}$ that $\text{rank}(\mathcal{L}) = 0 \iff \mathcal{L} = 0$, and its composition with $F$ is also called $\text{rank}$.

• If $u : \mathcal{X} \to \mathcal{X}'$ is a morphism in $\mathcal{C}$ that $F(u)$ is an isomorphism, then $\deg(\mathcal{X}) \leq \deg(\mathcal{X}')$ with equality iff $u$ is an isomorphism.

**Cor. (II.2.3.2).**

• We are free to choose the "kernel" for $u$ that $F(u)$ is surjection.

• The subobjects of subobjects are subobjects, by axiom 3.

**Def. (II.2.3.3) (Saturation).** Let $\mathcal{X}'$ be a subobject of $\mathcal{X}$, then we let $\bar{\mathcal{X}}'$ denote the strict subobject of $\mathcal{X}$ corresponding to the subobject $F(\mathcal{X}')$ of $F(\mathcal{X})$, called the \textbf{saturation} of $\mathcal{X}'$. The saturation satisfies:

\[ \cdot \]

\[ \cdot \]

\[ \cdot \]

Proof: Cf.\cite{Jon20}P3.

**Prop. (II.2.3.4).** Every morphism $f : \mathcal{X} \to \mathcal{Y}$ has a kernel and a image in $\mathcal{C}$, and $0 \to \text{Ker} f \to \mathcal{X} \to \text{Im} f \to 0$ is an exact sequence.

Proof: Cf.\cite{Jon20}P3.

**Prop. (II.2.3.5) (HN-Formalism on the Category of Filtered Vector Spaces).** If $L \triangleleft K$ is a field extension, there is a category $\text{VectFil}_{L/K}$ consisting of $(V, \text{Fil})$ where $V$ is a $K$-vector space and $\text{Fil}$ is a finite filtration on $V \otimes_K L$. It is an exact category by declaring exact sequences be those induce exact sequences on the gradeds.

The generic fiber functor is $\text{VectFil}_{L/K} \to \text{Vect}_K : (V, \text{Fil}) \mapsto V$, and rank is as usual, the \textbf{Hodge-Tate degree} is defined to be $t_H((V, \text{Fil})) = \sum_i \dim_K \text{gr}^i(V \otimes_K L)$. This is a HN-filtration.

Proof: The axioms can be directly checked, notice a filtration $W_n$ of a filtration $V_n$ is a strict object iff $W_k = W_n \cap V_k$.

**Def. (II.2.3.6) (Slope).** In a HN-formalism, the \textbf{slope} is defined to be $\text{slope}(E) = \frac{\deg(E)}{\text{rank}(E)}$.

$\mathcal{E}$ is called \textbf{semistable of slope} $\lambda$ iff $\text{slope}(\mathcal{E}) = \lambda$, and $\text{slope}(\mathcal{E}') \leq \lambda$ for any nonzero strict subobject $\mathcal{E}' \subset \mathcal{E}$.

**Prop. (II.2.3.7).** If $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be a short exact sequence in $\mathcal{C}$, then:

• If two of them have the same slope, then so does the third.

• If two of them have different slope, then we know the ordering of these slopes.

Proof: Just notice that the degree and rank are all additive functions.
**Prop. (II.2.3.8).** If $\mathcal{E}$ is semistable of slope $\lambda$, then for any morphism $u : \mathcal{E} \to \mathcal{E}'$ that $F(u)$ is surjective, $\mu(\mathcal{E}') \geq \lambda$.

*Proof:* Take the kernel of $F(u)$ in $\mathcal{A}$, which corresponds to a strict object $\mathcal{E}'$ of $\mathcal{E}$, and $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}' \to 0$ is exact, so we can use (II.2.3.19). □

**Cor. (II.2.3.9).** If $\mathcal{E}, \mathcal{F}$ are semistable of slopes $\lambda > \mu$, then $\text{Hom}_C(\mathcal{E},\mathcal{F}) = 0$.

*Proof:* Notice $F$ is faithful. □

**Prop. (II.2.3.10) (Semistable Objects).** If $f : \mathcal{E} \to \mathcal{F}$ be a map of vector bundles of the same slope $\lambda$, then $\text{Ker}(f)$ and $\text{Coker}(f)$ are all semistable vector bundles of slope $\lambda$, and if $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is exact and $\mathcal{E}',\mathcal{E}''$ are semistable of slope $\lambda$, then so does $\mathcal{E}$.

*Proof:* Use $F(f)$ to find the "coimage" $A$ and the "image" $B$ of $f$, then there is a map from $F(A)$ to $F(B)$ which is an isomorphism, but they have the same degree and rank, thus $A \cong B$ by the last axiom. And the image must has slope $\lambda$. Then $\text{Ker}(f),\text{Coker}(f)$ all can be defined, and they have the same slope $\lambda$ by (II.2.3.19).

$\text{Ker}(f)$ is semistable because strict subobjects of $\text{Ker}(f)$ are also strict subobjects of $\mathcal{E}$ (II.2.3.2). And for $\text{Coker}(f)$, if it is not semistable, choose $\mathcal{F}' \subset \text{Coker}(f)$ that has slope $\geq \lambda$, let $\mathcal{F}'$ be the inverse image, then $0 \to \text{Im}(f) \to \mathcal{F}' \to \mathcal{F}' \to 0$, then by (II.2.3.19) slope($\mathcal{F}'$) $> \lambda$, contradicting the semi-stability of $\mathcal{F}$.

For the extension, slope($\mathcal{E}$) $= \lambda$ by (II.2.3.19), and for a strict subobject $\mathcal{F}$ of $\mathcal{E}$, then we can find $\mathcal{F}',\mathcal{F}''$ be strict objects of $\mathcal{E}'$, $\mathcal{E}''$ respectively that there is an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, which shows slope($\mathcal{F}$) $\leq \lambda$, so $\mathcal{E}$ is semistable. □

**Lemma (II.2.3.11) (Final Subobjects of Maximal Slope).** Let $X$ be an object of $\mathcal{C}$ and $X',X''$ its subobjects of maximal slope, then $X' + X''$ and $X' \cap X''$ are also of maximal slope.


**Def. (II.2.3.12).** Given an object $X$ of $\mathcal{C}$, consider the following condition on a nonzero subobject $X'$ of $X$:

For all subobjects $X''$ of $X$ properly containing $X'$, $\mu(X'') < \mu(X')$.

**Def. (II.2.3.13) (SCSS).** Let $X'$ be a subobject of $X$, then the following conditions are equivalent:

- $X'$ satisfies condition (II.2.3.12) and is semistable.
- $X'$ satisfies condition (II.2.3.12) and is of maximal slope.
- $X'$ is the final object of $X$ of maximal slope.

If $X'$ satisfies these equivalent conditions and $X' \neq X$, then $X$ is called a strongly contradicting semi-stability or SCSS of $X$.


**Prop. (II.2.3.14).** Every object $X$ of $\mathcal{C}$ admits a SCSS.

**Def. (II.2.3.15) (Harder-Narasimhan Filtration).** Let $\mathcal{E} \in \mathcal{C}$, a chain of strict objects $0 \subset \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_m = \mathcal{E}$ is called a Harder-Narasimhan filtration iff each quotient $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable of slope $\lambda_i$ and $\lambda_1 > \lambda_2 > \ldots > \lambda_m$.
Prop. (II.2.3.16). Every object $E \in \mathcal{C}$ has a unique functorial Harder-Narasimhan filtration.

Proof: For uniqueness: if there are two filtrations, it suffices to show that $E'_1 = E_1$, because notice by (II.2.3.2) $E_i$ is a strict subobjects of $E_j$ for any $i < j$, so we finish by induction on the length of the filtration and considering $E/E_1$.

For this, firstly $\lambda_1 = \lambda'_1$, suppose the contrary and $\lambda_1 > \lambda'_1$, then $\lambda_1 > \lambda'_1$ for each $i$, so $\text{Hom}(E_1, E_1'/E_{i-1}) = 0$ for each $i$ by (II.2.3.32), so by induction $\text{Hom}(E_1, E) = 0$, contradiction.

Next by the same reason as in the proof above, $E_1 \hookrightarrow E$ has image in $E'_1$, and the reverse is true for $E'_1$, so $E_1 \cong E'_1$ in $E$.

For existence: Use induction on rank($E$). If $E$ is semistable, then we finish. Otherwise, there is a strict subobject $F$ and $0 \to F \to E \to G \to 0$ that slope($F$) > slope($E$), so rank($F$), rank($G$) < rank($E$). Now by induction $F$ and $G$ has HN-filtration, thus by argument as above, we see that $E$ cannot have strict subobject with slope bigger than slopes appearing in the HN-filtration of $F, G$. So if we choose a strict subobject of $E_1$ of maximal rank among the strict subobjects of maximal slope, we claim the subobjects of $E/E_1$ all have slopes smaller than slope($E_1$): if some slope($G$) ≥ slope($E_1$), consider its inverse image $G'$, then $0 \to E_1 \to G' \to G \to 0$, thus slope($G'$) ≥ slope($E_1$) and has bigger rank, contradiction. So we can use induction on $E/E_1$. □

HN-Polygons

Harder-Narasimhan Categories

This subsection is unnecessary. Main references are [Harder-Narasimhan Categories]

Def. (II.2.3.17) (Harder-Narasimhan Categories). A Harder-Narasimhan category consists of a geometric exact category $(\mathcal{C}, E, A)$ (I.11.1.16) consists of

1. A function $\text{deg}: \text{Ob}(\mathcal{C}_A) \to \mathbb{R}$ that is additive w.r.t short exact sequences in $E_A$.

2. A function $\text{rank}: \text{Ob}(\mathcal{C}) \to \mathbb{N}$ on $A$ that is additive w.r.t short exact sequences in $E$, rank($X$) = 0 $\iff$ $X = 0$.

The slope of a nonzero object $X$ is defined to be $\mu(X) = \text{deg}(X)/\text{rank} X$. And $X$ is called semistable of slope $\lambda$ iff $\mu(X) = \lambda$, and $\mu(X') \leq \lambda$ for any nonzero geometric subobject $X' \subset X$.

And the category satisfies the following axiom:

- **NH**: For any nonzero geometric object $X$, there exists a geometric subobject $X_{des} \subset X$ that

$$\mu(X_{des}) = \sup\{\mu(Y)\mid Y \text{ is a non-zero geometric subobject of } X\}$$

and moreover, any nonzero geometric subobject $Z$ of $X$ that $\mu(Z) = \mu(X_{des})$ is a geometric subobject of $X_{des}$.

Notice that $X_{des}$ is semistable and unique up to isomorphisms, called the destabilization of $X$.

Cor. (II.2.3.18). A geometric object of rank 1 is semistable.

Proof: For any nonzero geometric subobject $X' \subset X$, rank $X' = \text{rank} X = 1$, so rank $X/X' = 0$ hence $X/X' = 0$ and $X' \cong X$. Then clearly $\mu(X') = \mu(X)$ and $X$ is semistable. □

Cor. (II.2.3.19). If $0 \to E' \to E \to E'' \to 0$ be a short exact sequence in $E_A$, then:

- If two of them have the same slope, then so does the third.

- If two of them have different slope, then we know the ordering of these slopes.
Prop. (II.2.3.20) (Abelian Categories as Harder-Narasimhan Categories). Let \((C, \mathcal{E}, A)\) be a geometric exact category with functions \(\deg\) and \(\text{rank}\), and \(C\) is an Abelian category, \(\mathcal{E}\) is the set of short exact sequences, then \((C, \mathcal{E}, A, \deg, \text{rank})\) is a Harder-Narasimhan category.

Proof: We need to check HN: induct on rank \(X\): The condition is clear when \(X\) is semistable, in particular when rank \(X = 1\) (II.2.3.18), and when \(X\) is not semistable, let \(Y\) be a geometric subobject of \(X\) that \(\mu(X') > \mu(X)\) and rank \(X'\) is maximal, then by induction hypothesis, there is a destabilization \(Y_{\text{des}}\), and we want to show \(X'_{\text{des}}\) is just \(X_{\text{des}}\): Let \(Y\) be a nonzero geometric subobject of \(X\). If \(Y\) is a geometric subobject of \(X'\), then \(\mu(Y) \leq \mu(X'_{\text{des}})\). If \(Y\) is not a geometric subobject of \(X'\), then \(Y + X'\) is greater than \(X'\), and rank \((Y + X') > \text{rank} X'\), so by maximality, \(\mu(Y + X') \leq \mu(X) < \mu(X')\). Moreover, there is an exact sequence

\[
0 \to Y \cap X' \to Y \oplus X' \to Y + X' \to 0
\]

so

\[
\deg Y = \deg(Y \cap X') + \deg(Y + X') - \deg(X') \\
< \mu(X'_{\text{des}}) \text{rank}(Y \cap X') + \mu(X') \text{rank}(Y + X') - \text{rank}(X')) \\
\leq \mu(X'_{\text{des}})(\text{rank}(Y \cap X') + \text{rank}(Y + X') - \text{rank}(X')) \\
= \mu(X'_{\text{des}}) \text{rank}(Y).
\]

In particular, if \(\mu(Y) = \mu(X'_{\text{des}})\), then \(Y\) must be a geometric subobject of \(X'\) hence a geometric subobject of \(X'_{\text{des}}\), so HN holds for \(X\). \(\square\)

Def. (II.2.3.21) (Harder-Narasimhan Filtration). Let \(\mathcal{X}\) be a nonzero geometric object, a chain of admissible monomorphisms \(0 = X_0 \to X_1 \to \ldots \to X_m = X\) is called a Harder-Narasimhan filtration of \(X\) iff each quotient \(\mathcal{E}_i/\mathcal{E}_{i-1}\) are all semistable of slope \(\lambda_i\) and \(\lambda_1 > \lambda_2 > \ldots > \lambda_m\), called the slopes associated to \(X\).

Prop. (II.2.3.22) (Harder-Narasimhan Filtrations Exist). Let \((C, \mathcal{E}, A, \deg, \text{rank})\) be a Harder-Narasimhan category, then the Harder-Narasimhan filtration exists for any nonzero geometric object \(X\).

Proof: We induct on the rank of \(X\): if \(X\) is semistable, this is clear, so we are done in the case rank \(X = 1\) (II.2.3.18). We choose \(X_1 = X_{\text{des}}\), then \(X_{\text{des}}\) is semistable and \(X' = X/X_{\text{des}} \neq 0\). Now rank \(X' < \text{rank} X\), we can apply induction hypothesis to obtain a HN-filtration \(0 = X'_1 \xrightarrow{f'_1} X'_2 \to \ldots \to X'_{n-1} \xrightarrow{f'_{n-1}} X'_n = X'\). Now let \(X_i = X \times X' X'_i\) (exists by Ex6(I.11.1.13)), then \(X_1 = X'_{\text{des}}\). Since \(X \to X'\) is an admissible epimorphism, \(X_i \to X'_i\) are also admissible epimorphisms, and there are Cartesian diagrams

\[
\begin{array}{ccc}
X_i & \xrightarrow{f_i} & X_{i+1} \\
\downarrow \pi_i & & \downarrow \pi_{i+1} \\
X'_i & \xrightarrow{f'_i} & X'_{i+1}
\end{array}
\]
and \(f_i\) are monomorphisms, so \(f_i\) is the kernel of \(X_{i+1} \to X'_{i+1}/X'_i\), which is an admissible epimorphism, so \(f_i\) is admissible. There are natural isomorphisms \(\varphi_i : X_{i+1}/X_i \to X'_{i+1}/X'_i\) (Cf.[Harder-Narasimhan Filtrations]), and axiom A6(I.11.1.16) shows \(\varphi\) is compatible with the geometric structure, so \(X_{i+1}/X_i\) are all semistable, and notice

\[
\mu(X_2/X_1) = \frac{\text{rank}(X_2)\mu(X_2) - \text{rank}(X_1)\mu(X_1)}{\text{rank}(X_2) - \text{rank}(X_1)} < \mu(X_1)
\]

so \(0 \to X_1 \to X_2 \to \ldots \to X_m = X\) is a HN-filtration for \(X\).

Prop. (II.2.3.23) (HN-Formalism for Filtrations in an Abelian Category). Let \(\mathcal{C}\) be an Abelian category and \(\mathcal{E}\) the set of all short exact sequences in \(\mathcal{C}\), \(A(X)\) is the set of isomorphism classes of filtrations on \(X\), then \((\mathcal{C}, \mathcal{E}, A)\) is a geometric exact category, by(I.11.1.20), given any additive rank function on \(\mathcal{C}\), and define a degree function for any filtration \(F = (X_\lambda)\) as

\[
\text{deg}(F) = \int R \lambda(d \text{rank} X_\lambda),
\]

then deg is additive w.r.t short exact sequences of filtrations, and \((\mathcal{C}, \mathcal{E}, A, \text{deg}, \text{rank})\) is a Harder-Narasimhan filtration.

Then a filtration is semistable iff it has only one jump. Then the HN-filtration of a filtration is just the jump set ordered decreasingly.

Prop. (II.2.3.24) (HN Formalism for Vector Spaces with Two Norms). The vector spaces with two norms is a Harder-Narasimhan category, Cf.[Harder-Narasimhan Filtrations, P9].

Prop. (II.2.3.25) (HN-Formalism for Torsion-Free Sheaves). The category of torsion-free sheaves on a geometrically normal projective variety of dimension \(d \geq 1\) over a field \(K\) is a Harder-Narasimhan category. Cf.[Harder-Narasimhan Filtrations, P10].

Prop. (II.2.3.26) (HH-Formalism for Hermitian Adelic Bundle). Let \(K\) be a number field, then the category of Hermitian adelic bundle over \(K\) is a Harder-Narasimhan category. Cf.[Harder-Narasimhan Filtrations, P10].

Prop. (II.2.3.27) (HN Formalism for Filtered Vector Spaces Field). If \(L/K\) is a field extension, there is a category \(\text{VectFil}_{L/K}\) consisting of \((V, \text{Fil})\) where \(V\) is a \(K\)-vector space and \(\text{Fil}\) is a (finite) filtration of vectors spaces over \(L\) on \(V \otimes_K L\). It is a geometric exact category by(I.11.1.20) and a Harder-Narasimhan category by(II.2.3.21).

The rank is as usual, and the degree is defined to be

\[
\text{deg}((V, \text{Fil})) = \int R \lambda(d \text{rank} V_\lambda)(II.2.3.23)
\]

Slope Inequalities and Functoriality

Def. (II.2.3.28) (Additional Conditions). In this subsubsection, we assume the Harder-Narasimhan filtration satisfies the following axiom that is

Def. (II.2.3.29) (Slope Inequality Axioms). To show the functoriality of the Harder-Narasimhan filtration, we need the following axiom:
• (SI): If $X_1, X_2$ are two semistable geometric objects that $\mu(X_1) > \mu(X_2)$, then there are no non-zero morphism from $X_1$ to $X_2$ compatible with the geometric structures (I.11.1.17).

**Prop. (II.2.3.30).** If SI holds, then

**Prop. (II.2.3.31).** If $E$ is semistable of slope $\lambda$, then for any morphism $u : E \rightarrow E''$ that $F(u)$ is an isomorphism, slope($E''$) $\geq \lambda$.

**Proof:** Take the kernel of $F(u)$ in $A$, which corresponds to a strict object $E'$ of $E$, and $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ is exact, so we can use (II.2.3.19). □

**Cor. (II.2.3.32).** If $E, F$ are semistable of slopes $\lambda > \mu$, then $\text{Hom}_C(E, F) = 0$.

**Prop. (II.2.3.33) (Semistable Vector Bundles Form a Weak Serre Subcategory).** If $f : E \rightarrow F$ be a map of vector bundles of the same slope $\lambda$, then $\text{Ker}(f)$ and $\text{Coker}(f)$ are all semistable vector bundles of slope $\lambda$, and if $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ is exact and $E', E''$ are semistable of slope $\lambda$, then so does $E$.

**Proof:** Use $F(f)$ to find the "coimage" $A$ and the "image" $B$ of $f$, then there is a map from $F(A)$ to $F(B)$ which is an isomorphism, but they have the same degree and rank, thus $A \cong B$ by the last axiom. And the image must has slope $\lambda$. Then $\text{Ker}(f), \text{Coker}(f)$ all can be defined, and they have the same slope $\lambda$ by (II.2.3.19).

$\text{Ker}(f)$ is semistable because strict subobjects of $\text{Ker}(f)$ are also strict subobjects of $E$ (II.2.3.2). And for $\text{Coker}(f)$, if it is not semistable, choose $F' \subset \text{Coker}(f)$ that has slope $> \lambda$, let $F'$ be the inverse image, then $0 \rightarrow \text{Im}(f) \rightarrow F' \rightarrow F' \rightarrow 0$, then by (II.2.3.19) slope($F'$) $> \lambda$, contradicting the semi-stability of $F$.

For the extension, slope($E$) = $\lambda$ by (II.2.3.19), and for a strict subobject $F$ of $E$, then we can find $F', F''$ be strict objects of $E', E''$ respectively that there is an exact sequence $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$, which shows slope($F$) $\leq \lambda$, so $E$ is semistable. □

### 4 Tannakian Categories

Let $k$ be a field.

**Def. (II.2.4.1) (Fiber Functor).** Let $C$ be a $k$-linear Abelian tensor category, then a fiber functor on $C$ with values in a $k$-algebra $R$ is a $k$-linear exact faithful tensor functor $\eta : C \rightarrow \text{Mod}_R$ that takes values in the subcategory Proj$_R$.

**Def. (II.2.4.2) (Tannakian Category).** A Tannakian category is a rigid tensor category $C$ that $\text{End}(1) = k$ together with a fiber functor (II.2.4.1) with values in some $k$-algebra $R$.

**Def. (II.2.4.3) (Neutral Tannakian Categories).** A neutral Tannakian category is a Tannakian category that the fiber functor has values in $k$. By (V.9.9.9), such a category is equivalent to $\text{Rep}_k(G)$ for some affine group scheme $G$.
II.3 Derived Categories

Main references are [?], should consult [Sta].

1 Triangulated Category

Def. (II.3.1.1). A triangulated category is an additive category with a $T$: additive automorphism and an isomorphism class of distinguished triangles satisfying the following axioms:

TR1) $X \xrightarrow{id} X \rightarrow 0 \rightarrow X[1]$ is distinguished. Any morphism $X \xrightarrow{u} Y$ can be completed to a distinguished triangle $X \xrightarrow{u} Y \rightarrow C(u) \rightarrow X$.

TR2) A triangle $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ is distinguished iff $Y \rightarrow Z \rightarrow X[1] \rightarrow Y[1]$ is distinguished.

TR3) Any two consecutive morphisms of two distinguished class can be extended to a morphism of distinguished class.

TR4) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms, then there are maps $C(f) \rightarrow C(gf), C(gf) \rightarrow C(g)$ by TR3, then $C(f) \rightarrow C(gf) \rightarrow C(gf) \rightarrow C(g) \rightarrow C(f)[1]$ is distinguished.

Def. (II.3.1.2) (Exact Functors). A functor from a triangulated category to an Abelian category is called (co)homological iff it maps a distinguished triangle to an exact sequence.

Conversely, A functor from an Abelian category to a triangulated category is called $\delta$-functor iff it functorially maps an exact sequence to a distinguished triangle.

A functor between two triangulated category is called exact iff it preserves $-[1]$ and maps distinguished triangle to a distinguished triangle.

Prop. (II.3.1.3). For a distinguished category, $\text{Hom}(-,C)$ and $\text{Hom}(C,-)$ is (co)homological. In particular, composition of consecutive maps in a distinguished triangle is 0, (Easily from TR1 and TR3).

Thus the extension of TR3 of two isomorphisms is an isomorphism (by 5-lemma, $\text{Hom}(C,X) \rightarrow \text{Hom}(C,X')$ is an isomorphism, then use Yoneda). Hence the completion in TR2 is unique(up to non-unique isomorphism) by TR3.

Prop. (II.3.1.4). In a triangulated category $\mathcal{D}$, any commutative diagram

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X' & \longrightarrow & Y'
\end{array}
\]

can be extended to a diagram

\[
\begin{array}{cccccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X'' & \longrightarrow & Y'' & \longrightarrow & Z'' & \longrightarrow & X''[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\end{array}
\]
where the lower right is anti-commutative.

**Proof:** Let \((X, Y, Z), (X', Y', Z'), (X, X', X''), (Y, Y', Y''), (X, Y, A)\) be distinguished triangles, then we can find maps \(a : Z \to A, b : A \to Y', a' : X'' \to A, b' : A \to Z\) by TR3(II.3.1.1). Then TR4 says \((Z \to A, Y'')\) is distinguished.

Now let \((X'', Y'', Z'')\) be distinguished, then we use TR4 again to \((X'', A, Y'')\), then \((Z'', Z', Z[1])\) is distinguished, thus so does \((Z \to Z' \to Z'')\).

Now it is left to verify the anti-commutativity of the rightdown square, for this, Cf.[Sta]05R0. □

**Prop. (II.3.1.5) (\(K^*(A)\) is Triangulated).** For Abelian category \(A\), the categories \(K^*(A)\) with distinguished triangles(I.11.4.2) is triangulated, and they are all subcategories of \(K(A)\). This is hard to verify, but it solves every problem. Cf[Gelfand P246][[Sta]014S]. And an additive functor will induce exact functor between \(K^*\) because distinguished is split.

**Localization of Triangulated Category**

**Compactly Generatedness and Brown Representability**

**Prop. (II.3.1.6).** Let \(D\) be a triangulated category, then the compact objects of \(D\) form a Karoubian, saturated, strictly full triangulated category of \(D\).

**Proof:** Cf.[Sta]09QH]. □

**Def. (II.3.1.7) (Compactly Generated).** Let \(D\) be a triangulated category with arbitrary direct sums, then \(D\) is said to be compactly generated if there exists a set of compact objects that \(\oplus E_i\) generates \(D\)(I.11.2.22).

**Derived Limits and Colimits**

**Def. (II.3.1.8) (Derived (Co)Limits).** Let \(D\) be a triangulated category, and \((K_n, f_n)\) is an inverse system of objects in \(D\), then an object \(K\) is called the derived colimit of it iff there \(\oplus K_n\) exists and there is a distinguished triangle

\[
\oplus K_n \to \oplus K_n \to K \to \oplus K_n[1]
\]

where the first map is given by \((1 - f_n)\). By TR1, the derived colimit exists as long as \(\oplus K_n\) exists, and by TR3(II.3.1.3), the colimit is unique if it exists. And by TR3 again a morphism of systems induces a morphism of colimits.

The definition of derived limit is dual.

**Prop. (II.3.1.9) (Cofinality of Hocolim).** Let \(D\) be a triangulated category and \((K_n, f_n)\) be a system, if \(0 \leq n_0 < n_1 < \ldots\) be a sequence of integers, then there is an isomorphism \(hocolim K_{n_i} \to hocolim K_n\).

**Proof:** Cf.[Sta]0CRJ]. □

**Lemma (II.3.1.10).** Let \(\mathcal{A}\) be an Abelian category with countable products and enough injectives, then the derived limit \(R\lim\) for any inverse system in \(D^+(\mathcal{A})\) exists.

**Proof:** It suffices to show \(\prod K_n^\bullet\) exists in \(D(\mathcal{A})\). But every \(K_n^\bullet\) has a \(K\)-injective resolution \(I_n^\bullet\), by(I.11.5.9)(I.11.5.13). And then \(\prod K_n^\bullet\) is represented by \(\prod I_n^\bullet\), by(I.11.5.15). □
Def. (II.3.1.11) ($\text{Rlim}$). $\text{Rlim}$ on an Abelian category $\mathcal{A}$ with countable products and enough injectives is defined to be the right derived functor of $\text{lim} : \mathcal{A}^\mathbb{N} \to \mathcal{A}$. Equivalently, it is just the derived limit (II.3.1.8) (II.3.1.10) in $D(\mathcal{A})$ restricted to the case where each $K_n$ is discrete.

Prop. (II.3.1.12). Let $\text{Rlim} \mathcal{A}$ exists on $\mathcal{A}$, if $K_n, f_n$ is a system of objects in $D^+(\mathcal{A})$, then there are exact sequences

$$0 \to R^1 \lim (H^m(K_n), f_n) \to R^{n+1} \lim (K_n, f_n) \to \lim (H^m(K_n), f_n) \to \cdots$$

Immediately from the definition (II.3.1.8).

2 Derived Category

Def. (II.3.2.1). The derived category $D(\mathcal{A})$ of an Abelian category $\mathcal{A}$ represents the universal property that any functor to a category $\mathcal{A} \to \mathcal{C}$ s.t. quasi-isomorphisms is mapped to isomorphisms uniquely factors through $D(\mathcal{A})$.

It can be defined as the localization of quasi-isomorphisms, but the class of quasi-isomorphisms is not localizing. But one can prove the quasi-isomorphisms in $K(\mathcal{A})$ is localizing and the localization by quasi-isomorphisms of $K(\mathcal{A})$ is equivalent to $D(\mathcal{A})$. Cf. [Gelfand P159]

Notice that equivalent roofs induce the same map on homology, so the cohomology functor can be regarded defined on $D(\mathcal{A})$.

$$\mathcal{A} \to K(\mathcal{A}) \to K(\mathcal{A})[S^{-1}] = D(\mathcal{A}) \xrightarrow{H^*} \mathcal{A}.$$  

Prop. (II.3.2.2). The category $D^*(\mathcal{A})$ are the localized category of $K^*(\mathcal{A})$ at the class of quasi-isomorphisms respectively. The morphisms in $D^*(\mathcal{A})$ is of the form $t \circ s^{-1}$. (look at the homology map they induced).

Prop. (II.3.2.3). If $\mathcal{B}$ is a full subcategory that $S \cap \mathcal{B}$ is a localizing category of $\mathcal{B}$ and any $s \in S$ can be 'denominated' in one given side (any one is OK) into $\mathcal{B}$, then $\mathcal{B}[S \cap \mathcal{B}^{-1}]$ is a full subcategory of $\mathcal{C}[S^{-1}]$. The proof is easy, use left roof or right roof.

Prop. (II.3.2.4). $K$ is a triangulated category and a localizing class $S$ compatible with $T$, i.e. $s \in S \iff T(s) \in S$ and the extension in $TR3$ of $f, g$ in $S$ is in $S$. Then the localizing category $K[S^{-1}]$ is triangulated.

Cor. (II.3.2.5) (Derived Category is Triangulated). $D(\mathcal{A})$ is a triangulated category. The distinguished triangle is the one defined in (I.11.4.1), and for a distinguished triangle, the long exact sequence exists, (I.11.4.1). In other words, $H^0$ is a cohomological functor for $D(\mathcal{A})$.

Prop. (II.3.2.6) (Derived Category of Weak Serre Subcategory). Let $\mathcal{A} \subset \mathcal{B}$ be a weak Serre subcategory and suppose that any object in $\mathcal{A}$ can be embedded in an object in $\mathcal{A}$ that is injective in $\mathcal{B}$, then

Proof:
Prop. (II.3.2.7). The natural inclusion $A \subset D(A)$ embeds $A$ as a full subcategory of $D(A)$, and $H^0$ is just the left adjoint.

An object $K \in D(A)$ is called discrete if it is in the essential image of this embedding.

Proof: Compare with (II.3.2.7).

Def. (II.3.2.8) (Perfect Complex). Let $A$ be an Abelian category, a perfect complex in $D(A)$ is a complex that is equivalent to a bounded complex.

Operations on the Derived Category

Lemma (II.3.2.9) (Direct Sum). If $A$ is an Abelian category that has exact countable direct sums, then $D(A)$ has countable direct sums given by term-wise direct sums.

Proof: A system of morphisms $K_i^* \to L^*$ is a system of quasiisos $M_i^* \to K_i^*$ and $M_i \to L^*$. Then by hypothesis $\oplus M_i^* \to \oplus K_i^*$ is a quasi-iso, thus defines a morphism $\oplus K_i^* \to L^*$. It can be verified that this morphism is unique.

Lemma (II.3.2.10) (Termwise Colimit as Hocolim). Let $A$ be an Abelian category, $L_n^*$ be a system of complexes of $A$. Assume colimits over $\mathbb{N}$ exists and are exact over $A$, then the termwise colimit $L^*$ is a derived colimit in $D(A)$.

Proof: We have an exact sequence

$$0 \to \oplus L_n^* \to \oplus L_n^* \to L^* \to 0$$

and the termwise direct sum is the direct sum in $D(A)$ by (II.3.2.9), and then $L^*$ is a derived colimit, by (I.11.4.1).

3 Acyclic Elements and Derived Functors

Prop. (II.3.3.1) ($F$-Acyclic Objects and Adapted Class). For a left exact $F$, an object is called $F$-acyclic if it is sufficiently large and $F$ maps acyclic objects in $\text{Comp}^+(\mathcal{R})$ to acyclic objects. A

Injectives are $F$-acyclic for all left exact $F$ because if $I^* \in \text{Comp}^+(\mathcal{R})$, then $\text{id}: I^* \to I^*$ is homotopic to 0, by (I.11.5.13).

When $RF$ exists, an object $X$ is $F$-acyclic iff $R^iF(X) = 0$ for all $i > 0$. Then: there is an adapted class of $F$ iff the class of $F$-acyclic objects $Z$ is sufficiently large.

If this is the case, then adapted class of $F$ are exactly sufficiently large subclass of $Z$, and $Z$ contains all injectives, Cf.[Gelfand P195].

Prop. (II.3.3.2) (Acyclic Criterion). Let $F$ be a left exact functor from an Abelian category $C$ of enough injectives to another Abelian category, $T$ is a class of objects of $C$ that satisfies:

- $T$ is sufficiently large.
- Cokernel of maps between elements of $T$ is in $T$ and $0 \to F(A) \to F(A') \to F(\text{Coker}) \to 0$ is exact. (To use induction).

Then every object of $T$ is $F$-acyclic.

Proof: Cf.[Sta]05T8.
**Prop. (II.3.3.3).** For a class of objects \( \mathcal{R} \) in \( \mathcal{A} \) stable under finite direct sum and are adapted to a left exact functor \( F \), i.e. \( \text{Kom}^+(\mathcal{R}) \) is \( F \)-acyclic and every object in \( \mathcal{A} \) is a subobject of an object from \( \mathcal{R} \). Just need to verify the condition of (II.3.2.3). Similarly for the opposite category.

And in this case \( K^+(\mathcal{R})[S^{-1}_R] \) is equivalent to \( D^+(\mathcal{A}) \).

**Proof:** The hard part is to prove every complex in \( K^+(\mathcal{A}) \) is quasi-isomorphic to a complex in \( K^+(\mathcal{R}) \), for this, use direct construction. Cf.[Gelfand P187]. \( \square \)

**Prop. (II.3.3.4).** By(I.11.5.13), \( K^+(\mathcal{I}) \) is a saturated subcategory of \( D^+(\mathcal{A}) \). And if \( \mathcal{A} \) has enough injectives, this is an equivalence of categories. (We only need to verify that the localization of \( K^+(\mathcal{I}) \) is itself, using the last proposition). In particular, this applies to Grothendieck categories.

**Proof:** Cf.[Gelfand P179]. \( \square \)

**Prop. (II.3.3.5).** By(I.11.5.13), if \( \mathcal{A} \) contains sufficiently many injectives, then injective objects are adapted to any left exact functor \( F \). (Because id on acyclic injective complexes is homotopic to 0 by the lemma).

**Def. (II.3.3.6) (Total Derived Functor via Adapted Class).** The right derived functor \( R^F : D^+(\mathcal{A}) \to D^+(\mathcal{B}) \) for an additive functor \( F \) between Abelian categories is defined by the following universal property:

- \( R^F \) is exact and there is a natural transformation
  \[ \varepsilon_F : Q_B K^+ F \to R^F Q_A. \]
  and any other exact \( G : D^+(\mathcal{A}) \to D^+(\mathcal{B}) \) and a similar transformation must factor through \( \varepsilon_F \) uniquely. Thus this \( R^F \) is unique up to natural isomorphism.

- If a left exact functor \( F \) between Abelian categories has an adapted class \( \mathcal{R} \), then by preceding proposition, \( K^+(\mathcal{R})[S^{-1}_R] \) is equivalent to \( D^+(\mathcal{A}) \), then the derived functor exists, and it is just \( F^+ \) on \( K^+(\mathcal{R}) \), Cf.[Gelfand P188]. ? See also [Sta05TA].

**Remark (II.3.3.7).** Notice there is a more general derived functor that use inductive limits in \( \hat{\mathcal{A}} \) that it maps \( D^+(\mathcal{A}) \) to \( \text{Ind}(D^+(\mathcal{B})) \), and if it has image in the subcategory of representable objects, then it coincide with \( R^F \). Similarly for right exact functor \( F \). (This is easy to check) Cf.[Gelfand P198].

Yet there is another way to just look at the derived functors, it is the hypercohomology of the Cartan-Eilenberg resolution of complexes(II.3.3.13).

**Prop. (II.3.3.8).** \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) be Abelian categories with enough injectives in \( \mathcal{A}, \mathcal{B} \) and \( F : \mathcal{A} \to \mathcal{B}, G : \mathcal{B} \to \mathcal{C} \) are left exact functors. If \( F(\mathcal{I}) \subset R_{\mathcal{B}} \) for \( \mathcal{I} \) injective, then \( R(G \circ F) \to R G \circ R F \) is an isomorphism . (Because the definition of \( RF \) is just \( F^+ \) on \( K^+(\mathcal{I}) \)).

**Def. (II.3.3.9) (Universal \( \delta \)-Functors).** A universal \( \delta \)-functor between Abelian categories is one that any natural transformation from \( T^0 \) to another \( \delta \)-functor will generate a \( \delta \)-map. A effaceable \( \delta \)-functor is one that for any \( n > 0 \) and any object \( A \), there is an injection \( A \to B \) that \( T^n(A) \to T^n(B) = 0 \).

**Prop. (II.3.3.10) (Grothendieck).** A \( \delta \)-functor is universal if it is effaceable.
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Proof: We construct by induction on $n$. Choose a $0 \to A \to B \to C \to 0$ such that $T^{n+1}(A) \to T^{n+1}(B) = 0$ then there is an isomorphism $T^{n+1}(A) \cong \text{Coker}(T^n(B) \to T^n(C))$, and so we can construct the map on $T^{n+1}$ induces by

$$\text{Coker}(T^n(B) \to T^n(C)) \to \text{Coker}(G^n(B) \to G^n(C)) \to G^{n+1}(A).$$

This can be verified to be a $\delta$ map. $\square$

Prop. (II.3.3.11). The derived functors form a universal $\delta$-functor (when it exists).

Proof: It is a $\delta$ functor by (II.3.2.5), it is universal by (II.3.3.10). $\square$

Prop. (II.3.3.12). Derived functor commutes with filtered colimits, when $B$ is an Grothendieck category, this is by $AB5$.

Prop. (II.3.3.13) (Hypercohomology). Given an Abelian category $A$ with enough injectives, $B$ a complete Abelian category, $F : A \to B$ a left exact functor, and $K \in K(A)$, we can define the right hyper-derived functor of $F$ at $K$ as $\mathbb{R}F(K) = \text{Tot}^\Pi F(P) \in K(B)$ where $K \to P$ is a Cartan-Eilenberg resolution of $K$. and the hypercohomologies of $F$ at $K$ as $\mathbb{R}F(K) = H^n(\text{Tot}^\Pi F(P))$.

Dually we can define the left hyper-derived functor and hyperhomologies.

For complexes in $K^+(A)$, there is no restriction and the right derived-hyper functor descends to a functor from $D^+(A) \to D^+(B)$.

When the Abelian category $A$ satisfies $AB3^*$ and $AB4^*$, i.e. the direct product is exact, then $\text{Tot}^\Pi$ of the Cartan-Eilenberg resolution of any complex is a quasi-isomorphism to it by the dual of??. (Take horizontal filtration, $AB4^*$ assures it collapse).

4 (Co)Homological Dimension

Prop. (II.3.4.1). If $A$ has enough projectives, then the projective dimension of an object $X$ is the length of projective resolutions. (Use resolution and long sequence).

Prop. (II.3.4.2) (Hilbert Theorem). For an Abelian category $A$, the category $A[T]$ is an Abelian category. If $A$ has enough projectives and have infinite direct sum, then $\text{dhp}_{A[T]}(X,t) \leq \text{dhp}_A(X) + 1$ and equality with $t = 0$.

Cor. (II.3.4.3). The Categories $Ab$ and $K[X]$-mod have homological dimension 1. $K[X_1, \ldots, X_k]$ has homological dimension $k$.

5 Spectral Sequence

Reference for this section is [Weibel Ch5]. All the definition below is dual for homology and cohomology, just rotate the diagram 180 degree.

We work in an Abelian category.

Def. (II.3.5.1). A convergent Spectral Sequence is a three-dimensional arrange of entries $E^{p,q}_r$ that:

1. $d_r : E^{p,q}_r \to E^{p+r,q-r+1}_r$ that $d_r d_r = 0.$
2. $H^{p,q}(E^{p,q}_r) \cong E^{p,q}_{r+1}. \text{ And } E^{p,q}_r \text{ has a direct limit } E^{p,q}_\infty.$
3. There is a complex $E^*$ and a decreasing bounded filtration $F^pE^n$ on each $E^n$ and $E^{p,q}_\infty \cong F^pE^{p+q}/F^{p+1}E^{p+q}.$
we can use five lemma and induction to show that spectral sequence is regular, then the spectral sequence weakly converges to Prop. (II.3.5.7) (Complete convergence).

Prop. (II.3.5.6) (Classical Convergence).

A spectral sequence is said to weakly converges to $E^\bullet$ if there is a filtration

$$\cdots \subset F^k H^n \subset F^{k-1} H^n \subset \cdots \subset F^s H^n \subset \cdots \subset H^n$$

that $E_\infty^{pq} \cong F^p H^{p+q}/F^{p+1} H^{p+q}$.

A spectral sequence approaches $E^\bullet$ if it weakly converges to $E^\bullet$.

A spectral sequence converges to $E^\bullet$ if it approaches $E^\bullet$, it is regular, and $E^n = \varprojlim (E^n/F^n E^n)$.

If a first quadrant spectral sequence converges to $E^\bullet$, then the morphisms $E_0^{n,0} \to E_\infty^{n,0} \subset E^n$ and $E^n \to E_\infty^{0,n} \to E_0^{0,n}$ are called the edge morphisms.

Prop. (II.3.5.3) (Notation for Filtrations on Homological Complexes). Let $C_\bullet$ be a complex and $\cdots \subset F_{p-1}C \subset F_p C \subset \cdots \subset C$ be filtrations of complexes. Then it is called exhaustive if $C = \cup F_p C$. It is called Hausdorff if $\cap F_p C = 0$. It is called complete if $C = \varprojlim C/F_p C$.

Def. (II.3.5.4) (Spectral Sequence of a Filtered Complex). For a complex $K^\bullet$ and a filtration $F^p K^n$ on $K^n$, we have a natural spectral sequence

$$E_1^{pq} = H^{p+q}(F^p E^{p+q}/F^{p+1} E^{p+q}), \quad E^n = H^n(K^\bullet), \quad F^p E^n = H^n(F^p K^\bullet).$$

For a morphism of filtered complexes that are isomorphism for some $r$, induction on the exact sequence $0 \to F^p E^n \to F^{p+1} E^n \to E_\infty^{p,n-p}$ and use five-lemma shows it induces isomorphism on $H^s E$.

Prop. (II.3.5.5) (Comparison Theorem). For a morphism $F$ between two convergent spectral sequences, if it is an isomorphism for some $r$, then it induce isomorphism on the infinite homologies.

Proof: Clearly $F$ induces isomorphisms on $E_\infty^{pq}$. Because there are exact sequence

$$0 \to F^{p+1} H^n \to F^p H^n \to E_\infty^{p,n-p} \to 0$$

we can use five lemma and induction to show that $F$ induces isomorphisms on $F^p H^n/F^s H^n$. Then because $H^n = \cup F^p H^n$, we can take colimit to show $F$ induces isomorphisms on $H^n/F^s H^n$, then take inverse limits, we are done. $\square$

Prop. (II.3.5.6) (Classical Convergence). If the filtration on a complex $C_\bullet$ is bounded below and exhaustive for all $C_n$, then there is a spectral sequence that is also bounded below and converges to $H_\bullet(C_\bullet)$.

Proof: Cf [Gelfand P203] for cohomological case and [Weibel P135] for homological case. $\square$

Prop. (II.3.5.7) (Complete Convergence). If the filtration is complete, exhaustive and the spectral sequence is regular, then the spectral sequence weakly converges to $H_\bullet(C_\bullet)$. And if it is also bounded above, then it converges to $H_\bullet(C_\bullet)$. 

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Proof: Cf. [Weibel, P139].

There are two examples, the stupid filtration and the canonical filtration, the canonical filtration is natural and factors through $D(A)$.

Prop. (II.3.5.8) (Spectral Sequence of a Double Complex). A double complex has two natural filtration of the total complex, they defines two spectral sequence, one has

$$E_{2,x}^{p,q} = H_x^p(H_y^q(L^{••}))$$

and the other has

$$E_{2,y}^{p,q} = H_y^p(H_x^q(L^{••})).$$

Cf. [Gelfand P209]. In fact under reflection, there is only one spectral sequence. For the horizontal filtration, the differential goes vertical first, for the vertical filtration, the differential goes horizontal first. The differential goes one way, the convergence goes reversely.

If both the filtration is finite and bounded, in particular if $E$ is in the first quadrant, then they both converges to $H^n(E)$, this will generate important consequences.

Cor. (II.3.5.9). If a double complex in the first quadrant has its all column acyclic (3rd-quadrant pointing), then the total complex is acyclic. Thus a morphism of double complex inducing quasi-isomorphism on each column induces a quasi-isomorphism on the total complex.

Prop. (II.3.5.10) (Horizontal Filtration). For a second-quadrant-free homology double complex, the filtration is bounded below and exhaustive for $\text{Tot}^{\oplus}$, so the classical convergence (II.3.5.6) applies and there is a convergence

$$E_2^{p,q} = H_p^h H_q^{\oplus}(C) \Rightarrow H_{p+q}^{\oplus}(\text{Tot}^{\oplus}(C)).$$

For a fourth-quadrant free homology double complex, the filtration is complete and exhaustive for $\text{Tot}^{\Pi}$, so the complete convergence (II.3.5.7) applies and there is a weak convergence

$$E_2^{p,q} = H_p^h H_q^{\Pi}(C) \Rightarrow H_{p+q}^{\Pi}(\text{Tot}^{\Pi}(C)).$$

Cor. (II.3.5.11) (Grothendieck Spectral Sequence). If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be Abelian categories and $\mathcal{A}, \mathcal{B}$ have enough injectives, and $F : \mathcal{A} \to \mathcal{B}, G : \mathcal{B} \to \mathcal{C}$ are left exact functors. If $\mathcal{R}_A$ is adapted to $F$, $\mathcal{R}_B$ is adapted to $G$ and $F(I_A) \subseteq \mathcal{R}_B$, then for any $X \in K^+(\mathcal{A})$, there is a spectral sequence with $E_2^{p,q} = R^n G(R^n F(X))$ (to upper left) that converges to $E^n = R^n (G \circ F)(X)$. And this spectral sequence is functorial in $X$.

In particular, this applies to $F$ is a right adjoint and its left adjoint is exact, then we may choose $\mathcal{R}_A = \mathcal{I}_A$ and $\mathcal{R}_B = \mathcal{I}_B$.

Proof: Let $K = F(I_X) = RF(X)$, and choose the CE resolution of $K$ (I.11.5.9), because the resolutions for $B^i \to Z^i \to H^i$ and $Z^i \to K^i \to B^{i+1}$ split and $G$ is additive, we have

$$H_2^{q,•}(G(L^{••})) = G(H_2^{q,•}(L^{••})) = RG(H^q(K))$$

So

$$E_2^{p,q} = H_2^p(H_2^{q,•}(L^{••})) = R^n G(H^q(K)) = R^n G(R^n F(X))$$
and

\[ E^\bullet = RG(Tot(L)) = G(Tot(L)) = RG(K) = RG \circ RF(X) = R(G \circ F)(X) \text{(II.3.3.8)}. \]

\[ \square \]

**Cor. (II.3.5.12).** The low degree parts read:

\[ 0 \to R^1G(F(A)) \to R^1(G \circ F)(A) \to G(R^1F(A)) \to R^2(G(F(A)) \to R^2(G \circ F)(A). \]

(Check definition). More generally, if \( R^p G(R^q F(A)) = 0, 0 < q < n, \) then

\[ R^m G(F(A)) \cong R^m(G \circ F)(A) \quad m < n \]

And

\[ 0 \to R^n G(F(A)) \to R^n(G \circ F)(A) \to G(R^n F(A)) \to R^{n+1}(G(F(A)) \to R^{n+1}(G \circ F)(A). \]

**Remark (II.3.5.13).** The Grothendieck spectral sequence is tremendously important.

**Cor. (II.3.5.14) (Spectral Sequence for Hypercohomologies).** For chain complex \( K \)

in \( K^+(A) \) and a left exact functor \( F, \) the CE resolution will generate two spectral sequences by (II.3.5.10):

\[ E^{p,q}_{2,x} = H^p_x(R^q F(A)) \Rightarrow \mathbb{R}^{p+q}(A), \text{when } A \text{ is bounded below} \]

\[ E^{p,q}_{2,y} = (R^p F)(H^q(A)) \Rightarrow \mathbb{R}^{p+q}(A), \text{weakly convergent} \]

where the RHS is the hypercohomologies (II.3.3.13).

6 **T-Structures**

7 **Examples**
II.4 Differential Graded Algebras

Main references are [Ker] and [Sta].

Def. (II.4.0.1) (Differential Graded Algebras). A differential graded algebra or DGA is a chain complex $A^n$ of $R$-modules with $R$-linear maps $A^n \times A^m \to A^{n+m}$ that
\[ d(ab) = d(a)b + (-1)^n ad(b). \]
that makes $\oplus A^n$ into an associative and unital $R$-algebra.

Notice the first condition is equivalent to giving a map $\text{Tor}(A \times_R A) \to A$.

For a differential algebra $A^n$, a right differential module is defined naturally. The tensor operation gives a closed symmetric monoidal structure $M_A$.

Notice a usual $R$-algebra $A$ can be seen as a differential graded algebra as $A^0 = A$ and $A^n = 0$ for $n > 0$.

And as in the case of chain complexes, the category of differential modules over $A$ can be given a derived category.

Def. (II.4.0.2). A differential graded algebra $A^n$ is called commutative if $ab = (-1)^{\deg(a) \deg(b)} ba$. It is called strictly commutative if moreover $a^2 = 0$ for $\deg(a)$ odd.

Def. (II.4.0.3). For two differential graded algebras $A, B$, the tensor graded algebra $A^n \otimes B^m$ is given by the $\text{Tor}(A^n \otimes_R B^m)$.

1 dg-Categories

Def. (II.4.1.1) (dg-Categories). Given a DGA $A$, a dg-category over $A$ is a category enriched over the monoidal category $M_A$ (II.4.0.1). Let $\mathcal{dgCat}_A$ denote the category of small dg-categories over $A$ where morphisms are given by monoidal functors.

Def. (II.4.1.2) (Going Back). Because $H^0$ and $Z^0$ are right-lax monoidal functors from $\text{Ch}(R)$ to $\text{Mod}(R)$, given a dg-category $C$, by transferring, we can get categories $H^0(C)$ and $Z^0(C)$ enriched over $\text{Mod}(A^0)$.

Def. (II.4.1.3) (Equivalences). A morphism between dg-categories are called an equivalence if induces quasi-isomorphisms on all hom-complexes.

Prop. (II.4.1.4) (Model Category of dg-Categories). There is a cofibrantly generated model category on $\mathcal{dgCat}_A$, where weak equivalences are quasi-equivalences and the fibrations are morphisms $F : A \to B$ that:
- induces component-wise surjections on hom-complexes.
- given an isomorphism $g : F(X) \to Y \in H^0(B)$, there is an isomorphism in $H^0(A)$ lifting $g$.

This monoidal structure is induced from that of the case $A = R$ and the right-lax monoidal functor $\text{Ch}(R) \to \mathcal{M}(A)$ given by $M^\bullet \mapsto A \otimes M$.

II.5 Model Categories

Main references are [Model Category and Simplicial Methods, Goerss], [Model Categories, Kan-tor], [Homotopy Theories and Model Categories, Dwyer/Spalinski], [Higher Topos Theory, Lurie].

Prop. (II.5.0.1). If $C$ is a class of morphisms in a category, we denote $l(C)$ the set of morphisms that has left lifting property w.r.t. all morphisms in $C$, and $r(C)$ the set of morphisms that have right lifting property w.r.t. all morphisms in $C$. Then $l(C)$ is stable under pushout and $r(C)$ is stable under base change.

Def. (II.5.0.2) (Model Structure). A model structure on a category $C$ is three classes of morphisms: fibrations, cofibrations and weak equivalences that satisfy the following axioms:

- M1 $C$ has finite limits and colimits.
- M2 (two out of three)If two of $f, g, fg$ is weak equivalence, then so is the third.
- M3 (retracts) Fibrations, cofibrations and weak equivalences are closed under retract.
- M4 (lifting property)We have a lifting property with a cofibration $i$ and fibration $p$ when either of them is a weak equivalence.
- M5 (factorization)Any map $f$ can be factored as $pi$ where $i$ is trivial cofibration and $p$ is a fibration, and also as $pi$ where $i$ is a cofibration and $p$ is a trivial fibration.

Remark (II.5.0.3). Notice the axioms are symmetric in fibrations and cofibrations, thus the opposite category $C^{op}$ has a natural model structure. So whenever we write a theorem, we should always remember its dual counterpart.

Lemma (II.5.0.4) (Closedness). A model category satisfies retraction axiom iff:

- fibration= $r$(trivial cofibrations),
- cofibration= $l$(trivial fibrations),
- weak equivalence= $uv$, where $v \in l$(fibrations) and $u \in r$(cofibrations).

Proof: If these are satisfied, retraction axiom is easy: A retract satisfies the same lifting properties. Hence retraction of a (co)fibration is a (co)fibration. For retracts weak equivalences, Cf.[Quillen, Homotopical Algebra, Chap5.2].

Conversely, using (II.1.6.7), we first factorize a $p = f \circ i$, where $i$ is a trivial cofibration, then because $p \in r(i)$, pis a retraction of $f$ hence a fibration. And similarly for cofibrations and weak equivalences.

Cor. (II.5.0.5). In a model category,

- trivial fibrations= $r$(cofibrations),
- trivial cofibrations= $l$(fibrations).

Proof: The proof is the same as that of (II.5.0.4).

Cor. (II.5.0.6). In a model category, the class of (trivial)fibrations is stable under base change and the class of (trivial)cofibrations is stable under cobase change.

Prop. (II.5.0.7). Let $p$ be a fibration in $C_{cf}$, then $p \in r(Cof)$ iff $\gamma(p)$ is an isomorphism, Cf.[Quillen 5.2]. So if conditions of (II.5.0.4) are satisfied (i.e. $C$ is a closed model category), $\gamma(f)$ is an isomorphism iff $f$ is a weak equivalence by the charaterization of weak-equivalence of (II.5.0.4).
II.5. MODEL CATEGORIES

Proof:

Def. (II.5.0.8) (Proper). A model category is called left proper if weak equivalences are stable under co-base change by cofibrations. Dually it is called right proper if weak equivalences are stable under base change by fibrations.

Lemma (II.5.0.9) (Cofibration is Left Proper). For a pushout diagram

\[
\begin{array}{ccc}
A & \xrightarrow{i} & B \\
\downarrow{j} & & \downarrow{j'} \\
A' & \xrightarrow{i'} & B'
\end{array}
\]

in a model category \(\mathcal{C}\), if \(i\) is cofibration and \(A, A'\) are cofibrant, and \(j\) is weak equivalence, then \(j'\) is also weak equivalence.

Proof: It suffices to show \(j'\) is an isomorphism in \(\text{Ho}(\mathcal{C})\). For this, by Yoneda lemma, it suffices to show \(\text{Hom}_{\text{Ho}(\mathcal{C})}(B', Z) \cong \text{Hom}_{\text{Ho}(\mathcal{C})}(B, Z)\) for any fibrant object \(Z\).

For surjectivity, by (II.5.1.18), it suffices to show \(\pi(B', Z) \cong \pi(B, Z)\). Given a map \(f : B \to Z\), because \(j\) is weak equivalence, there is a map \(g : A' \to Z\) that \(g \circ j \sim f \circ i\). Then by (II.5.1.10), there is a \(f' \sim f\) that \(f' \circ i = g \circ j\), which determines a morphism \(B' \to Z\).

For injectivity, if \(P\) is a path object that \(H : B \to P\) induces a homotopy between maps \(s \circ j', s' \circ j'\), then we need to extend this homotopy to \(H' : B' \to P\), and the method is the argument is the same as above.

1 Homotopies

Def. (II.5.1.1) (Cylinder Objects). A cylinder object for an object \(X\) is an object \(X \land I\) which gives a factorization of the natural map \(X \coprod X \to X\) as \(X \coprod X \xrightarrow{j} X \land I \xrightarrow{\partial_i} X\), where \(j \in W\). It is called a good cylinder object if \(i\) is a cofibration, and very good cylinder object if \(j\) is trivial fibration. By factorization axiom, every object has a very good cylinder object.

There are two natural morphisms \(X \to X \land I\), denoted by \(\partial_0\) and \(\partial_1\).

Dually we can define path object \(Y^I\) for \(Y\), and every object has a very good path object.

Prop. (II.5.1.2). If \(A\) is cofibrant and \(A \land I\) is a cylinder object for \(A\), then \(\partial_i : A \to A \times I\) are trivial cofibrations.

Proof: Because it’s pushout of \(\varnothing \to A\) and \(\sigma \circ \partial_i = \text{id}_A\).

Cor. (II.5.1.3). If \(f \sim^l g\), then \(f\) is a weak equivalence iff \(g\) is a weak equivalence.

Proof: This is because \(f = H \circ \partial_0, g = H \circ \partial_1\), and we can use (II.5.1.2).

Def. (II.5.1.4) (Homotopies). Two morphisms \(f, g : X \to Y\) are called (good/very good) left homotopic, denoted by \(f \sim^l g\) iff there is a (good/very good) cylinder object \(X \coprod X \to X \land I\) with \(X \land I \to Y\) that induce \((f, g) : X \coprod X \to Y\). Dually for right homotopies. And we denote by \(\pi^l(A, B) / \pi^r(A, B)\) the equivalence classes of \(\text{Hom}(A, B)\) under the equivalence relation generated by left/right homotopies.

If \(f, g \in \text{Hom}(X, Y)\) and \(\varphi \in \text{Hom}(Y, Z)\) that \(\varphi \circ f = \varphi \circ g\), then \(f, g\) is called left homotopic over \(Z\) if there is a homotopy \(H\) of \(f \sim^l g\) that \(\varphi H\) is the trivial homotopy. Dually for right homotopic under \(X\).
Lemma (II.5.1.5) (Very Good Homotopies). For \( f, g \in \text{Hom}(X,Y) \), if \( f \sim^l g \), then \( f, g \) are good left homotopic. And if \( Y \) if fibrant, then \( f, g \) are moreover very good left homotopic.

**Proof:** The first assertion is each, just choose a factorization of the cylinder object \( X \coprod X \xrightarrow{j} X \land I \) that \( i \) is cofibrant. If \( Y \) is fibrant, then we further factorize \( X \land I' \xrightarrow{j} X \land I'' \xrightarrow{i} X \), where \( i \) is cofibrant and \( j \) is trivial fibration, then by two out of three, \( i \) is also trivial cofibration, and it suffices to extend the homotopy \( X \land I' \to Y \) to \( X \land I'' \to Y \), and this is because \( Y \) is fibrant. \( \square \)

Prop. (II.5.1.6) (Homotopy is Equivalence Relation). If \( A \) is cofibrant, then the left homotopy is an equivalence relation on \( \text{Hom}(A,B) \).

**Proof:** Reflexivity and symmetry is trivial, the only problem is transitivity, so we construct a glueing \( A \land I'' \) as the pushout of \( \partial_1 : A \to A \land I \) and \( \partial_0 : A \to A \land I' \). \( A \land I'' \to A \) is a weak equivalence by the universal property and (II.5.0.4), so this is a cylinder object. The rest is easy. \( \square \)

Prop. (II.5.1.7) (Properties of Left Homotopies). If \( A \) is cofibrant and \( f, g \in \text{Hom}(A,B) \), then

1. If \( f, g \) are right homotopic, then \( s \to B^I \) can be chosen to be trivial Cof.
2. If \( f, g \) are right homotopic, then so does \( uf \sim ug \) or \( fv \sim gv \). Thus if \( A \) is cofibrant, there is a composition map: \( \pi^r(A,B) \times \pi^r(B,C) \to \pi^r(A,C) \).
3. For any trivial fibration \( X \to Y \), \( \pi^l(A,X) \to \pi^l(A,Y) \) is a bijection. And dual arguments hold for fibrant objects.

**Proof:**

1. factorize \( B \to B^I \) to \( B \to B'' \to B^I \) where \( B \to B'' \in T\text{Cof} \) and \( B'' \to B^I \in W \), so \( B'' \) is also a cylinder object and the homotopy \( A \to B^I \) can be lifted to \( A \to B'' \).

2. there is a diagram \( \xymatrix{ B \ar[r]^{ssu} \ar[d]_{s} & C^I \ar[d]^{(d_0,d_1)} \ar[r] & \times C } \) which has a lifting \( \varphi \), then composed with \( A \to B^I \) will give the desired homotopy.

3. the map is well-defined, it is surjective because of lifting property, and it is injective because \( A \coprod A \to A \times I \in Cof \) so the homotopy can be lifted to \( X \). Cf.[Homotopy Theories and Model Categories, P20, 21]. \( \square \)

Lemma (II.5.1.8) (Left and Right Homotopies). If \( X \) is cofibrant, then for \( f, g \in \text{Hom}(X,Y) \), if \( f \sim^l g \), then \( f \sim^r g \). And the dual conclusion holds for \( Y \) fibrant.

In particular, for morphisms between bifibrant objects, left and right homotopic are equivalent.

**Proof:** Consider a cylinder object \( j : X \land I \to X \) for \( X \) and a path object for \( Y \). Suppose \( f, g \) are left homotopic via a map \( H : X \land I \to Y \), then we consider the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow^{d_0} & & \downarrow \ \\
X \land I & \xrightarrow{(f \circ j) \times H} & Y \times Y
\end{array}
\]
Then it can be solved by some \( \tilde{H} \) because \( \partial_1 \) is trivial cofibration\(^{\text{II.5.1.2}}\), and then it can be checked \( H \circ \partial_1 \) gives the desired right homotopy. \( \square \)

**Prop. (II.5.1.9) (Whitehead’s Theorem).** Let \( X, Y \in \mathcal{C} \) be bifibrant, then a map \( f : X \to Y \) is an equivalence iff there is a \( g : X \to Y \) that \( fg \) and \( gf \) are homotopic to id.

**Proof:** Cf.[Homotopy Theories and Model Categories, P23]. \( \square \)

**Prop. (II.5.1.10) (Lifting Criterion).** Let \( \mathcal{C} \) be a model category and \( i : A \to B \) be a cofibration between cofibrant objects, and \( X \) is fibrant, \( g : B \to X, f : A \to X \) satisfies \( g \circ i \sim f \), then there is a \( g' \sim g \) that \( g' \circ i = f \).

**Proof:** Choose a good cylinder object \( C(A) \) for \( A \) and factorize

\[
C(A) \coprod_{A \coprod A} (B \coprod B) \to C(B) \to B
\]

where the first map is cofibration and the second is trivial fibration, then \( C(B) \) is a good cylinder object for \( B \).

The homotopy is given by a map \( C(A) \coprod_A B \to X \), and we check \( (A \coprod A) B \to C(B) \) is a trivial cofibration\(^{\text{II.5.0.6}}\) and \( C(A) \coprod_A B \to C(A) \coprod_A \coprod_A (B \coprod B) \) is a cofibration as a cobase change of \( A \to B \) and \( C(A) \coprod_A \coprod_A (B \coprod B) \to C(B) \) is a cofibration by definition), so the homotopy extends to a homotopy \( C(B) \to X \), which is a homotopy between \( g \) and some \( g' \) and \( g' \circ i = f \). \( \square \)

**Def. (II.5.1.11).** Let \( C_c, C_f, C_{cf} \) denote the full subcategory of cofibrant, fibrant and cofibrant-fibrant objects. And we define \( \pi C_c \) as the category module right homotopy equivalence between morphisms, dually for \( \pi C_f \).

Notice\(^{\text{II.5.1.7}}\) assures \( \pi C_c, \pi C_f \) are truly categories.

Notice for \( C_{cf} \), left homotopy is equivalent to right homotopy by\(^{\text{II.5.1.8}}\), so \( \pi C_{cf} \) is full subcategory for both \( \pi C_c \) and \( \pi C_f \).

**Lemma (II.5.1.12) (Fibrant and Cofibrant Replacement).** For an object \( X \) in a model category \( \mathcal{C} \), the axioms show there is a cofibrant object \( QX \) and a trivial fibration \( QX \to X \). Also there is a fibrant object \( RX \) and a trivial cofibration \( X \to RX \). We fix choices of \( Q, R \) that is identity on bifibrant objects, and consider it a mapping from \( \mathcal{C} \) to \( \mathcal{C} \).

Then given any morphism \( f : X \to Y \), there is a morphism \( \tilde{f} : QX \to QY \) lifting \( f \), and \( \tilde{f} \) depends up to left and right homotopy only on \( f \). And if \( Y \) is fibrant, then it depends up to left and right homotopy only on the left homotopy classes of \( f \).

Dually assertions also holds, so we have functors: \( Q : \mathcal{C} \to \pi C_c \) and \( R : \mathcal{C} \to \pi C_f \).

**Proof:** The existence of the lifting follows from the fact \( QX \) is fibrant and \( QY \to Y \) is trivial fibration. The uniqueness of left homotopy follows from\(^{\text{II.5.1.7}}\), and also right homotopy, because \( QX \) is cofibrant and use\(^{\text{II.5.1.8}}\). For the last assertion, notice when \( Y \) is fibrant,\(^{\text{II.5.1.7}}\) shows the left homotopy class of \( QX \to Y \) is determined, and use\(^{\text{II.5.1.7}}\) again, the class of \( \tilde{f} \) is also determined. \( \square \)

**Cor. (II.5.1.13).** The restrictions define functors \( Q' : \pi C_c \to \pi C_{cf} \) and \( R' : \pi C_f \to \pi C_{cf} \).
Def. (II.5.1.14) (Homotopy Category). For any model category $\mathcal{C}$, we construct a homotopy category $\operatorname{Ho}(\mathcal{C})$ whose objects are the same as $\mathcal{C}$, but $\operatorname{Hom}_{\operatorname{Ho}(\mathcal{C})}(X,Y) = \pi(C_{\mathcal{C}} f(RQX, RQY))$.

There is a functor $\gamma : \mathcal{C} \to \operatorname{Ho}(\mathcal{C})$ by sending $X$ to $RQX$, by (II.5.1.12).

Prop. (II.5.1.15) (Weak Equivalence and Isomorphisms). A morphism in $\mathcal{C}$ maps to an isomorphism in $\operatorname{Ho}(\mathcal{C})$ iff it is a weak equivalence. The morphisms in $\operatorname{Ho}(\mathcal{C})$ are generated by the image of morphisms in $\mathcal{C}$ and the inverse of images of weak equivalences in $\mathcal{C}$.

Proof: If $f \in \mathcal{C}$ is a weak equivalence, then $f' = RQ(f)$ is also a weak equivalence, by two out of three lemma. Then Whitehead theorem (II.5.1.9) shows $f'$ is an isomorphism in $\pi(C_{\mathcal{C}} f)$ hence in $\operatorname{Ho}(\mathcal{C})$. Conversely, if $f'$ has an inverse in $\operatorname{Ho}(\mathcal{C})$, then $f'$ is a weak equivalence by Whitehead (II.5.1.9) again, and so is $f$.

For the last assertion, just notice $\operatorname{Hom}(RQX, RQY) \to \operatorname{Hom}_{\operatorname{Ho}(\mathcal{C})}((X,Y)$ is a surjection, and $X \to RQX, Y \to RQY$ are weak equivalence hence are isomorphisms in $\operatorname{Ho}(\mathcal{C})$.

Cor. (II.5.1.16). If $F, G : \operatorname{Ho}(\mathcal{C}) \to \mathcal{D}$ are two functors and $t : F \circ \gamma \to G \circ \gamma$ is a natural transformation, then $t$ also gives a natural transformation $F \to G$.

Proof: This is because the objects of $\operatorname{Ho}(\mathcal{C})$ are the same as that of $\mathcal{C}$, and the morphisms are generated by $\gamma(f)$ and $\gamma(g)^{-1}$ where $g$ is a weak equivalence. Then the desired transformation commutative diagrams commute.

Lemma (II.5.1.17). Let $\mathcal{C}$ be a model category and $F : \mathcal{C} \to \mathcal{D}$ be a functor taking weak equivalences to isomorphisms, then if $f \sim^l g$ or $f \sim^r g$, $F(f) = F(g)$.

Proof: We only prove for left homotopy and the right homotopy is dual: given the cylinder object $A \land I$, just need to prove that $F(\partial_0) = F(\partial_1)$.

Prop. (II.5.1.18). Suppose $A$ is cofibrant and $X$ is fibrant, then the map $\gamma : \operatorname{Hom}(A, X) \to \operatorname{Hom}_{\operatorname{Ho}(\mathcal{C})}(A, X)$ is surjective, and induces a bijection $\pi(A, X) \cong \operatorname{Hom}_{\operatorname{Ho}(\mathcal{C})}(A, X)$.

Proof: (II.5.1.17) shows $\gamma$ identifies homotopic maps. Consider the following commutative diagram:

\[
\begin{array}{ccc}
\pi(RA, QX) & \longrightarrow & \pi(A, X) \\
\downarrow_{\gamma} & & \downarrow_{\gamma} \\
\operatorname{Hom}_{\operatorname{Ho}(\mathcal{C})}(RA, QX) & \longrightarrow & \operatorname{Hom}_{\operatorname{Ho}(\mathcal{C})}(A, X)
\end{array}
\]

The second vertical arrow is isomorphism by (II.5.1.15), the first arrow is isomorphism by (II.5.1.7). The left vertical arrow is identity by construction, so the right vertical arrow is also isomorphism.

Prop. (II.5.1.19) (Homotopy Category as Localizing Category). Let $\mathcal{C}$ be a model category and $W$ the class of weak equivalences, then the functor $\gamma : \mathcal{C} \to \operatorname{Ho}(\mathcal{C})$ is the localizing category of $\mathcal{C}$ w.r.t $S$.

Proof: Cf.[Homotopy Theories and Model Categories, P29].
2 Quillen Functors and Derived Functors

Def. (II.5.2.1) (Quillen Adjunctions). An adjoint pair of functors \((F, G)\) between model categories is called a Quillen adjunction if \(F\) preserves cofibrations and \(G\) preserves fibrations. By adjointness and (II.5.0.4), in fact \(F\) preserves also trivial cofibrations and \(G\) preserves trivial fibrations.

It is called a Quillen equivalence if for any cofibrant object \(C\) and fibrant object \(D\), a map \(C \to G(D)\) is a weak equivalence iff the adjoint map \(F(C) \to D\) is a weak equivalence.

Prop. (II.5.2.2) (Quillen). The geometrization functor and the singular complex functor defines a Quillen equivalence.

\( | - | : S\Delta \xrightarrow{\cong} CG : S(-)\)

where the RHS is Serre-Quillen model category.

Proof: □

Def. (II.5.2.3) (Derived Functors). Let \(\mathcal{C}\) be a model category and \(F : \mathcal{C} \to \mathcal{D}\) a functor, then a left derived functor of \(F\) is a left Kan extension of \(F\) along \(\gamma : \mathcal{C} \to \text{Ho}(\mathcal{C})\).

Dually we can define right derived functors.

Prop. (II.5.2.4) (Existence of Derived Functors). In the situation of (II.5.2.3), if \(F\) maps weak equivalences between cofibrant objects to isomorphisms in \(\mathcal{D}\), then the left derived functor \((LF, t)\) exists, and for each cofibrant object \(X\), the morphism \(t_X : LF(X) \to F(X)\) is an isomorphism.

Dually for the right derived functor case.

Proof: Cf. [Homotopy Theories and Model Categories, P42]. □

Lemma (II.5.2.5). Let \(\mathcal{C}\) be a model category and \(F : \mathcal{C}_c \to \mathcal{D}\) be a functor that maps trivial cofibrations in \(\mathcal{C}_c\) to isomorphisms, then \(F\) maps right-homotopic morphisms to the same morphism.

Proof: Let \(H : A \to B^I\) be a right homotopy between \(f\) and \(g\), where \(B^I\) is a very good path object (II.5.1.5), then \(B \to B^I\) is a trivial cofibration, and thus mapped by \(F\) to an isomorphism. Then we can show \(F(\partial_0) = F(\partial_1)\), and then \(F(f) = F(g)\). □

Def. (II.5.2.6) (Total Left Derived Functors). Let \(F : \mathcal{C} \to \mathcal{D}\) be a morphism of model categories, then a total derived functor

\[LF : \text{Ho}(\mathcal{C}) \to \text{Ho}(\mathcal{D})\]

is defined to be a left derived functor of the morphism \(\mathcal{C} \to \mathcal{D} \to \text{Ho}(\mathcal{D})\).

Lemma (II.5.2.7) (Brown). Let \(F\) be a morphism of model categories that maps trivial cofibration between cofibrant objects to weak equivalences, then it preserves weak equivalences between cofibrant objects.

Proof: If \(f : A \to B\) is a weak equivalence between cofibrant objects, then we can factor the morphism \((f, \text{id}) : A \coprod B \to B\) as \(A \coprod B \xrightarrow{q} C \xrightarrow{p} B\) that \(q\) is cofibration and \(p\) is trivial fibration. It can be shown that \(q \circ \partial_i : B \to C\) are trivial fibrations and \(C\) is cofibrant, thus \(F(q \circ \partial_i)\) are weak equivalences, and hence also \(F(p)\) is weak equivalence and so does \(F(f)\). □
Prop. (II.5.2.8) (Total Derived Functors and Quillen Equivalence). If $(F, G)$ is a pair of Quillen functors between two model categories $C, D$, then total derived functors

$$LF : \text{Ho}(C) \rightleftarrows \text{Ho}(D) : RG$$

exists and form an adjoint pair. And if $(F, G)$ is a Quillen equivalence, then $(LF, RG)$ defines an equivalence of homotopy categories.

Proof: By (II.5.2.4) and its dual and (II.5.2.7), the total derived functors $LF, RG$ exist.

Next for $A$ cofibrant in $C$ and $X$ fibrant in $D$, we can show the adjunction map $\text{Hom}(A, G(X)) \cong \text{Hom}(F(A), X)$ preserves homotopy equivalence relations (II.5.1.8) and induces an isomorphism $\pi(A, G(X)) \cong \pi(F(A), X)$: If $H : A \land I \to X$ is a good homotopy between $f, g$, then $A \land I$ is cofibrant, and because $F$ preserves colimits and because of (II.5.2.7), $F(A \land I)$ is cylinder object for $F(A)$, thus $f^\flat \sim g^\flat$. A dual argument shows the converse.

Now for any $A \in C$ and $X \in D$, there is a bijection

$$\text{Hom}_{\text{Ho}(C)}(A, RG(X)) \cong \pi(QA, G(SX)) \cong \pi(F(QA), SX) \cong \text{Hom}_{\text{Ho}(D)}(LFA, X)$$

where the first isomorphism is due to the fact $QA \to RQA$ is trivial cofibration and $G(SX)$ is fibrant, thus we can use (II.5.1.7), dually for the last isomorphism.

Finally if $(F, G)$ is a Quillen equivalence, then consider the unit map:

$$A \to RG(LF(A)).$$

If $A$ is cofibrant, then this is $A \to G(SF(A))$ which is a weak equivalence because $F(A) \to SF(A)$ does, so it is an isomorphism in $\text{Ho}(C)$. Now any object in $\text{Ho}(C)$ is isomorphic to a cofibrant object, we know the unit map is an isomorphism. Dually the counit map is an isomorphism, thus $LF, GF$ are a pair of equivalences.

3 Combinatorial Model Structure

Cf. [HTT, A.2.6]

Def. (II.5.3.1) (Combinatorial Model Category). A cofibrantly generated model category is a model category $C$ that

- there is a set $I$ of generating cofibrations that generates the class of cofibrations as the minimal weakly saturated class containing $I$.
- there is a set $J$ of generating trivial cofibrations that generates the class of trivial cofibrations as the minimal weakly saturated class containing $J$.

A combinatorial model category is a cofibrantly generated and presentable model category.

4 Generating new Model Categories

Prop. (II.5.4.1). If $C$ is a model category and $A$ is an object, then the undercategory $C_{A/}$ and the overcategory $C_{/A}$ have natural model structures.

Prop. (II.5.4.2) (transfer Model Structure via Left Adjoint). Let

$$F : C \rightleftarrows D : G$$

be an adjoint pair of categories and
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• $\mathcal{C}, \mathcal{D}$ are complete and cocomplete,
• $\mathcal{C}$ is a cofibrantly generated model category,
• $\mathcal{C}, \mathcal{D}$ are presentable categories and $G$ is an accessible functor.

• If we define a morphism $f$ in $\mathcal{B}$ a fibration/weak equivalent iff $G(f)$ is a fibration/weak equivalence, and a cofibration iff it has left lifting property w.r.t. trivial fibrations, then $\mathcal{B}$ has a path object factorization and a fibrant replacement operator.

Then this defines a cofibrantly generated model category on $\mathcal{B}$, and makes $(F, G)$ a Quillen adjunction.

Proof: It suffices to show that the factorization property holds: In fact, it suffices to show if $I, J$ are generating classes of cofibrations and trivial cofibrations, then $F(I), F(J)$ are generating classes of cofibrations and trivial cofibrations. But this suffices to show $F(J)$ is are weak equivalence. For this, show that any morphism in $\mathcal{B}$ that has left lifting properties w.r.t. all fibrations is a weak equivalence using hypothesis item4.

□

5 Diagram Categories and Homotopy (Co)Limits

Cf.[HTT, A.3.3, A.3.5].

Def. (II.5.5.1). Let $\mathcal{C}$ be a small $\mathcal{S}$-enriched category and $\mathcal{A}$ a combinatorial $\mathcal{S}$-enriched model category, then a natural transformation $\alpha : F \to G$ in $\mathcal{A}^\mathcal{C}$ is called

• an injective cofibration if $F(C) \to G(C)$ is a cofibration in $\mathcal{A}$ for each $C$.
• a projective fibration if $F(C) \to G(C)$ is a fibration in $\mathcal{A}$ for each $C$.
• a weak equivalence if $F(C) \to G(C)$ is a weak equivalence in $\mathcal{A}$ for each $C$.

Prop. (II.5.5.2). Let $\mathcal{S}$ be an excellent model category and $\mathcal{A}$ a combinatorial $\mathcal{S}$-enriched model category, then there are two model structures on $\mathcal{A}^\mathcal{C}$:

• The projective model category determined by the projective cofibrations and weak equivalences,
• The injective model category determined by the injective cofibrations and weak equivalences.

Proof: Cf.[HTT, P868].

□

Prop. (II.5.5.3). Let $\mathcal{C}$ be a model category and let

$$
\begin{array}{ccc}
A & \xrightarrow{j} & A_1 \\
\downarrow^i & & \downarrow \\
A_0 & \longrightarrow & \bigoplus_A A_1
\end{array}
$$

be a pushout diagram, then it is a homotopy pushout diagram if either of the following is satisfied:

• $j$ is a cofibration and $A, A_0$ are cofibrant.
• $j$ is a cofibration and $\mathcal{C}$ is left proper.

Proof: □
6 Enriched and Monoidal Model Categories

Def. (II.5.6.1) (Left Quillen Bifunctor). Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be model categories, then a functor $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is called a left Quillen bifunctor if

- For any cofibrations $i: A \to A' \in \mathcal{A}, B \to B' \in \mathcal{B}$, the induced map
  $i \wedge j : F(A', B) \coprod_{F(A, B)} F(A, B') \to F(A', B')$

  is a cofibration in $\mathcal{C}$.
- $F$ preserves small colimits separably in each variables.

Def. (II.5.6.2) (Monoidal Model Category). A monoidal model category is a monoidal category $\mathcal{S}$ equipped with a model structure that:

- The tensor product $\otimes: \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ is a left Quillen bifunctor.
- The unit objects $1 \in \mathcal{S}$ is cofibrant.
- The monoidal structure is closed.

Def. (II.5.6.3) (Enriched Model Category). Given a monoidal model category $\mathcal{S}$, an $\mathcal{S}$-enriched model category is an $\mathcal{S}$-enriched category $\mathcal{A}$ with a model structure satisfying:

- $\mathcal{A}$ is tensored and cotensored over $\mathcal{S}$.
- the tensor product $\mathcal{A} \times \mathcal{S} \to \mathcal{S}$ is a left Quillen bifunctor (II.5.6.1).

Prop. (II.5.6.4). The second condition in (II.5.6.3) is equivalent to the following: For a cofibration $i: U \to V$ and a fibration $p: X \to Y$, the induced map

$$\text{Map}(V, X) \xrightarrow{(i^*, p_*)} \text{Map}(U, X) \times_{\text{Map}(U, Y)} \text{Map}(V, Y)$$

is a fibration in $\mathcal{S}$, and trivial fibration if any of $i, p$ is weak equivalence.

Proof: Use the adjunction relations to write it out. \qed

Def. (II.5.6.5) (Fibrant Enriched Categories). Let $\mathcal{A}$ be a $\mathcal{S}$-enriched category, the we denote $\mathcal{A}^\circ$ the subcategory of bifibrant objects of $\mathcal{A}$, which is also a $\mathcal{S}$-enriched category.

Prop. (II.5.6.6). Let $\mathcal{C}, \mathcal{D}$ be $\mathcal{S}$-enriched model categories and

$$F: \mathcal{C} \rightleftarrows \mathcal{D}: G$$

is a Quillen adjunction of underlying model categories. Assume every objects of $\mathcal{C}$ is cofibrant and the maps $\beta_{X,S}: S \otimes F(X) \to F(S \otimes X)$ is a weak equivalence for $X \in \mathcal{C}, S \in \mathcal{S}$ cofibrant, then the following are equivalent:

- $(F, G)$ is a Quillen equivalence.
- $G$ determines a weak equivalence of the underlying $\mathcal{S}$-enriched categories $\mathcal{D}^\circ \to \mathcal{C}^\circ$.

Proof: Cf.[HTT, P853]. \qed

Lemma (II.5.6.7). Let $\mathcal{S} = \text{Set}_\Delta$, then the map $\beta_{X,S}$ is a weak equivalence for every cofibrant object $X \in \mathcal{C}$.
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**Proof:** Cf.[HTT, P853]. □

**Prop. (II.5.6.8).** Let $\mathcal{C}, \mathcal{D}$ be simplicial model categories and $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a Quillen equivalence, and every element of $\mathcal{C}$ is cofibrant, then $G$ induces an equivalence of $\infty$-categories $N(\mathcal{D}^\circ) \cong N(\mathcal{C}^\circ)$.

**Proof:** This is because $G : \mathcal{D}^\circ \to \mathcal{C}^\circ$ is an equivalence of simplicial categories, by (II.5.6.6) and (II.5.6.7), and then (II.6.4.14) shows $N(G) : N(\mathcal{D}^\circ) \to N(\mathcal{C}^\circ)$ is an equivalence of $\infty$-categories. □

7 **$S$-enriched Categories**

**Def. (II.5.7.1) (Homotopy Category).** Let $S$ be a monoidal model category, then there is a natural monoidal structure on the homotopy category $hS$ (II.5.1.14), and the functor $S \to hS$ is monoidal, thus we can transfer from a category $C$ enriched over $S$ to an $hS$-enriched category, called the **homotopy category** of $C$.

**Def. (II.5.7.2) (Weak Equivalences).** Let $S$ be a monoidal model category and $\text{Cat}_S$ be the category of categories enriched over $S$, then a morphism in $\text{Cat}_S$ is called a **weak equivalence** if it induced an isomorphism of their homotopy categories.

**Def. (II.5.7.3) (Generating Cofibrations in $\text{Cat}_S$).** Let $S$ be a monoidal model category, $A$ is an object of $S$, then we can denote $[1]_A$ the $S$-enriched category consists of objects $\{X, Y\}$ and that $\text{Hom}(X, X) = \text{Hom}(Y, Y) = 1_S$, $\text{Hom}(X, Y) = A$, $\text{Hom}(Y, X) = \emptyset$. And if $1_S$ is the initial object of $S$, then we denote $[1]_1$ by $[1]_S$. Also we denote $[0]_S$ the $S$-enriched category consisting of one element and the morphism space is $1_S$.

Let $\widetilde{[1]}_S$ be the category consisting of two objects $\{X, Y\}$ that $\text{Hom}(Z_1, Z_2) = 1_S$ for any $Z_1, Z_2 \in S$.

Let $C_0$ be the class of morphisms in $\text{Cat}_S$ consisting of

- $\emptyset \to [0]_S$.
- The induced map $[1]_S \to [1]_{S'}$ where $S \to S'$ ranges over a generating class of cofibrations of $S$.

**Prop. (II.5.7.4) (Model Category on $\text{Cat}_S$).** Let $S$ be a combinatorial monoidal model category that every object of $S$ is cofibrant and the collection of weak equivalences of $S$ is stable under filtered colimits, then there exists a left proper combinatorial model structure on $\text{Cat}_S$ that

- The class of cofibrations in $\text{Cat}_S$ is the smallest weakly saturated class generated by $C_0$ defined in (II.5.7.3),
- The weak equivalences are as defined in (II.5.7.2).

**Proof:** Cf.[HTT, P856]. □

**Cor. (II.5.7.5).** Let $f : S \rightleftarrows S' : g$ be a Quillen adjunction between monoidal model categories satisfying conditions in (II.5.7.4), then they induces a Quillen adjunction $F : \text{Cat}_S \rightleftarrows \text{Cat}_{S'} : G$

and this is a Quillen equivalence if $(f, g)$ is.
**Proof:**

\[\square\]

**Def. (II.5.7.6) (Local Fibrations).** Let \( C \) be an \( S \)-enriched category where \( S \) is a monoidal model category, then a morphism \( f \in C \) is called an **equivalence** if it maps to an isomorphism in \( hC \).

\( C \) is called **locally fibrant** if for any \( X, Y \in C \), the mapping space \( \text{Map}(X, Y) \) is fibrant in \( S \).

An \( S \)-enriched functor \( F : C \to C' \) is called a **local fibration** if the following conditions are satisfied:

- for any \( X, Y \in C \), the induced map \( \text{Map}(X, Y) \to \text{Map}(FX, FY) \) is a fibration in \( S \).
- the induced map \( hC \to hC' \) is a quasi-fibration of categories.

**Def. (II.5.7.7) (Invertibility Hypothesis).** We say a monoidal model category \( S \) satisfies the **invertibility hypothesis** if: For any cofibrant morphism \([1]_S \to C(II.5.7.3)\) of \( S \)-enriched categories, and maps to a morphism \( f \) which is invertible in the homotopy category \( hC \), take the pushout:

\[
\begin{array}{c}
[1]_S \\
\downarrow \\
[1]_S \\
\end{array}
\xrightarrow{i}
\begin{array}{c}
C \\
\downarrow \\
C\langle f^{-1} \rangle \\
\end{array}
\]

then \( j \) is a weak equivalence of \( S \)-enriched categories (II.5.7.2).

**Def. (II.5.7.8) (Excellent Model Category).** An **excellent model category** is a monoidal model category \( S \) that The monoidal structure is symmetric.

- \( S \) is combinatorial,
- Every monomorphism in \( S \) is a cofibration, and the collection of cofibrations is stable under products,
- The class of weak equivalences in \( S \) is stable under filtered colimits,
- \( S \) satisfies the invertibility condition (II.5.7.7)

**Lemma (II.5.7.9).** Let \( T : S \to S' \) be a monoidal functor between monoidal model categories satisfies axioms besides invertibility hypothesis, that is also a left Quillen functor, then if \( S' \) satisfies invertibility hypothesis, so does \( S \).

**Proof:** Cf.[HTT, P862]. \( \square \)

**Prop. (II.5.7.10) (Dwyer, Kan).** The category of simplicial sets is an excellent model category with the Kan model structure and the Cartesian monoidal structures.

**Proof:**

\[\square\]

**Prop. (II.5.7.11) (Fibration and Local Fibration).** If \( S \) is an excellent model category, then

- An \( S \)-enriched category \( C \) is a fibrant object in \( \text{Cat}_S \) iff it is locally fibrant (II.5.7.6).
- Let \( F : C \to D \) be an \( S \)-enriched functor and \( D \) is fibrant in \( \text{Cat}_S \), then \( F \) is fibrant in \( \text{Cat}_S \) iff it is a local fibration.

**Proof:** Cf.[HTT, P863]. \( \square \)
8 Examples

Prop. (II.5.8.1) (Kan Model Structure). The category of Simplicial sets $\text{Set}_\Delta$ is a combinatorial left and right proper model category with
- Weak homotopy equivalences as weak equivalences.
- Inclusions as cofibrations,
- Kan fibrations as fibrations,

Proof: Cf.[Jardine P62].

Prop. (II.5.8.2). $\text{Sets}_\Delta$ is a monoidal model category w.r.t the Cartesian product and the Kan model structure.

Proof: 

Prop. (II.5.8.3) ($q$-Model Structure). For a unital ring $R$, then the category $\text{Ch}_{n \geq 0} R$ has the structure of a model category with a morphism $f : M \to N$ being
- a weak equivalence if $H_n(f)$ is isomorphism for any $n$.
- a fibration if $M_n \to N_n$ is surjective for any $n \geq 1$.
- a cofibration if $M_n \to N_n$ is injective with projective cokernel for any $n \geq 0$.

Proof: [Model category and simplicial methods, P5] or [Homotopy Theories and Model Categories].

Prop. (II.5.8.4) (Serre-Quillen). The category $\text{Top}$ can be given a Serre-Quillen model structure with
- Weak equivalences: weak homotopy equivalence,
- Fibrations: Serre fibrations,
- Cofibrations: Retracts of morphisms $X \to Y$ where $Y$ is obtained from $X$ by attaching cells.

And this restricts to a model category on the category $\text{CGH}$ of compactly generated weak Hausdorff spaces.

Proof: See(IX.4.6.27).

Prop. (II.5.8.5) (Hurewicz-Strm). The category $\text{Top}$ can be given a Hurewicz-Strm model structure with
- Weak equivalences: homotopy equivalences.
- Cofibrations: closed Hurewicz cofibrations.
- Fibrations: Hurewitz fibrations.

Proof: See(IX.4.6.30).

Prop. (II.5.8.6) (Derived Category Model). If $\mathcal{A}$ is an Abelian category with enough injectives, then $K^+(\mathcal{A})$ is a model category with
- Weak equivalence: quasi-isomorphisms,
- Fibration: epimorphisms with Ker in $K^+(\mathcal{I})$,
- Cofibration: monomorphisms.
Proof:

Prop. (II.5.8.7) (Joyal). The

Prop. (II.5.8.8) (Reedy Model Structures). Cf.[HTT, A.2.9].
II.6 Simplicial Homotopy Theory

Main references are [Jardine Simplicial Homotopy Theory], [Lur09].

1 Simplicial Objects

Def. (II.6.1.1) (Simplex Category). The simplex category $\Delta$ consists of simplicial objects $[n]$ for each $n \geq 0$ and there maps are order-preserving maps.

$\Delta$ has a subcategory $\Delta_+$ consisting of the same objects but the morphisms are all surjective order-preserving maps.

For a category $\mathcal{A}$, a simplicial object in $\mathcal{A}$ is a functor from $\Delta^{op} \to \mathcal{A}$. A cosimplicial object in $\mathcal{A}$ is a functor from $\Delta \to \mathcal{A}$.

Given a simplicial or cosimplicial object in $\mathcal{A}$, its underlying degeneracy map is defined to be

Def. (II.6.1.2). $\Delta^n$ is the simplicial set $\Delta^n([m]) = \operatorname{Hom}([m], [n])$.

Def. (II.6.1.3) (Augmentation). If $X$ is a simplicial object in a category, then an augmentation of $X$ is a morphism $d: A \to X$ that $dd_0 = dd_1$. In case $\mathcal{C}$ is $\text{Mod}_R$, this is equivalent to a morphism $\pi_0(X) \to A$.

Def. (II.6.1.4) (s-Free Simplicial Objects). A simplicial object in a category $\mathcal{C}$ is called $s$-free if the underlying category $\Delta_+^{op} \to \mathcal{C}$ is $\Delta_+^{op}$-free (II.1.2.16).

Equivalently, an $s$-free object is a simplicial object $X$ that there are objects $Z_n \in \mathcal{C}$ that $X_n = \coprod_{\varphi: [n] \to [k]} \varphi^* Z_k$. Moreover, a simplicial morphism of simplicial objects are called $s$-free if the underlying diagram $X_+ \to Y_+$ is of the form $X_+ \to X_+ \coprod Y_0$ where $Y_0$ is $s$-free.

Prop. (II.6.1.5) ($s\mathcal{C}$ is a Simplicial Category). Let $s\mathcal{C}$ be the category of simplicial $\mathcal{C}$-objects, then it can be made into a simplicial category which is also tensored and cotensored (II.1.5.6) over $\mathcal{S}et_\Delta$.

Proof: We define first an action of $\mathcal{S}et_\Delta$ on $s\mathcal{C}$:

$$\otimes : \mathcal{S}et_\Delta \times s\mathcal{C} \to s\mathcal{C} : (K \otimes X)_n = \coprod_{K_n} X_n,$$

with the simplicial maps determined by that of $K$ and $X$.

Also there is a action of $\mathcal{S}et_\Delta^{op}$ on $s\mathcal{C}$:

$$(-)^- : \mathcal{S}et_\Delta^{op} \times s\mathcal{C} \to s\mathcal{S} : (K \otimes X)_n = \coprod_{K_n} X_n$$

with the simplicial maps determined by that of $K$ and $X$.

Then there is an adjointness

$$\operatorname{Hom}_{s\mathcal{C}}(K \otimes X, Y) \cong \operatorname{Hom}_{s\mathcal{C}}(X, Y^K)$$

for any simplicial set $K$.

Next we define $\operatorname{Map}_{s\mathcal{C}}(X, Y) \subset \mathcal{S}et_\Delta$ as $(\operatorname{Map}_{s\mathcal{C}}(X, Y))_n = \operatorname{Hom}_{s\mathcal{C}}(X \otimes \Delta^n, Y)$, then there are functorial isomorphisms

$$\operatorname{Map}_{s\mathcal{C}}(K \otimes X, Y) \cong \operatorname{Map}_{s\mathcal{C}}(X, Y^K) \cong \operatorname{Map}_{\mathcal{S}et_\Delta}(K, \operatorname{Map}_{s\mathcal{C}}(X, Y))$$

(easy to check). So we are done. \qed
2 Topological Categories

Def. (II.6.2.1). A topological category is a category that is enriched over the category \( \mathcal{CG} \) of compactly generated and Hausdorff spaces. The category of topological categories is denoted by \( \mathcal{Cat}_{top} \).

Two topological categories is called **strongly equivalent** if they are equivalent as enriched categories.

Def. (II.6.2.2) (Homotopy Category of Spaces). Given a topological category \( \mathcal{C} \), the homotopy category \( h\mathcal{C} \) of \( \mathcal{C} \) is defined to be the category transferred from \( \mathcal{C}(II.1.5.3) \) by the right-lax monoidal functor \( \pi_0(II.1.4.12) \).

Def. (II.6.2.3) (Homotopy Category of Spaces). Let \( \mathcal{C} \) be the category of CW complexes that the morphisms are given the compact-open topology, then its homotopy category \( \mathcal{H} \) is called the **homotopy category of spaces**.

3 Simplicial Sets

Simplicial Set

Remark (II.6.3.1). The fact that any simplicial set \( X \) is a colimit of \( \Delta^n \) (II.1.2.13) is important in proving properties of constructions of simplicial set.

Def. (II.6.3.2) (Objects, Morphisms and Equivalences). Given a simplicial set \( S \), its objects are just maps \( \Delta^0 \to S \), and its morphisms are maps \( \Delta^1 \to S \).

Def. (II.6.3.3) (Nerves). The nerve of a category \( \mathcal{C} \) is a simplicial set with \( (N(\mathcal{C}))_n = \text{Hom}([n], \mathcal{C}) \), i.e. composable arrows of morphisms of length \( n \). It is a fully faithful functor from the category of small categories to the category of simplicial sets.

Prop. (II.6.3.4) (Natural Transformation and Homotopy). A natural transformation will induce homotopic nerve map. thus a pair of adjoint functors will induce a simplicial homotopy between their nerve.

Prop. (II.6.3.5). The nerve construction is a fully faithful functor from \( \text{Cat} \to sSets \).

Prop. (II.6.3.6) (Characterization for Nerves). For any simplicial set \( K \), there is a small category \( \mathcal{C} \) that \( K \cong N(\mathcal{C}) \) iff for each diagram

\[
\begin{array}{ccc}
\Lambda^n & \to & K \\
\downarrow & & \\
\Delta^n & \rightrightarrows & \\
\end{array}
\]

there exists uniquely a dotted arrow.

Proof: [HTT, P9]. \( \square \)

Def. (II.6.3.7) \((\text{Sets}_\Delta \text{ and } \mathcal{CG})\). The geometrization of a simplicial object \( X \) is

\[
|X| = \lim_{\Delta^n \to X} \Delta_n.
\]
The **singular complex functor** maps a topological space $Y$ to a simplicial set

$$(\text{Sing}(Y))_n = \text{Hom}(\Delta_n, Y).$$

The geometrization functor is left adjoint to the singular functor as functors between the categories $\mathcal{C}G$ and $\text{Sets}_\Delta$ (use colimit definition of $X$). This is just the Kan extension in (II.1.2.14).

Moreover, the geometrization as a functor from $\text{Sets}_\Delta \to \mathcal{C}G$ preserves finite limits. Cf. [Jardine P9].

The three kinds of geometrization of a bisimplicial set is the same: geometrization the diagonal simplicial set, the twice geometrization of left (resp. right) simplicial set.

**Proof:**

**Def. (II.6.3.8) (Weak Homotopy Equivalence).** A morphism of simplicial sets $S \to T$ is called a **weak homotopy equivalence** if the induced map $|S| \to |T|$ is a weak homotopy equivalence.

**Prop. (II.6.3.9) (Quillen).** The geometrization functor and the singular complex functor defines a pair of Quillen functors between model categories:

$$|-| : \text{Sets}_\Delta \rightleftarrows \mathcal{C}G : (-)$$

**Proof:** See (II.5.2.2). □

**Cor. (II.6.3.10).** The localized category of $\mathcal{C}G$ and $\text{Sets}_\Delta$ at weak homotopy equivalence classes are the same, and it is just the homotopy category of spaces $\mathcal{H}$, by (II.5.2.8).

**Constructing Simplicial Sets**

**Def. (II.6.3.11) (Joins).** Let $S, S'$ be simplicial sets, then the **join** of $S, S'$ is defined to be the simplicial set that for all any finite ordered set $J$,

$$(S \star S')(J) = \bigcup_{J = I \coprod J'} S(I) \times S(I').$$

where $I, I'$ satisfies $i < i'$ for any $i \in I, i' \in I'$. And the glueing is natural.

Join clearly commutes with colimits.

**Prop. (II.6.3.12) (Monoidal Structure of $\text{Sets}_\Delta$).** The join operation makes the category of sets a monoidal category.

**Def. (II.6.3.13) (Cones).** For a simplicial set $K$, the **left/right cone** of $K$ are defined to be the join $K^\circ = \Delta^0 \star K$ and $K^\vee = K \star \Delta^0$.

For a map $f : X \to S$, the **left/right cone** of $f$ are defined to be $S \coprod_X X^\circ$ and $S \coprod_X X^\vee$.

**Def. (II.6.3.14) (Overcategories).** Let $p : K \to S$ be a morphism of simplicial sets, then there is a simplicial set $S/p$ that

$$\text{Hom}(Y, S/p) \cong \text{Hom}_p(Y \star K, S).$$

Where $\text{Hom}_p$ indicates that we only consider morphisms that $f|_K = p$. And dually we can define the **undercategories**.

**Proof:** We just define $(S/p)_n = \text{Hom}_p(\Delta^n \star K, S)$, then the condition holds for $\Delta^n$, and use the fact every simplicial set is a colimit of $\Delta^n$s (II.6.3.1), and both sides commutes with colimits. □
Prop. (II.6.3.15) (Generating Simplicial Sets). Let $\mathcal{U}$ be a collection of simplicial sets that:

- $\mathcal{U}$ is stable under isomorphisms,
- $\mathcal{U}$ is stable under disjoint union,
- $\Delta^n \subset \mathcal{U}$ for any $n$,
- If there is a pushout diagram $X \to X'$ and $X, X', Y \in \mathcal{U}$ and $f$ is a monomorphism, then $Y' \in \mathcal{U}$,
- suppose we are given a sequence of monomorphisms between objects in $\mathcal{U}$ indexed over $\mathbb{N}$, then the colimit belongs to $\mathcal{U}$.

**Proof:** Use induction on the dimension of $S$, and notice $S$ is glued together using their simplexes. □

Fibrations and Anodynes

Def. (II.6.3.16) (Fibrations). A morphism of simplicial sets is called a

- Kan fibration iff it has right lifting property w.r.t all $\Lambda_i^k \to \Delta^n$.
- left fibration iff it has right lifting property w.r.t. all inclusions $\Lambda_i^0 \subset \Delta^n, 0 \leq i < n$.
- right fibration iff it has right lifting property w.r.t all inclusions $\Lambda_i^n \subset \Delta^n, 0 < i \leq n$.
- inner fibration iff it has right lifting property w.r.t all inclusions $\Lambda_i^n \subset \Delta^n, 0 < i < n$.

A morphism of simplicial set is called a

- anodyne iff it has left lifting property w.r.t. all Kan fibrations.
- left anodyne iff it has left lifting property w.r.t. all left fibrations.
- right anodyne iff it has left lifting property w.r.t all right fibrations.
- inner anodyne iff it has right lifting property w.r.t all inner fibrations.

So a morphism between topological spaces $X \to Y$ is a Serre fibration iff $S(X) \to S(Y)$ is a Kan fibration(II.6.3.7).

Def. (II.6.3.17) (Trivial Fibration). A morphism $X \to S$ of simplicial sets that has right lifting property w.r.t. all inclusions $\partial \Delta^n \to \Delta^n$ is called a trivial fibration.

A cofibration is a morphism that has left lifting property w.r.t all trivial fibrations. By(II.1.6.10) a cofibration of simplicial sets is just an inclusion.

Lemma (II.6.3.18) (Join and Anodynes). If $f : A_0 \subset A$ and $g : B_0 \subset B$ are inclusions of simplicial sets, then

$$h : (A_0 \star B) \coprod_{A_0 \star B_0} (A \star B_0) \subset A \star B$$

is a(n)

- inner anodyne if either $f$ is right anodyne or $g$ is left anodyne, then
- left anodyne if $f$ is left anodyne.
II.6. SIMPLICIAL HOMOTOPY THEORY

Proof: 1: By symmetry we assume \( f \) is right anodyne. Notice the class of all morphisms \( f \) that the conclusion holds is weakly saturated because \( \star \) commutes with colimits, so it suffice to check for \( f : \Lambda^n_k \subset \Delta^n \). Then similarly it suffices to check for \( g : \partial \Delta^n \subset \Delta^m \), but then the inclusion is just \( \Lambda^{m+n+1}_j \subset \Delta^{m+n+1} \), which is left anodyne.

2 is similar. \( \square \)

Prop. (II.6.3.19) (Anodynes).
- The saturated class generated by either of the following three classes of monomorphisms are both left anodynes:
  1. \( \Lambda^n_k \subset \Delta^n, \ 0 \leq k < n \).
  2. \( (\Delta^m \times \{0\}) \coprod (\partial \Delta^m \times \Delta^1) \subset \Delta^m \times \Delta^1 \).
  3. \( (S' \times \{0\}) \coprod (S \times \Delta^1) \subset S' \times \Delta^1 \), where \( S \subset S' \).

Similar conclusion holds for right anodynes and together implies the similar conclusion for anodynes.

Proof:
- 2 and 3 are equivalence because any inclusion comes from attaching cells (II.6.3.3).
- Now inclusions in 2 are compositions of pushouts of inclusions \( \Lambda^{n+1}_k \subset \Delta^{n+1} \), where \( 0 \leq k \leq n \), thus it is generated by 1. Conversely, \( \Lambda^n_k \subset \Delta^n \) is a retract of \( (\Delta^n \times \{0\}) \coprod (\Lambda^n_k \times \Delta^1) \subset \Delta^n \times \Delta^1 \):
  Cf. [HTT, P64].

Cor. (II.6.3.20) (Products and Anodynes). Let \( A \subset A' \) be a(n) left(inner) anodyne and \( B \subset B' \), then so does the induced map

\[
(A \times B') \coprod_{A \times B} (A' \times B) \to A' \times B'.
\]

Proof: For left anodyne, the proof is similar to that of (II.6.3.18), just check for classes 3 of (II.6.3.19), and use the fact

\[
(S' \times \Delta^1) \times B \coprod (S' \times \{0\}) \coprod (S \times \Delta^1) \times B' \to (S' \times \Delta^1) \times B'
\]

is just

\[
(S' \times B \coprod S \times B') \times \Delta^1 \coprod (S' \times B') \times \{0\} \to (S' \times B') \times \Delta^1.
\]

which is left(inner) anodyne. And similarly for inner anodynes. \( \square \)

Left Fibration

Remark (II.6.3.21) (Left and Right Fibrations Dual). The theory of left fibrations is dual to the theory of right fibrations, thus we don’t study right fibrations.

Prop. (II.6.3.22) (Left Fibration and CoFibered in Groupoids). Let \( F : \mathcal{C} \to \mathcal{D} \) be a functor, then \( \mathcal{C} \) is a category cofibered in groupoids over \( \mathcal{D} \), iff the induced functor \( N(F) : N(\mathcal{C}) \to N(\mathcal{D}) \) is a left fibration of simplicial sets.
Proof: By (II.6.3.6), $N(F)$ is an inner fibration, thus it suffices to check for $\Delta^n \to \Delta^n$. For $n = 1$, this is the definition of cofibered category (II.1.7.10), and $n = 2$ is just the surjectivity of the map defining CoCartesian arrows (II.1.7.7), and $n = 3$ is equivalent to the injectivity of the map defining Cocartesian arrows. And for $n > 3$, then extension is automatic for nerves.

□

Remark (II.6.3.23) (Right Fibrations and Fibered Categories). The (left)right fibration is the $\infty$-categorial analogue of (co)fibered categories in usual category theory.

Remark (II.6.3.24). Given a left fibration $X \to S$ is more or less similar to given a functor from the homotopy category $hS$ to the $\infty$-category $\mathcal{H}$ of spaces.

Proof: Cf. [HTT, P58].

□

Prop. (II.6.3.25) (Over(Under)categories and Fibrations). Given a digram of simplicial sets:

$$A \subset B \xrightarrow{p} X \xrightarrow{q} S.$$ 

Let $r = q \circ p$, $p_0 = p|_A$, $r_0 = r|_A$, and $q$ is an inner fibration, then

- the induced map $X_{p/} \to X_{p_0/} \times_{S_{r_0/}} S_{r/}$ is a left fibration. And dual argument holds for overcategories.
- If $q$ is a left fibration or $A \subset B$ is right anodyne, then $X_{p/} \to X_{p_0/} \times_{S_{r_0/}} S_{r/}$ is a trivial fibration.
- If $q$ is moreover a left fibration, then the induced map $X_{p/} \to X_{p_0/} \times_{S_{r_0/}} S_{r/}$ is a left fibration.

Proof: These just follow from (II.6.3.18).

□

Prop. (II.6.3.26) (Homotopy Extension Lifting Property). Let $p : X \to S$ be a map of simplicial sets and $i : A \subset B$, consider the map

$$q : X^B \to X^A \times_{S^A} S^B.$$ 

- If $p$ is a left fibration, then $q$ is a left fibration.
- If $p$ is a left fibration $i$ is a left anodyne, then $q$ is a trivial fibration.
- If $i : \{0\} \subset \Delta^1$, then $p$ is a left fibration iff $q$ is a trivial fibration.
- If $p$ is an inner fibration, then $q$ is an inner fibration.
- If $p$ is an inner fibration and $i$ is an inner anodyne, then $q$ is a trivial fibration.

Proof: First notice that a right lifting of $q$ w.r.t a map $Z \to Z'$ is equivalent to a right lifting of $p$ w.r.t the map $Z \times B \coprod Z' \times A \to Z' \times B$. Then the conclusions follow from (II.6.3.20) and (II.6.3.19).

□

Prop. (II.6.3.27) (Homotopy Section and Left Fibration). Let $X \to S$ be a map of simplicial sets and $s : S \to X$ be section of $p$, and let $h \in \text{Hom}_S(X \times \Delta^1, X)$ that $h|_{X \times \{0\}} = s \circ p$ and $h|_{X \times \{1\}} = \text{id}$, then $s$ is a left anodyne.

Proof: Cf. [HTT, P65].

□
Prop. (II.6.3.28) (Left Fibration and Functor to Spaces). Let \( X \to S \) be a left fibration, then the fibers are all Kan complexes by (II.7.2.5), and for any edge \( f : s \to s' \in S \), we can solve the lifting diagram

\[
\begin{array}{ccc}
\{0\} \times X_s & \longrightarrow & X \\
\downarrow & & \downarrow \ \\
\Delta^1 \times X_s & \longrightarrow & \Delta^1 \\
\end{array}
\]

because the left hand side is left anodyne by (II.6.3.20), thus getting a morphism \( f! : X_s \to X_{s'} \).

Then this determines a functor from \( hS \) to \( \mathcal{H} \) the homotopy category of spaces.

Proof: First notice \( f! \) is uniquely defined up to homotopy: given to dotted arrow solving the diagram, we can use the lifting property w.r.t. the map

\[
\Delta^1 \times \partial \Delta^1 \times X_s \to \Delta^1 \times \Delta^1 \times X_s
\]

which is a left anodyne by (II.6.3.20), to get a homotopy between them.

Next if \( \eta \in \text{Hom}_H(K,X_s), \eta' \in \text{Hom}_H(K,X_{s'}) \), then \( \eta' = f! \circ \eta \) iff there is a map \( q : K \times \Delta^1 \to X \) that \( q \circ p \) is given by the mapping \( X_s \times \Delta^1 \to \Delta^1 \overset{f \to}{\to} S \), and \( p|_{K \times \{0\}}, p|_{K \times \{1\}} \) have homotopy types \( \eta, \eta' \) resp..

Then for any \( g \circ f \cong h \in S \), which is depicted by a 2-complex, we can use the left anodyne \( X_u \times \{0\} \subset X_u \times \Delta^2 \) to get a morphism \( p : X_u \times \Delta^2 \to X \), then \( p|_{X_u \times \{1\}} \cong f! \), \( p|_{X_u \times \{2\}} \cong h! \), and the map \( p|_{X_u \times \Delta(1,2)} \) witnesses the fact \( g! \circ f! \cong h! \).

Kan Fibrations

Remark (II.6.3.29) (Kan Complexes and Groupoids). Kan complexes are \( \infty \)-categorical analogy of groupoids, by (II.7.2.5).

Lemma (II.6.3.30). If a left fibration \( p : S \to T \) is a weak homotopy equivalence of Kan complexes, then it is a surjection on vertices.

Proof: Because it is homotopy equivalence, for any \( t \in T \), there is a morphism \( p(s) \to t \) for some \( s \in S \) (Kan complex used), thus it lifts to a morphism in \( S \), so surjective on vertices.

Lemma (II.6.3.31) (Fibrations of Kan Complexes). If \( S \to T \) is a left fibration and \( T \) is a Kan complex, then \( p \) is a Kan fibration.

Proof: Firstly \( S \) is a Kan complex. Let \( A \subset B \) be anodyne morphisms, we need to show \( p : S^B \to S^A \times_{T^A} T^B \) is surjective on vertices. Since \( S,T \) are complexes, \( S^B \to S^A \) and \( T^B \to T^A \) are trivial fibrations by (II.6.3.26). Cf.[HTT, P66].

Notice this is an immediate consequence of (II.6.3.36), because the homotopy category of a Kan complex is a groupoid, by (II.7.2.5), so \( f_1 \) must be isomorphisms.

Prop. (II.6.3.32) (Examples of Kan Complexes).

- The singular complex of topological space is a Kan complex.
- The nerve of a topological space is a Kan complex.
- The nerve of a category is a Kan complex iff the category is a groupoid, by (II.6.3.22).
Prop. (II.6.3.33). The bar resolution $BG$ is a Kan fibration for every group $G$.

Proof: □

Prop. (II.6.3.34). A principal $G$ fibration, i.e. $X \to X/G$ where $X$ is a simplicial object of $G$-sets that $G$ acts freely on $X_n$, is a Kan fibration.

Lemma (II.6.3.35). Let $p : S \to T$ be a left fibration that all the fibers are contractible, then $p$ is a trivial Kan fibration.

Proof: By duality, it suffices to prove for right fibrations. Because fiber is nonempty, it has right lifting property w.r.t $\emptyset \subset \Delta^0$, and for $n > 0$, let $\partial \Delta^n \to S$ be any map, then to show the lifting property, we may take fiber product and assume $T = \Delta^n$, thus $S$ is an $\infty$-category. Cf.[HTT, P66]. □

Prop. (II.6.3.36) (Characterization of Kan Fibrations). Let $p : S \to T$ be a left fibration of simplicial sets, then $p$ is a Kan extension iff the morphism $f!$ defined in (II.6.3.28) is an isomorphism in $\mathcal{H}$ for any morphism $f \in T$.

Proof: Cf.[HTT, P66]. □

4 Simplicial Categories & Model Categories

Def. (II.6.4.1) (Simplicial Category). The category $\mathcal{Cat}_\Delta$ of simplicial categories consists of categories enriched over the Cartesian monoidal category $Sets_\Delta$ of simplicial sets.

Def. (II.6.4.2) (Homotopy Category). There are singular complex functor and geometrization functor that induce isomorphism of $h(Sets_\Delta) \cong h(CG)$ by (II.6.3.10), thus the theory of simplicial categories and topological categories are the same.

Simplicial Nerves

Def. (II.6.4.3) (Thickened Finite Ordered Sets). Let $J$ be a finite ordered set, define a simplicial category $\mathcal{C}[\Delta^J]$ as follows:

- The objects are elements of $J$.
- For $i, j \in J$, $\text{Mor}(i, j) = N(P_{i,j})$, where $P_{i,j}$ is the partially ordered set of subsets $I \subset J$ that $i, j \in I$.
- The morphism is induced by the inclusion of partially ordered sets $P_{i,j} \times P_{j,k} \subset P_{i,k}$.

Prop. (II.6.4.4) ($\mathcal{C}$).

- There is a unique functor $\mathcal{C}[\Delta^n] \to \Delta^n$, and it is an equivalence of simplicial categories, because each morphism space in $\mathcal{C}[\Delta^n]$ is a cube.
- The mapping $J \mapsto \mathcal{C}[\Delta^J]$ is a functor from $\Delta$ to $\mathcal{Cat}_\Delta$. In other words, $\mathcal{C}$ defines a cosimplicial object in $\mathcal{Cat}_\Delta$.
- $\mathcal{C}$ extends naturally to a colimit-preserving morphism $Sets_\Delta \to \mathcal{Cat}_\Delta$, by (II.6.3.1).
Def. (II.6.4.5) (Simplicial Nerve). For a simplicial category \( \mathcal{C} \), we define the simplicial nerve to be the simplicial set that
\[
(N(\mathcal{C}))_n = \text{Hom}_{\mathcal{C}}(\Delta[n], \mathcal{C}).
\]

If \( \mathcal{C} \) is a topological category, then define the topological nerve to be the simplicial nerve of \( \text{Sing}(\mathcal{C}) \).

By the definition and the adjointness of (II.6.3.7), the nerve functor \( N \) is right adjoint to the functor \( \mathcal{C}[\cdot] \) or \( \mathcal{C}[\cdot] \).

Remark (II.6.4.6). It should be checked that to give a 2-complex in \( N(\mathcal{C}) \) is equivalent to giving morphisms \( f, g, h \in \mathcal{C} \) and a path from \( g \circ f \) to \( h \).

Prop. (II.6.4.7) (Nerve and Simplicial Nerve). If we forget the enriched structure on \( hS \), then the usual nerve \( N \) and \( h \) determines a adjoint between \( \mathcal{C} \text{at} \) and \( \mathcal{C} \text{at}_{\Delta} \).

Proof: Notice that the usual nerve is a special case of simplicial nerve by the inclusion \( \mathcal{C} \text{at} \subset \mathcal{C} \text{at}_{\Delta} \).
And also the inclusion \( \mathcal{C} \text{at} \subset \mathcal{C} \text{at}_{\Delta} \) is left adjoint to the functor \( \pi_0 \).

Prop. (II.6.4.8) (Kan Fibrations Nerve Inner Fibrations). Let \( F : \mathcal{C} \to \mathcal{D} \) be a map of simplicial categories that for any \( C, C' \in \mathcal{C} \), \( \text{Map}(C, C') \to \text{Map}(F(C), F(C')) \) is a Kan fibration, then the induced map of simplicial sets \( N(\mathcal{C}) \to N(\mathcal{D}) \) is an inner fibration.

Proof: This is because a lifting of \( N(\mathcal{C}) \to N(\mathcal{D}) \) w.r.t. \( \Delta^n \) is equivalent to a lifting of \( \mathcal{C} \to \mathcal{D} \) w.r.t. \( \mathcal{C}[\Delta^n] \subset \mathcal{C}[\Delta[n]] \). But this lifting is equivalent to a lifting of \( \text{Map}(F(0), F(n)) \to \text{Map}(F'(0), F'(n)) \) w.r.t. the anodyne map \( \text{Map}_{\mathcal{C}[\Delta^n]} \subset \text{Map}_{\mathcal{C}[\Delta[n]]} \), which is a cube removing the interior and a face.

Cor. (II.6.4.9). The topological nerve of a topological category \( \mathcal{C} \) is an \( \infty \)-category, as the singular complex of a topological space is always a Kan complex, by (II.6.3.32).

Def. (II.6.4.10) (Homotopy Category). For a simplicial set \( S \), the homotopy category \( hS \) is defined to be the homotopy category (II.6.4.2) of the simplicial category \( \mathcal{C}[S] \). A map of simplicial sets is called a categorical equivalence if their homotopy categories are equivalent as \( \mathfrak{h} \)-enriched categories.

Remark (II.6.4.11). It is immediate that \( S \cong T \) categorically iff \( \mathcal{C}[S] \cong \mathcal{C}[T] \) iff \( ||\mathcal{C}[S]|| = ||\mathcal{C}[T]|| \).

Lemma (II.6.4.12). Let \( \mathcal{C} \) be a topological category, then the counit map \( |\mathcal{C}(N(\mathcal{C}))| \cong \mathcal{C}(\text{II.6.4.5}) \) is an equivalence of topological categories.

Proof: By the Quillen equivalence between \( \mathcal{C} \text{at}_{\Delta} \) and \( \mathcal{C} \text{at} \) (II.5.2.2), this follows from (II.6.4.13), as by (II.5.7.11) and (II.6.3.32), \( \text{Sing}(\mathcal{C}) \) is a fibrant simplicial category.

Lemma (II.6.4.13). Let \( \mathcal{C} \) be a fibrant simplicial category, then the counit map \( u : \text{Map}_{\mathcal{C}[N(\mathcal{C})]}(x, y) \to \text{Map}_{\mathcal{C}}(x, y) \) is a weak homotopy equivalence of simplicial sets.

Proof: Cf. [HTT, P72].

Prop. (II.6.4.14) (Topological Category and \( \infty \)-Category Equivalent). The adjoints functors \( N \) and \( |\mathcal{C}[\cdot]| \) define an equivalence of the category of topological categories and \( \infty \)-categories modulo equivalence.
Proof: It suffices to show the units and counits are equivalences:

\[ \mathcal{C}[N(C)] \cong \mathcal{C}, \quad S \mapsto N(|\mathcal{C}[S]|). \]

The first is (II.6.4.12), and the second follows from the first by remark (II.6.4.11).

□

Prop. (II.6.4.15). For simplicial sets \( S, S' \), the natural map \( \mathcal{C}[S \times S'] \to \mathcal{C}[S] \times \mathcal{C}[S'] \) is an equivalence of simplicial categories.

Proof: If \( S, S' \) are nerves of fibrant simplicial categories \( \mathcal{C}, \mathcal{C}' \), then we have a diagram \( \mathcal{C}[S \times S'] \to \mathcal{C}[S] \times \mathcal{C}[S'] \to \mathcal{C} \times \mathcal{C}' \). Then by the two out of three axiom, the assertion follows from the fact that for any fibrant simplicial category \( \mathcal{D} \), \( \mathcal{C}[N(\mathcal{D})] \to \mathcal{D} \) is an equivalence (II.6.4.12).

Now for general \( S, S' \), we can find a categorical equivalences \( S \to N(\mathcal{C}[S]) = T \), and then \( S \times S' \to T \times T' \) is also categorical equivalence by (II.6.4.15), and we are done, by (II.6.5.16).

□

Simplicial Model Categories

Def. (II.6.4.16) (Simplicial Model Category). A simplicial model category is a \( \text{Set}_{\Delta} \)-enriched model category (II.5.6.3).

Prop. (II.6.4.17) (Simplicial Model Category Criterion). Let \( \mathcal{C} \) be a simplicial category that is equipped with a model structure that every object of \( \mathcal{C} \) is cofibrant and the collection of weak equivalences in stable under filtered colimits, then \( \mathcal{C} \) is a simplicial model category iff the following conditions holds:

- \( \mathcal{C} \) is both tensored and cotensored over \( \text{Set}_{\Delta} \).
- Given a cofibration of simplicial sets \( i : K \to L \) and a cofibration \( C \to D \in \mathcal{C} \), the induced map \( (C \otimes L) \coprod_{C \otimes K} D \otimes K \to D \otimes L \) is a cofibration in \( \mathcal{C} \).
- The natural map \( C \otimes \Delta^n \to C \otimes \Delta^0 \cong C \) is a weak equivalence in \( \mathcal{C} \).

Proof: Cf. [HTT, P850].

□

Prop. (II.6.4.18). Let \( \mathcal{C} \) be a simplicial model category and \( X \) cofibrant and \( Y \) fibrant, then \( K = \text{Map}(X, Y) \) is a Kan complex, and there is a canonical bijection \( \pi_0 K \cong \text{Hom}_{\text{Ho}(\mathcal{C})}(X, Y) \).

Proof: Use (II.5.6.4).

□

Prop. (II.6.4.19).

5 Covariant Model Structure

Prop. (II.6.5.1) (Covariant Model Structure). Let \( S \) be a simplicial set, then we call a map \( X \to Y \in (\text{Sets}_{\Delta})_S \)

- **covariant cofibration** if it is a monomorphism.
- **covariant equivalence** if the induced map \( X^\circ \coprod_X S \to Y^\circ \coprod_Y S \) is a categorical equivalence (II.6.4.10).

Then these define a left proper combinatorial model structure on \( (\text{Sets}_{\Delta})_S \).

Proof: Cf. [HTT, P69].

□

Lemma (II.6.5.2). Every left anodyne map in \( (\text{Sets}_{\Delta})_S \) is a covariant equivalence.
Prop. (II.6.5.3) (Covariant Model Structure). \((\text{Set}_\Delta)_{/S}\) is a simplicial model category with the contravariant model structure and the simplicial structure where

\[
\text{Map}(X,Y) = Y^X \times_{S^X} \{\varphi\} \in \text{Set}_\Delta
\]

where \(\varphi : X \to S\) is the structure map.

\textit{Proof:} We use (II.6.4.17), it suffices to check that \(X \times \Delta^n \to X \times \Delta^0\) is a covariant equivalence. But it has a section, which is a left anodyne by (II.6.3.20), thus it is a covariant equivalence by (II.6.5.2). □

Cor. (II.6.5.4) (Contravariant Model Structure). Let \(S\) be a simplicial set, then the covariant model is usually not self-dual, and we can define a \textbf{contravariant model structure} as follows:

- A \textbf{contravariant cofibration} is a monomorphism of simplicial sets.
- \(f\) is a \textbf{contravariant equivalence} in \((\text{Set})_{/S}\) iff \(f^{op}\) is a covariant equivalence in \((\text{Set})_{/S}^{op}\).
- \(f\) is a \textbf{contravariant fibration} in \((\text{Set})_{/S}\) iff \(f^{op}\) is a covariant fibration in \((\text{Set})_{/S}^{op}\).

Prop. (II.6.5.5) (Base Change). Let \(S \to S'\) be a map of simplicial sets, then the forgetful functor and base change functor \(j_!\), \(j^*\) defines a Quillen adjunction of covariant models:

\[
j_!: (\text{Set}_\Delta)_{/S} \rightleftarrows (\text{Set}_\Delta)_{/S'} : j^*
\]

\textit{Proof:} it is clearly a pair of adjoints, and \(j_!\) preserves cofibrations. \(j_!\) also preserves covariant equivalences: Cf.[HTT, P71]. Thus it is a Quillen adjunction. □

Lemma (II.6.5.6). Let \(S' \subset S\) be simplicial sets, let \(p : X \to S\) be a map and \(q : Y \to S\) be a right fibration. Let \(X' = X \times_S S', Y' = Y \times_S S'\), then the restriction map

\[
\varphi : \text{Map}_{(\text{Set}_\Delta)_{/S}}(X,Y) \to \text{Map}_{(\text{Set}_\Delta)_{/S'}}(X',Y')
\]

is a Kan fibration.

\textit{Proof:} Firstly it is a right fibration because it has right lifting property w.r.t. right anodyne inclusion \(A \to B\): this is because \((A \times X') \amalg (A \times X) \subset B \times X\) is also a right anodyne (II.6.3.20). Next we apply this to the inclusion \(\emptyset \subset S'\) to see that \(\text{Map}_{(\text{Set}_\Delta)_{/S'}}(X',Y')\) is a Kan complex (II.7.2.5), and then \(\varphi\) is a Kan fibration by (II.6.3.31). □

Lemma (II.6.5.7). Let \(p : X \to S\) be an object of \((\text{Set}_\Delta)_{/S}\), then \(p\) is a right fibration iff it is a covariant fibrant object in \((\text{Set}_\Delta)_{/S}\).

\textit{Proof:} Cf.[HTT, P85]. □

Def. (II.6.5.8) (Pointwise Equivalence). Let \(X \to Y\) be a map in \(RFib(S)\), then \(f\) is called a \textbf{pointwise equivalence} iff the induced map \(X_s \to Y_s\) is a homotopy equivalence of Kan complexes (II.7.2.5) for any \(s \in S\).

Lemma (II.6.5.9). Let \(f : X \to Y\) be a morphism in \(RFib(S)\), then the following are equivalent:

- \(f\) is a pointwise equivalence.
• $f$ is an equivalence in the simplicial category $(\text{Set})_{/S}$.
• For any $A \in (\text{Set})_{/S}$, $f$ induces a homotopy equivalence of Kan complexes:
  $\text{Map}_{(\text{Set}_\Delta)_{/S}}(A,X) \cong \text{Map}_{(\text{Set}_\Delta)_{/S}}(A,Y)$.

**Proof:** Cf.[HTT, P82].

**Prop. (II.6.5.10) (Equivalences).** In the situation of (II.6.5.8), $f$ is a pointwise equivalence iff it is a contravariant equivalence iff it is a categorical equivalence.

**Proof:** Cf.[HTT, 2.2.3.13, 3.3.1.5].

**Prop. (II.6.5.11) (Cotravariant Fibration as Right Fibration).** Let $f : X \to Y$ be a map in $\text{RFib}(S)$, then $f$ is a contravariant fibration in $(\text{Set}_\Delta)_{/S}$ iff it is a right fibration.

**Proof:** Cf.[HTT, P86].

**Straightening and Unstraightening**

**Def. (II.6.5.12) (Straightening and Unstraightening).** Fix a simplicial set $S$, a simplicial category $\mathcal{C}$ and a functor $\mathcal{C}[S] \to \mathcal{C}^{op}$. Given an object $X \in (\text{Set}_\Delta)_{/S}$, let $v$ be the cone point of $X^p$. Then the simplicial category $M = \mathcal{C}[X^p] \coprod_{\mathcal{C}[X]} \mathcal{C}^{op}$ can be viewed as a correspondence? between $\mathcal{C}^{op}$ and $\Delta^0$, thus giving a simplicial functor
  
  $$St_\phi X : \mathcal{C} \to \text{Set}_\Delta : C \mapsto \text{Map}_M(C,v).$$

Then $St_\phi$ is called the **straightening functor** associated to $\phi$. And we denote by $St_\phi$ the functor $St_\phi$ where $\phi : \text{id}_{\mathcal{C}[S]}$.

By the adjoint functor theorem (II.1.1.24), $St_\phi$ has a left adjoint called **unstraightening functor** $Un_\phi$.

**Prop. (II.6.5.13).** Let $S$ be a simplicial set, $\mathcal{C}$ a simplicial category and $\phi : \mathcal{C}[S] \to \mathcal{C}^{op}$ a simplicial functor, then the straightening and unstraightening functor determines a Quillen adjunction

$$St_\phi : (\text{Set}_\Delta)_{/S} \rightleftarrows \text{Set}_\Delta^\phi : Un_\phi$$

determines a Quillen adjunction, where the LHS has the contravariant model structure and the RHS has the projective model structure. And if $\phi$ is an equivalence of simplicial categories, then $(St_\phi, Un_\phi)$ is a Quillen equivalence.

**Proof:**

**Unstraightening of Right Fibrations**

**Prop. (II.6.5.14).** For every simplicial set $S$, the unstraightening $Un_S$ induces an equivalence of simplicial categories

$$(\text{Set}_\Delta^\mathcal{C}[S]^{op})^\circ \to \text{RFib}(S).$$

(II.5.6.5), where the RHS is the category of fibrations $X \to S$.

**Proof:** Cf.[HTT, P83].
Joyal Model Structure

Lemma (II.6.5.15) (Inner Anodyne is Categorical Equivalence). Every inner anodyne map $A \to B$ of simplicial sets is a categorical equivalence.

Proof: The class of morphisms $f$ that $\mathcal{C}(f)$ is a trivial cofibration is weakly saturated (because $\mathcal{C}$ is a left adjoint (II.6.4.5) and (II.5.0.5)), then it suffices to check for $\Lambda^n_j \subset \Delta^n$. Then $\mathcal{C}[\Lambda^n_j] \subset \mathcal{C}[\Delta^n]$ is a pushout of $[i]_K \subset [1]_{(\Delta^n)^{n-1}}$, where $K$ is obtained form $(\Delta^n)^{n-1}$ by moving a face and the interior. Thus it is a trivial cofibration (II.5.7.4).

Prop. (II.6.5.16) (Joyal Model Structure). There is a left proper combinatorial model structure called Joyal model structure on $\text{Set}_\Delta$, with:

- Cofibrations: monomorphisms.
- Weak equivalences: categorical equivalences defined in (II.6.4.10).
- Fibrations: categorical fibrations which has the right lifting property w.r.t. trivial cofibrations.

And the adjoint functors $(\mathcal{C}, N)$ determines a Quillen equivalence between $\text{Set}_\Delta$ and $\text{Cat}_\Delta$.

Proof: Cf.[HTT, P89].

Firstly we show $\mathcal{C}$ preserves cofibrations. It suffices to show that $\mathcal{C}[\partial \Delta^n] \subset \mathcal{C}[\Delta^n]$ is a cofibration (II.5.7.4). But notice this two simplicial category only differ at $\text{Hom}_{\mathcal{C}[\partial \Delta^n]}(0, n)$ is the boundary of the simplicial cube $(\Delta^n)^{n-1} \cong \text{Hom}_{\mathcal{C}[\Delta^n]}(0, n)$, thus the inclusion is a pushout of the inclusion $[1]_{(\Delta^n)^{n-1}} \subset [1]_{\partial \Delta^n}$, which is a cofibration by (II.5.7.3).

Left properness is clear (II.5.0.9). $\mathcal{C}$ preserves weak equivalences by (II.6.4.10) and (II.5.7.4), so $(\mathcal{C}, N)$ is a Quillen adjunction. To show it is a Quillen equivalence: It suffices to check for each simplicial set $S$ and fibrant simplicial category $\mathcal{C}$, a map $u : S \to N(\mathcal{C})$ is a categorical equivalence if the adjoint map $v : \mathcal{C}[S] \to \mathcal{C}$ is an equivalence of simplicial categories. But $v$ factors as

$$\mathcal{C}[S] \xrightarrow{\mathcal{C}[u]} \mathcal{C}[N(\mathcal{C})] \xrightarrow{w} \mathcal{C}$$

and the counit map $w$ is an equivalence by (II.6.4.13).

Cor. (II.6.5.17). If $f : A \to B$ is a categorical equivalence of simplicial sets and $K$ is a simplicial set, then $A \times K \to B \times K$ is also a categorical equivalence.

Proof: Choose a factorization $B \to Q$, which is an inner anodyne and $Q$ is an $\infty$-category, by small object argument (II.1.6.8), then $B \times K \to Q \times K$ is also an inner anodyne map (II.6.3.20), hence a categorical equivalence (II.6.5.15), so we can assume $B$ is an $\infty$-category. Similarly we can reduce to the case $A, K$ are also $\infty$-categories.

For the rest, Cf.[HTT, P92].

Prop. (II.6.5.18) ($\infty$-Category Fibrant in Joyal Model). A simplicial set $C$ is fibrant in the Joyal model structure if it is an $\infty$-category.

Proof: Fibrant objects are $\infty$-categories, by (II.6.5.15). For the converse, fix an $\infty$-category and an inclusion $A \subset B$, given a map $A \to \mathcal{C}$, the inclusion $\mathcal{C} \subset C \coprod_A B$ is also a categorical equivalence because Joyal model structure is left proper (II.6.5.16), thus $\mathcal{C}$ is a retract of $\mathcal{C} \coprod_A B$ by (II.6.5.19), which gives an extension $B \to \mathcal{C}$.
Lemma (II.6.5.19). Let $\mathcal{C} \subset \mathcal{D}$ be an inclusion of simplicial sets that is also a categorical equivalence, and $\mathcal{C}$ is also an $\infty$-category, then $\mathcal{C}$ is a retract of $\mathcal{D}$.

Proof: Include $\mathcal{D}$ into an $\infty$-category by small object argument and (II.6.5.15), we may assume $\mathcal{D}$ is also an $\infty$-category, then we finish by applying (II.7.2.20) for $A = \mathcal{C}$ and $B = \mathcal{D}$. □

6 Cartesian Fibrations

Remark (II.6.6.1). The theory of Cartesian fibrations is an analogue of the theory of fibered categories.7

Def. (II.6.6.2) ($p$-Cartesian). Let $p : X \to S$ be an inner fibration and $f$ is an edge of $X$, then $f$ is called a $p$-Cartesian if the induced map

$$X_f \to X_{/y} \times_{S(p(y))} S_{p(f)}$$

is a trivial Kan fibration.

Remark (II.6.6.3). For an ordinary category $\mathcal{C}$ and a map $p : N(\mathcal{C}) \to \Delta^1$, then a morphism $f \in \mathcal{C}$ is $p$-Cartesian iff it is Cartesian in the usual sense.

Prop. (II.6.6.4) (Characterization of Cartesian Fibrations). Let $p : \mathcal{C} \to \mathcal{D}$ be an inner fibration of $\infty$-categories, then an edge $f : Y \to Z \in \mathcal{C}$ is $p$-Cartesian iff for every object $X \in \mathcal{C}$, there is a Cartesian diagram

$$\begin{array}{ccc}
\text{Map}(X,Y) & \longrightarrow & \text{Map}(X,Z) \\
\downarrow & & \downarrow \\
\text{Map}(p(X),p(Y)) & \longrightarrow & \text{Map}(p(X),p(Z))
\end{array}$$

Proof: Cf.[HTT, P131]. □

Cartesian Fibrations

Def. (II.6.6.5) (Cartesian Fibration). A Cartesian fibration is an inner anodyne map $p : X \to S$ that for any edge $f : x \to y \in S$ and a vertex $\bar{y}$ mapping to $y$, there is a $p$-Cartesian edge $\bar{f}$ with $p(\bar{f}) = f$. The dual of a Cartesian fibration is called a coCartesian fibration.

Prop. (II.6.6.6). The class of Cartesian fibrations is stable under compositions and base change.

Prop. (II.6.6.7) (Cartesian Fibration and Right Fibration). Let $p : X \to S$ be an inner fibration, then the following are equivalent:

- $p$ is a Cartesian fibration and the every edge of $X$ is $p$-Cartesian.
- $p$ is a right fibration.
- $p$ is a Cartesian fibration and every fiber $X_s$ is a Kan complex.

Proof: Cf.[HTT, P122]. □

Def. (II.6.6.8) (Locally Cartesian Fibration). A map $X \to S$ of simplicial sets is called a locally Cartesian fibration if it is an inner fibration and for every edge $\Delta^1 \to S$, the pullback $X \times_S \Delta^1 \to \Delta^1$ is a Cartesian fibration.
Prop. (II.6.6.9) (Cartesian and Locally Cartesian). Let $p : X \to S$ be a locally Cartesian fibration, then the following are equivalent:

- $p$ is a Cartesian fibration.
- Given a composition $fg \simeq h$ in the homotopy category, if $f,g$ are both locally $p$-Cartesian, then $h$ is also locally $p$-Cartesian.
- Every locally $p$-Cartesian edge in $X$ is $p$-Cartesian.

Proof: Cf.[HTT, P124]. □

Prop. (II.6.6.10). Given maps of $\infty$-categories: $C \xrightarrow{p} D \xrightarrow{q} E$, if $q,q \circ p$ are both locally Cartesian fibrations and $p$ maps locally $(q \circ p)$-Cartesian maps to locally $q$-Cartesian maps and for any $Z \in E$, $p$ induces a categorial equivalence $C_Z \to D_Z$, then $p$ is a categorical equivalence.

Proof: Cf.[HTT, P132]. □

Prop. (II.6.6.11). Categorical equivalences between $\infty$-categories are stable under base change of Cartesian fibrations of $\infty$-categories.

Proof: Cf.[HTT, P132]. □

7 Cyclic Homology Theory (欧阳恩林)

Combinatorial Category

Def. (II.6.7.1). The Segal category $\text{Fin}_*$ is the category of pointed finite sets. A morphism is called inert iff $|f^{-1}([i])| = 1$ for all $i \neq \ast$. It is called active iff $f^{-1}([\ast]) = \{\ast\}$.

A morphism can be uniquely factorized as a composition $gh$, where $h$ is inert and $g$ is active.

Prop. (II.6.7.2). There is a morphism $\text{Cut} : \Delta^{op} \to \text{Fin}_*$ where we interpret $[n] \in \text{Fin}_*$ as the set of cut in $[n]$, and

$$\text{Cut}(\alpha)(i) = \begin{cases} j & \text{if there are } j \text{ s.t. } \alpha(j - 1) < i \leq \alpha(j) \\ 0 & \text{otherwise} \end{cases}$$

Prop. (II.6.7.3). The category of functors from the $E_{\infty} = \text{Fin}_*$ to $\mathcal{C}at$ that

$$X([n]) \xrightarrow{\prod_{i=2}^{n} X(\rho)} \prod_{i=1}^{n} X([1]) \quad n \geq 0$$

and $X([0])$ is the final object, is equivalent to the category of symmetric unital monoidal categories with base category $X([1])$. (Because the commutativity of morphisms encodes the fact that the tensor action is symmetric).

Similarly, the category of functors from the $\Delta^{op}$ to $\mathcal{C}at$ that

$$X([n]) \xrightarrow{\prod_{i=2}^{n} X(\rho)} \prod_{i=1}^{n} X([1]) \quad n \geq 0$$

is equivalent to the category of symmetric unital monoidal categories $(X([1]))$. And it is symmetric iff it factors through $\text{Cut} : \Delta^{op} \to \text{Fin}_*$. 
Def. (II.6.7.4). The Conne cyclic category \( \Delta_C \) is a category containing \( \Delta \) that \( \text{Aut}_{\Delta_C}([n]) \) is \( C_{n+1} \). And every morphism \([n] \to [m] \) in \( \Delta_C \) can be uniquely written as the form \( \varphi g \), where \( \varphi \in \text{Hom}_\Delta([n],[m]) \) and \( g \in \text{Aut}_{\Delta_C}([n]) \).

\( \Delta_C^{\text{op}} \) is isomorphic to \( \Delta_C \). Cf. [杨恩林循环同调 P31], thus \( \Delta \) and \( \Delta^{\text{op}} \) are all subcategories of \( \Delta_C \).

Def. (II.6.7.5). The category \( \Delta_S \) is the category that \( \text{Aut}_{\Delta_S}([n]) \cong S^n \) and every morphism \([n] \to [m] \) in \( \Delta_S \) can be uniquely written as the form \( \varphi g \), where \( \varphi \in \text{Hom}_\Delta([n],[m]) \) and \( g \in \text{Aut}_{\Delta_S}([n]) \).

Def. (II.6.7.6). For a category \( C \), a cyclic object in \( C \) is a functor \( \Delta_C^{\text{op}} \to C \).

For example, the functor that maps \([n] \) to \( C_{n+1} \) and the functor maps to the pull back of the order of the cyclic, is a cyclic object.

**Hochschild Homology (Jeremy Hahn)**

Def. (II.6.7.7) (Hochschild Homology Group). Let \( R \) be a commutative ring and \( A \) a flat \( R \)-algebra, \( A^{\text{env}} = A \otimes_R A^{\text{op}} \). Then an \( A^{\text{env}} \)-module is equivalent to an \( (A,A) \)-bimodule.

If \( M \) is an \( (A,A) \)-bimodule, then we define the Hochschild homology group \( \text{HH}_n(A/R,M) = \text{Tor}^{A^{\text{env}}}_n(M,A) \). And also we denote \( \text{HH}_n(A/R) = \text{HH}_n(A/R,M) \).

\( \text{HH}_0(A,M) \) is a \( Z(A) \) module by the action of \( Z(A) \) on \( M \) and \( \text{HH}_n \) defines a functor \( \text{Alg}_R \to \text{Mod}_R \).

Def. (II.6.7.8) (Hochschild Complex). Let \( R \) be a commutative ring and \( A \) a flat \( R \)-algebra, we define \( A^{\text{env}} = A \otimes_R A^{\text{op}} \), and \( \text{HH}(A/R) = A \otimes_{A^{\text{env}}} A \in \text{D}(A) \).

Def. (II.6.7.9) (Flat Case). For a flat \( R \)-algebra \( A \) and a \( (A,A) \)-bimodule \( M \), there is a simplicial module \( C(A,M) \) called the Hochschild complex of \( A \) with coefficient in \( M \), with \( M_n = M \otimes A^n \)

\[
\begin{align*}
&d_i(m,a_1,\ldots,a_n) = \\
&\quad \begin{cases} 
(m_0a_1,a_2,\ldots,a_n) & i = 0 \\
(a_0m_0,a_1,\ldots,a_{n-1}) & i = n \\
(m_0,a_1,\ldots,a_i,a_{i+1},\ldots,a_n) & \text{otherwise}
\end{cases}
\end{align*}
\]

\[
s_j(m,a_1,\ldots,a_n) = (m,a_1,\ldots,a_{j-1},1,a_{j+1},\ldots,a_n)
\]

The homology group of the Moore complex associated to the Hochschild complex is just \( \text{HH}_n(A,M) \). The Moore complex is of the form

\[\ldots \to M \otimes A \otimes A \otimes A \xrightarrow{\partial_3} M \otimes A \otimes A \xrightarrow{\partial_2} M \otimes A \otimes A \xrightarrow{\partial_1} M \xrightarrow{\partial_0} 0 \to 0 \to \ldots\]

where

\[
\partial_1(m \otimes a) = ma - am, \quad \partial_2(m \otimes a_1 \otimes a_2) = ma_1 \otimes a_2 - m \otimes a_1a - 2 + a_2m \otimes a_1
\]

\[
\partial_3(m \otimes a_1 \otimes a_2 \otimes a_3) = ma - 1 \otimes a_2 \otimes a_3 - m \otimes a_1a_2 \otimes a_3 + m \otimes a_1 \otimes a_2a_3 - a_3m \otimes a_1 \otimes a_2.
\]

**Proof:** \[\square\]

**Example (II.6.7.10).**

- \( \text{HH}_n(R,R) = R \) if \( n = 0 \) and \( 0 \) otherwise.
- \( \text{HH}_0(A/R,A/R) = A^{\text{op}} \).
- If \( A \) is commutative, \( \text{HH}_1(A,A) \cong \Omega^1_{A/R} \) giving by \( a \otimes x \mapsto adx \) by (1.7.3.4).
For a symmetric \((A, A)\)-module \(M\), thus we have \(H_1(A, M) = M \otimes_A A^{ab}\) and \(H_1(A, M) = M \otimes_A \Omega^1_{A/R}\). And if \(M\) is flat, \(H_n(A, M) = M \otimes_A H_n(A, A)\).

\[ HH_n(R[X]/R) = \begin{cases} R[X] & n = 0 \\ \Omega^1_{R[X]/R} & n = 1 \\ 0 & \text{otherwise} \end{cases} \]

**Example (II.6.7.11)** \((HH(\mathbb{F}_p/\mathbb{Z}))\). Because \(\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p = (\mathbb{F}_p \xrightarrow{0} \mathbb{F}_p)\).

\[ HH_*(\mathbb{F}_p, \mathbb{Z}) \cong \mathbb{F}_p[X_1, X_2, \ldots]/(X_i X_j = \binom{i+j}{i} X_{i+j}) \]

where \(\deg(X_i) = 2i, \partial X_i = 0\).

**Prop. (II.6.7.12).** Suppose \(A, B\) are \(R\)-algebras, then

\[ HH(A \otimes^L_R B/R) = HH(A/R) \otimes^L_R HH(B/R). \]

**Cor. (II.6.7.13).** \(HH(R[X_1, \ldots, X_n]/R) = \Omega^*_R[X_1, \ldots, X_n]/R\).

**Prop. (II.6.7.14).** If \(A\) is a commutative \(R\)-algebra, then \(HH(A/R)\) is naturally a commutative dga. In particular, \(HH_*(A/R)\) is a graded ring.

**Prop. (II.6.7.15) (Spectral Sequence).** For a commutative ring \(A\) and a symmetric \(A\)-bimodule \(M\), there is a spectral sequence

\[ E_2^{pq} = \text{Tor}^R_p(H_p(A, A), M) \Rightarrow H_{p+q}(A, M). \]

**Hochschild Homology**

**Prop. (II.6.7.16) (Hochschild-Kostant-Rosenberg).** The isomorphism \(\Omega^1_{A/k} \cong HH_1(A)\) extends to a graded ring map

\[ \Psi : \Omega^*_A/k \to H_*(A, A) \]

. If \(A/k\) be smooth algebra and \(R\) Noetherian, then \(\Psi\) is an isomorphism of graded algebra. Cf.[Weibel P322], [阳恩林循环同调 P133].

**Def. (II.6.7.17) (Tsigan’s Double Complex).** For a cyclic objec \(M\) in an Abelian category, let \(t_*\) be the cyclic morphism and \(\partial_n = \sum_{i=0}^n (-1)^i d_i, \partial'_n = \sum_{i=0}^{n-1} (-1)^i d_i, N_n = \sum_{k=0}^n (-1)^n t_n^k\), then there is a double complex \(CC(M)\):
That the column are 2-cyclic. Cf.[Weibel P337]. The first column is called the Hochschild complex of $M$: $C^h(M)$, the second column is called acyclic complex of $M$ (II.6.7.18) $C^a(M)$. And we can even augment a cokernel column on the left, which is the complex of $M$ modulo the cyclic action, called the Conne complex $C^\lambda(M)$.

We define the Cyclic Homotopy Group $HC_n(M) = H_n(Tot CC(M))$ and when $M$ is the cyclic module $C(A)$ (II.6.7.9), denote $CC(C(A)) = CC(A)$, $HC_n(A) = HC_n(C(A))$.

**Lemma (II.6.7.18).** The second column is exact and $h = t_{n+1}s_n$ is a null-homotopy. Cf.[阳恩林循环同调 P122].

**Lemma (II.6.7.19).** Notice the rows are in fact a group homology $\text{Hom}(\mathbb{Z}/(n + 1)\mathbb{Z}, M_n)$, thus when $Q \in R$, we have the rows are acyclic because the group homology is killed by $|G|??$, thus $HC_*(M) \cong H^\lambda_*(M)$ are isomorphisms by spectral sequence.

**Prop. (II.6.7.20) (Conne SBI Sequence).** For a cyclic module $M$, there is a long exact sequence

$$\cdots \to HH_n(M) \to HC_n(M) \to HC_{n-2}(M) \to HH_{n-1}(M) \to \cdots$$

**Proof:** shift the diagram 2 column right, then there is an exact sequence of double complexes and notice the second column is exact (II.6.7.18), thus we have the kernel is quasi-isomorphic to $C^h(M)$. So the sequence follows. □

**Cor. (II.6.7.21).** $HC_0(A) = HH_0(A) = A^{ab}$.

When $A$ is commutative, $HC_1(A) = \text{Coker}(HC_0(A) \xrightarrow{B} HH_1(A)) = \Omega^1_{A/R}/dA$ as a $R$ module, because we can verify that $B(a) = a \otimes 1 - 1 \otimes A$.

**Cor. (II.6.7.22).** For a morphism of two cyclic objects, $HH_*(M) \cong HH_*(M')$ iff $HC_*(M) \cong HC_*(M')$. (Use five lemma).

**Def. (II.6.7.23).** A mixed complex $(M, b, B)$ is a complex with $b : M_n \to M_{n-1}$ and $B : M_n \to M_{n+1}$ that makes $M$ into a double chain complex. And there is a Conne double complex associated with this mixed complex. And similarly there is a same SBI sequence associated to the following diagram:

$$\begin{array}{cccc}
M_2 & \xleftarrow{B} & M_1 & \xleftarrow{B} & M_0 \\
M_1 & \xleftarrow{b} & M_0 \\
C_1 & \xleftarrow{b} & C_0 \\
C_0 & \xleftarrow{b} & C_0
\end{array}$$

From a cyclic object $M$, we notice that the 2k-th column is acyclic (II.6.7.18), thus there is a snake-like connection homomorphism $B$ that makes $M$ into a mixed complex $BM$. Cf.[Weibel P344]. And the Conne double complex will compute the same cyclic homology with previous defined cyclic homology, Cf.[Weible P345].

Notice for this $B, B_*$ on homology is exactly the composition $BI$. 
Prop. (II.6.7.24). Let $R$ be a unital commutative ring and $A$ is a commutative $R$-algebra and $M$ is a $A$-module, then there is a natural morphism
\[ M \otimes_A \Omega^n_{A/R} \xrightarrow{\varepsilon_n} H_n(A, M) \xrightarrow{\pi_n} M \otimes_A \Omega^n_{A/R}, \]
such that $\pi_n \circ \varepsilon_n = n!$.
We first define a map $\varepsilon_n : M \otimes A^n \rightarrow H_n(A, M)$ that
\[ \varepsilon_n(m, a_1, \ldots, a_n) = \sum \text{sgn}(\sigma)(m, a_{\sigma(1)}, \ldots, a_{\sigma(n)}) \]
then define $\varepsilon_n(m \otimes x a_1 \wedge \cdots \wedge d a_n) = \varepsilon_n(m, a_1, \ldots, a_n)$. And we verify that this map is well-defined and maps into $Z_n(C(A, M))$, Cf.[阳恩林循环同调 P99].
Then we define $\pi_n(m, a_1, \ldots, a_n) = m \otimes a_1 \wedge \cdots \wedge d a_n$ and verify easily that this vanish on $B_n(C(A, M))$. And it is easy to verify $\pi_n \circ \varepsilon_n = n!$.

Prop. (II.6.7.25). When $A$ is a unital $R$-algebra, there is a commutative diagram
\[
\begin{array}{c}
\Omega^n_{A/R} \xrightarrow{(n+1)d} \Omega^{n+1}_{A/R} \\
\varepsilon_n \downarrow \quad \varepsilon_n+1 \downarrow \quad \pi_n+1 \\
H^*_n(A) \xrightarrow{B} H^*_n+1(A)
\end{array}
\]
Proof: We notice $B = (1 - (-1)^n t)sN$:
\[ (m, a_1, \ldots, a_n) \mapsto \sum_{i=0}^{n} (-1)^i n(1, a_i, \ldots, a_n, m, a_1, \ldots, a_i_a) - \sum_{i=0}^{n} (-1)^i (a_i, 1, a_i+1, \ldots, a_n, m, a_1, \ldots, a_i-1). \]
Cf.[阳恩林循环同调 P128].

Cor. (II.6.7.26). For a commutative unital $R$-algebra $A$, there is a functorial $\varepsilon_n : \Omega^n_{A/R}/d\Omega^{n-1}_{A/R} \rightarrow HC_n(A)$ making the following diagram commutative:
\[
\begin{array}{c}
\cdots \xrightarrow{0} \Omega^{-1}/d\Omega^{-2} \xrightarrow{d} \Omega^{-1} \xrightarrow{0} \Omega^{-2}/d\Omega^{-3} \xrightarrow{d} \cdots \\
\varepsilon_{n-1} \downarrow \quad \varepsilon_n \downarrow \quad \varepsilon_{n-2} \downarrow \\
HC_{n-1} \xrightarrow{B} HH_n \xrightarrow{l} HC_n \xrightarrow{s} HC_{n-2} \xrightarrow{B} \cdots
\end{array}
\]
which is induced by the cokernel. Cf.[阳恩林循环同调 P130]. When $Q \in R$, $\varepsilon_n$ is a split injection.

Prop. (II.6.7.27). When $Q \in R$, $\frac{1}{2}\pi_n$ induces a morphism of mixed complexes $(BA, \partial, B) \rightarrow \Omega^n_{A/R}$ by (II.6.7.24), thus there is a natural map
\[ HC_n(A) \rightarrow \Omega^n_{A/R}/d\Omega^{n-1}_{A/R} \bigoplus_{i>0} H^*_{dR}(A). \]

Prop. (II.6.7.28) (Morita Invariance). $Tr : HH_*(M_r(A), M_r(M)) \cong HH_*(A, M)$ by the trace and inclusion functors. Cf.[阳恩林循环同调 Morita Invariance]. In particular, there is an isomorphism $HH_*(M_r(A)) \cong HH_*(A)$, thus also $HC_*(M_r(A)) \cong HC_*(A)$ by (II.6.7.20).

Prop. (II.6.7.29) (Karoubi). $BG$ is a cyclic group, and then the cyclic homology group $HC_n(G, A) \cong \bigoplus_{k \geq 0} H_{n-2k}(G, A)$. Cf.[Weibel P339].
Simplicial Homotopy

Prop. (II.6.7.30). For a Kan fibration $X$, there can be defined a homotopy groups $\pi_n$ that they agree with $\pi_i(|X|)$ thus also $\pi_i(S|X|)$, Cf.[Weibel P263]. Thus we see that $|BG|$ is truly the Eilenberg-Maclane spaces $BG$. 
II.7 Higher Topos Theory

1 Introduction

Remark (II.7.1.1). This subsection is a place that contains all the heuristic ideas leading to \(\infty\)-category. It should be deleted after the completion of this section.

Axiom (II.7.1.2) (Grothendieck’s Homotopy Hypothesis). Spaces and \(\infty\)-groupoids should be the same.

Axiom (II.7.1.3). Kan complexes and \(\infty\)-groupoids should be the same.

2 \(\infty\)-Categories

Def. (II.7.2.1) (\(\infty\)-Category). An \(\infty\)-category is a simplicial set that has lifting property w.r.t all \(\Lambda^n_i \to \Delta^n\), where \(0 < i < n\).

Prop. (II.7.2.2) (Characterization of \(\infty\)-Categories). \(C\) is an \(\infty\)-category iff the restriction map

\[
\text{Map}(\Delta^2, C) \to \text{Map}(\Lambda^2_1, C)
\]

is a trivial Kan fibration.

Proof: This follows immediately from (II.6.3.19).

Def. (II.7.2.3) (Equivalences). Two morphisms to an \(\infty\)-category are called homotopic if they define the same morphism in the homotopy category (II.6.4.10). An equivalence is a morphism that defined an isomorphism in the homotopy category.

Def. (II.7.2.4) (\(\infty\)-Groupoid). An \(\infty\)-groupoid is an \(\infty\)-category that \(hC\) is a groupoid (II.6.4.10).

Prop. (II.7.2.5) (\(\infty\)-Groupoids and Kan Complex). Let \(C\) be a simplicial set, then the following are equivalent:

- \(C\) is an \(\infty\)-groupoid.
- \(C \to \Delta^0\) is a left fibration.
- \(C \to \Delta^0\) is a right fibration.
- \(C\) is a Kan complex.

Proof: 1, 2 are equivalent by (II.7.2.10), and dually 1, 3 are equivalent, and 4 = 2 + 3.

Prop. (II.7.2.6) (Join). The join of two \(\infty\)-categories is also an \(\infty\)-category.

Proof: Given a morphism \(p : \Lambda^n_i \to S \ast S'\), if the image is in \(S\) or \(S'\), then it can be extended to \(\Delta^n\) by hypothesis. Then we may assume that it maps \(\{0, \ldots, j\}\) into \(S\) and \(\{j + 1, \ldots, n\}\) into \(S'\), then we restrict \(p\) to get a morphism \(\Delta^{[0, \ldots, j]} \to S, \Delta^{[j+1, \ldots, n]} \to S', \) which determines a map \(\Delta^n \to S \ast S'\) extending \(p\).

Prop. (II.7.2.7) (Homotopic Maps). If \(X\) is an \(\infty\)-category, then so does \(X^B\) for any simplicial set \(B\), by (II.6.3.26).

And we can call two maps in \(\text{Map}(B, X)\) homotopic if they are homotopic as vertices in \(X^B\) (II.7.2.3).
Lemma (II.7.2.8). Let \( p: \mathcal{C} \to \mathcal{D} \) be a left fibration of \( \infty \)-categories and \( f: X \to Y \) be a morphism that \( p(f) \) is an equivalence in \( \mathcal{D} \), then \( f \) is an equivalence in \( \mathcal{D} \). Compare with (II.1.7.8).

Proof: Let \( \overline{g} \) be a homotopy inverse to \( f \), then there is a 2-complex

\[
\begin{array}{ccc}
p(f) & \rightarrow & p(Y) \\
p(X) & \rightarrow & p(X)
\end{array}
\]

and by left fibration property lifts to a 2-complex

\[
\begin{array}{ccc}
f & \rightarrow & Y \\
g & \downarrow & \downarrow \\
X & \rightarrow & X
\end{array}
\]

So \( f \) admits a left homotopy inverse, and by the same reason, \( g \) admits a left homotopy inverse, thus \( g \) has a left homotopy inverse, and it can be chosen to be \( f \).

Lemma (II.7.2.9) (Equivalence Lifts via Left Fibrations). Let \( p: \mathcal{C} \to \mathcal{D} \) be a left fibration of \( \infty \)-categories, if \( X \in \mathcal{D} \) and \( Y \in \mathcal{C} \) and \( f: X \to p(Y) \) is an equivalence, then \( f \) can be lifted to a morphism in \( \mathcal{C} \). (Which is also an equivalence by (II.7.2.8)).

Proof: □

Prop. (II.7.2.10) (Equivalence and Left Extension). Let \( \mathcal{C} \) be an \( \infty \)-category and \( \varphi \) an edge, then \( \varphi \) is an equivalence iff for any \( n \geq 2 \) and every map \( \Lambda^n_0 \to \mathcal{C} \) that \( f_0|\Delta^{(0,1)} = \varphi \), there exists an extension of \( f_0 \) to \( \Delta^n \).

Proof: If \( \varphi \) is an equivalence, then consider the diagram

\[
\begin{array}{ccc}
\{0\} & \rightarrow & \mathcal{C}/\Delta^{n-2} \\
\downarrow & \nearrow \varphi' \\
\Delta^1 & \rightarrow & \mathcal{C}/\partial\Delta^{n-2}
\end{array}
\]

Because \( \mathcal{C}/\partial\Delta^{n-2} \to \mathcal{C} \) is right fibration (II.6.3.25), and by the dual of (II.7.2.8), \( \varphi' \) is an equivalence, thus by the dual of (II.7.2.9) the dotted arrow exists.

Conversely, if the condition holds, then we can use a diagram \( \Lambda^n_0 \to \mathcal{C} \) to find a morphism \( \psi \) that \( \psi \circ \varphi \cong \text{id} \), and we can also use a diagram \( \Lambda^n_0 \to \mathcal{C} \) to witness the fact \( \varphi \circ \psi \cong \text{id} \), so \( \varphi \) is an equivalence. □

Cor. (II.7.2.11). An equivalence in \( \text{Map}(K, \mathcal{C}) \) is equivalent to a map \( K \times \Delta^1 \to \mathcal{C} \) that \( \{x\} \times \Delta \) are all mapped to equivalences in \( \mathcal{C} \).

Proof: Cf.[HTT, P106]. □

Def. (II.7.2.12) (Space of Morphisms). For vertices \( x, y \) in a simplicial set \( S \), we want to defines a representative for \( \text{Map}_{\partial S}(x, y) \) other than \( \text{Map}_{\partial S}(x, y) \). We define the space of right morphisms

\[
\text{Hom}_S^R(x, y) = S_{/y} \times_S \{x\}.
\]
The definition is not symmetric, instead, we define the **space of left morphisms** \( \text{Hom}^L_S(x,y) = (\text{Hom}^R_{S^{op}}(x,y))^{op} \).

Also we define \( \text{Hom}_S(x,y) = \{x\} \times_S S^{\Delta^1} \times_S \{y\} \), then there are natural inclusions:

\[
\text{Hom}^R_S(x,y) \hookrightarrow \text{Hom}_S(x,y) \hookleftarrow \text{Hom}^L_S(x,y).
\]

**Prop. (II.7.2.13).** If \( \mathcal{C} \) is an \( \infty \)-category and \( x,y \in \mathcal{C} \), then \( \text{Hom}^R_{\mathcal{C}}(x,y) \) is a Kan complex.

*Proof:* This is obvious, because the right lifting diagram w.r.t. \( \Lambda^j_n \subset \Delta^n, 0 < j \leq n \) is equivalent to an extension \( \Lambda^j_n \star \Delta^0 \subset \Delta^{n+1} \) that satisfies \( \tilde{u}|_{\Delta^0} = x \). It can be solved by a two-step extension where the first is by identity extension and then extend using inner fibration property. \( \square \)

**Prop. (II.7.2.14) (Space of Morphisms Homotopic).** When \( S \) is an \( \infty \)-category, the inclusions defined in (II.7.2.12) are homotopy equivalences.

*Proof:* Cf. [HTT, 4.2.1.8]. \( \square \)

**Prop. (II.7.2.15).** Let \( p : \mathcal{C} \to \mathcal{D} \) be an inner fibration of \( \infty \)-categories, then the the induced maps on the spaces of right morphisms are Kan fibrations.

*Proof:* Since \( p \) is an inner fibration, the induced map \( \tilde{\varphi} : \mathcal{C}/Y \to \mathcal{D}/p(Y) \times_{\mathcal{D}} \mathcal{C} \) is a right fibration by (II.6.3.25), and the morphism on \( \text{Hom}^R_{\mathcal{C}}(X,Y) \) is obtained from \( \tilde{\varphi} \) by restricting to fiber over \( X \), thus also a right fibration. And by (II.7.2.13) and (II.6.3.31). \( \square \)

**Prop. (II.7.2.16) (Mapping Space).** Let \( \mathcal{C}, \mathcal{D} \) be \( \infty \)-categories and \( K, K' \) simplicial sets, then

- \( \text{Map}(K, \mathcal{C}) \) is also an \( \infty \)-category.
- If \( f : \mathcal{C} \to \mathcal{D} \) is a categorical equivalence, then the induced map \( \text{Map}(K, \mathcal{C}) \to \text{Map}(K, \mathcal{D}) \) is also a categorical equivalence.
- If \( g : K \to K' \) is a categorical equivalence, then the induced map \( \text{Map}(K', \mathcal{C}) \to \text{Map}(K, \mathcal{C}) \) is also a categorical equivalence.

*Proof:* 1 follows from (II.6.3.20). 2, 3 follows from [HTT, P94]. \( \square \)

**Prop. (II.7.2.17).** Let \( \mathcal{C} \) be an \( \infty \)-category and \( p : K \to \mathcal{C} \) be an morphism, then the projection \( \mathcal{C}_p/ \to \mathcal{C} \) is a left fibration. In particular, \( \mathcal{C}_p/ \) is itself an \( \infty \)-category.

*Proof:* We use the proposition (II.6.3.25) in case \( S = \Delta^0, A = \emptyset, X = \mathcal{C} \). \( \square \)

**Lemma (II.7.2.18).** Let \( \mathcal{C} \to \mathcal{D} \) be a fully faithful map of \( \infty \)-categories and a diagram \( K \to \mathcal{C} \), the map of Kan complexes (II.7.2.17)

\[
\mathcal{C}_{j/} \times_{\mathcal{C}} \{x\} \to \mathcal{D}_{p_{j/}} \times_{\mathcal{D}} \{p(x)\}
\]

is a homotopy equivalence.

*Proof:* Cf. [HTT, P134]. \( \square \)

**Prop. (II.7.2.19) (Invariance of Undercategories).** Let \( p : \mathcal{C} \to \mathcal{D} \) be a weak equivalence of \( \infty \)-categories and let \( j : K \to \mathcal{C} \) be a map, then the induced map \( \mathcal{C}_{j/} \to \mathcal{D}_{p_{j/}} \) is a categorical equivalence.
Proof: There is a factorization \( C_{ij} \xrightarrow{f} D_{pj} \times_D C \xrightarrow{g} D_{pj} \). (II.7.2.18) (II.7.2.17) shows \( C_{ij} \) and \( D_{pj} \times_D C \) are fiberwise equivalent left fibrations over \( C \), thus by (II.6.6.7) and (II.6.6.10), \( f \) is a categorical equivalence. Also, \( g \) is a categorical equivalence by (II.6.6.11). So we are done. \( \Box \)

Prop. (II.7.2.20) (Lifting of Homotopies). Let \( p : C \to D \) be a categorical equivalence of \( \infty \)-categories and \( A \subset B \) be an inclusion of simplicial sets. Let \( f_0 : A \to C, g : B \to D \) be any maps that \( h_0 : A \times \Delta^1 \to D \) be an equivalence between \( g|_A \) and \( p \circ f_0 \), then there exists a map \( B \to C \) and an equivalence \( h : B \times \Delta^1 \to D \) between \( g \) and \( p \circ f \) that \( h_0 = h|_{A \times \Delta^1} \).

Proof: Working with simplexes, it suffices to prove for \( A = \partial \Delta^n \subset B = \Delta^n \). The case \( n = 0 \) is true because categorical equivalence is essentially surjective. For \( n > 0 \), we need to construct \( h \) from \( h|_{\Delta^n \times \{0\}} \coprod_{\partial \Delta^n \times \Delta^1} \Lambda_{k+1} \). For \( k \neq 0 \), the extension is clear because \( D \) is \( \infty \)-category, and for \( k = 0 \), we need to use [HTT, P136]. \( \Box \)

Def. (II.7.2.21) (Correspondence). Let \( C, D \) be \( \infty \)-categories, an correspondence between \( C, D \) is defined to be an \( \infty \)-category \( M \) and a map \( M \to \Delta^1 \subset C \cong M_0 \) and \( D \cong M_1 \).

Prop. (II.7.2.22) (Equivalent Definition of Adjoints). Cf. [HTT, P5.2.2.8].

Prop. (II.7.2.23). Left adjoints preserves all colimits and right adjoints preserves all limits.

Def. (II.7.2.24) (Localization Category). A full subcategory \( C_0 \) of \( C \) is called a localization of \( C \) if the inclusion has a left adjoint.

\( n \)-Categories

Def. (II.7.2.25) (\( n \)-Categories). Let \( \mathcal{C} \) be a simplicial set and \( n \geq -1 \), then \( \mathcal{C} \) is called an \( n \)-category if it is an \( \infty \)-category and:
- Given any maps \( f, f' : \Delta^n \to \mathcal{C} \) that are homotopic (II.7.2.7) relative to \( \partial \Delta^n \), then \( f = f' \).
- For any \( m > n \) and maps \( f, f' : \Delta^m \to \mathcal{C} \) that coincide on \( \partial \Delta^m \), then \( f = f' \).

Also \( \mathcal{C} \) is called an \((-2\))-category if it is isomorphic to \( \Delta^0 \).

The definition of an \( n \)-category is equivalent to the following: if \( f, f' : K \to \mathcal{C} \) satisfies \( f|_{sk^n K} \) is homotopic to \( f'|_{sk^n K} \) relative to \( sk^{n-1} K \), then \( f = f' \).

Prop. (II.7.2.26). A \((-1\))-category is seen to be isomorphic to \( \emptyset \) or \( \Delta^0 \). A 0-category is equivalent to the nerve of a partially ordered set.

Prop. (II.7.2.27). If \( \mathcal{C} \) is an \( n \)-category and \( m > n + 1 \), then the restriction map \( \text{Hom}(\Delta^m, \mathcal{C}) \to \text{Hom}(\partial \Delta^m, \mathcal{C}) \) is bijective. (use \( \infty \)-category property to extend).

Prop. (II.7.2.28) (Usual Category as 1-Category). For a simplicial set \( S \), the following are equivalent:
- \( u : S \to N(hS) \) (II.6.4.7) is an isomorphism of simplicial sets.
- There is a small category \( \mathcal{C} \) that \( S \cong N(\mathcal{C}) \).
- \( S \) is a 1-category.
Proof: It suffices to show $3 \rightarrow 1$: we induct on the dimension: $n = 0$ is trivial and $n = 1$ follows from the definition of 1-category (II.7.2.25). For $n > 1$, the injectivity of $u$ follows from induction hypothesis and (II.7.2.25), and for surjectivity, for a map $\Delta^n \rightarrow N(hS)$, choose $0 < i < n$ and let lift $\Lambda^n_i$ to $S$, then use the fact $S$ is an $\infty$-category to lift to $\Delta^n$, and now it coincide on $N(hS)$ because it is a nerve of a category.

Prop. (II.7.2.29). If $\mathcal{C}$ is an $n$-category, then for any simplicial set $X$, $\mathcal{C}^X$ is also an $n$-category.

Proof: This is because $sk^p(K \times X) \subset sk^p(K) \times X$ for any simplicial set $K$ and integer $p$, and use (II.7.2.25). □

Prop. (II.7.2.30). Let $n \geq 1$ and $\mathcal{C}$ an $\infty$-category, then $\mathcal{C}$ is an $n$-category iff it satisfies the unique lifting property w.r.t. the inclusion $\Lambda^m_i \subset \Delta^m$, where $0 < i < m$.

Proof: Cf. [HTT, P109]. □

Def. (II.7.2.31) ($n$-Truncated Kan Complex). Let $X$ be a Kan complex and $k \geq -1$, then a Kan complex is called $k$-truncated if for every $i > k$ and every point $x \in X$, we have $\pi_i(X, x) \cong \ast$. And it is called $(-2)$-truncated if it is contractible.

Prop. (II.7.2.32). A $(-1)$-truncated Kan complex is either empty or contractible. A $0$-contractible Kan complex is a Kan complex that $X \rightarrow \pi_0(X)$ is a homotopy equivalence.

Proof: ?. □

Prop. (II.7.2.33) (Equivalent to an $n$-Category). Let $\mathcal{C}$ be an $\infty$-category and $n \geq -1$, then the following conditions are equivalent:

- There is a minimal model $\mathcal{C}' \subset \mathcal{C}$ that is $n$-truncated.
- $\mathcal{C}$ is categorically equivalent to an $n$-truncated category.
- For any $X, Y \in \mathcal{C}$, the mapping space $\text{Map}(X, Y)$ is $(n - 1)$-truncated.

Proof: Cf. [HTT, P112]. □

Cor. (II.7.2.34). A Kan complex is categorically equivalent to an $n$-category iff it is $n$-truncated.

Proof: Cf. [HTT, P113]. □

Cor. (II.7.2.35). Let $\mathcal{C}$ be an $\infty$-category and $K$ a simplicial set, if $\text{Map}(C, D)$ is $n$-truncated for any objects $C, D \in \mathcal{C}$, then the $\infty$-category $\mathcal{C}^K$ has the same property.

Proof: Cf. [HTT, P114]. □

3 $\infty$-Category of $\infty$-Categories

4 Limits and Colimits

5 Presentable and Accessible $\infty$-Categories

Def. (II.7.5.1) (Presentable $\infty$-Category). An $\infty$-category is called presentable if it is accessible and admits small limits.

Prop. (II.7.5.2). A presentable $\infty$-category also admits all small limits, Cf. [HTT, P5.5.2.4].
Prop. (II.7.5.3). An $\infty$-category is presentable iff it is an accessible localization of an $\infty$-category of presheaves, Cf. [HTT, P5.5.1.1].

Prop. (II.7.5.4) (Adjoint Functor Theorem). Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable $\infty$-categories, then

- $F$ admits a right adjoint iff it preserves small colimits.
- $F$ admits a left adjoint iff it preserves small limits and $\kappa$-filtered colimits for some regular cardinal $\kappa$.

Proof: [HTT, 5.5.2.9].

6 $\infty$-Algebras

7 Topological Cyclic Homology(Scholze)
II.8 Stable ∞-Categories

1 Stable ∞-Categories

Main References are [HTT, Chap 8].

Def. (II.8.1.1) (Stable ∞-Category). Stable ∞-category is a "linearized ∞-category" for doing geometry. A Stable ∞-category is an ∞-category that
- There exists a zero element.
- Every morphism $g$ in $C$ admits a fiber and cofiber over the zero object.
- A triangle in $C$ is a fiber sequence iff it is a cofiber sequence.

Def. (II.8.1.2) (Suspensions and Loops).

Prop. (II.8.1.3). The suspension and loop are functors from $C \to C$ that are mutually inverse equivalences.

Proof: Cf.[HTT, 4.3.1.5]. □

Prop. (II.8.1.4) (Stable ∞-Categories and Triangulated Categories). The distinguished triangles in $hC$ endow $hC$ with the classical structure of triangulated category.

Prop. (II.8.1.5) (HA.1.1.3.4). A stable ∞-category admits all finite limits and colimits. And pullback squares and pushout squares coincide.

Prop. (II.8.1.6). Let $C$ be a pointed ∞-category, then the following are equivalent:
- $C$ is stable.
- $C$ admits finite colimits, and the suspension functor $\Sigma : C \to C$ is an equivalence.

Proof: Cf.[HA, 1.4.2.27]. □

Def. (II.8.1.7) (Exact Functors). A functor $F : C \to D$ between stable ∞-categories is called exact if the following equivalent definitions hold:
- $F$ preserves fiber sequences.
- $F$ is left exact.
- $F$ is right exact.

Prop. (II.8.1.8). Let $Cat^Ex_\infty \subset Cat_\infty$ be the subcategory of all stable ∞-categories and exact functors, then it admits all small limits and small filtered colimits, and they are preserved by the inclusion.

Proof: Cf.[HA1.1.4.4., 1.1.4.6.] □

Def. (II.8.1.9) (T-Structure). A $T$-structure on a stable ∞-category $C$ is a pair of full subcategories $C_{\geq 0}, C_{\leq 0}$ that

Prop. (II.8.1.10). For any $n \in \mathbb{Z}$, $C_{\leq n} \subset C$ is a localization, thus admits a left adjoint $\tau_{\leq n}$, called the truncation functor. Dually for $\tau_{\geq n}$.

Proof: Cf.[HA, P1.2.1.5]. □

Def. (II.8.1.11) (Heart). The heart $C^\heartsuit \subset C$ is defined to be the full subcategory of $C_{\leq 0} \cap C_{\geq 0} \subset C$.

2 Monoidal ∞-Categories
Chapter III

Representation Theory

III.1 Representation Theory

For representations, the theory of semisimple algebras should be kept in mind.

1 Linear Representations

Def. (III.1.1.1) (Character). Let $\rho : G \to GL(n, V)$ be a linear representation of f.d. Then the character of $\chi_{\rho}$ is defined to be $\chi_{\rho}(g) = tr(\rho(g))$.

Prop. (III.1.1.2) (Schur’s lemma). If $\pi$ is an at most countable dimensional irreducible $\mathbb{C}$-representation of an algebra $A$, then $\text{End}(V) \cong \mathbb{C}$. In particular this holds for $\dim A$ countable.

Proof: First the dimension of $\dim \mathbb{C}\text{End}(V)$ is at most countable, because $V$ is acyclic by irreducibly, so $\dim \mathbb{C}\text{End}(V) \leq \dim \mathbb{C} V$. And $\text{End}(V)$ is a skew field, by irreducibility. So the result follows from (I.2.3.24).

Prop. (III.1.1.3). Any f.g.(i.e. f.g. over $F[G]$) representation of a group has an irreducible quotient.

Proof: Use Zorn’s lemma for the set of proper $G$-subspaces of $U$, the combine of a chain of proper $G$-space is proper, because it is f.g., so it has a maximal proper $G$-space, so the quotient is irreducible.

Prop. (III.1.1.4) ((Co)Induced Representation). Following (I.2.4.14), for a group $G$ and a subgroup $H$, we define the induced/coinduced representation to be of $\mathbb{Z}[G] \otimes \mathbb{Z}[H]$ and $\text{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], -)$ from $\mathbb{Z}[H]$ – Mod to $\mathbb{Z}[G]$ – Mod, denoted by Ind, ind.

I.e., if $H$ acts on $V$ by $\rho$, then $\text{Ind}(\rho)$ is the space $\bigoplus_{\gamma \in G/H} V_{\gamma}$ where $V_{\gamma} \cong V \in \text{Mod}_H$, and that for $v_{\gamma} \in V_{\gamma}$, $(\text{Ind}(\rho)g)v_{\gamma} = \rho(h)v_{\gamma'} \in V_{\gamma'}$ where $g\gamma = \gamma' h$ that $\gamma' \in G/H, h \in H$.

And $\text{ind}(\rho)$ is the space of functions $f$ from $G$ to $V$ that satisfies $f(hg) = \rho(h)f(g)$, and for $f \in \text{ind}(\rho)$, $(\text{ind}(\rho)g)f(g') = f(g'g)$ (I.2.4.13).

If $[G : H]$ is finite, then induced is the same as coinduced. In this case, $\dim \text{Ind}_H^G(V) = [G : H] \dim V$. Notice these two are often mistaken when only one appears. It is my fault. So please distinguish them.
Choose a set of left coset representatives $\Gamma$ of $G/H$, then $\Gamma^{-1}$ is a set of right coset representatives for $H \backslash G$. Now $\mathbb{Z}[G] = \bigoplus_{\gamma \in \Gamma} \mathbb{Z}[H]_{\gamma^{-1}}$, and we define the space $V_{\gamma} \cong V$ of maps from $\mathbb{Z}[G]$ to $V$ determined by: $v_{\gamma}(g) = \rho(g\gamma)v \in V$ if $g \in H\gamma^{-1}$ and 0 otherwise.

Then $\text{ind}(\rho)(V) = \bigoplus_{\gamma \in \Gamma} V_{\gamma}$, that for $v_{\gamma} \in V_{\gamma}$, $(\text{ind}(\rho)g)v_{\gamma}$ is the map $\mathbb{Z}[G]$ to $V$ determined by: $(\text{ind}(\rho)g)v_{\gamma}(g) = \rho(g\gamma'h)v \in V$ if $g'h \in H\gamma'^{-1}$, and 0 otherwise, where $g\gamma = \gamma'h$. Then $\text{ind}(\rho)g)v_{\gamma} = \rho(h)v_{\gamma'}$, which is the same as the formula for $\text{Ind}(\rho)$ as in (III.1.1.4). So the map $v_{\gamma} \mapsto v_{\gamma} : \text{Ind}(\rho) \to \text{ind}(\rho)$ is an isomorphism in $\mathbb{Z}[G] - \text{Mod}$. □

**Prop. (III.1.1.5)** (Clifford’s Theorem). If $\rho : G \to GL(V)$ is a semisimple representation and $H$ is a normal subgroup of $G$, then $\rho|_H$ is also semisimple.

**Proof:** Use definition (I.4.1.4), we reduce to the case $\rho$ is simple. Now an $H$-subrepresentation is simple iff it has no proper $H$-subrepresentations, so clearly $G$ maps a simple $H$-subrepresentation to another simple $H$-subrepresentation. So if $W$ is the sum of all simple $H$-subrepresentations, then $G$ preserves $W$, which shows $W = V$, and $V$ is $H$-semisimple by (I.4.1.4). □

### 2 Linear Representation of Finite Groups

Basic references are [Ser77].

The representations in this subsection is assumed to be of char0, in particular, over a subfield of $\mathbb{C}$.

**Remark (III.1.2.1).** Because a finite group is compact, all results of compact groups is applicable to a finite group, see 4.

**Cor. (III.1.2.2)** (Representations Determined by Characters). A representation of $G$ over $\mathbb{C}$ is determined by its character, by (X.6.4.25).

**Cor. (III.1.2.3)**. The characters $\chi_i$ of irreducible representations of $G$ form a basis of $ZL^2(G)$ by (X.6.4.24). Also, for $s \in G$, let $c(s)$ be the number of elements in the conjugacy class of $s$, then:

- $\int_{G} |\chi_i(s)|^2 = g/c(s)$.
- If $t$ is not conjugate to $s$, then $\sum_i \chi_i(s)\bar{\chi}_i(t) = 0$.

**Proof:** The last assertion follows from the first one, if you consider a matrix with conjugacy classes as column and characters as rows, and place $\sqrt{c(s)/g}\chi_i(s)$ in the entries, then it is an orthogonal matrix. □

**Cor. (III.1.2.4)** (Number of Representations). If $G$ is a finite group, then the cardinality of $\hat{G}$ is equal to the number of conjugates of $G$, and $\sum_{\pi \in \hat{G}} d_{\pi}^2 = |G|$.

**Proof:** Both $\{\chi_{\pi}\}$ and the characteristic functions of the conjugate classes of $G$. And the second assertion follows from the Peter-Weyl theorem (X.6.4.18) as $\sum_{\pi \in \hat{G}} d_{\pi}^2$ is the dimension of $L^2(G)$. □

**Cor. (III.1.2.5)**. $G$ is Abelian iff every irreducible representation of $G$ is of dimension 1.

**Proof:** This follows immediately from the equation $\sum_{\pi \in \hat{G}} d_{\pi}^2 = |G|$ (III.1.2.4), as $G$ is Abelian iff it has $|G|$ conjugacy classes iff $|\hat{G}| = |G|$ iff $d_{\pi} = 1$ for any $\pi$. □

**Prop. (III.1.2.6)**. If $G$ is a finite $p$-group and $A$ is a nonzero $p$-torsion $G$-module, then $A^G \neq 0$.

**Proof:** We may consider $A$ generated by a single element. Because $A$ is $p$-torsion, $|A| = p^n$ for some $n$. Now consider the orbit, then if the orbit is not a single element, then its order is divisible by $p$, so $|A^G|$ is divisible by $p$. But 0 is fixed, so $A^G \neq 0$. □
III.1. REPRESENTATION THEORY

Group Algebra $\mathbb{C}[G]$

Prop. (III.1.2.7) (Maschke’s Theorem). If $F$ is a field of char $p$ and $G$ is a finite group of order prime to $p$, then for any representation $U$ of $F[G]$ and a submodule $V$, there exists a complement of $V$ in $U$.

Proof: Choose an arbitrary projection $\pi$ of $U$ to $V$, and let $\rho(v) = 1/|G| \sum g^{-1} \pi(g(v))$, then it can be checked $\rho$ commutes with $G$-actions, thus its kernel is also a $G$-modules, and it is identity on $V$, so $U = V \oplus k \ker \rho$. □

Cor. (III.1.2.8) (Totally Decomposable). Any such representation of $G$ is a direct sum of irreducible representations.

Prop. (III.1.2.9) (Brauer-Nesbitt). For a finite group $G$, if two finite dimensional semisimple representations over a field has the same char poly for every element $g$ of $G$, then they are isomorphic.

Proof: Just use the irreducible representations are orthogonal and that they have the same and for char $p$, we can use divide by $p$ and the char poly becomes $p$-th power and we can do this forever, contradiction. □

Prop. (III.1.2.10). Integral properties of characters.

Prop. (III.1.2.11). The degree of the irreducible representations of $G$ divides the order of $G$.

Proof: □

Prop. (III.1.2.12) (Burnside’s Theorem). Any group of order $n$ that $n$ has only two groups are solvable.

Proof: □

Induced Representations and Mackey Theory

Prop. (III.1.2.13) (Character of Induced Representations). Character of induced representations, Cf.[Serre, P30].

Prop. (III.1.2.14) (Mackey’s Restriction Theorem). Let $H,K$ be subgroups of a finite group $G$, $(\tau,W)$ is a representation of $H$. If $s \in G$, then we can define a new representation of $H_s = sHs^{-1} \cap K$ as $\tau^s(g) = \tau(s^{-1}gs)$. Then

$$\text{res}_K^G \text{ind}_H^G \tau \cong \bigoplus_{s \in H \backslash G/ K} \text{ind}_{H_s}^K \tau^s.$$ 

Proof: □

Cor. (III.1.2.15) (Mackey’s Intertwining Theorem). Let $H,K$ be subgroups of a finite group $G$, $(\sigma,U)$ is a representation of $K$ and $(\tau,W)$ is a representation of $H$, define $\tau^s$ as in(III.1.2.14), then

$$\text{Hom}_G(\text{ind}_K^G \sigma, \text{ind}_H^G \tau) \cong \bigoplus_{s \in H \backslash G/ K} \text{Hom}_{H_s}(\sigma, \tau^s).$$
Proof: This is a direct consequence of Mackey’s restriction theorem and Frobenius reciprocity (X.6.5.4):

\[ \text{Hom}_G(\text{ind}_K^G \sigma, \text{ind}_H^G \tau) \cong \text{Hom}_K(\sigma, \text{res}_K^G \text{ind}_H^G \tau) \cong \bigoplus_{s \in H \setminus G/K} \text{Hom}_K(\sigma, \text{ind}_K^H s \tau^s) \cong \bigoplus_{s \in H \setminus G/K} \text{Hom}_{H_s}(\sigma, \tau^s). \]

\[\square\]

Prop. (III.1.2.16) (Mackey). Let \( G \) be a finite group and \( H_1, H_2 \) be two subgroups and \((\pi_1, V_1)\) be representations of \( H_i \) respectively, Then the space \( \text{Hom}_G(\text{ind}_{H_1}^G (V_1), \text{ind}_{H_2}^G (V_2)) \) is isomorphic to the space \( D \) of functions \( \Delta : G \to \text{Hom}_C(V_1, V_2) \) that satisfies

\[ \Delta(h_2gh_1) = \pi_2(h_2) \Delta(g) \pi_1(h_1) \]

where for each such \( \Delta \), the corresponding \( L \in \text{Hom}_G(\text{ind}_{H_1}^G (V_1), \text{ind}_{H_2}^G (V_2)) \) is given by

\[ L : \text{Hom}_G(\text{ind}_{H_1}^G (V_1), \text{ind}_{H_2}^G (V_2)) : f_1 \mapsto \Delta \ast f_1 \]

where \( (\Delta \ast f_1)(x) = \frac{1}{|G|} \sum_{g \in G} \Delta(xg^{-1})(f(g)) \).

Proof: Firstly given a \( \Delta \in D \), then it is clear if \( f_1 \in \text{ind}_{H_1}^G (V_1) \), then \( \Delta \ast f_1 \in \text{ind}_{H_2}^G (V_2) \), and \( \Delta \ast f \) is linear in \( f \), so there is a linear map \( D \to \text{Hom}_G(\text{ind}_{H_1}^G (V_1), \text{ind}_{H_2}^G (V_2)) \).

The converse of this map is constructed by mapping a \( L \in \text{Hom}_G(\text{ind}_{H_1}^G (V_1), \text{ind}_{H_2}^G (V_2)) \) to a \( \Delta : G \to \text{Hom}_C(V_1, V_2) \) that

\[ \Delta(g)v = [G : H_1]L(f_{g^{-1}v})(1), \]

where \( f_{g,v}(x) = \pi_1(h)v \) if \( x = hg \) for some \( h \in H_1 \), and \( f_{g,v} = 0 \) otherwise.

Firstly \( f_{g,v} \in \text{ind}_{H_1}^G (V_1) \) is clear, and \( \Delta(g) \in D \) because \( \Delta(g)v = \Delta(g)\pi_1(h_1)v \) and \( \Delta(h_2g) = \pi_2(h_2)\Delta(g) \). In the process, it should be checked that \( f_{h_1^{-1}g^{-1}v} = f_{g^{-1}, \pi_1(h_1)v}, f_{g^{-1}h_2^{-1}v}(x) = f_{g^{-1}, v}(xh_2) \).

And we check these two mapping are inverse to each other, this is not hard, it is formal. \[\square\]

Cor. (III.1.2.17) (Convolution Structure). In this isomorphism, if \( H_3 \) is a third subgroup, then we have

\[ \Delta_{13}(L_{23} \circ L_{12}) = \Delta_{23}(L_{23}) \ast \Delta_{12}(L_{12}) \]

where the convolution is clear.

Proof: It suffices to show

\[ (L_{23} \circ L_{12})(f_1) = (\Delta_{23}(L_{23}) \ast \Delta_{12}(L_{12})) \ast f_1. \]

But this is clear, because by the clear associativity of convolution,

\[ (\Delta_{23}(L_{23}) \ast \Delta_{12}(L_{12})) \ast f_1 = \Delta_{23}(L_{23}) \ast (\Delta_{12}(L_{12}) \ast f_1) = \Delta_{23}(L_{23}) \ast L_{12}f_1 = L_{23}L_{12}f_1. \]

\[\square\]
Rationality Problems

Def. (III.1.2.18) (Ring $R_K(G)$). We want to consider the representations over a subfield $K$ of $\mathbb{C}$.

Let $R_K(G)$ be the $\mathbb{Z}$-module generated by the characters of the representations of $G$ over $K$, then it is a subring of $R(G) = R_{\mathbb{C}}(G)$. And define the $\mathbb{Z}$-module $\overline{R}_K(G)$ to be the elements of $R(G)$ with values in $K$. Clearly $R_K(G) \subset \overline{R}_K(G)$.

Prop. (III.1.2.19) (Induction and Restriction Morphism). Let $H$ be a subgroup of $G$, then the induction induces an Abelian group homomorphism $R(H) \to R(G)$, and restriction induces a ring homomorphism $R(G) \to R(H)$. The formula $\text{Ind}(\varphi \cdot \text{res}(\psi)) = \text{Ind}(\varphi) \cdot \psi$ shows the image of Ind is an ideal of $R(G)$. Also by Frobenius reciprocity (X.6.5.4), Ind and Res are dual to each other:

$$\langle \varphi, \text{res} \psi \rangle_H = \langle \text{ind} \varphi, \psi \rangle_G.$$

Prop. (III.1.2.20). Let $\rho_i$ be the isomorphism classes of all irreducible linear representations of $G$ over $K$ and $\chi_i$ there characters. Then

- $\chi_i$ form a basis of $R_K(G)$.
- $\chi_i$ are mutually orthogonal.

Proof:

Cor. (III.1.2.21). A representation of $G$ over $\mathbb{C}$ is realizable over $K$ iff its character belongs to $R_K(G)$.

Proof: One direction is trivial, for the other, if $\chi \in R_K(G)$, then $\chi = \sum n_i \chi_i$, and $\langle \chi, \chi_i \rangle = n_i \langle \chi_i, \chi_i \rangle$. As $\langle \chi, \chi_i \rangle \geq 0$ as they are representations of $G$, we have $n_i \geq 0$, thus $\rho = \sum n_i \rho_i$ is realizable over $K$, by (III.1.2.2).

Def. (III.1.2.22) (Schur Indices). $K[G]$ is called quasisplit if the $D_i$ are all commutative, or equivalently, all $m_i = 1$.

Prop. (III.1.2.23). If $L/K$ is finite and $L[G]$ is quasisplit, then $[L : K]$ is divisible by each of the Schur indices $m_i$.

Proof:

Prop. (III.1.2.24). The characters $\psi = \chi_i/m_i$ form a basis of $\overline{R}_K(G)$.

Proof:

Cor. (III.1.2.25). $R_K(G) = \overline{R}_K(G)$ iff $K[G]$ is quasisplit.

Cor. (III.1.2.26) (Brauer). If $m$ is the least common multiples of the orders of the elements of $G$ and $K$ contains the $m$-th roots of unity, then $R_K(G) = R(G)$.

Proof:

Cor. (III.1.2.27). If $m$ is the least common multiples of the orders of the elements of $G$, then all the Schur indices of $G$ over any field $K$ divides the Euler function $\varphi(m)$.
Proof: This follows from (III.1.2.26) and (III.1.2.23) by considering the field $K[\mu_m]$ over $K$. □

Def. (III.1.2.28) (Galois-Action on $G$). Let $L = K(\mu_m)$, where $m$ divides the order of any element in $G$, then $L/K$ is Galois and $G/L/K = \Gamma_K$ is a subgroup of $(\mathbb{Z}/m\mathbb{Z})^*$. Then this group can act on $G$ by $\sigma_t(x) = x^n$ as a set, and we call two elements $s, s' \in G$ $\Gamma_K$-conjugate iff they are in the same $\Gamma_K$ orbits of $G$.

Prop. (III.1.2.29). A class function $f$ on $G$ with values in $L$ belongs to $K \otimes \mathbb{Z} R(G)$ iff
$$\sigma_t(f(s)) = f(s^t)$$
for $\sigma_t \in \Gamma_K$ and $s \in G$.

Proof: Cf. [Serre, P95]. □

Cor. (III.1.2.30). A class function $f$ on $G$ with values in $K$ belongs to $K \otimes \mathbb{Z} R_K(G)$ iff it is constant on the $\Gamma_K$-orbits of $G$.

Proof: Because the □

Prop. (III.1.2.31). For a finite group $G$, all representations of $G$ has characters in $\mathbb{Q}$ iff it all representations have characters in $G$, iff every two element generating the same subgroup of $G$ is conjugate.

Proof: Cf. [Serre, P103]. □

Cor. (III.1.2.32). representations of $S_n$ all has characteristic in $\mathbb{Z}$.

Artin’s Theorem & Brauer’s Theorem

Prop. (III.1.2.33) (Generalized Artin Theorem). Let $X$ be a family of subgroups of a finite group $G$. Let $\text{Ind} : \bigoplus_{H \in X} R_K(H) \rightarrow R_K(G)$ be the ring homomorphism induced by induction, then the following properties are equivalent:

- $G$ is the union of conjugates of the subgroups in $X$.
- the cokernel of $\text{Ind}$ is finite.

Proof: $2 \rightarrow 1$: By the character of induced representations (III.1.2.13), any function in the image of $\text{Ind}$ vanishes outside the union of conjugates of the subgroups in $X$, so if this is not $G$, then the cokernel cannot by finite.

$1 \rightarrow 2$: Notice the duality of $\text{Ind}$ and $\text{Res}$ (III.1.2.19), it suffices to show that $\text{Res}$ is injective, but this is clear. □

Cor. (III.1.2.34) (Artin Theorem). Choose $X$ as the family of cyclic subgroups of $G$, then every character of $G$ is a rational combination of characters induced from cyclic subgroups of $G$.

Proof: □

Prop. (III.1.2.35) (Brauer’s Theorem).

Proof: □

Prop. (III.1.2.36) (Generalized Brauer’s Theorem).

Proof: □
Important representations

Prop. (III.1.2.37) \((Q_8)\). There is a 2-dimensional representation of the quadratic group \( Q_8 = \{\pm1, \pm i, \pm j, \pm k\} \):

\[
i \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad j \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad k \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

Prop. (III.1.2.38). There is a representation of \( S_n \) on the \( n-1 \)-dimensional hypersurface \( \sum x_i = 0 \).

3 Modular Representations

Prop. (III.1.3.1). The only irreducible representation of a \( p \)-group over a a field of char \( p \) is the trivial representation.

Proof: For any \( v \in V \), consider the additive subgroup generated by \( g(s)v \), then it is a finite group of prime power order. Then [I.3.5.4] shows it has a element other than 0 fixed by all \( G \), thus it is not irreducible unless trivial representation.

□

4 Topological Groups

Def. (III.1.4.1). Let \( G \) be a topological group and \( V \) a TVS, then a representation of \( G \) on \( V \) is a group action that \( G \times V \to V \) is continuous.

Locally Compact Groups

Cf. X.6.

5 \((\mathfrak{g}, K)\)-Modules

Real Reductive Groups

Def. (III.1.5.1) (Admissible Representation). Let \( G \) be a connected real reductive group (which is relevant, thus the complex representations of \( G \) and \( G(\mathbb{R}) \) are the same [IX.8.8.8]), let \( K \subset G(\mathbb{R}) \) be a maximal compact subgroup. Consider \( V^\infty, V^\rho, V^{K-fin} \) as in [X.6.4.8] [IX.8.9.6].

A representation \( V \) of \( G \) is called admissible if for any \( \rho \), \( V^\rho \) is of f.d.

Prop. (III.1.5.2). For any \( \rho \), \( V^\infty \cap V^\rho \) is dense in \( V^\rho \).

Proof: 

Cor. (III.1.5.3). If \( V \) is admissible, then \( V^{K-fin} \subset V^\infty \) by [IX.8.9.10].

Prop. (III.1.5.4). If \( V \) is admissible, then \( V^{K-fin} \) is a \((\mathfrak{g}, K)\)-module (III.1.5.8), and the map \( V \mapsto V^{K-fin} \) induces a functor

\[ \text{Rep}(G)_{adm} \to (\mathfrak{g}, K) - \text{mod}_{adm}. \]

And we call two admissible representations \( V_1, V_2 \) of \( G \) infinitesimal equivalent iff they are isomorphic after this functor.
Proof: By (III.1.5.3), \( \mathfrak{g} \) can act on \( V^{K-fin} \), and \( U(\mathfrak{g}) \) fixes \( V^{K-fin} \): if \( f \in V^{K-fin} \), let \( R \) be a f.d. \( K \)-subspace of \( V \) containing \( f \), then \( \mathfrak{g} f \in R \). Let \( R_1 \) be the f.d. vector space spanned by \( \mathfrak{g} R \), then \( R_1 \) is invariant under \( \mathfrak{g} \): for \( X \in \mathfrak{g}, Y \in \mathfrak{g}, \varphi \in R \)

\[
X(Y \varphi) = [X,Y] \varphi + Y(X \varphi) \in R_1
\]

so \( R_1 \subset V^{K-fin} \), so \( \mathfrak{g} \) fixes \( V^{K-fin} \).

Also we check

\[
T_k T_\eta T_k^{-1} = T_{Ad_k \eta}
\]

which is by definition, and the second condition in (IX.8.9.10) is also obvious. \( \square \)

Lemma (III.1.5.5). Let \( V \) be an admissible representation of \( G \), and \( v \in V^{K-fin} \), then for any \( \eta \in V^* \), the function \( g \mapsto \eta(g(v)) \) is real analytic.

Proof: Cf. [Gaitsgory P37]. \( \square \)

Prop. (III.1.5.6) (Rep(G) and (\( \mathfrak{g}, K \))-Modules).

- If \( V_1, V_2 \) be two admissible representations of \( G \), if \( S : V_1 \to V_2 \) is a continuous map of TVS. Assume \( S(V_1^{K-fin}) \subset V_2^{K-fin} \) and induces a \( (\mathfrak{g}, K) \)-module map, then the initial \( S \) is a map of \( G \)-representations.

- If \( V \in \text{Rep}(G)_{adm}, M = V^{K-fin} \), then the functors

\[
(V_1 \subset V) \mapsto (V_1)^{K-fin} \subset M; \quad (M_1 \subset M) \mapsto M_1 \subset V
\]

induces mutually inverse bijections between closed \( G \)-subrepresentations of \( V \) and \( (\mathfrak{g}, K) \)-submodules of \( M \).

Proof: 1: It suffices to show for \( v_1 \in V^{K-fin}, T_g S(v_1) ST_g(v_1) \). So by Hahn-Banach it suffices to show for any \( \eta \in V_2^* \),

\[
\eta(T_g S(v_1)) = \eta(ST_g(v_1)).
\]

Both sides are analytic in \( g \) by (III.1.5.5), so it suffices to show all their derivatives at 1 are equal, and use the fact \( \pi_0(K) \to \pi_0(G) \) is surjective (IX.8.8.11). And the derivatives equal because \( S \) commutes with \( \mathfrak{g} \)-action.

2: Firstly \( M_1 \) is a \( G \)-representation: because \( M_1 = ((M_1)^{K-fin})^{K-fin} \) by Hahn-Banach. so it suffices to show for \( v_1 \in M_1, \eta(g(v_1)) = 0 \) for any \( \eta \in (M_1)^{K-fin} \). Then this uses analyticity (III.1.5.5) as above and the fact \( M_1 \) is a \( (\mathfrak{g}, K) \)-subrepresentation.

For the bijection, notice \( V_1^{K-fin} \) is dense in \( V_1 \) by (X.6.4.19). Conversely, for a submodule \( M_1 \), it suffices to show the image of \( T_{\xi,\rho,H_{lao}} (M_1) \subset M_1^{K-fin} \) by (X.6.4.17). However \( T_{\xi,\rho,H_{lao}} (M_1) \in M_1^{K-fin} \) by (X.6.4.17), so this is true by continuity. \( \square \)

Cor. (III.1.5.7) (Irreducibility of \( (\mathfrak{g}, K) \)-Modules). An admissible \( G \)-representation \( V \) is irreducible iff \( V^{K-fin} \) is irreducible as \( (\mathfrak{g}, K) \)-modules.

\( (\mathfrak{g}, K) \)-Modules

Def. (III.1.5.8) \( (\mathfrak{g}, K) \)-Modules. If \( G \) is a Lie group and \( K \subset G \) be a maximal compact Lie subgroup (IX.8.8.11), then \( K \) acts on \( \mathfrak{g} \). Then a \( (\mathfrak{g}, K) \)-Module is a \( \mathbb{C} \)-vector space that has a \( K \)-finite action and a \( \mathfrak{g} \) action that satisfies:
- for $k \in K, \eta \in \mathfrak{g}$, $T_k T_\eta T_{k^{-1}} = T_{\text{Ad}_k(\eta)}$.
- The action of $\mathfrak{t}$ on $V$ induced by $K$ agrees with the restriction of the action of $\mathfrak{g}$.

A $(\mathfrak{g}, K)$-module is called \textbf{admissible} iff every $V^\rho$ part is of f.d. as $K$-representation.

**Def. (III.1.5.9).** If $M$ is an admissible $(\mathfrak{g}, K)$-module, then we can define its \textbf{algebraic dual} as

$$(M^*)^\text{alg} = \bigoplus_\rho (M^\rho)^*,$$

which is the $K$-finite part of the usual dual of $M$.

**$(\mathfrak{g}, K)$-Modules and $\mathfrak{g}$-Modules**

**Def. (III.1.5.10).** Let $K_0$ be the unital component of $K$, then there are forgetful functors

$$(\mathfrak{g}, K) - \text{mod} \to (\mathfrak{g}, K_0) - \text{mod} \to \mathfrak{g} - \text{mod}.$$ 

**Prop. (III.1.5.11).** The functor $(\mathfrak{g}, K) - \text{mod} \to \mathfrak{g} - \text{mod}$ is fully faithful, and its essential image is stable under taking submodules.

**Proof:** Cf.[Gaitsgory P39].

**Cor. (III.1.5.12).** If $M \in (\mathfrak{g}, K) - \text{mod}$ is irreducible as a $\mathfrak{g}$-module, then it is irreducible.

**Prop. (III.1.5.13).** The functor $(\mathfrak{g}, K) - \text{mod} \to \mathfrak{g} - \text{mod}$ sends f.g. objects to f.g. objects.

**Proof:** By (III.1.5.11), it suffices to consider the functor $(\mathfrak{g}, K) - \text{mod} \to (\mathfrak{g}, K_0) - \text{mod}$. Let $M$ be f.g. $(\mathfrak{g}, K)$-module, and $\cup M_i = M$ be a chain of $(\mathfrak{g}, K_0)$-submodules. Pick $k \in K$ for each element of $\pi_0(K)(\text{f.m.})$, then each $M'_i = \sum_k k(M_i)$ is a $(\mathfrak{g}, K)$-submodule, thus $M'_i = M$ for some $i$. Now we can choose $j$ large that $k(M_i) \in M_j$ for any $k$, then $M_j = M$ (because we may choose $M_i$ be f.g. $(\mathfrak{g}, K_0)$-modules).

**Cor. (III.1.5.14).** The category $(\mathfrak{g}, K) - \text{mod}$ is Noetherian.

**Proof:** If $M \in (\mathfrak{g}, K) - \text{mod}$ is f.g. and $M_1 \subseteq M$, then $M$ is f.g. as $\mathfrak{g}$-module, then $M_1$ is f.g. as $\mathfrak{g}$-module by (I.12.8.16). So clearly it is also f.g. as a $(\mathfrak{g}, K)$-module.

**Prop. (III.1.5.15).** For an irreducible $(\mathfrak{g}, K)$-module $M$, the underlying $\mathfrak{g}$-module is a direct sum of f.m. irreducibles.

**Proof:** By (III.1.5.11), it suffices to prove $M$ is a direct sum of f.m. irreducible $(\mathfrak{g}, K_0)$-modules. $M$ is f.g. as a $(\mathfrak{g}, K_0)$-modules by the proof of (III.1.5.13), so it has a maximal submodule $M'$ that $N = M/M'$ is irreducible. Pick $k \in K$ for each component of $K$, consider

$$M'' = \cap_k k(M')$$

which is a proper $(\mathfrak{g}, K)$-submodule of $M$, so it is 0. Hence the map

$$M \to \oplus (N)^k$$

is injective, where $N^k$ is $N$ twisted by conjugate action of $k$, so it is a submodule of a semisimple-module, thus semisimple.
Properties of \((\mathfrak{g}, K)\)-Modules

Cor. (III.1.5.16) (Schur’s Lemma). Schur’s lemma holds for irreducible \((\mathfrak{g}, K)\)-modules.

Proof: It suffice to show any endomorphism \(S\) of an irreducible \((\mathfrak{g}, K)\)-module \(M\) has an eigenvalue. But \(S\) preserves \(M^\rho\) for any \(\rho\), and \(M^\rho\) is of f.d, thus it has an eigenvalue over \(\mathbb{C}\).

Cor. (III.1.5.17) (Irreducible Unitary Representation Determined by Finite Part). If \(V_1, V_2\) are two irreducible unitary representations of \(G\) that are infinitesimal equivalent, then they are isomorphic.

Proof: Firstly they are admissible by (III.1.6.13), so we can talk about their corresponding \((\mathfrak{g}, K)\)-modules \(M_i\), then \(V_i\) are the Hilbert space completion of \(M_i\) by (III.1.5.6).

Now \(M_i\) has Hermitian forms, so \(M_i \cong (M^*)^{alg}\), and if \(S : M_1 \cong M_2\), then \(S^*S\) is an automorphism of \(M_1\), thus by (III.1.5.7)(III.1.5.16) it is a scalar map, so after a scalar change, we may assume \(S\) preserves Hermitian structure thus induces an isomorphism of vector spaces \(V_1 \cong V_2\), so by (III.1.5.6) it is an isomorphism of \(G\)-representations.

Prop. (III.1.5.18). Any irreducible \((\mathfrak{g}, K)\)-module has a Banach space structure.

Proof:

Action of \(Z(U(\mathfrak{g}))\)

Prop. (III.1.5.19). Let \(M\) be an admissible \((\mathfrak{g}, K)\)-module, then

\[ M \cong \bigoplus_{\chi \in \text{Spec}(Z(\mathfrak{g}))} M_\chi \]

s.t. \(Z(\mathfrak{g})\) acts on each \(M_\chi\) with a generalized character \(\chi\).

Now let \((\mathfrak{g}, K) - \text{Mod}_\chi\) be the full subcategory of \((\mathfrak{g}, K)\)-modules on which \(Z(\mathfrak{g})\) acts with a generalized character \(\chi\). Cf.[Gaitsgory P42].

Proof: \(Z(\mathfrak{g})\) commutes with \(G\) thus \(K\) action, so it preserves each \(M^\rho\), which are of f.d..

Prop. (III.1.5.20). The category \((\mathfrak{g}, K) - \text{Mod}_\chi\) has only f.m. isomorphism classes of irreducible objects.

Proof: Cf.[Gaitsgory].

Prop. (III.1.5.21). If \(M\) is a f.g. \((\mathfrak{g}, K)\)-module, then for any \(\rho\) of \(K\), \(M^\rho\) is f.g. over \(Z(\mathfrak{g})\).

Proof: Cf.[Gaitsgory P43].

Prop. (III.1.5.22). For \(M \in (\mathfrak{g}, K) - \text{mod}_\chi\), the following are equivalent:

- \(M\) is f.g.,
- \(M\) is of finite length,
- \(M\) is admissible.

Proof: 2 \(\rightarrow\) 1 is trivial, 1 \(\rightarrow\) 3 is by (III.1.5.21).

For 3 \(\rightarrow\) 2: Use (III.1.5.20), there are only f.m. irreducible classes \(\rho_\alpha\), let \(\rho = \oplus_\alpha \rho_\alpha\), then if there is a chain of length \(n\), then there are at least \(n\) linearly independent morphisms in \(\text{Hom}_K(\rho, M)\). Thus \(n\) is bounded, because \(\dim_K \text{Hom}_K(\rho, M)\) is finite because \(M\) is admissible.
Cor. (III.1.5.23). The category \((\mathfrak{g},K) - \text{mod}_\chi\) is Artinian(I.11.2.20).

Cor. (III.1.5.24). Every irreducible \((\mathfrak{g},K)\)-module is admissible.

Proof: Firstly irreducible module are in \((\mathfrak{g},K) - \text{mod}_\chi\) for some \(\chi\), and then use the proposition and(III.1.5.15). \(\square\)

Cor. (III.1.5.25) (Harish-Chandra Modules). For a \((\mathfrak{g},K)\)-module, the following conditions are equivalent:

- \(M\) is f.g. and admissible.
- \(M\) is f.g. and its support over \(\text{Spec}(\mathbb{Z}(\mathfrak{g}))\) is finite.
- \(M\) is admissible and its support over \(\text{Spec}(\mathbb{Z}(\mathfrak{g}))\) is finite.
- \(M\) is of finite length.

Then such modules are called a Harish-Chandra module.

Proof: \(?\) \(\square\)

6 Unitary Representations

Lemma (III.1.6.1) (Auxiliary Compact Supported Function Approximation). Let \(G\) be a locally compact Lie group and \(K\) a compact subgroup. If \(\mathcal{H}\) is a unitary representation of \(G\) on a Hilbert space, and let \(f \neq 0 \in \mathcal{H}\), then for any \(\varepsilon > 0\), there is a \(\varphi \in C^\infty_c(G)\) s.t. \(\pi(\varphi)\) is self-adjoint and \(|\varphi(\rho)f - f| < \varepsilon\).

Moreover, if \(f \in \mathcal{H}_\xi\) which is the decomposition part for \(K\), we can assume \(\varphi(kg) = \varphi(gk) = \xi(k)^{-1}\varphi(g)\). In particular if \(\mathcal{H}_\xi\) is f.d., we find a \(\varphi\) that \(\pi(\varphi)f = f\), by(III.2.5.3).

Proof: By continuity, there is a nbhd \(H\) of 1 that \(|\pi(\varphi)f - f| < \varepsilon\), then we can choose a \(\varphi\) positive real valued with support in \(U\) with integral 1, then \(|\pi(\varphi)f - f| < \varepsilon\) by(X.4.3.22). We can also choose \(\varphi(g) = \varphi(g^{-1})\), then \(\pi(\varphi)\) is self-adjoint.

For the second case, notice first there is a nbhd \(V\) of 1 that \(kVk^{-1} \in U\) for any \(k \in K\)(IX.1.12.6), so let \(\varphi_1\) be a positive real valued function supported in \(V\), and let

\[
\varphi_0(g) = \int_K \varphi_1(kgk^{-1})dk
\]

then \(\varphi_0\) is supported in \(U\) and \(\varphi(kgk^{-1}) = \varphi_0(g)\) for any \(k \in K\). Assume now that \(\pi(k_\theta) = e^{ik\theta}f\), then we can use(X.6.1.30) for \(P = G\) to see that

\[
\pi(\varphi_0)f = \int_G \varphi_0(h)\pi(h)f dh = \int_G \int_K \varphi_0(hk)\pi(hk)f dk dh = \int_G \int_K \xi(k)\varphi_0(hk)dk\pi(h)f dh = \pi(\varphi)f
\]

where

\[
\varphi(g) = \int_K \xi(k)\varphi_0(gk)dk = \int_K \xi(k)\varphi_0(kg)dk
\]

so \(\varphi(k) = \varphi(gk) = \xi^{-1}(k)\varphi(g)\) as required. \(\square\)
Trace Formula for Compact Quotient

Prop. (III.1.6.2) \((L^2(\Gamma\backslash G)\) Totally Decomposable). Let \(G\) be a unimodular locally compact topological group and \(\Gamma \subset G\) be a discrete subgroup that \(\Gamma\backslash G\) is compact, then the space \(L^2(\Gamma\backslash G)\) decomposes as

\[
L^2(\Gamma\backslash G) = \bigoplus_{\pi \in G} m_\pi V_\pi
\]

that each \(m_\pi\) is finite.

Proof: Let \(\Sigma\) be the set of sums of irreducible invariant subspaces of \(L^2(\Gamma\backslash G, \chi)\) that is mutually orthogonal, then choose by Zorn’s lemma a maximal one in \(\Sigma\), and we prove the orthogonal complement \(H = 0\) otherwise we construct an irreducible subspace of \(H\).

Let \(f \neq 0 \in H\), choose by (III.1.6.1) and (III.2.5.3) a \(\varphi \in C_c^\infty(G)\) that \(\rho(\varphi)\) is compact self-adjoint and \(\rho(\varphi)f \neq 0\). So \(\rho(\varphi)\) has a non-zero eigenvalue and the eigenspace \(L\) is of f.d..

Let \(L_0\) be a minimal nonzero subspace of \(L\) that is an intersection of \(L\) with a nonzero closed invariant subspace of \(\mathcal{H}\), and let \(V\) be the intersection of all closed invariant subspaces \(W\) of \(H\) that \(L_0 = L \cap W\). We show \(V\) is irreducible, if not, then \(V = V_1 \cap V_2\), and if \(0 \neq f_0 \in L_0\), then \(f_0 = f_1 + f_2\) and both \(f_1, f_2\) are eigenfunctions of \(\rho(\varphi)\) of eigenvalue \(\lambda\). Now if \(f_1 \neq 0\), then by minimality, \(V_1 \cap L = L_0\).

The finiteness of \(m_\pi\) follows from the fact that \(\rho(f)\) is Hilbert-Schmidt for every \(f \in C_c(G)\)(III.1.6.4).

Prop. (III.1.6.3). Given a locally compact Hausdorff space and a discrete subgroup \(\Gamma\), the right regular action of \(G\) extends to a continuous unitary representation of \(G\) on \(L^2(\Gamma\backslash G)\).

Proof: Because we can approximate \(f \in L^2(\Gamma\backslash G)\) by compactly supported continuous functions, then the \(G\) action is uniformly continuous.

Prop. (III.1.6.4). If \(\Gamma \subset G\) is a discrete subgroup that \(\Gamma\backslash G\) is compact, consider the right action of \(G\) on \(L^2(\Gamma\backslash G)\)(III.1.6.3), let \(\varphi \in C_c(G)\), then \(\varphi\) can act on \(L^2(\Gamma\backslash G, \chi)\) by (X.4.3.24), and:

- \(\rho(\varphi)\) is an integration operator, in particular Hilbert-Schmidt and compact.
- If \(\varphi(g^{-1}) = \overline{\varphi(g)}\), then \(\rho(\varphi)\) is self-adjoint.

Proof: 1:

\[
(\rho(\varphi)f)(g) = \int_G f(h)\varphi(g^{-1}h)dh = \int_{\Gamma\backslash G_1} \sum_{\gamma \in \Gamma} f(\gamma h)\varphi(g^{-1}\gamma h)dh = \int_{\Gamma\backslash G_1} f(h)K_\varphi(g, h)dh
\]

where

\[
K_\varphi(g, h) = \sum_{\gamma \in \Gamma} \varphi(g^{-1}\gamma h).
\]

Because \(\varphi\) is compactly supported, this is a smooth function in \(g\) and \(h\), in particular square integrable on \(\Gamma\backslash G\) compact. And \(\rho(\varphi)(f)(g)\) is smooth in \(g\) because \(f \in L_1(\Gamma\backslash G_1, \chi)\) as \(\Gamma\backslash G_1\) is compact, and \(K(g, h)\) is smooth in \(g\).

2 is easy.

Prop. (III.1.6.5) (Trace of \(\rho(f)\)). If \(\varphi = \varphi_1 \ast \varphi_2\) where \(\varphi_i \in C_c(\Gamma\backslash G)\), then \(\rho(\varphi) = \rho(\varphi_1)\rho(\varphi_2)(X.4.3.24)\) and hence a trace class (X.5.5.7). And its integral kernel is

\[
K_\varphi(x, y) = \int_{\Gamma\backslash G} K_{\varphi_1}(x, z)K_{\varphi_2}(z, y)dz.
\]
and
\[ \text{tr} \rho(f) = \int_{\Gamma \backslash G} K_\varphi(x,x)dx = \int_{\Gamma \backslash G} \int_{\Gamma \backslash G} K_{\varphi_1}(x,y)K_{\varphi_2}(y,x)dxdy. \]

**Proof:** This follows from (X.5.5.10). \qed

**Cor. (III.1.6.6)** *(the Geometric Side of Trace Formula).* If \( G \) is unimodular, let \( f = f_1 \ast f_2 \), \( c(\gamma) \) be a representative for the conjugacy classes of \( \Gamma \), then
\[ \text{tr}(\rho(f)) = \sum_{\gamma \in c(\Gamma)} \int_{\Gamma \backslash G} f(x^{-1}\gamma x)dx. \]

And if \( G_{\gamma} \) is unimodular for every \( \gamma \in \Gamma \), then
\[ \text{tr}(\rho(f)) = \sum_{\gamma \in c(\Gamma)} V(\Gamma_{\gamma} \backslash G) \int_{\Gamma_{\gamma} \backslash G} f(x^{-1}\gamma x)dx. \]

**Proof:**
\[ K_f(x, x) = \sum_{\gamma \in \Gamma} f(x^{-1}\gamma x) = \sum_{\gamma \in c(\Gamma)} \sum_{\delta \in \Gamma_{\gamma} \backslash \Gamma} f(x^{-1}\delta^{-1}\gamma \delta x), \]
so we have
\[ \int_{\Gamma \backslash G} K_f(x, x)dx = \int_{\Gamma \backslash G} \sum_{\gamma \in c(\Gamma)} \sum_{\delta \in \Gamma_{\gamma} \backslash \Gamma} f(x^{-1}\delta^{-1}\gamma \delta x) = \sum_{\gamma \in c(\Gamma)} \int_{\Gamma_{\gamma} \backslash G} f(x^{-1}\gamma x)dx. \]
\qed

**Cor. (III.1.6.7)** *(trace Formula for \( \Gamma \backslash G \) Compact).* Let \( G \) be a unimodular locally compact topological group and \( f = f_1 \ast f_2 \) where \( f_i \in C_c(G) \), and \( \Gamma \) be a discrete subgroup of \( G \) with \( \Gamma \backslash G \) compact and \( G_{\gamma} \) is unimodular for every \( \gamma \in \Gamma \), then \( \rho(f) \) is a trace class with
\[ \sum_{\pi \in \hat{G}} m_{\pi} \text{tr}(\pi(f)) = \text{tr}(\rho(f)) = \sum_{\gamma \in c(\Gamma)} V(\Gamma_{\gamma} \backslash G) \int_{\Gamma_{\gamma} \backslash G} f(x^{-1}\gamma x)dx. \]

**Proof:** Follows from (III.2.5.5) and (III.1.6.6). \qed

**Unitary Irreducible Representation is Admissible**

\( G \) appearing in this subsection are assumed to be a Lie group.

**Prop. (III.1.6.8).** If \( V \) is an irreducible unitary representation of \( G \), then the image of the induced action of \( \text{Meas}_c(G) \) is dense in \( \text{End}(V) \) in the strong topology (X.3.3.4).

**Proof:** This follows immediately from the von Neumann theorem (X.5.3.14) and Schur’s lemma (X.6.2.5): if we denote the algebra generated by \( \text{Meas}_c(G) \) by \( A \), then
\[ \overline{A} = (A^\Gamma)^c = (C)^c = \text{End}(V). \]
\qed
Prop. (III.1.6.9). If $V$ is a representation of $G$ that the image of the induced action of $Meas_c(G)$ is dense in $\text{End}(V)$ in the strong topology, then

$$\dim(V^\rho) \leq \dim(\rho)^2.$$ 

Proof: Follows directly from the following two lemmas (III.1.6.11)(III.1.6.12). \hfill \square

Cor. (III.1.6.10) (Irreducible Unitary Representation is Admissible). For any irreducible unitary representation of $G$, the $K$-finite part is an irreducible admissible $(\mathfrak{g},K)$-module, and $\dim(V^\rho) \leq \dim(\rho)^2$.

Proof: Follows directly from (III.1.5.7), (III.1.6.8) and (III.1.6.9). \hfill \square

Lemma (III.1.6.11). For any $\rho \in \text{Irrep}(K)$, let $A_\rho = \xi_\rho \cdot Meas_c(G) \cdot \xi_\rho$, this is an algebra that acts on $V^\rho$ by (X.6.4.17). Then there exists a family of f.d. representations $\pi$ of $A_\rho$ that:

- Each $\pi$ is of dimension $\leq n = \dim(\rho)^2$.
- For every element $a \in A_\rho$, there exists a $\pi$ that $\pi(a)$ is non-trivial.

Proof: Consider the set of all irreducible f.d. representations of $G$, and $\pi^\rho$ their $\rho$-isotopic parts. Then these are representations of $A_\rho$, and for any $\varphi \in Meas_c(G)$, there is a $\pi$ that $T_\varphi \neq \text{Id}$, and each $\pi^\rho$ has dimension $\leq \dim(\rho)^2$. Cf. [Gaitsgory P46]. \hfill \square

Lemma (III.1.6.12). If $A$ is an associative algebra equipped with a family of f.d. modules satisfying conditions in (III.1.6.11), then if $V$ is a representation of $A$ that the image of $A$ is dense in $\text{End}(V)$ in the strong topology, then $\dim V \leq n$.

Proof: For an associated algebra $A$, consider the minimal integer $r$ that the property $P(r)$:

$$\sum_{\sigma \in \Sigma_r} \text{sgn}(\sigma)a_{\sigma(1)} \cdots a_{\sigma(r)} = 0$$

for any $a_1, \ldots, a_r$, then Amitsur-Levitski showed that for $A = GL(n, \mathbb{C})$, $r = 2n$ (I.12.11.7).

Now the condition of $A$ in (III.1.6.11) shows that $P_{2n}$ is true for $A$. If $\dim V \geq n + 1$, then the image of $A$ satisfies $P(2n)$, so also $\text{End}(V)$ satisfies $P(2n)$ because $A$ is dense in $\text{End}(V)$. But $V$ contains a subgroup $GL(n + 1, \mathbb{C})$, so it cannot satisfy $P(2n)$ by (I.12.11.7), contradiction. \hfill \square

Cor. (III.1.6.13). Let $V$ be an irreducible unitary representation of $G$, then for any $\rho \in \text{Irrep}(K)$,

$$\dim(V^\rho) \leq \dim(\rho)^2.$$ 

In particular, every unitary irreducible representation of $G$ is admissible.

Proof: Directly from Lemmas (III.1.6.8) and (III.1.6.9) above, as the action of $A_\rho$ on $V^\rho$ is also have dense image in the strong topology. \hfill \square

Prop. (III.1.6.14). If $M$ is an irreducible $(\mathfrak{g}, K)$-module equipped with an invariant inner product $((km_1, km_2) = (m_1, m_2), (\xi m_1, m_2) + (m_1, \xi m_2) = 0)$, then the Hilbert space completion of $M$ carries a unique unitary $G$-representation s.t. $V^K_{\text{fin}} = M$ as $(\mathfrak{g}, K)$-modules.
Proof: By (III.1.6.15), the Hermitian form can be extended continuously to the Banach space completion of $M$, It suffices to prove the extended Hermitian form is continuous, because then we can choose its completion w.r.t. $(-,-)$.

For the invariance, consider $f(g) = (gm_1, m_2) - (m_1, g^{-1}m_2)$, then notice $(a,-)$ are continuous functional on $V$, thus by (III.1.5.5) and similar analytic method as in (III.1.5.6) using the invariance of inner product.

Lemma (III.1.6.15). Situation as in (III.1.6.14), $M$ has a Banach norm that $(m, m) \leq ||m||^2$.

Proof: (III.1.5.18) shows $M$ does have a Banach norm. Then let $M \cong V^{K-f\text{in}}$ and $M^{\text{alg}} \cong (V^*)^{K-f\text{in}}$. However the Hermitian form induces $M \cong M^{\text{alg}}$, thus we can form

$$M \xrightarrow{\Delta} M \oplus M \rightarrow V \oplus V^*$$

let $V'$ be the closure of the image of $M$, then it is a $G$-representation by (III.1.5.6), and then

$$(m, m) \leq ||i_1(m)||||i_2(m)|| \leq (||i_1(m)|| + ||i_2(m)||)^2.$$ 

Cor. (III.1.6.16). The above proposition (III.1.6.14) is true for $M$ admissible.

7 $l$-adic Galois Representations

$G_K$ denotes the separable Galois group of $K$.

Def. (III.1.7.1). A Galois representation is a continuous representation of the Galois group $G_K$.

Prop. (III.1.7.2). The most common representation is the cyclotomic representation $G_k \rightarrow \mathbb{Z}_l^*$, where $(l, p) = 1$. And for any representation of $G_k$ over a module $V$ over a $\mathbb{Z}_l$-algebra $R$, we can associate the representation $V(n)$ which is the twist with the cyclotomic representation.

Prop. (III.1.7.3) (Continuous Action of Compact Group Stable Lattice). Let $\Gamma$ be a compact group and let $\rho: \Gamma \rightarrow GL_n(\mathbb{Q}_l)$ be a continuous homomorphism, then there exists a finite extension $L/\mathbb{Q}_p$ that $\rho(\Gamma) \subset GL_n(L)$, and up to conjugation, it is $\rho(\Gamma) \subset GL_n(O_L)$.

Proof: Notice $\rho(\Gamma)$ is compact and Hausdorff, so by Baire category theorem, now that $GL_n(L)$ is closed in $GL_n(\mathbb{Q}_l)$ for all $L/\mathbb{Q}_p$ finite, and all this extensions are countable by primitive element theorem, so there is an $L$ that $\rho(\Gamma) \cap GL_n(L)$ contains an open subset of $\rho(\Gamma)$, so it is an open subgroup, thus of finite index, hence by adding all the coset representations into $L$, we get an $L'$ finite.

For the second assertion, notice $\rho(\Gamma)$ is compact in $GL_n(L)$, thus by (III.4.2.3), it is conjugate to $GL_n(O_L)$.

Prop. (III.1.7.4). If $\rho: \Gamma \rightarrow GL_n(k) = GL(V)$ is a representation, then it has a filtration $0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$ where $V_{i+1}/V_i$ is irreducible, then there is a semisimplification of $\rho$, which is $\rho^{ss} = \oplus V_{i+1}/V_i$.

Prop. (III.1.7.5). If $k = L/\mathbb{Q}_p$ finite, we may take a $\Gamma$-stable $O_L$-lattice $\Gamma$, then the residual representation $\overline{\sigma}_L$ is defined by $\Gamma \rightarrow GL(\Lambda/\pi\Lambda)$. Then the semisimplification of $\overline{\sigma}_L$ is independent of $\Lambda$ chosen.
Prop. (III.1.7.6) (Brauer-Nesbitt). If two representations satisfies they have the same char polynomial or that char $k = 0$ or char $k > n$ and trace is the same, then their semisimplification are the same.

Proof: The proof is not hard, use the Artin-Wedderburn theorem, and the fact the representation may not by semisimple. □

8 Locally Profinite Groups

Basics

Prop. (III.1.8.1) (Smooth Representations). A representation of a locally profinite group $G$ on a complex vector space is called smooth iff it is continuous w.r.t the discrete topology.

The category $\mathcal{M}(G)$ of smooth representations is a full Abelian subcategory of the category of continuous $G$-modules, and there is a right adjoint to the forgetful functor:

$$V^\infty = \bigcup_{K \subset G \text{ compact open}} V^K$$

So it preserves injectives and $\mathcal{M}(G)$ has enough injectives.

Def. (III.1.8.2) (Equivariant Sheaf). Let $G$ be a locally profinite group acting on a locally profinite space $X$, let $p: G \otimes X \to X$ be the projection and $a: G \times X \to X$ be the action, then a equivariant sheaf on $X$ is a pair $(\mathcal{F}, \rho)$, where $\mathcal{F}$ is a sheaf on $X$ and $\rho$ is an isomorphism of sheaves $p^*(\mathcal{F}) \cong a^*(\mathcal{F})$ that:

- $\rho$ is identity on $e \otimes X$.
- $p_{23}^* \rho \circ (id_G \times a)^* \rho = (m \times id_X)^* \rho$ on $G \times G \times X$.

Prop. (III.1.8.3). If $X$ is a pt with the trivial $G$-action, then the equivariant sheaves on $X$ is equivalent to a representation of $G$ on $X$.

Proof: For any equivariant sheaf on $X$, the pullback are just locally constant functions of $G$ with value in $V$. Then $\rho$ on each stalk $g$ defines an action of $g$ on $V$. Compatibility with the $G$-action shows that this is a group action. And consider the stalk at $e$, because $\rho$ is id at $e$, for each $v$, there is an open nbhd $U$ that $\rho = id$ on $U$, thus it is smooth. The converse is obvious. □

Structure Sheaf and Distributions

Def. (III.1.8.4) (Structure Sheaf and Distributions). For a locally profinite group $X$, the structure sheaf $C^\infty_c(X)$ on a locally profinite space $X$ is defined to be the constant sheaf $\underline{C}$ of $X$.

The space $C^\infty_c(X) = C_c(X)$ of test functions on a locally profinite space $X$ is the set of locally constant continuous functions with compact supports.

The space $S^*$ of distribution on $X$ is a linear functional on $C^\infty_c(X)$.

Prop. (III.1.8.5). Any $\varphi \in C^\infty_c(G)$ is $K$-bi-invariant under some compact open subgroup $K$.

Proof: $\varphi$ must be of the form $\sum a_i \chi_{U_i}$, where each $U_i$ is open constant. Then for each element $x \in U_i$, there is an open compact group $K_x$ that $K_x x \subset U_i \cup xK_x \subset U_i$, by (X.6.1.41). Now because Supp $f$ is compact, f.m. of the $U_x x \cap xU_x$ covers Supp $f$, thus we consider their intersection $\cap U_{x_i}$, which is a compact open subgroup $K_0$ that $\varphi$ is $K_0$-bi-invariant. □
Prop. (III.1.8.6). If \( X \) is locally profinite space and \( \mathcal{F} \) is a \( C^\infty \) sheaf, then for any open subset \( U \subset X \), \( Z = X - U \), there is an exact sequence:

\[
0 \rightarrow \Gamma_c(U, \mathcal{F}) \rightarrow \Gamma_c(X, \mathcal{F}) \rightarrow \Gamma_c(Z, \mathcal{F}) \rightarrow 0.
\]

So if we define the space of distributions on \( \mathcal{F} \) on \( X \) \( \mathcal{D}(X, \mathcal{F}) \) to be the space of linear functional on \( \Gamma_c(X, \mathcal{F}) \), then there is an exact sequence

\[
0 \rightarrow \mathcal{D}(Z, \mathcal{F}) \rightarrow \mathcal{D}(X, \mathcal{F}) \rightarrow \mathcal{D}(U, \mathcal{F}) \rightarrow 0.
\]

Proof: \( \square \) Cf. [Bump, P448].

Cor. (III.1.8.7). For \( X \) locally profinite and \( U \subset X \) open, \( Z = X - U \), there is an exact sequence:

\[
0 \rightarrow C^\infty_c(U) \rightarrow C^\infty_c(X) \rightarrow C^\infty_c(Z) \rightarrow 0,
\]

Thus also an exact sequence

\[
0 \rightarrow \mathcal{S}^*(Z) \rightarrow \mathcal{S}^*(X) \rightarrow \mathcal{S}^*(U) \rightarrow 0,
\]

Proof: The first is the exact sequence (III.1.8.6) applied to the constant sheaf (structure sheaf of \( X \)). The second is the dual of the first. \( \square \)

Prop. (III.1.8.8). If \( X, Y \) are both locally profinite, then

\[
C^\infty_c(X \times Y) = C^\infty_c(X) \otimes C^\infty_c(Y)
\]

Proof: Because the subspaces of the form \( U \times V \) for \( U, V \) open form a subsbasis of \( X \times Y \). \( \square \)

Lemma (III.1.8.9). Let \( G \) be locally profinite and \( H \) a closed subgroup, then \( G/H \) is locally profinite space by (X.6.1.42). Then the projection \( P : C_c(G) \rightarrow C_c(G/H) \) defined in (X.6.1.28) restricts to a projection \( P^\infty : C^\infty_c(G) \rightarrow C^\infty_c(G/H) \), and it is surjective.

Proof: Firstly \( P \) maps \( C^\infty_c(G) \) into \( C^\infty_c(G/H) \) because if \( \varphi \) is left invariant under \( K \), then \( P\varphi \) is also left invariant under \( K \). For the surjectivity, let \( \varphi \in C^\infty_c(G/H) \), then \( V = \text{Supp} \varphi \) is open compact, then by (X.6.1.43) there is an open compact subspace \( V \) such that \( p(V) = U \), then we can define

\[
\psi(x) = \chi_V(x)\varphi(p(x))/P(\chi_V)(p(x)),
\]

then it is supported on \( V \) and \( P(\psi) = \varphi \). Also, it is locally constant, as is easily verified. \( \square \)

Lemma (III.1.8.10) (Left Invariant Distribution). Let \( G \) be a locally profinite group and \( T \) is a left invariant distribution on \( G \), then it is the restriction of a Haar measure.

Proof: Any element \( f \in C^\infty_c(G) \) is of the form \( \sum a_i \lambda(h_i)e_K \) for some \( K \), by (III.1.8.5). Because \( T \) is left-invariant, \( T(f) = \sum a_i T(e_K) \), and \( \int f d\mu = \sum a_i \). Thus to show \( T \) is a multiple of \( d\mu \), it suffices to show \( T(e_K) \) is independent of \( K \).

If \( K_1 \leq K_2 \), then \( K_2 = \prod_{i=1}^n a_i K_1 \), where \( n = [K_2 : K_1] \), so \( e_{K_2} = \mu(K_1)/\mu(K_2) \sum_{i=1}^n \lambda(a_i)e_{K_1} \). Also \( \mu(K_2) = n\mu(K_1) \) by left invariance. Now it is clear \( T(e_{K_1}) = T(e_{K_2}) \) by left invariance of \( T \). Then for any two \( K, K' \), we can find an open compact group \( K'' \) in their intersection, thus \( T(e_K) = T(e_{K''}) \). \( \square \)
Prop. (III.1.8.11) (Invariant Quotient Distribution). Let \( G \) be a locally profinite group and \( H \) a closed subgroup. If there exists a left \( G \)-invariant distribution \( D \) on \( G/H \), then \( G/H \) admits a \( G \)-invariant measure (X.6.1.29), and \( D \) is the restriction.

Proof: Consider the distribution \( D \circ P^\infty \), which is a left invariant measure on \( G \), (III.1.8.10) shows that it is a Haar measure on \( G \). Then the surjectivity of \( P^\infty \) (III.1.8.9) shows \( D \) is a quotient measure on \( G/H \).

Prop. (III.1.8.12) (\( C^\infty_c(X) \)-Module and \( C^\infty(X) \)-Sheaves). Let \( \mathcal{F} \) be a \( C^\infty \)-sheaf on \( X \), then the space \( \Gamma_c(X, \mathcal{F}) \) is a \( C^\infty_c(X) \)-module, and this defines an equivalence of categories between the category of non-degenerate \( C^\infty_c(X) \)-modules and \( C^\infty(X) \)-sheaves.

Notice that in this case, a \( C^\infty_c(X) \)-module \( M \) being non-degenerate is equivalent to: for any \( m \in M \), there is a compact open subset \( U \) that \( \chi_U m = m \).

Proof: For any non-degenerate \( C^\infty_c(X) \)-module \( M \), define a sheaf \( \mathcal{F}_M \) of the compatible stalks in \( \prod_{x \in X} M/M(x) \), where

\[
M(x) = \{ m \in M | fm = 0 \text{ for some } f \in C^\infty_c(X), f(x) \neq 0 \}.
\]

Then we show these two functors are inverse to each other: One direction is clear, as the sheaf \( \mathcal{F} \) is just the sheaf of compatible stalks in \( \prod_{x \in X} \mathcal{F}_x \). For the other direction, any element in \( \Gamma_c(X, \mathcal{F}_M) \) is induced from \( m_i \) on \( U_i \), where \( U_i \) are pairwise disjoint compact open subset. This follows from (IX.1.4.7). These elements are isomorphic to \( C^\infty_c(X) \) \( M \) by mapping the above element to \( \sum \chi_{U_i} m_i \). Then the non-degeneracy of \( M \) shows \( \Gamma_c(X, \mathcal{F}_M) \cong M \).

Remark (III.1.8.13). Notice that \( M(x) \) can be equivalently defined to be the space spanned by the elements

\[
\{ gm | g \in C^\infty_c(X), g(x) = 0, m \in M \}.
\]

Prop. (III.1.8.14) (Bernstein-Zelevinsky). If \( p : X \to Y \) is a continuous map of locally profinite groups, \( \mathcal{F} \) be a cosmooth \( C^\infty \)-sheaf on \( X \)??, Let \( G \) be a group acting on \( X \) and the sheaf \( \mathcal{F} \) that \( p(gx) = p(x) \), and \( \chi \) a character of \( G \). Then

- Let \( \Gamma_c(X, \mathcal{F})(\chi) \) be the \( C^\infty_c(X) \)-submodule of \( \Gamma_c(X, \mathcal{F}) \) generated by \( gf - \chi(g)^{-1} f, g \in G, f \in \Gamma_c(X, \mathcal{F}) \). Then \( \Gamma_c(X, \mathcal{F})/\Gamma_c(X, \mathcal{F})(\chi) \) is a non-degenerate \( C^\infty_c(Y) \)-module by composing \( \varphi \), so we can define \( \mathcal{G} \) the sheaf on \( Y \) corresponding to this submodule by (III.1.8.12). Then if \( y \in Y \) and \( Z = p^{-1}(y) \), the stalk

\[
\mathcal{G}_y \cong \Gamma_c(Z, \mathcal{F})/\Gamma_c(Z, \mathcal{F})(\chi).
\]

- Assume there are no non-zero distributions \( D \in \mathcal{D}(p^{-1}(y), \mathcal{F}|_{p^{-1}(y)}) \) that satisfies \( gD = \chi(g)D \) for any \( y \in Y \), then no such \( D \) exists in \( \mathcal{D}(X, \mathcal{F}) \).

Proof: 1: Firstly \( \Gamma_c(X, \mathcal{F})/\Gamma_c(X, \mathcal{F})(\chi) \) is a \( C^\infty_c(Y) \)-module because \( \varphi \circ p \) fixes \( \Gamma_c(X, \mathcal{F})(\chi) \):

\[
(\varphi \circ p)(gf - \chi(g)^{-1} f) = g((\varphi \circ p)f) - \chi(g)^{-1}((\varphi \circ p)f),
\]

which uses the condition \( p(gx) = p(x) \). The non-degeneracy is also clear, by (III.1.8.12).

Secondly \( \Gamma_c(Z, \mathcal{F}) \cong \Gamma_c(X, \mathcal{F})/\Gamma_c(U, \mathcal{F}) \) by (III.1.8.6), so \( \Gamma_c(Z, \mathcal{F})(\chi) \) is isomorphic the quotient of \( \Gamma_c(X, \mathcal{F}) \) by \( \Gamma_c(U, \mathcal{F}) \) and \( \Gamma_c(X, \mathcal{F})(\chi) \), because \( \Gamma_c(U, \mathcal{F}) \) is stable under action of \( f \mapsto gf - \chi(g)^{-1} f \) (this uses the condition \( p(gx) = p(x) \)).
We claim the space $\Gamma_c(U, \mathcal{F})$ is the space $L$ generated by elements of the form $(\varphi \circ p)f$, where $\varphi \in C_c^\infty(Y), \varphi(y) = 0, f \in \Gamma_c(X, \mathcal{F})$: $L$ is clearly contained in $\Gamma_c(U, \mathcal{F})$, and if $f \in \Gamma_c(U, \mathcal{F})$, then $\text{Supp } f$ is compact and disjoint from $Z$, so there is an open compact subset $U \subset Y$ containing $p(\text{Supp } f)$ but not $y$. Let $\varphi = \chi_U$, then $f = (\chi_U \circ p)f \in L$.

Then by (III.1.8.12), the stalk $\mathcal{G}_y$ is isomorphic to $M/M(y)$, where $M = \Gamma_c(X, \mathcal{F})/\Gamma_c(X, \mathcal{F})(\chi)$. So $M(y)$ is just the image of the space $L$ in $M$ by (III.1.8.13), hence $\mathcal{G}_y$ is exactly $\Gamma_c(Z, \mathcal{F})/\Gamma_c(Z, \mathcal{F})(\chi)$.

2: this follows from 1, as $gD = \chi(g)D$ is just saying $D(g^{-1}f - \chi(g)f) = 0$, or that $D$ annihilates $\Gamma_c(Z, \mathcal{F})/\Gamma_c(Z, \mathcal{F})(\chi) = \mathcal{G}_y$ by item 1. So this is equivalent to $\mathcal{G}_y = 0$ for any $y$, which is equivalent to $G = 0$, as $\mathcal{G}$ is a sheaf.

**Cor. (III.1.8.15) (Invariant Distribution On Orbits).** Let $\gamma$ be an action of a locally profinite group $G$ on a locally profinite space $X$ and a $C^\infty(X)$-sheaf. Assume the action is constructible (IX.1.12.21) and there are no $G$-invariant distribution on any $G$-orbit in $X$, then there are no non-zero $G$-invariant distribution on $X$.

**Proof:** Firstly, by (IX.1.12.22) and (IX.1.4.6) any orbit is locally profinite. If there is a $G$-invariant distribution on $X$, we may change $X$ to $\text{Supp } T$, which is $G$-invariant, thus by (IX.1.12.22) there is an open subset $U \subset X$ that $G$ acts regularly, thus we reduce to the regular action case.

Then we can consider $X \to X/G$, $X/G$ is locally profinite by (IX.1.12.10), so Bernstein-Zelevinsky (III.1.8.14) can be used.

**Prop. (III.1.8.16) (Gelfand-Kazhdan).** If $G$ is a locally profinite group, and $\gamma$ is an action of $G$ on a locally profinite space $X$, $\sigma$ is a homeomorphism $X \cong X$, $\mathcal{F}$ is a $C^\infty$-sheaf on $X$, and we assume:

- $\gamma$ is constructible,
- for each $g \in G$, there is a $g^\sigma \in G$ that $\gamma(g)\sigma = \sigma\gamma(g^\sigma)$.
- For some $n \geq 0$ and $g_0 \in G$, $\gamma^n = \gamma(g_0)$.
- If there is a non-zero $G$-invariant $\mathcal{F}$-distribution $T$ on a $G$-orbit $S$, then $\sigma(S) = S$ and $\sigma(T) = T$.

Then any $G$-invariant distribution on $X$ is invariant under $\sigma$.

**Proof:** Let $T$ be a $G$-invariant $\mathcal{F}$-distribution that $\sigma T \neq T$, then $n > 1$, and for any $n$-th root of unity $\xi$, consider $T_\xi = \sum \xi^{-i}T_i(T)$. Then

$$\sigma T_\xi = \xi T_\xi, \quad \sum \xi T_\xi = nT, \quad \sum \xi T_\xi = n\sigma(T).$$

so $\sum (\xi - 1)T_\xi = n(\sigma(T) - T) \neq 0$, which shows there is a root $\xi \neq 1$ that $T_\xi \neq 0$. Notice $T_\xi$ is $G$-invariant by condition 2. Consider the action $\sigma_x i = \xi \cdot \sigma$, then $T_\xi$ is invariant under $\sigma_x$.

Let $G'$ be the semi-direct product of $G$ with $\sigma$, under the action of $\sigma_x^{-1}g\sigma = g'$, then $G'$ is locally profinite and acts on $X, \mathcal{F}$. Clearly this action is also constructible.

Now for any $G'$-orbit $S'$, we prove there are no $G'$-invariant distribution on $S'$, because it is in priori $G$-invariant, so some distribution exists on some $G$-orbit $S \subset S'$, but then condition 4 shows $\sigma$ fixes $S$, then $S = S'$ and $\sigma(T) = T$. But $\sigma(T) = T$, contradiction. Finally (III.1.8.15) shows there are no $G'$-invariant distribution on $X$, contradicting $T_\xi$.

**Cor. (III.1.8.17) (Gelfand-Kazhdan).** If $G$ is a locally profinite group that is $\sigma$-compact, and $\gamma$ is an action of $G$ on a locally profinite space $X$, $\sigma$ is a homeomorphism $X \cong X$, and we assume:
• \( \gamma \) is constructible,
• for each \( g \in G \), there is a \( g^\sigma \in G \) that \( \gamma(g)\sigma = \sigma\gamma(g^\sigma) \).
• For some \( n \geq 0 \) and \( g_0 \in G \), \( \sigma^n = \gamma(g_0) \).
• \( \sigma \) preserves \( G \)-orbits.
Then any \( G \)-invariant distribution on \( X \) is invariant under \( \sigma \).

**Proof:** We use (III.1.8.17) and take \( F \) to be just \( C^\infty(X) \). Then we need to check condition 4: for any \( G \)-orbit \( S \) of \( X \) let \( s \in S \) and \( \text{Stab}(s) = H \), \( S \cong G/H \) by (IX.1.12.22) and (X.6.1.44). Then (III.1.8.11) shows \( T \) is just the \( G \)-invariant measure on \( G/H : T\varphi = \int_{G/H} \varphi(\gamma(g)s) \). Now condition 4 and 2 shows \( \sigma T \) is also \( G \)-invariant, thus \( \sigma T = cT \). Clearly \( c \geq 0 \), and condition 3 shows \( c^n = 1 \), thus \( c = 1 \). \( \square \)

**Hecke Algebra**

This is a continuation of 4.

**Prop. (III.1.8.18).** Let \( (\pi, V) \) be a non-zero smooth representation of a locally profinite group \( G \), then the following are equivalent by (I.4.4.4) and (III.4.2.9):
• \( \pi \) is irreducible.
• \( V \) is simple as \( \mathcal{H} \)-module.
• \( V^{K_0} \) is either zero or simple as a \( \mathcal{H}_{K_0} \)-module for all open compact subgroups \( K_0 \) of \( G \).

**Cor. (III.1.8.19).** \( (-)^\infty \) is exact, this is because \( (-)^K \) do: it is left exact clearly, and it is right exact because it is the image of \( e_K \). Then use filtered colimits.

**Admissible Representations**

**Def. (III.1.8.20) (Contragradient Representation).** For a smooth representation \( V \) of \( G \), the **contragradient representation** \( \tilde{V} \) is the smooth part of \( V^* = (V^*)^\infty \).

**Prop. (III.1.8.21).**
• For any compact open subset \( K \) of \( G \), \( V^K = (\tilde{V})^K \).
• \( \text{Hom}_G(V, \tilde{W}) = \text{Hom}(W, \tilde{V}) \).
• \( V \hookrightarrow \tilde{V} \).

**Proof:**
1: Using (III.4.2.9), because \( (\tilde{V})^K = V^{*K} \). There is a homomorphism \( V^{*K} \rightarrow V^{K*} \), it is injective, because if \( f(v) = 0 \) for each \( v \in V^K \), then \( f(w) = f(\epsilon_K v) = 0 \). It is also injective, because for each \( f \in V^{K*} \), the inverse image is \( g(w) = g(\epsilon_K w) \).
2: \( \text{Hom}(V, \tilde{W}) = \text{Hom}(V, W^*) = \text{Hom}(V \otimes W, \mathbb{C}) \).
3: by the proof of item 1,
\[
\tilde{V} = \cup_K ((\cup_K V^{*K})^K) = \cup_K ((\cup_K V^{*K})^{K*}) = \cup_K (V^{*K*}) = \cup_K (V^{K*})
\]
So the filtered colimits of the injections \( V^K \rightarrow (V^K)^* \) gives an injection \( \cup_K (V^{K*}) \). \( \square \)

**Cor. (III.1.8.22) (Contragradient-Functor-Exact).** The contragradient functor \( V \mapsto \tilde{V} \) is exact. Because \( (-)^* \), \( (-)^\infty \) are all exact (III.1.8.19).
Cor. (III.1.8.23). If \( P \) is projective in \( \mathcal{M}(G) \), then \( \tilde{P} \) is injective in \( M(G) \).

Proof: \( \text{Hom}(X, \tilde{P}) = \text{Hom}(P, \tilde{X}) \), and notice that the contragradient functor is exact (III.1.8.22). \( \square \)

Def. (III.1.8.24) (Admissible Representation). A smooth representation is called admissible iff for any compact open subgroup \( K \) of \( G \), \( V^K \) is of f.d.

A representation is admissible iff \( V \cong \tilde{V} \). In particular, the contragradient of an admissible representation is admissible.

Proof: If \( V^K \) is of f.d. for each \( K \), then by the proof of item 3 of (III.1.8.21), \( V \cong \tilde{V} \). Conversely, if \( V \cong \tilde{V} \), then \( V^K \cong V^{K*} \), thus \( V \) must be finite, by (I.1.3.3). \( \square \)

Prop. (III.1.8.25) (Decomposition of Admissible Representations). Let \( K \) be a compact subgroup of \( G \), then any smooth representation of \( G \) decomposes as

\[
V = \bigoplus_{\rho \in \hat{K}} V^\rho,
\]

and \( V \) is admissible iff each \( V^\rho \) are all of f.d. In particular, this shows the two notations of admissible (locally compact group and locally profinite groups) are compatible for smooth representations.

Proof: Firstly \( V \subseteq \sum_{\rho \in \hat{K}} V(\rho) \), because any \( v \in V \) is fixed by some compact open subgroup \( K_0 \) of \( K \), and we can choose \( K_0 \) to be normal in \( K \) by (IX.1.12.6), so for \( \Gamma = K/K_0 \),

\[
v \in V^{K_0} = \bigoplus_{\rho \in \hat{\Gamma}} V(\rho) \subseteq \sum_{\rho \in \hat{K}} V(\rho).
\]

Also this sum is direct, because otherwise \( \sum_{\rho \in S} c_{\rho} v_{\rho} = 0 \), but let \( K_0 \) be the intersection of kernels of \( \rho \), then this is an equation of elements in representations of \( \Gamma = K/K_0 \) finite, so contradicting (III.1.2.8).

If \( \pi \) is admissible, then \( V(\rho) \subseteq V^{K_{\rho}} \) is of f.d.. Conversely, if \( V \) is not admissible, then \( V^{K_0} \) is of infinite dimensional for some \( K_0 \) compact compact normal, so \( V^{K_0} \) decomposes as direct sums of \( V(\rho) \) for \( \rho \in \hat{K}/\hat{K}_0 \), thus one of these space must be of infinite dimensional. \( \square \)

Remark (III.1.8.26). WARNING: The decomposition theorem of compact groups cannot be directly used, as representation may not be of f.d. so not unitarizable.

Prop. (III.1.8.27) (Representations of Product Group). If \( G_1, G_2 \) are all locally profinite groups and \((\pi_i, M_i)\) are irreducible admissible representations of \( G_i \), then \( M_1 \otimes M_2 \) is an irreducible admissible representation of \( G_1 \times G_2 \), and all representations of \( G_1 \times G_2 \) comes like this.

Proof: By (III.4.2.9) and (I.4.4.13), this follows if we have \( \mathcal{H}_{G_1 \otimes G_2} \cong \mathcal{H}_{G_1} \otimes \mathcal{H}_{G_2} \). And this fact is easily deduced from (III.1.8.8). \( \square \)

Def. (III.1.8.28) (Character of Admissible Representations). Let \((\pi, V)\) be an admissible representation of \( G \), then for any \( \varphi \in \mathcal{H} \), \( \varphi \in \mathcal{H}_K \) for some compact open subset of \( G \), by (III.4.2.6), so \( \text{Im}(\pi(\varphi)) \subseteq V^K \), which is of f.d., so we can define the trace of \( \varphi \) as \( \text{tr}(\pi(\varphi)|V^K) \). Notice this is independent of \( K \) chosen by linear algebra reasons. And this defines a distribution on \( \mathcal{H} : \varphi \mapsto \text{tr}(\varphi) \), called the character of \( V \).
Prop. (III.1.8.29) (Smooth Irreducible Representation is Admissible). Every smooth irreducible representation is admissible. In fact, this is true for general connected reductive group.

Proof: □

Cor. (III.1.8.30). \( \mathcal{M}(G) \) has enough injectives.

Proof: As \( \mathcal{M}(G) \) has enough projectives (I.4.4.6)(III.4.2.9), there is a surjection \( P \to \tilde{X} \), thus an injection \( \tilde{X} \to \tilde{P} \) (III.1.8.22). Now \( X \to \tilde{X} \) by (III.1.8.21). □

Induced Representations

Def. (III.1.8.31) (Smooth Induced Representation). Let \( G \) be a locally profinite group and \( H \) is a closed subgroup, \( (\pi,V) \) be a smooth representation of \( H \), we can define the smooth induced representation \( \text{Ind}_H^G \) as the smooth vectors in the right \( G \)-representation on the space of functions \( f \) on \( G \) with values in \( H \) that

\[
f(hg) = \sqrt{\frac{\Delta_H(h)}{\Delta_G(h)}} \pi(h)f(g).
\]

Notice this is similar to that of unitary representations of locally compact groups, in (X.6.5.3).

Prop. (III.1.8.32) (Smooth Frobenius Reciprocity). If \( G \) is compact and \( H \) is a closed subgroup, \( (\pi,W) \) a smooth representation of \( G \), \( (\rho,V) \) a smooth representation of \( H \), then

\[
C(\pi, \text{ind}_H^G(\rho)) = C(\pi|_H, \rho \otimes \Delta_G^{-1} \Delta_H).
\]

Proof: Given a \( \Phi : W \to \text{ind}_H^G V \), we have a map \( \varphi : W \to V : \varphi(w) = \Phi(w)(1) \), then it is verified that \( \varphi \in C(\pi|_H, \rho \otimes \Delta_G^{-1} \Delta_H) \). Conversely, if \( (\varphi : W \to V) \in C(\pi|_H, \rho \otimes \Delta_G^{-1} \Delta_H) \) is given, we can define \( \Phi; W \to \text{Ind}_H^G V : \Phi(w)(g) = \varphi(\pi(g)w) \), then \( \Phi(w) \in \text{Ind}_H^G V \) and this is linear in \( w \) (remember to check smoothness). Finally, it is easily verified these maps are inverse to each other. □

Irreducible Admissible Representations

Prop. (III.1.8.33) (Separation Lemma). If \( G \) is a \( \sigma \)-compact locally profinite group, then for any \( 0 \neq h \in \mathcal{H}(G) \), there is an irreducible representation \( \rho \) that \( \rho(h) \neq 0 \).

Proof: Cf.[Bernstein, P20]. □

Prop. (III.1.8.34) (Shur’s Lemma).

- If \( G \) is a \( \sigma \)-compact locally profinite group, then any irreducible representation \( V \) is of at most countable dimension, thus \( \text{End}_G(V) = \mathbb{C} \) by (I.2.3.24).
- If \( G \) is a locally profinite group, then any irreducible admissible representation \( V \) of \( G \) satisfies \( \text{End}_G(V) = \mathbb{C} \).

Proof: 1: If it is of at most finite dimension because if \( \xi \) generate \( V \), then notice its stabilizer is compact open, and \( G \) is \( \sigma \)-compact, so \( V \) is at most countable.

2: Let \( K_0 \) be a small open compact subgroup that \( V^{K_0} \neq 0 \), then \( V^{K_0} \) is of f.d. and preserved under \( T \in \text{End}_G(V) \), thus \( T \) has an eigenvalue \( c \), thus \( T = cf \) on \( V \). □
Prop. (III.1.8.35). Let \((\pi_1, V_1), (\pi_2, V_2)\) are two irreducible admissible representations of a locally profinite group \(G\). If \(V_1^K \cong V_2^K\) as \(\mathcal{H}_K\)-module for any compact open subgroup \(K\) of \(G\), then \(\pi_1 \cong \pi_2\). This follows immediately from (III.4.2.9) and (I.4.4.5).

Prop. (III.1.8.36) (Characters Determine Representations). Let \((\pi_1, V_1), (\pi_2, V_2)\) are two irreducible admissible representations of a locally profinite group \(G\). If the characters (III.1.8.28) of \(\pi_1, \pi_2\) are the same, then the two representations are equivalent.

Proof: The hypothesis together with (I.4.1.13) shows \(V_1^K \cong V_2^K\) as a \(\mathcal{H}_K\)-module for any compact open subgroup \(K\) of \(G\). Then (III.1.8.35) shows \(\pi_1 \cong \pi_2\) as \(\mathcal{H}_G\)-modules, thus isomorphic as \(G\)-modules, by (III.4.2.9).

Compact Representations

Def. (III.1.8.37). A smooth representation of a locally profinite group \(G\) is called compact iff for every \(\xi \in V\) and every open compact subgroup \(K \subset G\), the function \(D_{\xi, K} : g \mapsto \pi(\epsilon_K)\pi(g^{-1})\xi\) has compact support.

Prop. (III.1.8.38). If \(\xi \in V, \tilde{\xi} \in \tilde{V}\), then the function \(m_{\tilde{\xi}, \xi}(g) = \tilde{\xi}(\pi(g^{-1})\xi)\) is called the matrix coefficients. Then \(V\) is compact iff every matrix coefficient is compactly supported.

Proof: Cf. [Bernstein P22].

□
III.2 Automorphic Representations of $GL(2, \mathbb{R})(\text{Bump})$

Main References are [Automorphic Forms and Representations, Bump] and [A Course given by Liang Xiao].

1 Setups

Def. (III.2.1.1) (General Notations). $\mathcal{H}$ is the Poincaré’s upper plane with measure $y^{-2}dx\,dy$ (III.2.1.3), $G = GL(2, \mathbb{R})^+$, $G_1 = SL(2, \mathbb{R})$, $K = SO(2, \mathbb{R})$, $B$ the upper triangular matrices. Then $G$ acts on $\mathcal{H}$ through linear fractional transformation (IX.9.6.6) and fixes the measure.

We will denote $a, b, c, d$ the linear functionals on $M(2, \mathbb{R})$ that for $\gamma \in M(2, \mathbb{R})$, $\gamma = \begin{bmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{bmatrix}$.

We will consider right regular action $\rho$ of $G$ on $C^\infty(G)$, and also the left regular action $\lambda$. We will write $dX$ for $X \in g$ as the representation of Lie algebra of $G$ via $\rho$, then it commutes with $\lambda$.

So it induces a map of $U(g)$ to the ring of left $G$-invariant differential operators on $G$ (IX.8.9.2).

Also we will use the Lie algebra notations (I.12.2.11):

$H = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $R = \frac{1}{2} \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$, $L = \frac{1}{2} \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}$, $W = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = iH$.

and the Casimir element in $Z = Z(U(g))$:

$\Delta = -\frac{1}{4}(H^2 + 2RL + 2LR)$.

Prop. (III.2.1.2) (Classification of Transformations). Let $\alpha \in SL(2, \mathbb{R})$ and $\alpha \neq I$, then by Jordan decomposition, $\alpha$ is conjugate to one of the matrix of the two following types:

$\begin{bmatrix} \lambda & 1 \\ \lambda & \mu \end{bmatrix}, \begin{bmatrix} \lambda & 1 \\ \mu & \lambda \end{bmatrix}, \lambda \neq \mu$

according as it has repeated eigenvalues or distinct eigenvalues. In the first case, $\alpha$ is called parabolic, and in the second case, if $|\lambda/\mu| = 1$, it is called elliptic. If $\lambda/\mu$ is real and positive, it is called parabolic, and called loxodromic otherwise.

- If $\alpha \in SL_2(\mathbb{R})$ is parabolic, then it has a unique real eigenvector, which means it has a unique fixed point in $\mathbb{R} \cup \infty$.
- If $\alpha \in SL_2(\mathbb{R})$ is elliptic, then $\alpha$ has two conjugate eigenvectors, which means it has exactly one fixed point $z$ in $\mathcal{H}$, and a second fixed point, namely $\bar{z}$, in the lower half plane.
- If $\alpha \in SL_2(\mathbb{R})$ is hyperbolic, then it has two real eigenvectors, which means it has two distinct fixed points in $\mathbb{R} \cup \infty$.

Prop. (III.2.1.3) (Iwasawa-Decomposition). Every element of $G$ has a unique representation of the form (IX.8.6.3):

$g = \begin{bmatrix} u & y^{1/2} \\ u & xy^{-1/2} \end{bmatrix} k_\theta$

where $k_\theta = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$. So by (X.6.1.30) and (X.6.1.12) the Haar measure is calculated to be $dg = \frac{1}{2\pi} \frac{du}{u} \frac{dx}{y^2} \, d\theta$, and it is unimodular by (X.6.1.15). (Notice the upper triangular matrix group $B \subset GL(2, \mathbb{R})^+$ is not unimodular).
III.2. AUTOMORPHIC REPRESENTATIONS OF $GL(2, \mathbb{R})(\text{BUMP})$

Proof: For the calculation of Haar measure, notice that it suffices to calculate for $u, x, y$ and it is

$$
\frac{d(uy^{1/2})d(uy^{-1/2})d(uxy^{-1/2})}{(uy^{1/2})^2uy^{-1/2}} = \frac{dxd(uy^{1/2})d(uy^{-1/2})}{u^2y} = \frac{dxdydu}{uy^2}
$$

□

Cor. (III.2.1.4). Every element of $G_1$ has a unique representation of the form

$$
g = \begin{bmatrix} e^{\frac{\theta}{2}} & 0 \\ 0 & e^{-\frac{\theta}{2}} \end{bmatrix} \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} k_\theta.
$$

And the Haar measure is given by $dg = \frac{1}{2\pi}dudxd\theta$ by similar calculation, which is unimodular.

Prop. (III.2.1.5). In the coordinate (III.2.1.3), we have the following equation:

$$
R = e^{2i\theta}(iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{1}{2i} \frac{\partial}{\partial \theta}), \quad L = e^{-2i\theta}(-iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - \frac{1}{2i} \frac{\partial}{\partial \theta}), \quad H = -i \frac{\partial}{\partial \theta}
$$

So in particular

$$
\Delta = d\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + y \frac{\partial^2}{\partial x \partial \theta}.
$$

Proof: $H = -iW$, and $\exp(tW) = k_t = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$, so $dW = \partial/\partial \theta$ is clear.

For $dR$, first notice that

$$
k_\theta \exp(tR) = \exp(e^{2i\theta}R)k_\theta,
$$

as

$$
k_\theta \exp(tR)k_{-\theta} = C^{-1} \begin{bmatrix} e^{i\theta} & e^{-i\theta} \\ e^{-i\theta} & e^{i\theta} \end{bmatrix} \exp(te) \begin{bmatrix} e^{-i\theta} & e^{i\theta} \\ e^{i\theta} & e^{-i\theta} \end{bmatrix} C = C^{-1} \begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-i\theta} & e^{i\theta} \\ e^{i\theta} & e^{-i\theta} \end{bmatrix} C = C^{-1} \begin{bmatrix} 1 & e^{2i\theta}t \\ 1 & 1 \end{bmatrix} C = \exp(e^{2i\theta}tR)
$$

where $C$ is the Cayley transformation, notation as in (I.12.2.11).

Now

$$
(dRf)(g) = \frac{d}{dt}f(bk_\theta \exp(tR)) = \frac{d}{dt}f(b \exp(e^{2i\theta}tR)k_\theta) = e^{2i\theta} \frac{d}{dt}f(b \exp(tR)k_\theta)
$$

Then notice

$$
R = 1/2H + \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix} + 1/2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \exp(tR) \sim k_{t/2} + \begin{bmatrix} 1 & it \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} e^{t/2} & \cdot \\ \cdot & e^{-t/2} \end{bmatrix}
$$

so we get the desired result.

For $dL$ the calculation is similar to that of $dR$. □
Def. (III.2.1.6) (Γ Setting). Let Γ be a discrete subgroup of $G$ that the volume of $\Gamma \backslash \mathcal{H}$ is finite, we may also assume that $-1 \in \Gamma \subset SL(2, \mathbb{R})$.

Let $\chi$ be a character of $\Gamma$, $\omega$ be a character of the center $Z(\mathbb{R}) \subset G$ (the scalar matrices). Assume that $\omega(-1) = \chi(-1)$.

Def. (III.2.1.7) (Cusps). A cusp of $\Gamma$ is a point in $\mathbb{P}^1(\mathbb{R})$ whose stabilizer in $\Gamma$ contains a non-trivial parabolic element (III.2.1.2).

Let $\infty$ be a cusp, then $\{\pm 1\} \Gamma_{\infty} = \{\pm 1\} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ for some $r > 1$. Then if $\Gamma_{\infty} = \langle - \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \rangle$, then it is called an irreducible cusp, otherwise it is called a regular cusp. Similarly, for any other cusp $a = \xi(\infty)$, where $\xi \in SL(2, \mathbb{R})$, we called $a$ is regular/irregular cusp if $\infty$ is regular/irregular cusp w.r.t. $\Gamma' = \xi^{-1} \Gamma \xi$.

Def. (III.2.1.8) (Form Spaces). Let $C^\infty(\Gamma \backslash G, \chi, \omega)$ be the space of smooth functions $F : G \to \mathbb{C}$ that

$$F(\gamma g) = \chi(\gamma) F(g), \quad \gamma \in \Gamma, g \in G,$$

$$F(z g) = \omega(z) F(g), \quad z \in Z(\mathbb{R}), g \in G.$$

Let the subspace $C^\infty_c(\Gamma \backslash G, \chi, \omega)$ be those functions $f$ that $|f|$ is compactly supported in $G/Z(\mathbb{R})$. Similarly we define $C(\Gamma \backslash G, \chi, \omega)$ and $C_c(\Gamma \backslash G, \chi, \omega)$.

Def. (III.2.1.9) (Arithmetic Automorphic Forms). Let the space of automorphic forms $\mathcal{A}(\Gamma \backslash G, \chi, \omega)$ be the subspace of $C^\infty(\Gamma \backslash G, \chi, \omega)$ consisting of $K$-finite and $Z$-finite and satisfies the condition of moderate growth:

$$|F(g)| < C||g||^N$$

for some $C, N > 0$, where the height $||g||^2 = \frac{a^2 + b^2 + c^2 + d^2}{2|ad - bc|}(|ad - bc|^2 + |ad - bc|^{-2})$.

Def. (III.2.1.10) (Cuspidal Forms). If $\infty$ is a cusp of $\Gamma$, then $\{\pm 1\} \Gamma_{\infty} = \{\pm \tau\} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$, so a $F \in \mathcal{A}(\Gamma \backslash G, \chi, \omega)$ is called cuspidal at $\infty$ iff $\chi(\tau) \neq 1$ or

$$\int_0^r F\left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx = 0.$$

for any $g \in G$. Notice this is independent of $r$ chosen.

More generally, if $a$ is a cusp of $\Gamma$, then choose $\xi \in SL(2, \mathbb{R})$ that $\xi(\infty) = a$, then for $F \in \mathcal{A}(\Gamma \backslash G, \chi, \omega)$, $F'(g) = F(\xi g) \in \mathcal{A}(\Gamma' \backslash G, \chi', \omega)$, where $\Gamma' = \xi^{-1} \Gamma \xi, \chi'(\gamma') = \chi(\xi \gamma' \xi^{-1})$ (Because left and right actions commute). Then $F$ is called cuspidal at $a$ iff $F'$ is cuspidal at $\infty$.

The subspace of cuspidal forms is denoted by $\mathcal{A}_0(\Gamma \backslash G, \chi, \omega) \subset \mathcal{A}(\Gamma \backslash G, \chi, \omega)$.

$\Gamma \backslash \mathcal{H}$

Def. (III.2.1.11) (Right Weight Action). There are right actions of $GL(2, \mathbb{R})^+$ on $C^\infty(\mathcal{H})$ defined to by

$$(f|k)g)(z) = \left( \frac{cz + d}{|cz + d|^k} \right)^k f\left( \frac{az + b}{cz + d} \right)$$

Proof: It is an action because?
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Def. (III.2.1.12) (Holomorphic Right Weight Action). Besides the right weight action (III.2.1.11), there is another family of actions:

\[ f[\gamma]_k(z) = \deg(\gamma)^{k/2}(cz + d)^{-k}f\left(\frac{az + d}{cz + d}\right). \]

Proof: This is an action because? \hfill \square

Lemma (III.2.1.13). If $f$ is a holomorphic function on $\mathcal{H}$ and $\gamma \in SL_2(\mathbb{R})$ satisfies $\gamma^n \neq 1$ for any $n \neq 0$ and $f[\gamma]_k = f$, then $f = 0$.

Proof: Use a Cayley transformation $\mathcal{H} \to \mathbb{D}$ to map the fixed point to origin, then $\gamma$ corresponds to $\text{diag}(\alpha, \alpha^{-1})$, where $\alpha$ is not a root of unity. Then if $f(z) = \sum a_n z^n$, the formula $f[\gamma]_k = f$ says $a_n \alpha^{2n+k} = a_n$. Because $\alpha^{2n+k} \neq 1$ for any $n$, $a_n = 0$ for any $n$, thus $f = 0$. \hfill \square

Prop. (III.2.1.14) (Poincare Fundamental Domain). Fundamental domain for $\Gamma \subset GL(2, \mathbb{R})$ acting on $\mathcal{H}$ is defined in (X.6.1.25). There is a well-shaped open subset $F$ with a set $F \subset F' \subset F$ that $F'$ is a fundamental domain for $\Gamma$ acting on $\mathcal{H}$.

Notice that if $\mathcal{F}$ is a fundamental domain for $\Gamma$ in $\mathcal{H} \cong SL_2(\mathbb{R})/SO(2, \mathbb{R})$, then the inverse image of $\mathcal{F}$ is a fundamental domain for $\Gamma$ in $SL_2(\mathbb{R})$, because a.e. $z \in \mathcal{H}$ is not fixed by any $\gamma \in \Gamma$.

Proof: Choose a $z \in \mathcal{H}$ which is not fixed by any $\gamma \in \Gamma$, then for any $\gamma \in \Gamma$, draw the circle of points that have the same distance to $z$ and $\gamma(z)$, then the intersection of all the part containing $z$ is a fundamental domain. \hfill \square

Prop. (III.2.1.15) (Siegel). If a Poincare fundamental domain $\Omega$ has finite area, then $\partial \Omega$ is a union of f.m. geodesics, and $\partial \overline{\Omega} \cap X$ is finite, and $\Gamma(\partial \overline{\Omega} \cap X)$ is the set of cusps for $\Gamma$.

Proof: The volume of $\Omega$ has a relation with the angles of geodesics of $\Omega$, so it can only have f.m. vertices with interior angle $< 0.9\pi$. And the $\Gamma$-orbits of these angles intersect $\overline{\Omega}$ at f.m. vertices, which is because they consists of $\Gamma$-conjugates with the same distance with $z_0$, and a compact set meets f.m. geodesics of $\Omega$, because $\Gamma z_0$ is discrete in $\mathcal{H}$. In particular, $\Gamma$ has f.m. vertices in the boundary.

Now if $\Gamma$ has infinitely many vertices, there is a vertex $b$ that all its $\Gamma$-conjugates have angles $> 0.9\pi$, then consider the $\Gamma$-conjugates of $\Omega$ with a vertex $b$, then all their angle at $b > 0.9\pi$, but this cannot be possible geometrically.

For any cusp $x$ of $\Gamma$, suppose the Poincare fundamental domain is defined using $z$, notice there is a $\gamma \in \Gamma$ that $d(\gamma x, z)$ attains minimum, then $\gamma x$ must by in the boundary of $\Gamma$, as $d(\gamma_0 z, \gamma x) = d(z, \gamma x)$, and no other $z$ are closer to $\gamma x$. \hfill \square

Prop. (III.2.1.16) (Siegel Set and Fundamental Set). Siegel set is a nicely shaped substitutes for a fundamental domain:

- Let $a_1, \ldots, a_n \in \mathbb{R} \cup \{\infty\}$ be a representation of the $\Gamma$-orbits of cusps of $\Gamma$(III.2.1.15), let $\xi \in SL(2, \mathbb{R})$ be chosen that $\xi(a_i) = \infty$. If $c > 0, d > 0$ be chosen suitably, then the set $\cup \xi_i^{-1}F_{c,d}$ contains a fundamental domain of $\Gamma$.
- Suppose $\infty$ is a cusp of $\Gamma$, then if $d$ is large enough, then $F_d^\infty$ contains a fundamental domain for $\Gamma$. 

Proof: 1: $\xi_i \Gamma \xi_i^{-1}$ contains a unipotent subgroup generated by \[
\begin{bmatrix}
1 & \delta_i \\
0 & 1
\end{bmatrix},
\] so if $d \geq \delta_i$, then $\xi_i^{-1} F_{c,d}$ contains a nbhd of the cusp $a_i$ in the fundamental domain $F$ of $\Gamma$. So $F - \cup \xi_i^{-1} F_{c,d}$ is precompact in $\mathcal{H}$. Now if $c = 0, d = \infty$, then $F - \cup \xi_i^{-1} F_{c,d} = \emptyset$, then this is true for some $c, d$.

For 2: because $\infty$ is a cusp, we may assume $F \in \mathcal{H} \cap \{x > 0\}$. If $d$ is large, then $d$ will contain each of the pieces $\xi_i^{-1} F_{c,d} \cap F$ in item 1.

**Def. (III.2.1.17) (Form Spaces).** if $\Gamma$ is a discrete subgroup of $G_1$, let $C^\infty(\Gamma \setminus \mathcal{H}, \chi, k)$ be the space of smooth functions on $\mathcal{H}$ that

$$f|_k \gamma = \chi(\gamma)f, \quad \gamma \in \Gamma.$$ 

And elements in $C^\infty(\Gamma \setminus \mathcal{H}, 1, 0)$ are called **automorphic functions**.

**Prop. (III.2.1.18) (Inner Product on Form Spaces).** Notice that if $f, g \in C^\infty(\Gamma \setminus \mathcal{H}, \chi, k)$, then $f \bar{g}$ is invariant under action of $\Gamma$, so when $\Gamma \setminus \mathcal{H}$ is compact, $C^\infty(\Gamma \setminus \mathcal{H}, \chi, k)$ can be given the inner product (III.2.1.1):

$$\langle f, g \rangle = \int_{\Gamma \setminus \mathcal{H}} f(z) \bar{g}(z) \frac{dx dy}{y^2}.$$ 

and we can define the Hilbert space $L^2(\Gamma \setminus \mathcal{H}, \chi, k)$ as the Hilbert space completion of $C^\infty(\Gamma \setminus \mathcal{H}, \chi, k)$.

**Def. (III.2.1.19) (Behavior at Cusps).** If $\infty$ is a cusp of $\Gamma$, then $\Gamma$ contains some $\tau_r = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}$ for $r > 0$, then a function $f(x + iy)$ on $\mathcal{H}$ is called

- **of moderate growth** at $\infty$ iff $|f(x + iy)|$ is bounded by a polynomial function of $y$.
- **decay rapidly** at $\infty$ iff $|f(x + iy)| \leq y^{-N}$ for some $N > 0$.
- **cuspidal** at $\infty$ iff either $\chi(\tau_r) \neq 1$ or $\int_0^r f(z + u) du = 0$ for any $z \in \mathcal{H}$.

If $f$ is meromorphic on $\mathcal{H}$ then we have a Fourier expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i z/r} = \sum_{n=-\infty}^{\infty} a_n q^n = T(q).$$

Then $f$ is called **meromorphic/holomorphic/vanishes** at the cusp $\infty$ iff $T(q)$ does at 0.

The general cusp case is reduced to the $\infty$ case the same way as in (III.2.1.10). Notice this is independent of possible $r$ chosen.

**$L^2$-Spaces**

**Def. (III.2.1.20) ($L^2$-Spaces).** Denote by $C^\infty(\Gamma \setminus G, \chi)$ the space $C^\infty(\Gamma \setminus G, \chi, \omega)$ where $\omega$ is trivial on $Z^+(\mathbb{R})$ and $\omega(-1) = \chi(-1)$.

$L^2(\Gamma \setminus G, \chi)$ the space of functions on $G$ that is square integrable on $\Gamma \setminus G_1$ (because it can descend) with the quotient Haar measure (III.2.1.3) (notice that the absolute value descents to $\Gamma \setminus G_1$) that satisfies conditions in (III.2.1.8), and also the subspace $L^2_0(\Gamma \setminus G, \chi)$ consisting of cuspidal elements, where cuspidality is defined the same way as in (III.2.1.10) but in the sense that holds for a.e. $g$.

**Prop. (III.2.1.21).** The space $C_c(\Gamma \setminus G, \chi)$ is dense in $L^2(\Gamma \setminus G, \chi)$. 
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Proof: Firstly we show $C_c(\Gamma \setminus G, \chi)$ is dense in $L^2(\Gamma \setminus G, \chi)$. Let $\mathcal{F}$ be a Poincaré fundamental domain for $\Gamma$ in $SL_2(\mathbb{R})$ (III.2.1.14), then elements in $C_c(\mathcal{F})$ can be extended to elements of $C_c(\Gamma \setminus G, \chi)$, and in this way, $L^2(\Gamma \setminus G, \chi)$ is identified with $L^2(\mathcal{F})$, so the claim follows from (X.1.6.5).

Cor. (III.2.1.22). The right regular action of $G$ extends to a continuous unitary representation of $L^2(\Gamma \setminus G, \chi)$. $L^2(\Gamma \setminus G, \chi)$ is invariant under this representation.

Proof: We must verify continuity, and this is clear using the proposition because we can choose a compact supported function $f$ to approximate, then the right action is uniformly continuous. The unitarity is clear. The invariance of $L^2(\Gamma \setminus G, \chi)$ is clear.

Remark (III.2.1.23). Can this be extended to arbitrary locally compact group $G$? Compare with (III.1.6.3).

2 Technicalities

Prop. (III.2.2.1).

$$k_\theta \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 y_2 D(\theta)^{-1} & \xi D(\theta)^{-1} \\ \xi D(\theta)^{-1} & D(\theta) \end{bmatrix} k_{\theta'}.$$  

where 

$$\theta' = \arctan\left(\frac{y_1 \tan \theta}{y_2}\right), \quad D(\theta) = \sqrt{y_1^2 \sin^2 \theta + y_2^2 \cos^2 \theta}, \quad \xi = (y_2^2 - y_1^2) \sin \theta \cos \theta.$$

Proof: Hint: find $\theta'$ first.

Prop. (III.2.2.2) (Gelfand). Let $G = GL(n, \mathbb{R})^+, K = SO(n)$, denote $C_c^\infty(K \setminus G/K)$ to be the smooth functions $\varphi \in C^\infty(G)$ such that $\varphi(k_1 g k_2) = \varphi(g)$, then $C_c^\infty(K \setminus G/K)$ is commutative.

Proof: Consider the map $\varphi \mapsto \hat{\varphi}: \hat{\varphi}(g) = \varphi(g')$, then it is an anti-involution of $C_c^\infty(K \setminus G/K)$:

$$(\hat{\varphi_1} \ast \hat{\varphi_2})(g) = \int_G \varphi_1(g'h) \varphi_2(h^{-1}) dh = \int_G \hat{\varphi_1}(h) \varphi_2(h^{-1}) dh = \int_G \hat{\varphi_2}(h) \hat{\varphi_1}(h^{-1}) dh = (\hat{\varphi_2} \ast \hat{\varphi_1})(g)$$

But we find $\hat{\varphi} = \varphi$, because we can use (IX.8.6.1), $\varphi(g) = \varphi(d) = \hat{\varphi}(d) = \hat{\varphi}(g)$.

Prop. (III.2.2.3). Let $G = GL(2, \mathbb{R})^+, K = SO(2)$, let $\sigma$ be a character of $K$, then $C_c^\infty(K \setminus G/K, \sigma)$ is commutative.

Proof: The proof is the same as that of (III.2.2.2), but modified as

$$\hat{\varphi}(g) = \varphi\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right).$$

Prop. (III.2.2.4) (Harish-Chandra Theorem (Dimension 2 Case)). Let $G = SL(2, \mathbb{R})$, $f \in C^\infty(G)$ be both $K$-finite and $Z$-finite, then

- $f$ is analytic.
- $U(g) f$ is an admissible $(g, K)$-module.
• There exists $\alpha \in C^\infty_c(G)$ that
  \[ \alpha(kgk^{-1}) = \alpha(g), \quad f * \alpha = f. \]
  Moreover if $|f(g)| < C||g||^N$ where $||g||$ is induced from the Euclidean norm of $\mathbb{R}^4$, then all $U(g)f$ satisfies similar equalities with the same $N$.

**Proof:** Because $W$ lies in the Lie algebra of $K$ and $W = iH$, the hypothesis implies $Rf$ is f.d., where $R = \mathbb{C}[\Delta, H]$. Let $V$ be the smallest closed $G$-invariant subspace of $C^\infty(G)$ containing $f$ and let $V_0 = U(g)f$.

We first prove that
  \[ V_0 = \bigoplus_{n=-\infty}^{\infty} (V_0 \cap V(n)). \]
  Notice there is a continuous projection of $V$ onto $V(n)$:
  \[ E_n \varphi(g) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} \varphi(gk_\theta) d\theta, \]
  because $f$ is $K$-finite, there is an $N$ s.t. $f = \sum_{n=-N}^{N} E_n f$. Then notice any $Df$ is also $K$-finite because $D$ is combination of polynomials in $R, L, H$ and $R, L, H$ shift the weights, so the LHS is contained in the RHS. It’s left to show now $E_n Df \in V_0$ for any $n$: $E_n Df$ can be extracted using Lagrange polynomial in $H$, so it is clearly in $V_0$.

Next we show $V_0 \cap V(n)$ is of f.d.: Let $f_1, \ldots, f_k$ be a basis of $Rf$, because each $f_k$ is $K$-finite, so if we use the decomposition of $U(\mathfrak{sl}_2(\mathbb{C}))(I.12.8.24)$, only the $R, L$ shifting of the $E_n f$ will be considered, and clearly each $V_0 \cap V(n)$ is of f.d.

Now we show $f$ is analytic: because $f$ is $Z$-finite, there is an equation $P(\Delta)f = 0$ where $P$ is a monic function, and because $\Delta$ commutes with $E_k$ and $f$ is $K$-finite, $P(\Delta)E_k f = 0$. Now the $P(\Delta)E_k f = P(\Delta_k)E_k f$ by(III.2.1.5) and(III.2.3.1), and $P(\Delta_k)$ is an elliptic operator, so $E_k f$ is analytic by(X.8.8.5).

Now we show $V$ is the closure of $V_0$: Suppose not, then by Hahn-Banach there is a non-zero continuous linear functional $\Lambda$ on $V$ that $\Lambda(V_0) = 0$. If $F \in C^\infty(G)$, let $\varphi_F(g) = \Lambda(\rho(g)F)$, and let $\varphi = \varphi_F$. Clearly $dX \varphi_F = \varphi_{dX_F}$, so $D \varphi_F = \varphi_{DF_F}$, which implies $\varphi$ is $Z$-finite and smooth, and also $K$-finite because $f$ does. So by what we have proved, $\varphi$ is analytic. But now $\varphi$ is analytic and $D \varphi(1) = \varphi_{D_f}(1) = 0$ because $D f \in V_0$, so $\varphi = 0$ by Taylor expansion. So $\Lambda(\rho(g)f) = 0$ for any $g$, contradiction because $\rho(g)f$ is dense in $V$.

So actually $V(n) \subset V_0$, because $E_n V_0 = V_0 \cap V(n) \subset V(n)$ is dense, and $V_0 \cap V(n)$ is of f.d., so $V(n) \subset V_0 = V_0(X.4.1.8)$ and $V_0 = \oplus V(n)$ is an admissible $(g, K)$-module.

Let $J$ be the convolution algebra(because $G$ is unimodular) of functions $\alpha$ that $\alpha(kgk^{-1}) = \alpha(g)$, then it can be checked that convolution $\ast \alpha$ commutes with action of $K$, so $f \ast J$ is in the same $K$-type space as $f$, thus $K$-finite and in a f.d. space.

Now we can approximate $f$ by $f \ast J$: choose a Dirac sequence $\{\alpha_n\} \subset C^\infty_c(G)$, we may replace $\{\alpha_n\}$ by the function $\beta_n(g) = \int_K \alpha_n(k^{-1}gk)dk$ to obtain a Dirac sequence in $J$. Then $f \ast \alpha_n \rightarrow f$ uniformly on compact sets. But $f \ast J$ is f.d., so there are some $f \ast \alpha = f$.

Finally for the growth estimate, it suffices to check for $D \in \mathfrak{g}$. Then $dX(f) = dX(f \ast \alpha) = f \ast (dX\alpha)$, from which the estimate is clear. \qed
Cor. (III.2.2.5) (Automorphic Forms Generate Admissible \((g,K)\)-Modules). The space \(A(\Gamma \backslash G, \chi, \omega)\) and \(A_0(\Gamma \backslash G, \chi, \omega)\) are stable under the action of \(U(g)\), and for \(f \in A(\Gamma \backslash G, \chi, \omega)\), \(f\) is analytic and \(U(g)f\) is an admissible \((g,K)\)-module.

Moreover, if \(f\) satisfies condition of moderate growth (III.2.1.9) and \(D \in U(g)\), then \(Df\) satisfies similar conditions with the same constant \(N\).

Proof: It suffices to prove for \(A(\Gamma \backslash G, \chi, \omega)\) because the cuspidality condition is clearly preserved by right action.

We want to use Harish-Chandra theorem (III.2.4). Now the condition of \(\omega\) shows \(|f|\) is constant on each \(Z(\mathbb{R})\)-coset of \(G\), and \(I\) acts by a scalar on \(f\), so we know the \(Z\)-finite condition is compatible with that of (III.2.4). Also the condition of moderate growth is compatible because the minimal \(|g|\) in a \(Z(\mathbb{R})\)-orbit is achieved when \(\text{deg } g = 1\) (III.2.1.9). So all the assertion follows from that of (III.2.4) and its proof.

\[\Box\]

3 Maass Forms and Representations

Maass Forms

Def. (III.2.3.1) (Maass Operator). A Maass differential operators on \(C^\infty(H)\) is defined to be

\[ R_k = (z - \bar{z}) \frac{\partial^2}{\partial z^2} + \frac{k}{2} = iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{k}{2}, \quad L_k = -(z - \bar{z}) \frac{\partial^2}{\partial \bar{z}^2} - \frac{k}{2} = -iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - \frac{k}{2} \]

and

\[ \Delta_k = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \frac{\partial}{\partial x} = -L_{k+2}R_k - \frac{k}{2} \left( 1 + \frac{k}{2} \right) = -L_{k-2}L_k + \frac{k}{2} \left( 1 - \frac{k}{2} \right) \]

When \(G = \Gamma \backslash G\) is compact, \(\Delta_k\) is a symmetric operator on \(L^2(H)\) with domain \(C_c^\infty(H)\).

Proof: For the formula: ?

For the symmetry: composed with the measure, the order 2 part is just the ordinary Laplacian, and the order 1 part becomes \(iy^{-1} \frac{\partial}{\partial x}\), then notice

\[ \int_{\Gamma \backslash G} iy^{-1} \left( f \frac{\partial f}{\partial x} + \frac{\partial \bar{f}}{\partial x} \right) dx dy = i \int_{\Gamma \backslash G} d(y^{-1} f \bar{f} dy) = 0 \]

\[\Box\]

Prop. (III.2.3.2) (Maass Operator and Weight Action). For \(f \in C^\infty(H), g \in G\),

\[ (R_k f)|_{k+2g} = R_k(f|_{k}g), \quad (L_k f)|_{k-2g} = L_k(f|_{k}g), \quad (\Delta_k f)|_{k}g = \Delta_k(f|_{k}g) \]

Proof: For \(R_k\), because scalar doesn’t matter, we may assume \(g \in SL_2(\mathbb{R})\), and let \(w = \frac{az+b}{cz+d}\), then

\[ \frac{\partial}{\partial z} = \frac{\partial w}{\partial z} \frac{\partial}{\partial w} = (cz+d)^{-2} \frac{\partial}{\partial w}, \quad (w - \bar{w}) = \frac{z - \bar{z}}{|cz+d|^2}. \]

So

\[ (w - \bar{w}) \frac{\partial}{\partial w} = \left( \frac{(cz+d)^2}{|cz+d|^2} \right) (z - \bar{z}) \frac{\partial}{\partial z}. \]
And for any smooth function \( \varphi \in C^\infty(\mathcal{H}) \),
\[
(z - \overline{z}) \frac{\partial}{\partial z} \left( \left( \frac{cz + d}{cz + d} \right)^k \varphi \right) = (z - \overline{z}) \left( \frac{cz + d}{cz + d} \right)^k \frac{\partial \varphi}{\partial z} + \frac{k}{2} \left[ \left( \frac{cz + d}{cz + d} \right)^{k+2} - \left( \frac{cz + d}{cz + d} \right)^k \right].
\]
This is because
\[
\frac{\partial}{\partial z} \left( \frac{cz + d}{cz + d} \right)^k = \frac{c}{2} \frac{cz + d}{cz + d} \left( \frac{cz + d}{cz + d} \right)^k = -\frac{k}{2} c(z - \overline{z}) \left( \frac{cz + d}{cz + d} \right)^k = \frac{k}{2} \left[ \left( \frac{cz + d}{cz + d} \right)^{k+2} - \left( \frac{cz + d}{cz + d} \right)^k \right].
\]
Thus,
\[
R_k(f|kg) = [(z - \overline{z}) \frac{\partial}{\partial z} + \frac{k}{2}] \left( \frac{cz + d}{cz + d} \right)^k f(w) = [(z - \overline{z}) \frac{\partial}{\partial z} \left( \frac{cz + d}{cz + d} \right)^k \frac{\partial \varphi}{\partial z} + \frac{k}{2} \left( \frac{cz + d}{cz + d} \right)^{k+2} f(w) = \left( \frac{cz + d}{cz + d} \right)^{k+2} ([w - \overline{w}) \frac{\partial}{\partial w} + \frac{k}{2}] f(w) = ((R_k f)|_{k+2} g)(z).
\]

The case of \( L_k \) is similar, and \( \Delta_k \) follows. \( \square \)

**Cor. (III.2.3.3).** The operator \( R_k, L_k, \Delta_k \) maps functions between \( C^\infty(\Gamma \backslash \mathcal{H}, \chi, k) \) and arises and decreases weights respectively.

**Def. (III.2.3.4) (Maass Forms).** A Maass Form of weight \( k \) is an element in \( C^\infty(\Gamma \backslash \mathcal{H}, \chi, k)(III.2.1.17) \) that is an eigenform for \( \Delta_k \) (of eigenvalue \( \lambda \)) and is of moderate growth at cusps of \( \Gamma(III.2.1.19) \).

**Def. (III.2.3.5) (Holomorphic Modular Forms as Maass Forms).** Let \( \Gamma \backslash \mathcal{H} \) be compact, then if \( f \) is a holomorphic modular form for \( \Gamma \) of weight \( k > 0 \), then \( y^{k/2} f(z) \) is a Maass form in \( C^\infty(\Gamma \backslash \mathcal{H}, 1, k) \) with eigenvalue \( \frac{k}{2} (1 - \frac{k}{2}) \) of \( \Delta_k \) and annihilated by \( L_k \).

Conversely, if a Maass form for \( \Gamma \) of weight \( k \) can be annihilated by \( L_k \), then it comes from a modular form like above. Moreover, \( f(z) \) is a cusp form iff \( y^{k/2} f(z) \) is cuspidal.

**Proof:** Direct calculation shows
\[
L_k(y^{k/2} f(z)) = 2i y^{(k+2)/2} \frac{\partial}{\partial \overline{z}} f(z), \text{ and by (X.2.2.5),}
\]
\[
\text{Im}(z)^{k/2} f(z)_{| \gamma} = \left( \frac{cz + d}{cz + d} \right)^k \text{Im}(\gamma z)^{k/2} f(\gamma z) = \left( \frac{cz + d}{cz + d} \right)^k \text{Im}(z)^{k/2} f(\gamma z)
\]
so their invariant properties are compatible. Also, if \( f \) is a holomorphic modular form, then \( y^{k/2} f \) is bounded by a polynomial of \( y \) at \( \infty \), and conversely, if \( f(q) \) is bounded by a polynomial of \( \log(1/|q|) \), then ?? shows \( f \) is holomorphic at \( \infty \). And the cuspidal condition
\[
\int_0^\infty \text{Im}(z)^{k/2} f(z + u) du = \text{Im}(z)^{k/2} \int_0^\infty f(z + u) du = 0
\]
just means \( f \) is a cusp form. \( \square \)
III.2. AUTOMORPHIC REPRESENTATIONS OF $GL(2, \mathbb{R})(BUMP)$

Comparison between $\Gamma \backslash \mathcal{H}$ and $\Gamma \backslash G$

Prop. (III.2.3.6) (Two Form Spaces Equal). Let $L^2(\Gamma \backslash G, \chi, k)$ be the subspace of $L^2(\Gamma \backslash G, \chi)$ consisting of functions $F$ that $ho(k_g)F = e^{ik\theta}F$. Then there is an isomorphism of Hilbert spaces

$$\sigma_k : L^2(\Gamma \backslash \mathcal{H}, \chi, k) \cong L^2(\Gamma \backslash G, \chi, k) : (\sigma_k f)(g) = (f|_{\mathbb{R}})(i).$$

And we have (III.2.3.1):

$$\sigma_{k+2}R_k = dR\sigma_k, \quad \sigma_{k-2}L_k = dL\sigma_k, \quad \sigma_k\Delta_k = \Delta\sigma_k.$$

Also the behavior at the cusps are compatible.

Proof: Check the left action of $\gamma$ and $Z^+(\mathbb{R})$, and it can be verified that the inverse of $\sigma_k$ is given by $f(z) = F(0, x, y, 0).$

More precisely, if coordinates (III.2.1.3),

$$F(u, x, y, \theta) = f(x + iy)e^{ik\theta}, \quad f(z) = F(0, x, y, 0),$$

so the behavior as cusps are clearly compatible. Then check the $\Gamma$-invariance (III.2.1.17) and (III.2.1.8). Finally check that $\sigma_k$ preserves inner product, which is by (III.2.1.18) and (III.2.1.3).

The equations between $R_k, L_k$ and $R, L$ are easily verified. $\square$

Cor. (III.2.3.7) (Maass Forms as Admissible $(g, K)$-Modules). If $f$ is a Maass form of weight $k$, then $\sigma_k(f) \in C^\infty(\Gamma \backslash G, \chi, k)$ (III.2.3.6), and is a eigenform of $\Delta$. In particular it is $K$-finite and $Z$-finite, so by (III.2.2.5) generates an admissible $(g, K)$-module.

4 Irreducible Unitary Representations of $GL(2, \mathbb{R})^+$

$(g, K)$-Modules of $GL(2, \mathbb{R})$

Prop. (III.2.4.1) (Lie Theory). Let $V$ be an irreducible admissible $(g, K)$-module for $GL(2, \mathbb{R})^+$, then

- $V_k$ is the space of all vectors $x \in V$ that $Hx = kx$.
- If $x \in V^k$, then $Rx \in V^{k+2}, Lx \in V^{k-2}$.
- If $0 \neq x \in V^k$, then $\mathbb{C}x = V^k, \mathbb{C}R^n x = V^{k+2n}, \mathbb{C}L^n V^{k-2n}$ and

$$V = \mathbb{C}x \oplus \bigoplus_{n>0} \mathbb{C}R^n x \oplus \bigoplus_{n>0} \mathbb{C}L^n x.$$

- Suppose $\Delta = \lambda$ on $V$, then if $x \in V^k$, then

$$LRx = (-\lambda - \frac{k}{2}(1 + \frac{k}{2}))x, \quad RLx = (-\lambda + \frac{k}{2}(1 - \frac{k}{2}))x.$$

Proof: 1: Let $W = iH$. If $x \in V^k$, then

$$Wx = \frac{d}{dt} \pi(e^{itW})x = \frac{d}{dt} \pi(k_xt)x = \frac{d}{dt} e^{ikt}x = ikx.$$
Thus $Hx = kx$. And we know $V$ decomposes as direct sums of representations of $K$, thus the result follows.

2: clear from(I.12.2.11).

3: Because the RHS is a $g$-submodule, by representation of $sl_2(\mathbb{C})$ (III.6.1.11). And it is also a $K$-subrepresentation, by item1.

4: By(III.6.1.11).

Cor. (III.2.4.2) (Non-Discrete Case). For any $\lambda, \mu \in \mathbb{C}$ that $\lambda \neq \frac{k}{2}(1 + \frac{k}{2})$ for $k$ even/odd, there exists at most one isomorphism class of irreducible admissible even/odd $(g, K)$-module $V$ on which $\Delta, I$ acts by $\lambda, \mu$ respectively, and the $K$-type is one vector $f_k$ for each $k \in \mathbb{Z}$.

Proof: This follows from the classification of representation of $sl_2(\mathbb{C})$ (III.6.1.11). Notice the action of $K$ is controlled by(III.2.4.1) item1.

Cor. (III.2.4.3) (Discrete Case). Let $k \geq 1$ be an integer and $\lambda = \frac{k}{2}(1 + \frac{k}{2})$. Let $V$ be an irreducible admissible $(g, K)$-module with parity equals $k$, Let $\Sigma$ be the $K$-types of $V$, then $\Sigma$ is one of the following sets:

- $\Sigma^+(k) = \{l \in \mathbb{Z}|l \equiv k \ mod \ 2, l \geq k\}
- \Sigma^-(k) = \{l \in \mathbb{Z}|l \equiv k \ mod \ 2, l \leq -k\}
- \Sigma^0(k) = \{l \in \mathbb{Z}|l \equiv k \ mod \ 2, -k < l < k\}$

And there are at most one isomorphism class with each $\Sigma$.

Proof: This follows from the classification of representation of $sl_2(\mathbb{C})$ (III.6.1.11). Notice the action of $K$ is controlled by(III.2.4.1) item1.

Def. (III.2.4.4) ($H(s_1, s_2, \varepsilon)$). If $\lambda \geq 1/4$, let $s - 1/2$ be the square root of $1/4 - \lambda$ which is imaginary, and let $s_1, s_2$ be determined that $\mu = s_1 + s_2, s = \frac{1}{2}(s_1 - s_2 + 1)$.

Consider the 1-dimensional representation $\sigma$ of $B(\mathbb{R})^+$ that

$$\sigma\left(\begin{bmatrix} y_1 & x \\ y_2 \end{bmatrix}\right) = \text{sgn}(y_1)^{\varepsilon}|y_1|^{s_1}|y_2|^{s_2},$$

If $s_1, s_2$ are purely imaginary (i.e. $\mu$ is purely imaginary), then this representation is unitary, and we can consider the induced representation on $GL(2, \mathbb{R})^+$ (X.6.5.3), then it is a unitary representation of $GL(2, \mathbb{R})^+$.

In this particular case, we may dual the induce process to get a right action version of $GL(2, \mathbb{R})^+$, then in fact?

$$f\left(\begin{bmatrix} y_1 & x \\ y_2 \end{bmatrix} g\right) = \text{sgn}(y_1)|y_1|^{s_1+1/2}|y_2|^{s_2-1/2}f(g)$$

and

$$(f_1, f_2) = \frac{1}{2\pi} \int_{0}^{2\pi} f_1(k_\theta)f_2(k_\theta) d\theta.$$

so $H(s_1, s_2, \varepsilon)$ is identical to $L^2[-\pi/2, \pi/2]$ by Iwasawa decomposition(III.2.1.3). Then its $K$-finite vectors can be determined, which are sums of

$$f_1(g) = u^{s_1+s_2}y^\varepsilon e^{i\theta}, l \equiv \varepsilon \ mod \ 2.$$
Prop. (III.2.4.5). We can define the regular action of $G$ on $H(s_1, s_2, \varepsilon)$ (but may not be unitary), and the subspace $H^\infty(s_1, s_2, \varepsilon)$ is the space of smooth vectors for this representation.

Proof: It suffices to define the regular action of $G$ on $H^\infty(s_1, s_2, \varepsilon)$ and show it is a bounded operator, so that it can be extended to $H(s_1, s_2, \varepsilon)$ by continuity.

By Cartan decomposition (IX.8.6.1), $G$ is generated by $K$ and diagonal matrices of positive entries. $\rho(K)$ clearly preserves the inner product, so it suffices to consider these matrices.

By (III.2.2.1) and the calculation $d\theta' = y_1 y_2 D(\theta)^{-2} d\theta$, we have
\[
\int_0^{2\pi} |\pi(\text{diag}(y_1, y_2))f(k_0)|^2 d\theta = (y_1 y_2)^{s_1 - 1/2} \int_0^{2\pi} D(\theta)^{-s_1 + s_2 + 1} |f(k_0)|^2 d\theta',
\]
where
\[\theta' = \arctan\left(\frac{y_1}{y_2} \tan \theta\right), \quad D(\theta) = \sqrt{y_1^2 \sin^2 \theta + y_2^2 \cos^2 \theta}.
\]
$D(\theta)$ is bounded above and below, thus $\pi(\text{diag}(y_1, y_2))$ is a bounded operator. Also to show this representation is continuous, it suffices to show for $|f|_2$ small and $y_1, y_2$ small, $|\pi(\text{diag}(y_1, y_2))f|_2$ is small. And this is also a consequence of the above formula. \(\square\)

Lemma (III.2.4.6). For $f \in H(s_1, s_2, \varepsilon)$ as in (III.2.4.4), we have
\[H f_\lambda = l f_\lambda, \quad R f_\lambda = (s + \frac{1}{2}) f_{l+2}, L f_\lambda = (s - \frac{1}{2}) f_{l-2}, \Delta f_\lambda = \lambda f_\lambda, \quad I f_\lambda = \mu f_\lambda\]

Proof: Clear from (III.2.1.5) and definition of $f_\lambda$ (III.2.4.4). \(\square\)

Prop. (III.2.4.7) (Existence of $(g,K)$-Modules). Let $s = \frac{1}{2} (s_1 + s_2 + 1), \lambda = s (1 - s), \mu = (s_1 + s_2)$, then (subquotients) of the $(g,K)$-module $\mathfrak{H}$ of $H(s_1, s_2, \varepsilon)$ afford classes in (III.2.4.2) and (III.2.4.3). More precisely, $\Delta$ and $I$ acts by scalars $\lambda, \mu$ respectively, and
- If $s$ is not of the form $k/2, k \equiv \varepsilon \mod 2$, then $\mathfrak{H}$ is irreducible.
- If $s \geq 1/2$ and $s = \frac{k}{2}$ where $k \geq 1$ is an integer that $k \equiv \varepsilon \mod 2$, then $\mathfrak{H}$ has two irreducible invariant subspaces $\mathfrak{H}_+, \mathfrak{H}_-$ with $K$-types $\Sigma_+, \Sigma_-$ respectively, and the quotient $\mathfrak{H}_+/\mathfrak{H}_+ \oplus \mathfrak{H}_-$ is irreducible and has $K$-type $\Sigma^0(k)$.
- If $s \leq 1/2$ and $s = 1 - \frac{k}{2}$ where $k \geq 1$ is an integer that $k \equiv \varepsilon \mod 2$. Then $\mathfrak{H}$ has an invariant subspace $\mathfrak{H}_0$ with $K$-types $\Sigma^0(k)$ and the quotient $\mathfrak{H}/\mathfrak{H}_0$ decomposes into two irreducible invariant subspaces $\mathfrak{H}_+, \mathfrak{H}_-$ with $K$-types $\Sigma_+, \Sigma_-$ respectively.

Proof: The action of $H, R, L, \Delta, I$ is all clear from (III.2.4.6), and the decomposition and irreducibility is all clear from the representation theory of $sl_2$ and (III.2.4.2),(III.2.4.3). \(\square\)

Prop. (III.2.4.8) (List of Irreducible Admissible $(g,K)$-Modules for $GL(2, \mathbb{R})^+$). Every irreducible admissible $(g,K)$-module may be realized as the space of $K$-finite vectors in an admissible representation of $G$ on a Hilbert space. Let $\lambda, \mu \in \mathbb{C}$, and $\varepsilon = 0, 1$.
- If $\lambda$ is not of the form $\frac{k}{2} (1 - \frac{k}{2})$, where $k \equiv \varepsilon \mod 2$, then there exists a unique irreducible admissible $(g,K)$-module of parity $\varepsilon$ on which $\Delta, I$ acts by scalars $\lambda, \mu$, denoted by $P_{\mu}(\lambda, \varepsilon)$. These are called the principal series.
- If $\lambda = \frac{k}{2} (1 - \frac{k}{2})$ for some $1 \leq k \equiv \varepsilon \mod 2$, then there exists three (two for $k = 1$) irreducible admissible $(g,K)$-module of parity $\varepsilon$ on which $\Delta, I$ acts by scalars $\lambda, \mu$. Their $K$-types are $\Sigma^+, \Sigma^0$ respectively. The $K$-types $\Sigma^+$ are denoted by $D^+_{\mu}(k)$. If $k > 1$, $D^+_{\mu}(k)$ are called discrete series and for $k = 1$ they are called limits of discrete series.
Proof: This is a consequence of (III.2.4.2)(III.2.4.3) and (III.2.4.7). □

Cor. (III.2.4.9). Let $G = GL(2, \mathbb{R})^+$, $K = SO(2)$, $(\pi, V)$ be an irreducible admissible $(\mathfrak{g}, K)$-module, then the contragradient $\hat{\pi}$ is isomorphic to the $(\mathfrak{g}, K)$-module $g \mapsto \pi(g^{-1})$.

Proof: This is a consequence of the classification of irreducible $(\mathfrak{g}, K)$-modules (III.2.4.8), where the $K$-type are unchanged, so they are isomorphic. □

Prop. (III.2.4.10) (List of Irreducible Admissible $(\mathfrak{g}, K)$-Modules for $GL(2, \mathbb{R})$). Every irreducible admissible $(\mathfrak{g}, K)$-module may be realized as the space of $K$-finite vectors in an admissible representation of $G$ on a Hilbert space. Let $\lambda, \mu \in \mathbb{C}$, and $\varepsilon = 0, 1$.

- The f.d. representations are obtained by tensoring the symmetric powers of the standard representation of $G$ with the 1-dimensional representation of the form $\chi \circ \det$.
- If $\chi_1, \chi_2$ are characters of $\mathbb{R}^+$ that $\chi_1 \chi_2^{-1}$ is not of the form $y \mapsto \text{sgn}(y)^\varepsilon |y|^{k-1}$, where $k \equiv \varepsilon \mod 2$, then there is a irreducible $(\mathfrak{g}, O(2))$-module $\pi(\chi_1, \chi_2)$.
- If $\mu$ is a real number and $k \geq 1$ is an integer, then there are representations $D_\mu(k)$, called discrete series if $k \geq 2$ and limits of discrete series if $k = 1$.

Proof: Cf. [Bump P219]?.

Unitarizability of $(\mathfrak{g}, K)$-Modules

Lemma (III.2.4.11) (Finite Dimensional Case). The only irreducible f.d. unitary representations of the group $GL(n, \mathbb{R})^+$ are the 1-dimensional characters $g \mapsto \det(g)^r$ where $r$ is purely imaginary.

Proof: Such a representation defines a continuous map of $GL(n, \mathbb{R})^+$ into the compact unitary group $U(n)$. Now it induces a Lie algebra map $\mathfrak{sl}_n(\mathbb{R}) \to \mathfrak{u}_m$. This map must be trivial because otherwise this is an embedding because $\mathfrak{sl}_n(\mathbb{R})$ is simple. But this is impossible because the adjoint action of $\mathfrak{sl}_n(\mathbb{R})$ has real eigenvalues but the adjoint action of $\mathfrak{u}_m$ are all purely imaginary by (I.12.5.4). So the action is trivial on $SL(n, \mathbb{R})^+$, so induces an irreducible representation of $\det(g)$, which is clearly 1-dimensional. □

Lemma (III.2.4.12). Because for a unitary representation $\mathcal{H}$ of $G$, for $X \in \mathfrak{g}$, we have

$$(Xu, v) = -(u, Xv),$$

so $(Xv, w) = -(v, \overline{X}w)$ when complexified. So

$$(Rv, w) = -(v, Lw)$$

for any $v, w \in \mathcal{H}$, by (III.2.1.1).

Lemma (III.2.4.13) (Principal Series). For the principal series $P_\mu(\lambda, \varepsilon)$ of $GL(2, \mathbb{R})^+$, there exists an irreducible unitary representation in this class if $\mu$ is purely imaginary and $\lambda \geq 1/4$ real.

Proof: Consider the unitary representation $H(s_1, s_2, \varepsilon)$ defined in (III.2.4.4), then it is irreducible and its class is $P_\mu(\lambda, \varepsilon)$ by (III.2.4.7) and (III.1.5.7). □

Lemma (III.2.4.14) (Possibilities of Unitary Representations). Let $\mathcal{H}$ be a unitary representation of $GL(2, \mathbb{R})^+$. Assume $\Delta, I$ acts by scalars $\lambda, \mu$ respectively, then
• $\mu$ is purely imaginary and $\lambda$ is real.
• If the $(g, K)$-module type of $H$ is a principal series $P_\mu(\lambda, \varepsilon)$, then $\lambda > 0$, and if $\varepsilon = 1$, $\lambda > 1/4$.

Proof: 1: This follows from (III.2.4.12), as action of $I$ is skew-symmetric and action of $\Delta$ is symmetric.

2: By (III.2.4.2), $V^k \neq 0$ for $k \equiv \varepsilon \mod 2$, let $f_k \in V^k$. Because $-4\Delta-H^2+2H = 4RL(III.2.1.1)$, take $k = \varepsilon$, then

$(-4\lambda - \varepsilon^2 + 2\varepsilon)f_\varepsilon = 4RLf_\varepsilon.$

But by (III.2.4.12),

$(4RLf_\varepsilon, f_\varepsilon) = -4(Lf_\varepsilon, Lf_\varepsilon) < 0$

thus $4\lambda > 2\varepsilon - \varepsilon^2$. □

Cor. (III.2.4.15) (Reduction of $\mu$). The infinitesimal equivalence class of representations $P_\mu(\lambda, \varepsilon)$ or $D^\pm(\mu)$ contains an irreducible unitary representation iff $\mu$ is purely imaginary and the corresponding class $P(\lambda, \varepsilon)$ or $D^\pm(\lambda)$ contains an irreducible unitary representation.

Proof: $\mu$ must be purely imaginary by the proposition. And we may tensoring a unitary representation by a deg($g$)$_r$, it is also unitary iff $r$ is purely imaginary, and this increases the value of action of $I$ by $2r$ and doesn’t affect $\mu$ and $\varepsilon$ because $\Delta$ has nothing to do with $u(III.2.1.5)$. □

Prop. (III.2.4.16) (Intertwining Integral). Let $s = \frac{1}{2}(s_1 - s_2 + 1)$, define for $f \in V$,

$$M(s) : H^\infty(s_1, s_2, \varepsilon) \rightarrow H^\infty(s_2, s_1, \varepsilon) : (M(s)f)(g) = \int_{N(F)} f(w_0ug) du.$$  

Then if $\text{Re}(s_1 - s_2) > 0$, the integral is absolutely convergent, and commutes with the action of $G$.

Proof: Replacing $f$ with $\rho(h)f$, we see that the convergence of $M(s)(f)(h)$ is equivalent to the convergence of $M(s)(\rho(h)f)$, so we assume $h = 1$.

We use the identity

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \Delta_x^{-1} & -x\Delta_x^{-1} \\ \Delta_x & \Delta_x \end{bmatrix} k_{\theta_x}$$

similar to (III.2.2.1) where

$$\Delta_x = \sqrt{1 + x^2}, \quad \theta(x) = \arctan(-1/x).$$

Then

$$(M(s)f)(1) = \int_{-\infty}^{\infty} (1 + x^2)^{-s} f(k_{\theta(x)}) dx$$

which convergences for $s > 1/2$, that is $\text{Re}(s_1 - s_2) > 0$.

To show $M(s)f \in H(s_2, s_1, \varepsilon)$, we check

$$(M(s)f)(\begin{bmatrix} 1 & x \\ \varepsilon & 1 \end{bmatrix} g) = (M(s)f)(g), \quad (M(s)f)(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} g) = \text{sgn}(y_1)^\varepsilon |y_1|^{s_2 + \frac{1}{2}} |y_2|^{s_1 - \frac{1}{2}} (M(s)f)(g).$$

The first one is an easy consequence of change of variable, for the second,

$$(M(s)f)(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} g) = \int f(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & x \\ \varepsilon & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} g) dx.$$
\[= \int f\left[ \begin{array}{cc} 1 & -1 \\ \frac{y_1}{y_2} & 1 \end{array} \right] \left[ \begin{array}{cc} y_1^{-1}y_2x \\ 1 \end{array} \right] g) dx \]

\[= \frac{y_1}{y_2} \int f\left[ \begin{array}{cc} y_1 & 1 \\ \frac{y_2}{1} & 1 \end{array} \right] \left[ \begin{array}{cc} -1 \\ 1 \end{array} \right] \left[ \begin{array}{cc} x & 1 \\ 1 & 1 \end{array} \right] g) dx \]

\[= \frac{y_1}{y_2} \text{sgn}(y_1)^\varepsilon |y_1|^{s_2 - \frac{1}{2}} |y_2|^{s_1 + \frac{1}{2}} \int f\left[ \begin{array}{cc} -1 \\ 1 \end{array} \right] \left[ \begin{array}{cc} x & 1 \\ 1 & 1 \end{array} \right] g) dx \]

\[= \text{sgn}(y_1)^\varepsilon |y_1|^{s_2 + \frac{1}{2}} |y_2|^{s_1 - \frac{1}{2}} (M(s)f)(g) \]

To check \(M(s)f\) is smooth, notice that the restriction of \(M(s)f\) to \(K\) equals

\[(M(s)f)(k_i) = \int_{-\infty}^{\infty} (1 + x^2)^{-s} f(k_{\theta(x)+t}) dx.\]

The convergence is uniform in \(t\), thus is smooth in \(t\).

The commutativity of \(M(s)\) with \(G\)-action is immediate, because left and right action commutes.

**Prop. (III.2.4.17).** Let \(f_{k,s}\) be the function \(f_k\) in \(H(s_1, s_2, \varepsilon)(\text{III.2.4.4})\) where \(\text{Re}(s_1 - s_2) > 0\) and \(s = \frac{1}{2}(s_1 - s_2 + 1)\), then

\[M(s)f_{k,s} = (-i)^k \sqrt{\pi} \frac{\Gamma(s)\Gamma(s - \frac{1}{2})}{\Gamma(s + k)\Gamma(s - \frac{k}{2})} f_{k,1-s}\]

**Proof:** Because \(M(s)\) commutes with \(G\)-action, \(M(s)f_{k,s}\) is a multiple of \(f_{k,1-s}\), thus it suffices to calculate \((M(s)f_{k,s})(1)\), which is

\[(M(s)f)(1) = \int_{-\infty}^{\infty} (1 + x^2)^{-s} e^{ik\theta(x)} dx, \quad \theta(x) = \arctan(-1/x)\]

by (III.2.4.16).

This integral then is calculated to be the expression above, by (X.1.8.1).

**Lemma (III.2.4.18) (Complementary Series).** For \(\mu\) purely imaginary and \(0 < \lambda < 1/4\) and \(\varepsilon = 0\), there exists an irreducible unitary representation in this class of the \((\mathfrak{g}, K)\)-module \(P_\mu(\lambda, 0)\).

**Proof:** Let \(s_1, s_2\) be complex numbers, we construct first a Hermitian pairing

\[H(s_1, s_2, \varepsilon)_{\text{fin}} \times H(-\overline{s_1}, -\overline{s_2}, \varepsilon)_{\text{fin}} \to \mathbb{C} : (f, g') \mapsto \int_K f(k)\overline{g'(k)} dk\]

which is invariant under action of \(G\) by (X.6.5.5). Now let \(s_2 = -\overline{s_1}\), then \(s = \frac{1}{2}(s_1 - s_2 + 1)\) is real and \(\mu = s_1 + s_2\) is purely imaginary. Then composing this pairing with \(\overline{\varepsilon} M(s) : H(s_1, s_2, \varepsilon)_{\text{fin}} \to H(s_2, s_1, \varepsilon)_{\text{fin}} = H(-\overline{s_1}, -\overline{s_2}, \varepsilon)_{\text{fin}}\), then

\[(f, \overline{g'}) = \int_K f(k)\overline{\varepsilon M(s)f'(k)} dk,\]

is \(G\)-invariant. We will show that it is positive definite if \(\varepsilon = 0\) and \(1/2 < s < 1\).
III.2. AUTOMORPHIC REPRESENTATIONS OF $GL(2, \mathbb{R})(BUMP)$

It can be seen from (III.2.4.17) that an orthogonal basis for $H(s_1, s_2, 0)_{fin}$ under this pairing is $f_{k,s}$ for $k$ even. And by (III.2.4.17),

$$(f_{k,s}, f_{k,s}) = (-1)^{k/2} \sqrt{\pi} \frac{\Gamma(s)\Gamma(s - \frac{1}{2})}{\Gamma(s + \frac{k}{2})\Gamma(s - \frac{k}{2})},$$

which is positive for $1/2 < s < 1$. Now we obtain a unitary representation of $G$ on the Hilbert completion of this space (X.6.2.9).

Now we have constructed a unitary representation in the infinitesimal equivalence class $P_\mu(\lambda, 0)$ with $\lambda = s(1 - s)$, $\frac{1}{2}(s_1 - s_1 + 1)$, so any $0 < \lambda < 1/4$ is possible. □

Lemma (III.2.4.19). For any integer $k$, there is a bijection between holomorphic functions $\varphi$ on $H$ and smooth functions $\Phi$ on $GL(2, \mathbb{R})^+$ that is invariant under $Z(\mathbb{R})^+$ and

$$\Phi(gk\theta) = e^{ik\theta} \Phi(g), \quad L\Phi = 0.$$  

Proof: The bijection is given by

$$\varphi(z) = y^{-k/2} \Phi\left(y \begin{bmatrix} x \\ 1 \end{bmatrix}\right), \quad \Phi(g) = ((y^{k/2}\varphi)[g])_k i.$$

and the proof is a combination of formal calculation in (III.2.3.5) and (III.2.3.6) forgetting $\Gamma$:

$$L_k(y^{k/2}f(z)) = -(z - \overline{z}) \frac{\partial}{\partial \overline{z}} - \frac{k}{2} (y^{k/2}f(z)) = -2iy^{(k+2)/2} \frac{\partial}{\partial \overline{z}} f(z).$$

□

Lemma (III.2.4.20) (Discrete Series). if $k > 1$, then there exists a unitary representation in the infinitesimal equivalence class $D^\pm(k)$, more precisely,

- Let $L^2(H, \mu_k)$ be the $L^2$-space of holomorphic functions $f$ on the upper plane $H$ w.r.t the measure $\mu_k = y^k dx dy / y^2$ (X.2.6.17). Then the left action

$$\pi_k(g)f = f[g^{-1}]_k.$$

of $G$ is unitary and this representation $\pi_k$ is in the infinitesimal equivalence class $D^-(k)$.

- Consider the automorphic of $GL(2, \mathbb{R})^+$:

$$\iota\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$$

then the representation $\pi_k \circ \iota$ is in the infinitesimal equivalence class $D^+(k)$

Proof: The second one follows from the first one, as $\iota$ interchanges the action of $K$ thus the $K$-types.

For the first one, firstly it is a unitary representation: for $z' = g(z) = x' + iy'$, we have

$$y' = \frac{ad - bc}{|cz + d|^2 y}.$$
and $\mu_z = \mu_{z'}$, thus

$$||\pi(g^{-1})f||^2 = \int_H |f(z')|^2 \frac{(ad - bc)^k}{|cz + d|^{2k}} y^k \mu_z = \int_H |f(z')|^2 (y')^k \mu_{z'} = ||f||^2.$$ 

For the infinitesimal equivalence class, we consider the orthogonal basis $\varphi_n = (\frac{z-i}{z+i})^n (\frac{2i}{z+i})^k$ and prove

$$\pi(k^{-1})\varphi_n = e^{2\pi i (k+2n)\theta} \varphi_n.$$ 

Then this will determine the $K$-type of $\pi_k$. This can be proven by direct calculation, Cf. [Ngo, P39].

**Cor. (III.2.4.21).** By (III.2.4.19) and the measure $\mu_k$ we choose, it is clear that there is an isometry between $L^2(H, \mu_k)$ and a subspace of $L^2(G\lhd Z)$ that is compatible with the left $G$ action on $L^2(G)$, but the left and right action on $L^2(G/Z)$ is isomorphic, as $f(t) \mapsto f(t^{-1})$ intertwine them, because $G/Z$ is unimodular. So this representation is **square integrable**, i.e. it can be embedded in $L^2(G/Z)$.

**Lemma (III.2.4.22) (Limits of Discrete Series).** There exists a unitary representation in the infinitesimal equivalence class $D^\pm (1)$.

**Proof:** These two classes already appear in the unitary representation $H(0, 0, 1)$, by (III.2.4.7). □

**Prop. (III.2.4.23) (List of Irreducible Unitary Representations of $GL(2, \mathbb{R})^+$).**

- The 1-dimensional representation $g \mapsto |\text{deg}(g)|^\mu$, where $\mu$ is purely imaginary.
- The unitary principal series $P_\mu(\lambda, \varepsilon)$, where $\mu$ is purely imaginary, $\varepsilon = 0, 1$ and $\lambda \geq 1/4$.
- The complementary series representations $P_\mu(\lambda, 0)$ where $\mu$ is purely imaginary and $0 < \lambda < 1/4$.
- The holomorphic discrete series ($k \geq 2$) and limits of discrete series ($k = 1$) $D^\pm_\mu(k)$, where $\mu$ is purely imaginary.

Notice each of these infinitesimal equivalence classes of irreducible representations has a unique representative that is a unitary representation by (III.1.5.17).

**Proof:** By (III.1.5.7) and (III.1.5.17), the conclusion follows from the classification of $(g, K)$-modules (III.2.4.8) and determining which infinitesimal class has a unitary representative, which follows from (III.2.4.11)(III.2.4.14), (III.2.4.13)(III.2.4.15), (III.2.4.18)(III.2.4.20), (III.2.4.22). □

**Cor. (III.2.4.24).** The only spherical unitary representation of $GL(2, \mathbb{R})^+$ (i.e. $V^K \neq \emptyset$) is the principal series.

**Whittaker Models**

**Def. (III.2.4.25) (Whittaker Function Space).** Let $\psi$ be an additive character on $\mathbb{R}$, denote $W$ the space of smooth functions on $GL(2, \mathbb{R})^+$ that satisfies

$$W \left[ \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right] g = \psi(x) W(g).$$

A function $f \in W$ is called:
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- of **moderate growth** iff in coordinate (III.2.1.3), \( f \) is bounded by a polynomial in \( y \) as \( y \to \infty \).
- **rapidly decreasing** iff in the same coordinate \( y^N f \to 0 \) as \( y \to \infty \) for every \( N > 0 \).

**Lemma (III.2.4.26).** Let \( \mu, \lambda \in \mathbb{C} \) and \( k \in \mathbb{Z} \). Let \( W(\lambda, \mu, k) \) be the space of functions \( f \in W(\text{III.2.4.25}) \) on \( G \) s.t. \( \Delta f = \lambda f \), \( I f = \mu f \), \( f \in (C^\infty(\mathbb{G}))^k \) and \( f \) is of moderate growth, then \( W(\lambda, \mu, k) \) is 1-dimensional, and functions in this space are actually rapidly decreasing and analytic.

Moreover, the operators \( R, L \) map \( W(\lambda, \mu, k) \) into \( W(\lambda, \mu, k + 2), W(\lambda, \mu, k - 2) \) respectively.

**Proof:** For \( f \in W(\lambda, \mu, k) \), in coordinate (III.2.1.3), we have

\[
f(y) = u^k \psi(x) e^{ik\theta} w(y), \quad w(y) = f\left(\begin{bmatrix} y^{1/2} \\ y^{-1/2} \end{bmatrix}\right)
\]

Thus it suffices to study the behavior of \( w(y) \). By the expression of \( \Delta \) (III.2.1.5), if \( \psi(x) = e^{iax} \), then

\[
w'' + \left(-a^2 \right) + \frac{k}{2y} + \frac{\lambda}{y^2} w = 0
\]

And this is the Whittaker’s equation, and the only moderate growth function is rapidly decreasing and analytic. This is by direct calculation in [A course of Modern Analysis, Whittaker/Watson(1927)]?.

For the action of \( R, L \), they preserve \( W \) because they are right actions. And \( Rf, Lf \) have the same eigenvalue of \( \Delta, I \) because \( \Delta, I \) are in the center of \( U(\mathfrak{g}) \). They shift the weight by (III.2.3.6) and (III.2.3.2).

**Prop. (III.2.4.27) (Uniqueness of Whittaker Models for GL(2, R)).** Let \( (\pi, V) \) be an irreducible admissible \((\mathfrak{g}, K)\)-module for \( G = GL(2, \mathbb{R}) \) or \( GL(2, \mathbb{C}) \), then there exists at most one space \( W(\pi, \psi) \in W \) consisting of \( K \)-finite functions \( f \in W \) that is of moderate growth, and is invariant under the action of \( U(\mathfrak{g}) \) and \( K \), that is infinitesimal equivalent to \( (\pi, V) \).

Functions in \( W(\pi, \psi) \) are actually rapidly decreasing and analytic. The space \( W(\pi, \psi) \) is called the **Whittaker model** of \( \pi \), if it exists.

**Proof:** By (III.1.5.16), \( \Delta, I \) acts by scalars \( \lambda, \mu \) on \( V \). By (III.2.4.1) or (III.2.4.10) if \( V^k \neq 0 \), then \( \dim V^k = 1 \). If \( V^k = 0 \), then the image of \( V^k \) under the isomorphism with \( W(\pi, \psi) \) is in the space \( W(\lambda, \mu, k) \). Thus \( W(\pi, \psi) \) is the direct sum of the \( W(\lambda, \mu, k) \) for all \( k \) that \( V^k \neq 0 \), so uniquely determined by (III.2.4.26). And the rapid decreasing and analytic properties are also consequences of (III.2.4.26).

**Prop. (III.2.4.28) (Whittaker Models for GL(n, \mathbb{C})).** This result is true if \( \mathbb{R} \) is replaced by \( \mathbb{C} \).

**Proof:** Cf. [Automorphic Forms on GL(2), Jacquet/Langlands (1970) Thm5.3. P232].

**Prop. (III.2.4.29) (Shalika’s Local Multiplicity One Theorem).** Let \( F = \mathbb{R} \) or \( \mathbb{C} \). Let \( \psi \) be a character of \( F \). We define a character \( \psi_N \) on the space of upper triangular unipotent matrices in \( GL(n, F) \) by

\[
\varphi_N(u) = \psi(\sum_{i=1}^{n-1} u_{i,i+1}).
\]

Given a unitary irreducible representation \( V \) of \( GL(n, F) \), a **Whittaker functional** on \( V^\infty \) is a continuous linear functional \( \lambda \) on \( V^\infty \) that \( \lambda(\pi(u)x) = \psi_N(u)\lambda(x) \) for all \( u \in N(F), x \in V^\infty \).

Then the dimension of the space of Whittaker functionals on \( V^\infty \) is at most 1-dimensional.
Proof:

\[ (R_k f, g) = (f, -L_{k+2} g). \]

\[ \Delta_k = -L_{k+2} R_k - \frac{k}{2} (1 + \frac{k}{2}) \] is symmetric on \( L^2(\Gamma \backslash H, \chi, k) \) (unbounded and defined only on \( C^\infty(\Gamma \backslash H, \chi, k) \) now).

Proof: Let \( \omega = y^{-1} f(z) \overline{g(z)} dz \). It can be shown by definition and using (X.2.2.5) that \( \omega(\gamma z) = \omega(z) \), so \( \omega \) descends to a differential form on \( \Gamma \backslash H \), so

\[
0 = \int_{\Gamma \backslash H} d(y^{-1} f(z) \overline{g(z)} dz) = \int_{\Gamma \backslash H} \left[ \frac{\partial}{\partial y} (y^{-1} f \overline{g}) + i \frac{\partial}{\partial x} (y^{-1} f \overline{g}) \right] dx \wedge dy
= \int_{\Gamma \backslash H} \left[ (iy \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}) \overline{g} - (iy \frac{\partial g}{\partial x} - y \frac{\partial g}{\partial y}) f - f \overline{g} \right] \frac{dxdy}{y^2}
= \int_{\Gamma \backslash H} |(R_k f) \overline{g} + f(L_{k+2} g)| \frac{dxdy}{y^2}.
\]

so the conclusion follows.

\[ \square \]

5 The Spectral Problem

The Spectral Problems for \( \Gamma \backslash H \) Compact

Def. (III.2.5.1). In this subsection we assume \( \Gamma \backslash G_1 \) is compact, or equivalently \( \Gamma \backslash H \) is compact, because \( K \) is compact. This condition makes \( C^\infty(\Gamma \backslash G, \chi) = L^2(\Gamma \backslash G, \chi) \), and will make the decomposition having only discrete parts (III.2.5.5).

Prop. (III.2.5.2). For \( f \in C^\infty(\Gamma \backslash H, \chi, k), g \in C^\infty(\Gamma \backslash H, \chi, k + 2) \),

\[
(R_k f, g) = (f, -L_{k+2} g).
\]

In particular, \( \Delta_k = -L_{k+2} R_k - \frac{k}{2} (1 + \frac{k}{2}) \) is symmetric on \( L^2(\Gamma \backslash H, \chi, k) \). (unbounded and defined only on \( C^\infty(\Gamma \backslash H, \chi, k) \) now).

Proof: Let \( \omega = y^{-1} f(z) \overline{g(z)} dz \). It can be shown by definition and using (X.2.2.5) that \( \omega(\gamma z) = \omega(z) \), so \( \omega \) descends to a differential form on \( \Gamma \backslash H \), so

\[
0 = \int_{\Gamma \backslash H} d(y^{-1} f(z) \overline{g(z)} dz) = \int_{\Gamma \backslash H} \left[ \frac{\partial}{\partial y} (y^{-1} f \overline{g}) + i \frac{\partial}{\partial x} (y^{-1} f \overline{g}) \right] dx \wedge dy
= \int_{\Gamma \backslash H} \left[ (iy \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}) \overline{g} - (iy \frac{\partial g}{\partial x} - y \frac{\partial g}{\partial y}) f - f \overline{g} \right] \frac{dxdy}{y^2}
= \int_{\Gamma \backslash H} |(R_k f) \overline{g} + f(L_{k+2} g)| \frac{dxdy}{y^2}.
\]

so the conclusion follows.

\[ \square \]

Prop. (III.2.5.3). Consider the right action of \( G \) on \( L^2(\Gamma \backslash G, \chi)(III.2.12) \), let \( \varphi \in C^\infty_c(G) \), then \( \varphi \) can act on \( L^2(\Gamma \backslash G, \chi) \) by (X.4.3.24), and:

- \( \rho(\varphi) \) is an integration operator, in particular Hilbert-Schmidt and compact. And \( \text{Im}(\rho(\varphi)) \subset C^\infty(\Gamma \backslash G, \chi) \).
- If \( \varphi(g^{-1}) = \overline{\varphi(g)} \), then \( \rho(\varphi) \) is self-adjoint.
- If \( \varphi(kg) = e^{-tk\theta} \varphi(g) \), then \( \text{Im}(\rho(\varphi)) \subset C^\infty(\Gamma \backslash G, \chi, k) \).

Compare with (III.1.6.4).

Proof: 1:

\[
(\rho(\varphi) f)(g) = \int_G f(h) \varphi(g^{-1} h) dh = \int_{\Gamma \backslash G_1} \int_{\Gamma \backslash G} f(\gamma h) \overline{\varphi(g^{-1} \gamma h)} \frac{du}{u} dh = \int_{\Gamma \backslash G_1} f(h) K(g, h) dh
\]

where

\[
K(g, h) = \int_{\Gamma \backslash G} \chi(\gamma) \overline{\varphi(g^{-1} \gamma h)} \frac{du}{u}.
\]

Because \( \varphi \) is compactly supported, this is a smooth function in \( g \) and \( h \), in particular square integrable on \( \Gamma \backslash G \) compact. And \( \rho(\varphi)(f)(g) \) is smooth in \( g \) because \( f \in L^1(\Gamma \backslash G_1, \chi) \) as \( \Gamma \backslash G_1 \) is compact, and \( K(g, h) \) is smooth in \( g \).

2, 3 is easy.

\[ \square \]
Cor. (III.2.5.4). Let $H$ be a nonzero closed $G$-subrepresentation of $L^2(\Gamma \backslash G, \chi)$, then $H$ decomposes as $\oplus_k H_k$ w.r.t. action of $SO(2)$. And if $H_k \neq 0$, then $\Delta$ has a nonzero eigenvector in $H_k \cap C^\infty(\Gamma \backslash G, \chi)$.

Proof: The decomposition is clear from (X.6.4.4). It’s left to show $\Delta$ has an eigenvalue in $H_k \cap C^\infty(\Gamma \backslash G, \chi)$. By lemma (III.1.6.1) above, for $f_0 \in H_k$, there is a $\varphi \in C^\infty_c(G)$ s.t. $\rho(\varphi)f_0 \neq 0$, and $\varphi(\xi g) = e^{-ik\theta} \varphi(g)$. So (III.2.5.3) shows $\rho(\varphi)$ maps $H$ into $H_k \cap C^\infty(\Gamma \backslash G, \chi)$ and induces a compact self-adjoint operator on $H_k$. So we can choose a f.d. eigenspace of it. Notice $\Delta$ commutes with the action $\rho(\varphi)$, so $\Delta$ fixes this eigenspace, thus it has an eigenvalue on $H_k \cap C^\infty(\Gamma \backslash G, \chi)$. □

Prop. (III.2.5.5) ($L^2(\Gamma \backslash G, \chi)$ Totally Decomposable). The space $L^2(\Gamma \backslash G, \chi)$ decomposes into a Hilbert space direct sum of subspaces that are invariant and irreducible under the right regular action $\rho$.

Proof: Compare with the proof of (III.1.6.2).

Let $\Sigma$ be the set of sums of irreducible invariant subspaces of $L^2(\Gamma \backslash G, \chi)$ that is mutually orthogonal. Then choose by Zorn’s lemma a maximal one in $\Sigma$, and we prove the orthogonal complement $H = 0$ otherwise we construct an irreducible subspace of $H$.

Let $f \neq 0 \in H$, choose by (III.1.6.1) and (III.2.5.3) a $\varphi \in C^\infty_c(G)$ that $\rho(\varphi)$ is compact self-adjoint and $\rho(\varphi)f \neq 0$. So $\rho(\varphi)$ has a non-zero eigenvalue and the eigenspace $L$ is of f.d.

Let $L_0$ be a minimal nonzero subspace of $L$ that is an intersection of $L$ with a nonzero closed invariant subspace of $\mathcal{H}$, and let $V$ be the intersection of all closed invariant subspaces $W$ of $H$ that $L_0 = L \cap W$. We show $V$ is irreducible, if not, then $V = V_1 \cap V_2$, and if $0 \neq f_0 \in L_0$, then $f_0 = f_1 + f_2$ and both $f_1, f_2$ are eigenfunctions of $\rho(\varphi)$ of eigenvalue $\lambda$. Now if $f_1 \neq 0$, then by minimality, $V_1 \cap L = L_0$. □

Lemma (III.2.5.6). Let $\sigma$ be the character on $K$ that $\sigma(k_0) = e^{-ik\theta}, C^\infty_c(K \backslash G/K, \sigma)$ is commutative by (III.2.2.3), let $\xi$ be a character of it. Let $H(\xi)$ be the subspace of $f \in L^2(\Gamma \backslash G, \chi, k)$ that $\pi(\varphi)f = \xi(\varphi)f$ for $\varphi \in C^\infty_c(K \backslash G/K, \sigma)$.

Then $H(\xi)$ are of f.d. and different $H(\xi), H(\eta)$ are orthogonal that $\oplus_\chi H(\chi) = L^2(\Gamma \backslash G, \chi, k)$.

Proof: Suppose $0 \neq f \in H(\xi)$, then by (III.1.6.1), we can find a $\varphi \in C^\infty_c(K \backslash G/K, \sigma)$ s.t. $\rho(\varphi)f \neq 0$. And by hypothesis $\rho(\varphi)f = \xi(\varphi)f$, thus $\xi(\varphi) \neq 0$, and $f$ is an eigenvalue of $\rho(\varphi)$ which is compact and self-adjoint, so the $\xi(\varphi)$-eigenspace of $\rho(\varphi)$ is f.d. and $H(\varphi)$ is contained in this space, thus f.d.

To show the orthogonality, choose $\varphi \in C^\infty_c(K \backslash G/K, \sigma)$ that $\xi(\varphi) \neq \eta(\varphi)$. Considering $\varphi = \varphi_1 + i\varphi_2$, where $\rho(\varphi_1), \rho(\varphi_2)$ are both self-adjoint, then we mays assume $\varphi$ is self-adjoint. Then $H(\xi), H(\eta)$ are contained in different eigenspaces of $\rho(\varphi)$, so they are orthogonal.

Finally for the direct sum, it suffices to show that if $f$ is orthogonal to all $H(\xi)$, then $f = 0$. Given $f$, let $\varphi_0 \in C^\infty_c(K \backslash G/K, \sigma)$ be chosen that $\rho(\varphi_0)f$ is near $f$ that $\rho(\varphi_0)f, f$ are not orthogonal (III.1.6.1).

Consider the eigenspace decomposition of $\rho(\varphi_0)$ on $L^2(\Gamma \backslash G, \chi, k)$, then $f = f_0 + f_1 + f_2 + \ldots$, then $\rho(\varphi_0)f = \lambda_1 f + \lambda_2 f + \ldots$. Because $f$ is not orthogonal to $\rho(\varphi_0)f$, thus $f_i$ is not orthogonal to $f$ for some $i \geq 1$. Let $V$ be the $\lambda_i$-eigenvector of $\rho(\varphi_0)$, then $V$ is f.d. and invariant under $\rho(\varphi)$ for all $\varphi \in C^\infty_c(K \backslash G/K, \sigma)$ because $C^\infty_c(K \backslash G/K, \sigma)$ is commutative (III.2.2.3). So $V$ is a direct sum of elements of the spaces $H(\xi)$, so $V$ is orthogonal to $f$, contradiction. □

Prop. (III.2.5.7). The space $L^2(\Gamma \backslash \mathcal{H}, \chi, k)$ decomposes into a Hilbert space direct sum of eigenspaces for $\Delta_k$. 

Proof: By (III.2.3.6), it suffices to prove for \( L^2(\Gamma \backslash G, \chi, k) \) and \( \Delta \). Because \( \Delta \) are in the center of \( U(\mathfrak{g}) \), \( H(\xi) \) are all \( \Delta \)-invariant. So we finish by the lemma (III.2.5.6), as each \( H(\xi) \) is f.d. so \( \Delta \) is a self-adjoint operator on \( H(\xi) \), because \( C_c^\infty(\Gamma \backslash G, \chi) \) is dense (III.2.1.21). So it is a direct sum of \( \Delta \)-eigenspaces.

\[ \Box \]

Cor. (III.2.5.8) (Automorphic Representation is Admissible). For \( n = 2 \), Let \( (\pi, \mathcal{H}) \) be an irreducible subrepresentation of \( L^2(\Gamma \backslash G, \chi) \) as in (III.2.5.5). If \( \mathcal{H}_k \) is the \( k \)-type part of this representation w.r.t. \( K \), then \( \dim \mathcal{H}_k \leq 1 \). In particular, \( \pi \) is admissible.

Proof: This is an immediate consequence of (III.1.6.10). \[ \Box \]

Prop. (III.2.5.9).
- The eigenvalues \( \lambda_i \) of \( \Delta_k \) on \( L^2(\Gamma \backslash \mathcal{H}, \chi, k) \) tends to \( \infty \), and satisfies \( \sum \lambda_i^{-2} < \infty \).
- The laplacian \( \Delta_k \) has an extension to a self-adjoint operator on the Hilbert space \( L^2(\Gamma \backslash G, \chi, k) \).

Proof: Cf. [Bump P185]. \[ \Box \]

Prop. (III.2.5.10) (Main Theorem). Let \( \chi(-1) = (-1)^\varepsilon \), \( \varepsilon = 0, 1 \). Now (III.2.5.5) shows the representation \( \mathcal{H} = L^2(\Gamma \backslash G, \chi) \) decomposes into Hilbert space direct sums of irreducible representations, and \( \Delta \) acts as real scalars on each irreducible subspace (III.2.4.8), and \( \mu \) acts by \( 0 \). So comparing the classification of representations of \( G \) (III.2.4.23), we can list what representation can appear in it by looking at eigenvalues \( \lambda \) of \( \Delta \):

- There is only one f.d. irreducible unitary subrepresentation of \( G \) occurring in \( \mathcal{H} \), the trivial representation.
- If \( \lambda \neq \frac{k}{2} (1 - \frac{k}{2}) \), where \( k \equiv \varepsilon \mod 2 \), then this subrepresentation is isomorphic to \( P(\lambda, \varepsilon) \). And let \( k' \equiv \varepsilon \mod 2 \) be any integer, then the multiplicity of \( P(\lambda, \varepsilon) \) is equal to the multiplicity of the eigenvalue \( \lambda \) of \( \Delta_{k'} \) on \( L^2(\Gamma \backslash \mathcal{H}, \chi, k') \) because \( L^2(\Gamma \backslash \mathcal{H}, \chi, k') \equiv L^2(\Gamma \backslash G, \chi, k') \) by (III.2.3.6).
- If \( \lambda = \frac{k}{2} (1 - \frac{k}{2}) \), where \( k \equiv \varepsilon \mod 2 \), then this subrepresentation is isomorphic to either \( D^+(k) \) or \( D^-(k) \), and \( D^+(k) \) have the same multiplicity in \( \mathcal{H} \), equal to the dimension \( \dim(M_k(\Gamma, \chi))(IV.5.1.9) \) of holomorphic modular forms of weight \( k \).

Proof: Only the relation with modular forms need proving. By (III.2.4.3), the multiplicity of \( D^+(k) \) equals the dimension of the \( \frac{k}{2} (1 - \frac{k}{2}) \)-eigenspace of \( \Delta_k \) on \( L^2(\Gamma \backslash \mathcal{H}, \chi, k) \), and any \( f \) in this eigenspace is annihilated by \( L_k \) by (III.2.3.1)(III.2.3.3). But (III.2.3.5) shows the dimension of space of functions annihilated by \( L_k \) equals the dimension of modular forms of weight \( k \). Notice now complex conjugation interchanges \( L^2(\Gamma \backslash \mathcal{H}, \chi, k) \) and \( L^2(\Gamma \backslash \mathcal{H}, \chi, -k) \) and \( \Delta_k = \Delta_{-k}(III.2.3.1) \), so the multiplicity of \( D^+(k) \) and \( D^-(k) \) equal. \[ \Box \]

The Spectral Problems for Non-Compact Case

Prop. (III.2.5.11) (Gelfand,Graev and Piatetski-Shapiro). Let \( \varphi \in C_c^\infty(G) \), then
- There exists a constant \( C(\varphi) \) that for all \( f \in L_1^2(\Gamma \backslash G, \chi, k), \) we have \( ||\rho(\varphi)f||_{C(G)} \leq C(\varphi)||f||_2 \).
- \( \rho(\varphi) \) is a compact operator on \( L_1^2(\Gamma \backslash G, \chi, \omega) \). Notice this generalizes (III.2.5.3).
III.2. AUTOMORPHIC REPRESENTATIONS OF $GL(2, \mathbb{R})(\text{BUMP})$

Proof: We may assume $\Gamma$ has cusps, otherwise this is proved in (III.2.5.3). Conjugating $\Gamma$ by an element of $SL(2, \mathbb{R})$, we may assume that $\infty$ is a cusp for $\Gamma$, and $\Gamma_\infty$ is generated by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then it suffices to prove that

$$\sup_{g \in G_{c,d}} |\rho(\varphi)f(g)| \leq C_0||f||_2,$$

because we can do the same for other cusps of $\Gamma$, and use (III.2.1.16) to show that $\sup_{g \in F} |\rho(\varphi)f(g)| \leq C_0||f||_2$, hence for all $g \in \mathcal{H}$.

Now let $\varphi_\omega(g) = \int_{\mathcal{R}^*} \varphi(\rho g\omega(z)) dz$, then

$$\rho(\varphi)f(g) = \int_{Z(\mathbb{R}) \backslash G} f(gh) \varphi_\omega(h) dh$$
$$= \int_{Z(\mathbb{R}) \backslash G} f(h) \varphi_\omega(g^{-1}h) dh$$
$$= \int_{\Gamma_\infty Z(\mathbb{R}) \backslash G} \sum_{\gamma \in \Gamma_\infty} f(\gamma h) \varphi_\omega(g^{-1}\gamma h) dh$$
$$= \int_{\Gamma_\infty Z(\mathbb{R}) \backslash G} K(g,h) f(h) dh,$$

where $K(g,h) = \sum_{\gamma \in \Gamma_\infty} \chi(\gamma) \varphi_\omega(g^{-1}\gamma h)$. Then we can estimate the kernel $K(g,h)$ and show it decays rapidly for $h$ when $g$ is fixed, and this will give the desired result, Cf.[Bump, P286].

2: Let $X(\Gamma)$ be the space obtained by compactifying $\Gamma\backslash G_1$ by adjoining cusps, and let $\Sigma$ be the image of the unit ball in $L^2_0(\Gamma\backslash G, \chi, \omega)$ under $\rho(\varphi)$, then we can extend each $|\rho(\varphi)f|$ to $X(\Gamma)$ that it vanish at the cusps, by the corollary to item1 below (III.2.5.12), then $\Sigma$ is bounded in the $L^\infty$-norm by item1, and it is also equicontinuous, because its derivatives can be bounded uniformly for $f$:

$$(X\rho(\varphi)f)(g) = \rho(\varphi_X)f(g), \quad \varphi_X(g) = \frac{d}{dt} \varphi(\exp(-tX)g)$$

so the conclusion of item1 applied to $\varphi_X$'s shows the uniformly boundedness. Now Arzela-Ascoli shows that $\Sigma$ is precompact in $C(X(\Gamma))$, thus also in $L^2(X(\Gamma))$.

□

Cor. (III.2.5.12). If $\varphi \in C_c^\infty(\Gamma\backslash G), f \in L^2_0(\Gamma\backslash G, \chi, \omega)$, then $\rho(\varphi)f$ is smooth and rapidly decreasing at cusps.

In particular, any automorphic cuspidal form decay rapidly at cusps.

Proof: This is implicit in the proof of (III.2.5.11) above.

□

Prop. (III.2.5.13) ($L^2_0(\Gamma\backslash G, \chi, \omega)$ Totally Decomposable). The space $L^2_0(\Gamma\backslash G, \chi, \omega)$ decomposes into a Hilbert space direct sum of irreducible representations of $G$, and if $H$ is such a subrepresentation, then the $K$-finite vectors in $H$ is dense in $H$, and form an irreducible admissible $(\mathfrak{g}, K)$-module contained in $\mathcal{A}_0(\Gamma\backslash G, \chi, \omega)$.

Proof: The proof is exactly the same as that of (III.2.5.5), but where we use (III.2.5.11) in place of (III.2.5.3). Lemma (III.1.6.1) is indispensable.

The rest is general, by (III.1.6.10). We only need to show that $H^k \subset \mathcal{A}_0(\Gamma\backslash G, \chi, \omega)$ for any $k$. For this, if $0 \neq f \in H_k$, choose by (III.1.6.1) a function $\varphi \in C_c^\infty(G)$ that $\rho(\varphi)f \neq 0$, thus we can assume $\rho(\varphi)(f) = f$ because $\dim H^k \leq 1$ by (III.1.6.9). Then by (III.2.5.12) $f$ is smooth and decay rapidly, and it is clearly $K$-finite and $\mathcal{Z}$-finite, thus $f \in \mathcal{A}_0(\Gamma\backslash G, \chi, \omega)$.

□
Cor. (III.2.5.14). Let \( A(\Gamma \backslash G, \chi, \omega, \lambda) \) be the \( \lambda \)-eigenspace of \( \Delta \) on \( A(\Gamma, \chi, \omega) \), and \( A(\Gamma \backslash G, \chi, \omega, \lambda, k) \) be the \( k \)-part in \( K \)-decomposition of \( A(\Gamma \backslash G, \chi, \omega, \lambda) \), and \( A_0(\Gamma \backslash G, \chi, \omega, \lambda, k) \) the cuspidal part. Then

- The multiplicity of any irreducible unitary representation of \( G \) appearing in \( L^2_0(\Gamma \backslash G, \chi, \omega) \) is finite.
- \( A_0(\Gamma \backslash G, \chi, \omega, \lambda, k) \) is of f.d.

Proof:

1: Let \((\pi, V)\) be an irreducible unitary representation of \( G \), let \( k \) be chosen that \( V^k \neq 0 \), let \( 0 \neq \xi \in V^k \), then by (III.1.6.1) there is a \( \varphi \in C_c^\infty(G) \) that \( \pi(\varphi)\xi = \xi \) (because \( V^k \) is of f.d.) and \( \pi(\varphi) \) is self-adjoint. Now consider for any continuous linear map \( T : V \to L^2_0(\Gamma \backslash G, \chi, \omega) \), \( T\xi \) lies in the \( 1 \)-eigenspace of the compact self-adjoint operator \( \rho(\varphi) \), which is compact by (III.2.5.11), which is of f.d.. Of course, \( T \) is determined by \( T\xi \) because \( V \) is irreducible, so these \( T \) form a f.d. vector space, so the multiplicity is finite.

2: This follows from the first, because by (III.2.5.13) the \( K \)-finite parts of an irreducible subrepresentation of \( L^2_0(\Gamma \backslash G, \chi, \omega) \) are just the space of cuspidal automorphic forms contained in it, and \( \lambda, k \) determined the action of \( \Delta, I \), so there are at most 2 representation of \( G \) that satisfy these, by classification in (III.2.4.8) and (III.1.5.17), and they appear for finite multiplicity by item 1, and for each of them, the \( k \)-part is of f.d., because they are admissible (III.1.6.10). So the conclusion follows.

6 Trace Formulae

\( \Gamma \backslash G \) Compact case

Lemma (III.2.6.1). Let \( \Gamma \backslash G \) be compact, \( \rho \) be the principal series \( P(\lambda, 0) \) (III.2.4.13) of \( G \), and \( s = \frac{1}{2}(s_1 - s_2 + 1), \lambda = s(1 - s), \mu = (s_1 + s_2) \), then for any \( f \in C_c^\infty(K \backslash G/K) \), \( \rho(f) \in V^K_\rho \), which has dimension 1, so \( f \) is a trace class and

\[
\text{tr}(\rho(f)) = \int \int f \left[ \begin{array}{cc} e^{u/2} & x \\ 0 & e^{-u/2} \end{array} \right] e^{-u^2} dudx.
\]

Proof: The trace of \( \rho(f) \) is just the scalar by which \( \rho(f) \) acts on a non-zero vector of \( \rho^K \). Take \( \varphi \in \rho^K \subset H(s_1, s_2, 0) \) normalized that \( \varphi(I) = 1 \), then

\[
(\rho(f)\varphi)(I) = \int_G f(g)\varphi(g) dg = \int_K \int_A \int_N f(ank)\varphi(ank) dAdN d\theta = \int_A \int_N f(an)\varphi(an) dAdN
\]

\( \Box \)
III.3 Automorphic Representations

Main references are [Bum98]. Remember to do the problems.

Def. (III.3.0.1) (Notations). We will use the notation that \( F \) is a Global field, \( A \) is the Adele group of \( F \) and \( A^\ast \) is the Idele group of \( F \). \( A_f \) is the ring of finite Adeles, \( F_\infty = \prod_{v \in S_\infty} F_v \), and \( A = F_\infty A_f \).

Denote \( w_0 = \begin{bmatrix} 1 & -1 \end{bmatrix}, w_1 = \begin{bmatrix} -1 & 1 \end{bmatrix} \).

1 Tate’s thesis

Main references are [Fourier Analysis in Number Fields and Hecke’s Zeta Functions Tate], [Tate’s Thesis Poonen] and [Fourier Analysis on Number Fields].

Topology And Measure of Local Field

Remark (III.3.1.1) (Notations). The basis setting is that \( K \) is a local field, and we renormalize the valuation as follows:

\[
|k| = \begin{cases} 
|k| & K = \mathbb{R} \\
|k|^2 & K = \mathbb{C} \\
\frac{1}{(NP)^v(k)} & K \text{ non-Archimedean}
\end{cases}
\]

where \( NP \) is the number of elements of the residue field of \( K \).

\( O \) is the ring of integers of \( K \), \( \pi \) is a uniformizer of \( K \).

\( \delta \) is the differential of \( K \) when \( K \) is a number field, or \( \delta = \{ x \mid \text{tr}(\text{res}(xO)) = 0 \}^{-1} \) when \( K \) is a function field.

Prop. (III.3.1.2) (Self Duality of Topological Fields). If \( X \) is a non-trivial character on the additive group \( K^+ \), then for any \( \eta \in K^+ \), \( \xi \mapsto X(\eta \xi) \) is also a character, and \( F_X : \eta \mapsto (\xi \mapsto X(\eta \xi)) \)

is an isomorphism of topological groups of \( K^+ \) and \( \hat{K}^+ \). In fact, this character \( X \) does exist, by the lemma (III.3.1.5) below.

Proof: First this is clearly a homomorphism of groups, and it is injective, because if \( X(\eta \xi) = 1 \) for all \( \xi \), then \( \eta K^+ \neq K^+ \) (because \( X \) is nontrivial), so \( \eta = 0 \).

Now the image of \( F \) is dense, because if \( X(\eta \xi) = 1 \) for all \( \eta \), then \( \xi = 0 \), so \( \text{Im}(F)^{\perp} = 1 \). Now \( (H^+)^{\perp} = H \) for \( H \) closed (X.6.3.22) and use Pontryagin duality (X.6.3.19), so \( \text{Im} f \) is dense in \( \hat{G} \).

Now \( F \) is open and continuous, because: for any \( B \in G \) compact, there is a nbhd \( V \) of 0 that \( |X(V) - 1| < \varepsilon \), so there is a nbhd \( V' \) that \( V' B \subseteq V \), so if \( \eta \in V \), \( |X(\eta B) - 1| < \varepsilon \), so \( F \) is continuous (\( \hat{K}^+ \) has the compact-open topology). And if we choose \( \xi_0 \) that \( X(\xi_0) \neq 1 \), then choose \( B = B(0, \frac{\varepsilon_0}{2}) \) compact, if \( |X(\eta B) - 1| < |X(\xi_0) - 1| \), then \( \xi_0 \notin \eta B \), which means that \( |\eta| < \varepsilon \). This means \( F(B(0, \varepsilon)) \) contains \( V(B, |X(\xi_0) - 1|) \), so \( F \) is open.

So the image of \( F \) is a locally compact subgroup of \( \hat{G} \), so by (X.6.1.6) it is closed, hence equals \( G \) as it is dense, so \( F \) is surjective, and is an isomorphism. \( \square \)
Prop. (III.3.1.3). For a Haar measure $\mu$ on $K^+$, $d\mu(\alpha\xi) = |\alpha|d\mu(\xi)$.

Proof: If $K = \mathbb{R}, \mathbb{C}$, this is routine calculation. If $K$ is non-Archimedean, then by the translation invariance of $\mu$, $\mu(\alpha\mathcal{O}) = \frac{\mu(\mathcal{O})}{N(\alpha)} = |\alpha|\mu(\mathcal{O})$.

Def. (III.3.1.4) (Self-Adjoint Haar Measure on $K^+$). Now by Fourier inversion (X.6.3.15), a Haar measure $d\mu$ on $K^+$ corresponds to a Haar measure $d\alpha$ on $\hat{K}^+$, but now by the isomorphism $F : K^+ \to \hat{K}^+$, $d\alpha$ corresponds to a measure $\hat{d}\alpha$ on $K^+$. Now anyway, there is a unique $d\mu$ that $d\mu = \hat{d}\alpha$, and this is called the self-dual Haar measure on $K^+$.

In other words, by (X.6.3.15), this is equivalent $f(\eta) = \int_K \hat{f}(\eta)X(\xi\eta)d\mu(\eta)$.

From now on, we fix a character of $K^+$, and let $dx$ be the self-dual measure Haar measure on $K^+$ w.r.t. $\psi$.

Lemma (III.3.1.5) (Canonical Character of Local Fields). Consider the base field $k$ of $K$(topologically), which is $\mathbb{R}, \mathbb{Q}_p$ or $\mathbb{F}_p((t))$ by Ostrowski(I.9.3.14). Now let

$$\lambda(x) = \begin{cases} -x \mod 1 & k = \mathbb{R} \\ 
\text{a rational number } \lambda(x) \text{ that } \lambda(x) - x \in \mathbb{Z}_p \text{ in } \mathbb{Q}/\mathbb{Z} & k = \mathbb{Q}_p \\
\frac{a-1}{p} = \text{res}(x)/p & k = \mathbb{F}_p((t))
\end{cases}$$

Then $\lambda$ is a continuous additive function on $k$. Now let

$$\Lambda(x) = \begin{cases} \lambda(\text{tr}_{K/k}(x)) & \text{number field case} \\
\lambda(\text{tr}_{K/k}(x\omega_\nu)) & \text{function field case, where } \omega \text{ is a chosen global meromorphic form on } X.
\end{cases}$$

And $X(x) = e^{2\pi i \Lambda(x)}$. Notice that this is just a rigorous definition of the character $e^{2\pi i \text{tr}_{K/k}(x)}$.

Cor. (III.3.1.6). $F(\eta) = e^{2\pi i \Lambda(\eta\xi)}$ is trivial on $\mathcal{O}_K$ is equivalent to $\xi \in \delta^{-1}$, the different of $K/k$.

In other words, adopting the isomorphism of(III.3.1.2), $\mathcal{O}^\perp = \delta, \delta^\perp = \mathcal{O}$.

Proof: Because $\Lambda(\eta\mathcal{O}) = 0$ iff $\text{tr}_{K/k}(\eta\mathcal{O}) \subset \mathcal{O}_k$, which is equivalent to $\eta \in \delta^{-1}$.

Prop. (III.3.1.7) (Canonical Self-Adjoint Haar Measures). We can calculate the self-adjoint Haar measure w.r.t. the canonical character on $K^+$ (III.3.1.5) as follows:

$$d\mu = \begin{cases} dm & K = \mathbb{R} \\
2dm & K = \mathbb{C} \\
\text{the measure that } \mu(\mathcal{O}) = \frac{1}{N(\delta)^{1/2}} & \text{others}
\end{cases}$$

Proof: We only calculate for the $p$-adic fields?.

Let $f(\eta) = I_{\mathcal{O}}(\xi)$, then

$$\hat{f}(F(\eta)) = \int_{\mathcal{O}} e^{-2\pi i \Lambda(\eta)}d\mu(\xi) = \begin{cases} \mu(\mathcal{O}) & \eta \in \delta^{-1} \\
0 & \text{otherwise}
\end{cases} = \mu(\mathcal{O})I_{\delta^{-1}}(\eta)$$

By (III.3.1.6) and (X.6.4.1). So

$$I_{\mathcal{O}}(\xi) = \int_G \hat{f}(F(\eta))(\xi, F(\eta))d\mu(\eta) = \int_{\delta^{-1}} \mu(\mathcal{O})e^{2\pi i \Lambda(\eta)}d\mu(\eta) = \mu(\mathcal{O})\mu(\delta^{-1})I_{\mathcal{O}}(\eta).$$

So $\mu(\mathcal{O})\mu(\delta^{-1}) = N(\delta)\mu(\mathcal{O})^2 = 1$, which shows the desired result.
Def. (III.3.1.8). The multiplicative group $K^*$ is also locally compact group. For a quasi-character $c$ of $K^*$, it is called \textbf{unramified} iff $c(\alpha) = 1$ whenever $|\alpha| = 1$.

An unramified quasi-character on $K^*$ is all of the form $|\cdot|^s$ for $s \in \mathbb{C}$.

\textbf{Proof:} An unramified quasi-character is equivalent to a continuous group homomorphism from $\text{val}(K^*) \to \mathbb{Z}$. But $\text{val}(K^*)$ must be isomorphic to $\mathbb{Z}$ or $\mathbb{R}$, so the assertion follows from (X.6.3.3). \hfill $\square$

Cor. (III.3.1.9) (Quasi-Character of $K^*$). Let $U$ be the elements of $K^*$ of norm 1, then there is a continuous morphism from $K^*$ to $U$: $\bar{\alpha} = \alpha/\pi^{v(\alpha)}$ when $\alpha$ is non-Archimedean or $\bar{\alpha} = \alpha/|\alpha|$ if $K$ is Archimedean. So any quasi-character $c$ is of the form $c(\alpha) = c(\bar{\alpha})$ times an unramified quasi-character, which is of the form $|\cdot|^s$, where $\Re(s)$ is called the \textbf{exponent} of $c$. Now $U$ is a compact group, so continuous quasi-characters $\tilde{c}$ on it must be a character.

Now for a character $\tilde{c}$ of $U$, if $K$ is non-Archimedean, then by continuity, there is a minimum $v$ that $\tilde{c}(1 + p^v) = 1$, and $p^v$ is called the \textbf{conductor} of $\tilde{c}$.

Def. (III.3.1.10) (Haar Measure on $K^*$). Notice that if $g(\alpha) \in C_c(K^*)$, then $\frac{g(\alpha)}{|\alpha|} \in C_c(K^+ \setminus 0)$, so if we define $\Phi(g) = \int_{K^+ \setminus 0} g(\xi) |\xi|^{-1} d\xi$, then

$$\Phi(\alpha g) = \int_{K^+ \setminus 0} g(\xi) |a\xi|^{-1} a d\xi = \int_{K^+ \setminus 0} g(\alpha \xi) |\alpha \xi|^{-1} d\xi = \Phi(\alpha) \Phi(g).$$

By (III.3.1.3). So By Riesz representation, there is a Haar measure $d_1^\tau \alpha$ on $K^*$ that $\int_{K^*} g(\alpha) d_1^\tau \alpha = \int_{K^+ \setminus 0} g(\xi) |\xi|^{-1} d\xi$, for any $g \in C_c(K^*)$.

But when $K$ is non-Archimedean, renormalize $d_1^\tau \alpha = (1 - \frac{1}{Np})^{-1} d_1^\tau \alpha$.

The reason behind this normalization is when $d\xi$ is the canonical measure (III.3.1.7), we want to make $V(U) = (N\delta)^{-1/2}$:

$$\int_{\mathcal{O} \setminus 0} d\xi = \sum_{k=0}^{\infty} \int_{\pi^k U} d\xi = (1 + \frac{1}{Np} + \frac{1}{Np^2} + \ldots) \int_U d\xi = \frac{1}{1 - \frac{1}{Np}} \int_U d\xi$$

so

$$\int_U d_1^\tau \alpha = \int_U |\xi|^{-1} d\xi = \frac{Np - 1}{Np} \int_{\mathcal{O} \setminus 0} d\xi = \frac{Np - 1}{Np} (N\delta)^{-1/2} (III.3.1.7)$$

\textbf{Local $\zeta$-function & Functional Equations}

Remark (III.3.1.11) (Good Functions). In this section, we work over the set $I$ of \textbf{good functions} that satisfy the following properties:

- $f, \hat{f} \in L^1(K^*)$.
- $f(\alpha)|\alpha|^\sigma, \hat{f}(\alpha)|\alpha|^\sigma \in L^1(K^*)$ for $\sigma > 0$.

Clearly $I$ is stable under Fourier transform.

Def. (III.3.1.12) (Local Zeta Function). Fix a character $\psi$ on $K^+$ and a self-dual Haar measure $dx$ w.r.t. $\psi$. The zeta function for $f$, a Hecke character $\chi$, $Re s > 0$

$$\zeta(s, \chi, f) = \int_{K^*} f(\alpha)\chi(\alpha)|\alpha|^sd\alpha$$

\textbf{Lemma (III.3.1.13) (Zeta Function Holomorphic Part).} For $Re s > 0$, the local zeta function $\zeta(s, \chi, f)$ (III.3.1.12) for a fixed $f$ is a holomorphic function.
Proof: To prove it is holomorphic, use (X.2.3.7), it suffices to show that for any loop $\gamma$ near $c$,
\[ \int_{\gamma} \int_{K^*} f(\alpha) \chi(\alpha)|\alpha|^s d\alpha ds = 0. \]
But the integrand converges uniformly because $\text{Re } s > 0$ and the hypothesis of $f$, so interchanging the integration, it is 0. □

**Lemma (III.3.1.14).** For any $f, g \in I$,
\[ \frac{\zeta(f, c)}{\zeta(\hat{f}, \hat{c})} = \frac{\zeta(g, c)}{\zeta(\hat{g}, \hat{c})}, \]
where $\hat{c}$ is the conjugate of $c$.

Proof:
\[ \zeta(f, c) \zeta(\hat{g}, \hat{c}) = \int_{K^*} f(\alpha)c(\alpha) \int_{K^*} \hat{g}(\beta)c^{-1}(\beta)|\beta|d\beta = \int \int f(\alpha)\hat{g}(\beta)c(\alpha\beta^{-1})|\beta|d\alpha d\beta \]
by Fubini.
\[ = \int \int f(\alpha)\hat{g}(\alpha\beta)|\alpha\beta^{-1}|d\alpha d\beta = \int \int f(\alpha)\hat{g}(\alpha\beta)|\alpha|d\alpha|\beta|c(\beta^{-1})d\beta \]
And notice
\[ \int_{K^*} f(\alpha)\hat{g}(\alpha\beta)|\alpha|d\alpha = C\int_{K^*} f(\xi)\hat{g}(\xi\beta)d\xi = C\int_{K^*} f(\xi)\hat{g}(\xi \beta)d\xi = C\int \int f(\xi)g(\eta)e^{-2\pi i\Lambda(\xi \beta \eta)}d\eta d\xi \]
which is clearly symmetric in $f$ and $g$. So the conclusion follows. □

**Prop. (III.3.1.15) (Local Functional Equation).** The local zeta function $\zeta(s, \chi, f)(\text{III.3.1.12})$ can be extended to a meromorphic function to all $s \in \mathbb{C}$, and there is a function $\gamma(s, \chi, \psi)$, meromorphic for $s$ and independent of $f$, such that
\[ \zeta(1 - s, \chi^{-1}, \hat{f}) = \gamma(s, \chi, \psi)\zeta(s, \chi, f) \]
for any $f \in I$.

Proof: The strategy is that we only need to calculate explicitly for a good function $f$ and $c$ that $0 < \sigma(c) < 1$ that $\rho(c) = \zeta(f, c)/\zeta(\hat{f}, \hat{c})$ extends to a meromorphic function on all equivalent class of $c$, then the lemma (III.3.1.14) shows $\rho(c)$ is independent of $f$, so $\zeta(f, c) = \rho(c)\zeta(\hat{f}, \hat{c})$ holds for any $f \in I$.

For the calculation of $\rho$, Cf.[Tate Thesis P316]. □

**Prop. (III.3.1.16).** For any $s_0 \in \mathbb{C}$, we can choose a function $f \in S(K)$ that $\zeta(s, \chi, f)$ is neither zero nor pole at $s_0$. In fact, when $K$ is non-Archimedean, we can even choose $f$ that $\zeta(s, \chi, f) = 1$.

Proof: Cf.[Bum98]P265. □

**Prop. (III.3.1.17).**
- $\gamma(1 - s, \chi^{-1}, \psi) = \chi(-1)/\gamma(s, \chi, \psi)$. 
• $\rho(c) = c(-1)\overline{\rho(c)}$

Proof: 1:

$$\zeta(f, c) = \rho(c)\zeta(\hat{f}, \hat{c}) = \rho(c)\rho(\overline{\hat{c}})\zeta(\hat{f}, c) = \rho(c)\rho(\overline{c})c(-1)\zeta(f, c)$$

2:

$$\zeta(f, c) = \rho(c)\zeta(\hat{f}, \hat{c}), \quad \zeta(\hat{f}, \hat{c}) = \rho(\overline{\hat{c}})\zeta(\hat{f}, \overline{c})$$

And

$$\overline{\hat{f}(\xi)} = \int \overline{\hat{f}(\eta)}e^{-2\pi i \Lambda(\xi\eta)}d\eta = \int \hat{f}(\eta)e^{2\pi i \Lambda(\xi\eta)}d\eta = \overline{\hat{f}(-\xi)}$$

so

$$\rho(\overline{c})\zeta(\hat{f}, \overline{c}) = \rho(\overline{c})c(-1)\zeta(\hat{f}, \overline{c}) = \rho(\overline{c})c(-1)\hat{\zeta}(\hat{f}, c)$$

Thus $\rho(\overline{c}) = c(-1)\overline{\rho(c)}$. \qed

Cor. (III.3.1.18). Because when $\sigma(c) = \frac{1}{2}$, $\hat{c} = \overline{c}$, we have $|\rho(c)| = 1$ in this case.

Def. (III.3.1.19) (Local L-Factors). The local factor for a Hecke character $\chi$ and $s \in \mathbb{C}$ is define to be

$$L(s, \chi) = \begin{cases} (1 - \chi(\pi)(Np)^{-s})^{-1} & \chi \text{ unramified} \\ 1 & \chi \text{ ramified} \end{cases}$$

in the non-Archimedean case, and

$$L(s) = \begin{cases} \pi^{-(s+\varepsilon)/2}\Gamma((s+\varepsilon)/2) & K = \mathbb{R}, \chi(x) = (x/|x|)^{\varepsilon}, \varepsilon = 0, 1, \\ 2(2\pi)^{s+v+|k|/2}\Gamma(s + v + |k|/2) & K = \mathbb{C}, \chi(x) = |x|^v(x/|x|)^k, v \in i\mathbb{R}, k \in \mathbb{Z}. \end{cases}$$

Prop. (III.3.1.20). $\frac{\zeta(s, \chi, f)}{L(s, \chi)}$ is holomorphic for all $s$, and if $K$ is non-Archimedean, $\zeta(s, \chi, f)$ is a rational function of $(Nq)^{-s}$.

There exists a choice of $f \in \mathcal{S}(K)$ that $\frac{\zeta(s, \chi, f)}{L(s, \chi)}$ is of the form $ab^s$ for $a \in \mathbb{C}^*, b \in \mathbb{R}$.

Proof: Cf.[Bum98]P272. \qed

Prop. (III.3.1.21) (Local $\varepsilon$-Factors). In situation of (III.3.1.15), there exists a non-vanishing holomorphic function $\varepsilon(\chi, \psi)$ that

$$\frac{\zeta(1-s, \chi^{-1}, f)}{L(1-s, \chi^{-1})} = \varepsilon(s, \chi, \psi)\frac{\zeta(s, \chi, f)}{L(s, \chi)},$$

and $\frac{\zeta(s, \chi)}{L(s, \chi)}$ is holomorphic. Moreover, $\varepsilon(s, \chi, \psi)$ is of the form $ab^s$ for $a \in \mathbb{C}^*, b \in \mathbb{R}$. And $\varepsilon(\chi, \psi) = 1$ if $K$ is non-Archimedean, $\chi$ is unramified and the conductor of $\psi$ is $\mathcal{O}_K$.

Proof: This all follows from the direct calculation of $\rho$ in [Tate Thesis P316]. Cf.[Bum98]P274. \qed
Global case: Poisson Formula and Riemann-Roch

**Def. (III.3.1.22).** In this subsubsection, let $K$ be a global field and $A = A_K$. We use notations in (IV.2.4.10).

**Def. (III.3.1.23).** Let $\psi = \prod_v \psi_v$ be a non-trivial Hecke character of $A_K/K$, then (III.3.1.2) shows $\psi$ induces a canonical isomorphism

$$A \cong \hat{A} : \eta \mapsto (\xi \mapsto \psi(\eta \xi)),$$

and we choose the corresponding self-dual Haar measure $dx$, then the Fourier transform on $A$ is then defined to be:

$$\hat{f}(y) = \int_A f(x) \overline{\psi(xy)} dx, \quad f(x) = \int_A \hat{f}(y) \psi(xy) dy.$$

**Prop. (III.3.1.24).** $K^\perp = K$, i.e. $\psi(xy) = 0, \forall y \in K \iff x \in K$.

*Proof:* Because $K^\perp \cong \widehat{A/K}$ and $A/K$ is compact (IV.2.4.14), $K^\perp$ is discrete (X.6.3.7) and contains $K$. So $K^\perp/K$ is discrete hence finite in $A/K$. But $K^\perp$ is clearly a vector space over $K$, thus $K^\perp = K$ must be true, because $|K| = \infty$. □

**Def. (III.3.1.25) (Global Canonical Character).** Let the global canonical character be given by

$$X(x) = e^{2\pi i \Lambda(x)},$$

where $\Lambda(x) = \sum_p \Lambda_p(x_p)$ (III.3.1.5), notice this is definable because $x_p \in \mathcal{O}_p$ a.e. $p$, thus $\Lambda_p(x_p) = 1$ a.e..

Then $X$ is a Hecke character, i.e. $\Lambda(\xi) \subset \mathbb{Z}, \forall \xi \in K$, i.e. $\varphi(K) = 1$.

*Proof:* In the number field case,

$$\Lambda(\xi) = \sum_p \sum_{p|p} \lambda_p(Tr_{p\mid p}(\xi)) = \sum_p \lambda_p(Tr_{K/Q}(\xi))$$

so to show $\lambda$ is an integer, it suffices to show $\Lambda(a)$ is a $q$-adic integer for any $q$ and any $a \in \mathbb{Q}$, but for this, notice

$$\sum_p \lambda_p(x) = \sum_{p \neq q, \infty} \lambda_p(x) + \lambda_q(x) - x$$

is a $q$-adic integer, by definition (III.3.1.5).

In the function field case, this follows from the fact that the sum of residues of a meromorphic 1-form is 0?.

**Def. (III.3.1.26) (Tamagawa Measure).** Let $dx = \prod x_p$ be the restricted product measure on $A(IV.2.4.5)$, where $dx_p$ is the self-dual measure w.r.t $X_p$ in (III.3.1.7), then $dx$ is the self-dual Haar measure w.r.t the global canonical character $X$ by (IV.2.4.8), called the Tamagawa measure on $A$.

Then in this measure, the Fourier transform is given by:

$$\hat{f}(y) = \int_A f(x) e^{-2\pi i \Lambda(xy)} dx, \quad f(x) = \int_A \hat{f}(y) e^{2\pi i \Lambda(xy)} dy.$$

**Prop. (III.3.1.27).** The Volume of $V^*/K^*$, Cf.[Tate Thesis, P337].
Lemma (III.3.1.28) (Poisson Formula). If $F \in L^1(A)$ and $\sum_{\xi \in K} |\hat{f}(\xi)| < \infty$, then in the self-dual Haar measure

$$\sum_{\xi \in K} \hat{f}(\xi) = \sum_{\xi \in K} f(\xi).$$

and $V(A/K) = 1$.

Proof: By (III.3.1.24), this is a special case of (X.6.3.24). In fact, we know it is true for a constant $V(A/K)$, but it is symmetric, so $V(A/K)^2 = 1$. □

Prop. (III.3.1.29) (Riemann-Roch). If $f(ax) \in L^1(A)$ and $\sum_{\xi \in K} |\hat{f}(a\xi)| < \infty$ for any idele $a \in A^\ast$, then for any idele $a$,

$$\frac{1}{|a|} \sum_{\xi \in K} \hat{f}(\xi/a) = \sum_{\xi \in K} f(a\xi).$$

Proof: Consider $g(x) = f(ax)$, then

$$\tilde{g}(x) = \int_A f(a\eta)e^{-2\pi i \Lambda(x\eta)}d\eta = \frac{1}{|a|} \int_A f(\eta)e^{-2\pi i \Lambda(x\eta/a)}d\eta = \frac{1}{|a|} \hat{f}(x/a).$$

Then apply Poisson formula (III.3.1.28) to $g$. □

Def. (III.3.1.30) (Schwartz-Bruhat Function). For a local field $F$, we define the set $S(F)$ of Schwartz-Bruhat function on $F$ as Schwartz functions on $F$ if $F = \mathbb{R}, \mathbb{C}$ (X.4.2.1) and locally continuous constant functions of compact support if $F$ is non-Archimedean.

For a global field $F$, define $S(A)$ to be the restricted tensor product $\prod'_v S(F_v)$, where $f_0^v = \chi_{O_v}$.

Lemma (III.3.1.31). The Fourier transform of a Schwartz function is a Schwartz function.

Proof: ? □

Prop. (III.3.1.32). The Schwartz-Bruhat functions satisfies the condition in (III.3.1.29).

Proof: $\int_A |f| = \prod_v \int_{F_v} |f_v|dx_v < \infty$, noticing (III.3.1.7) and $N(\delta) = 1$ a.e. $v$? . And for any Schwartz function $f$ and any $x$, the set of $k$ that $f_v(x_v+k_v) \neq 0$ is $v$-bounded for $v$ non-Archimedean and $|k|_v \leq 1$ a.e., so in the function case, these $k$ are finite because $K$ is discrete in $A$.

And in the number field case, these $k$ is contained in some fractional ideal $I$, but $I$ is then a lattice in $F_\infty$ by Minkowski theory, so

$$\sum_{\xi \in K} |f(x + \xi)| \leq C \sum_{x \in K} \prod_{p \in S_\infty} |f_p(x + \xi)|$$

but $f_p$ is an Archimedean Schwartz function, thus this is absolutely convergent.

Now we showed that $\int_{x \in K} \hat{f}(\xi) < \infty$, because $\hat{f} \in S(A)$, by (III.3.1.31). □

Prop. (III.3.1.33) (True Riemann-Roch For Function Fields). Cf.[Handwritten Notes].
Analytic Continuation and Functional Equation of $\zeta$-Functions

Prop. (III.3.1.34) (Good Functions). Consider the family $I$ of good functions on $A$ that

- $f(x) \in L^1(A)$, $\hat{f}(x) \in L^1(A)$ and $f, \hat{f}$ is continuous.
- $\sum_{\xi \in K} f(a(x + \xi))$ and $\sum_{\xi \in K} \hat{f}(a(x + \xi))$ converges uniformly absolutely on compact sets of $A$.
- $f(a)|a|^\sigma, \hat{f}(a)|a|^\sigma \in L^1(A^*)$ for $\sigma > 1$.

Then $I$ contains all the Schwartz functions.

Proof: $1, 2$ is the same as in (III.3.1.32), for $3$: $\int_{A^*} |f||a|^\sigma da = \prod_p \int_{K_p^* \cap \mathcal{O}_p} |a_p|^\sigma da_p$, and for a.e. $p$,

$$\int_{K_p^*} |f_p||a|^\sigma da_p = \int_{K_p^* \cap \mathcal{O}_p} |a_p|^\sigma da_p = \frac{1}{1 - (Np)^\sigma} \int_{U_p} da_p = \frac{1}{1 - (Np)^\sigma}.$$

So it suffices to show $\prod_{p \notin S_\infty} \frac{1}{1 - (Np)^\sigma} < +\infty$. For a number field of degree $d$, this is bounded by

$$\prod_p (1 - p^{-\sigma})^{-d} = (\sum_{n \geq 1} n^{-\sigma})^d < \infty.$$

And for a function field, let the number of irreducible polynomial modulo $p$ be $N_n$, then

$$\prod_{p \notin S_\infty} \frac{1}{1 - (Np)^\sigma} = \prod_{n \geq 1} (\frac{1}{1 - q^{-n\sigma}})^{N_n}$$

So it is convergent iff $\sum_{n \geq 1} N_n q^{-n\sigma}$ is convergent, but this is bounded by

$$\sum_{n \geq 1} q^n q^{-n\sigma} = \sum_{n \geq 1} q^{-n(\sigma - 1)} < \infty.$$

Prop. (III.3.1.35) (Main Theorem). If $f$ is a good function and $\chi$ is a Hecke character, $s \in \mathbb{C}$, we can define a global zeta function

$$\zeta(s, \chi, f) = \int_{A^*} f(a)\chi(a)|a|^s d^\times a,$$

which is defined for $\text{Re } s > 1$ and is holomorphic (The same as in (III.3.1.13)).

Then this function can be extended to a meromorphic function for all $s$, and it has poles iff $\chi(x) = |x|^\lambda$ for some $\lambda \in i\mathbb{R}$. In this case, it has has poles at

$$s = \begin{cases} 1 - \lambda, \lambda & F \text{ is a number field} \\ 1 - \lambda + 2\pi ni/\log(Np), -\lambda + 2\pi ni/\log(Np) & F \text{ is a function field} \end{cases}$$

with respectively residue $-k f(0)$ and $k \hat{f}(0)$, where $k = V(A^*_F/K^*)$. And we have functional equations

$$\zeta(s\chi, f) = \zeta(1 - s, \chi^{-1}, \hat{f}).$$
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**Proof:** Consider the exact sequence

$$1 \to A_1^* \to A^* \to |A|^* \to 1$$

let $A_1^*$ be the inverse image of $t \in |A|^*$, where $|A|^* = \mathbb{R}_{>0}$ or $q^\mathbb{Z}$, with Haar measure $dt/t$ or log $q$ times the counting measure (also denoted $dt/t$), and choose Haar measure $d^* x$ on $A_1^*$ compatible with $d a$ and $dt/t$, i.e.

$$\int_{|A|^*} \int_{A_1^*} f(a_t x) d^* x \, \frac{dt}{t} = \int_{A^*} f(a) da.$$

Then the measure $d^* x$ and the counting measure on $K^*$ induces a measure $d^* x$ on $A_1^*/K^*$, in particular we have

$$\int_{A_1^*/K^*} f(x) d^* x = \int_{A_1^*/K^*} f(a_t x) d^* x = \int_{A_1^*/K^*} f(a_t x) d^* x = \int_{A_1^*/K^*} f_1(x) d^* x.$$

Denote $\zeta_t(f, c) = \int_{A_1^*} f(x) c(x) d^* x$, then

$$\zeta(f, c) = \int_0^1 \zeta_t(f, c) \, \frac{dt}{t} + \int_1^\infty \zeta_t(f, c) \, \frac{dt}{t} = J + I$$

where if $|A|^* = q^\mathbb{Z}$ the value at 1 is counted half-half at this two part. Now for the $I$-part, if $\sigma(c)$ is smaller, it is smaller, thus $I$ extends to a holomorphic function to the whole Riemann surface. For the $J$-part, by lemmas (III.3.1.37) (III.3.1.36),

$$J = \int_0^1 \zeta_{1/t}(\hat{f}, \hat{c}) \, \frac{dt}{t} + \left[ \int_0^1 (k \hat{f}(0)(1/t)^{1-s} - k f(0)t^s) \, \frac{dt}{t} \right] \delta_{c, |.|}$$

$$= \int_1^\infty \zeta(\hat{f}, \hat{c}) \, \frac{dt}{t} + \left[ \int_0^1 (k \hat{f}(0)t^{s-1} - k f(0)t^s) \, \frac{dt}{t} \right] \delta_{c, |.|}$$

$$= I(f, c) + I(\hat{f}, \hat{c}) + k \delta_{c, |.|} \left[ \int_0^1 (\hat{f}(0)t^{s-1} - f(0)t^s) \, \frac{dt}{t} \right]$$

So it can be extended, and the final part is

$$\frac{k \hat{f}(0)}{s-1} \frac{k f(0)}{s}$$

when $F$ is number field, and when $F$ is function field, it equals

$$k \log q \{ \hat{f}(0)(-\frac{1}{2} + \sum_{n=0}^{\infty} (q^{-n})^{s-1}) - f(0)(-\frac{1}{2} + \sum_{n=0}^{\infty} (q^{-n})^s) \} = \frac{k \log q}{2} \left\{ \hat{f}(0) \frac{1}{1-q^{-s-1}} + f(0) \frac{1+q^s}{1-q^s} \right\}.$$

Now clearly $\xi(f, c) = \xi(\hat{f}, \hat{c})$, and it has the desired residues at 1 and $|.|$. □

**Lemma (III.3.1.36).** \( \int_{A_1^*/K^*} c(x) d^* x = k t^s \) if $c = |.|^s$ and 0 otherwise.

**Proof:** \( \int_{A_1^*/K^*} c(x) d^* x = c(t) \int_{A_1^*/K^*} c(x) d^* x, A_1^*/K^* \) is compact (IV.2.4.17) and $c$ is trivial on $A_1^*/K^*$ iff $c = |.|^s$, thus we can use (X.6.4.1). □

**Lemma (III.3.1.37).** \( \zeta_t(f, c) + f(0) \int_{A_1^*/K^*} c(x) d^* x = \zeta_{1/t}(\hat{f}, \hat{c}) + \hat{f}(0) \int_{A_1^*/K^*} \hat{c}(x) d^* x. \)
Proof:
\[
\zeta_t(f,c) + f(0) \int_{A^*_t/K^*} c(x)d^*x = \int_{A^*_t/k^*} \left( \sum_{a \in K^*} f(ax) \right) c(ax)d^*x + \int_{A^*_t/K^*} f(0)c(x)d^*x \\
= \int_{A^*_t/K^*} \left( \sum_{a \in K^*} f(ax) \right) c(x)d^*x \\
= \int_{A^*_t/K^*} \frac{1}{|x|} \left( \sum_{a \in K} \hat{f}(a/x) \right) c(x)d^*x \\
= \int_{A^*_t/K^*} \left( \sum_{a \in K} \hat{f}(ay) \right) |y|c^{-1}(y)d^*y \\
= \xi_{1/t}(\hat{f}, \hat{c}) + \hat{f}(0) \int_{A^*_t/K^*} \hat{c}(x)d^*x
\]

\[
\square
\]

**Hecke L-Functions**

**Prop. (III.3.1.38) (Hecke L-Functions).** Recall the definition of local L-factor in (III.3.1.19), for a Hecke character \( \chi = \prod_v \chi_v \) on \( A^*/K^* \) and \( s \in \mathbb{C} \), we define the **global Hecke L-function** as

\[
L(s, \chi) = \prod_v L(s, \chi_v)
\]

which converges for \( \text{Re } s > 1 \) and has a meromorphic continuation to all \( s \in \mathbb{C} \).

Also, we can define the **global \( \varepsilon \)-factor** as

\[
\varepsilon(s, \chi) = \prod_v \varepsilon(s, \chi_v, \psi_v).
\]

All but f.m. of the product equals 1 by (III.3.1.21), and they are of the form \( ab^s \), so \( \varepsilon(s, \chi) \) is also of the form \( ab^s \), where \( a \in \mathbb{C}^*, b \in \mathbb{R} \). In particular, it is holomorphic and non-vanishing. The fact \( \varepsilon(s, \chi) \) is independent of \( \psi \) can be seen from (III.3.1.39).

**Prop. (III.3.1.39) (Functional Equation of Hecke L-Functions).** The Hecke L-function has meromorphic continuation to all \( s \), and it has poles iff \( \chi(x) = |x|^\lambda \) for some \( \lambda \in i\mathbb{R} \). In this case, it has has poles at

\[
s = \begin{cases} 
1 - \lambda, \lambda \\
1 - \lambda + 2\pi ni/\log(Np), -\lambda + 2\pi ni/\log(Np)
\end{cases}
\]

\( F \) is a number field

\( F \) is a function field

Moreover, there is a functional equation

\[
L(s, \chi) = \varepsilon(s, \chi)L(1 - s, \chi^{-1}).
\]

**Proof:** Cf.[Bum98]P275.

\[
\square
\]

**Prop. (III.3.1.40).** Cf.[GTM 186 P7.6, 7.6, 7.8].
Dirichlet Characters

Def. (III.3.1.41) (Dirichlet Character). Let $N \in \mathbb{Z}_+$, a Dirichlet character modulo $N$ is a multiplicative function $\mathbb{Z} \to \mathbb{C}$ that is periodic of period $N$, and satisfies

$$ |\chi(n)| = \begin{cases} 1 & (n, N) = 1 \\ 0 & \text{otherwise} \end{cases}. $$

A primitive Dirichlet character modulo $N$ is a Dirichlet character modulo $N$ that is not a Dirichlet character modulo $N'$ for any other $0 < N' < N$.

Def. (III.3.1.42) (Gauss Sum). Let $\chi$ be a primitive Dirichlet character mod $N$, then the Gauss sum of $\chi$ is defined to be

$$ \tau(\chi) = \sum_{n \mod N} \chi(n)e^{2\pi i n/N}. $$

Prop. (III.3.1.43). $\tau(\chi) = \chi(-1)\overline{\tau(\chi)}$.

Prop. (III.3.1.44). $\sum_{n \mod N} \chi(n)e^{2\pi inm/N} = \chi(m)\tau(\chi)$.

Proof: If $(m, N) = 1$, then this follows from

$$ \tau(\chi) = \sum_{n \mod N} \chi(mn)e^{2\pi i mn/N} = \chi(m) \sum_{n \mod N} \chi(n)e^{2\pi i mn/N}. $$

If $(m, N) \neq 1$, then we need to show the LHS is 0. Let $m = dN, N = dN_1$. Because $\chi$ is primitive character mod $N$, there is some $c \equiv 1 \mod N_1$ that $\chi(c) \neq 1$, otherwise $\chi$ is defined mod $N_1$.

Notice

$$ \sum_{n \mod N} \chi(n)e^{2\pi i nm/N} = \sum_{r \mod N_1} \left\{ \sum_{n \mod N, n \equiv r \mod N_1} \chi(n) \right\} e^{2\pi i r m/N_1} $$

But $r \mapsto cr$ is a permutation of $\{n \mod N, n \equiv r \mod N_1\}$, thus

$$ \sum_{n \mod N, n \equiv r \mod N_1} \chi(n) = \sum_{n \mod N, n \equiv r \mod N_1} \chi(cn) = \chi(c) \sum_{n \mod N, n \equiv r \mod N_1} \chi(n) $$

which means this sum vanishes. \qed

Prop. (III.3.1.45). $|\tau(\chi)|^2 = N$.

Proof: For any $m$,

$$ |\sum_{n \mod N} \chi(n)e^{2\pi i nm/N}|^2 = \sum_{(n_1n_2, N) = 1} \chi(n_1)\overline{\chi(n_2)}e^{2\pi i (n_1-n_2)m/N}, $$

summing over $m \in (\mathbb{Z}/N\mathbb{Z})^*$,

$$ \varphi(N)|\tau(\chi)|^2 = \sum_{m \mod N} \sum_{(n_1n_2, N) = 1} \chi(n_1)\overline{\chi(n_2)}e^{2\pi i (n_1-n_2)m/N} = \sum_{n_1 \equiv n_2 \mod N, (n_1n_2, N) = 1} N = \varphi(N)N \quad \Box $$
Cor. (III.3.1.46). From this and (III.3.1.44) and also (III.3.1.43), we get that
\[ \chi(n) = \frac{\chi(1)\tau(\chi)}{N} \sum_{m \mod N} \chi(m)e^{2\pi inm/N}. \]

Def. (III.3.1.47) (Theta Function). Let \( \chi \) be a primitive character modulo \( N \) that \( \chi(-1) = 1 \), define
\[ \theta_{\chi}(t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \chi(n)e^{-\pi n^2t} = \frac{1}{2} \chi(0) + \sum_{n=1}^{\infty} \chi(n)e^{-\pi n^2t}. \]

Prop. (III.3.1.48) (Jacobi’s Triple Product Formula). For \( 0 < |q| < 1, |x| \neq 0 \),
\[ \sum_{n=-\infty}^{\infty} q^{n^2}x^n = \prod_{n=1}^{\infty} (1 - q^n)(1 + q^{2n-1}x)(1 + q^{2n-1}x^{-1}). \]

Proof: Cf. [Bump, Ex1.3.2]. \( \square \)

Cor. (III.3.1.49) (Dedekind Eta Function). Substitute \( q = q^{3/2} \) and \( x = -q^{-1/2} \), we get:
\[ \sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2+n)/2} = \prod_{n=1}^{\infty} (1 - q^{3n})(1 - q^{3n-1})(1 - q^{3n-2}) = \prod_{n=1}^{\infty} (1 - q^n). \]

By completing the square,
\[ \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n^2+1)^2/24} = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \]

the last term is known as the Dedekind eta function \( \eta(z) = e^{2\pi iz/24} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}) \).

Dirichlet L-Functions

Def. (III.3.1.50) (Dirichlet L-Functions). Let \( \chi \) be a Dirichlet character, we define the Dirichlet L-function to be
\[ L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \]
where \( \text{Re} \, s > 1 \).

In particular, \( L(s, 1) = \zeta(s) \) is called the Riemann Zeta function.

Prop. (III.3.1.51) (Functional Equation for Dirichlet Characters). Let \( \chi(-1) = (-1)^{\varepsilon} \), then we can define
\[ \Lambda(s, \chi) = \pi^{-(s+\varepsilon)/2} \Gamma\left(\frac{s + \varepsilon}{2}\right)L(s, \chi), \]
then this is just the Hecke L-function attached to the Hecke character corresponding to \( \chi(IV.2.4.13)(III.3.1.38) \). Thus \( L(s, \chi) \) can be extended to a meromorphic function for all \( s \in \mathbb{C} \), and it has simple poles at \( s = 0 \) or \( 1 \) when \( \chi = 1 \), and analytic otherwise.

Moreover, there are functional equations
\[ \Lambda(s, \chi) = (-i)^{\varepsilon} \tau(\chi) N^{-s} \Lambda(1 - s, \chi^{-1}), \]
In other words,
\[ \varepsilon(s, \chi) = (-i)^{\varepsilon} \tau(\chi) N^{-s}. \]
III.3. AUTOMORPHIC REPRESENTATIONS


Prop. (III.3.1.52). For \( k \geq 1 \), \( \zeta(2k) = \frac{2^{2k}}{(2k)!} B_{2k} \pi^{2k} \).

Proof: Because
\[
\cot(z) = i + \frac{2i}{e^{2iz} - 1},
\]
\[
z \cot(z) = 1 - \sum_{k=1}^{\infty} B_{2k} \frac{2^{2k} z^{2k}}{(2k)!}
\]
where \( B_k \) are Bernoulli numbers\(\text{(X.2.5.5)}\). But also
\[
z \cot(z) = 1 - 2 \sum_{k=1}^{\infty} \zeta(2k) \frac{z^{2k}}{\pi^{2k}}.
\]
by\(\text{(X.2.5.7)}\), thus the assertion follows. □

Prop. (III.3.1.53) (Leibniz Formula for \( \pi \)).

\[
1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}.
\]

Proof: Use the arctan integration formula. □

2 Automorphic Representations

Cf.[Wee Teck Gan’s Notes]

Def. (III.3.2.1) \( (L^2\text{-Space}) \). Define \( L^2(GL(n,F)\backslash GL(n,A),\omega) \) the space of all measurable functions \( \varphi \) on \( GL(n,A) \) that satisfies
\[
\varphi(zg) = \omega(z) \varphi(g), \quad z \in A^*, \quad \varphi(\gamma g) = \varphi(g), \quad \gamma \in GL(n,F)
\]
and square integrable modulo the center:
\[
\int_{Z(A)GL(n,F)\backslash GL(n,A)} |\varphi(g)|^2 dg < \infty.
\]

Then the right action of \( GL(n,A) \) on \( L^2(GL(n,F)\backslash GL(n,A)) \) is continuous?.

Def. (III.3.2.2) (Cuspidality). For a \( \varphi \in L^2(GL(n,F)\backslash GL(n,A),\omega) \), \( \varphi \) is called cuspidal iff
\[
\int_{Mr \times s(F)\backslash Mr \times s(A)} \varphi\left[\begin{bmatrix} I_r & X \\ I_s \end{bmatrix} g\right] dX = 0
\]
a.e. \( g \) for any \( r+s = n \). The closed space of all cuspidal elements in \( L^2(GL(n,F)\backslash GL(n,A),\omega) \) is denoted by \( L^2_0(GL(n,F)\backslash GL(n,A),\omega) \).

Def. (III.3.2.3) (Adelic Automorphic Forms). We denote by \( \mathcal{A}(GL(n,F)\backslash GL(n,A),\omega) \) the space of automorphic forms consisting of function on \( GL(n,A) \) that is
- smooth and satisfies \( \varphi(zg) = \omega(z) \varphi(g) \), \( z \in A^* \).
• $K$-finite.
• $Z$-finite, where $Z$ is the the product of $Z(U(\mathfrak{gl}(n, F_v)))$ where $v$ is an Archimedean place.
• of moderate growth, in the sense that $|f(g)| < C||g||^N$, where the height $||g||$ is defined to be the product of local heights $||g||_v$, defined by (III.2.1.9) in Archimedean place and defined by $\max\{\|a_{ij}\|, |\deg(g)|^{-1}\}$ in non-Archimedean place. Notice in non-Archimedean place $||g||_v = 1$ for $g \in GL(m, \mathcal{O}_v)$, so it is definable.

And the space $A_\emptyset(GL(n, F)\backslash GL(n, A), \omega)$ of cusp forms of automorphic forms that is cuspidal in sense of (III.3.2.2). $A(GL(n, F)\backslash GL(n, A), \omega)$ is a $(\mathfrak{g}_\infty, K_\infty)$-module by (III.2.2.5) in the $n$-dimensional case, and it is stable under right translation by $GL(n, A_f)$, because of (III.3.3.2).

**Prop. (III.3.2.4).** A $K$-finite and $Z$-finite vector in $L^2(GL(n, F)\backslash GL(n, A), \omega)$ is analytic and automatically of moderate growth.

**Proof:** Have something to do with (III.2.5.12)? \qed

**Def. (III.3.2.5) (Admissible Representations).** A smooth representation of $GL(n, A)$ is defined to be a commuting representation of a smooth $GL(n, A_f)$ structure and a $(\mathfrak{g}_\infty, K_\infty)$-structure (III.3.3.3).

If $(\pi, V)$ is a representation of $GL(2, A)$, then it induces a representation of $K = K_f \cdot K_\infty$. Then this representation is called admissible iff every vector is $K$-finite, and for any irreducible representation $\rho$ of $K$, $V^\rho$ is of f.d. By (III.1.8.25), it can be checked that this is equivalent to the respective admissibility property.

**Def. (III.3.2.6) (Automorphic Representations).** Then $A(GL(n, F)\backslash GL(n, A), \omega)$ affords a representation of $GL(n, A)$ (III.3.2.3), and we define an automorphic representation to be an irreducible representation of $GL(n, A)$ that can be realized as a quotient of subrepresentation of $A(GL(n, F)\backslash GL(n, A), \omega)$, and an automorphic cuspidal representation to be an irreducible representation of $GL(n, A)$ that can be realized as a subrepresentation of $A_\emptyset(GL(n, F)\backslash GL(n, A), \omega)$.

3 Automorphic Representations of $GL(n, A)$

Main references are [Bun98]Chap3.

**Basics**

**Def. (III.3.3.1) (Notations).** Let $F$ be a global field, we use the notation as in 4.

$G = GL(n, A)$ is identified with the restricted product $\prod_v(GL(n, F_v), GL(n, \mathcal{O}_v))$, where the $\mathcal{O}_v$ are only defined for non-Archimedean places. $GL(n, \mathcal{O}_v)$ are the maximal compact subgroup of $GL(n, F_v)$ by (III.4.2.3).

Similarly define $SL(n, A)$ and $T_1(A) = \begin{bmatrix} A & \vdots \\ \vdots & 1 \end{bmatrix}$.

$\omega$ be a Hecke character (IV.2.4.11) on $A^*/F^*$.

**Lemma (III.3.3.2).** For a non-Archimedean place $v$, Any left coset $GL(n, \mathcal{O}_v)a$ is contained in f.m. right coset of $GL(n, \mathcal{O}_v)$.

**Proof:** This is because $GL(n, \mathcal{O}_v)$ is compact open in $GL(n, K_v)$. \qed
III.3. AUTOMORPHIC REPRESENTATIONS

Def. (III.3.3.3) (Maximal Compact Subgroup). Fix the following compact subgroup of $GL(n, A)$:

$$K = \prod_v K_v, \quad K_v = \begin{cases} O(n) & v \text{ real} \\ U(n) & v \text{ complex} \\ GL(n, \mathcal{O}_v) & v \text{ non-Archimedean} \end{cases}.$$  

then $K$ is the maximal compact subgroup of $GL(n, A)$ in the sense that any compact subgroup can conjugate into $K$.

Let $K_{\infty} = \prod_{v \in S_{\infty}} K_v$, and $g_{\infty} = \prod_{v \in S_{\infty}} \mathfrak{gl}(n, F_v)$, then we can define $(g_{\infty}, K_{\infty})$-modules.

Proof: It is maximal because every factor does: $GL(n, \mathcal{O}_v)$ is proved in (III.3.3.1), and the Archimedean case is proved in (IX.8.5.8). \hfill \Box

Prop. (III.3.3.4). $GL(n, A)$ is unimodular.

Proof: This is because $GL(n, K)$ is unimodular for any local field $K$ and because we can calculate the restricted product measure by (IV.2.4.5). \hfill \Box

Prop. (III.3.3.5) (Strong Approximation). Let $F$ be a global number field, then

- $SL(n, A_{\infty})SL(n, F)$ is dense in $SL(n, A)$.

- Let $K_0$ be an open compact subgroup of $GL(n, A_f)$, Assume the image of $K_0$ under the determinant map is $\prod_{v \notin S_{\infty}} \mathcal{O}_v^\prime$, then the cardinality of $GL(n, F)GL(n, A_{\infty})/GL(n, A)/K_0$

is equal to the class number of $F$.

Proof: Cf. [Humphreys, Arithmetic Groups (1980)]. \hfill \Box

Def. (III.3.3.6) (Congruence Subgroup). Define the congruence subgroup $K_0(N)\backslash GL(2, A_f)$ as follows: $K_0(N) = \prod_{v \notin S_{\infty}} K_0(N)_v$, where

$$K_0(N)_v = \begin{cases} GL(2, \mathcal{O}_v) & p_v \nmid N \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathcal{O}_v), \quad c \equiv 0 \mod N \quad p_v | N \end{cases}.$$  

Prop. (III.3.3.7). If $F = \mathbb{Q}$, then the inclusion induces a homeomorphism

$$\Gamma_0(N)\backslash SL(2, \mathbb{R}) \cong GL(2, \mathbb{Q})Z(A) \backslash GL(2, A)/K_0(N)$$

Proof: Because the class number of $\mathbb{Q}$ is 1, item 2 of (III.3.3.5) shows

$$GL(2, A) = GL(2, \mathbb{Q})GL(2, \mathbb{R})K_0(N) = GL(2, \mathbb{Q})GL(2, \mathbb{R})^\dagger K_0(N),$$

so the map

$$GL(2, \mathbb{R})^\dagger \to GL(2, \mathbb{Q})\backslash GL(2, A)/K_0(N)$$

is surjective. Now if $g_{\infty}$ and $g_{\infty}$ has the same image, then $g_{\infty}^f = \gamma g_{\infty} k_0$, so $g_{\infty}^f = \gamma g_{\infty} \gamma^{-1}$. Then $\gamma_{\infty}$ has positive determinant and belongs to $\Gamma_0(N)$. Thus there is a bijection

$$\Gamma_0(N)\backslash GL(2, \mathbb{R})^\dagger \cong GL(2, \mathbb{Q})\backslash GL(2, A)/K_0(N).$$

Now consider the center, $A^* = \mathbb{R}_+^* \mathbb{Q}^* \prod_{v \notin S_{\infty}} \mathcal{O}_v^\star$, so $Z(A) = Z(\mathbb{R})^+Z(\mathbb{Q})(Z(A) \cap K_0(N))$ because we can adjust by a scalar. Hence the conclusion is true. \hfill \Box
Cor. (III.3.3.8). The quotient space \( GL(n,F)Z(A)\backslash GL(n,A) \) has finite measure.

**Proof:** For the general case, Cf.[Humphreys, Arithmetic Groups, (1980)].

Because \( K_0(N) \) is compact, it suffices to prove that \( GL(n,F)Z(A)\backslash GL(n,A)/K_0(N) \) has finite measure (because \( GL(n,F) \) and \( GL(n,A) \) are both unimodular, the measure is compatible). But this space is homeomorphic to \( \Gamma_0(N)\backslash SL(2,\mathbb{R}) \), which has finite measure because \( \Gamma(1) \) does and it is of finite index in \( \Gamma(1) \).

**Def. (III.3.3.9) (Global Siegel Sets).** For \( c,d > 0 \), we define the **global Siegel set** \( \mathcal{G}_{c,d} \) as the set of all Adeles of the form \( (g_v) \), where \( g_{\infty} \) is the Archimedean Siegel set \( \mathcal{G}_{c,d} \) and \( g_v \in K_v \) for all non-Archimedean places. And denote \( \overline{\mathcal{G}_{c,d}} \) its image in \( Z(A)\backslash GL(2,A) \).

**Prop. (III.3.3.10).** For \( c,d \) suitable chosen, \( GL(2,A) = GL(2,F)\mathcal{G}_{c,d} \).

**Proof:** We prove only for \( F = \mathbb{Q} \): This is true for \( c \leq \sqrt{3}/2 \) and \( d \geq 1 \) because of the shape of the fundamental domain of \( GL(2,\mathbb{R}) \) for \( SL(2,\mathbb{Z}) \).

**Spectral Problem**

**Lemma (III.3.3.11) (Auxiliary Compact Supported Function Approximation).** If \( \mathcal{H} \) is a unitary representation of \( G \) on a Hilbert space, and let \( f \neq 0 \in \mathcal{H} \), then for any \( \varepsilon > 0 \), there is a \( \varphi \in C_c^\infty(G) \) s.t. \( \pi(\varphi) \) is self-adjoint and \( ||\varphi(\rho)f - f|| < \varepsilon \).

**Proof:** The same as the first part of the proof of(III.1.6.1), notice the condition of smoothness in this case can be achieved.

**Prop. (III.3.3.12) (Gelfand,Graev and Piatetski-Shapiro).** Let \( G = GL(n,A), \varphi \in C_c^\infty(G) \), then

- There exists a constant \( C(\varphi) \) that for all \( f \in L_0^2(GL(n,F)\backslash GL(n,A),\omega) \), we have \( ||\rho(\varphi)f||_{C(G)} \leq C(\varphi)||f||_2 \).

- \( \rho(\varphi) \) is a compact operator on \( L_0^2(GL(n,F)\backslash GL(n,A),\omega) \).

**Proof:** The proof is the same as that of(III.2.5.11), but use global Siegel sets(III.3.3.9), Cf.[Bump, P297].

**Cor. (III.3.3.13) \( (L_0^2(GL(n,F)\backslash GL(n,A),\omega) \) Totally Decomposable).** The space \( L_0^2(GL(n,F)\backslash GL(n,A),\omega) \) decomposes into a Hilbert space direct sum of irreducible invariant subspaces over \( GL(n,A) \).

**Proof:** The proof is exactly the same as(III.2.5.5), but where we use(III.3.3.12) in place of(III.2.5.3) and lemma(III.3.3.11) in place of lemma(III.1.6.1).

**Prop. (III.3.3.14) (Irreducible Automorphic Cuspidal Representations Admissible).** If \( (\pi,V) \) is an irreducible constituent of the decomposition of \( \mathcal{H} = L_0^2(GL(n,F)\backslash GL(n,A),\omega) \) over \( G \) in(III.3.3.13), then the \( K \)-finite vectors in \( \mathcal{H} \) is dense, and form an irreducible admissible \( GL(n,A) \)-module contained in \( A_0(GL(n,F)\backslash GL(n,A),\omega) \).

**Proof:** We only prove for \( n = 2 \) and \( F = \mathbb{Q} \):

We must prove \( V^\rho \) is of f.d. for any irreducible representation \( (\rho,V_\rho) \) of \( K \). By(III.3.4.6), \( \rho = \otimes_v \rho_v \), where \( \rho_v \) is trivial a.e.. Then there exists an open subgroup \( K_{0,f} \subset \prod_{v \notin S_\infty} K_v \) that \( V_\rho \) is fixed by under \( K_{0,f} \).
We want to consider $V^{K_0, f}(\rho_\infty)$ and to show it is of f.d.

Notice $GL(2, A) = GL(2, \mathbb{Q})GL(2, \mathbb{R})^+ \backslash GL(2, A)/K_f$ by strong approximation (III.3.3.5), and $K_{0, f}$ is of finite index in $K_f$, thus

$$GL(2, \mathbb{Q})GL(2, \mathbb{R})^+ \backslash GL(2, A)/K_{0, f}$$

is a finite set, and let $g_i$ be a set of representatives that $g_i \in GL(2, A_f)$. Now for any $\varphi \in V^{K_{0, f}}$, $\varphi$ is determined by the $h$ functions $\Phi_i(g_\infty = \varphi(g_\infty g_i)$ on $GL(2, \mathbb{R})^+$. So it suffices to prove for $\Phi_i$.

Consider the projection $\Gamma$ of $GL(2, \mathbb{Q}) \cap (GL(2, \mathbb{R})^+ K_{0, f})$ which is a group of finite index in $SL(2, \mathbb{Z})$. Then $\Phi_i$ is invariant under $\Gamma$ and $Z$-finite and of moderate growth, as $\varphi$ is, so $\Phi_i \in \mathcal{A}(\Gamma \backslash GL(2, \mathbb{R})^+ , 1, \omega_\infty)$, so the $\rho_\infty$ part of this space is of f.d. by (III.2.5.13).

The rest is general by (III.1.6.10), and for the containment in $A_0$ by a similar argument as in (III.2.5.13) using a similar lemma as lemma (III.2.5.12). □

Cor. (III.3.3.15). The space $A_0(GL(n, F) \backslash GL(n, A), \omega)$ decomposes into a direct sum of irreducible admissible $GL(n, A)$-modules, by (III.3.3.14) and (III.3.3.13).

Adelization of Classical Automorphic Forms

Def. (III.3.3.16).

Prop. (III.3.3.17) (Hecke Eigenform Gives Automorphic Representations). If $F$ is a Hecke eigenform, then $\varphi$ lies in an irreducible subspace of $L_0^2(GL(n, F) \backslash GL(n, A), \omega)$, thus giving an automorphic representation.

Proof: □

4 Tensor Product Theorem

Def. (III.3.4.1) (Restricted Tensor Representation). Given a set of locally compact groups $G_v$ and a.e. their compact subgroups $K_v$, and $(\rho_v, V_v)$ representations of $G_v$. If $\epsilon_v^0 \in V_v^{K_v}$ are given for a.e. $v$, then we can define the restricted tensor representation of $(\rho_v, V_v)$ of $\prod (G_v, K_v)$ on $\otimes_v V_v$ by

$$(\otimes_v \rho_v)(g_v) \xi_v = \otimes_v \rho_v(g_v) \xi_v.$$ 

Def. (III.3.4.2) (Global Hecke Algebra). Let $\mathcal{H}_{GL(n, F_v)}$ be the Hecke algebras constructed in (III.4.2.6) and (I.4.4.15). As $\mathcal{H}_{GL(n, F_v)}$ has a spherical idempotent $\epsilon_v^0 = \epsilon_{K_v}$, (III.3.2.5), we can define the global Hecke algebra $\mathcal{H}_{GL(n, A)}$ as the restricted tensor product of $\mathcal{H}_{GL(n, F_v)}$ w.r.t $\epsilon_v^0$.

By definition of representation in (III.3.2.6) and (I.4.4.14)(II.4.2.9), the category of representations of $GL(n, A)$ is equivalent to the category of representations of $\mathcal{H}_{GL(n, A)}$.

Prop. (III.3.4.3) (Tensor Product Theorem). Let $(V, \pi)$ be an irreducible admissible representation of $GL(n, A)$, then there exists for each Archimedean place $v$ of $F$ an irreducible admissible $(g_\infty, K_v)$-module $(\pi_v, V_v)$, and for each non-Archimedean place $v$ an irreducible admissible representation $(\pi_v, V_v)$ of $GL(n, F_v)$ that for a.e. $v$, $V_v$ contains a non-zero $K_v$-fixed vector $\xi_v^0$, and $\pi$ is the restricted tensor product of the representations $\pi_v$.

Proof: By considering the global Hecke algebra, this follows immediately from (III.3.4.2) and (I.4.4.16)(II.4.2.9) □
Cor. (III.3.4.4) (Contragradient Representations). In view of this theorem, for any irreducible admissible representation \( \rho = \otimes_v \rho_v \) of \( GL(n, A) \), we can define the contragradient of \( \rho \) as \( \hat{\rho} = \otimes_v \hat{\rho}_v \), where \( \hat{\rho}_v \) is the contragradient of \( \rho_v \) (III.1.8.20).

Cor. (III.3.4.5) (Contragradient of Automorphic Cuspidal Representations). If \( (\pi, V) \) is an automorphic cuspidal representation of \( GL(n, A) \) with central character \( \omega \), then so is its contragradient \( (\hat{\pi}, \hat{V}) \). If \( V \subset \mathcal{A}_0(GL(n, F) \backslash GL(n, A), \omega) \), then its contragradient can be chosen to be the space of all functions \( g \mapsto \varphi(g^{-t}) \).

**Proof:** We only prove for \( n = 2, F \) totally real (the problem is the Archimedean places).

Let \( \hat{V} \) be the space of functions of the form \( \hat{\varphi}(g) = \varphi(g^{-t}) \), then the right action of \( G \) on \( \hat{V} \) corresponds back to the restriction of the right action composed with the automorphism \( g \mapsto g^{-t} \). Thus the result follows from (III.4.2.12) and its Archimedean analogy for \( (g_\infty, K_\infty) \)-modules (III.2.4.9).

Cor. (III.3.4.6) (Irreducible Representations of \( K \)). Let \( \rho \) be an irreducible f.d. representation of \( K \), then there exist f.d. representations \( (\rho_v, V_0) \) of \( K_v \) that for a.e. \( v, \rho_v \) is trivial representation and \( \xi_v^0 \in V_0 \) is a vector that \( \rho \) is isomorphic to the restricted tensor product \( \otimes_v \rho_v \) w.r.t. \( \xi_v^0 \).

**Proof:** This is similar to the proof of (III.3.4.3).

5 Whittaker Models

Def. (III.5.1) (Whittaker Models). The notion of Whittaker model and Whittaker functional is defined the same as in the real case (III.2.4.27) (III.2.4.29).

Prop. (III.5.2) (Decomposition of Whittaker Functional). Let \( F \) be a function field and \( \pi \) is an irreducible admissible representation of \( GL(2, A) \), which is of the form \( \otimes_v \pi_v \) w.r.t \( \xi_v^0 \) \( K_v \)-fixed a.e. and \( \pi_v \) irreducible admissible as in the tensor product theorem (III.3.4.3). If \( \Lambda \) is a Whittaker functional on \( V \), then for each place \( v \) of \( F \) there is a Whittaker functional \( \Lambda_v \) on \( V_v \) that \( \Lambda_v(\xi_v^0) = 1 \) for a.e. \( v \) and

\[
\Lambda(\otimes_v \xi_v) = \prod_v \Lambda_v(\xi_v).
\]

The space of Whittaker functionals on \( V \) has dimension at most 1.

**Proof:** If \( \Lambda \neq 0 \), then it is non-zero on some pure tensor \( \xi_v = \otimes_v \xi_v^0 \). Then the restriction \( \Lambda_v \) of \( \Lambda \) on each \( V_v \) is a Whittaker functional. We will prove \( \Lambda(\otimes_v \xi_v) = \prod_v \Lambda_v(\xi_v) \) by induction on the number of places \( S \) that \( \xi_v \neq \xi_v^0 \). If \( S = \emptyset \), then trivial. Now choose \( w \in S \), consider the functional

\[
x \mapsto \Lambda(x_w \otimes (\bigotimes_{v \neq w} \xi_v))
\]

is a Whittaker functional on \( V_v \), so it equals a multiple of \( \Lambda_v \) by (III.4.3.2), so we can induct.

The uniqueness (dimension 1) of \( \Lambda \) now follows from the uniqueness of \( \Lambda_v \).

**Prop.** (III.5.3) (Global uniqueness of Whittaker Models). Let \( (\pi, V) \) be an irreducible admissible representation of \( GL(2, A) \), then \( (\pi, V) \) has a Whittaker model w.r.t. \( \psi \) if each \( (\pi_v, V_v) \) (III.3.4.3) has a Whittaker model \( \mathcal{W}(\pi_v, \psi_v) \). If this is the case, then \( \mathcal{W}(\pi, \psi) \) is unique and consists of linear combinations of functions of the form \( W(g) = \prod_v W_v(g_v) \) where \( W_v = W_v^0 \) a.e. \( v \) and \( W_v^0 \) are the (unique (III.4.2.10)) normalized spherical element of \( \mathcal{W}_v \) that \( W_v^0(GL(2, O_v)) = 1 \).
Proof: Let $(\pi, V) = \otimes'_v (\pi_v, V_v)$ w.r.t a.e. spherical vectors $\xi_v^0$ by tensor product theorem (III.3.4.3), and $(\pi_v, V_v)$ are spherical a.e. $v$.

Firstly if every $(\pi_v, V_v)$ has a Whittaker model, the $W$ in the proposition is truly a Whittaker model: $W(\pi, \psi)$ consists of smooth $K$-finite functions of moderate growth is clear by how they are defined, and there is a canonical isomorphism of $V$ onto $W(\pi, \psi)$ by letting $(W_\xi)_v = W_v^0$ if $\xi_v = \xi_v^0$ and

$$W_\xi(g) = \prod_v W_{v, \xi_v}(g).$$

Secondly if $W$ is a Whittaker model for $(\pi, V)$, denote $\xi \mapsto W_\xi$ the isomorphism of $V$ onto $W$. Consider the restriction of $W$ to $GL(n, F)$ (using $\xi$), then they are clearly Whittaker models for $(\pi_v, V_v)$ thus unique (III.4.2.3) (III.2.4.27), so $W_v^0$ exists a.e. uniquely. Now we prove $W$ is of the form we said above, this will prove uniqueness.

Then we need to prove

$$W_\xi(g) = \prod_v W_{v, \xi_v}(g_v)$$

We only need to prove for $\xi = \xi_v^0$, because $W, W(\pi_v, \psi_v)$ are both irreducible. And also we can assume $g_v = 1$, a.e., because $(W_\xi)_v(g_v) = W_{v, \xi_v}(g_v) = 1$, a.e. Then we are in the finite case and we can multiply by scalars at f.m. $v$ s.t. this equation is true and nonzero for some $g$, then we use induction, as in the proof of (III.3.5.2).

$\Box$

Prop. (III.3.5.4) (Existence of Whittaker Models for Automorphic Cuspidal Representations). If $(\pi, V)$ is an automorphic cuspidal representation of $GL(2, A)$ in $A_0(GL(2, F) \backslash GL(2, A), \omega)$, for $\varphi \in V$, define

$$W_\varphi(g) = \int_{A/F} \varphi\left(\begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix} g\right) \psi(-x) dx,$$

then these functions is a Whittaker model $W(\pi, \psi)$.

And we have a Fourier expansion formula:

$$\varphi(g) = \sum_{\alpha \in F^*} W_\varphi\left(\begin{bmatrix} \alpha \\ 1 \end{bmatrix} g\right).$$

Proof: For any consider the continuous function $F(x) = \varphi\left(\begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix} g\right)$, then it is a function on $A/K$, which is continuous, and $A/K$ is compact (IV.2.4.14), so by Fourier inversion formula (X.6.3.15),

$$F(x) = \sum_{\alpha \in F} C(\alpha) \psi(\alpha x), \quad C(\alpha) = \int_{A/F} \varphi\left(\begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix} g\right) \psi(-\alpha x) dx.$$

Now $C(0) = 0$ because $\varphi$ is cuspidal, and if $\alpha \in F^*$,

$$C(\alpha) = \int_{A/F} \varphi\left(\begin{bmatrix} \alpha \\ 1 \end{bmatrix} \left[\begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix} g\right] \psi(-\alpha x) dx = \int_{A/F} \varphi\left(\begin{bmatrix} 1 & \alpha x \\ 1 & 1 \end{bmatrix} \left[\begin{bmatrix} \alpha \\ 1 \end{bmatrix} g\right] \psi(-\alpha x) dx = W_\varphi\left(\begin{bmatrix} \alpha \\ 1 \end{bmatrix} g\right).$$

So if we let $x = 1$, then we get $\varphi(g) = \sum_{\alpha \in F^*} W_\varphi\left(\begin{bmatrix} \alpha \\ 1 \end{bmatrix} g\right)$. 

Now we show \( \{ W_\varphi \} \) is Whittaker model: they all satisfies the equation by construction, and because \( W_\chi \varphi = X W_\varphi, \rho(g)W_\varphi = W_{\rho(g)\varphi} \), it is clear that this space is invariant under action of \( GL(2, A_f) \) and \( (g_\infty, K_\infty) \), and also it is of moderate growth in \( y \) because \( \varphi \) does (III.3.2.3), and it consists of \( K \)-finite vectors because \( V \) is admissible (III.3.3.14).

\[ \square \]

**Cor. (III.3.5.5).** A function \( \varphi \) in an automorphic cuspidal representation is rapidly decreasing, because of the Fourier expansion formula, and the fact that Whittaker functions decay rapidly.

**Prop. (III.3.5.6) (Strong Multiplicity One Theorem).** Let \( (\pi, V), (\pi', V') \) be two irreducible admissible subrepresentations of \( A_0(GL(n, F)\backslash GL(nA), \omega) \), assume that \( \pi_v \cong \pi'_v \) for all Archimedean places and a.e. non-Archimedean places, then \( V = V' \).

**Proof:** We only prove for \( n = 2 \).

Firstly if \( \pi_v \cong \pi'_v \) for every place \( v \), then their corresponding Whittaker model is the same (multiplied by a scalar) by (III.3.5.3). Then by (III.3.5.4) we have a Fourier expansion formula

\[
\varphi(g) = \sum_{\alpha \in F^*} W_\varphi \left( \begin{array}{c} \alpha \\ 1 \end{array} \right) g.
\]

Thus \( V = V' \).

In case that \( \pi_v \cong \pi'_v \) only outside a finite set \( S \), we choose \( W_v, W'_v \) local Whittaker functions for \( \pi_v, \pi'_v \) that: if \( v \notin S, W_v = W'_v \) and \( W_v \) is the unique \( K_v \) fixed function normalized that \( W_v(K_v) = 1 \) a.e. \( v \), and if \( v \in S \), they are chosen that

\[
F(y) = W_v \left( \begin{array}{c} y \\ 1 \end{array} \right) = W'_v \left( \begin{array}{c} y \\ 1 \end{array} \right) \in C_c^\infty(F_v^*),
\]

which is possible by (III.4.7.2). And we define

\[
\varphi(g) = \sum_{\alpha \in F^*} W_\varphi \left( \begin{array}{c} \alpha \\ 1 \end{array} \right) g, \quad \varphi'(g) = \sum_{\alpha \in F^*} W'_\varphi \left( \begin{array}{c} \alpha \\ 1 \end{array} \right) g,
\]

\[
W(g) = \prod_v W_v(g_v), \quad W'(g) = \prod_v W'_v(g_v).
\]

as in (III.3.5.4).

Then we claim \( \varphi = \varphi' \) on \( GL(2, A) \): Firstly \( W, W' \) are simultaneously right invariant under some open subgroup \( K_0 \subset GL(2, A_f) \) by our construction, and \( \varphi \in V, \varphi' \in V' \) are automorphic, thus \( \varphi = \varphi' \) on \( GL(2, F)T_1(A)GL(2, F_\infty)K_0 \), but this group is just \( GL(2, A) \) because by strong approximation (III.3.3.5) \( SL(n, F)SL(2, F_\infty) \) is dense in \( SL(n, A) \), thus \( SL(n, F)SL(2, F_\infty)K_0 \) contains \( SL(n, A) \) and \( T_1(A) \) maps subjectively via determinant onto \( A^* \), so \( GL(2, F)T_1(A)GL(2, F_\infty)K_0 = GL(2, A) \).

So we have a \( \varphi \in V \cap V' \), meaning \( V = V' \).

\[ \square \]

6 Global Functional Equations

**Cuspidal Automorphic Forms**

**Def. (III.3.6.1) (L-function).** Let \( (\pi, V) \) be an automorphic cuspidal representation of \( GL(2, A) \), assume its central character \( \omega \) is unitary. Now \( \pi = \otimes_v \pi_v \), and let \( S \) be a finite set of places that if \( v \notin S, \pi_v \) is spherical, then define the partial L-function as

\[
L_S(s, \pi, \xi) = \prod_{v \notin S} L_v(s, \pi_v, \xi_v).
\]
Def. (III.3.6.2) (Zeta-Integral). Consider the function
\[ Z(s, \varphi) = \int_{A^*/F^*} \varphi\left( \begin{bmatrix} y & \varepsilon \end{bmatrix} \right) |y|^{s-1/2} d^* y \]
then if \( \text{Re}(s) > 3/2 \)
\[ Z(s, \varphi) = \int_{A^*} W_\varphi\left( \begin{bmatrix} y & \varepsilon \end{bmatrix} \right) |y|^{s-1/2} d^* y \]
Because the latter is absolutely convergent and use(III.3.5.4)(X.6.1.29). In fact, this integral decomposes as a product of local factors \( \prod_v Z_v(s, W_v) \), where
\[ Z_v(s, W_v) = \int_{F_v^*} (W_v)_{\varphi_v}\left( \begin{bmatrix} y_v & \varepsilon \end{bmatrix} \right) |y_v|^{s-1/2} d^* y_v. \]
Cf.[Bump, P335].
More generally, we can consider the twisting
\[ Z(s, \varphi, \xi) = \int_{A^*} W_\varphi\left( \begin{bmatrix} y & \varepsilon \end{bmatrix} \right) |y|^{s-1/2} \xi(y) d^* y \]
where \( \xi \) is a unitary Hecke character.

Def. (III.3.6.3) (Nonramified Places). In the situation of(III.3.6.1), given a cuspidal function \( \varphi \in V \), we call a place \( v \) of \( F \) nonramified if \( v \) is non-Archimedean, \( \varphi_v \) is spherical in \( \pi_v \), and the Whittaker function \( (W_v)_{\varphi_v} \) is normalized that \( (W_v)_{\varphi_v}(K_v) = 1 \). This condition is true for a.e. \( v \).

Prop. (III.3.6.4). If \( v \) is nonramified in the sense of(III.3.6.3), then for \( s \) sufficiently large,
\[ Z_v(s, W_v, \psi_v) = L_v(s, \pi_v, \psi_v). \]
Proof: We assume \( \psi_v = 0 \), and the general case is similar. There is an explicit formula for \( W_v \) in terms of the Satake parameters \( \alpha_1, \alpha_2 \): if \( \text{ord}_v(y) = m \), then
\[ W_v\left( \begin{bmatrix} y & \varepsilon \end{bmatrix} \right) = \begin{cases} q^{-m/2} \frac{\alpha_1^{m+1} - \alpha_2^{m+1}}{\alpha_1 - \alpha_2} & m \geq 0 \\ 0 & \text{otherwise} \end{cases} \]
so the integration of \( Z_v(s, \varphi_v) \) is
\[ Z_v(s, W_v) = \sum_{m=0}^{\infty} q^{-m/2} \frac{\alpha_1^{m+1} - \alpha_2^{m+1}}{\alpha_1 - \alpha_2} q^{m/2 - m s} = \frac{1}{(1 - \alpha_1 q^{-s})(1 - \alpha_2 q^{-s})} = L(s, \pi) \]

Prop. (III.3.6.5). In spite of the convergence problem of the product of local zeta integrals, the global zeta integral is absolutely convergent for any \( s \) as \( \varphi \) decay rapidly both at \( \infty \) and at 0. Because \( \varphi \) is automorphic, we have
\[ Z(s, \varphi, \xi) = \int_{A^*/F^*} \varphi(w_1 \begin{bmatrix} y & \varepsilon \end{bmatrix}) |y|^{s-1/2} \xi(y) d^* y \]
where the last one is by direct calculation and the fact that
\[ \hat{\pi} \]
for \( v \) is the contragradient representation.

**Prop. (III.3.6.6) (Local Functional Equation).** The local zeta integral \( Z_v(s, W_v, \xi_v) \), defined in (III.3.6.2), then it has a meromorphic continuation to all \( s \), and there exists a meromorphic function \( \gamma_v(x, \pi_v, \xi_v, \psi_v) \) s.t.

\[
Z_v(1 - s, \pi_v(w_1)W_v, \xi_v^{-1}\omega_v^{-1}) = \gamma_v(x, \pi_v, \xi_v, \psi_v)Z_v(s, W_v, \xi_v).
\]

**Proof:** Cf. [Bump, P339].

**Prop. (III.3.6.7) (Global Functional Equation).** Notation as in (III.3.6.1), if \( S \) contains all the places \( v \) that is not nonramified, then \( L_S(s, \pi, \xi) \) has meromorphic continuation to all \( s \in \mathbb{C} \), and satisfies the functional equation:

\[
L_S(s, \pi, \xi) = [\prod_{v \in S} \gamma_v(s, \pi_v, \xi_v)\omega_v)]L_S(1 - s, \pi, \xi)^{-1}
\]

where \( \pi \) is the contragradient representation.

**Proof:** We will prove that both sides are equal to the meromorphic function

\[
S = [\prod_{v \in S} Z_v(s, W_v, \xi_v)^{-1}]Z(s, \varphi, \xi)
\]

for suitable \( s \). This function is meromorphic by (III.3.6.6) and (III.3.6.5).

Firstly, if \( \text{Re}(s) \) is large, then \( S \) equals the LHS by (III.3.6.4) and (III.3.6.2).

Secondly, if \( - \text{Re}(s) \) is large, then \( S \) equals

\[
[\prod_{v \in S} Z_v(s, W_v, \xi_v)^{-1}Z_v(1 - s, \pi(w_1)W_v, \xi_v^{-1}\omega_v^{-1})] \prod_{v \not\in S} Z_v(1 - s, \pi(w_1)W_v, \xi_v^{-1}\omega_v^{-1})
\]

then the \( v \in S \) part is just \( \gamma_v(s, \pi_v, \xi_v) \) by (III.3.6.6), and the for \( v \not\in S \), \( v \) is nonramified, so \( \pi(w_1)W_v = W_v \), and

\[
Z_v(1 - s, \pi(w_1)W_v, \xi_v^{-1}\omega_v^{-1}) = L_v(s, \pi_v, \omega_v^{-1}\xi - v) = L_v(s, \pi_v, \xi_v^{-1})
\]

where the last one is by direct calculation and the fact \( \alpha_1^{-1}, \alpha_2^{-1} \) are the Satake parameters of \( \pi \).

**Eisenstein Series**

**Rankin-Selberg Method**

**Def. (III.3.6.8).** Let \( (\pi_1, V_1), (\pi_2, V_2) \) be automorphic cuspidal representations of \( GL(2, \mathbb{A}) \) with central characters \( \omega_1, \omega_2 \), then by restricted product theorem, we can fix a finite set \( S \) of places that for \( v \not\in S \), \( \pi_{1v}, \pi_{2v} \) are both spherical, and the conductor of the canonical character \( \psi_v \) is \( O_v \). Then

**7 Global Langlands Correspondence**

**L-Groups**
III.4 Representations of GL(2) over p-adic Fields

Def. (III.4.0.1) (Notations for General Field). \( G = GL(n, F) \) and \( B(F) \) is the Borel subgroup consisting of upper triangular matrices in \( G \), \( N(F) \) the group of unipotent upper triangular matrices in \( G \), \( T(F) \) the group of diagonal matrices, \( Z(F) \) the group of scalar matrices.

Denote \( w^0 \) the matrix \( \sum_{i=1}^{n} e_{i,n-i} \).

If \( n = 2 \), let \( T_1(F) = \begin{bmatrix} F & 1 \\ 1 & 1 \end{bmatrix} \), and denote

\[
\begin{align*}
t(y) &= \begin{bmatrix} y & -1 \\ 1 & 1 \end{bmatrix}, \\
n(z) &= \begin{bmatrix} 1 & z \\ 1 & 1 \end{bmatrix}, \\
w_1 &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \\
w_0 &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},
\end{align*}
\]

following 4.

\( \psi \) is a non-trivial character of \( F \), and there is a character \( \psi_N \) on \( N(F) \) given by \( \psi_N(g) = \sum_i g_{i,i+1} \).

1 GL(2) over Finite Fields

Principal Series Representations

Def. (III.4.1.1) (Notations). In this section, let \( G = GL(n, F) \), \( F = \mathbb{F}_q \) a finite field.

Prop. (III.4.1.2). Let \((\pi, V)\) a f.d. representation of \( G \), then

- if the representation \((\pi_1, V)\) is defined by \( \pi_1(g) = \pi(g^{-1}) \), then \( \pi_1 \cong \pi \).

- if \( n = 2 \) and \((\pi, V)\) is irreducible, let \( \omega \) be the central character of \( \pi \), and if \((\pi_2, V)\) is defined by \( \pi_2(g) = \omega(\deg g)^{-1} \pi(g) \), then \( \pi_2 \cong \pi \).

Proof: The proof of (III.4.2.12) applies to this case, noticing a finite group is profinite hence locally profinite. \( \square \)

Lemma (III.4.1.3). Let \( \chi_1, \chi_2, \mu_1, \mu_2 \) be characters of \( F^* \), consider the principal representations \( \mathcal{B}(\chi_1, \chi_2), \mathcal{B}(\mu_1, \mu_2) \) of \( GL(2, F) \) defined in (III.4.4.1), then

\[
\dim \text{Hom}_{GL(2,F)}(\mathcal{B}(\chi_1, \chi_2), \mathcal{B}(\mu_1, \mu_2)) = \delta_{\chi_1,\mu_1} \delta_{\chi_2,\mu_2} + \delta_{\chi_1,\mu_2} \delta_{\chi_2,\mu_1}.
\]

Proof: Let \( \chi, \mu \) be characters of \( B(F) \) defined as in (III.4.4.1), then by (III.1.2.16), the dimension is just the dimension of space of functions \( \Delta : GL(2, F) \to \mathbb{C} \) that

\[
\Delta(b_2gb_1) = \mu(b_1)\Delta(g)\chi(b_1), \quad b_1, b_2 \in B(F).
\]

Then by the Bruhat decomposition (IX.8.6.4), \( \Delta \) is determined by its values on 1 and \( w_0 \). Notice that if \( \chi_1 \neq \mu_1 \) or \( \chi_2 \neq \mu_2 \), we can use this condition to show \( \Delta(1) = 0 \), and if \( \chi_1 \neq \mu_2 \) or \( \chi_2 \neq \mu_1 \), we can use this condition to show \( \Delta(w_0) = 0 \), and also the construction of \( \Delta \) is clear in other cases. \( \square \)

Prop. (III.4.1.4) (Principal Series Representations). Let \( \chi_1, \chi_2, \mu_1, \mu_2 \) be characters of \( F^* \), then \( \mathcal{B}(\chi_1, \chi_2) \) is an irreducible representation of degree \( q = |F| + 1 \) unless \( \chi_1 = \chi_2 \), in which case it is the direct sum of two irreducible representations having degree 1 and \( q \). And \( \mathcal{B}(\chi_1, \chi_2) \cong \mathcal{B}(\mu_1, \mu_2) \) iff \( \{\chi_1, \chi_2\} = \{\mu_1, \mu_2\} \).
Proof: Use (III.4.1.3), then
\[
\dim \text{End}_{\text{GL}(2, F)}(B(\chi_1, \chi_2)) = 1 + \delta_{\chi_1, \chi_2}.
\]
Now by Peter-Weyl, if a representation \( V \) is isomorphic to \( \sum d_i \pi_i \), where \( \pi_i \) are irreducible, then \( \dim G(V) = \sum d_i^2 (X.6.4.6) \). Then we now \( B(\chi_1, \chi_2) \) decomposes into two representations if \( \chi_1 = \chi_2 \) and is irreducible if \( \chi_1 \neq \chi_2 \).

In case \( \chi_1 = \chi_2 \), there is an invariant subspace of dimension 1, generated by the function \( f(g) = \chi(\deg g) \), so the rest representation is of dimension \( q \), because \( G(2, F)/B(F) = q + 1 \).

Remark (III.4.1.8). The last relation evaluated at \( y = 1 \) shows \( \hat{\Phi}(x) = \Phi(-x) \), the Fourier inversion formula.

Proof: Use (III.4.1.3), then
\[
\dim \text{End}_{\text{GL}(2, F)}(B(\chi_1, \chi_2)) = 1 + \delta_{\chi_1, \chi_2}.
\]
Now by Peter-Weyl, if a representation \( V \) is isomorphic to \( \sum d_i \pi_i \), where \( \pi_i \) are irreducible, then \( \dim G(V) = \sum d_i^2 (X.6.4.6) \). Then we now \( B(\chi_1, \chi_2) \) decomposes into two representations if \( \chi_1 = \chi_2 \) and is irreducible if \( \chi_1 \neq \chi_2 \).

In case \( \chi_1 = \chi_2 \), there is an invariant subspace of dimension 1, generated by the function \( f(g) = \chi(\deg g) \), so the rest representation is of dimension \( q \), because \( G(2, F)/B(F) = q + 1 \).

Remark (III.4.1.8). The last relation evaluated at \( y = 1 \) shows \( \hat{\Phi}(x) = \Phi(-x) \), the Fourier inversion formula.
III.4. REPRESENTATIONS OF $GL(2)$ OVER $P$-ADIC FIELDS

Prop. (III.4.1.9) (Weil Representation for $GL(2, F)$). Define

$$W(\chi) = \{ \phi \in W | \phi(yx) = \chi(y)\phi(x), y \in E_1^* \},$$

then by considering the order of $E_1^*$, $\dim W(\chi) = q + \varepsilon$. Also it is verified that $W(\chi)$ is stable under the Weil representation of $SL(2, F)$ (III.4.1.7).

Now we want to extend this representation to $GL(2, F)$ by defining

$$(\omega \bigg[ \begin{array}{cc} a \\ 1 \end{array} \bigg] \phi)(x) = \chi(b)\phi(bx)$$

where $b$ is arbitrary that $N(b) = a$.

Proof: It must be shown that this is truly a representation of $GL(2, F)$, it suffices to show that

$$\omega \bigg[ \begin{array}{cc} a \\ 1 \end{array} \bigg] \omega(g)\omega \bigg[ \begin{array}{cc} a^{-1} \\ 1 \end{array} \bigg] = \omega \bigg[ \begin{array}{cc} a \\ 1 \end{array} \bigg] g \bigg[ \begin{array}{cc} a^{-1} \\ 1 \end{array} \bigg],$$

where $g$ is a generator of $SL(2, F)$. This is clear for $g = t(a)$, and also clear for $g = n(z)$. For $g = w_1$, it suffices to check

$$\omega \bigg[ \begin{array}{cc} a \\ 1 \end{array} \bigg] \circ \hat{\omega} \bigg[ \begin{array}{cc} a^{-1} \\ 1 \end{array} \bigg] \phi(x) = \hat{\phi}(ax)$$

which is subtle but also clear. \qed

Prop. (III.4.1.10) (Split Weil Representation). In the split case, the character $\chi$ of $E^*$ is of the form $\chi((x, y)) = \chi_1(x)\chi_2(y)$, and then condition in (III.4.1.6) just says $\chi_1 \neq \chi_2$. Then:

$$(\pi(\chi), W(\chi)) \cong \mathcal{B}(\chi_2, \chi_1)$$

Proof: The intertwining operator is given by

$$L : W(\chi) \to \mathcal{B}(\chi_2, \chi_1) : (L\phi)(g) = (\omega(g)\phi)((1, 0)).$$

Firstly $L\phi \in \mathcal{B}(\chi_2, \chi_1)$ by direct verification for $t(a), T_1(F)$ and $n(z)$. Then it is an intertwining operator is clear.

Then this is an isomorphism because both are of dimension $q + 1$ (III.4.1.9) and $\mathcal{B}(\chi_2, \chi_1)$ is irreducible (III.4.1.4). \qed

Def. (III.4.1.11) (Cuspidal Representation). Call a f.d. representation $(\pi, V)$ of $GL(2, F)$ cuspidal representation if there exists no nonzero linear functional $l$ on $V$ that

$$l(\pi(n(z))v) = l(v)$$

for all $v \in V, n(z) \in N(F)$.

Notice in this finite case, this is equivalent to $V$ has no $N(F)$-fixed vector, because the contra-gradient is well-known and the trivial isotropic part correspond, unlike the $p$-adic case.

Lemma (III.4.1.12). If $(\pi, V)$ is a cuspidal representation of $GL(2, F)$, then the dimension of $V$ is a multiple of $q - 1$. 

Proof: Because \( N(F) \cong F \), any character of \( N(F) \) is of the form \( \psi_a(n(x)) = \psi(ax) \). Now decompose the contragradient representation \( V^* \) of \( G \) into isotypic parts of \( N(F): V^* = \oplus_{a \in F} V^*(a) \), then the hypothesis implies \( V^*(0) = 0 \).

Notice that the group \( T_1(F) \) acts transitively on the spaces \( V^*(a), a \neq 0 \) by \( \hat{\pi}( \begin{bmatrix} t \\ 1 \end{bmatrix} )l, \) because

\[
\begin{bmatrix} t \\ 1 \end{bmatrix} \begin{bmatrix} 1 & x/t \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix}.
\]

So the dimension of \( V \) is a multiple of \( q - 1 \). \( \square \)

**Prop. (III.4.1.13) (Anisotropic Representation).** In the anisotropic case, the Weil representation \((\pi(\chi), W(\chi))\) is cuspidal and irreducible.

**Proof:** Suppose it is not cuspidal, then it contains a non-zero \( N(F) \) fixed vector \( \Phi_0(\text{III.4.1.11}) \), which means \( \Phi_0(x) = w(n(z)\Phi_0)(x) = \psi(zN(x))\Phi(x) \). Now \( \Phi_0(0) = 0 \) because \( \chi \) is nontrivial on \( E^* \), and if \( x \neq 0 \), then there is a \( z \) that \( \psi(zN(x)) \neq 1 \) because \( \psi \) is non-trivial, so \( \Phi_0(x) = 0 \) also, so \( \Phi_0 = 0 \), contradiction.

Finally, subrepresentation of cuspidal representation is cuspidal by (III.4.1.11), then \((\pi(\chi), W(\chi))\) is irreducible, by the fact it is of dimension \( q - 1 \) (III.4.1.9) and lemma (III.4.1.12). \( \square \)

**Prop. (III.4.1.14) (Classification of Representations of \( GL(2, F) \)).** There is a list of all irreducible representations of \( GL(2, F) \):

- \( q - 1 \) 1-dimensional representations \( \chi(\deg g) \), where \( \chi \) is a character of \( \mathbb{F}^* \).
- \( \frac{(q-1)(q-2)}{2} \) principal series representations of dimension \( q + 1 \).
- \( q - 1 \) Steinberg representations (with twists) of dimension \( q \).
- \( \frac{q(q-1)}{2} \) cuspidal Weil representations of dimension \( q - 1 \).

**Proof:** All these are irreducible representations by (III.4.1.4)(III.4.1.5) and (III.4.1.13). It suffices to show they are not isomorphic. Notice different kind of representation have different dimensions, thus it suffices to compare the same representations.

For principal series representations \( B(\chi_1, \chi_2), \chi_1 \neq \chi_2 \), their isomorphisms are known by (III.4.1.4). For Steinberg representations, twisting are clearly different. For cuspidal Weil representations, there are \( q^d - q \) way of choosing \( \chi(\text{III.4.1.6}) \) and \( ? \) Cf. [Local Langlands For \( GL(2) \)].

Finally, they are all the irreducible representations by the fact \( \sum d_a^2 = |G| = (q - 1)^2 q(q + 1) \) (III.1.2.4). \( \square \)

**Whittaker Models**

**Prop. (III.4.1.15) (Uniqueness of Whittaker Models).** Let \( \mathcal{G} \) be the representation of \( GL(2, F) \) induced from the character \( \psi_N(n(z)) = \psi(z) \) on \( N(F) \), then it is multiplicity-free, and every irreducible representation of dimension \( > 1 \) occurs in it. Notice that this is just the existence and uniqueness of Whittaker models.

**Proof:** To show multiplicity free, it suffices to show \( \text{End}_{GL(2, F)}(\mathcal{G}) \) is commutative, by Shur’s lemma (X.6.2.5). Then we use (III.1.2.16), which says this ring is isomorphic to the ring of functions \( \Delta \) on \( G \) that

\[
\Delta(n_2gn_1) = \psi_N(n_2)\Delta(g)\psi_N(n_1).
\]
where the multiplication is convolution (III.1.2.17).

Notice it follows from the the Bruhat Decomposition (IX.8.6.4) that the double coset 
\( N(F) \backslash GL(2, F) / N(F) \) is uniquely represented by matrices with exactly two non-zero entries. Then it is clear a diagonal coset can support a function \( \Delta \) iff it is a scalar multiple of \( I \). In other words, the representatives are 
\[
\begin{bmatrix}
  a \\
  a
\end{bmatrix}, 
\begin{bmatrix}
  b \\
  c
\end{bmatrix}. 
\]

Consider the involution of \( G \) given by 
\[
\iota(g) = w_1 g^t w_1^{-1},
\]
then it is an anti-involution of \( G \) and it induces isomorphism on \( N(F) \), so it induces an anti-involution of order two on the ring of functions \( \Delta \) by \( \iota(\Delta)(g) = \Delta(\iota(g)) \). Notice this is an anti-involution because of (X.6.1.20) and the fact a finite group is unimodular. But this anti-involution fixes the representatives as above, so it is in fact identity on these \( \Delta \), which proves the convolution is commutative.

For the last assertion, just notice the dimension of \( G \) is \( (q-1)(q^2-1) \), and the sum dimensions of irreducible representations of dimension \( > 1 \) is just \( (q-1)(q^2-1) \) by (III.4.1.14). □

Cor. (III.4.1.16). By Frobenius reciprocity (X.6.5.4) implies that the space of Whittaker functionals, defined as in (III.2.4.29) is of dimension 1 for any irreducible representation of dimension \( > 1 \).

2 Representations of \( GL(n) \) over \( p \)-adic Local Fields

References are [Representations of \( p \)-adic Groups Bernstein]. [Bump, Automorphic Forms and Representations, Chap4].

Def. (III.4.2.1) (Notations). Let \( F \) be a \( p \)-adic local field, \( G = GL(n, F), K = GL(n, O) \). Notice they are all locally profinite groups.

Def. (III.4.2.2) (Notations). Let \( F \) be a \( p \)-adic local field and \( O \) the ring of integers in \( F \), \( p \) the maximal ideal in \( O \), and \( \varphi \) a uniformizer of \( p \).

\[
G = GL(n, F), K = GL(n, O), B(F) \text{ is the subgroup of upper triangular matrices. Notice they are all locally profinite.}
\]

We can define subgroups \( N_k(F) \) of \( N(F) \) as \( N_k(F) \) is the group of unipotent matrices that \( v(e_{ij}) \geq k(i-j) \). Then \( \cup_{k>0} N_k(F) = N(F) \).

Fix a non-trivial character \( \psi \) of \( F \).

Notice the modular function of \( B(F) \) is \( \Delta(\begin{bmatrix}
  x & y \\
  z & \end{bmatrix}) = \sqrt{\frac{x}{z}} \) (use (X.6.1.12)).

Prop. (III.4.2.3). \( K \) is the maximal compact subgroup of \( G \), and any compact subgroup of \( G \) is conjugate to \( K \).

Proof: \( GL(n, O) \) is compact because it is a union of \( |k_v|-1 \) many \( SL(n, O) \), which is compact.

For maximality, it suffices to find an \( O_L \)-lattice that is stable under \( \Gamma \)-action. Notice \( \rho(\Gamma) \cap GL_n(O_L) \) is open in \( \rho(\Gamma) \), thus is of finite index, so taking the coset representation, it is a lattice that is stable under \( \Gamma \). □

Prop. (III.4.2.4) (Iwasawa Decomposition). \( G = B(F)K \), in particular, \( B(F) \backslash G \) is compact.
Proof: Prove by induction on \( n \). \( n = 1 \) is clear. Given a \( g \in GL(n, F) \), we consider its bottom row, as let \( x_{n} \) be the term of minimal multiplicative valuation, then we can right multiply by a permutation matrix \( w \in K \) that \( x_{n} \) is of minimal valuation. Now right multiply by a matrix \( k \), which is \( 1 \) on the diagonal and \( k_{n} = -x_{n}^{-1} \) on the bottom row, then \( k \in K \), and \( g k \) has bottom row \( = e_{n} \). Thus we can induct and find a \( k_{0} \in K \) that \( g k_{0} \in B(F) \).

Prop. (III.4.2.5) \((p\text{-adic Cartan Decomposition})\). A complete set of double coset representatives for \( K \backslash GL(n, F) / K \) consists of diagonal matrices with entries in \( \pi^{\mathbb{Z}} \), and the power in non-increasing order.

Proof: This follows directly from Smith normal form (IX.8.6.5).

Hecke Algebra

Def. (III.4.2.6) (Hecke Algebras of Locally Profinite Groups). The algebra \( \mathcal{H}(G) \) of test functions on a locally profinite group \( G \) under the convolution of measures is an algebra, called the Hecke algebra of \( G \). And for a compact open subgroup \( K \) of \( G \), \( \mathcal{H}_{K} \) is the subspace of \( K \)-bi-invariant functions in \( \mathcal{H}(G) \).

Notice \( \mathcal{H} = \bigcup_{K} \mathcal{H}_{K} \) by (III.1.8.5). Also \( \mathcal{H}_{K} \) has a unit \( e_{K} = \mu(K)^{-1} \chi_{K} \). This is easily verified.

Then \( \mathcal{H}_{G} \) is an idempotented algebra (I.4.4.2).

Proof: Define the set \( E \) of idempotents in \( \mathcal{H}(G) \) as \( e_{K} \), where \( K \) is compact open in \( G \). The fact that \( \mathcal{H} \) is an idempotented algebra follows from (III.4.2.7) and (III.4.2.8).

Prop. (III.4.2.7) (Point Measure). Consider the point measure \( \delta_{g} \) for \( g \in G \), it is not an element in \( \mathcal{H}(G) \), but it can left convolute on \( \mathcal{H}(G) \): \( \delta_{g} * \varphi(x) = \varphi(g^{-1}x) \).

\( e_{K}, \delta_{g} \) convolution relations.

Proof:

Prop. (III.4.2.8). \( \mathcal{H}_{K} = e_{K} * \mathcal{H} * e_{K} = \mathcal{H}[e_{K}] \).

Proof: Notice by (III.4.2.7), functions in \( e_{K} * \mathcal{H} * e_{K} \) is clearly \( K \)-bi-invariant. For the other direction, notice if \( \varphi \) is left and right \( K \)-invariant, then \( \varphi = e_{K} * \varphi = \varphi * e_{K} = e_{K} * \varphi * e_{K} \).

Prop. (III.4.2.9) (Equivalence of Representations Locally Profinite Case). For a smooth representation \( (\pi, V) \) of \( G \), for any \( v, g \mapsto \pi(g)v \) can be regarded as a locally constant function with value in \( V \), thus for any \( \varphi \in \mathcal{H}_{G} \), \( g \mapsto \varphi(g)\pi(g)v \) is locally constant with compact support, thus we can define a representation of the Hecke algebra \( \mathcal{H}_{G} \) by

\[
\pi(\varphi)v = \int_{G} \varphi(g)\pi(g)v \, dg
\]

which just has nothing to do with integration, and this is compatible with convolution by formal reason. Then this is a smooth \( \mathcal{H}_{G} \)-module, and this gives an equivalence between the category of smooth(admissible) representations of \( \mathcal{H}(G) \) and the category of smooth(admissible) representations of \( G \).

Proof: For any \( v \in V \), there is an open compact subgroup \( K \) that \( v \in V[e_{K}] \), thus \( e_{K}v = v \), so \( V \) is smooth. For the equivalence of categories, for any \( \mathcal{H}(G) \)-module \( V \), we can define a \( G \)-action by linearly extending the action \( \pi(g)e_{K}v = (\delta_{g} * e_{K})v \). Notice by associativity of representation and smoothness, this is well-defined on all of \( V \), and is a representation of \( G \). Also it is continuous because \( v = e_{K}v \) for some \( K \) thus \( \pi(g)v = (\delta_{g} * e_{K})v = e_{K}v = v \) for any \( g \in K \). Finally these two functors are inverse to each other is also easily checked.
Prop. (III.4.2.10) (Spherical Idempotent). The Hecke algebra $H_{GL(n,F)}$ has a spherical idempotent (I.4.4.7) $e_K$, where the anti-involution is given by transposition.

Then by (III.4.2.9), an irreducible admissible representation of $GL(n,F)$ is spherical iff it contains a $K$-fixed vector. And the dimension of spherical vectors $\leq 1$ by (I.4.4.9).

**Proof:** For the invariance of $H[e_K]$, notice that $H[e_K] = H_K$ is the subspace of $K$-bi-invariant functions on $G$, but we have the $p$-adic Cartan decomposition (III.4.2.5), so the value of $\varphi \in H[e_K]$ are determined by restriction on the diagonal matrices, but they are invariant under transposition. This shows $e_K$ is spherical. □

Prop. (III.4.2.11) (Transpose Invariant Distribution). If $D$ is a distribution on $G$ that is invariant under conjugation, then it is also invariant under transpose.

**Proof:** This follows from (III.1.8.17), as we look at the conjugate action of $G$ on itself, with $\sigma$ being the transposition. Conjugate action is constructive, by (V.9.8.23), and $g\sigma = g^{-t}$, and a matrix is conjugate to its transpose (I.1.4.19). □

Prop. (III.4.2.12) (Gelfand-Kazhdan). Let $G = GL(n,F)$ and $(\pi,V)$ is an irreducible admissible representation of $G$, then:

- If $\pi_1$ is defined by $\pi_1(g) = \pi(g^{-t})$, then $\hat{\pi} \cong \pi_1$.
- suppose $n = 2$ and $\omega$ be the central character of $\pi$, then if $\pi_2$ is defined by $\pi_2(g) = \omega(\deg g)^{-1}\pi(g)$, then $\hat{\pi} \cong \pi_2$.

**Proof:** 1: It is clear that the character of a representation is conjugation invariant, thus by (III.4.2.11) it is transpose invariant.

Now the character of $\pi_1$ is $\chi_1(\varphi) = \chi(\varphi''')$, where $\varphi'''(g) = \varphi(g^{-t})$, and this equals $\chi(\varphi')$ where $\varphi'(g) = \varphi(g^{-1})$, because the character is transpose invariant. It is also clear that $\varphi' = \pi(\varphi)^t$ on a finite space $V^K$, then $\hat{\chi}(\varphi) = \chi(\varphi') = \chi_1(\varphi)$, so by (III.1.8.36) $\pi \cong \pi_1$.

2: If $n = 2$, we use further the property that if $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$g^{-1} = (\deg g)^{-1} \begin{bmatrix} d & -b \\ c & a \end{bmatrix} = (\deg g)^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} g^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}^{-1}$$

then the assertion is clear using item1. □

Cor. (III.4.2.13) (Contragradient of Spherical Representation). If $V$ is a spherical representation of $GL(2,F)$, then the contragradient representation $\hat{V}$ is also spherical.

**Proof:** By the proposition, it suffices to prove for $\pi_1$, but $K$ is stable under transposition, thus this is clear. □

Cor. (III.4.2.14). Let $\pi$ be an admissible representation of $GL(n,F)$, then $\pi$ is irreducible iff $\hat{\pi}$ is irreducible.

**Proof:** If either of them is irreducible, then the other is composing with the conjugation, so all irreducible, because $GL(n,F)^{-t} = GL(n,F)$. □
Finite dimensional Representations

**Prop. (III.4.2.15).** A finite dimensional irreducible admissible representation of $GL(2, V)$ is of dimension 1.

**Proof:** By the no-small-subgroup argument, the kernel of this representation contains an open normal subgroup. But any open normal subgroup of $GL(2, V)$ contains $SL(2, V)$, as it contains all $N(F), \text{diag}(t, t^{-1})$, and also $w_0$ because any matrix with $c \neq 0$ will generate $w_0$ by multiplying $N(F)$.

□

**Lemma (III.4.2.16) (2-Dimensional Representation of $GL(1)$).** There are only two families of isomorphism classes of 2-dimensional representations of $F^*$:

- $t \mapsto \text{diag}(\xi(t), \xi'(t))$, where $\xi, \xi'$ are two quasi-characters of $F^*$.

- $t \mapsto \xi(t) \begin{bmatrix} 1 & v(t) \\ 0 & 1 \end{bmatrix}$, where $\xi$ is a quasi-character of $F^*$.

**Proof:** As $F^*$ is commutative, there exists a 1-dimensional invariant subspace spanned by $x$, on which $F^*$ acts by a quasi-character $\xi$, and consider the quotient space, on which $F^*$ acts by a quasi-character $\xi'$. Choose $y$ that is linearly-independent of $x$, then $\rho(t)y = \xi'(t)y + \lambda(t)x$, and

$$\lambda(tu) = \xi'(u)\lambda(t) + \lambda(u)\xi(t)$$

which is symmetric in $t, u$.

If $\xi \neq \xi'$,

$$\lambda(t)(\xi(u) - \xi'(u)) = \lambda(u)(\xi(t) - \xi'(t))$$

therefore $\lambda(t) = C(\xi(t) - \xi'(t))$, so $z = y - Cx$ is fixed by $\rho(F^*)$.

If $\xi = \xi'$, then $\lambda/\xi$ is an additive character of $F^*$, thus it is trivial on $O^*$, so $\lambda(t) = cv(t)$.

□

3 Whittaker Model and Jacquet Functor

**Prop. (III.4.3.1) (Transpose Invariant Distribution).** If $\Delta$ is a distribution in $D(GL(n, F))$ that satisfies

$$\lambda(u)\Delta = \psi_N(u)^{-1}\Delta, \quad \rho(u)\Delta = \psi_N(u)\Delta,$$

where $\psi$ is defined in (III.2.4.29), then $\Delta$ is stable under involution $\iota: GL(n, F) \rightarrow GL(n, F) : \iota(g) = w_0^g w_0^0$.

**Proof:** Firstly notice that $\iota$ fixes $N(F)$, and $\psi_N(\iota(g)) = \psi_N(g)$, so $\iota(\Delta)$ also satisfies these equations, so we can replace $\Delta$ by $\Delta - \iota(\Delta)$, then assume $\iota(\Delta) = -\Delta$ and prove $\Delta = 0$.

Consider the group $G$ that is a semi-direct product

$$1 \rightarrow N(F) \times N(F) \rightarrow G \rightarrow F_2$$

and $\iota \in F_2$ acts on $N(F) \times N(F)$ by $(u_1, u_2) \mapsto (\iota(u_2)^{-1}, \iota(u_1)^{-1})$.

Define a character $\chi$ on $G$ by $\chi((u_1, u_2)) = \psi_N(u_1)^{-1}\psi_N(u_2), \chi(\iota) = -1$, then $G$ acts on $GL(n, F)$ by

$$\sigma((u_1, u_2)) = \lambda(u_1)\rho(u_2), \quad \sigma(\iota) = \iota$$

then the conditions are summarized into a single condition:

$$\sigma(g)\Delta = \chi(g)\Delta.$$
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We only prove for $n = 2$.

Consider the action of $N(F) \times N(F)$ on $GL(2, F)$ by left-right action, then we can use (III.1.8.16) and (III.1.8.17), because the action is constructive, by (V.9.8.23), and $tN(F)t = N(F)$, and $t$ preserves orbits except for $\text{diag}\{a, d\}, a \neq d$.

But there are no desired distribution on this orbit: this orbit is homeomorphic to $N(F)$ via $u \mapsto u\text{diag}(a, d)$, and the distribution is transferred to a left invariant distribution, thus by (III.1.8.10) it is just the Haar measure

$$\Delta(f) = c_1 \int_{N(F)} f(u \begin{bmatrix} a & b \\ d \\ \end{bmatrix}) \psi_N(u) du.$$ 

We notice using a right-invariant version of (III.1.8.10) that

$$\Delta(f) = c_2 \int_{N(F)} f(\begin{bmatrix} a & b \\ d \\ \end{bmatrix} u \psi_N(u) du = c_2 \int_{N(F)} f(u \begin{bmatrix} a & b \\ d \\ \end{bmatrix}) \psi_N(\begin{bmatrix} a & b \\ d \\ \end{bmatrix}^{-1} u \begin{bmatrix} a & b \\ d \\ \end{bmatrix}) du$$

as $N(F)$ is unimodular. Notice now $c_1 \psi_N(u) = c_2 \psi_N(\begin{bmatrix} a & b \\ d \\ \end{bmatrix}^{-1} u \begin{bmatrix} a & b \\ d \\ \end{bmatrix})$ cannot happen for all $u$, as this implies $c_1 = c_2$ by choosing $u = I$, and then $\psi(x) = \psi(ax/d)$, which is impossible by (III.3.1.2). So if some $u$ this is not equal, then we find a function supported at a nbhd of $u$, then this two distributions cannot be equal. \hfill $\square$

Prop. (III.4.3.2) (Local Multiplicity One Theorem). Define a Whittaker functional as in (III.2.4.29), then for an irreducible admissible representation $(\pi, V)$ of $GL(n, F)$, the space of Whittaker functionals has dimension $\leq 1$.

Proof: Define another representation $\pi'(g) = \pi(\iota(g)^{-1})$, then this representation is isomorphic to $\pi_1$ defined in (III.4.2.12), which is isomorphic to the contragradient of $\pi$, so there is a pairing on $V$ that

$$(\pi(g)\xi, \eta) = (\xi, \pi(\iota(g))\eta).$$

Now for any smooth functional $\Lambda$, there is an element $[\Lambda]$ that $(\xi, [\Lambda]) = \Lambda(\xi)$.

Now for any linear functional $\Lambda$ on $V$ and $\varphi \in \mathcal{H}_G$, we can define another smooth linear function $(\Lambda * \varphi)(\xi) = \Lambda(\pi(\varphi)\xi)$. Then clearly $\varphi * (\varphi_1 * \varphi_2) = (\Lambda * \varphi_1) * \varphi_2$. We need the following lemma:

Lemma (III.4.3.3).

- $\pi(g)[\Lambda * \varphi] = [\Lambda * \rho(\iota(g)^{-1})\varphi].$
- If $L$ is a smooth functional, $[L * \varphi] = \pi(\iota(\varphi))[L].$
- If $L$ is a Whittaker functional, $[\Lambda * \lambda(u)\varphi] = \psi_N(u)[\Lambda * \varphi].$

Proof: 1:

$$(\xi, \pi(g)[\Lambda * \varphi]) = (\pi(\iota(g))\xi, [\Lambda * \varphi]) = (\Lambda * \varphi)(\pi(\iota(g))\xi) = \int_G \Lambda(\pi(h)\pi(\iota(g))\varphi(\xi)dh = \int_G \Lambda(\pi(h)\varphi(\xi)h^u(\iota(g)^{-1}))dh = \Lambda(\varphi(\xi)h^u(\iota(g)^{-1})).$$

2: $(\xi, [L * \varphi]) = (L * \varphi)(\xi) = L(\pi(\varphi)\xi) = (\pi(\varphi)\xi, [L]) = (\xi, \pi(\iota(\varphi))[L]).$ 3:

$$(\xi, [\Lambda * \lambda(u)\varphi]) = (\Lambda * \lambda(u)\varphi)(\xi) = \int_G \Lambda(\pi(g)\varphi(\xi)u^{-1}g)dg$$
Now if Λ₁, Λ₂ are two Whittaker functionals, we will show they are propositional: we define a distribution Δ on G that Δ(φ) = Λ₁([Λ₁ * φ]), then by the lemma above, (III.4.3.1) can be applied to Δ so we have Δ = \iota(Δ).

Next we show for any linear functional Λ, V = \{[Λ * φ]|φ ∈ \mathcal{H}\}: Notice the RHS is G-invariant by (III.4.3.3), and it is not empty by smoothness technique. To go further, need another lemma:

Lemma (III.4.3.4). If φ ∈ \mathcal{H} satisfies Λ₁ * φ = 0, then Λ₂ * φ = 0.

Proof: Firstly, all Λ₁ * π(g)φ = 0, which follows from (III.4.3.3) item1. Hence,

\[
Λ₂([Λ₁ * λ(g)φ]) = Δ(\iota(ρ(\iota(g)^{-1})φ)) = Δ(ρ(\iota(g)^{-1})φ) = Λ₂([Λ₁ * ρ(\iota(g)^{-1})φ]) = 0.
\]

hence by linearity, for any σ ∈ \mathcal{H}, Λ₂([Λ₁ * σ * \iota(φ)]) = 0, which by (III.4.3.3) item2 is equivalent to Λ₂(π(φ)[Λ₁ * φ]) = 0 = (Λ₂ * φ)[Λ₁ * φ], because Λ₁ * σ is smooth. But we know [Λ * φ] can be any v ∈ V, thus Λ₂ * φ = 0.

By the lemma, we can define a map T : V → V : T([Λ₁ * φ]) = [Λ₂ * φ], which is a G-homomorphism by (III.4.3.3), and it is defined on all of V, so Tξ = cξ for some c, then we see Λ₂ = cΛ₁, by a smoothness technique.

Cor. (III.4.3.5) (Local Multiplicity One Equivalent Form). Define Whittaker function space W the same as in (III.2.4.25), then for any irreducible admissible representation (π, V) of GL(n, F), there exists at most one GL(n, F)-subrepresentation W(π, ψ) ⊂ W that is isomorphic to (π, V).

Proof: A Whittaker model is equivalent to a GL(n, F)-homomorphism V → Ind_{N(F)}^G(ψ_N), which by smooth Frobenius reciprocity (III.1.8.32) is equivalent to a N(F)-homomorphism V → ψ_N, which is just a Whittaker functional. So this proposition is equivalent to (III.4.3.2).

Jacquet Functor and the Existence of Whittaker Model

Def. (III.4.3.6) (Jacquet Functor). Let (π, V) be a B(F) module, then regard it as a N(F)-module, and form the coinvariants J(V) = V/V_N. Then J(V) is acted on by T(F), because the set \{π(u)v - v|u ∈ N(F), v ∈ V\} is invariant under T(V)-action as T(F) normalizes N(F). Then the Jacquet functor is the functor V → J(V) from the category of smooth B(F)-modules to the category of smooth T(V)-modules.

Also given the character ψ_N of N(F), we can define similarly the twisted Jacquet functor J_ψ mapping V to the quotient space J(V) = V/(π(u)v - ψ_N(u)v) = V/V_N,ψ, but as a Z(F)-module.

Lemma (III.4.3.7). Let v ∈ V, then

\begin{itemize}
  \item v ∈ V_N iff for sufficiently large n, \int_{N^{-n}(F)} π(u)vdu = 0.
  \item v ∈ V_N,ψ iff for sufficiently large n, \int_{N^{-n}(F)} \overline{ψ_N(u)π(u)v}du = 0.
\end{itemize}

Proof: A more general version is available in Zelevinsky?.
1: If \( v = \pi(u)w - w \), then this integral is 0 when \( v \in N_{-n}(F) \), then the integral is 0 as \( N_{-n}(F) \) is compact thus unimodular. Conversely, if this integral is 0, notice \( \pi(N_m(F))v = v \) for some \( m \) large, then we have

\[
\int_{\alpha \in N_{-n}(F)/N_m(F)} \pi(\alpha)v = 0
\]

Then it is visible that \( v \) is a multiple of sums of \( v - \pi(\alpha)v, \alpha \in N_{-n}(F)/N_m(F) \), thus \( v \in V_N \).

2: this is literally the same as that of 1, make sure to find an \( m \) large that \( \psi_N(N_m(F)) = 1 \). □

Prop. (III.4.3.8) (Jacquet Functor is Exact). \( J \) and \( J_\psi \) are both exact functors.

Proof: The right exactness is clear as \( J \) is a left adjoint. For the left adjointness, it suffices to show that if \( V' \subset V \), then \( V'_N = V' \cap V_N \). But this is clear from the characterization (III.4.3.7). □

Prop. (III.4.3.9). If \( (\pi, V) \) is an irreducible admissible representation of \( GL(n, F) \), then \( \dim J_\psi(V) \leq 1 \).

Proof: This is because a functional is a Whittaker functional if it annihilates \( V_{N,\psi} \), thus equivalent to a functional on \( J(V) \). Then the assertion follows from (III.4.3.2). □

Def. (III.4.3.10) (Sheaf on \( F \) associated to \( V \)). From now we consider \( n = 2 \).

Recall in Tate’s thesis, we know the given character \( \psi \) of \( F \) induces an isomorphism of \( F \) with its dual (III.3.1.2), then the Fourier transform defines an equivalence of \( C_c^{\infty}(F) \cong \mathcal{H}_F \) (X.6.3.9).

As \( N(F) \cong F \), for any representation \( V \) of \( N(F) \), we can define a \( C_c^{\infty}(F) \)-module by

\[
\varphi v = \int_F \hat{\varphi}(x) \pi(\begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix})v \, dx.
\]

Then this module is non-degenerate: Let \( p^a \) be the conductor of \( \psi \), then the Fourier transform of \( \chi_{p^{-k}} \) is \( V(p^{-k})\chi_{p^{n+k}} \). Then for \( k \) large, \( \chi_{p^{-k}}v = V(p^{-k})V(p^{n+k})v \), thus \( v \in C_c^{\infty}(F)V \). Now, we can define \( \mathcal{S}(F) \) the sheaf associated to \( V \), as in (III.1.8.12).

Cor. (III.4.3.11). Let \( V \) be a smooth \( B(F) \)-sheaf and let \( a \in F \), then the stalk

\[
\mathcal{S}(V)_a \cong \begin{cases} J(V) & a = 0 \\ J_\psi(V) \cong J_\psi(V) & a \neq 0 \end{cases}
\]

Proof: By the definition of the stalk, it is \( V \) modulo the subgroup consisting of elements \( v \) that \( \chi_U \cdot v = 0 \), where \( U = a + p^k \) for large \( k \). Back the definition of \( \mathcal{S}(V) \), consider the Fourier transform

\[
\overline{\chi_{a+p^k}(x)} = \overline{\psi(ax)}V(p^k)\chi_{p^{n-k}}(x)
\]

where \( p^n \) is the conductor of \( \psi \). Thus

\[
\chi_{a+p^k}v = C \int_{p^{-n-k}} \overline{\psi(ax)} \pi(\begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix})v \, dx = 0
\]

for large \( k \), which is equivalent to \( v \in V_{N,\psi_a} \) by (III.4.3.7), thus the result follows. □

Cor. (III.4.3.12) (Existence of Whittaker Functional). If \( (\pi, V) \) is a smooth representation of \( GL(2, F) \), then it has a Whittaker functional, unless it factors through the determinantal map. In particular, it is of 1-dimensional if it is irreducible.
Proof: The existence of a Whittaker functional is equivalent to the fact $J_\psi(V) \neq 0$, which is the stalk of the sheaf $S(V)$. If it vanishes, then $S(V)$ is a skyscraper sheaf by (III.4.3.11), and by the correspondence of $S(V)$ and $V$ (III.1.8.12), $V$ equals $\Gamma(F, S(V)) = J(V)$, thus $N(F)$ acts trivially on $V$.

So also all the conjugates of $N(F)$ acts trivially, so $SL(2,F)$ acts trivially, by 4, thus the representation factors through the quotient $F^*$, the rest is clear. \qed

Def. (III.4.3.13) (Kirillov Model). For an irreducible admissible representation of $G = GL(2,F)$ that has a Whittaker functional $\Lambda$, then it has a Whittaker model $W$ consisting of functions $W_v(g) = \Lambda(\pi(g)v)$, and we define the Kirillov Model $K$ of $V$ as the space of functions $F^* \rightarrow \mathbb{C}$ that

$$\varphi_v(a) = W_v\left(\begin{bmatrix} a & \cdot \\ c & d \end{bmatrix}\right).$$

Then $G$ acts on $K$ by acting on the subscript. This representation is isomorphic to $V$ because $V$ is irreducible.

**Iwahori Subgroup**

Def. (III.4.3.14) (Iwahori Subgroup). Let $a$ be an ideal of $\mathcal{O}$, then we can define

$$K_0(a) = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K | c \equiv 0 \mod a \}, \quad K_1(a) = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K | c \equiv 0, a \equiv d \equiv 1 \mod a \}.$$ 

In particular, $K_0(p)$ are called the Iwahori subgroup, a vector is called Iwahori fixed iff it is $K_0(p)$-fixed.

Prop. (III.4.3.15) (Iwahori Factorization). Define

$$N(a) = \{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} | x \in a \}, \quad N_-(a) = \{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} | x \in a \}, \quad T(a) = \{ \begin{bmatrix} t_1 & t_2 \\ 1 & 1 \end{bmatrix} | t_1, t_2 \in \mathcal{O}^*, t_1, t_2 \equiv 1 \mod a \}.$$ 

Then if $a$ is a proper ideal,

$$K_0(a) = N_-(a)T(\mathcal{O})N(\mathcal{O}), \quad K_1(a) = N_-(a)T(a)N(\mathcal{O}).$$

Denote $K_0 = K_0(a)$ or $K_1(a)$, and $T_0 = T(\mathcal{O})$ or $T(a)$ respectively, then we can use (X.6.1.30) to decompose the Haar measure on $K_0$ as

$$\int_{K_0} \varphi(k) dk = \int_{N_-(a)} \int_{T_0} \int_{N(\mathcal{O})} \varphi(n_{-t_0}n) dndt_0dn_-.$$

The proof is easy.

Cor. (III.4.3.16) (Iwahori-Bruhat Decomposition).

$$GL(2,F) = B(F)K_0(p) \prod B(F)w_0K_0(p).$$
III.4. REPRESENTATIONS OF $GL(2)$ OVER $P$-ADIC FIELDS

**Proof:** By pulling the Bruhat decomposition of $GL(2, \mathcal{O}/p)$ to $K = GL(2, \mathcal{O})$, we have

$$K = K_0(p) \cup K_0(p)w_0K_0(p).$$

The Iwahori factorization shows $K_0(p) = (K_0(p) \cap B(F)) \cap N_-(p)$. Then this implies

$$K = K_0(p) \cup (K_0(p) \cap B(F))w_0K_0(p),$$

because $w_0^{-1}N_-(p)w_0 \in K_0(p)$. Then Iwasawa decomposition $G = BK$ gives the desired result. Notice this decomposition is clearly disjoint. \(\Box\)

**Lemma (III.4.3.17) (Jacquet).** If $(\pi, V)$ is a smooth representation of $GL(2, F)$, then using the notation as in (III.4.3.15) where $a$ is a proper ideal, $V^{K_0}$ and $V^{N_-(a)T_0}$ have the same image in the Jacquet module $J(V)$.

**Proof:** One inclusion is trivial, for the other, if $x \in V^{N_-(a)T_0}$ then $x_1 = \int_{K_0} \pi(k)xdk$ lies in $V^{K_0}$, so it suffices to show $x$ and $x_1$ have the same image in $J(V)$. But

$$x_1 = \int_{N_-(a)} \int_{T_0} \int_{N(O)} \pi(nt_0n_-)dnt_0dn_- = \int_{N(O)} \pi(n)xdn$$

and notice $p(\pi(n)x) = p(x)$. \(\Box\)

**Prop. (III.4.3.18) (Jacquet Module Admissibility).** Let $(\pi, V)$ be a smooth representation of $GL(2, F)$, then

- Using the notation as in (III.4.3.15), if $a$ is a proper ideal, then the projection map induces a surjection of $V^{K_0} \to J(V)^{T_0}$.
- If $(\pi, V)$ is admissible, then $J(V)$ is also admissible representation of $T(F)$.

**Proof:** The second follows from the first because we can choose $T_0 = T(a)$ to be arbitrarily small, then it is of f.d., thus admissible.

For the first, notice that any $x \in J(V)^{T_0}$ is an image of a $x_1 \in V^{T_0}$, because $p(\pi(t)x) = \pi(t)p(x) = p(x)$, thus we can choose $x_1 = \frac{1}{v^{T_0}} \int_{T_0} \pi(t)xdt$.

Thus for any f.d. subspace $\overline{U}$ of $J(V)^{T_0}$, we can find a f.d. $U \subset V^{T_0}$ that is mapped isomorphically onto $\overline{U}$. Now $U$ is fixed by some $N_-(p^n)$ for $n$ large, so $U$ is fixed by $N_-(p^n)T_0$. Notice $a = p^n$ for some $m$, and

$$\pi(d)N_-(p^n)T_0\pi(d)^{-1} = N_-(a)T_0, \quad d = \begin{bmatrix} a^{n-m} & 1 \\ 0 & 1 \end{bmatrix},$$

so $\pi(d)U$ is stabilized by $N_-(a)T_0$. Hence by lemma (III.4.3.17), $\pi(d)p(U) = p(\pi(d)U) \subset p(V^{K_0})$, so the dimension of $\overline{U}$ is bounded by dimension of $V^{K_0}$, so we can choose $\overline{U}$ just to be $J(V)^{T_0}$. Now $\pi(d)$ commutes with $T(F)$, so we have $\pi(d)J(V)^{T_0} = J(V)^{T_0} \subset p(V^{K_0})$. The reverse containment is clear. \(\Box\)

4 **Principal Series Representations**

**Def. (III.4.4.1) (Principal Series Representations).** Let $F$ be a locally profinite field, $B(F)$ be the Borel set of upper triangular matrices in $GL(2, F)$. Let $\chi_1, \chi_2$ be quasi-characters of $F^*$, then define two quasi-characters $\chi, \chi'$ on $B(F)$:

$$\chi \left( \begin{bmatrix} y_1 & * \\ y_2 & * \end{bmatrix} \right) = \chi_1(y_1)\chi_2(y_2), \quad \chi' \left( \begin{bmatrix} y_1 & * \\ y_2 & * \end{bmatrix} \right) = \chi_2(y_1)\chi_1(y_2).$$

4 **Principal Series Representations**

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$$\chi \left( \begin{bmatrix} y_1 & * \\ y_2 & * \end{bmatrix} \right) = \chi_1(y_1)\chi_2(y_2), \quad \chi' \left( \begin{bmatrix} y_1 & * \\ y_2 & * \end{bmatrix} \right) = \chi_2(y_1)\chi_1(y_2).$$
Then define the principal series representation of $G$ as

$$B(\chi_1, \chi_2) = \text{Ind}_{B(F)}^{GL(2, F)}(\chi) \quad (\text{III.1.8.31})$$

whenever it is irreducible, and its isomorphism class is denoted by $\pi(\chi_1, \chi_2)$.

**Lemma (III.4.4.2) (Where Does This map Come From?)**: We have a map $P : C_c^\infty(GL(2, F)) \to B(\chi_1, \chi_2)$:

$$(P\varphi)(g) = \int_{B(F)} \varphi(b^{-1}g)(\Delta_B^{-1}\chi)(b)db$$

Then this map is intertwining and surjective. Moreover, we have

$$P(\lambda(b)^{-1}\varphi) = (\delta^{-1/2}\chi)(b)P(\varphi), \quad b \in B(F)$$

**Proof**: It is easily verified to be a map and it is intertwining. For the surjectivity, if $f \in V$, define $\varphi = \chi_K f$, then $P\varphi = V(K \cap B(F))f$: it suffices to check this on $K$ by Iwasawa decomposition (III.4.2.4), and this is clear. The last equation is also clear.

**Prop. (III.4.4.3)**. The representation $B(\chi_1, \chi_2)$ admits at most one Whittaker functional.

**Proof**: Let $\Lambda : V \to \mathbb{C}$ be a Whittaker functional, then we define a distribution $\Delta$ on $GL(2, F)$ as $\Delta(\varphi) = \Lambda(P\varphi)$. Then

$$\lambda(b)\Delta = (\delta^{-1/2}\chi)(b)\Delta, \quad b \in B(F), \quad \rho(n)\Delta = \psi_N(n)^{-1}\Delta, \quad n \in N(F)$$

by (III.4.4.2). Because $P$ is surjective (III.4.4.2), it suffices to show that these $\Delta$ are unique up to scalar.

Consider the left-right action of $B(F) \times N(F)$ on $GL(2, F)$, then there are two orbits by Bruhat decomposition, $B(F)$ and $GL(2, F) - B(F)$. Then the distribution on $GL(2, F) - B(F)$ must be of the form

$$\Delta(\varphi) = c \int_{B(F)} \int_{N(F)} \varphi(bw_0n^{-1})\psi_N(n)(\delta^{1/2}\chi^{-1})(b)dbdn$$

by (III.1.8.10).

As for $B(F)$, the same reasoning shows we have a formula for $\Delta$, but it doesn’t satisfy the second condition, so there are no distribution on $B(F)$. Finally, (III.1.8.6) gives us the result.

**Prop. (III.4.4.4)**. The contragradient of $B(\chi_1, \chi_2)$ is $B(\chi_1^{-1}, \chi_2^{-1})$.

**Proof**: If $f \in B(\chi_1, \chi_2), f' \in B(\chi_1^{-1}, \chi_2^{-1})$, then the pairing $(f, f') = \int_K f(k)f'(k)dk$ is $G$-invariant by (X.6.5.5), so this defines a smooth functional $l_f$, and it is non-degenerate, the mapping $f' \mapsto l_{f'}$ is injective (by letting $f = 1/f'$ on a $B(F)$-orbit that $f'$ is non-zero) from $B(\chi_1^{-1}, \chi_2^{-1})$ to $B(\chi_1, \chi_2).$

Now by symmetry the other side is also injective, and we are done because $\widetilde{V} = \text{V(III.1.8.24)}$.

**Prop. (III.4.4.5) (Principal Series)**. $B(\chi_1, \chi_2)$ is irreducible except the following two cases:

- If $\chi_1\chi_2^{-1} = |\cdot|^{-1}$, then $B(\chi_1, \chi_2)$ has a 1-dimensional invariant subspace and the quotient representation is irreducible.
- If $\chi_1\chi_2^{-1} = |\cdot|$, then $B(\chi_1, \chi_2)$ has an irreducible 1-dimensional invariant subspace of codimension 1.
Proof: 2 is dual to 1, as for 1: Firstly we assume $\mathcal{B}(\chi_1, \chi_2)$ has a 1-dimensional subspace, then if $V = \{f\}$ is an invariant subspace, then $\pi(g)f = \rho(\det(g))f$ for some quasi-character $\rho$ of $F^*$. Now consider the fact $f \in \mathcal{B}(\chi_1, \chi_2)$, take $b = \text{diag}(y, y^{-1})$, then $(\delta^{1/2}\chi)(b) = 1$, showing $\chi_1\chi_2^{-1} = |\cdot|^{-1}$.

Next we prove non-trivial subspace is of dimension or codimension 1 if they exist: Let $V'$ be the invariant subspace and $V''$ the quotient space, by the exactness of Jacquet functor (III.4.3.8) and (III.4.3.9), at least one of $J_\psi(V'), J_\psi(V'')$ vanishes. If $J_\psi(V') = 0$, then by (III.4.3.12) it factors through the determinant map, thus it has a 1-dimensional invariant space. If $J_\psi(V'') = 0$, then we can use (III.4.4.4) and (III.1.8.22) to dualize.

Finally, if $\chi_1\chi_2^{-1} = |\cdot|^{-1}$, let $\chi = \chi' |\cdot|^{-1/2}$, then $f(g) = \chi(\det(g))$ is an invariant 1-dimensional subspace, and the quotient representation is irreducible by considering Jacquet functor again. And the second case follow by duality. □

**Prop. (III.4.4.6)**. $\text{Hom}(\mathcal{B}(\chi_1, \chi_2), \mathcal{B}(\mu_1, \mu_2)) \neq 0$ only if $\{\chi_1, \chi_2\} = \{\mu_1, \mu_2\}$.

**Proof**: By smooth Frobenius reciprocity (III.1.8.32),

$$\text{Hom}_G(\mathcal{B}(\chi_1, \chi_2), \mathcal{B}(\mu_1, \mu_2)) \cong \text{Hom}_{B(F)}(\mathcal{B}(\chi_1, \chi_2), \delta^{1/2}\mu).$$

The proof below is similar to that of (III.4.4.3): For such a map $\Lambda$, we define a distribution $\Delta(\varphi) = \Lambda(P(\varphi))$, then

$$\lambda(b)\Delta = (\delta^{-1/2}\chi)(b)\Delta, \rho(b)\Delta = (\Delta^{-1/2}\mu^{-1})(b)\Delta, b \in B(F).$$

So by the exact sequence (III.1.8.6), such distribution exists on either $B(F)$ or $GL(2, F) \backslash B(F)$.

If such a distribution exists on $GL(2, F) \backslash B(F)$, then noticing $\rho(n)\Delta = \Delta$ for $n \in N(F)$, by (III.1.8.10)

$$\Delta(\varphi) = \int_{B(F)} \int_{N(F)} \varphi(bw_0n^{-1})(\Delta^{-1/2}\chi^{-1})(b)dbdn,$$

then we apply $\rho(t)$ with $t$ diagonal, then

$$(\Delta^{-1/2}\mu^{-1})(t)\Delta(\varphi) = (\rho(t)\Delta)(\varphi) = \int_{N(F)} \int_{B(F)} \varphi((bw_0t^{-1}w_0^{-1})w_0(tnt^{-1})^{-1})(\Delta^{-1/2}\chi^{-1})(b)dbdn$$

$$= \Delta(t)^{-1}(\Delta^{-1/2}\chi^{-1})(w_0tw_0^{-1}).$$

Notice that $\Delta(t) = \Delta(w_0tw_0^{-1})^{-1}$, thus $\mu(t) = \chi(w_0tw_0^{-1})$, which means $\chi_i = \mu_{i+1}$.

Similarly, if such a distribution exists on $B(F)$, then

$$\Delta = \int_{B(F)} \varphi(b)(\Delta^{-1/2}\chi^{-1})(b)db.$$

We apply $\rho(b)$, then $\Delta^{-1/2}\chi^{-1}(b) = (\Delta^{-1/2}\mu^{-1})(b)$, so $\chi_i = \mu_i$. □

**Prop. (III.4.4.7)** (Intertwining Integral). Let $\xi_i$ be two unitary characters of $F^*$, and $\chi_i = |\cdot|^h\xi_i$, $(\pi, V) = \mathcal{B}(\chi_1, \chi_2), (\pi', V') = \mathcal{B}(\chi_2, \chi_1)$. Define for $f \in V$,

$$Mf : GL(2, F) \to \mathbb{C} : Mf(g) = \int_{N(F)} f(w_0ug)du$$

then if $\text{Re}(s_1 - s_2) > 0$, the integral is absolutely convergent, $Mf \in V'$, and $M$ is a nonzero intertwining, so $V \cong V'$ if they are both isomorphic.
Proof: Should compare this proof with (III.2.4.16).

For the convergence, it suffices to check for \( \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \), \( |x| \) large, but then

\[
f( \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} ) = f( \begin{bmatrix} x^{-1} & -1 \\ x^{-1} & 1 \end{bmatrix} ) = |x|^{-1} (\chi_1^{-1} \chi_2)(x) f( \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} )
\]

and because \( f \) is locally constant, when \( |x| \) is large, \( f( \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} ) = f(g) \), thus the convergence is dominated by

\[
\int_{|x| > qN} |x|^{-1} (\chi_1^{-1} \chi_2)(x) dx = \int_{|x| > qN} |x|^{-s_1+s_2-1} dx,
\]

which converges for \( \Re(s_1 - s_2) > 0 \).

To show \( Mf \in V' \), we need to check

\[
Mf( \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} ) = Mf(g), \quad Mf( \begin{bmatrix} y_1 & y_2 \\ 0 & 1 \end{bmatrix} ) = |y_1/y_2|^{1/2} \chi_2(y_1) \chi_1(y_2) Mf(g).
\]

The first is trivial and for the second:

\[
Mf( \begin{bmatrix} y_1 & y_2 \\ 0 & 1 \end{bmatrix} ) = \int_F f( \begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix} ) \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_2 & iy_1^{-1}x \\ 1 & 1 \end{bmatrix} g dx
\]

so it is clear.

Finally \( M \) is clearly intertwining, and it is not trivial by looking at the function \( f \):

\[
f(x) = |y_1/y_2|^{1/2} \chi_1(y_1) \chi_2(y_2) \phi(x), \quad \text{where } g = \begin{bmatrix} y_1 & z \\ y_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}
\]

and vanish if \( g \in B(F) \). Notice the representation of \( g \) is unique by Bruhat decomposition. Then \( Mf(1) = 1 \). \( \square \)

Prop. (III.4.4.8) (Analytic Continuation of Intertwining Integral). Let \( \xi_i \) be two unitary characters of \( F^* \), \( \chi = |\cdot|^{\nu_1} \xi_1, (\pi_{s_1,s_2}, V_{s_1,s_2}) = B(\chi_1, \chi_2), (\pi'_{s_2,s_1}, V'_{s_2,s_1}) = B(\chi_2, \chi_1) \).

Notice that by Iwasawa decomposition, an arbitrary \( f \in B(\chi_1, \chi_2) \) is determined by its restriction on \( K \), and if a function \( f_0 \) on \( K \) satisfies

\[
f_0( \begin{bmatrix} y_1 & x \\ y_2 & 1 \end{bmatrix} ) = \xi_1(y_1) \xi_2(y_2) f_0(k), y_1, y_2 \in O^*,
\]

then \( f_0 \) can extends uniquely to an element \( f_{s_1,s_2} \in V_{s_1,s_2} \) for any \( s_1, s_2 \), called the flat sections of \( f_0 \).

Then the intertwining integral \( Mf_{s_1,s_2} \) defined in (III.4.4.7) has an analytic continuation to all \( s_1, s_2 \) that \( \chi_1 \neq \chi_2 \), and defines a nonzero operator \( V_{s_1,s_2} \to V'_{s_2,s_1} \).

Proof: The proof is parallel to that of (III.4.4.7).

\[
Mf_{s_1,s_2}(g) = \int_{|x| \leq qN} f_{s_1,s_2}( \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} ) dx + \int_{|x| > qN+1} |x|^{-s_1+s_2-1} (\xi_1^{-1} \xi_2)(x) dx f_{s_1,s_2}(g)
\]
the first term can easily be extended, and the second term vanishes if $\xi_1^{-1}\xi_2$ ramifies, and equals something like a multiple of $\frac{(\chi_1^{-1}\chi_2(\sigma))^{n+1}}{1-\chi_1^{-1}\chi_2(\sigma)}$, which extends unless $\chi_1 = \chi_2$.

For the intertwining property, it is because equalities maintain along analytic continuation, in particular, it is non-trivial as $M_{f_{s_1,s_2}}(1) = 1$.

**Cor. (III.4.4.9).** If $\mathcal{B}(\chi_1,\chi_2)$ is irreducible, then $\mathcal{B}(\chi_1,\chi_2) \cong \mathcal{B}(\chi_2,\chi_1)$. So $\pi(\chi_1,\chi_2) \cong \pi(\chi_2,\chi_1)$, $\sigma(\chi_1,\chi_2) \cong \sigma(\chi_2,\chi_1)$, and there are no other isomorphisms between these representations.

**Proof:** It suffices to show there are no other isomorphisms, and this is by (III.4.4.6).

**Prop. (III.4.4.10) (General Principal Representation).** More generally, we can define a representation $\mathcal{B}(\chi_1,\ldots,\chi_n)$ of $\text{GL}(n,F)$, $F$ is a local or global field. Then $\mathcal{B}(\chi_1,\ldots,\chi_n)$ is irreducible unless $\chi_i\chi_j^{-1} = |x|$ for some $i \neq j$. And in any case, the composition factors of $\mathcal{B}(\chi_1,\ldots,\chi_n)$ is independent of the order of $\chi_k$.

**Proof:** Cf. [Bump, P378].

**Prop. (III.4.4.11) (Jacquet Module of $\mathcal{B}(\chi_1,\chi_2)$).** Let $\chi, \chi'$ be quasi-characters of $F^*$, and let $\chi, \chi'$ be the quasi-characters of $T(F)$ that

$$\chi(t_1/ t_2) = \chi_1(t_1)\chi_2(t_2), \quad \chi'(t_1/ t_2) = \chi_2(t_1)\chi_1(t_2).$$

Then the representation of $T(F)$ on the Jacquet module of $\mathcal{B}(\chi_1,\chi_2)$ is equivalent to the

$$t \mapsto \left\{ \begin{array}{cl} (\Delta^{1/2}\chi)(t) & , \chi_1 \neq 2 \\ (\Delta^{1/2}\chi')(t) & , \chi_1 = \chi_2 = \chi \end{array} \right.$$ 

**Proof:** Firstly $J(V)$ is of dimension 2, or equivalently there are two $N(F)$-invariant functionals on $V$. The following proof is similar to that of (III.4.4.3): given such a functional $\Lambda$, we can define a distribution $\Delta(\varphi) = \Lambda(P\varphi)$, where $P$ is defined in (III.4.4.2), then because of Bruhat decomposition and (III.1.8.10), the distribution is determined on $B(F)$ and $GL(2,F)\backslash B(F)$ as in (III.4.4.2), thus there are at most two such functional. For the existence, we can take $\Lambda_1(\varphi) = \varphi(1)$, and

$$\Lambda_2(\varphi) = \int_F (\varphi(\begin{bmatrix} 1 & -1 \\ x & 1 \end{bmatrix}) - h(x)\varphi(1))dx$$

where $h(x) = |x|^{-1}(\chi_1^{-1}\chi_2)(x)$ if $|x| > 1$ and 0 otherwise. Notice by formula in the proof of (III.4.4.7), this integral is compactly supported for any $\varphi$, and it is $N(F)$-invariant, as $\int_F h(x+a) - h(x)dx = 0$.

To show these two functional are linearly independent, we consider $f_2$ the function defined in (III.4.4.7), then $\Lambda_1(f_2) = 0, \Lambda_2(f_2) = 1$, and for $f_1$, Cf. [Bump, Ex.5.3]?.

Next we consider $J(V)$ as a 2-dimensional $T(F)$-representation must be of the two forms in (III.4.2.16), it suffices to distinguish these two cases.

Denote $V = \mathcal{B}(\chi_1,\chi_2)$. Consider for any quasi-character $\mu$ of $T(F)$,

$$\text{Hom}_{T(F)}(J(V), \Delta^{1/2}\mu) \cong \text{Hom}_{B(F)}(V, \Delta^{1/2}\mu) \cong \text{Hom}_{GL(2,F)}(V, \mathcal{B}(\mu_1,\mu_2))$$
(III.4.4.6)(III.4.4.9) can be used, so if \( \chi_1, \chi_2 \), there are two \( \mu \) that can make the Hom group non-vanish, so it is the first case. If \( \chi_1 = \chi_2 \), then \( V \) is irreducible, and there is only one \( \mu \) that makes this Hom group non-vanish, and the Hom group is of dimension 1 by Schur’s lemma, so it is the second case.

**Prop. (III.4.4.12).** Let \((\pi, V)\) be an irreducible admissible representation of \( GL(2, F) \), then the dimension of the Jacquet module of \( V \) is at most 2, and if it is nonzero, then \( \pi \) is isomorphic to a subrepresentation of \( \mathcal{B}(\chi_1, \chi_2) \).

**Proof:** Cf. [Bump, P512].

**Prop. (III.4.4.13) (Whittaker functional of \( \mathcal{B}(\chi_1, \chi_2) \)).** There is a Whittaker functional on \( \mathcal{B}(\chi_1, \chi_2) \) defined by

\[
\Lambda(f) = \int_F f(w_0 \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}) \psi(-x) dx.
\]

This integral is absolutely convergent if \( \chi \) is dominant, by method the same as in the proof of (III.4.4.7). And this can also be extended to all \( \chi \) (as flat section (III.4.4.8)) by defining

\[
\Lambda(f) = \lim_{k \to \infty} \int_{p^{-k}} f(w_0 \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}) \psi(-x) dx.
\]

This makes sense because it stabilize as \( k \to \infty \): when \( k \) is large,

\[
\int_{p^{-k-1}p^{-k}} f(w_0 \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}) \psi(-x) dx = \int_{p^{-k-1}p^{-k}} \psi(-x) dx q^{-k-1}(\chi_1^{-1}\chi_2)(\varpi)^{-k-1} f(1)
\]

which vanishes as \( \psi \) is an additive character.

5 Spherical Representations

**Prop. (III.4.5.1) (Hecke Operators).** Let \( T(p^k) \in \mathcal{H}_K \) be the characteristic function of the set of all \( g \in M_2(O) \) that the ideal generated by \( \det g \) is \( p^k \), and \( R(p) \) the characteristic function of \( \varpi K \), then for \( k \geq 1 \),

\[
T(p)T(p^k) = T(p^{k+1}) + qR(p)T(p^{k-1}).
\]

Notice the algebra is given by convolution.

**Proof:** Both sides are supported on matrices whose determinant generate \( p^{k+1} \), so by Cartan decomposition (III.4.2.5), it suffices to check on \( \text{diag}(\varpi^{k+1-r}, \varpi^r) \).

It is clear

\[
T(p^{k+1})(\begin{bmatrix} \varpi^{k+1-r} & \\ \varpi^r \end{bmatrix}) = 1, \quad 0 \leq r \leq k + 1, \quad R(p)T(p^{k-1})(\begin{bmatrix} \varpi^{k+1-r} & \\ \varpi^r \end{bmatrix}) = \begin{cases} 0 & r = 0, k + 1 \\ q & 1 \leq r \leq k \end{cases}
\]

And notice that

\[
K \begin{bmatrix} \varpi & \\ 1 & \end{bmatrix} K = \begin{bmatrix} 1 & \\ \varpi & \end{bmatrix} K \prod_b \prod_{b \equiv_p 1} \begin{bmatrix} \varpi & b \\ 1 & \end{bmatrix} K,
\]
so
\[
(T(p) \ast T(p^k))(g) = \int_{GL(2,F)} T(p)(h)T(p^k)(h^{-1}g)dh = \int_K \begin{bmatrix} \varpi & 0 \\ 0 & 1 \end{bmatrix} K T(p^k)(h^{-1}g)dh
\]
\[
= T(p^k)\left(\begin{bmatrix} 1 & 0 \\ 0 & \varpi \end{bmatrix}^{-1} g\right) + \sum_{b \mod p} T(p^k)\left(\begin{bmatrix} \varpi & b \\ 0 & 1 \end{bmatrix}^{-1} g\right)
\]

Then we can calculate the Smith form and conclude that
\[
(T(p) \ast T(p^k))\left(\begin{bmatrix} \varpi^{k+1-r} & \varpi^r \\ \varpi^r & \varpi^{k+1-r} \end{bmatrix}\right) = \begin{cases} 1 & r = 0, k + 1 \\ q + 1 & 1 \leq r \leq k \end{cases}
\]

\[\square\]

**Prop. (III.4.5.2).** The spherical Hecke algebra \(\mathcal{H}_K\) is generated by \(T(p), R(p)\) and \(R(p)^{-1}\). Notice it is commutative by (III.4.2.10)(1.4.4.7), and \(R(p)\) is invertible.

**Proof:** By Cartan decomposition (III.4.2.5), a basis for \(\mathcal{H}_K\) consists of characteristic functions of \(K \left[ \begin{array}{cc} \varpi^n & 0 \\ 0 & \varpi^n \end{array} \right] K, n \geq m\), which equals \(R(p)^m\) times the characteristic function of \(K \left[ \begin{array}{cc} \varpi^{m-n} & 0 \\ 0 & 1 \end{array} \right] K\), and this is just \(T(p^{n-m}) - R(p)R(p^{n-m-1})\). So it suffices to show \(T(p^k)\) is generated by \(T(p), R(p)\), which follows from (III.4.5.1).

**Def. (III.4.5.3) (Normalized Spherical Vector).** Let \(\chi_1, \chi_2\) be nonramified quasi-characters of \(F^*\), then \(B(\chi_1, \chi_2)\) contains a \(K\)-fixed vector \(\varphi_K\) that is defined to be

\[
\varphi_K(bk) = \Delta^{1/2} \chi(b), b \in B(F), k \in K
\]

Notice this is well-defined, because for \(u \in B(F) \cap K, \Delta^{1/2} \chi(u) = 1\). \(\varphi_K\) is \(K\)-spherical, and we refer to it the \textit{normalized spherical vector} in \(V\).

Notice when \(\chi_2 = \chi_1 \cdot |\cdot|\), this spherical vector just spans the 1-dimensional invariant subspace in (III.4.4.5).

**Prop. (III.4.5.4) (Satake Isomorphism).** Let \(\chi_1, \ldots, \chi_n\) be nonramified, then the representation \(B(\chi_1, \ldots, \chi_n)\) contains a spherical vector. It has a unique composition factor that is spherical, and every spherical representation of \(GL(n, F)\) arises in this way.

In this case, define \(\alpha_i = \chi_i(\varpi)\) is \(\chi_i\) is unramified and 0 otherwise. We call the two numbers \(\alpha_1, \alpha_2\) the \textbf{Satake parameters} of \(\pi(\chi_1, \chi_2)\).

When \(n = 2\), the quasi-character \(\chi\) on \(T(F)\) is called \textbf{regular} if \(\chi_1 \neq \chi_2\), and \textbf{dominant} if \(|\alpha_1/\alpha_2| < 1\).

**Proof:** Cf.[Bump, P378].

**Cor. (III.4.5.5).** When \(n = 2\), let \(\chi_1, \chi_2\) be two nonramified quasi-characters of \(F^*\), \(\alpha_i = \chi_i(\varpi)\), then the Hecke operators (III.4.5.1) acts by

\[
T(p)\varphi_K = q^{1/2}(\alpha_1 + \alpha_2)\varphi_K, \quad R(p) = \alpha_1 \alpha_2 \varphi_K.
\]
Proof: We can prove directly for $n = 2$: Because $T(p), R(p) \in H_K$, there image are spherical, thus are scalars of $\varphi_K$, because spherical vectors are unique (I.4.4.9). Then it suffices to calculate the value at 1: Using the representative of $K \begin{bmatrix} \overline{\omega} & \gamma \\ 1 & 1 \end{bmatrix} K$ over $K$ as in the proof of (III.4.5.1),

$$
(T(p)\varphi_K)(1) = \int_K \begin{bmatrix} \overline{\omega} & \gamma \\ 1 & 1 \end{bmatrix} K \varphi_K(g)dg = \sum_{\gamma \in K} \varphi_K(\lambda) = (\Delta^{1/2}\chi)\begin{bmatrix} 1 \\ \overline{\omega} \end{bmatrix} + q(\Delta^{1/2}\chi)\begin{bmatrix} \overline{\omega} \\ 1 \end{bmatrix} = q^{1/2}(\alpha_1 + \alpha_2)
$$

and similarly for $R(p)$. □

Cor. (III.4.5.6) (Spherical Representation of $GL(2, F)$). Every irreducible admissible spherical representation $(\pi, V)$ of $GL(2, F)$ is either 1-dimensional with nonramified character through $\det(g)$, or isomorphic to $\pi(\chi_1, \chi_2)(III.4.4.1)$ for some unramified quasi-characters $\chi_1, \chi_2$.

Proof: Let $\xi$ be the character of $H_K$ associated to $(\pi, V)$, and $\lambda, \mu$ be the eigenvalue of $T(p), R(p)$. Notice $R(p)$ is invertible (III.4.5.2), so $\mu \neq 0$. Let $\alpha_i$ be the roots of the equation $X^2 - q^{-1/2}\lambda X + \mu = 0$, and $\chi_i$ be nonramified quasi-characters of $F^*$ that $\chi_i(\overline{\omega}) = \alpha_i$. Then by (III.4.5.5), the character of the Hecke algebra associated to $B(\chi_1, \chi_2)$ coincides with that of $(\pi, V)$.

So if $B(\chi_1, \chi_2)$ is irreducible, then by (I.4.4.10), $V \cong B(\chi_1, \chi_2)$. And if $B(\chi_1, \chi_2)$ is not irreducible, then we may assume $\chi_2 = \chi_1 \cdot | \cdot |$, so $\varphi_K$ spans an invariant subspace of $B(\chi_1, \chi_2)(III.4.5.3)$, so by (I.4.4.10) again $(\pi, V)$ is isomorphic to this representation, and $\pi(g)\varphi_K = \chi(\det(g))\varphi_K$, where $\chi = \chi_1 \cdot | \cdot |^{1/2}$. □

Prop. (III.4.5.7) (Intertwining Operator on Spherical Vector). For $\chi_1, \chi_2$ nonramified with Satake parameter $\alpha_i$, we have

$$M\varphi_{K, \chi} = \frac{1 - q^{-1}\alpha_1\alpha_2^{-1}}{1 - \alpha_1\alpha_2^{-1}} \varphi_{K, \chi'}.$$

Proof: Clearly this equation is true up to scalar because $M$ is intertwining, so it suffices to calculate $(M\varphi_K)(1)$. For this, we assume $\chi$ is dominant (III.4.5.4) because we can use analytic continuation. Then

$$(M\varphi_K)(1) = \int_F \varphi_K(w_0 \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix})dx.$$

The integral is 1 on $\mathcal{O}$, and for $m > 0$, on $p^{-m}\mathfrak{p}^{-m+1}$, by (III.4.4.7), it equals $q^{-m}\alpha_1^m\alpha_2^{-m}q^m(1 - q^{-1})$, so the total sum is $1 + (1 - q^{-1})\alpha_1\alpha_2/(1 - \alpha_1\alpha_2)$. □

Prop. (III.4.5.8). If $\chi_1, \chi_2$ are nonramified, $V = B(\chi_1, \chi_2)$, $K_0(p)$ is the Iwahori subgroup (III.4.3.14), then the composition $V^{K_0(p)} \hookrightarrow V \rightarrow J(V)$ is an isomorphism.
III.4. REPRESENTATIONS OF GL(2) OVER P-ADIC FIELDS

*Proof:* Firstly notice $V^{K_0(p)}$ has dimension $\leq 2$, because of the decomposition (III.4.3.16) and definition, and $J(V)$ has dimension 2 by (III.4.4.11), so it suffices to show the map is surjective. The image is just $J(V)^{T(O)}$ by (III.4.3.18). But by (III.4.4.11) and the fact $\chi_i$ are nonramified, all of $J(V)$ are $T(O)$-fixed, thus $J(V)^{T(O)} = J(V)$. \hfill $\Box$

**Cor. (III.4.5.9) (Casselman Basis).** When $\chi_1 \neq \chi_2$, we can easily find a basis of $V^{K_0(p)}$, that is

$$L_1(\varphi) = \varphi(1), \quad L_0(\varphi) = (M\varphi)(1)$$

where $M$ is intertwining integral defined in (III.4.4.8). These are $N(F)$-invariant, thus define functionals on $J(V) \cong V^{K_0(p)}$. They are linearly independent checked on $T(F)$.

Then the dual basis $\varphi_0, \varphi_1 \in V^{K_0(p)}$ are called the **Casselman basis**. There is a simple basis of $\varphi_0$: using Iwahori-Bruhat decomposition (III.4.3.16),

$$\varphi_0(g) = \begin{cases} (\Delta^{1/2} \chi)(b) & g \in B(F)w_0K_0(p) \\ 0 & \text{otherwise} \end{cases}$$

*Proof:* It is easily verified that this formula is well-defined and defines a Iwahori-fixed vector, so it suffices to valuate it by $L_i$. Clearly $L_1(\varphi_0) = 0$, and for $L_0(\varphi_0)$, notice that

$$w_0 \begin{bmatrix} 1 \\ x \\ 1 \end{bmatrix} \in B(F)w_0K_0(p)$$

iff $x \in O$, so $L_0(\varphi_0) = 1$. \hfill $\Box$

**Lemma (III.4.5.10).** When $\chi_1 \neq \chi_2$ nonramified, the function

$$F_m(g) = \int_O \varphi_K(g \begin{bmatrix} 1 \\ x \\ 1 \end{bmatrix}) a_m dx$$

is Iwahori-fixed and

$$F_m = q^{-m/2} \alpha_2^m M(\varphi_K)(1) \varphi_0 + q^{-m/2} \alpha_1^m \varphi_1$$

*Proof:* $F_m$ is Iwahori fixed because it equals

$$\int_{K_0(p)} \pi(ka_m) \varphi_K dk = \int_{N-} \int_{T(O)} \int_{N(O)} \varphi_K(gnt_0n-a_m) dndt_0dn \quad \text{(III.4.3.15)}$$

noticing that $a_m^{-1}t_0n-a_m \in K$, so the integrand is independent of $t_0, n_\sim$. And

$$c_1 = L_1(F_m) = F_m(1) = \varphi_K(a_m) = (\Delta^{1/2} \chi)(a_m) = q^{-m/2} \alpha_1^m.$$ 

Similarly, for $\chi$ dominant,

$$c_0 = L_0(F_m) = \int_{N(F)} F_m(w_0n) dn = \int_{N(F)} \int_{N(O)} \varphi_K(w_0nn'a_m) dn'dn.$$ 

By Fubini, the $n'$ can be omitted, thus it equals $(M\varphi_K)(a_m) = (\Delta^{1/2} \chi')(a_m)$. \hfill $\Box$
Spherical Whittaker Function and Spherical Functions

Def. (III.4.5.11). The spherical Whittaker function is just the spherical vector $W_0$ in the Whittaker model, or equivalently, $W_0(g) = \Lambda(\pi(g)\varphi_K)$, where $\Lambda$ is a Whittaker functional and $\varphi_K$ is a spherical vector.

Prop. (III.4.5.12) (Calculating $W_0$). We may assume that the conductor of $\psi$ is $\mathcal{O}$, because any other character is of the form $x \mapsto \psi(ax)$, and we want to calculate $W_0$. Notice that

$$W_0\left(\begin{array}{ccc} 1 & x & z \\ 1 & 1 & z \\ & & 1 \end{array}\right)gk = \psi(x)\omega(z)W_0(g), \ z \in F^*, \ k \in K$$

where $\omega = \chi_1\chi_2$, so it suffices to compute $W_0\left(\begin{array}{ccc} \varpi^m & & \\ & 1 & \\ & & 1 \end{array}\right)$.

$$W'_0(g) = \int_F f(w_0\left(\begin{array}{ccc} 1 & x & \\ 1 & 1 & \\ & & 1 \end{array}\right)g)\psi(-ax)dx = |a|^{-1/2}\chi_2(a)W_0\left(\begin{array}{ccc} a & \\ & 1 & \\ & & 1 \end{array}\right)g).$$

For $\mathcal{B}(\chi_1,\chi_2)$ with Satake parameters $\alpha_1, \alpha_2$, the spherical Whittaker function satisfies

$$W_0(1) = 1 - q^{-1}\alpha_1\alpha_2^{-1}, \ W_0(a_m) = 0$$

where $a_m = \left(\begin{array}{ccc} \varpi^m & & \\ & 1 & \\ & & 1 \end{array}\right)$, $m < 0$.

Proof: As in the proof of (III.4.4.7), because $\varphi_K$ is $K$-invariant, we have

$$\int_{\mathcal{O}} \varphi_K(w_0\left(\begin{array}{ccc} 1 & x & \\ 1 & 1 & \\ & & 1 \end{array}\right))\psi(-x)dx = 1,$$

$$\int_{p^{-1}\mathcal{O}} \varphi_K(w_0\left(\begin{array}{ccc} 1 & x & \\ 1 & 1 & \\ & & 1 \end{array}\right))\psi(-x)dx = q^{-1}\alpha_1\alpha_2^{-1}\int_{p^{-1}\mathcal{O}} \psi(-x)dx = -q^{-1}\alpha_1\alpha_2^{-1},$$

$$\int_{p^{-k-1}\mathcal{O}} \varphi_K(w_0\left(\begin{array}{ccc} 1 & x & \\ 1 & 1 & \\ & & 1 \end{array}\right))\psi(-x)dx = q^{-k-1}\alpha_1\alpha_2^{-1}\int_{p^{-k-1}\mathcal{O}} \psi(-x)dx = 0, \ k \geq 1$$

For $W_0(a_m)$, choose $x \in \mathcal{O}$ that $\psi(\varpi^mx) \neq 1$, then

$$W_0(a_m) = W_0(a_m\left(\begin{array}{ccc} 1 & x & \\ 1 & 1 & \\ & & 1 \end{array}\right)) = W_0\left(\begin{array}{ccc} \varpi^m & & \\ & 1 & \\ & & 1 \end{array}\right)a_m = \psi(\varpi^mx)W_0(a_m)$$

so $W_0(a_m) = 0$. □

Lemma (III.4.5.13). For fixed $g$, $W_0(g)$ is an analytic function of $\alpha_1, \alpha_2$, and $(1 - q^{-1}\alpha_1\alpha_2^{-1})^{-1}W_0(g)$ is symmetric in $\alpha_1, \alpha_2$.

Proof: By (III.4.5.3) and (III.4.4.13). Then to check these two expression are equal, it suffices to check for $\alpha_1\alpha_2^{-1} \neq q$ or $q^{-1}$. In this case, $\mathcal{B}(\chi_1,\chi_2)$ is irreducible, and $\mathcal{B}(\chi_1,\chi_2) \cong \mathcal{B}(\chi_2,\chi_1)$, and they have the same Whittaker model up to scalar, and by the calculation in (III.4.5.12) they are equal. □
Prop. (III.4.5.14) (Calculating $W_0$).

\[(1 - q^{-1}\alpha_1\alpha_2^{-1})^{-1}W_0(a_m) = q^{-m/2}\frac{\alpha_1^{m+1} - \alpha_2^{m+1}}{\alpha_1 - \alpha_2}, \quad m \geq 0\]

Proof: Firstly, when $\chi$ is dominant,

\[W(a_m) = \int_F F_m(w_0 \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] )\psi(-x)dx,\]

by (III.4.4.13)(III.4.10) and a change of parameter. Then use (III.4.5.10), we see that

\[W(a_m) = C_1 q^{-m/2}\alpha_1^m + C_0 q^{-m/2}\alpha_2^m\]

where

\[C_0 = (M\varphi_K)(1) \int_F \varphi_0(w_0 \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] )\psi(-x)dx\]

And \(\int_F \varphi_0(w_0 \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] )\psi(-x)dx = 1\), by the same consideration as in (III.4.5.9).

Finally, \((M\varphi_K)(1)\) can be calculated by (III.4.5.7), and the requirement of (III.4.5.13) will determine \(C_1\).

Remark (III.4.5.15). This formula has a generalization, Cf.[The Unramified Principal Series of $p$-adic Groups 2, Casselman/Shalika].

Prop. (III.4.5.16) (Spherical Function). For a spherical irreducible admissible representation \((\pi,V)\) of $GL(2,F)$, its contragradient $\hat{V}$ is also spherical (I.4.4.9). Then we define the spherical function

\[\sigma(g) = (\pi(g)v, \hat{v})\]

which is bi-invariant under $K$-action By (III.4.5.6).

By (III.4.5.6), \((\pi,V)\) is of the form $\chi(\det(g))$ or $\pi(\chi_1, \chi_2)$ for $\pi_i$ nonramified. We only consider the latter interesting case, for which there is a spherical functional $\hat{v} : \varphi \mapsto \int_K \varphi(k)dk$, and it is 1 on the normalized spherical vector $v$ (III.4.5.3), and equals

\[= \int_K (\pi(g)\varphi_K)(k)dk = \int_K \varphi_K(kg)dk\]

in this case.

Prop. (III.4.5.17) (Macdonald Formula). The spherical function of $\pi(\chi_1, \chi_2)$ behave well under $Z(F)$-action and is $K$-biinvariant, so in order to compute it, it suffices to compute its value on $a_m$. We have:

\[\sigma(a_m) = q^{-m/2} \frac{1 - q^{-1}\alpha_2\alpha_1^{-1}}{1 - \alpha_2\alpha_1^{-1}} + q^{-1} - q^{-1}\alpha_1\alpha_2^{-1} \frac{1}{1 - \alpha_1\alpha_2^{-1}}\]

Proof: First notice that

\[\int_K F_m(k)dk = \int_K \int_O \varphi_K(k \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] a_m)dxdk = \int_K \varphi_K(ka_m)dk\]
by a change of variable. Then by (III.4.5.10), this equals

$$\int_K \varphi_0(k)dk q^{-m/2} \alpha_2^{-1} \frac{1 - q^{-1} \alpha_1^{-1}}{1 - \alpha_1 \alpha_2^{-1}} + \int_K \varphi_1(k)dk q^{-m/2} \alpha_1^{-1}.$$ 

Next we calculate $\int_K \varphi_0(k)dk$ directly: By (III.4.5.9), this equals the volume of $K \cap (K_0(p)w_0K_0(p)) = K_0(p)w_0K_0(p)$. $K/K_0(p) \cong GL(2, \mathbb{F}_q)/B(\mathbb{F}_q)$ has cardinality $q + 1$, and by pulling back the Bruhat decomposition of $GL(2, \mathbb{F}_q)$, $K_0(p)w_0K_0(p)$ consists of $q$ left cosets of $K_0(p)$. So

$$\int_K \varphi_0(k)dk = \frac{q}{1 + q}.$$ 

Finally the expression is symmetric in $\alpha_1, \alpha_2$, because the spherical vectors in $V, \hat{V}$ are unique (I.4.4.9), and $B(\chi_1, \chi_2) = B(\chi_2, \chi_1)$. Also, the expression is a combination of $\alpha_i^m$ and the coefficient is independent of $m$, so it can be determined as above. □

6 Unitarizable, Supercuspidal and Weil Representation

Unitarizable Principal Series Representations

**Prop. (III.4.6.1).** If

**Lemma (III.4.6.2) (Possibility of Unitarization of Principal Series).** If $B(\chi_1, \chi_2)$ admits an invariant non-degenerate Hermitian pairing, then either $\chi_1, \chi_2$ are all unitary or $\chi_1 = \overline{\chi_2}^{-1}$.

**Proof:** There is an anti-linear $GL(2, F)$-map $B(\chi_1, \chi_2) \to B(\overline{\chi_1}, \overline{\chi_2})$ which is conjugation, so if there is a $GL(2, F)$-invariant Hermitian pairing on $B(\chi_1, \chi_2)$, $(f_1, f_2) \mapsto (f_1, \overline{f_2})$ will be a non-degenerate $GL(2, F)$-invariant bilinear pairing

$$B(\chi_1, \chi_2) \times B(\overline{\chi_1}, \overline{\chi_2}) \to \mathbb{C}.$$ 

So $B(\chi_1, \chi_2) \cong B(\overline{\chi_1}^{-1}, \overline{\chi_2}^{-1})$, so $\chi_i$ is unitary or $\chi_1 = \overline{\chi_2}^{-1}$. □

**Lemma (III.4.6.3).** If $\chi_1, \chi_2$ are unitary, then $B(\chi_1, \chi_2)$ is unitarizable.

**Proof:** In this case the representation $\chi$ of $B(F)$ is unitary, so by the compatibility of definition of induced representation with that of (X.6.5.3), we see $B(\chi_1, \chi_2)$ is unitarizable. □

**Lemma (III.4.6.4).** Let $\chi_s = \chi_0 |^{-s}$, where $\chi_0$ is a unitary character of $F$. If $s \neq 0, 1/2$ is a real number (so that $B(\chi_s, \chi_{-s})$ is irreducible), then $B(\chi_s, \chi_{-s})$ is unitarizable iff $-1/2 < s < 1/2$.

**Proof:** Because $B(\chi_s, \chi_{-s}) = \chi_0 \otimes B(|^{-s}|, |^{-s}|)$, we can assume $\chi_0 = 1$. Let $M_s : B(\chi_s, \chi_{-s}) \to B(\chi_{-s}, \chi_s)$ be the intertwining integral (III.4.4.8), then we see the sesquilinear pairing

$$(f_1, f_2) = \int_K (M_s f_1)(k) \overline{f_2(k)}dk$$

is $G$-invariant and non-degenerate by the proof of (III.4.4.4). For an irreducible representation, such a pairing must be unique, so we are reduced to checking this representation is positive/negative definite.

Consider the Iwahori-fixed vector $f_0 = \Delta^{s+1/2}(b)$ for $g = bk$ as defined in (III.4.5.9), then as $s$ varies, $f_0$ forms a flat section. We calculate $(f_0, f_0)$ for $s > 0$ and then use continuation:
In this case, 
\[(f_0, f_0) = V(K_0(p)) \int_F f_0(w_0 \begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix}) dx\]
and 
\[\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \in B(F)K_0(p)\]
iff \(x \not\in \mathcal{O}\), in which case 
\[\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} x^{-1} & -1 \\ x & 1 \end{bmatrix} \begin{bmatrix} 1 & x^{-1} \\ 1 & 1 \end{bmatrix} \].

So 
\[(f_0, f_0) = \frac{1}{q+1} \int_{|x| > 1} |x|^{-1-2s} dx = \frac{1-q^{-1}}{1+q} \frac{q^{-2s}}{1-q^{-2s}}\]

We then consider the spherical vector \(\varphi_K(\text{III.4.5.3})\), then (III.4.5.7) shows 
\[(\varphi_K, \varphi_K) = \frac{1-q^{-1-2s}}{1-q^{-2s}}\]
which is positive if \(|s| > 1/2\). So if \(B(\chi_s, \chi_{-s})\) is unitarizable, \(|s| < 1/2\).

Now for \(|s| < 1/2\), we show \(B(\chi_s, \chi_{-s})\) is unitary: Modify the intertwining operator \(M_s^* = (1-q^{-2s})M_s\), then it is definable for \(s = 0\), by calculation in (III.4.4.8), so the modified Hermitian product 
\[(\cdot, \cdot)^* = (1-q^{-2s})(\cdot, \cdot)\]
is defined at \(s = 0\), and is positive/negative definite, because in this case it is irreducible and unitarizable (III.4.6.3). The eigenvalue of this Hermitian form deforms continuously, and it is never zero because it is non-degenerate as said above (only when \(s \neq 1/2\) because we need \(M_s\) to be isomorphism), so it is always definite and \(B(\chi_s, \chi_{-s})\) is unitarizable. \(\square\)

**Prop. (III.4.6.5) (Unitarizable Principal Series).** An irreducible representation \(B(\chi_1, \chi_2)\) is unitary iff either \(\chi_1, \chi_2\) are all unitary, called **tempered principal series representations** or there is a unitary character \(\chi_0\) and \(-1/2 < s < 1/2\) that \(\chi_1 = \chi_0|\cdot|^s, \chi_1 = \chi_0|\cdot|^{-s}\), called **complementary series representations**.

**Proof:** Follows immediately from the lemmas above (III.4.6.3)(III.4.6.2)(III.4.6.4). \(\square\)

**Supercuspidal Representation**

**Def. (III.4.6.6) (Supercuspidal Representation).** If \((\pi, V)\) is an irreducible admissible representation of \(GL(n, F)\), then \(\pi\) is called **supercuspidal** iff \(J(V) = 0\).

**7 Kirillov Model and Local Functional Equation**

**Def. (III.4.7.1).** If \((\pi, V)\) is an irreducible admissible representation of \(GL(2, F)\) that admits a Whittaker model, we can define a **local L-function** \(L(s, \pi)\) as follows:

- If \((\pi, V)\) is supercuspidal, \(L(s, \pi) = 1\).
• If \((\pi, V) = \pi(\chi_1, \chi_2)\) is an irreducible principal representation,

\[
L(s, \pi) = \frac{1}{(1 - \alpha_1 q^{-s})(1 - \alpha_2 q^{-s})},
\]

where \(\alpha_i\) are the Satake parameters (III.4.5.4).

• If \((\pi, V) = \sigma(\chi_1, \chi_2)\), where \(\chi_1 \chi_2^{-1} = |\cdot|^{-1}\), then

\[
L(s, \pi) = \frac{1}{1 - \alpha_2 q^{-s}},
\]

where \(\alpha_2\) is the Satake parameter of \(\chi_2\).

It follows from (III.4.4.12) and (III.4.4.5) that any irreducible representation of \(GL(2, \mathbb{F})\) is one of the form above. Also when \(\xi\) is a quasi-character of \(F^*\), define

\[
L(s, \pi, \xi) = L(s, \pi \otimes \xi).
\]

Prop. (III.4.7.2). Cf. [Bump, Thm4.7.1].

8 Local Langlands Correspondence

The basic object of LLC are the Weil group and its representations.

A representation \(\rho\) of \(W_K\) is called \(F\)-semisimple iff \(\rho(\text{Frob})\) is diagonalizable.

Cor. (III.4.8.1) (LLC for \(GL_1\)).

Local class field theory told us that \(W_K^{ab}\) is isometric to \(K^*\). And notice by Schur’s lemma, any smooth representation of \(K^*\) is 1-dimensional and factors through some \(U_k\).

And a Weil-Deligne representation is now a continuous \(W_K^{ab} \rightarrow C^*\). But it must factor through some \(U_K\), so these two are equivalent.

most \(l\)-adic representation of \(G_K\) comes from étale cohomology.

LLC for \(GL_2\)

Thm. (III.4.8.2) (LLC for \(GL_n\)). The set of

irreducible smooth, admissible representations of \(GL_n(K)\)

corresponds to

\(n\)-dimensional \(F\)-semisimple Weil-Deligne representations of \(W_K\).
III.5 Automorphic Forms Beyond $GL_2$

1 Automorphic Forms on Unitary Groups

Main references are [Automorphic Forms on Unitary Groups, Eischen].

Unitary groups provide a particularly fruitful setting in which to work. Unitary groups have associated Shimura varieties, which provide convenient structure for studying algebraic aspects of automorphic forms (which, in turn, arise as sections of a vector bundle over Shimura varieties). We have substantial results about Galois representations associated to automorphic forms on unitary groups (e.g. [Ski12, Che04, Che09, CH13, Har10]). In addition, we have convenient representations of the L-functions associated automorphic forms on unitary groups, which are useful both for proving analytic properties and for extracting algebraic information (and even p-adic properties, as seen in [EHLS20]). Working with unitary groups has enabled major developments (which go far beyond the scope of these lectures but several of which are mentioned here as motivation for learning about automorphic forms on unitary groups), including a proof of the main conjecture of Iwasawa Theory for $GL_2$ [SU14] and the rationality of special values of certain automorphic L-functions (including [Shi00, Har97, Har08, Har84, Bou15]), as well as progress toward cases of the Bloch–Kato conjecture (including [SU06, Klo09, Klo15, Wan19]), and the Gan–Gross–Prasad conjecture (many recent developments, including [Xue14, Xue19, Zha14, Liu14, Yun11, JZ20, He17, BP20, BPLZZ21]).

2 Quaternionic Modular Forms

Main references are [Modular Forms on Exceptional Groups, Pollack].

3 Theta Correspondence

Classical Theta Functions

Def. (III.5.3.1) (Poisson Summation for Lattices). Let $V$ be a vector space of dimension $n$ with an Haar measure $\mu$, $\Gamma \subset V$ a full lattice, and $\Gamma' \subset V'$ be its $\mathbb{Z}$-dual, Let $V = \mu(V/\Gamma)$, then for any Schwartz function $f \in S(V)$,

$$\sum_{x \in \Gamma} f(x) = \frac{1}{V} \sum_{y \in \Gamma'} \hat{f}(y).$$

Proof: This is just??.

Def. (III.5.3.2) (Theta function). Let $\Gamma \subset V$ be a real inner product space with an Haar measure $\mu$ normalized that for an orthonormal basis $e_i$ of $V$, $V/\mathbb{Z}\{e_i\}$ has volume 1, then we can identify $V$ with $V'$ by this inner product. Let $\Gamma$ be a full lattice, then its $\mathbb{Z}$-dual $\Gamma'$ is identified with a lattice in $V$ that $(x, y) \in \mathbb{Z}$ for any $x \in \Gamma, y \in \Gamma'$.

The theta function $\theta_\Gamma(z)$ is defined to be

$$\theta_\Gamma(z) = \sum_{x \in \Gamma} q^{-(x,x)/2} = \sum_{x \in \Gamma} e^{-\pi i z (x,x)} \text{Im}(z) > 0,$$

and

$$\Theta_\Gamma(t) = \theta_\Gamma(it) = \sum_{x \in \Gamma} e^{-\pi t (x,x)}$$
Cor. (III.5.3.3). With the notation as in (III.5.3.2), the theta function satisfies

$$
\Theta_\Gamma(t) = \frac{t^{-n/2}}{\mu(V/\Gamma)} \Theta_\Gamma(t^{-1})
$$

Proof: Notice that $\Theta_{2\Gamma}(t) = \Theta_\Gamma(s^2 t)$, so this formula follows from (III.5.3.1) applied to $t^{-1/2} \Gamma$ and $f(x) = e^{-\pi(x,x)}$. 

Def. (III.5.3.4) (Self-Dual Lattices). Situation as in (III.5.3.2), a self-Dual Lattice is a lattice $\Gamma$ in $V$ that $V' = V$. Equivalently, if $\{f_i\}$ is a Z-basis of $\Gamma$, then the matrix $A = ((e_i, e_j))$ is a matrix with integer coefficients and determinant 1. The last equivalence is because $\Gamma' \subset \Gamma$ equals $\Gamma$ if $\mu(V/\Gamma) = \mu(V/\Gamma')$, but this is equivalent to $\mu(V/\Gamma) = 1$, because $\mu(V/\Gamma) \cdot \mu(V/\Gamma') = 1$.

A self-dual lattice is called even iff $(x, x) \in 2\mathbb{Z}$ for any $x \in \Gamma$.

Example (III.5.3.5) ($E_{6k}$). Let $V = \mathbb{R}^{6k}$ with the canonical inner product, denote $E_{6k}$ the set of vectors $\sum_i x_i e_i$ in $V$ that

$$2x_i \in \mathbb{Z}, x_i - x_j \in \mathbb{Z}, \sum_{i=1}^{6k} x_i \in 2\mathbb{Z}.$$ 

Notice $E_8$ is just the Z-span of the root system $E_8$ (I.14.1.28), and the root system just consists of all vectors in $E_8$ of length $\sqrt{2}$.

Then $E_{6k}$ is a self-dual and even.

Cor. (III.5.3.6). Let $k \geq 2$, then all the vectors in $E_{6k}$ of length $\sqrt{2}$ are $\{\pm e_i \pm e_j | i \neq j\}$.

Remark (III.5.3.7). For more examples of self-dual lattices, see [Ser73] Chap5.

Prop. (III.5.3.8) (Theta Function for Self-Dual Even Lattices). Let $\Gamma \subset V$ be a self-dual even lattice (III.5.3.4), then the dimension $n$ of $V$ is divisible by 8, and the theta function $\theta_\Gamma(z)$ is a modular form for $\Gamma(1)$ of weight $n/2$.

Proof: We first show that

$$\theta_\Gamma(-1/z) = (-iz)^{n/2} \theta_\Gamma(z)$$

and because both sides are holomorphic functions on $\mathcal{H}$, it suffices to show this for $z = it, t > 0$. Thus it suffices to show

$$\Theta_\Gamma(t^{-1}) = t^{-n/2} \Theta_\Gamma(t).$$

And this is just (III.5.3.3), because $\Gamma$ is self-dual (III.5.3.4).

Then $\theta_\Gamma(ST)_{n/2} = (-i)^n/2 \theta_\Gamma(IV.5.1.5)$, but $(ST)^3 = 1$, so $(-i)^{3n/2} = 1$, so $8 \| n$, and $\theta_\Gamma \in M_{n/2}(\Gamma(1))$. 

Prop. (III.5.3.9) (Theta Function for Non-Even Self-Dual Lattices). If we consider theta function for non-even self-dual lattices in $\mathbb{R}^n$, then we get a modular form of weight $n/2$ w.r.t. the subgroup of $SL(2,\mathbb{Z})$ generated by the elements $S$ and $T$. This image of this subgroup has index 3 in $PSL(2,\mathbb{Z})$, and it has two cusps, thus two Eisenstein series.

In particular, we can apply this to the lattice $\{e_i\}$, and use this information to obtain formula giving the number of ways to represent an integer into a sum of $n$ squares.

Proof: 


III.6 Representations of Semisimple Lie Algebras and Category \( \mathcal{O} \)

1 Semisimple Representations

Lemma (III.6.1.1). If \( \mathfrak{g} \) is a semisimple Lie algebra over \( k \), then every 1-dimensional representation of \( \mathfrak{g} \) is trivial.

Proof: Such a representation vanishes at \([\mathfrak{g},\mathfrak{g}]\), which equals \( \mathfrak{g} \) by (I.12.2.4).

Prop. (III.6.1.2) (Weyl). For a Lie algebra \( \mathfrak{g} \) over a field \( k \),

- If the adjoint representation \( \mathfrak{g} \to \mathfrak{gl}_\mathfrak{g} \) is semisimple, then \( \mathfrak{g} \) is semisimple.
- If \( \mathfrak{g} \) is semisimple and \( k \) has characteristic 0, then \( \text{Rep}(\mathfrak{g}) \) is semisimple.

Proof: If the adjoint representation of \( \mathfrak{g} \) is semisimple, then every ideal of \( \mathfrak{g} \) has a complement, thus if \( \mathfrak{g} \) is not semisimple, it has a dimension 1 quotient. But notice the Lie algebra \( k \) of dimension 1 has non-semisimple representations, for example \( c \mapsto \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \), contradiction.

Semisimplicity of a Lie algebra is invariant under base change, also does simplicity of the category of representations (III.6.1.4), so we can assume that \( k \) is alg.closed. Now we need to show that any proper submodule \( W \) of a \( \mathfrak{g} \)-module \( V \) has a complement.

Assume first that \( \dim V/W = 1 \) and \( W \) is a simple \( \mathfrak{g} \)-module. This implies \( \mathfrak{g} \) acts trivially on \( V/W \) by (III.6.1.1). Let \( c_V \) be the Casimir element of \( V \) (I.12.9.11), then \( c_V \) is also trivial on \( V/W \). And \( c_V \) acts as a nonzero scalar on \( W \) as \( W \) is simple, by (I.12.9.11). Then the kernel of \( c_V \) is of 1-dimensional, and is a \( \mathfrak{g} \)-complement of \( W \) in \( V \).

Next if \( \dim V/W \) but \( W \) is not simple \( \mathfrak{g} \)-module, then there is a submodule \( W' \subset W \). By induction, the \( \mathfrak{g} \)-submodule \( W/W' \) has a complement \( V'/W' \) in \( V/W' \). Then \( V'/W' \) has dimension 1, thus by induction, \( V' = W' \oplus L \) for some 1-dimensional \( \mathfrak{g} \)-module \( L \). Then \( L \) is complementary to \( W \) in \( V \).

Finally for the general case, let \( \mathfrak{g} \) acts on \( \text{Hom}_k(V,W) \), consider the subspaces \( V_1, W_1 \) of \( \text{Hom}_k(V,W) \), where \( V_1 \) is the subspace of maps that restriction to \( W \) is a constant multiple of identity, and \( W_1 \) is the subspaces of \( W \) consisting of maps vanishing on \( W \). They are both \( \mathfrak{g} \)-modules and \( \dim V_1/W_1 = 1 \). Then the above case shows \( V_1 = W_1 \oplus L \) for some 1-dimensional \( \mathfrak{g} \)-module \( L \).

Because \( \mathfrak{g} \) acts trivially on \( L \) (III.6.1.1), this means \( L = \mathbb{F} f \) consists of \( \mathfrak{g} \)-homomorphisms. But \( f|_W \) is non-zero constant, so the kernel of \( f \) is a complement of \( W \) in \( V \).

Cor. (III.6.1.3). Let \((V,\rho)\) be a representation of a semisimple Lie algebra \( \mathfrak{g} \) and \( f : \mathfrak{g} \to V \) a linear map that

\[
f([x,y]) = \rho(x)f(y) - \rho(y)f(x),
\]

then there exists a \( v_0 \in V \) that \( f(x) = \rho(x)v_0 \).

Proof: The condition on \( f \) is equivalent to \((f,\rho) : \mathfrak{g} \to \mathfrak{gl}(V)\) (I.12.1.10) is a homomorphism of Lie algebras. And this induces a representation \( \rho' \) of \( \mathfrak{g} \) on \( V' = V \oplus k \) that \( \rho'(x)(V') \subset V \) for all \( x \in \mathfrak{g} \).

Because \( \mathfrak{g} \) is semisimple, there is a line \( L \subset V' \) that \( V' = V \oplus L \) and \( \mathfrak{g} \) acts trivially on \( L \) (III.6.1.1).

In other words, there is a vector \((-v_0,1)\) that \( \rho'(x)(-v_0,1) = 0 \) for all \( x \in \mathfrak{g} \). So \( f(x) = \rho(x)(v_0) \) for all \( x \). So we are done.

Prop. (III.6.1.4) (Semisimplicity and Extension). Let \( \mathfrak{g} \) be a Lie algebra over a field \( k \). If \( \text{Rep}(\mathfrak{g}_K) \) is semisimple for some extension field \( K/k \), then also is \( \text{Rep}(\mathfrak{g}) \).
Proof: This is because for any representation \((V, \rho)\) of \(g\), \(K \otimes_k \text{End}(\rho) \cong \text{End}(\rho_K)\), because this is true for \(\text{End}(V)\), and being \(g\)-equivariant is a linear condition. Then we can use(I.4.1.25) and(I.4.1.26).

Prop. (III.6.1.5) (Semisimple Representations). The following conditions on a representation \(g \to \mathfrak{gl}_V\) are equivalent:

- \(\rho\) is semisimple.
- \(\rho(g)\) is reductive and its center consists of semisimple endomorphisms.
- \(\rho(\mathfrak{r})\) consists of semisimple endomorphisms.
- The restriction of \(\rho\) to \(\mathfrak{r}\) is semisimple.

Proof: 1 \(\to\) 2: If \(\rho\) is semisimple, then \(\rho(g)\) is reductive by(I.12.4.4). Its center consists of semisimple endomorphisms by \([\text{Mil}13]\)P60?.

2 \(\to\) 3: This is because if \(\rho(g)\) is reductive, then its center equals its radical and contains \(\rho(\mathfrak{r})\), so \(\rho(\mathfrak{r})\) consists of semisimple endomorphisms.

3 \(\to\) 4: \([\text{Mil}13]\)P60?.

4 \(\to\) 1: \([\text{Mil}13]\)P60?.

\(\square\)

Cor. (III.6.1.6). Let \(\rho\) and \(\rho'\) be representations of \(g\). If \(\rho\) and \(\rho'\) are semisimple, then so are \(\rho \otimes \rho'\) and \(\rho \vee \rho'\).

In particular, the category \(\text{Rep}^{ss}(g)\) of semisimple representations of a Lie algebra \(g\) form a Tannakian category, thus there is an algebraic group \(G\) that \(\text{Rep}^{ss}(g) = \text{Rep}(G)\).

Proof: Use the third criterion, for any \(x \in \mathfrak{r}\), as \(\rho(x), \rho'(x)\) are semisimple, so is \(\rho(x) \otimes \rho'(x)\), so \(\rho \otimes \rho'\) is also semisimple by(III.6.1.5).

\(\square\)

representations of \(\mathfrak{sl}_2(\mathbb{C})\)

Def. (III.6.1.7) (Primitive Element). Let \(V\) be a \(\mathfrak{sl}_2\)-module, an element \(v \in V\) is called primitive of weight \(\lambda\) if it is non-zero and \(Xe = 0, He = \lambda e\).

Prop. (III.6.1.8). Every non-zero f.d. \(\mathfrak{sl}_2\)-module contains a primitive element.

Proof: An element \(e\) is primitive iff the line generated by \(e\) is stable under the action of \(\{X, H\}\): if \(Xe = \lambda e\) and \(He = \mu e\), then using the \([H, X] = 2X\), we see that \(2\lambda 0\), thus \(\lambda = 0\), and \(e\) is primitive. So each f.d. \(\mathfrak{sl}_2\)-module contains a primitive element, by Lie’s theorem(I.12.1.23).

\(\square\)

Prop. (III.6.1.9) (Submodule Generated by Primitive Element). Let \(V\) be a \(\mathfrak{sl}_2\)-module and \(e \in V\) a primitive element of weight \(\lambda\). Let \(e_n = Y^n e / n!\), and \(e_{-1} = 0\), then we have

\[He_n = (\lambda - 2n)e_n, \quad Ye_n = (n + 1)e_{n+1}, \quad Xe_n = (\lambda - n + 1)e_{n-1} .\]

Proof: By induction on \(n\),

\[HY^n e = ([H, Y] + YH)Y^{n-1} e = (\lambda - 2(n - 1) - 2)Y^n e = (\lambda - 2n)e .\]

\(Ye_n = (n + 1)e_{n+1}\) is obvious.

And

\[nXe_n = XYe_{n-1} = [X, Y]e_{n-1} + YXe_{n-1} .\]
\[ = H e_{n-1} + (\lambda - n + 2)Y e_{n-2} \]
\[ = (\lambda - 2n + 2 + (\lambda - n + 2)(n - 1))e_{n-1} \]
\[ = n(\lambda - n + 1)e_{n-1} \]

\[ \square \]

**Cor. (III.6.1.10).** Only two cases arise: either
- The elements \( \{e_n\} \) are linearly independent.
- The elements \( e_0, e_1, \ldots, e_m \) are linearly independent, and \( e_{m+1} = e_{m+2} = \ldots = 0 \), and weight \( \lambda \) of \( e \) equals \( m \).

And if \( V \) is f.d., then case 1 cannot happen, and the subspace \( W \) generated by \( e_0, \ldots, e_m \) is a \( \mathfrak{g} \)-module and it is irreducible.

**Proof:** Because each \( e_i \) has different eigenvalue under action of \( H \), thus if they are all nonzero, then they are linearly independent. If \( e_0, e_1, \ldots, e_m \) are linearly independent, and \( e_{m+1} = e_{m+2} = \ldots = 0 \), then by the proposition,
\[ X e_{m+1} = (\lambda - m)e_m \]
and \( e_{m+1} = 0 \) with \( e_m \neq 0 \), thus \( \lambda = m \). Now the formulas in (III.6.1.9) shows that \( W \) is a \( \mathfrak{g} \)-module. And if \( W' \subset W \) is a subspace invariant under \( \mathfrak{g} \), then it contains some eigenvalues \( e_k \) of \( H \), and then the formulas in (III.6.1.9) shows it contains all \( e_0, e_1, \ldots, e_m \), thus \( W' = W \). Thus \( W \) is irreducible. \[ \square \]

**Prop. (III.6.1.11) (Irreducible Representations of \( \mathfrak{sl}_2(\mathbb{C}) \)).** Let \( W_m = \{e_0, \ldots, e_m\} \) be a \( m + 1 \)-dimensional vector space and \( \mathfrak{sl}_2 \) acts on \( W_m \) by
\[ H e_n = (\lambda - 2n)e_n, \quad Y e_n = (n + 1)e_{n+1}, \quad X e_n = (\lambda - n + 1)e_{n-1}. \]

Then \( W_n \) is a f.d. irreducible representation of \( \mathfrak{sl}_2 \), and any f.d representation of \( \mathfrak{sl}_2 \) of dimension \( m + 1 \) is isomorphic to one of \( W_m \).

**Proof:** The first assertion follows from (III.6.1.10) and the fact \( e_0 \) is a primitive element. For the second assertion, notice any f.d. representation \( W \) of \( \mathfrak{g} \) contains a primitive element (III.6.1.8) thus by (III.6.1.10) generates an irreducible \( \mathfrak{sl}_2 \)-submodule \( W_n \), thus this submodule equals \( W \), and \( n + 1 = m + 1 \), thus \( n = m \). \[ \square \]

**Cor. (III.6.1.12).** \( W_0 \) is the just trivial action of \( \mathfrak{sl}_2 \), \( W_1 \) is isomorphic to the natural action of \( \mathfrak{sl}_2 \) on \( \mathbb{C}^2 \), and \( W_2 \) is isomorphic to the adjoint action of \( \mathfrak{sl}_2 \) on itself.

**Proof:** In fact, \( W_1 \) can be identified with the vector space \( \mathbb{C}\{x, y\} \) where
\[ H x = x, H y = -y, Y x = y, Y y = 0, X y = x, X x = 0. \]

Then the \( m \)-th symmetric tensor of \( W_1 \) is isomorphic to the vector space of polynomials in \( x, y \) of degree \( m \), and by (I.12.9.2),
\[ H(C_m^k y^k x^{m-k}) = (m - 2k)(C_m^k y^k x^{m-k}), \]
\[ Y(C_m^k y^k x^{m-k}) = (k + 1)(C_m^{k+1} y^{k+1} x^{m-k-1}), \]
\[ X(C_m^k y^k x^{m-k}) = (m - k + 1)(C_m^{k-1} y^{k-1} x^{m-k+1}). \]

So it is isomorphic to \( W_{m+1} \). \[ \square \]
Cor. (III.6.1.13) (Representations of $\mathfrak{sl}_2(\mathbb{K})$). Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$,

- Any f.d. representation $V$ of $\mathfrak{sl}_2(\mathbb{K})$ is isomorphic to a direct sum of $W_m$s, and thus

- The endomorphism on $V$ induced by $H$ is diagonalizable and with integral eigenvalues. If $\pm n$ are eigenvalues of $H$, then so are $n - 2, n - 4, \ldots, 4 - n, 2 - n$.

- For any $n \geq 0$, the linear maps

$$Y^n : V^n \to V^{-n}, X^n : V^{-n} \to V^n$$

are isomorphisms. In particular, $V^{-n}$ and $V^n$ have the same dimensions.

Proof: by Weyl’s theorem (III.6.1.2), any representation of $\mathfrak{sl}_2(\mathbb{K})$ is isomorphic to a direct sum of irreducible representations, and the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ clearly come from real representations of $\mathfrak{sl}_2(\mathbb{R})$, thus irreducible representations of $\mathfrak{sl}_2(\mathbb{R})$ must be of the same form (otherwise tensor with $\mathbb{C}$ and decompose, and use conjugation).

Because we can assume $V$ is one of $W_m$, so the other assertions are clear. □

Lemma (III.6.1.14) (Central Characters). Let $(\rho, V)$ be a representation of $\mathfrak{sl}_2(\mathbb{C})$, we define a character $\chi_V : \mathbb{C} \to \mathbb{C} : \chi_V(z) = \text{tr}(\exp(\text{ad}(zH)))$. Then

$$\chi_V \otimes \chi_W(z) = \chi_V(z) \chi_W(z), \quad \chi_V \oplus \chi_W(z) = \chi_V(z) + \chi_W(z),$$

and

$$\chi_W(z) = e^{nz} + e^{(n-2)z} + \ldots + e^{-nz} = \frac{e^{(n+1)z} - e^{-(n+1)z}}{e^z - e^{-z}}.$$  

Notice these functions are linearly independent, so by (III.6.1.13), representations of $\mathfrak{sl}_2(\mathbb{C})$ are determined by their characters.

Prop. (III.6.1.15) (Clebsch-Gordan Rule). The tensor products of representations of $\mathfrak{sl}_2(\mathbb{C})$ satisfy

$$W_m \otimes W_n \cong \bigoplus_{i=0}^{\min\{m, n\}} W_{|m-n|+2i}.$$  

Proof: This follows from (III.6.1.14). □
Chapter IV

Number Theory

IV.1 $p$-adic Analysis

This section should only contain theorems that are only applicable to non-Archimedean valuations. Theorems that are applicable to both Archimedean and non-Archimedean valuations should be put into X.3.

As far as I know, all properties proved in Functional Analysis independent of complex analysis is applicable to the non-Archimedean case, and in fact, the goal of this section is to build an analytic theory parallel to complex analysis.

References are [Non-Archimedean Analysis Part A].

1 (Ultranormed) Valuation Theory

Ultranormed Rings

Def. (IV.1.1.1). A semi-normed group is a group with a non-Archimedean valuation, it is called a normed group iff the valuation has kernel 0, which is equivalent to Hausdorff.

A normed group is totally connected, because open balls at 0 are subgroups hence closed.

Def. (IV.1.1.2) (Normed Ring). A (semi-)normed ring is a (semi-)normed additive group that

- $|1| = 1$. or the valuation is trivial.
- $|ab| \leq |a||b|$.

A valued ring is a normed ring with $|ab| = |a||b|$. It is called degenerate if all non-zero valuation value $\geq 1$.

Prop. (IV.1.1.3). A valuation on a ring is non-Archimedean iff $\{|n|\}$ is bounded. Thus any valuation on a field with finite characteristic is non-Archimedean.

Prop. (IV.1.1.4). In a normed ring, every triangle is an acute isosceles triangle. (This is because the biggest is smaller than the maximal of the other two, thus the biggest two are equal). Hence we have, for a circle $B(O, r)$, any interior point $P$ is a center of circle, because $OP < r$.

Def. (IV.1.1.5). A normed ring $R$ is call a B-ring if elements of valuation 1 is invertible, it is called bald if there is a $\varepsilon$ that no elements has valuation in $(1 - \varepsilon, 1)$.
Def. (IV.1.1.6) (Uniform Rings). A non-Archimedean ring $A$ is called uniform if the set of topologically nilpotent elements are bounded in $A$.

Prop. (IV.1.1.7). If $K$ is a normed field with valuation ring $R$, the smallest subring containing a zero sequence $a_0, a_1, \ldots$ is bald.

Proof: Cf.[Formal and Rigid Geometry P25].

Def. (IV.1.1.8). An element $a$ in a normed ring $A$ is called topologically nilpotent iff $\lim a^n = 0$. The set of all topological nilpotent elements in $A$ are denoted by $\hat{A}$ or $A^0$.

Prop. (IV.1.1.9). $\hat{A}$ is a subgroup of $A^+$, which is multiplicatively closed. And $\hat{A}$ is Clopen in $A$. In particular, $\hat{A}$ is complete if $A$ is complete.

Proof: Cf.[Non-Archimedean analysis P27].

Prop. (IV.1.1.10) (Nakayama’s Lemma). If $A$ is complete normed ring and $M$ is a $A$-module, if there are f.m. elements $x_i$ of $M$ that $M = N + \sum x_iM$, then $M = N$.

Proof: The proof is verbatim as the proof of the usual Nakayama lemma.

Normed Modules

Def. (IV.1.1.11) (Ultranormed Module). A module $M$ over a normed ring $A$ is called normed module iff it is a normed additive group and $|ax| \leq |a||x|$ for $a \in A, x \in M$. If $A$ is valued and the equality always holds, we call it faithfully normed or valued module.

If $A$ is a valued field, any normed module is valued.

Prop. (IV.1.1.12) (Ultranormed Algebra). A normed algebra is an $A$ algebra $B$ with $A \to B$ bounded of norm 1.

Prop. (IV.1.1.13). For two valued module over $A$, if $A$ is non-degenerate, a morphisms is bounded iff it is continuous. This is because we can multiply by elements of $A$ to reduce to a nbhd of 0.

This applies to the case when $A$ contains a field where the valuation is non-trivial, because we can use(IV.1.1.11).

Def. (IV.1.1.14) (Completed Tensor Product). For two normed modules over a normed ring $R$, there is a complete normed $R$-module $M \hat{\otimes} N$ called the completed tensor product, satisfying the following universal properties: $M \times N \to M \hat{\otimes} N$ is bounded by 1, and for any complete normed $R$-module $T$ and a $R$-map $M \times N \to T$ bounded by $a$, then it factor through a $R$-map $M \hat{\otimes} N \to T$ bounded by $a$.

It satisfies many universal properties as you can imagine.

Proof: Cf.[Formal and Rigid Geometry P238].

Cor. (IV.1.1.15). By(IV.1.1.13), when $A$ is non-degenerate, then the amalgamated sum is just the fibered pushout when restricted to the category of complete valued module over $A$ with continuous maps as morphisms, because it satisfies the universal property.

Prop. (IV.1.1.16) (Amalgamated Sum). For two normed $R$-algebras there is an operation of amalgamated sum which satisfies universal properties similar to(IV.1.1.14). In fact, it is just the completed tensor product when seen as modules.

Proof: Cf.[Formal and Rigid Geometry P242].
Weakly Cartesian Space

Def. (IV.1.1.17). A normed \( K \)-vector space over a valued field \( K \) is called weakly Cartesian iff?

Prop. (IV.1.1.18). If \( K \) is a complete valued field, then each normed \( K \)-vector space \( V \) is weakly Cartesian.

Proof: Cf.[Non-Archimedean Analysis P92]. □

Extensions of Norms and Valuations

Def. (IV.1.1.19). If \( L \triangleleft K \) is a finite extension of valued field of degree \( n \), then \( v \) extends uniquely to \( w(\alpha) = \frac{1}{n}v(N_{L/K}(\alpha)) \), now we define the ramification degree as \((w(L^*) : v(K^*))\), and the inertia degree as the degree of the residue field extension.

Completeness

Prop. (IV.1.1.20) (Cauchy Sequence of Non-Archimedean field). For a sequence \( \sum a_i \) in a non-Archimedean field, it is a Cauchy sequence iff \( \lim a_i = 0 \).

In particular, convergent sequence are all absolutely convergent and for a Cauchy sequence not converging to 0, the valuations of the terms stabilize.

Proof: One way is easy, the other way, notice \( \sum_{i=1}^{j} a_i \leq \max_{i,j} |a_i| \). □

Prop. (IV.1.1.21) (Completion of a Field). The completion of a non-Archimedean field is preferred to choose the definition of Cauchy sequence, so we see by (IV.1.1.20) that \( v(\hat{K}) = v(K) \).

Prop. (IV.1.1.22). For a complete field \( K \) and any finite vector space \( L \), \( L \) has only one norm up to equivalence and it is complete.

Proof: Cf.[Formal and Rigid Geometry P230]. □

Prop. (IV.1.1.23). A complete valuation on a field can extend uniquely to a valuation on its alg.closure. And in the finite case, it is \( |\alpha| = |N(\alpha)|^\frac{1}{d} \). This is an immediate consequence of (IV.1.1.31) and (IV.1.1.28), and \( |\alpha| \leq 1 \) iff it is integral over valuation ring \( R \) of \( K \).

Prop. (IV.1.1.24). Any infinite separable algebraic extension of a complete field is never complete.

Proof: We use Krasner’s lemma (IV.1.1.32). By Ostrowski theorem (I.9.3.14), we can assume it is non-Archimedean, otherwise it cannot by infinite dimensional. Choose an infinite linearly independent basis of decreasing value rapidly enough, then we can see the field generated by the limit contains all the partial sums, contradiction. □

Prop. (IV.1.1.25). If \( K \) is alg.closed valued field, then its completion is also alg.closed.

Proof: Let \( L = (\hat{K})^{\text{alg}} \), then we can extend to a valuation on \( L \), now let \( f \) be a monic polynomial with coefficients in \( \hat{K} \), we show its root \( \alpha \in L \) can be approximated by elements in \( K \), now let \( g \) monic in \( K[X] \) be an approximation of \( f \) that \( |g(\alpha)| \leq \epsilon^n \), then there is a root \( \beta \) of \( g \) that \( |\alpha - \beta| < \epsilon \), and \( \beta \in K \) by alg.closedness. □

Prop. (IV.1.1.26). If \( F \) is a complete valued field, then \( F^{\text{sep}} \) is dense in \( F^{\text{alg}} \).

Proof: Assume \( F \) is non-Archimedean, then for \( y \in F^{\text{alg}} \), there is a \( n \) that \( y^{p^n} = \alpha \in F^{\text{sep}} \). We may assume \( |\alpha| \leq 1 \), then let \( \pi \) be an element that \( |\pi| < 1 \), then if \( y_i \) is a root of the separable polynomial \( Y^{p^n} - \pi^i Y - \alpha = 0 \), then \( (y - y_i)^{p^n} = \pi^i y_i \). So \( |y - y_i| \to 0 \). □
Henselian Value Field

Def. (IV.1.1.27). A valued field $K$ is called Henselian iff the valuation ring is a Henselian local ring(IV.6.10.1).

Prop. (IV.1.1.28). $K$ is Henselian iff the valuation of $K$ has a unique extension to any finite extension $L/K$.

Proof: Cf.[Algebraic Number Theory Neukirch P144]. □

Cor. (IV.1.1.29). Thus for a normal extension, $x$ and $\sigma(x)$ has the same valuation. Hence any polynomial in $K[X]$ has a decomposition into polynomials where all their roots has the same valuation.

Prop. (IV.1.1.30) (Hensel's Lemma Generalized). Let $K$ be a complete valued non-Archimedean field and $\mathcal{O}_K$ be the valuation ring. If $P, Q, R \in \mathcal{O}_K[X]$ and $0 \leq \lambda < 1$ that $\deg P = m + n, \deg Q = n, \deg R = m$, and

$$\deg(P - QR) \leq m + n - 1, \quad |P - QR|_G \leq \lambda |\text{res}(Q, R)|^2$$

Where $| - |_G$ is the induced Gauss norm on $K[X]$. Then there exist polynomials $U, V$ that

$$|U|_G, |V|_G \leq \lambda |\text{res}(Q, R)|^2, \deg U \leq n - 1, \deg V \leq m - 1$$

and $P = (Q + U)(R + V)$.

Proof: If $\rho = |\text{res}(Q, R)| = 0$, then $P = QR$. Otherwise, the map $\theta_{Q, R} : W_m \oplus W_n \to W_{m+n}$ is invertible(IV.2.2.12). Then we let $\varphi(U, V) = \theta_{Q,R}^{-1}(P - QR - UV)$, then If $U, V \in B(0, \lambda \rho)$, then $|\varphi(U, V)|_G \leq \lambda \rho$. And it can be proved $\varphi$ is a contraction map from $B(0, \lambda \rho)^2$ to itself with contraction factor $\lambda$, so it has a fixed point $(U, V)$ by(IX.1.8.7). So $QU + RV = P - QR - UV$. □

Cor. (IV.1.1.31) (Hensel's Lemma). Let $K$ be a complete valued non-Archimedean field and $A$ be the valuation ring. If $P(X) \in A[X]$ and $a_0$ is an element of $A$ s.t. $|P(a_0)/P'(a_0)|^2 = \varepsilon < 1$, then there exists an $\alpha \in A$ that $P(\alpha) = 0$ and $|\alpha - a_0| \leq |P(a_0)/P'(a_0)|$.

The usual form is when $|P'(a_0)| = 1$, in which case we can pass to the residue field. Equivalently, the valuation ring of a complete non-Archimedean field is a Henselian local ring.

Proof: Let $\lambda = |P(a_0)/P'(a_0)|$ and $\text{res} = |P'(a_0)|$. Notice If $P(X) = Q(X)(X - a_0) + P(\alpha)$, then $\text{res}(Q(X), X - a_0) = Q(a_0) = P'(a_0)$). □

Prop. (IV.1.1.32) (Krasner's Lemma). For a Henselian non-Archimedean field $K$, the if $\alpha, \beta \in K$ that $|\alpha - \beta| < |\alpha - \sigma(\alpha)|$ for all $\sigma$, then $K(\alpha, \beta)/K(\beta)$ is purely inseparable. So when $\alpha$ is separable over $K$, $K(\alpha) \subseteq K(\beta)$.

Proof: It suffice to prove that for all field morphism $\tau : K(\alpha, \beta) \to \overline{K}$ fixing $K(\beta)$, $\tau(\alpha) = \alpha$. This is because $|\tau(\alpha) - \beta| = |\alpha - \beta| < |\alpha - \sigma(\alpha)|$, thus $|\tau(\alpha) - \alpha| \leq \max\{|\tau(\alpha) - \beta|, |\beta - \alpha|\} < |\alpha - \sigma(\alpha)|$. □

Cor. (IV.1.1.33). If $f$ is a separable irreducible polynomial and $\alpha$ is a root, then for $g$ closed enough to $f$, there is a root $\beta$ of $g$ that $K(\beta) = K(\alpha)$. (Immediate consequence of(IX.9.3.15)).

Cor. (IV.1.1.34). Let $K$ be a non-Archimedean valued field with completion $\hat{K}$, then any finite separable extension $\mathcal{L}/\hat{K}$ is of the form $L\hat{K}$. (Because of Primitive element theorem).
Cor. (IV.1.1.35). If a Henselian field is dense in its alg.closure, then it is alg.closed.

Prop. (IV.1.1.36) (Kaplansky-Schilling). A field which is Henselian w.r.t two inequivalent valuation is separably closed, and separably closed field is Henselian w.r.t any valuation.

Proof: □

2 Ultranormed Banach Spaces

In this section, denote $K$ a complete non-Archimedean field(of rank $1$), $K^0 = \{x \in K||x| < 1\}$, and $|t| < 1$ a uniformizer.

Ultranormed $K$-modules

Prop. (IV.1.2.1). If $K$ is complete, then each normed $K$-module is weakly-Cartesian, Cf.[Non-Archimedean Analysis P92].

Cor. (IV.1.2.2). If $K$ is complete, any two valuation on a finite $K$-vector space are equivalent.

Proof: Cf.[Non-Archimedean Analysis P93]. □

Prop. (IV.1.2.3). If $V$ is a normed $Q_p$ vector space and $V_0 = \{x \in V||x| \leq 1\}$, then $\hat{V} \cong (V_0)_p[p^{-1}]$.

Ultranormed Banach Spaces

Def. (IV.1.2.4) (Ultranormed Banach Spaces). In the non-Archimedean case, an **ultranormed Banach algebra** is defined as in(X.3.4.2), but additionally $|a + b| \leq \max\{|a|, |b|\}$.

Def. (IV.1.2.5) (Uniform Banach Space). For a complete non-Archimedean field $K$ and a Banach algebra $R$, define $R^0$ to be the ring of **power bounded elements**. Then it is a subring, and it is open, as it contains the closed ball $B(0, 1)$.

Recall $R$ is called uniform if $R^0$ is itself bounded in $R$ (IV.1.1.6). Notice a uniform Banach space must be reduced, because a nilpotent element is clearly power bounded, and any scalar multiple of it is nilpotent.

Lemma (IV.1.2.6). Fix a uniformizer $t$ in a non-Archimedean complete field $K$, if $|K^*|$ is discrete, then if $A$ is a $t$-adically complete and $t$-torsion-free $K^0$-algebra, let $R = A[t^{-1}]$, then the norm

$$|f| = \inf\{|t|^n |f| \in t^n A\},$$

then this makes $R$ into a $K$-Banach space that the $t$-adic topology of $A$ is the same as the metric topology of $A$, so $A \subset R_{\leq 1} \subset R^0$.

Notice if $|K^*|$ is not discrete but there is a pseudo-uniformizer $t$ that has a compatible system of $p^n$-th roots, if $A$ is a $t$-adically complete and $t$-torsion-free $K^0$-algebra, let $R = A[t^{-1}]$, then the norm

$$|f| = \inf\{|t|^n |f| \in t^n A\},$$

then this makes $R$ into a $K$-Banach space that the $t$-adic topology of $A$ is the same as the metric topology of $A$, so $A \subset R_{\leq 1} \subset R^0$, and in this case $R^0 = A_\ast = \text{Hom}(t^{\frac{1}{p^n}} A),(1.16.2.2)$. 

Recall $R$ is called uniform if $R^0$ is itself bounded in $R$ (IV.1.1.6). Notice a uniform Banach space must be reduced, because a nilpotent element is clearly power bounded, and any scalar multiple of it is nilpotent.

Lemma (IV.1.2.6). Fix a uniformizer $t$ in a non-Archimedean complete field $K$, if $|K^*|$ is discrete, then if $A$ is a $t$-adically complete and $t$-torsion-free $K^0$-algebra, let $R = A[t^{-1}]$, then the norm

$$|f| = \inf\{|t|^n |f| \in t^n A\},$$

then this makes $R$ into a $K$-Banach space that the $t$-adic topology of $A$ is the same as the metric topology of $A$, so $A \subset R_{\leq 1} \subset R^0$.

Notice if $|K^*|$ is not discrete but there is a pseudo-uniformizer $t$ that has a compatible system of $p^n$-th roots, if $A$ is a $t$-adically complete and $t$-torsion-free $K^0$-algebra, let $R = A[t^{-1}]$, then the norm

$$|f| = \inf\{|t|^n |f| \in t^n A\},$$

then this makes $R$ into a $K$-Banach space that the $t$-adic topology of $A$ is the same as the metric topology of $A$, so $A \subset R_{\leq 1} \subset R^0$, and in this case $R^0 = A_\ast = \text{Hom}(t^{\frac{1}{p^n}} A),(1.16.2.2)$. 

Recall $R$ is called uniform if $R^0$ is itself bounded in $R$ (IV.1.1.6). Notice a uniform Banach space must be reduced, because a nilpotent element is clearly power bounded, and any scalar multiple of it is nilpotent.
Prop. (IV.1.2.7) (Uniform $K$-Banach Space and $K^0$-Algebra). Fix a pseudo uniformizer $t$ in a non-Archimedean complete field $K$, the following category are equivalent:

- The category $\mathcal{C}$ of uniform Banach $K$-algebras $R$.
- The category $\mathcal{D}_{\text{adic}}$ of $t$-adically complete and $t$-torsionfree $K^0$-algebras $A$ with $A$ totally integrally closed (I.5.5.1) in $A[t^{-1}]$.

Proof: The functor $F : \mathcal{C} \to \mathcal{D}_{\text{adic}} : R \to R^0 : R^0$ is open subring by (IV.1.2.5), and $R^0 \in B(0,r)$ for some $r > 0$ by uniformity. As $R$ is $K$-Banach, $\cap t^n B(0,r) = 0$, so $R^0$ is $t$-adically separated, and also it is complete. If $f^N \in t^{-k} R^0 \subset t^{-k} B(0,r]$, then clearly $f$ is power bounded thus $f \in R^0$, so $R^0$ is totally integrally closed in $R$. $R \to R^0$ is preserved by continuous mappings, so $F$ is truly a functor.

Conversely, lemma above (IV.1.2.6) shows $R = A[t^{-1}]$ is a $K$-Banach algebra, this is a functor $G : \mathcal{D}_{\text{adic}} \to \mathcal{C}$, and $A \subset R^0$. We show $A = R^0$, as this is equivalent to $FG \cong \text{id}$: as the $t$-adic topology and metric topology are the same (IV.1.2.6), if $t^c f^N \subset A$ for some $c$, thus $f$ is totally integral over $A$, thus $f \in A$ by $t$ic.

Finally, we need to show $GF \cong \text{id}$, which in fact that the given Banach algebra norm on $R$ is equivalent to the norm $| \cdot |'$ given in (IV.1.2.6) w.r.t $R^0$. $R_{\leq 1} \subset R^0 \subset R_{\leq c}$ by uniformity, and conversely, $R_{\leq 1} \subset R^0 \subset R_{\leq 1}$, thus this two norms are equivalent. \hfill $\Box$

Prop. (IV.1.2.8). Let $\varphi : A \to B$ be a $k$-homomorphism between $k$-Banach algebras that there is a family $\mathfrak{B}$ of ideals of $B$ that for each $b \in \mathfrak{B}$:

- $B$ is closed and $\varphi^{-1}(b)$ is closed in $A$.
- $\dim_k B/b < \infty$.
- $\cap_{b \in \mathfrak{B}} b = (0)$.

Then $\varphi$ is continuous.

Proof: Consider the map $A/\varphi^{-1}(b) \to B/b$ with the residue norms, Cf. [non-Archimedean analysis P167]. \hfill $\Box$

Cor. (IV.1.2.9). Let $\varphi : A \to B$ be a $k$-homomorphism between Noetherian $k$-Banach algebras that there is a family $\mathfrak{B}$ of ideals of $B$ that for each $b \in \mathfrak{B}$, $\dim_k B/b < \infty$ and $\cap_{b \in \mathfrak{B}} b = (0)$, then $\varphi$ is continuous. (Because the closedness condition is automatic by (IV.1.2.12)).

Cor. (IV.1.2.10). All complete $k$-algebra norms on a Noetherian $k$-algebra $B$ satisfying the condition of (IV.1.2.9) are equivalent.

Modules over $K$-Banach Spaces

Prop. (IV.1.2.11). If $M$ is a normed module over a $K$-Banach algebra $A$, if the completion of $M$ is a finite $A$-module, then $M$ is complete.

Proof: There are morphism $\pi : A^n \to \hat{M}$ that are surjective continuous, so by open mapping theorem (X.3.2.4), this map is open, so $\sum \hat{A}x_i = \pi(A^n)$ is a nbhd of $0$ in $\hat{M}$, because $\hat{A}$ is open (IV.1.1.9) and then $\hat{M} = M + \sum \hat{A}x_i$, because $M$ is dense in $\hat{M}$, then we are done by (IV.1.1.10). \hfill $\Box$

Cor. (IV.1.2.12) (Noetherian and Submodule Closed). For a complete normed module over a $K$-Banach algebra $A$, $M$ is Noetherian iff all submodules of $M$ are closed. In particular, $A$ is Noetherian iff all ideals of $A$ are closed.
Proof: If $M$ is Noetherian, then the completion of any submodule is finite over $A$, so it is complete hence closed by (IV.1.2.11). Conversely, if any ideal of $M$ is closed, then for a chain of ideals of $M : \cup M_i = M'$, $M'$ is complete hence Baire space by (IX.1.9.2), so some $M_i$ must contain a nbhd of $M'$, because it is an ideal, but then $M_i = M'$.

3 $p$-adic Analysis

Basic References are [p-adic Analysis Robert].

Prop. (IV.1.3.1). For $b \in \mathbb{Z}_p$, we can define a power series in $\mathbb{Z}_p[[T]]$ as the limit of $(1 + a)^{b_n}$ for $b_n \to b$ in $\mathbb{Z}_p$. So for $a \in \mathbb{C}_p$ with $v(a) > 0$, there can be defined an element $(1 + a)^b \in \mathbb{C}_p$, and we have $(1 + a)^b = \sum C_b^k a^k$.

Prop. (IV.1.3.2). $\overline{\mathbb{Q}_p}$ is isomorphic to $\mathbb{C}$ as a field, un canonically.

Cor. (IV.1.3.3). The $p$-adic valuation on $\mathbb{Q}$ can be extended to $\mathbb{R}$ un canonically.

Holomorphic functions

Def. (IV.1.3.4). For a $p$-adic field $L$, denote by $\mathcal{L}_L$ the set of Laurent series with coefficients in $L$, then the set of valuations that a Laurent series converges Conv$(f)$ is an interval of $[-\infty, +\infty]$. Let $\mathcal{A}(I)$ denote the set of elements in $L$ of valuation in $I$.

If $f$ is bounded at $r_1, r_2$, then it is convergent on $(r_1, r_2)$.

Def. (IV.1.3.5). Denote 
$\mathcal{L}_L[r_1, r_2] = \{f | f$ is convergent on $[r_1, r_2]\}$.
$\mathcal{L}_L(r_1, r_2) = \{f | f$ is convergent on $(r_1, r_2)\}$.
$\mathcal{L}_L[r_1, r_2] = \{f | f$ is convergent on $(r_1, r_2)$ and bounded at $r_1\}$.

$\mathcal{B}_L(I)$ is the subset of bounded functions. These are all rings under addition and multiplication.
And if we define $v^{(r)}(f)$ as the minimum of $v(a_n) + nr$, then it is a valuation on these rings.

Proof: Cf. [Foundations of Theory of $(\varphi, \Gamma)$-modules over the Robba Ring P31].

Def. (IV.1.3.6). If we set for $\mathcal{L}_L[r_1, r_2]$ the value of $v^{[r_1, r_2]}(f) = \min\{v^{(r_1)}(f), v^{(r_2)}(f)\}$, then this is a valuation on it.

Prop. (IV.1.3.7). $\mathcal{L}_L(\{r\})$ is complete under valuation $v^{(r)}$. Similarly the valuation $v^{[r_1, r_2]}(f)$ makes $\mathcal{L}_L[r_1, r_2]$ a Banach space unless $r_1 = r_2 = \infty$.

Proof: We let $r = 0$. For a Cauchy sequence of Laurent series, we see that each coefficient is a Cauchy sequence, hence converge to some element in $L$, so it converge term-wise to a Laurent series $f$, so it converse to $f$ in $v^{(r)}$.

Cor. (IV.1.3.8). We consider $\mathcal{L}_L(0, r]$, then it has a countable sequence of norms $v^{1/n,r}$, which makes it a locally convex space, and the last proposition shows that these valuations are complete, and a Cauchy sequence must converge to the term-wise limit, so $\mathcal{L}_L(0, r]$ is a complete Fréchet space in the Fréchet topology.

Cor. (IV.1.3.9). The same method shows that $\mathcal{L}_L(I)$ is a Fréchet space for any interval $I$. 
Def. (IV.1.3.10) (Robba Ring and Overconvergent Elements). We define $\mathcal{E}$ as the Laurent sequences that are bounded at 0 and $\lim_{n \to -\infty} v(a_n) = \infty$, and we define the overconvergent elements $\mathcal{E}^\dagger$ and Robba ring $\mathcal{R}$ as

$$\mathcal{E}^\dagger = \bigcup_{r > 0} \mathcal{L}_L[0, r], \quad \mathcal{R} = \bigcup_{r > 0} \mathcal{L}_L(0, r], \quad \mathcal{E}^\dagger \subset \mathcal{R}$$

and equip them with the final topology w.r.t. the Fréchet topologies on $\mathcal{L}_L(0, r]$. And denote by $\mathcal{E}^+ = \mathcal{E}^\dagger \cap L[[T]]$ and $\mathcal{R}^+ = \mathcal{R} \cap L[[T]]$.

For more properties of Robba ring, See [Foundations of Theory of $(\varphi, \Gamma)$-modules over the Robba Ring Chap4].

Def. (IV.1.3.11) (Newton Polygon). For a non-Archimedean valued field $K$ and a polynomial or power series $P(X) = a_0 + a_1X + \cdots + a_dX^d \in K[X]$, we denote by Newton polygon as the lower convex hull of the set of points $(0, v(a_0)), (1, v(a_1)), \ldots, (d, v(a_d))$.

Prop. (IV.1.3.12). For a non-Archimedean field $K$ the number of roots of $P$ in $\overline{K}$ with valuation $\lambda$ equals the horizontal width of the segment of Newton polynomial of $P$ of slope $-\lambda$.

Proof: We may assume $P$ is monic, then its coefficients are elementary polynomials of roots of $P$. And the conclusion follows as $K$ is non-Archimedean. □

For Newton polynomial of power series, see [Berger Galois Representations Chap3] and Reference [Zeros of Power Series over complete Valued Field Lazard].

Prop. (IV.1.3.13). If $I = [0, +\infty]$ and $f(X) \in \mathcal{H}(I)$, then the number of zeros of $f(X)$ in $\mathcal{A}(I)$ equals the length of the segment of $NP(f)$ whose slope is $-s$, and these roots gives a $P_s(X) \in K[X]$ that $f(X) = P_s(X)G(X)$, $G(X) \in \mathcal{H}(I)$.

Proof: Cf.[Zeros of Power Series over complete Valued Field Lazard]. □

Cor. (IV.1.3.14). If $f(X) \in \mathcal{H}(I)$, then $f(X) \in \mathcal{B}_L(I)$ iff it has f.m. zeros in $\mathcal{A}(I)$.

Proof: Let $r = \inf I$ and $s = \sup I$. First notice that $f \in \mathcal{L}_L(I)$ is in $\mathcal{B}(I)$ iff $v(a_n) + nr$ is bounded from below as $n \to +\infty$ and $v(a_n) + ns$ is bounded below as $n \to -\infty$. And from the graph of $NP(f)$, this is equivalent to $f$ has f.m. zeros in $\mathcal{A}(I)$. □

Prop. (IV.1.3.15). $\mathcal{H}(I)$ is a Bezout domain.
IV.2 Algebraic Number Theory

Main references are [Algebraic Number Theory Neukirch], should also include notes of Pete.L.Clark. [Neukirch Chap2.8, 2.9, 3.1, 3.2] should be added quickly, [Sen80].

1 Ramification Theory

In this subsection we study ramification theory of local fields, or more generally CDVRs.

Prop. (IV.2.1.1). If a prime $p$ splits completely in two separable extension $LM$ of $K$, then it also splits completely in the composite $LM$.

Proof: We use the language of valuation. The extension of a valuation $v$ of $K$ corresponds to the set of equivalent classes of algebra map from $L$ to $\overline{K_v}$ module conjugacy over $K_v$. So we only need to show that two different maps of $LM$ are not conjugate over $K_v$. But the restrict of them to $L$ or $M$ is different, thus not conjugate over $K_v$ by the assumption. □

Cor. (IV.2.1.2). A prime splits completely in a separable extension $L$ if it splits completely in the Galois closure $N$ of $L$.

Proof: This is because the Galois closure is the composite of the conjugates of $L$. But it also can be proven directly: Set $H = \text{Gal}(N/L)$, $P$ a prime of $N$ over $p$, then

$$H\backslash G/G_P \rightarrow \{\text{Primes of } L \text{ over } p\}, \quad H\sigma G_P \mapsto \sigma P \cap L$$

is a bijection. So it splits completely in $L \iff G_P$ is trivial $\iff$ it splits completely in $N$ by counting numbers. □

Prop. (IV.2.1.3). A prime $p$ splits in $\mathbb{Z}[\xi_n]$ iff $p \equiv 1(\text{mod } n)$.

Proof: First, if it splits, then $f = 1$, Because the ring of integers is $\mathbb{Z}[\xi_n]$, so $X^n - 1$ splits in $\mathbb{F}_p$ [IV.2.3.4], thus $p \equiv 1(\text{mod } n)$. And if $p \equiv 1(\text{mod } n)$, it is unramified and $X^n - 1$ splits in $\mathbb{F}_p$, so $f = 1$. □

Lemma (IV.2.1.4) (Extension is Monogenous). For a finite extension of CDVR, if the residue field extension $\lambda/k$ is separable, then there exists a $x \in \mathcal{O}_L$ that $\mathcal{O}_K[x] = \mathcal{O}_L$.

Proof: If $\pi$ is an element of $\lambda$ that generate $\lambda$ over $k$, by primitive element theorem, then let $\overline{f}$ be the minipoly of $\pi$, then let $f, x$ be lifting of them, then $f(x)$ is a uniformizer, otherwise, we now $f'(x)$ has valuation 0, so $f(x + \pi L)$ is a uniformizer. Now we see that $x^i f(x)^j$ is a basis of $\mathcal{O}_L$ over $\mathcal{O}_K$. □

Unramified Extension

Def. (IV.2.1.5). For $K$ a Henselian non-Archimedean valued field, $L/K$ a finite extension is called unramified iff the residue field extension $\lambda/k$ is separable and $[L:K] = [\lambda:k]$. Any algebraic extension is called unramified iff any finite extension is unramified.

This is compatible because unramified extensions form a distinguished class. So we can talk about the maximal unramified extension $T$ of $K$. 

Proof: It is faithfully transitive because the field extension degree is transitive, and for base change, as the residue field is separable, we let $\lambda = k[\overline{\alpha}]$, and choose a lift $\alpha \in \mathcal{O}_L$, the minipoly of $\alpha$ is $f(X) \in \mathcal{O}_K[X]$. Then we have

$$[\lambda : k] \leq \deg \overline{f} = \deg f = [K(\alpha) : K] \leq [L : K] = [\lambda : k]$$

So $L = K(\alpha)$ and $\overline{f}$ is the minipoly of $\overline{\alpha}$. Then $L' = K'(\alpha)$, and let $g(X)$ be the minipoly of $\alpha$ over $K'$, then $\overline{g}$ is a factor of $\overline{f}$ so separable, hence irreducible by Hensel’s lemma. Noe:

$$[\lambda' : k'] \leq [L' : K'] = \deg g = \deg \overline{g} = [k'(\alpha) : k'] \leq [\lambda' : k].$$

So $[\lambda' : k'] = [L' : K']$. □

Prop. (IV.2.1.6). The residue field of the maximal unramified extension $T/K$ is $\overline{k}$, and the value group is the same as $K$.

Proof: The first assertion is because for any separable polynomial, it has a lift which is irreducible has a root lifting $\overline{\alpha}$, contradicting the maximality. For the second, look at finite subextensions, then it results from the fundamental inequality(I.9.3.16). □

Tamely Ramified Extension

Def. (IV.2.1.7). For $K$ a Henselian non-Archimedean valued field, $L/K$ a finite extension is called tamely ramified iff the residue field extension is separable and $([L : T], p) = 1$, where $T$ is the maximal unramified subextension.

Prop. (IV.2.1.8). A finite extension $L/K$ is tamely unramified iff the extension is generated by radicals: $L = T(\sqrt[n]{\alpha_i})$, where $a_i \in L$, (WARNING: make sure if $a_i \in K$ or not?).

Proof: Cf.[Algebraic Number Theory Neukirch P155]. □

Prop. (IV.2.1.9). Tamely unramified extensions form a distinguished class, so we can talk about the maximal tamely unramified extensions.

Proof: Cf.[Algebraic Number Theory Neukirch P156]. □

Prop. (IV.2.1.10). The value field of tamely ramified extensions. Cf.[Neukirch P157].

Totally Ramified Extension

Def. (IV.2.1.11) (Eisenstein Polynomial). Let $K$ be a DVR with uniformizer $\pi$, an Eisenstein polynomial in $K[X]$ is a polynomial of the form

$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$$

where $a_i \in (\pi)$ for any $i$ and $a_0 \in (\pi) \backslash (\pi^2)$. 
Ramification Groups

**Prop. (IV.2.1.12).** For an extension valuation \( w \mid v \), the **decomposition group** is \( G_w(L/K) = \{ \sigma \in G(L/K) \mid w \circ \sigma = w \} \). The **decomposition field** \( Z_w \) is the fixed field of \( G_w \).

When \( w \) is non-Archimedean, we further define:

- The **inertia group** is \( I_w(L/K) = \{ \sigma \in G_w(L/K) \mid \sigma(x) \equiv x \mod \mathfrak{P} \} \). The **inertia field** \( T_w \) is the fixed field of \( I_w \).

- The **ramification group** is \( R_w(L/K) = \{ \sigma \in G_w(L/K) \mid \sigma(x)/x \equiv 1 \mod \mathfrak{P} \} \). The **ramification field** \( V_w \) is the fixed field of \( R_w \).

**Prop. (IV.2.1.13).** For a local field, the ramification degree \( e \) equals the order of inertia group \( \mid I_{L/K} \mid \).

**Prop. (IV.2.1.14).** When \( w \) is non-Archimedean, the residue field extension \( \lambda/k \) is normal and there is an exact sequence

\[
1 \to I_w \to G_w \to G(\lambda/k) \to 1.
\]

**Proof:** Cf.[Neukirch P172].

**Prop. (IV.2.1.15).** \( T_w/Z_w \) is the maximal unramified subextension of \( L/Z_w \).

**Proof:** Cf.[Neukirch P173].

**Prop. (IV.2.1.16).** \( V_w/Z_w \) is the maximal tamely ramified subextension of \( L/Z_w \).

**Proof:** Cf.[Neukirch P175].

**Higher Ramification Groups**

**Def. (IV.2.1.17) (Higher Ramifications).** For \( L/K \) be a finite Galois extension of CDVR, we define the **\( s \)-th ramification group** \( G_s(L/K) = \{ \sigma \in G \mid v_L(\sigma(x) - x) \geq s + 1 \text{ for all } a \in \mathcal{O}_L \} \).

Then we have \( G = G_{-1} \supset G_0 \supset G_1 \subset \ldots \) And \( G_0 \) is the inertia group.

When \( K \) has finite quotient field, then \( G_1 \) is the ramification group(one way is trivial, for the other, we use Teichmüller representatives, then \( R_w \) preserves all them, and \( \sigma(x) - x \equiv 0 \mod \mathfrak{P}^2 \) is true for \( \pi \), so it is true for all). In this case, we have

\[
G_s(L/K) = \{ \sigma \in G_0 \mid \frac{\sigma(\pi_L)}{\pi_L} \in U^{s+1}_L \}, \text{ for } s \geq 0.
\]

So there are injective morphism \( G_s/G_{s+1} \to U^s_L/U_L^{s+1} : \sigma \mapsto \sigma(\pi_L)/\pi_L \text{ for } s \geq 0, \text{(This is independent of } \pi_L \text{ chosen because units are mapped mod } U_L^{s+1}).

**Prop. (IV.2.1.18).** For local fields \( L/K \), if \( \sigma \) is in the inertia group, then

\[
v_L(\sigma(x)/x - 1) \geq v_L(\frac{\sigma(\pi_L)}{\pi_L} - 1) + \delta_{v_L(x),0}
\]

for any \( x \in \mathcal{O}_L \) and a uniformizer \( \pi_L \). Equality holds when \( v_L(x) = 1 \).
Proof: if \( L \) has residue field \( \mathbb{F}_q \), then any element of \( \mathcal{L} \) can be written as \( \sum \xi_n \pi_L^n \), where \( \xi_n \) are all \( q - 1 \)-th roots of unity. And because \( \sigma \) is inertia group, all \( q - 1 \)-th roots of unity are preserved, so \( \sigma(\xi_n \pi_L^n) = \xi_n \pi_L(\frac{\sigma(\pi_L)}{\pi_L} - 1) + \sigma(\pi_L)^{n-1} + \sigma(\pi_L)^{n-2} \pi_L + \ldots + \pi_L^{n-1} \) has valuation \( \geq v(\frac{\sigma(\pi_L)}{\pi_L} - 1) + n \). Thus the result. 

In the sequel, we assume that the residue field extension is separable, as to use the proposition (IV.2.1.4).

Lemma (IV.2.1.19). We define \( i_{L/K}(\sigma) = v_L(\sigma x - x) \), where \( x \) is the generator of \( \mathcal{O}_L/\mathcal{O}_K \).

If \( L/L'/K \) are Galois extensions that \( \epsilon \) is the ramification index of \( L/L' \). Then

\[
i^\epsilon_{L'/K}(\sigma') = \frac{1}{\epsilon} \sum_{\sigma|L'=\sigma'} i_{L/K}(\sigma).
\]

Proof: Cf. [Neukirch Algebraic Number Theory P178]. 

Def. (IV.2.1.20) (Upper Numbering). We define the Herbrand function \( \varphi_{L/K}(u) = \int_0^u \frac{dx}{(\log |x|)} \). It maps \( \{x \geq 1\} \) to itself and is strictly increasing.

If \( m \leq s < m + 1 \), then it is just \( \varphi_{L/K}(s) = \frac{1}{g_0}(g_1 + g_2 + \ldots + g_m + (s - m)g_{m+1}) \), where \( g_i = |G_i| \).

By a double counting, it is

\[
\varphi_{L/K}(s) = \frac{1}{g_0} \sum_{\sigma \in G} \min\{i_{L/K}(\sigma), s + 1\} - 1.
\]

The derivative of \( \varphi_{L/K} \) is \( \varphi'_{L/K}(s) = \frac{|G_s|}{g_0} \).

Let \( \psi_{L/K} \) be the inverse function of \( \varphi_{L/K} \), this is called the upper numbering.

Lemma (IV.2.1.21). For \( L/L'/K \) Galois extensions, one has \( G_s(L/K)H/H = G_t(L'/K) \), where \( t = \varphi_{L'/L}(s) \). Equivalently, \( G_s/H_s = (G/H)_{\varphi_{L'/L}(s)} \).

Proof: For \( \sigma' \in G(L'/K) \), we choose a inverse image \( \sigma \in G(L/K) \) of maximal \( i_{L/K}(\sigma) \), then \( i_{L'/K}(\sigma') - 1 = \varphi_{L'/L}(i_{L/K}(\sigma) - 1) \). To prove this, let \( i_{L/K}(\sigma) = m \), then we see \( i_{L/K}(\sigma \tau) = \min\{i_{L/K}(\tau), m\} \), so by (IV.2.1.19), \( i_{L'/K}(\sigma') = \frac{1}{\epsilon} \sum_{\tau \in H} \min\{i_{L/K}(\tau), m\} \). And \( e = |H_0| \) by (IV.2.1.13). So the assertion follows from (IV.2.1.20).

Now \( \sigma' \) is in the image of \( G_s \) is equivalent to \( i_{L/K}(\sigma) - 1 \geq s \iff \varphi_{L'/L}(i_{L/K}(\sigma) - 1) \geq \varphi_{L'/L}(s) \), which by what proved is equivalent to \( \sigma' \in G_t(L'/K) \).

Cor. (IV.2.1.22). For \( L/L'/K \) Galois extensions, \( \varphi_{L/K} = \varphi_{L'/L} \circ \varphi_{L'/L'} \), hence similar formula holds for \( \psi \).

Proof: By the proposition and multiplicity of ramification index \( e \), we get

\[
\frac{1}{e_{L/K}}|G_s| = \frac{1}{e_{L'/K}}|(G/H)_t| \cdot \frac{1}{e_{L'/L'}}|H_s|.
\]

where \( t = \varphi_{L'/L}(s) \), which is equivalent to the derivative \( \varphi'_{L/K}(s) = \varphi'_{L'/K}(t) \varphi'_{L'/L'}(s) = (\varphi_{L'/K} \circ \varphi_{L'/L'})'(s) \), and they are equal at 0, so the conclusion follows.
Prop. (IV.2.1.23) (Herbrand’s Theorem). For \( L/L'/K \) Galois extensions, \( G^t(L'/K) \) is the image of \( G^t(L/K) \) under the quotient.

Proof: Let \( r = \varphi_{L'/K}(t) \), by the above lemma and corollary,
\[
G^tH/H = G_{\varphi_{L/K}(t)}H/H = G'_{\varphi_{L'/L}(\psi_{L/K}(t))} = G'_{\varphi_{L'/L}(\psi_{L'/L}(r))} = G_{\varphi_{L'/K}(r)} = G^t(L'/K) = G^t(L'/K)
\]
\[\Box\]

Prop. (IV.2.1.24) (Hasse-Arf). For an Abelian extension of CDVRs \( L/K \) that the residue field extension is separable, the jump in the upper numbering of higher ramification group \( G^v \) must happen at integers. (Note: The proof in the case where \( K \) is a local field is much easier by Lubin-Tate group, See(IV.4.1.33).

Proof: The theorem is just saying that if \( G_s \neq G_{s+1} \) for \( s \) integer, then \( \varphi_{L/K}(s) \) is an integer.

This follows from the following lemma, because if \( G \) is not totally ramified, then we can change it to the Galois field of \( G_{0} \), this didn’t change anything by the definition of(IV.2.1.20), and the fact \( \varphi(0) = 0 \). And when \( G^v \neq G^{v+} \), then we consider splitting \( G/G^v \) into product of cyclic groups, thus there is one cyclic group \( H \) that the projection of \( G^v \) into \( H \) is not trivial. Now \( H \) is a Galois group of some \( L'/K \), and Herbrand’s theorem shows that \( H^v \neq H^{v+} \), hence \( v \) is an integer by the following lemma. \[\Box\]

Lemma (IV.2.1.25). For a cyclic totally ramified extension of CDVRs \( L/K \) that the residue field extension is separable, if \( \mu \) is the maximal integer that \( G_{\mu} \neq 1 \), then \( \varphi_{L/K}(G_{\mu}) \) is an integer.

Proof: Cf.[Serre Local Fields P94]. \[\Box\]

Remark (IV.2.1.26). An example: If \( F_n = \mathbb{Q}_p(\zeta_{p^n}) \), then

\[
G(F_n/\mathbb{Q}_p)_s = G(F_n/F_t) \quad \text{for} \quad p^t - 1 \leq s < p^{t+1} - 2.
\]

(This is because \( \zeta_{p^n} - 1 \) is a uniformizer of \( F_n \)). Thus \( G(F_n/\mathbb{Q}_p)^t = G(F_n/F_t) \).

Different and Discriminant

Def. (IV.2.1.27) (Different). Let \( L/K \) be a finite separable field extension with separable residue field extension, and \( \mathcal{O}_K \) is a Dedekind domain with integral closure \( \mathcal{O}_L \) in \( L \), there is a trace form on \( L: (x, y) \rightarrow \text{tr}(xy) \).

We define the dual module for a fractional ideal \( I \) as \( \hat{I} = \{ x \in L | \text{tr}(xI) \in \mathcal{O}_K \} \). This is truly a fractional ideal because if \( \alpha_i \in \mathcal{O}_L \) is a basis of \( L/K \), and let \( d = \det(\text{tr}(\alpha_i\alpha_j)) \), then for any \( a \in I \cap \mathcal{O}_L, ad\hat{I} \in \mathcal{O}_L \), because if \( x = \sum x_i\alpha_i \in \hat{I} \), then \( \sum ax_i \text{tr}(\alpha_i\alpha_j) = \text{tr}(x\alpha_j) \in \mathcal{O}_K \), so solve the equation shows \( dax_i \in \mathcal{O}_K \).

The different of \( K/L \) is defined to be \( \mathcal{D}_{L/K} = \mathcal{O}_L^{-1} \).

Prop. (IV.2.1.28) (Properties of Differents). Different is compatible with composition, localize to a prime ideal and completion.

Proof: Cf.[Neukirch Algebraic Number Theory P195]. \[\Box\]

Cor. (IV.2.1.29). \( \mathcal{D}_{L/K} = \prod_{\mathfrak{p}} \mathcal{D}_{L_{\mathfrak{p}}/K_{\mathfrak{p}}} \).
Prop. (IV.2.1.30). If \( \mathcal{O}_L = \mathcal{O}_K[\alpha] \), then \( \mathcal{D}_{L/K} = (f'(\alpha)) \), where \( f \) is the minipoly of \( \alpha \).

Proof: Let \( f = a_0 + a_1 X + \ldots + a_n X^n \) and \( f(X)/(x - \alpha) = b_0 + b_1 X + \ldots + b_{n-1} X^{n-1} \), and denote the roots of \( f \) be \( \alpha_i \), then
\[
\sum_{r=0}^{n} f(X) \frac{\alpha_i^r}{X - \alpha_i} f'(\alpha_i) = X^r
\]
for all \( r \) by Lagrange interpolation (I.2.2.2). This is equivalent to
\[
\text{tr}(\alpha_i^r b_j f'(\alpha_i)) = \delta_{ij}
\]
So \( \mathcal{D}_{L/K} = f'(\alpha)^{-1}(b_0 \mathcal{O}_K + b_1 \mathcal{O}_K + \ldots + b_{n-1} \mathcal{O}_K) \). Now the result follows if \( (b_0 \mathcal{O}_K + b_1 \mathcal{O}_K + \ldots + b_{n-1} \mathcal{O}_K) = \mathcal{O}_L \), which is easy to see if we write \( b_i \) as polynomials of \( \alpha \). \( \square \)

Cor. (IV.2.1.31). If \( L/K \) is finite extension of local fields, then
\[
v_L(\mathcal{D}_{L/K}) = \sum_{\sigma \in G, \sigma \neq 1} i_{L/K}(\sigma) = \int_{-1}^{\infty} (|G(L/K)|t - 1)dt.
\]
Notation as in (IV.2.1.19).

Prop. (IV.2.1.32). If \( L/K \) is a finite extension and if \( I \) is an ideal of \( \mathcal{O}_L \), then \( v_K(\text{tr}_{L/K}(I)) = |v_K(I \cdot \mathcal{D}_{L/K})| \).

Proof: By definition, \( \text{tr}_{L/K}(x \mathcal{O}_L) \subset \mathcal{O}_K \) iff \( x \in \mathcal{D}_{L/K}^{-1} \), thus \( \text{tr}_{L/K}(I) \subset J \) iff \( I \subset \mathcal{D}_{L/K}^{-1} J \), i.e. \( \text{tr}_{L/K}(I) \) is the smallest ideal \( J \) of \( \mathcal{O}_K \) that contains \( I \cdot \mathcal{D}_{L/K} \), thus the result. \( \square \)

Prop. (IV.2.1.33) (Ramification and Different). A prime ideal \( \mathfrak{p} \) of \( L \) is ramified over \( K \) if \( \mathfrak{p} \mid \mathcal{D}_{L/K} \).

Let \( e \) be the ramification of \( \mathfrak{p} \), then the power \( s \) of \( \mathfrak{p} \) in \( \mathcal{D}_{L/K} \) is
\[
\begin{align*}
s = e - 1 & \quad \mathfrak{p} \text{ tamely ramified} \\
e \leq s \leq e - 1 + v_{\mathfrak{p}}(e) & \quad \mathfrak{p} \text{ wildly ramified}
\end{align*}
\]

Proof: Cf.[Neukirch, P199]. \( \square \)

Prop. (IV.2.1.34) (Different as Annihilator of Kähler Differential). The different \( \mathcal{D}_{L/K} \) is the annihilator of \( \Omega_{\mathcal{O}_L/\mathcal{O}_K} \).

Proof: It suffices to show the exact sequence
\[
0 \to \mathcal{D}_{L/K} \to \mathcal{O}_L \to \Omega_{\mathcal{O}_L/\mathcal{O}_K},
\]
but because exactness is stalkwise (I.5.1.55), we can localized at a maximal ideal, then by (IV.2.1.4), \( \mathcal{O}_L = \mathcal{O}_K[x] \) is monogenous, thus \( \Omega_{\mathcal{O}_L/\mathcal{O}_K} \) is cyclic, and the annihilator of \( dx \) is \( f'(x) \). So by (IV.2.1.30) we are done. \( \square \)

Def. (IV.2.1.35) (Discriminant of a Basis). Let \( \alpha_i \) be a basis of a separable extension \( L/K \), then the discriminant \( d(\alpha_1, \ldots, \alpha_n) \) is defined to be \( \det(\sigma_i(\alpha_j))^2 \).

Notice \( \text{tr}_{L/K}(\alpha_i \alpha_j) = \sum_k \sigma_k(\alpha_i)\sigma_k(\alpha_j) \), thus \( (\text{tr}_{L/K}(\alpha_i \alpha_j)) \) is the product of the matrices \( (\sigma_k \alpha_i)^t \) and \( (\sigma_k \alpha_j) \), thus we also have
\[
d(\alpha_1, \ldots, \alpha_n) = \det(\text{tr}_{L/K}(\alpha_i \alpha_j)).
\]

Clearly, \( d(\alpha_1, \ldots, \alpha_n) \) is invariant under the Galois action of \( L/K \), thus it is an element of \( K \).
Def. (IV.2.1.36) (Discriminant). Let the situation the same as in (IV.2.1.27), the discriminant $\delta_{L/K}$ is defined to be the set of discriminants $d(\alpha_1, \ldots, \alpha_n)$ (IV.2.1.35), where $\alpha_i$ is a basis of $L/K$ that $\alpha_i \in \mathcal{O}_L$.

Because $d(\alpha_1, \ldots, \alpha_n) \subset \mathcal{O}_K$ and $\text{tr}_{L/K}$ is $\mathcal{O}_K$-linear, it is an ideal of $\mathcal{O}_K$.

Prop. (IV.2.1.37) (Different and Discriminant).

$$\delta_{L/K} = N_{L/K} \mathcal{D}_{L/K}.$$  

Proof: Cf. [Neukirch ANT, P201].

Cor. (IV.2.1.38). For a tower of fields $K \subset L \subset M$, we have

$$\delta_{M/K} = \delta_{L/K}^{[M:L]} \mathcal{N}_{L/K}(\delta_{M/L}).$$

Proof: Apply $N_{M/K} = N_{L/K} N_{M/L}$ to the equation $\mathcal{D}_{M/K} = \mathcal{D}_{M/L} \mathcal{D}_{L/K}$ (IV.2.1.28), we get

$$\delta_{M/K} = N_{L/K}(\delta_{M/L}) \mathcal{N}_{L/K}(\mathcal{D}_{L/K}^{[M:L]}) = \mathcal{N}_{L/K}(\delta_{M/L}) \delta_{L/K}^{[M:L]}.$$  

Cor. (IV.2.1.39) (Ramification and Discriminant). A prime ideal $p$ of $K$ ramifies in $L$ iff $p | \delta$. In particular, the extension is unramified iff $\delta = 1$.

Proof: This follows from (IV.2.1.33).

Cor. (IV.2.1.40). $\delta_{L/K} = \prod_p \delta_{L_p/K_p}$.

Proof: This follows from (IV.2.1.29) and (IV.2.1.37).

Minkowski Theory

Thm. (IV.2.1.41) (Finite Ramification is Rare). Let $K$ be a number field and $S$ a finite set of primes of $K$, then there are only f.m. field of a given degree $n$ that are unramified outside $S$.

More precisely, there are only f.m. extension field of bounded degrees and bounded discriminants.

Proof: The power of a prime $\mathfrak{P}$ in the discriminant is controlled by $n$ by (IV.2.1.33). Together with (IV.2.1.37), thus shows the power of $p$ in the discriminant of the extension is controlled by $n$, independent of the field. Also we can assume $\sqrt{-1} \in L$, because it multiply the discriminant with a bounded factor. So it suffices to prove there are only f.m. field extension with fixed degree and discriminant. By (IV.2.1.38), we can assume $K = \mathbb{Q}$.

For the rest, we use Minkowski’s theorem, Cf. [Neukirch, P203].

Prop. (IV.2.1.42) (Lower Bounds for Discriminant). The discriminant of a number field $K$ of degree $n$ satisfies

$$|d_K|^{1/2} \geq \frac{n^n}{n!} \left(\frac{\pi}{4}\right)^{n/2}.$$  

Proof: Cf. [Neukirch, P204].

Cor. (IV.2.1.43) (Hermit’s Theorem). There are only f.m. number fields of finite discriminant.

Proof: The above proposition shows the degree is controlled by the discriminant, thus the theorem follows from (IV.2.1.41).

Cor. (IV.2.1.44) (Minkowski’s Theorem). The discriminant of a number field $K \neq \mathbb{Q}$ is not $\pm 1$.

Cor. (IV.2.1.45). $\mathbb{Q}$ doesn’t have any unramified extensions, by (IV.2.1.39).
2 Local Fields

Def. (IV.2.2.1) (Local Fields). A local field is a field that is complete w.r.t. a discrete valuation and has a finite residue field.

For a local field, the normalized exponential valuation is denoted by $v_p$, and the normalized absolute valuation is defined by $|x|_p = q^{-v_p(x)}$, where $q = |k|$, where $k$ is the residue field.

Prop. (IV.2.2.2). For a local field, $O_K$ and $K$ are locally compact.

Proof: $O_K = \lim O/p^n$, because $O_K$ is a complete DVR, so it is profinite, hence closed and compact. $K$ is locally compact because for any $a$, $a + O_K$ is compact.

Prop. (IV.2.2.3) (Local Fields). The local fields are precisely the finite extensions of the field $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, called $p$-adic number field and $p$-adic function field respectively.

Proof: Cf. [Neukirch Algebraic Number Theory P135].

The Group Structure of Local Fields

Prop. (IV.2.2.4). For $m > 0$, there is an isomorphism $(-)^m : U^n \cong U^{n+v(m)}$ when $n$ is sufficiently large.

Proof: Let $m = u\pi^{v(m)}$. For surjectivity, we need to find $x$, that $1 + a\pi^{n+v(m)} = -(1 + x\pi^n)^m$. i.e.

$$-a + ux + \pi^{n-v(m)}f(x) = 0.$$ 

This has a solution $x$ by Hensel’s lemma.

Cor. (IV.2.2.5). $(K^*)^m$ is an open subgroup of $K^*$, and $\bigcap_m (K^*)^m = 1$. (Because if $a \in \bigcap_m (K^*)^m = 1$, then $a$ is a unit, thus $a \in \bigcap_m (U)^m = 1$, thus $a \in U^n$ for every $n$ thus $a = 1$).

Prop. (IV.2.2.6). $[K^* : (K^*)^m] = m \cdot |m|_p^{-1} \cdot |\mu_m(K)|$.

Proof: Use the multiplicative Herbrand quotient (I.11.4.9), $(K^* : (K^*)^m) = q_{0,m}(K^*)|\mu_m(K)|$. $q_{0,m}$ is additive, thus

$$q_{0,m}(K^*) = q_{0,m}(K/U)q_{0,m}(U/U^n)q_{0,m}(U^n).$$

$q_{0,m}(K/U) = m$, $q_{0,m}(U/U^n) = 1$ as $U/U^n$ is finite (I.11.4.11). It For $q_{0,m}(U^n)$, when $n$ is large, it equals $(U^n : U^{n+v(m)})$ by (IV.2.2.4), which is $|m|_p^{-1}$.

Prop. (IV.2.2.7) ($p$-adic Logarithm). For a $p$-adic number field $K$, there is a unique $p$-adic logarithm function $\log : K^* \to K$ that $\log(p) = 0$, and for $1 + x \in U^1$, it is defined to be

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots.$$ 

Moreover, for $n > \frac{e}{p-1}$, where $e$ is the ramification index of $K$, there is a map $\exp : \mathbb{F}_p^n \to U^n$ which is an inverse to $\log$ on $U^n$, so $U^n \cong \mathbb{F}_p^n$.

Proof: This is easy by a Newton polygon analysis, the slope of the Newton polygon is $\frac{1}{p^{e-1}(p-1)}$, which converges to 0, so this is definable for all $x$ that $v_p(x) > 0$.

For the exp, $v_p(n!) = \frac{n-c(n)}{p-1}$ by (XIV.1.2.1), so its Newton polygon is a single line with slope $\frac{1}{p-1}$, so it is definable for $v_p(x) \geq \frac{1}{p-1}$, which is equivalent to $x \in U^{[\frac{e}{p-1}]}$. That exp and log are converse to each other is just a formal calculus.
Remark (IV.2.2.8). In fact, this map can be extended to a function from $\mathbb{C}_p^*$ to $\mathbb{C}_p$.

Cor. (IV.2.2.9). For a local field $K$, $\mathcal{O}_K^*$ thus also $K^*$ are locally compact.

Proof: For $n$ large, $U^n \cong \mathfrak{p}^n$ is compact. 

Prop. (IV.2.2.10) (Multiplicative Group Structure). For a local field $K$,
- If char $K = 0$, then $\mathcal{O}_K^* \cong \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}/p^a\mathbb{Z} \oplus \mathbb{Z}^d$, where $d = [K : \mathbb{Q}_p]$.
- If char $K = p$, then $\mathcal{O}_K^* \cong \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}_p^N$.

Proof: Cf. [Neukirch P140]. 

Prop. (IV.2.2.11). Any automorphism of $\mathbb{R}$ is identity, and any automorphism of a $p$-adic number field is identity.

Proof: It suffices to show that an automorphism is continuous. For $\mathbb{R}$, this is because $a > 0 \iff a = b^2 \iff \sigma(a) = \sigma(b)^2 \iff \sigma(a) > 0$, and $\mathbb{Q}$ is dense in $\mathbb{R}$.

For a local field, we prove that $\sigma(\mathcal{O}_K^*) \subset \mathcal{O}_K^*$. $\mathcal{O}_K^*$ is characterized by the property that $\{n|y^n = x\}$ are infinite. This is because $x^p = a$ has a root for $a \in \mathcal{O}_K^*$ for $p$ large prime, by Henselian lemma. 

Ramifications of Cyclotomic Fields ($\mathbb{Z}_p$-Extensions)

Prop. (IV.2.2.12) (Unramified case). For $K$ a finite extension of $\mathbb{Q}_p$ of residue field $\mathbb{F}_q$, we consider $L = K(\zeta_n)/K$, where $(n, p) = 1$. Then it is unramified of degree $f$ where $f$ is the minimal number that $q^f \equiv 1$ mod $n$. And $\mathcal{O}_L = \mathcal{O}_K[\zeta_n]$.

Proof: $\zeta$ is a root of $\Phi_n |X^n - 1$, which is separable in $k$, so $\Phi$ and $\Phi$ are both irreducible of the same degree by Hensel’s lemma, so it is unramified, and $\lambda$ is the minimal extension of $\mathbb{F}_q$ that contains the $n$-th roots and are generated by it, thus the result by the theory of finite fields.

For the last assertion, notice it is unramified so $\mathcal{O}_L = \mathcal{O}_K[\zeta_n] + p\mathcal{O}_L$ hence the result follows from Nakayama. 

Cor. (IV.2.2.13). The maximal unramified extension of $K$ is generated by adjoining all $n$-th roots where $(n, p) = 1$. This is because there is an inclusion relation and their residue field $\mathbb{F}_p$ is already generated by roots of unity.

Prop. (IV.2.2.14) (Totally Ramified case). Consider $\mathbb{Q}_p$(other local fields behave different), we have the $\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}$ is totally ramified of degree $\varphi(p^n)$ the Galois group is $(\mathbb{Z}/p^n\mathbb{Z})^*$. The ring of valuation of $\mathbb{Q}(\zeta_{p^n})$ is $\mathbb{Z}_p[\zeta_{p^n}]$ and $1 - \zeta_{p^n}$ is a uniformizer.

Proof: $\zeta$ is a root of the polynomial $\Phi = X^{p^n-1}(p-1) + X^{p^n-2}(p-1) + \ldots + 1 = 0$, which equals $\frac{X^{p^n-1}}{X^{p^n-1}} \equiv (X - 1)p^{n-1}(p-1)$ mod $p$ and $\Phi(1) = p$, so $\Phi(X + 1)$ is a Eisenstein polynomial, hence irreducible. So $\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}$ is totally ramified of degree $p^{n-1}(p-1)$ and $N(1 - \zeta) = \prod(1 - \sigma(\zeta)) = \Phi(1) = p$, so it is a uniformizer. The ring of integer is generated by a uniformizer by (I.9.3.16) as the extension is totally ramified. 

Prop. (IV.2.2.15). The cyclotomic field $\mathbb{Q}[\zeta_n]$ has integral basis.

Proof: Cf. [Algebraic Number Theory Milne P98].
Prop. (IV.2.2.16) (Infinite Cyclotomic Field). For a $p$-adic number field $K$, let $K_n = K(\zeta_{p^n})$ and $K_\infty = \bigcup K_n$ and $F = \mathbb{Q}_p$. Let $\chi$ be the cyclotomic character, then $\chi(G_K)$ is an open subgroup of $\mathbb{Z}_p^*$, thus contains a $U_n$ for some $n$. Thus there is an isomorphism of groups: $\chi^{-1}(U_n) \cap G_K = \chi^{-1}(U_n + 1) \cap G_K \cong U_n/U_{n+1}$ which has order $p$, for $n$ large.

So $K_{n+1}/K_n$ is totally ramified of degree $p$, because $K_n = K \cdot F_n$, and its value group extension is of degree $p$, too.

And $\{|\{K_n : F_n\}|$ is decreasing and eventually equals to $[K_\infty : F_\infty]$. This is because its order equals $\chi^{-1}(U_n')/\chi^{-1}(U_n') \cap G_K \cong \chi^{-1}(U_n)G_K/G_K$, which is eventually $\mathrm{Ker}(\chi)G_K/G_K$, because $U_n \subset (\chi(G_K))$.

Cor. (IV.2.2.17). For $n$ large, if $x_1$ is a set of basis of $O_{K_n}$ over $O_{F_n}$, then they form a basis of $K_N$ over $F_N$ for all $N \geq n$. This is because it generate $K_N$ over $F_N$ and $[K_N : F_N] = [K_n : F_n]$.

Prop. (IV.2.2.18). $p^n v_p(D_{K_{n+1}/F_n})$ is bounded and eventually constant. In particular $v_p(D_{K_n/F_n})$ converges to $0$.

Proof: Cf.[Galois representation Berger P20].

Cor. (IV.2.2.19). If $L/K$ is a finite extension, then $\text{tr}_{L_\infty/K_\infty}(m_{L_\infty}) = m_{K_\infty}$.

Proof: By (IV.2.1.32) and the fact $G(L_\infty/K_\infty) \cong G(L/K)$ for $n$ large by (IV.2.2.16), we have $\text{tr}_{L_\infty/K_\infty}(m_L) = m_{L_n}$, where $m_L = |\nu_K(m_{L_n} D_{L_n/K_n})|$. By the above proposition, $c_n$ is bounded by a $c$. But if $x \in m_{K_\infty}$, $x \in m_{K_n}$ for $n$ large, so $x \in \text{tr}_{L_\infty/K_\infty}(m_{L_\infty})$.

Lemma (IV.2.2.20). For any $\delta > 0$, when $n$ is large, if $x \in O_{K_{n+1}}$ and $g \in G(K_{n+1}/K_n)$, $v_p(g(x) - x) \leq \frac{1}{p-1} - \delta$. In particular, $v_p(N_{K_{n+1}/K_n}(x) - x^p) \leq \frac{1}{p-1} - \delta$.

Proof: Choose a basis $e_i$ of $O_{K_n}/O_{F_n}$, then $e_i^*$ is a basis for $D_{K_n/F_n}$, and if $x_i = \text{tr}_{K_{n+1}/F_{n+1}}(xe_i)$, then $x_i \in O_{F_{n+1}}$ and $x = \sum x_i e_i$, by (IV.2.2.17), and we have by (IV.2.1.26), $v_p(g(x_i) - x_i) \geq 1/(p-1)$, so when $n$ is large, by (IV.2.2.18), $v_p(x_i) \geq -\delta$, so the require is satisfied.

Prop. (IV.2.2.21). if $\delta > 0$ and $I$ is the ideal of elements of valuation $\geq 1/(p-1) - \delta$, then for $n$ large, there is a map $x \mapsto x^p : O_{K_{n+1}} \cap I \cap O_{K_{n+1}} \to O_{K_n} \cap I \cap O_{K_n}$, and it is surjective.

Proof: For $n$ large, choose a uniformizer $\pi_{n+1}$ of $K_{n+1}$, then $\pi_n = N_{K_{n+1}/K_n}(\pi_{n+1})$ is the uniformizer of $K_n$ because it is totally ramified (IV.2.2.16), so any element $x \in O_{K_{n+1}}$ can be written as $\sum \pi_{n+1}^i [x_i]$, where $x_i \in k_{K_{n+1}} = k_{K_\infty}$. Then $x^p \equiv \sum \pi_{n+1}^{pi} [x_i]^p \equiv \sum \pi_{n}^{pi} [x_i]$ mod $I$ by the above proposition. And the surjection is verbatim.

Def. (IV.2.2.22) (Tate’s Normalized Trace). The function $R_n(x) = p^{-k} \text{tr}_{F_{n+k}/F_n}(x)$ is compatible with $k$ and defines a $F_n$-linear projection from $F_\infty$ to $F_n$, and it commutes with $G_F$ action, called the Tate’s normalized trace.

From (IV.2.2.23) it’s easily verified that $R_n(O_{F_{n+k}}) \subset O_{F_n}$, thus $R_n(\pi_n O_{F_{n+k}}) \subset \pi_n^k O_{F_n}$. So we have $v(R_n(x)) > v(x) - v(\pi_n)$. So $R_n$ extends by continuity to a map $R_n : \hat{F}_\infty \to F_n$. If $x \in F_\infty$, then $R_n(x) = x$ for $n$ large, thus $(R_n(x) \mapsto x$ for any $x \in F_\infty$.

Now for a finite extension $K/\mathbb{Q}_p$, for $n$ large, if $e_i$ is a set of basis of $O_{K_n}/O_{F_n}$, then for any $x \in O_{K_{n+k}}$, $x = \sum x_i e_i^*$, where $x_i = \text{tr}_{K_{n+k}/F_{n+k}}(xe_i) \in O_{F_{n+k}}$, as in the proof of (IV.2.2.20). So we can define $R_n(x) = \sum R_n(x_i) e_i^*$. Notice this is defined only for $n$ large, and is independent of $e_i$ chosen, and by the following lemma, it is continuous and extends to a $K_n$-linear projection $R_n : \hat{K}_\infty$ to $K_n$. 

CHAPTER IV. NUMBER THEORY
Lemma (IV.2.2.23). Let $k \geq 0$ and $n \geq 1$, then $R_n(\zeta_{p^{n+k}}^j) = 1$ for $j = 0$ and vanishes otherwise.

Proof: This is clear from the fact $\text{tr}_{F_{n+k}/F_n}(\zeta_{p^{n+k}}^j) = \zeta_{p^{n+k}}^j \sum_{\eta \in H} \eta^j$. □

Lemma (IV.2.2.24). For any $\delta > 0$, when $n$ is large, $v(R_n(x)) \geq v(x) - \delta$.

Proof: We have $v(x_i) > v(x) - v(\pi_{n+k})$ by $F_{n+k}$-linearity, and $v(R_n(x)) > v(x_i) - v(\pi_n)$ as in (IV.2.2.22), and $v(e^*_i) \geq -\delta$ when $n$ is large, by (IV.2.2.18). Thus the result. □

Prop. (IV.2.2.25) (Refinement of Hilbert’s Theorem90). There is a decomposition of $\hat{K}_{\infty} = \hat{K}_n \oplus X_n$, where $X_n = \text{Ker} R_n$. If $\delta > 0$, then for $n$ large, $\alpha \in \mathbb{Z}_p^*$ and $\gamma_n$ that $\chi(\gamma_n)$ is a topological generator of $\Gamma_F$, $1 - \alpha \gamma_n : X_n \to X_n$ (because $\gamma_n$ commutes with $R_n$) is invertible and

$$v_p((1 - \alpha \gamma_n)^{-1} x) \geq v_p(x) - 1/(p - 1) - \delta,$$

unless $\alpha = -1$ and $p = 2$, in which case it is only invertible on $X_{n+1}$.

Proof: As usual, $x_i$ is a basis of $\mathcal{O}_{K_n}/F_n$, then $x = \sum x_i e_i^*$, $x_i = \text{tr}_{K_n/F_n}(x e_i) \in \hat{F}_n$, and $R_n(x) = 0$. Then $(1 - \alpha \gamma_n)$ acts on $x_i$, so it reduce to the case $\hat{K} = \mathbb{Q}_p$.

Injectivity: If $\alpha = 1$, this is Ax-Sen-Tate theorem. In other situations, $(1 - \alpha \gamma_n)(R_{n+k}(x)) = 0$ for all $k \geq 0$, so $R_{n+k}(x) = \alpha^k \gamma_n^k(R_{n+k}(x)) = \alpha^k R_{n+k}(x)$, so $R_{n+k}(x) = 0$, hence $x = 0$ by continuity.

Surjectivity: Let $F^*_{n+k} = \oplus_{j=1,\ldots,\delta} F_n \zeta_{p^{n+k}}^j$, then $F_{n+k} = F^*_n \oplus F^*_{n+1} \oplus \ldots \oplus F^*_{n+k}$, and $F_{n+k} \cap X_n = F^*_{n+1} \oplus \ldots \oplus F^*_{n+k}$. Now if $x = \sum_{j=1,\ldots,\delta} x_j e_{p^{n+k}}^j$ with $x_j \in \mathcal{O}_{F_n}$, then

$$x = (1 - \alpha \gamma_n^{-1}) \sum_{j=1,\ldots,\delta} x_j \frac{\zeta^j_{p^{n+k}}}{1 - \alpha \gamma_n^{-1} \zeta^j_{p}}.$$

Now $v_p(1 - \alpha \gamma_n^{-1} \zeta^j_{p}) \leq 1/(p - 1)$, and

$$(1 - \alpha \gamma_n)^{-1} = \frac{1 - \alpha \gamma_n^{-1} \zeta^j_{p} \gamma_n^{-1}}{1 - \alpha \gamma_n^{-1} \zeta^j_{p}}(1 - \alpha \gamma_n^{-1} \zeta^j_{p} \gamma_n^{-1})^{-1},$$

so $\alpha_n : 1 - \alpha \gamma_n : F^*_{n+k} \to F^*_{n+k}$ is invertible and

$$v_p((1 - \alpha \gamma_n)^{-1} x) \geq v_p(x) - 1/(p - 1) - v_p(\zeta_{p^n} - 1)$$

holds. And the assertion holds by uniform continuity. □

Miscellaneous

Prop. (IV.2.2.26). $\sqrt{p}$ is contained in $\mathbb{Q}(\zeta_p)$. In fact, $(\sum_{\alpha=0}^{p-1} \zeta_p^2)^2 = p$.

Proof: □
3 Global Fields

Def. (IV.2.3.1) (Global Fields). A global field is a finite extension of \( \mathbb{Q} \) or \( \mathbb{F}_p((t)) \), without a valuation. The former is called a number field and the latter a function field.

Def. (IV.2.3.2) (Order). An order in a number field \( K \) is a subring \( \mathcal{O} \) of \( K \) that is a finite \( \mathbb{Z} \)-module and \( \mathcal{O} \otimes \mathbb{Q} = K \).

Prop. (IV.2.3.3). \( \mathbb{G}(\mathbb{Q}[\mu_n]^{\mathbb{Q}}) \oplus (\mathbb{Z}/n\mathbb{Z})^{*} \).

Proof: We choose a prime \( p \) prime to \( n \) and show that \( \mu_n^p \) is conjugate to \( \mu_n \).

Let \( X^n - 1 = f(X)h(X) \) with \( f(X) \) minimal polynomial of \( \mu_n \). If \( f(\mu_n^p) \neq 0 \), then \( h(\mu_n^p) = 0 \).

Thus \( h(X^p) = f(X)g(X) \). So module \( p \), \( X^n - 1 \) has a multi root, which is impossible. \( \Box \)

Prop. (IV.2.3.4). The ring of integers in a cyclotomic field is generated by the roots of identity.

Proof: First consider the case \( n \) a prime power. Because \( d(1, \zeta, \cdots, \zeta^{d-1}) = \pm 1 \), \( L \mathcal{O} \subset \mathbb{Z}[\zeta] \subset \mathcal{O} \).

Because \( p \) totally splits, \( \mathcal{O} = \mathbb{Z}[\zeta] + \pi \mathcal{O} \), thus \( \mathcal{O} = \mathbb{Z}[\zeta] + \pi^t \mathcal{O} \). Choose \( t = s\phi(n) \) yields \( \mathbb{Z}[\zeta] = \mathcal{O} \).

Then for different \( p \), the fields are disjoint and the discriminants are pairwise coprime, thus by (2.11) in Neukirch, the products of the integral basis form an integral basis. \( \Box \)

Prop. (IV.2.3.5) (Unit Theorem). If \( S \) is a finite set of primes containing all the infinite primes, the group \( \mathbb{G}(K_S) \) of elements of \( K^* \) that has only prime divisors in \( S \), is a f.g. group of rank \( |S| - 1 \).

Prop. (IV.2.3.6) (Class Number). The ideal class group is defined as the group of ideals in \( K \) quotients \( J_K \) the principal ideals, it has finite order, class the class number of \( K \).

Prop. (IV.2.3.7) (Hermite’s theorem). There exists only f.m. number fields with bounded discriminant.

Proof: Cf.[Neukirch Algebraic Number Theory P206]. \( \Box \)

Prop. (IV.2.3.8) (Minkowski’s theorem). The discriminant of a number field different from \( \mathbb{Q} \) is not \( \pm 1 \).

Proof: Cf.[Neukirch Algebraic Number Theory P207]. \( \Box \)

Cor. (IV.2.3.9). The field \( \mathbb{Q} \) doesn’t contain any unramified extensions.

Prop. (IV.2.3.10) (Strong Approximation Theorem).

4 Adele and Idele

Restricted Direct Product

Def. (IV.2.4.1) (Restricted Direct Product). Let \( \{p\} \) be a set of indices and given a family of LCA gps \( G_p \), and for a.e. \( p \) an open compact subgroup \( H_p \subset G_p \). Then the restricted direct product is defined to be

\[
G = \prod_{S \in \{p\}, |S| < \infty} \prod_{p \in S} G_p \times \prod_{p \not\in S} H_p
\]

given the colimit space topology. And we denote \( \prod_{p \in S} G_p \times \prod_{p \not\in S} H_p = G_S, \prod_{p \not\in S} H_p = G_S \).

This topology is stronger than the product topology of \( \prod_p G_p \). It has an open basis \( N = \prod_p N_p \), where \( N_p \) is open in \( G_p \) and \( N_p = H_p \) for a.e. \( p \). It is locally compact because every \( G_S \) does.
Prop. (IV.2.4.2). Every compact subset $N$ of $G$ is contained in a $\prod_p N_p$, where $N_p$ is compact and $N_p = H_p$ for a.e. $p$.

Proof: This is because $G_S$ is an open covering of $G$, and the union of f.m. $G_S$ is also of the form $G_S$. So $N$ is contained in some $G_S$, thus its projection in the $S$-coordinates is compact. □

Prop. (IV.2.4.3) (Quasi-Characters on $G$). Quasi-characters on $G$ are all of the form $\otimes_p c_p$ that $c_p$ is trivial on $H_p$ for a.e. $p$.

Proof: Let $c$ be a quasi-character, choose a nbhd of $1 \in U \subset \mathbb{C}$ that contains no subgroup, then $c^{-1}(U)$ has contains an open basis $\prod_{p \in S} N_p \times G_S$, where $N_p$ are open nbhds of 1, so $c(G_S) = 1$. Thus $c(a) = \prod_p c_p(a_p)$ is true for any $a \in G$.

Conversely, clearly $\otimes_p c_p$ is a quasi-character on $G$, it is continuous. □

Prop. (IV.2.4.4) (Dual of $G$). In each $\hat{G}_p$, by (X.6.3.7) $H_p$ are compact, so $\hat{H}_p = \hat{G}_p/H_p^\perp$ are discrete, so $H_p^\perp$ is open; $H_p$ are open, so $H_p^\perp = \hat{G}_p/\hat{H}_p$ are compact. So we can define the space $\prod'(\hat{G}_p,H_p^\perp)$.

Then the dual group $\hat{G} \cong \prod'(\hat{G}_p,H_p^\perp)$ as a topological group.

Proof: (IV.2.4.3) shows that this is an algebraic isomorphism, so it suffices to prove this is a topological homeomorphism (X.6.3.6):

For any compact $B \in G_1 = \prod_{p \in S} N_p \times \prod_{p \notin S} H_p$, for any $\varepsilon > 0$, if $c \in \prod_{p \in S} N'_p \times \prod_{p \notin S} H_p^\perp$, where $N'_p = \{ c_p | |c_p(N_p - 1)| < \varepsilon/|S| \}$, then $|c(B) - 1| < \varepsilon$.

Conversely, if $\varepsilon$ is small enough, then if $c(\prod_{p \in S} N_p \times_{p \notin S} H_p) - 1| < \varepsilon$, then $c \in \prod_{p \in S} N'_p \times_{p \notin S} H_p^\perp$, where $N'_p = \{ c_p | |c_p(N_p - 1)| < \varepsilon \}$ □

Prop. (IV.2.4.5) (Restricted Product Measure). Let measures $d\alpha_p$ be given on $G_p$ that $\alpha_p(H_p) = 1$ for a.e. $p$, define a Haar measure on $G$ as follows:

On $G_S$, $d\alpha_S = \prod_{p \in S} d\alpha_p \cdot \alpha^S$, where $\alpha^S$ is the product measure on $G_S$.

Then these can define a functional a positive left-invariant functional $I$ that $|I(f)| \leq ||f||$ for any $f$ that depends only on f.m. coordinates $p \in S$. Then Stone-Weierstrass theorem shows these functions are dense in $C(G)$, thus $I$ can be uniquely extended to a functional on $C(G)$, and this defines a Haar measure on $G$ by Riesz representation (X.1.1.10), denoted by $d\alpha = \prod_p d\alpha_p$, called the restricted product measure.

Prop. (IV.2.4.6). For a function $f$ on $G = \prod'(\hat{G}_p,H_p)$ measurable, if either $f \geq 0$ or $f \in L^1(G)$, then

$$\int_G f(a) da = \lim_{S \uparrow} \int_{G_S} f(a) da$$

as a net limit.

Proof: The second case follows from the first case, as $\int_G f = \lim_B f_B$ compact $f_B$ by monotone convergence theorem, and any $B$ compact is contained in some $G_S$(IV.2.4.2). □

Cor. (IV.2.4.7). If $f(a) = \prod_p f_p(a_p)$, where $f_p \in L_1(G_p)$ and $f_p = \chi_{H_p}$ a.e. $p$, then if

$$\prod_p \int_{G_p} |f_p(a_p)| da_p < \infty,$$

then $f \in L^1(G)$, and

$$\int_G f(a) da = \prod_p (\int f_p a_p da_p).$$
Def. (IV.2.4.8) (Dual Measure). Notice if \( f_p = \chi_{H_p} \), then
\[
\hat{f}_p(c_p) = \int_{H_p} c_p(a_p) d\alpha_p = d\alpha_p(H_p) \chi_{H_p^\perp}(c_p).
\]

So by Fourier transform (X.6.3.20), if \( d\alpha_p \) is the dual measure on \( \hat{G}_p \), then \( \chi_{H_p^\perp} = d\alpha_p(H_p) \chi_{H_p^\perp} \), which means \( d\alpha_p(H_p) = 1 \), a.e. \( p \), thus we can define a measure on \( \hat{G} \) as \( dc = \prod_p d\alpha_p \).

Then \( dc \) is the measure on \( \hat{G} \) dual to \( d\alpha \) on \( G \).

Proof: The duality is by the lemma below (IV.2.4.9), applied to both \( f \) and \( \hat{f} \). \( \square \)

Lemma (IV.2.4.9) (Fourier Transform on Product). If \( f_p \in B_1(G_p) \) and \( f_p = \chi_{H_p} \) a.e. \( p \), then \( f(a) = \prod_p f_p(a_p) \in B_1(G) \), and \( \hat{f}(c) = \prod_p \hat{f}_p(c_p) \).

Proof: For any character \( c \), because
\[
f(a)c(a) = \prod_p f_p(a_p)c_p(a_p)
\]
and every \( f_p(a_p)c_p(a_p) \in L^1(G_p) \). So (IV.2.4.7) applies and shows the equations. Similarly, because \( \hat{f}_p = \chi_{H_p^\perp} \) a.e. \( p \), we have \( \hat{f} \in L_1(\hat{G}) \), so \( f(a) \in B_1(G) \). \( \square \)

Adele and Idele

Def. (IV.2.4.10) (Notations). We fix some notation:

- \( K \) is a global field.
- \( S \) is a finite set of primes.
- \( S_\infty \) the set of Archimedean places of \( K \).
- The adele of \( F \) is defined to be \( A_F = \prod'_{p}(K_p, \mathcal{O}_p) \).
- The idele of \( F \) is defined to be \( I_F = A_F^* = \prod'_{p}(K_p^*, \mathcal{O}_p^*) \).
- The ideal class group \( C_K = I_K/K^* \).
- The finite adele \( A_f = \prod_{v \not\in S_\infty}(K_v, \mathcal{O}_v) \).
- The infinite adele \( A_\infty = \prod_{v \in S_\infty} F_v \).
- The group \( I_K^S = \prod_{p \in S} K_p^* \times \prod_{p \not\in S} U_p \) is called the group of \( S \)-ideles of \( K \).
- \( K^S = K^* \cap I_K^S \) is the set of \( S \)-units of \( K \).

Def. (IV.2.4.11) (Hecke Character). A Hecke character is a continuous character of \( C_K \).

Def. (IV.2.4.12) (Unramified). Let \( \chi \) be a Hecke character of \( C_K \), then for a.e. non-Archimedean places \( v \), \( \chi_v \) is trivial on \( \mathcal{O}_v^* \). For these \( v \), \( \chi \) is said to be unramified at \( v \).

Prop. (IV.2.4.13) (Hecke Characters and Dirichlet Characters).

- Any Hecke character of \( C_K \) can be uniquely written as the form \( \chi(x) = \chi_1(x)|x|^\lambda \), where \( \chi_1 \) is a Hecke character of finite order, and \( \lambda \) is purely imaginary.
• Let $\chi$ be a Hecke character of finite order of $C_Q$, then there is a positive integer $N$ s.t. the prime divisors of $N$ are precisely the non-Archimedean primes that $\chi$ is ramified, and a primitive Dirichlet character $\chi_0$ modulo $N$ (III.3.1.41) that if $v$ is a non-Archimedean place not dividing $N$, then $\chi_0(p_v) = \chi(p_v)$. This induces a bijection between Hecke characters of finite finite order and primitive Dirichlet characters.

Proof: [Bum98]P259.

Prop. (IV.2.4.14) ($K$ cocompact in Adele). $K$ is discrete in $A$ and $A/K$ is compact.

Proof: □

Prop. (IV.2.4.15) (Product Formula). If $a \in A^*$ is an Idele, then $|a| = 1$ if $a \in K^*$.

Proof: Consider the restricted product measure $d\mu$ on $A$, then clearly $d\mu(ax) = |a|d\mu(x)$, and multiplying by $a$ induces an isomorphism of $A/K$, but preserves the counting measure on $K$. But $A/K$ is compact (IV.2.4.14) thus has finite volume, so $|a| = 1$. □

Prop. (IV.2.4.16). $I_K$ is locally compact in the restricted product topology, and $K^*$ is a discrete subgroup of $I_K$, thus $C_K$ is also Hausdorff locally compact.

Proof: $K^*$ is discrete in $I_K$ because it is already discrete in $A_K$ (IV.2.4.14). □

Prop. (IV.2.4.17) ($K^*$ Cocompact in Kernel Idele). There is an absolute valuation on $I_K$ and it vanish on $K^*$, thus induce a valuation on $C_K$. Then he kernel $C^0_K$ is compact and $C_K = C^0_K \times \mathbb{R}^*_+$ (for number field).

Proof: Cf.[Neukirch P159]. □

Prop. (IV.2.4.18). We let $I^S_K$ be the group of Ideles that has unit as components at all primes except $S$. Then we have a canonical isomorphism

$$I_K/I^S_K \cong J_K, \quad I_K/I^S_K \cdot K^* \cong J_K/P_K.$$ 

Proof: The proof is easy, just cut out the infinite prime part of $a$. □

Prop. (IV.2.4.19). If $S$ is sufficiently large containing a $S_0$ then $I_K = I^S_K \cdot K^*$ hence $C_K = I^S_K \cdot K^*/K^*$.

Proof: The ideal class group is finite, hence we can find a finite set of representative for it. Only finite set of primes are involved in it, thus we let $S$ contain all these primes and infinite primes, then for any $a$, $\prod_{p | \infty} a_p = A_i \cdot (x)$, and $A_i \in I^S_K$, hence $a \in I^S_K \cdot K^*$. □

Prop. (IV.2.4.20). For a field extension $L/K$, $I_K \subset I_L$, and $I^L_L = I_K$, this is be the diagonal inclusion to all the primes above a given prime, and the action is by $\sigma a_p = \sigma a_{p^{-1}\sigma}$. This induces an inclusion $C_K \subset C_L$ and $C^L_L = C_K$. The last assertion uses long exact sequence and $H^1(G, L^*) = 0$.

Prop. (IV.2.4.21). If $L/K$ is a separable extension of global fields then $A_K \otimes_K L \cong A_L$.

Proof: Cf.[GTM186, P170]. □
IV.3 Profinite Cohomology

Basic Reference is Neukirch’s Wonderful book [Neukirch Class Field Theory 2015] and the giant book [Neukirch Cohomology of Number Fields]. More should be added to the discussion of CFT.

1 Group Cohomology

We usually consider finite group \( G \), at least it should be discrete.

Def. (IV.3.1.1). The group cohomology \( H^n(G, A) \) is the derived functor of the left exact functor \( H^0(G, A) = A^G = \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, A) \), so \( H^n(G, A) = \text{Ext}_{\mathbb{Z}[G]}^n(\mathbb{Z}, A) \).

The group homology \( H_n(G, A) \) is the derived functor of the right exact functor \( H^0(G, A) = \mathbb{A}^G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} A \), so \( H_n(G, A) = \text{Tor}_{\mathbb{Z}[G]}^n(\mathbb{Z}, A) \).

\( A^H \) is left exact from \( G\text{-mod} \) to \( G \triangleleft H \text{-mod} \) because it is right adjoint to the inclusion functor: \( \text{Hom}_G(X, A) = \text{Hom}_G(H, X, A^H) \) and it preserves injectives ?? Dually for \( A_H \).

Prop. (IV.3.1.2) (Serre-Hochschild Spectral Sequence). By Grothendieck Spectral sequence, the relation \( A^G = (A^H)^{G/H} \) gives us a spectral sequence \( E^{p,q} \) that

\[
E^{p,q}_2 = H^p(G/H, H^q(H, A)) \implies E^n = H^n(G, A).
\]

The lower parts give us:

\[
0 \to H^1(G/H, A^H) \xrightarrow{\text{inf}} H^1(G, A) \xrightarrow{\text{res}} H^1(H, A)^{G/H} \xrightarrow{\text{transgression}} H^2(G/H, A^H) \xrightarrow{\text{inf}} H^2(G, A).
\]

dually for homology group.

Moreover if \( H^k(H, A) = 0 \) for \( k = 1, \ldots, n-1 \), then the rows are blank, thus the above lower part can change to dimension \( n \).

Cor. (IV.3.1.3) (Hopf). If \( G = F/R \), \( F \) is free, then use the homology spectral sequence, \( H_2(G, \mathbb{Z}) \cong \frac{R[F,F]}{[F,R]} \). Cf.[Weibel P198].

Prop. (IV.3.1.4). For \( G = \mathbb{Z} \), we have a free resolution \( 0 \to \mathbb{Z}[t, t^{-1}] \xrightarrow{1-t} \mathbb{Z}[t, t^{-1}] \to \mathbb{Z} \to 0 \). In particular, thus \( H_n(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z} \) iff \( n = 0 \) and vanish otherwise.

Prop. (IV.3.1.5) (Tate Cohomology). Neukirch Constructed a standard resolution of the \( \mathbb{Z}[G] \)-module \( \mathbb{Z} \), Cf.[Neukirch CFT P13]:

\[
\cdots \leftarrow X_{-2} \leftarrow X_{-1} \leftarrow X_0 \leftarrow X_1 \leftarrow \cdots
\]

that \( X_q = X_{-q-1} \) are \( \mathbb{Z}[G] \)-module generated by \( q \)-cells \( (\sigma_1, \ldots, \sigma_q) \), \( X_0 = X_{-1} = \mathbb{Z}[G] \).

It then can be verified that for \( G \) finite, \( \text{Hom} \) from this resolution gives out the Tate cohomology

\[
H^n_T(G, A) = \begin{cases} 
H^n(G, A) & n \geq 1 \\
A^G/N_G A & n = 0 \\
N_G A/I_G A & n = -1 \\
H_{-1-n}(G, A) & n \leq -2
\end{cases}
\]

and \( H^n_T \) is a long exact sequence.
In particular, the Hom complex looks like:

$$\cdots \to A_{-2} \xrightarrow{\partial_{-1}} A_{-1} \xrightarrow{\partial_0} A_0 \xrightarrow{\partial_1} A_1 \xrightarrow{\partial_2} A_2 \to \cdots$$

where $A_{-1} = A_0 = A$ and $\partial_0 x = N_G x$, $(\partial_1 x)(\sigma) = \sigma x - x$,

$\partial_2(x)(\sigma_1, \sigma_2) = \sigma_1 x(\sigma_2) - x(\sigma_1 \sigma_2) + x(\sigma_1)$.

From now on, consider only Tate cohomology.

**Prop. (IV.3.1.6).**

$H^{-2}(G, \mathbb{Z}) = G^{ab}$, $H^{-1}(G, \mathbb{Z}) = 0$, $H^0(G, \mathbb{Z}) = \mathbb{Z}/|G|\mathbb{Z}$, $H^1(G, \mathbb{Z}) = 0$, $H^2(G, \mathbb{Z}) = \chi(G)$.

**Proof:** $H^0$ is trivial and $H^1(G, \mathbb{Z}) = H^0(G, \mathbb{Q}/\mathbb{Z}) = 0$, $H^2(G, \mathbb{Z}) = H^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$. $H^{-1}(G, \mathbb{Z}) = N_G \mathbb{Z}/I_G A = 0$.

For $H^{-2}(G, \mathbb{Z})$, use the dimension shifting $0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$, $= H^{-1}(G, I_G) = I_G / I_G^2$. And $G^{ab} \cong I_G / I_G^2$ by $\sigma \mapsto \sigma - 1$.

**Prop. (IV.3.1.7).** $H^n(\mathbb{Z}/n\mathbb{Z}, A) = A^G / NA$ for $n$ even and $H^n(\mathbb{Z}/n\mathbb{Z}, A) = NA / (\sigma - 1)A$ for $n$ odd.

**Prop. (IV.3.1.8).** For a finite group $G$, $|G|\cdot H^n(G, A) = 0$ for any $G$-module $A$. (True for $H^0$ and use dimension shifting). In particular, a divisible $G$-module $A$ has trivial cohomology.

**Prop. (IV.3.1.9).**

**Operations**

**Prop. (IV.3.1.10) (Dimension Shifting).** There are fundamental split exact sequence $0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$ and $0 \to \mathbb{Z} \to \mathbb{Z}[G] \to J_G \to 0$, thus $A_G = A / I_G A$. This can be used to tensor $A$ and define natural dimension shifting of cohomology $\delta$.

**Def. (IV.3.1.11).** The **inflation** is defined for $p \geq 0$ by composing with $G \to G/H$.

The **restriction** is the map $H^p(G, A) \to H^p(H, A)$ that is id when $q = 0$ and commutes with $\delta$.

The **corestriction** is the map $H^q(H, A) \to H^q(G, A)$ that maps $a$ to $N_{G/H} a$ when $q = 0$ and commutes with $\delta$.

**Prop. (IV.3.1.12).** cor $\circ$ res $= [G : H]$ for a subgroup $H$. (check at degree 0 and use dimension shifting).

**Prop. (IV.3.1.13).** For an isomorphism $(\sigma^*, \sigma)$ of a group and its cochain map in the sense that $\sigma^*(g)(\sigma(a)) = g(a)$, we have an isomorphism of Conjugation acts trivially on the group cohomology, because it does on $H^0$ because $H^0 = A^G$ fixed by $G$, and it commutes with dimension shifting. (Warning, if you count directly $a(\sigma^* \sigma^{-1}) - \sigma a(\tau)$, you won’t get 0, but a 1-coboundary).

**Prop. (IV.3.1.14) (Cup Product).** The cup product is defined by $C^p(X, A) \times C^q(X, B) \to C^{p+q}(X, A \otimes B)$:

$$(a \cup b)(\sigma_1, \ldots, \sigma_{p+q}) = a(\sigma_1, \ldots, \sigma_p) \otimes \sigma_1 \ldots \sigma_p b(\sigma_{p+1}, \ldots, \sigma_{p+q}).$$

It satisfies $\partial(a \cup b) = \partial(a) \cup b + (-1)^p a \cup \partial(b)$, thus defines $a$:

$$\cup: H^p(G, A) \times H^q(G, B) \to H^{p+q}(G, A \otimes B)$$

for $p, q \geq 0$. And in negative dimension this is also definable but not computable, Cf.[Neukirch Cohomology of Number Fields P42] or [Neukirch Class Field Theory 2015 P45].
\[ \begin{align*}
\cdot & \quad a \sim b = a \otimes b \text{ for } a \in H^0(G, A), b \in H^0(G, B).
\cdot & \quad \delta(a \sim b) = \delta a \sim b, \delta(a \sim b) = (-1)^p(a \sim \delta b) \text{ for } a \in H^p(G, A).
\cdot & \quad \sim \text{ is associative and skew-symmetric (follows from dimension shifting and the last one.}
\end{align*} \]

**Prop. (IV.3.1.15) (Duality and Cup Product).** Let \( 0 \to A' \overset{i}{\to} A \overset{j}{\to} A'' \to 0 \) and \( 0 \to B' \overset{\pi}{\to} B \overset{\varphi}{\to} B'' \to 0 \) be exact and there is a pairing \( \varphi : A \times B \to C \) that \( \varphi(A' \times A') = 0 \) hence induce a compatible pairing on \( A' \times B'' \) and \( A'' \times B' \), then we have
\[
\delta(\alpha) \sim \beta + (-1)^p \alpha \sim \delta(\beta) = 0
\]
for \( \alpha \in H^p(G, A'') \) and \( \beta \in H^q(G, B'') \).

*Proof:* Use the definition of \( \delta \), let \( a, b \) be the preimage of \( \alpha, \beta \) in \( A \) and \( B \), and \( i a' = \partial a, u b' = \partial b \), then \( \delta(a) \sim \beta + (-1)^p \alpha \sim \delta(\beta) = a' \sim v b + (-1)^p j a \sim b' = \partial a \sim b + (-1)^p a \sim \partial b = \partial(a \sim b) \) is a boundary. \( \square \)

**Prop. (IV.3.1.16).**
\[
\text{res}(a \sim \beta) = \text{res}(a) \sim \text{res}(b), \quad \text{cor}(\text{res}a \sim b) = a \sim \text{cor} b
\]
Cf.[Neukirch CFT P48].

**Prop. (IV.3.1.17).** Let \( \sigma \in G^{ab} = H^{-2}(G, \mathbb{Z}) \) and \( a_1 \in H^1(G, A), a_2 \in H^2(G, A) \), then
\[
a_1 \sim \sigma = a_1(\sigma), \quad a_2(\sigma) = \sum \tau a_2(\tau, \sigma).
\]
Cf.[Neukirch CFT P50, P51].

**Prop. (IV.3.1.18).** For cyclic group, the Tate cohomology is 2-cyclic.

*Proof:* There is an exact sequence \( 0 \to \mathbb{Z} \to \mathbb{Z}[G] \overset{\sigma}{\to} \mathbb{Z}[G] \to \mathbb{Z} \to 0 \), and this defines an isomorphism \( \delta^2 : H^0(G, \mathbb{Z}) \cong H^2(G, \mathbb{Z}) \). And this is also true for any \( A \) when tensored with it. The isomorphism is \( a \mapsto \delta^2 a = \delta(1) \sim a \). \( \square \)

**Prop. (IV.3.1.19) (Duality).** The cup product induces an isomorphism \( H^i(G, A^\vee) \cong (H^{-i-1}(G, A))^\vee \), i.e., \( H^n(G, A^\vee) \) and \( H_n(G, A) \) are dual to each other when \( n > 0 \), where \( A^\vee = \text{Hom}(A, \mathbb{Q}/\mathbb{Z}) \).

*Proof:* We only need to verify \( A^*G/N_G A^* \cong (N_G A/I_G A)^* \) and use dimension shifting. Should use the injectivity of \( \mathbb{Q}/\mathbb{Z} \) and the compatibility of cup product with dual. \( \square \)

**Cor. (IV.3.1.20).** When \( A \) is \( \mathbb{Z} \)-free, the cup product also induce an isomorphism \( H^i(G, \text{Hom}(A, \mathbb{Z})) \cong H^{-i}(G, A)^\vee \).

**Prop. (IV.3.1.21) (Theorem of Cohomological Triviality).** For a \( G \)-module \( A \), if there is a \( q \) s.t. \( H^q(g, A) = H^{q+1}(g, A) = 0 \) for all subgroups of \( G \), then \( H^p(g, A) = 0 \) for any \( p \) and subgroup \( g \). Cf.[Neukirch CFT P57].
Prop. (IV.3.1.22) (Tate’s Theorem). Assume $A$ is a $G$-module that $H^1(G, A) = 0$ and $H^2(g, A)$ is cyclic of order $|g|$ for any subgroup $g$ of $G$, then for a generator $a$ of $H^2(G, A)$, there is an isomorphism
\[ a ↣ H^q(G, \mathbb{Z}) \cong H^{q+2}(G, A) . \]

Cf.[Neikirch CFT P79].

Cor. (IV.3.1.23). In particular, by dimension shifting, if $A$ is a $G$-module that $H^1(G, A) = 0$ and $H^2(g, A)$ is cyclic of order $|g|$ for any subgroup $g$ of $G$ this gives an isomorphism:
\[ a ↣ H^q(G, \mathbb{Z}) \cong H^{q+2}(G, A) . \]

for a generator $a$ of $H^2(G, A)$, because cup product commutes with dimension shifting.

Miscellaneous

Prop. (IV.3.1.24) ((Schreier) $H^2$ and Extensions). For a $G$-module $A$, there is a correspondence of equivalence classes of extension of $G$ over $A$ that are compatible with the $G$ action and $H^2(G, A)$.

Proof: Cf.[Weibel P183]. In fact there are also interpretations of $H^3(G, A)$ as $0 \to A \to N \to E \to G$ under some equivalences. □

Prop. (IV.3.1.25). When $G$ is a cyclic group and $A$ is a $G$-module, let $f = \sigma - 1$, $g = 1 + \sigma + \ldots + \sigma^{n-1}$, then we can form a cyclic complex of order 2 and compute the Herbrand quotient(I.11.4.9).

In this case, $g_f.g$ is just $|H^0(G, A)|/|H^{-1}(G, A)|$. And by(I.11.4.11), if a $G$-morphism $A \to B$ has finite kernel and cokernel, then they have the same Herbrand quotient.

2 Cohomology of Profinite Groups

Prop. (IV.3.2.1) (Abelian Sheaves on $T_G$). If $G$ is profinite, the category of Abelian sheaves on the canonical topology $T_G$ of $G$-sets is equivalent to the category of $G$-modules, by Yoneda functor. The inverse map is $F \mapsto \lim \leftarrow F(G/H)$.

Proof: The task is to prove $F \cong h_{\lim \leftarrow F(G/H)}$. Cf.[Tamme P29].

The inverse of the Yoneda functor is the functor $F \mapsto F(G)$ as a left $G$-set where $g.s = F(g).s$. The task is to show that $F \cong h_{F(G)}$. For this, for any $U$ we consider the covering $\{ G \xrightarrow{\varphi} U \}$ where $\varphi_U(g) = gu$. Sheaf condition says
\[ F(U) \to \prod_{u \in U} F(G) \Rightarrow F(G \times_U G) \]
is exact, in other words, $F(U) \cong \text{Hom}_{G}(U, F(G))$. □

Prop. (IV.3.2.2) (Profinite Cohomology). The profinite cohomology is the derived functor of $A \to A^G$ in the Abelian category $C_G$(It has enough injectives by(III.1.8.1)). And
\[ H^*(G, A) \cong H^*(C(G, A)) \cong \lim \leftarrow H^*(G/U, A^U) \]
where $C(G, A)$ is the set of continuous cochain complex of morphisms from $G$ to $A$. Moreover, for the same reason, when $G = \lim \leftarrow G_i$, and $A = \lim \rightarrow A_i$, then
\[ H^*(G, A) \cong \lim \leftarrow H^*(G_i, A_i) . \]
Proof: The second is an isomorphism because $C^n(G, A) = \colim C^n(G/U, A^U)$ and direct limit is exact.

For the first, the $H^0$ obviously coincide, so it suffice to prove $H^*(C(G, A))$ form a universal $\delta$-functor. It is effaceable because $I^U$ is injective $G/U$-module.

For the last one, we need to check $C^n(G, A) = \lim_{\rightarrow} C^n(G_i, A_i)$. Notice $G$ has the profinite topology, thus must factor through some $G_i$, and the right through some $A_i$ because the image of a morphism from $G^n$ to $A$ has finite image. Thus the result follows.

Prop. (IV.3.2.3). cor \circ res = [G : H] for a subgroup $H$ is also true for profinite cohomology (IV.3.1.12), if $H$ is an open subgroup of $G$. This is because of (IV.3.2.2).

Prop. (IV.3.2.4). If $H$ is a closed subgroup of a profinite group $G$ that $[G : H]$ is relatively prime to $p$, then for any $G$-module $A$ and $i$, the restriction map $H^i(G, A) \to H^i(H, A)$ is injective on the $p$-primary part of $H^i(G, A)$.

Proof: $H^i(H, A) = \lim_{\rightarrow U} H^i(U, A)$ for open subgroups $U$ containing $H$, by (IV.3.2.2), and $H^i(G, A) \to H^i(U, A)$ is injective on the $p$-primary part by (IV.3.2.3), so it is injective.

Lemma (IV.3.2.5) (Shapiro).

\[ H_*(G, \ind^G_H(A)) \cong H_*(H, A), \quad H^*(G, \Coind^G_H(A)) \cong H^*(H, A) \]

because (co)induced is adjoint to exact functors, so it preserves injectives (projectives) and it is exact because $\mathbb{Z}[G]$ is free $\mathbb{Z}[H]$-module.

And in the finite case, this is also true for Tate cohomology using dimension shifting.

Prop. (IV.3.2.6) (Serre-Hochschild Spectral sequence). Same as the finite case (IV.3.1.2) also applies to profinite cohomology with $H$ closed normal in $G$.

Cohomological Dimensions

Def. (IV.3.2.7). The $p$-cohomological dimension $cd_p(G)$ of a profinite group $G$ is defined as the smallest integer $n$ that the $p$-primary part of $H^i(G, A)$ vanish for any torsion $G$-module $A$. The strict $p$-cohomological dimension $scd_p(G)$ of a profinite group $G$ is defined as the smallest integer $n$ that the $p$-primary part of $H^i(G, A)$ vanish for any $G$-module $A$.

The cohomological dimension $cd(G)$ is defined to be $\sup_p(cd_p(G))$. The strict cohomological dimension $scd(G)$ is defined to be $\sup_p(scd_p(G))$.

Prop. (IV.3.2.8). For a profinite group $G$, the following are equivalent:

- $cd_p(G) \leq n$.
- $H^i(G, A) = 0$ for any $i > n$ and any $p$-torsion $G$-module $A$.
- $H^{n+1}(G, A)$ for any simple $p$-torsion $G$-module $A$.

And if $G$ is pro-$p$, then it suffice to check $H^{n+1}(G, \mathbb{Z}/p\mathbb{Z}) = 0$

Proof: For any torsion $G$-module $A$, $A = \oplus_p A(p)$, so $H^i(G, A(p))$ is the $P$-primary part of $H^i(G, A)$, so 1, 2 are equivalent. For $3 \to 1$: use the fact cohomology commutes with colimits (IV.3.2.2), reduce to the case of $A$ finite, and then use use the quotient tower.

The last assertion is by (I.3.12.12).

Prop. (IV.3.2.9). $cd_p(G) \leq scd_p(G) \leq cd_p(G) + 1$. 

Proof: Let \( A_p = \ker(p : A \to A) \). There are exact sequences \( 0 \to A_p \to A \xrightarrow{p} pA \to 0 \) and \( 0 \to pA \to A \to A/pA \to 0 \). \( A_p \) and \( A/pA \) are \( p \)-torsion \( G \)-modules, so if \( i > \text{cd}_p(G) + 1 \), then \( H^i(G, A_p) \) and \( H^{i-1}(G, A/pA) \) vanish. So \( H^i(G, A) \xrightarrow{p} H^i(G < pA) \) and \( H^i(G, pA) \to H^i(G, A) \) are injections, so their composition \( H^i(G, A) \xrightarrow{p} H^i(G, A) \) is injective, showing \((H^i(G, A))_p = 0\), so \( \text{scd}_p(G) \leq \text{cd}_p(G) + 1 \).

Prop. (IV.3.2.10). For a closed subgroup \( H \) of a profinite group \( G \), \( \text{cd}_p(H) \leq \text{cd}_p(G) \) and \( \text{scd}_p(H) \leq \text{scd}_p(G) \), and if \([G : H]\) is relatively prime to \( p \), then equality holds.

Proof: The first is because of Shapiro’s lemma (IV.3.2.5). For the equality, use (IV.3.2.4).

Cor. (IV.3.2.11). \( \text{cd}_p(G) = \text{cd}_p(G_p) = \text{cd}(G_p) \), \( \text{scd}_p(G) = \text{scd}_p(G_p) = \text{scd}(G_p) \).

Prop. (IV.3.2.12). If \( H \) is a closed normal subgroup of \( G \), then \( \text{cd}_p(G) \leq \text{cd}_p(H) + \text{cd}_p(G/H) \), by Hochschild-Serre spectral sequence.

Prop. (IV.3.2.13). If \( K \) is a field of char \( p \), then \( \text{cd}_p(G(K_s/K)) = 0 \).

If \( H^2(G(K_s/L), K^s) = 0 \) for all \( L/K \) separable, then \( \text{cd}(G(K_s/K)) \leq 1 \). In particular \( H^i(G(K_s/K), K^s) = 0 \) for \( i \geq 1 \).

Proof: Let \( G_p \) be the Sylow \( p \)-subgroup of \( G(K_s/K) \) and \( M = K^GP \). There is an exact sequence \( 0 \to \mu_p \to K_s \xrightarrow{x^p-x} K_s \to 0 \), and combined with the fact that \( H^i(G_p, K_s) = H^i(G(K_s/M), K_s) = 0 \) for \( i \geq 1 \) (IV.3.3.1), so \( H^i(G_p, \mathbb{Z}/p\mathbb{Z}) = 0 \) for \( i \geq 2 \). Thus by (IV.3.2.8) and (IV.3.2.11), \( \text{cd}_p(G(K_s/K)) \leq 1 \).

For the second assertion, similarly, for \( l \neq p \), consider the kernel of \( x^l \), \( \mu_l \) of \( l \)-th roots of unity in \( K_s \), and \( H^2(G_l, \mu_l(K_s)) = \lim_{\to L} H^2(G(K_s/L), \mu_l(K_s)) = 0 \), so \( \text{cd}(G(K_s/K)) \leq 1 \). Then \( \text{cd}(G(K_s/K)) \leq 1 \), and \( \text{scd}(G(K_s/K)) \leq 2 \), so \( H^i(G(K_s/K), K^s) = 0 \) for \( i \geq 1 \).

Prop. (IV.3.2.14). For \( L/K \) field extension, \( \text{cd}_p(G(L_s/L)) \leq \text{cd}_p(G(K_s/K)) + \text{tr.deg}(L/K) \).

Proof: Cf. [Etale Cohomology Fulei P169].

Cor. (IV.3.2.15). If \( k \) is separably closed and \( K \) be a function field over \( k \), then \( \text{cd}(G(K_s/K)) \leq 1 \).

And if \( K \) is of char \( p > 0 \), \( H^2(G(K_s/K), K^s) \) is a \( p \)-torsion group.

Proof: Th first one is clear, for the second, for any \( l \neq p \), use the exact sequence \( \mu_l(K_s) \to K_s^* \xrightarrow{x \to x^l} K_s^* \to 0 \), then \( H^2(G(K_s/K), \mu_l(K_s)) = 0 \), and \( H^2(G(K_s/K), K^s) \xrightarrow{l} H^2(G(K_s/K), K^s) \) is injective. \( l \) is arbitrary, so \( H^2(G(K_s/K), K^s) \) must be a \( p \)-torsion group.

3 Galois Cohomology

References are [Neukirch Chap6]. Should include [Galois Cohomology Serre].

This subsection is not included in the following subsection about Galois/Profinite Cohomology because the \( G \)-groups may not be Abelian and it may not be endowed with the discrete topology.

Prop. (IV.3.3.1) (Hilbert’s Additive Satz 90). For \( L/K \) a Galois extension, \( H^n(L/K, \mathbb{Z}/l\mathbb{Z}) = 0 \) for \( n > 0 \), where \( L \) is equipped with the discrete topology.

Proof: Form the normal basis theorem??, for finite Galois extension \( L/K \), \( L \) is an induced module over \( K \), thus \( H^n(L, \mathbb{Z}/l\mathbb{Z}) = H^n(L, \mathbb{Z}/l\mathbb{Z}) = 0 \) for \( * \neq 0 \) and \( H^2(L, \mathbb{Z}/l\mathbb{Z}) = 0 \) by (IV.3.2.5).

Hence the same is true, for arbitrary Galois extension, when \( L \) is equipped with the discrete topology, the same as in the proof of (IV.3.3.7).

Prop. (IV.3.3.2) (Hilbert’s Multiplicative Satz 90). \( H^1(L/K, \mathbb{Z}/l\mathbb{Z}) = 0 \) for Galois extension \( L/K \), where \( L \) is equipped with the discrete topology, (follows from (IV.3.3.7)).
Non-Abelian Cohomology

Def. (IV.3.3.3) (Non-Abelian Cohomology). Let $G, M$ be topological groups, with a continuous action of $G$ on $M$, then we define $H^0(G, M) = M^G$.

We define $Z^1(G, M) = \text{continuous maps } x : G \to M$ such that

$$\sigma_1(x(\sigma_2)(x(\sigma_1\sigma_2))^{-1}x(\sigma_1) = 1 \quad \text{i.e.} \quad x(gh) = x(g)x(h).$$

If $x \in Z^1(G, M)$, then $x_m : \sigma \mapsto m^{-1}x(\sigma)\sigma(m) \in Z^1(G, M)$ too. This defines an equivalence relation on $Z^1(G, M)$, the equivalence classes are called $H^1(G, M)$. This is compatible with the commutative case.

Prop. (IV.3.3.4). Restriction map and inflation map is definable for $H^0$ and $H^1$, and $H^1(H, M)$ is a $G/H$-set where $G$ acts on $H^1(H, M)$ by $g(c)(h) = g(c(g^{-1}hg))$.

Prop. (IV.3.3.5). There is an exact sequence of pointed sets:

$$0 \to H^1(G/H, M^H) \xrightarrow{\text{inf}} H^1(G, M) \xrightarrow{\text{res}} H^1(H, M)^{G/H}.$$ 

Proof: First $\text{res}(H^1(G, M)) \subset H^1(H, M)^{G/H}$ because $g(c)(h) = c(g)^{-1}c(h)c(g))$ is checked so $g(c)$ is cohomologous to $c$.

$\text{res} \circ \text{inf} = 0$ is easy, if $\text{res}(c) = 0$, then $c$ is trivial on $H$, hence $c(gh) = c(g)$ and $h(c(g)) = c(hg) = c(g \cdot g^{-1}hg) = c(g)$, so $c$ is inflated from $H^1(G/H, M^H)$.

For the injectivity of $\text{inf}$. If $c(\overline{g}) = g^{-1}g(a)$, then $a \in M^H$, so it is a coboundary in $H^1(G/H, M^H)$.

Prop. (IV.3.3.6). Let $1 \to A \to B \to C \to 1$ be an exact sequence of $G$-groups, then there is a long exact sequence of pointed sets

$$1 \to A^G \to B^G \to C^G \xrightarrow{\delta} H^1(G, A) \to H^1(G, B) \to H^1(G, C) \xrightarrow{\Delta} H^2(G, A)$$

the last term is defined only when $A$ is in the center of $G$.

Where $\delta$ is defined as follows: for $c \in C^G$, let $b$ be an inverse image of $c$ in $B$, then $a_\sigma = b^{-1}\sigma(b) \in A$, and it defines a cocycle in $H^1(G, A)$, different choice differ by a coboundary, so it is well-defined.

$\Delta$ is defines as: for $c_\sigma$ a cocycle in $H^1(G, C)$, choose $b_\varsigma$ inverse images of $c_\varsigma$, then $a_{\sigma,\tau} = b_{\sigma}\sigma(b_\tau)b^{-1}_{\sigma\tau}$ is a cocycle in $H^2(G, A)$.

Proof: The verification of well-definedness of $\Delta$ is checked at [Serre Local Fields P124].

For the exactness at $C^G$, the definition of $\delta$ shows that $\delta(c) = 1$ iff there is an inverse image $b$ that $b^{-1}\sigma(b) = 1$ for all $\sigma$.

For the exactness at $H^1(G, A)$, $a_\sigma = b^{-1}\sigma(b)$ if $a_\sigma$ is in the image of $\delta$. Conversely, the image of $b$ in $C$ is in $C^G$, so it is in the image of $\delta$.

For the exactness at $H^1(G, B)$, one way is clear, and for the other, if $\pi(b_\sigma) = c^{-1}\sigma(c)$, then $t$ is an inverse image of $c$, then $tb_\sigma\sigma(t)^{-1}$ is a cocycle in $A$ cohomologous to $b_\sigma$.

For the exactness at $H^1(G, C)$, one way is clear, and if $b_\varsigma$ is an inverse image of $b_\varsigma$ and $a_{\sigma,\tau} = b_{\sigma}\sigma(b_\tau)b^{-1}_{\sigma\tau}$ is a coboundary, then it is $a_\sigma\sigma(a_\tau)a^{-1}_{\sigma\tau}$, so we change $b$ to $a^{-1}_{\sigma}b_\sigma$, as $A$ is in the center of $B$, this lifts $c$ to a cocycle in $B$.  

\qed
Prop. (IV.3.3.7) (Hilbert’s Theorem 90). For $L/K$ a Galois extension, $H^1(G(L/K), GL_n(L)) = 1$, where $L$ is equipped with the discrete topology.

Proof: We prove any cocycle is a coboundary, for this, notice any cocycle factor through a finite quotient, and the images of it is contained in a finite extension of $K$, hence it reduce to the case of $L/K$ finite.

By definition, this is equivalent to any $B$-semi-linear representation of $G$ free of finite rank is trivial, which is by (VIII.1.3.4).

Cor. (IV.3.3.8) (semi-linear representations). The proposition implies that any semi-linear $L$-representation of $G_{L/K}$ is trivial.

Cor. (IV.3.3.9). $H^1(G(L/K), SL_n(L)) = 1$. This is seen from the exact sequence $1 \to SL_n(L) \to GL_n(L) \to L^* \to 1$.

Continuous Cochain Complex

In this subsubsection cohomology of $G$-modules with non-discrete topology is studied. References are [Cohomology of Number Fields Neukirch Chap 2.7].

Prop. (IV.3.3.10). $H^*_cts(G, -)$ forms a long exact sequence for any $0 \to A \to B \to C \to 0$ of continuous $G$-modules.

Prop. (IV.3.3.11). If $A$ is a compact $G$-module which is an inverse limit of finite discrete $G$-modules $A_n$, then if $H^i(G, A_n)$ is finite for all $n$, then

$$H^{i+1}_{cts}(G, A) = \lim_{\leftarrow} H^{i+1}(G, A_n).$$

Proof: Cf.[Cohomology of Number Fields Neukirch P142].

Lemma (IV.3.3.12). Let $\pi$ be a topologically nilpotent element of $A$ which is complete in the $\pi$-adic topology and $\pi$ is not a zero-divisor, let $R = A/\pi A$ equipped with discrete topology. Let $G$ be a group which acts continuously on $A$ and fix $\pi$, then if $H^1(G, R)$ is trivial, then $H^1(G, A)$ is trivial, and if moreover $H^1(G, GL_n(R))$ is trivial, then $H^1(G, GL_n(A))$ is trivial.

Proof: Cf.[Galois Representations Berger P15].

Prop. (IV.3.3.13) (Cyclic Case). if $G$ is a topological cyclic group $\langle g \rangle$, then the map $H^1(G, M) \to M/(1 - g)$ is well-defined and injective. And when $M$ is profinite, $p$-adically complete, then the map is also surjective.

Proof: The surjection: there is only one choice: $c(g^r) = (1 + g + \ldots + g^{r-1})(m)$. And we need to verify that it is continuous. The case of $p$-adic can be deduced from profinite case, because $c(\gamma) \in p^{-k}M$ for some $k$, and $p^{-k}M$ is then profinite. For any finite quotient $N$ of $M$, there is a $k$ that $kM = 0$, and a $n$ that $g^n = id$ on $N$, so $c(g^{rn}) = 0$ on $N$, which shows $c$ is continuous.
Interpretation of $H^1$ and Torsors

**Def. (IV.3.3.14) ($A$-Torsors).** A $G$-set $X$ is a discrete set with a continuous $G$-action on $X$. Let $A$ be a $G$-group, an $A$-torso is a right $A$-action that is simply transitive and semi-linear in $G$.

**Prop. (IV.3.3.15) ($H^1$ and Torsors).** We have a canonical bijection of pointed sets: $H^1(G, A) \cong TORS(A)$.

**Proof:** Let $X$ be an $A$-torsor, choose $x \in X$, then $\sigma(x) = xa_\sigma$ for $a_\sigma \in A$. Now that $\sigma \mapsto a(\sigma)$ is checked to be a cocycle, and change of $x$ changes to $\sigma \mapsto b^{-1}a_\sigma \sigma(b)$. Conversely, for an $a \in H^1(G, A)$, we let $X = A$ be a right $A$-module, and let $\sigma'(x) = a_\sigma \sigma(x)$, i.e. regarding coming from $x = 1$, then this is an inverse map. □

**Prop. (IV.3.3.16) (Extension of Rings).**

**Prop. (IV.3.3.17).** There is an isomorphism of pointed sets $H^1(G, O(\varphi_L)) \cong E_{\varphi}(L/K)$.

**Proof:** Cf.[Neukirch Cohomology of Number Fields P346]. □

**Prop. (IV.3.3.18).** There is an isomorphism of pointed sets $H^1(G, PSL_n(L)) \cong BS_n(L/K)$, where $BS_n(L/K)$ is the isomorphism classes of Brauer-Severi varieties of dimension $n - 1$ that splits in $L$.

**Proof:** Cf.[Neukirch Cohomology of Number Fields P348]. □

4 Iwasawa Modules
IV.4  Cohomology of Number Fields

1  Class Field Theory

Abstract Class Field Theory

Def. (IV.4.1.1). A formation consists of a profinite group $G$ regarded as a Galois group $G(K)$ and a $G$-module $A$. It is called a field formation iff for any normal extension $L/K$, $G(L/K, A^L) = 0$.

For a field extension, by (IV.3.1.2), $\inf$ is an injection on $H^2$. We denote $H^2(K)$ as the profinite cohomology group $H^2(G, A) = \Br(K)$. Inflation should be thought of as inclusions.

It is called a class formation if moreover for every normal extension $L/K$, there is a canonical isomorphism $\inj_{L/K}: H^2(L/K) \rightarrow 1 + \frac{1}{[L : K]}Z \leq Z$, that is compatible with inflation and restriction in the sense that:

- If $N/L/K$ with $N/K$ and $L/K$ normal, then $\inj_{N/L/K} = \inj_{N/K} \mid H^2(L/K)$ via inflation.
- If $N/L/K$ with $N/L$ and $N/K$ normal, then $\inj_{N/L} \circ \res_L = [L : K] \cdot \inj_{N/K}$.

The element of $H^2$ that is mapped to $1 + \frac{1}{[L : K]} Z$ is called the fundamental class $u_{L/K}$.

Prop. (IV.4.1.2). $\inj$ also commutes with cor and conjugation:

$$\inj_{N/K}(\cor_K c) = \inj_{N/L} c, \quad \inj_{\sigma N/\sigma K}(\sigma^* c) = \inj(c).$$

The first is because $\inj$ commutes with res thus res is surjective, thus there is a $c'$ that $c = \res c'$. Because of cor res $= [L : k]$, we have $\cor_K(c) = c'_{[L : K]}$. Thus $\inj_{N/K}(\cor_K c) = [L : K] \inj_{N/K}(c') = \inj_{N/L}(\res_L c') = \inj_{N/L}(c)$.

For the conjugation, Cf. [Neukirch CFT P69].

Cor. (IV.4.1.3). From this we easily get that

$$u_{L/K} = (u_{N/K})^{[N:L]}, \quad \res_L(u_{N/K}) = u_{L/K}$$

$$\cor_K(u_{N/L}) = (u_{N/K})^{[L : K]}, \quad \sigma^*(u_{N/K}) = u_{\sigma N/\sigma K}.$$

Prop. (IV.4.1.4) (Main Theorem). Tate’s theorem (IV.3.1.22) tells us for a class formation, for $L/K$ normal extension, there is an isomorphism

$$u_{L/K} \rightarrow H^q(G_{L/K}, Z) \cong H^{q+2}(L/K).$$

Especially, for $q = -2$, there is a canonical isomorphism $G_{L/K}^{ab} \cong A_K/N_{L/K}A_L$ that its inverse is called reciprocity isomorphism and $A_K \rightarrow G_{L/K}^{ab}$ is called norm residue symbol $(-, L/K)$. This norm residue symbol also induce a universal residue symbol $(-, K)$ on the limit $G_{K}^{ab}$, i.e. the maximal Abelian extension of $K$.

Lemma (IV.4.1.5). Let $L/K$ be a normal extension, $a \in A_K$ and $\chi \in \chi(G_{L/K}^{ab}) = H^1(G_{L/K}, \mathbb{Q}/Z)$ is a character, then

$$\chi((a, L/K)) = \inj_{L/K}(a \rightsquigarrow \delta \chi) \equiv \frac{1}{[L : K]}Z \leq Z.$$ 

Proof: Cf. [Neukirch CFT P71].
Prop. (IV.4.1.6) (Properties of Inv). There are commutative diagrams:

\[
\begin{array}{ccc}
A_K & \rightarrow & G_{N/K}^a \\
\downarrow \text{id} & & \downarrow \pi \\
A_K & \rightarrow & G_{N/K}^a_L \\
\end{array}
\quad \begin{array}{ccc}
A_K & \rightarrow & G_{N/K}^a \\
\downarrow \text{Ver} & & \downarrow \sigma \\
A_L & \rightarrow & G_{N/L}^a_{\sigma K} \\
\end{array}
\quad \begin{array}{ccc}
A_K & \rightarrow & G_{N/K}^a \\
\downarrow \iota & & \downarrow \sigma^* \\
A_K & \rightarrow & G_{N/L}^a_{\sigma L/\sigma K} \\
\end{array}
\]

Where Ver is the transfer map defined in ??.

Proof: Cf.[Neukirch CFT P72]. □

Prop. (IV.4.1.7). For a finite normal extension \( L/K \), \( N_{L/K}A_L = N_{L/K}^a A_L^a \). This is because the quotient both correspond to \( G_{L/K}^a \). So class field theory doesn’t tell about non-Abelian extension.

Prop. (IV.4.1.8) (Norm Group and Abelian Extension). The map \( L \mapsto I_L = N_{L/K}A_L \) defines a inclusion reversing isomorphism between the lattice of Abelian extension \( L \) of \( K \) and the lattice of norm groups of \( A_K \), i.e.:

\[
I_{L_1 \cap L_2} = I_{L_1} \cap I_{L_2}, \quad I_{L_1 \cdot L_2} = I_{L_1} \cdot I_{L_2}.
\]

And any group that contains a norm group is a norm group.

Proof: By the first commutative diagram of inv, if \((a, L_1/K) = 0\), then \((a, L_1L_2/K)\) is trivial on \( G_{L_1/L_2} \), thus trivial on \( G_{L_1L_2/K} \), thus \( a \in I_{L_1L_2} \). So \( I_{L_1} \cap I_{L_2} \subset I_{L_1L_2} \), the other side is easy. the second is because \( |I_{L_1 \cap L_2}/I_{L_1}| = |G_{L_1/L_1 \cap L_2}| = |G_{L_1L_2}/L_2| = |I_{L_1}I_{L_2}/I_{L_1}| \). Also we deduce \( I_{L_1} \subset I_{L_2} \iff L_2 \subset L_1 \), thus by canonical isomorphism, groups containing \( N_{L/K}A_L \) are one-to-one correspondence with middle fields of \( L/K \) by counting numbers. □

Remark (IV.4.1.9). This shows the philosophy of CFT, i.e. the property of Abelian extensions of a field is can be read from its multiplicative group structure. Of course, determining and characterizing these norm groups requires some work.

Local Class Field Theory

The strategy is to first establish CFT for unramified extensions, then show that unramified extensions already cover \( H^2(K/K) \).

Lemma (IV.4.1.10). Let \( L/K \) be an unramified extension, then \( H^q(G_{L/K}, U_L) = 0 \) for all \( q \).

Proof: Cf.[Neukirch P83]. □

Prop. (IV.4.1.11). The unramified extensions of \( K \) forms a class formation.

Proof: We first define the inv map: use the exact sequence \( 1 \rightarrow U_L \rightarrow L^* \xrightarrow{\nu_L} \mathbb{Z} \rightarrow 0 \), using the lemma(IV.4.1.10), we have

\[
H^2(G_{L/K}, L^*) \xrightarrow{\nu_K} H^2(G_{L/K}, \mathbb{Z}) \xrightarrow{\delta^{-1}} H^1(G_{L/K}, \mathbb{Q}/\mathbb{Z}) = \chi(G_{L/K}).
\]

And there is an isomorphism \( \chi(G/K) \xrightarrow{\varphi} 1_{[L,K]} \mathbb{Z}/\mathbb{Z} \), where \( \varphi \) is the Frobenius which generate \( G_{L/K} \), and \( \varphi(\chi) = \chi(\varphi) \).

To verify this is a class formation, we should verify(IV.4.1.1), Cf.[Neukirch P85]. □
Prop. (IV.4.1.12). If $L/K$ is unramified, then $(a, L/K) = \varphi_{L/K}(a)$, Cf.[Neukirch CFT P86]. The same holds for $L$ replaces by $T$, in which case

$$1 \to U_K \to K^* \to G_{T/K} \to 0$$

is exact. Cf[Neukirch P88].

Proof: We use (IV.4.1.5), then $\chi(a, L/K) = \text{inv}_{L/K}(\bar{a} \sim \delta \chi) = \varphi \circ \delta^{-1} \circ v_K(\bar{a} \sim \delta \chi) = \varphi(\delta^{-1}(v_K(a)\delta \chi)) = \varphi(v_K(a)\chi) = v_K(a)\chi(\varphi_{L/K}) = \chi(\varphi_{L/K}(a))$, for any $\chi$. The second assertion follows from the last prop(IV.4.1.13). □

Cor. (IV.4.1.13). The norm group of an unramified extension of degree $f$ is

$$U_K \times \{\pi^f n | n = 0, 1, \ldots\}.$$  

(This follows from the proposition as the degree $f$ is the order of the Frobenius map).

Now we pass to ramified extensions.

Lemma (IV.4.1.14). If $L/K$ is normal, then $|H^2(L/K)|/|L : K|$.

Proof: Cf.[Neukirch CFT P89]. Should use the fact that $G_{L/K}$ is solvable and Herbrand quotient. □

Lemma (IV.4.1.15). If $L/K$ is a normal extension and $L'/K$ is another unramified extension of the same degree, then $H^2(L/K) = H^2(L'/K) \subset Br(K)$.

Proof: In view of (IV.4.1.14) and (IV.4.1.11), we only need to prove $H^2(L'/K) \subset H^2(L/K)$. For this, we let $N = LL'$, then there is an exact sequence (IV.3.1.2)

$$1 \to H^2(L/K) \to H^2(N/K) \xrightarrow{\text{res}_L} H^2(N/L)$$

then we only need to prove $\text{res}_L(c) = 0$, and this follows from $\text{inv}_{N/L}(\text{res}_L c) = 0$. This will follow, if we have

$$\text{inv}_{N/L}(\text{res}_L c) = [L : K] \cdot \text{inv}_{L'/K} c.$$  

This follows from the lemma below (IV.4.1.16). □

Lemma (IV.4.1.16). For two subextensions $L/K, L'/K$ in $M/L$ normal with $L'/K$ unramified extension, $N = LL'$, for $c \in H^2(L'/K)$,

$$\text{inv}_{N/L}(\text{res}_L c) = [L : K] \cdot \text{inv}_{L'/K} c.$$  

Proof: Cf[Neukirch CFT P90]. □

Prop. (IV.4.1.17). $(G_K, K^*)$ forms a class formation.

Proof: This almost follows from that of unramified extensions (IV.4.1.11). We verify axioms (IV.4.1.11) that inf is natural and commutes with res. It is natural because it is natural on unramified extensions, it commutes with res because we can assume $c \in H^2(L'/K)$ unramified and use (IV.4.1.16). □
Cor. (IV.4.1.18) (Main Theorem of Local Class Field Theory). Let $L/K$ be a normal extension, then the homomorphism

$$u_{L/K} : H^3(G_{L/K}, \mathbb{Z}) \cong H^{q+2}(L/K)$$

is an isomorphism.

Cor. (IV.4.1.19). $H^3(L/K) = 1, H^4(L/K) = \chi(G_{L/K})$, by (IV.3.1.6).

Cor. (IV.4.1.20). For a $p$-adic number field $K$, $Br(K) \cong \mathbb{Q}/\mathbb{Z}$.

Prop. (IV.4.1.21). By (IV.4.1.6), there is commutative diagrams

\[
\begin{array}{ccc}
K^* & \rightarrow & G_{N/K}^{ab} \\
\downarrow{id} & & \downarrow{\pi} \\
K^* & \rightarrow & G_{L/K}^{ab}
\end{array}
\quad \begin{array}{ccc}
K^* & \rightarrow & G_{N/K}^{ab} \\
\downarrow{N_{L/K}} & & \downarrow{i} \\
K^* & \rightarrow & L^*
\end{array}
\quad \begin{array}{ccc}
K^* & \rightarrow & G_{N/K}^{ab} \\
\downarrow{\text{Ver}} & & \downarrow{\sigma} \\
K^* & \rightarrow & \sigma^* L^*
\end{array}
\quad \begin{array}{ccc}
K^* & \rightarrow & G_{N/K}^{ab} \\
\downarrow{\sigma K^*} & & \downarrow{\sigma^*} \\
\sigma L^* & \rightarrow & G_{\sigma L/\sigma K}^{ab}
\end{array}
\]

Prop. (IV.4.1.22). For an Abelian extension $L/K$, the higher principal units $U^*_K$ are mapped under the higher ramification groups of $G_{L/K}$ under the upper numbering.

Prop. (IV.4.1.23). Getting things together, we get a universal norm residue map

$$K^* \xrightarrow{(-,K)} G_K^{ab}$$

It is injective because $(K^*)^n$ are all norm groups by (IV.4.1.25), so the kernel is there intersection with is 1 by (IV.2.2.5). It image is called the Weil group.

Now we want to characterize the norm groups of $K^*$.

Prop. (IV.4.1.24) (Norm Group and Abelian Extension). The map $L \mapsto I_L = N_{L/K}A_L$ defines a inclusion reversing isomorphism between the lattice of Abelian extension $L$ of $K$ and the lattice of norm groups of $A_K$, i.e.:

$$I_{L_1L_2} = I_{L_1} \cap I_{L_2}, \quad I_{L_1 \cap L_2} = I_{L_1} \cdot I_{L_2}.$$

And any group that contains a norm group is a norm group. This follows from (IV.4.1.8) and (IV.4.1.17).

Prop. (IV.4.1.25). The norm groups are precisely the open(closed) subgroups of finite index in $K^*$. In fact finite index are itself open because it contains $(K^*)^n$ which is open.

Proof: One part follows from (IV.4.1.24) and the fact that $(K^*)^n$ is open (IV.2.2.5). For the converse, we only need to prove $(K^*)^n$ is a norm group. This uses Kummer theory and Cf.[Nuekirch CFT P96].

Prop. (IV.4.1.26) (Norm Groups of Local Fields). The norm groups of $K^*$ are exactly the groups containing $U^n_K \times (\pi^f)$ for some $n, f$.

Proof: $U^n_K \times (\pi^f)$ is a norm group because it is closed of finite index. Conversely, any norm group contains some $U^n_K$ because it is open and contains some $(\pi^f)$ because it is of finite index.
Lubin-Tate Formal Group

This is a continuation of 2.

Prop. (IV.4.1.27). There is an isomorphism of $\mathcal{O}$-modules $\Lambda_{f,n} \cong \mathcal{O}/\pi^n\mathcal{O}$, Cf.[Neukirch CFT P101]. Thus the automorphism of $\Lambda_{f,n}$ is all of the form $uf$ for units, isomorphic to $U_K/U_K^n$.

So we can define a Tate module $TG = \varprojlim \text{Ker}(\pi_K^n)$, it is a free $\mathcal{O}_G$-module of rank 1.

Def. (IV.4.1.28). As $TG$ is a free $\mathcal{O}_G$-module of dimension 1, and $G_K$ acts on $TG$, there can be attached a Lubin-Tate character $\chi_K : G_K \to \mathcal{O}_K^*$ by $g(\alpha) = \frac{\chi_K(g)(\alpha)}{\alpha}$, this depends on $\pi_K$, but its restriction on $I_K$ doesn’t depend on $\pi_K$, and is just the local CFT isomorphism composed with $x \to x^{-1}$.

Proof: $[\chi_K(g)]$ is, by definition, the morphism that is id on $K^{ur}$ and $g$ on $L_\pi$. So it equals $g$ on all $K^{ab}$ iff $g$ is id on $K^{ur}$, that is, $g \in I_K$. So if $g \in I_K$, by local CFT, $(\chi(g))^{-1}$ corresponds to $g$, uniquely.

Prop. (IV.4.1.29). $G_{\pi,n} \cong \mathcal{O}_K^*/U_K^n$, thus we have $G_{\pi} \cong \mathcal{O}_K^*$. $L_{\pi,n}/K$ is Abelian totally ramified of degree $p^{n-1}(p-1)$ generated by a Eisenstein polynomial with constant coefficient $\pi$ so $\pi$ is in the norm group.

Proof: For this, first note Galois action induce an isomorphism on $\Lambda_{f,n}$, thus correspond to an element of $U_K/U_K^n$ by (IV.4.1.27), this is an injection because $\Lambda_{f,n}$ generate $L_{\pi,n}$. Then we use the canonical polynomial $f(Z) = \pi Z + Z^n$, $f^n = f^{n-1}(\varphi(n))$, where $\varphi(n)$ is a Eisenstein polynomial, thus $L_{\pi,n}/K$ is totally ramifies with $|G_{\pi,n}| = q^n(q-1) = |U_K/U_K^n|$, thus the result.

Prop. (IV.4.1.30) (Explicit Local Norm Residue Symbol). Now we can write the universal residue symbol little bit more explicitly. For $a = u\pi^n$, $(a, K)$ acts by $\varphi^m$ on $T$ and generated by the action $(u^{-1})_f$ on $\Lambda_{f,n}$ on $L_{\pi,n}$. Cf.[Neukirch CFT P106].

Thus the norm group of $L_{\pi,n}$ is just $U^n$ by (IV.4.1.29).

Cor. (IV.4.1.31). The norm groups of the totally ramified Abelian extension is precisely the groups that contains some $U_K^n \times (\pi)$ for some uniformizer $\pi$. And every totally ramified Abelian extension $L/K$ is contained in some $L_{\pi,n}$.

Proof: For any totally ramified extension, choose a uniformizer, then its norm is a uniformizer $\pi$ of $K$. And $N_{L/K}$ is open (as it contains $(K^*)^m$??.) Thus it contains some $U^n$. The rest follows from local CFT (IV.4.1.24).

Cor. (IV.4.1.32) (Maximal Abelian Extension of Local Fields). Let $L_\pi = \cup L_{\pi,n} = K(\Lambda_f)$, where $\Lambda_f = \cup \Lambda_{f,n}$, then $T \cdot L_\pi$ is the maximal extension of Abelian extension of $K$. Hence $G_{ab}^\pi = G_{T,K} \times G_{\pi}$. This follows immediately from (IV.4.1.26).

Cor. (IV.4.1.33) (Hasse-Arf). We can prove Hasse-Arf (IV.2.1.24) in the case where $K$ is a local field. This is because we already know the maximal Abelian extension, and $G(K^{ab}/T) \cong G(L_\pi/K) \cong \mathbb{Z}_p$ for which we know the Galois action well (IV.4.1.27)/(IV.4.1.29), so $i(\sigma) = v(\sigma(\alpha_n) - \alpha_n) = v([\sigma - 1](\alpha))$, which jumps at $U_K^n$ (the same pattern as $K = \mathbb{Q}_p$ (IV.2.1.26), thus the result.

Remark (IV.4.1.34). There is a concrete example. When $K = \mathbb{Q}_p$, we can choose $f(Z) = (1 + Z)^p - 1$, thus $L_{\pi,n}$ is just $\mathbb{Q}_p(\xi_p^n)$. And we have $\tau = (1 + Z)^r - 1$, thus we have

$$(a, \mathbb{Q}_p(\xi_p^n)/\mathbb{Q}_p)\zeta = \zeta^r$$

where $a = up^m$, and $r \equiv u^{-1} \mod p^n$. 
Global Class Field Theory

For Basic Notations regarding Idele and Adele, See(IV.2.4.10).

The ideal class group $C_K = I_K/K^*$ is a the main object of global class field theory. We will denote $H^q(G_{L/K}, C_L)$ by $H^q(L/K)$. $H^2(G_{L/K}, I_L)$ is the secondary object.

**Prop. (IV.4.1.35).** Let $\mathfrak{p}$ be a prime of $L$ lying over $p$, then $H^q(G, I^\mathfrak{p}_L) \cong H^q(G_{\mathfrak{p}}, L^\mathfrak{p}_L)$. If $p$ is a finite unramified prime of $L$, then $H^q(G, U^\mathfrak{p}_L) = 1$ for all $q$.

**Proof:** Notice $I^\mathfrak{p}_L = \bigcup S I^\mathfrak{p}_L$, then use the last proposition, notice group cohomology commutes with colimits(IV.3.2.2). □

**Cor. (IV.4.1.36).**

$$H^q(G, I^{S^\mathfrak{p}}_L) = \bigoplus_{p \in S} H^q(G_{\mathfrak{p}}, L^\mathfrak{p}_L), \quad H^q(G, I_L) = \bigoplus_{p} H^q(G_{\mathfrak{p}}, L^\mathfrak{p}_L).$$

And the isomorphism is natural, by restriction to components.

**Proof:** For this, just notice $I_L = \bigcup S I^S_L$, then use the last proposition, notice group cohomology commutes with colimits(IV.3.2.2). □

**Cor. (IV.4.1.37).** $H^1(G, I_L) = H^3(G, I_L) = 0$, by(IV.4.1.19).

**Cor. (IV.4.1.38).** An idele $a \in I_K$ is the norm of an idele $b$ in $I_L$ if each component $a_p$ is the norm of an element $b^\mathfrak{p}_p \in L^\mathfrak{p}_L$.

**Prop. (IV.4.1.39).** The decomposition commutes with inf, res and cor. Cf.[Neukirch CFT P125].

The strategy is to first establish CFT for cyclic extensions, then show they cover all $H^2(K/K)$.

**Lemma (IV.4.1.40).** For a cyclic extension $L/K$ of order $p$, $C_L$ is a Herbrand module with Herbrand quotient $h(C_L) = p$.

**Proof:** □

**Prop. (IV.4.1.41) (First Fundamental Inequality).** $(C_K : N_G C_L) \geq p$

**Prop. (IV.4.1.42).** If $K$ contains $p$-th roots of unity and $L/K$ is a cyclic extension of order $p$, then $(C_K : N_G C_L) \leq p$.

**Cor. (IV.4.1.43) (Second Fundamental Inequality).** If $L/K$ is a cyclic extension of order $p$, then $(C_K : N_G C_L) = p$.

**Cor. (IV.4.1.44) (Hass Norm Theorem).** For a cyclic extension $L/K$, an element $x \in K^*$ is a norm iff it is locally a norm everywhere.

**Proof:** Use the long exact sequence for $1 \to L^* \to I_L \to C_L \to 1$, we see that $H^0(G, L^*) \to H^0(G, I_L)$ is an injection, which is

$$0 \to K^*/N_L/K L^* \to \bigoplus_p K^*_p/N_{L/K^p} L^*_p.$$

In fact, by(IV.4.1.37), we say that this is equivalent to $H^1(G_{L/K}, C_L) = 1$, which is equivalent to second fundamental inequality. □
**Prop. (IV.4.1.45).** For $L/K$ normal extension, $|H^2(G, C_L)||L : K|$. 

*Proof:* Cf.[Neukirch P137].

**Prop. (IV.4.1.46).** Let $K$ be a finite algebraic number field, then

$$Br(K) = \bigcup_{L/K \text{ cyclic}} H^2(G_{L/K}, L^*), \quad H^2(G_{K/K}, I_K) = \bigcup_{L/K \text{ cyclic}} H^2(G_{L/K}, I_L).$$

*Proof:* Cf.[Neukirch P127].

Next we construct the Invariant map, first for $H^2(G_{L/K}, I_L)$, then for $H^2(G_{L/K}, C_L)$. 

**Def. (IV.4.1.47).** We define for $c = (c_p) \in H^2(G_{L/K}, I_L)$ by

$$\text{inv}_{L/K} c = \sum_p \text{inv}_{L_p/K_p} c_p.$$

For an Abelian extension $L/K$, we define for $a \in I_K$:

$$(a, L/K) = \prod_p (a_p, L_p/K_p) \in G_{L/K}.$$

**Prop. (IV.4.1.48).** If $c \in H^2(G_{L/K}, L^*)$, then $\text{inv}_{L/K} c = 0$. Cf.[Neukirch P141].

**Cor. (IV.4.1.49).** Now we can define the inv map for $C_K$ when . By the exact sequence $1 \to L^* \to I_L \to C_K \to 1$, we have

$$1 \to H^2(G_{G/K}, L^*) \to H^2(G_{L/K}, I_L) \to H^2(G_{L/K}, C_L) \to H^3(G_{L/K}, L^*)$$

The last one is 1 if $L/K$ is cyclic, thus by this proposition, inv is defined for $H^2(G_{L/K}, C_L)$.

**Prop. (IV.4.1.50) (Hasse’s Main Theorem).** For every finite algebraic number field $K$, there is a canonical exact sequence

$$1 \to Br(K) \to \bigoplus_p Br(K_p) \xrightarrow{\text{inv}_K} \mathbb{Q}/\mathbb{Z} \to 0$$

*Proof:* Cf.[Neukirch P146].

**Prop. (IV.4.1.51).** If $L/K$ is normal and $L'/K$ is cyclic and they have the same degree, then $H^2(L'/K) = H^2(L/K) \subset H^2(K/K)$.

**Cor. (IV.4.1.52).** $H^2(K/K) = \bigcup_{L/K \text{ cyclic}} H^2(L/K)$, thus the homomorphism $H^2(G_K, I_K) \to H^2(K/K)$ is surjective by (IV.4.1.49).

**Prop. (IV.4.1.53).** The inv map is defined for $H^2(K/K)$, and $\text{inv}_{L/K} : H^2(L/K) \to \frac{1}{[L:K]} \mathbb{Z}/\mathbb{Z}$ is an isomorphism for every normal extension $L/K$.

**Prop. (IV.4.1.54) (Main Theorem).** The formation $(G_K, C_K)$ is a class formation with the inv map.
Cor. (IV.4.1.55) (Artin’s Reciprocity Law). The cup product with the fundamental class in $H^2(L/K)$ defines an isomorphism reciprocity map

$$G^a_{ab/L/K} \cong H^{-2}(G_{L/K}; \mathbb{Z}) \to H^0(L/K) = C_K/N_{L/K}C_L.$$  

And the reverse map is called the norm residue symbol

$$1 \to N_{L/K}C_L \to C_K \xrightarrow{(-/L/K)} G^a_{ab/L/K} \to 1$$

Remark (IV.4.1.56). WARNING: we have already defined a norm residue map in (IV.4.1.47), they are compatible with that derived from CFT mechanism. i.e. local global correspondence and vanish on $K^*$.  

Proof: Cf.[Neukirch CFT P154]. □


Cor. (IV.4.1.58). By (IV.4.1.8), the map $L \mapsto I_L = N_{L/K}C_L$ defines an inclusion reversing isomorphism between the lattice of Abelian extension $L$ of $K$ and the lattice of norm groups of $C_K$, i.e.: 

$$I_{L_1 \cap L_2} = I_{L_1} \cap I_{L_2}, \quad I_{L_1 \cdot L_2} = I_{L_1} \cdot I_{L_2}.$$  

And any group that contains a norm group is a norm group.

Prop. (IV.4.1.59). Let $L/K$ be an Abelian extension, then $(a, L/K) = \prod_p (a_p, L_p/K_p)$

Proof: Cf.[Neukirch P154]. □

Prop. (IV.4.1.60) (Existence Theorem). The norm groups of $C_K$ are precisely the (open)closed subgroups of finite index.

Proof: Cf.[Neukirch P162]. □

Now we want to further characterize the norm groups of $C_K$ in an arithmetic way.

Def. (IV.4.1.61) (Notations). A modulus $\mathfrak{m}$ is a $\prod_p p^n$ that $n_p = 0$ a.e.  

{\mathfrak{m}} is the set of primes in $\mathfrak{m}$.

$$I^{\mathfrak{m}}_K = \{ a \in I_K | a \equiv 1 \pmod{\mathfrak{m}} \}.$$  

The congruence subgroup mod $\mathfrak{m}$ $C^\mathfrak{m}_K = I^\mathfrak{m}_K \cdot K^*/K^* \subset C_K$.

$C^\mathfrak{m}_K$ is a norm group by (IV.4.1.63), the Abelian class field $L/K$ associated with $C^\mathfrak{m}_K$ is called the ray class field mod $\mathfrak{m}$, so its Galois group is isomorphic to $C_K/C^\mathfrak{m}_K$.

Prop. (IV.4.1.62). For a field $K$, if $S$ is a finite set of primes that contains all the infinite primes and all the primes lying above the primes dividing $n$, and $I_K = I^S_K \cdot K^*$, then $C^S_K \cdot U^S_K$ is a norm group. If $K$ contains the $n$-th roots of unity, then it corresponds to the Kummer extension $T = K(\sqrt[n]{K^S}/K)$.

Prop. (IV.4.1.63). The norm groups $N_{L/K}$ of $C_K$ is precisely the groups containing some congruence subgroup $C^\mathfrak{m}_K$. Such $\mathfrak{m}$ are called a modulus of definition for $L/K$.

Proof: Cf.[Neukirch P164]. □
Prop. (IV.4.1.64). Getting things together, we get a universal norm residue symbol \( C_K \xrightarrow{(-K)} G_K^{ab} \), and its kernel is \( D_K = \cap_L N_{L/K} C_L \).

Then we have \( D_K = \cap C_K^\alpha \) and it is the connected component of \( 1 \in C_K \) and \( C_K/D_K \to G_K^{ab} \) is an isomorphism.

Proof: Cf.[Neukirch P167]. and [Class Field Theory Artin Tate Chap9]. \( \square \)

Prop. (IV.4.1.65). When \( K = \mathbb{Q} \) and \( \mathfrak{m} = m \cdot p_\infty \), then the ray class field mod \( \mathfrak{m} \) is \( \mathbb{Q}(\zeta_m) \).

Proof: Cf.[Neukirch P165]. \( \square \)

Cor. (IV.4.1.66) (Kronecker Theorem). Every Abelian extension of \( \mathbb{Q} \) is a subfield of \( \mathbb{Q}(\zeta_m) \) for some cyclotomic field.

Remark (IV.4.1.67). The ray class field mod 1 is important, it is the Hilbert class field of \( K \), its Galois group is isomorphic to \( C_K/C_K^1 \cong I_K/I_K^{S\infty} \cdot K^* \cong J_K/P_K \) by (IV.2.4.18). Its degree is equal to the ideal class number \( h \) of \( K \).

Next we investigate the relation of CFT with the decomposition of primes in extension fields.

Prop. (IV.4.1.68). If \( L/K \) is an Abelian extension, then \( N_{L/K} C_L \cap K^* = N_{L/K} p L^*_p \).

Proof: For the non-trivial part, notice if \( a \in N_{L/K} L^*_p \) is a norm times a \( a \in K^* \), then it is also norm at \( p \) by the multiplicative definition of the inv map (IV.4.1.47). \( \square \)

Cor. (IV.4.1.69). Let \( L/K \) be Abelian and \( N = N_{L/K} C_L \) be the norm group, then \( p \) is unramified in \( L \) iff \( U_p \subset N \) and \( p \) splits completely in \( L \) iff \( K^*_p \subset N_{L/K} \).

Cor. (IV.4.1.70) (Conductor). We can define the conductor \( f \) of \( L/K \) as the gcd of all \( \mathfrak{m} \) that \( C_K^\mathfrak{m} \in N_{L/K} \). Then all primes not in \( f \) are unramified and in particular, all primes not in \( \mathfrak{m} \) are unramified in \( C^\mathfrak{m} \).

Prop. (IV.4.1.71) (Ramification and Norm Group). Let \( L/K \) is an Abelian extension of degree \( n \) and \( p \) is an unramified prime ideal of \( K \) and \( \pi \) is a uniformizer, then if \( f \) is the smallest number that \( (\ldots, 1, \pi^f, 1, \ldots) \in N_{L/K} C_L \), then \( p \) factors in the extension \( L \) into \( r = n/f \) distinct primes of degree \( f \).

Proof: The degree the extension of \( p \) is just the order of the Frobenius automorphism of \( G_{\mathfrak{q}/p} \), which is just the order in \( G_{L/K} \cong C_K/N_{L/K} C_L \). The Frobenius of \( p \) correspond exactly to \( (\ldots, 1, \pi, 1, \ldots) \) by (IV.4.1.12), so the result follows. \( \square \)

Prop. (IV.4.1.72). The Hilbert class field is the maximal unramified extension of \( K \).

Prop. (IV.4.1.73) (Principal Ideal Theorem). In the Hilbert class field over \( K \), every ideal \( \mathfrak{a} \) of \( K \) becomes a principal ideal.

Proof: Cf.[Neukirch P171]. It should use a Finite Group theory theorem (I.3.8.3) \( \square \)

Next we interpret the conclusions of GCFT in the language of ideals.
Def. (IV.4.1.74) (Notations). \( J^m \) is the group of all ideals relatively prime to \( m \).

The ray mod \( m \) \( P^m \) is the group of all principal ideals \( (a) \) that \( a \equiv 1 \mod m \).

All subgroups of \( J^m/P^m \) are called ideal groups defined mod \( m \).

If \( L/K \) is an Abelian extension with a modulus of definition \( m \), then \( H^m = N_{L/K} J^m \cdot P^m \) is called the ideal group defined mod \( m \).

Def. (IV.4.1.75). We have a homomorphism \( J^m \rightarrow G_{L/K} \) called the Artin symbol \( (L/K) \). On primes \( p \), it maps a prime \( p \) which is unramified by (IV.4.1.70) to its local Frobenius automorphism in \( G_{\mathfrak{F}/p} \subset G_{L/K} \), which doesn’t depend on \( \mathfrak{P} \) because it is Abelian.

Lemma (IV.4.1.76). When \( m \) is a modulus of definition, the restriction to finite part defines isomorphism \( C/C^m \cong J^m/P^m \) and \( N_{L/K} C_L/C^m \cong H^m/P^m \).

Proof: Cf. [Neukirch CFT P176].

Prop. (IV.4.1.77) (classical Artin Reciprocity Law). If \( L/K \) is an Abelian extension and \( m \) is a modulus of definition, then there is a commutative diagram

\[
\begin{array}{ccc}
1 & \rightarrow & N_{L/K} C_L \\
 \downarrow & & \downarrow \\
1 & \rightarrow & H^m/P^m
\end{array}
\]

Thus the second row is exact by (IV.4.1.55), and \( G_{L/K} \cong J^m/H^m \).

Prop. (IV.4.1.78) (Ramification and Ideal Group). Let \( L/K \) is an Abelian extension of degree \( n \) with a modulus of definition \( m \) (e.g. the conductor) and \( p \) doesn’t divide \( m \). Then if \( f \) is the smallest number that \( p^f \in H^m \), then \( p \) factors in the extension \( L \) into \( r = n/f \) distinct primes of degree \( f \).

Proof: The degree the extension of \( p \) is just the order of the Frobenius automorphism of \( G_{\mathfrak{P}/p} \), which is just the order in \( G_{L/K} \cong J^m/H^m \). The Frobenius of \( p \) correspond exactly to \( p \) by (IV.4.1.12), so the result follows.

2 Cohomology of Local Fields

Def. (IV.4.2.1) (Notations). For an algebraic extension \( K/\mathbb{Q}_p \), we let \( G_K \) be \( G(\mathbb{Q}_p/K) \).

For a finite extension \( K/\mathbb{Q}_p, K_\infty \) is defined to be \( K \) adding all the \( p^n \)-th roots of unities.

\( H_K \) is defined to be \( G(\overline{\mathbb{Q}_p}/K_\infty), \Gamma_K = G_K/H_K \).

The cyclotomic character \( \chi \) is defined to be the multiplicative map \( G_K \rightarrow \mathbb{Z}_p^* \) that \( \sigma(\zeta) = \zeta^{\chi(\sigma)} \) for every \( \sigma \in G_K \) and \( \zeta \) a \( p^n \)-th root of unity. The kernel of \( \chi \) is \( H_K \), and it identifies \( \Gamma_{\mathbb{Q}_p} \) as \( \mathbb{Z}_p^* \) and \( \Gamma_K \) as an open subgroup of \( \mathbb{Z}_p^* \).

Prop. (IV.4.2.2). The profinite group \( \overline{\mathbb{Q}_p}^\text{tame} \) is \( \hat{\mathbb{Z}} \times \Delta_p \). Which is the profinite group generated by the relationship \( \sigma \tau \sigma^{-1} = \tau^p \), where \( \sigma \) is a lift of Frobenius. Which means that it is the limit of finite quotients of the group \( \langle \sigma \tau \sigma^{-1} = \tau^p \rangle \).

Proof: Cf. [Local Fields Clark].

3 Cohomology of Global Fields

4 Iwasawa Theory
IV.5  Modular Forms

Main References are [D-S16] and [Mil17c]. This section is a continuation of the discussion of Automorphic Forms in III.2 and use the same notations.

1  Modular Forms

**Lemma (IV.5.1.1).** There is a complete set of coset representatives for $SL(2,\mathbb{Z}) \backslash GL(2,\mathbb{Q}) / SL(2,\mathbb{Z})$ consisting of the diagonal matrices $\text{diag}(d_1, d_2)$, where $d_1, d_2 \in \mathbb{Q}$ and $d_1 / d_2$ is a positive integer.

*Proof:* This follows directly from (IX.8.6.5), noticing the sign. □

**Def. (IV.5.1.2) (Congruence Subgroup).** Let $\Gamma(N)$ be the inverse image of $1 \in SL(2,\mathbb{Z}/N)$ in the mapping $SL(2,\mathbb{Z}) \to SL(2,\mathbb{Z}/N)$, then a subgroup $\Gamma$ of $SL(2,\mathbb{Z})$ is called a congruence subgroup if it contains $\Gamma(N)$ for some $N$.

In particular, a congruence subgroup has finite index in $\Gamma(1)$.

**Prop. (IV.5.1.3) (Conjugate of Congruence Subgroup).** If $\Gamma$ is a congruence subgroup of level $N$ and $\alpha \in GL(\mathbb{Z})$ with $\det(\alpha) = D > 0$, then $\alpha^{-1} \Gamma \alpha$ is a congruence subgroup of level $DN$.

*Proof:* If $A$ is a matrix that $A \equiv 1 \mod DN$, then $\alpha(\Gamma - 1) \alpha^{-1} = \frac{1}{D} \alpha(\Gamma - 1) \alpha^* \equiv 1 \mod N$. Thus $\alpha^{-1} \Gamma \alpha$ contains $\Gamma(DN)$. □

**Lemma (IV.5.1.4).** The action of $\Gamma(1) = SL(2,\mathbb{Z})$ on $\mathcal{H}$ is properly discontinuous (IX.1.12.13).

*Proof:* Cf. [Bump P18]. □

**Prop. (IV.5.1.5).** The subset $F = \{ z \in \mathcal{H} | |\text{Re}(z)| < 1/2, |z| > 1 \}$ is a fundamental domain for $\Gamma(1) = SL_2(\mathbb{Z})$, and moreover, let $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $S^2 = 1$, $(ST)^3 = (TS)^3 = -1$, and

1. two elements $z, z'$ of $\mathcal{D}$ are equivalent under $\Gamma(1)$ iff
   
   (a) $\text{Re}(z) = -1/2$ and $z' = z + 1$, then $z' = T(z)$.
   
   (b) $|z| = 1$ and $z' = -\frac{1}{z}$, then $z' = S(z)$.

2. Let $z \in \mathcal{D}$, if the stabilizer of $z$ is not $\pm 1$, then
   
   (a) $z = i$ and $\text{Stab}(i) = \langle S \rangle$.
   
   (b) $z = \rho = e^{2\pi i/6}$ and $\text{Stab}(\rho) = \langle TS \rangle$.
   
   (c) $z = \omega$, and $\text{Stab}(\omega) = \langle ST \rangle$.

*Proof:* 1: Let $\Gamma'$ be the subgroup of $\Gamma(1)$ generated by $S$ and $T$. By (X.2.2.5),

$$\text{Im}(\gamma(z)) = \frac{\text{Im}(z)}{|cz + d|^2},$$

and there is a $\gamma \in \Gamma'$ that $|cz + d|$ attains the minimal value, then $\text{Im}(\gamma(z))$ attains a maximal value. Now there is a $n$ that $z' = \text{Re}(T^n \gamma(z)) \in [-1/2, 1/2]$. Now I claim that $|z'| \geq 1$, because otherwise

$$\text{Im}(Sz') = \text{Im}(-1/z') = \frac{\text{Im}(z')}{|z'|^2} > \text{Im}(z'),$$

and use the same notations.
2: Suppose \( z, z' \in \mathcal{D} \) are \( \Gamma \)-conjugate, we can assume \( \text{Im}(z) \leq \text{Im}(z') \). Suppose \( z' = \gamma(z) \) and \( z = x + iy \), then
\[
(cx + d)^2 + (cy)^2 \leq 1
\]. Then \(|c| < 2\). If \( c = 0 \), then \( d = \pm 1 \), and \( \gamma \) is a translation, thus we are in case 1. If \( c = 1 \), then \( d = 0 \), unless \( z = \rho \) or \( \omega \), and \( c = -1 \) case is similar. \( \square \)

**Def. (IV.5.1.6) (Elliptic Points).** A point \( z \in \mathcal{H} \) is called a **elliptic point** if it is the fixed point of an elliptic element \( \gamma \) of \( \Gamma(\text{III.2.1.2}) \).

**Prop. (IV.5.1.7).** Let \( \Gamma \) be a discrete subgroup of \( SL(2, \mathbb{R}) \) and \( z \) an elliptic point of \( \Gamma \), then the stabilizer \( \Gamma_z \) of \( z \) in \( \Gamma \) is a finite cyclic subgroup.

**Proof:** Because \( SL(2, \mathbb{R}) \) acts transitively on \( \mathcal{H} \), by conjugacy we can assume the elliptic point is \( i \) for another \( \Gamma' \). Then the stabilizer of \( i \) in \( SL(2, \mathbb{R}) \) is \( SO(2, \mathbb{R}) \cong S^1 \), and \( SO(2, \mathbb{R}) \cap \Gamma' \) is a compact and discrete subgroup, so it is finite cyclic. \( \square \)

**Prop. (IV.5.1.8) (Cusps and Elliptic Points for \( \Gamma(1) \)).**

- The cusps of \( \Gamma(1) \) are exactly \( \mathbb{P}^1(\mathbb{Q}) \), and each of them is \( \Gamma(1) \)-equivalent to \( \infty \).
- The elliptic points of \( \Gamma(1) \) are exactly those that are \( \Gamma(1) \)-conjugate to \( i \) or \( \rho = (1 + \sqrt{3})/2 \).

**Proof:**

1: Clearly \( \infty \) is fixed by the parabolic matrix \( T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \). Now for any \( m/n \) that \( m/n \) is coprime, there exists integers \( r, s \) that \( rm - sn = 1 \). Let \( \gamma = \begin{bmatrix} m & s \\ n & r \end{bmatrix} \), then \( \gamma(\infty) = m/n \) thus \( m/n \) is also a cusp. Conversely, every parabolic matrix is conjugate to \( T \), thus its fixed point is conjugate to \( \infty \), which means the fixed point is in \( \mathbb{Q} \cup \{ \infty \} \).

2: For the elliptic points of \( \Gamma(1) \), use (IV.5.1.5). \( \square \)

**Def. (IV.5.1.9) (Meromorphic Modular Forms).** Let \( k > 0 \) and \( \chi \) is a character of \( \Gamma \) that \( \chi(-1) = (-1)^k \), the space of (twisted) **meromorphic modular forms** \( A_k(\Gamma, \chi) \) is the space of all holomorphic functions \( \mathcal{H} \to \mathbb{C} \) that satisfies

- \( f \) is meromorphic.
- \( f[\gamma]_k = \chi(\gamma)f(\text{III.2.1.12}) \).
- \( f \) is meromorphic at the cusps (III.2.1.19).

And it is called a **holomorphic modular form** iff it is holomorphic and holomorphic at cusps (III.2.1.19).

Denote \( A_k(\Gamma, 1) = A_k(\Gamma) \), \( M_k(\Gamma, 1) = M_k(\Gamma) \), and moreover we denote by \( S_k(\Gamma, \chi) \) the set of **holomorphic cusp forms** consisting of all \( f \in M_k(\Gamma, \chi) \) that vanishes at the cusps (III.2.1.19), and denote \( S_k(\Gamma, 1) = S_k(\Gamma) \).

Let \( A = \oplus_{k \geq 0} A_k(\Gamma) \) be the graded ring of meromorphic modular forms, \( M(\Gamma) = \oplus_{k \geq 0} M_k(\Gamma) \) the graded algebra of holomorphic modular forms for \( \Gamma \), \( \otimes_{k \geq 0} S_k(\Gamma) \) is a graded ideal of \( M(\Gamma) \).

**Prop. (IV.5.1.10) (Integral Modular Forms).** If \( A \subset \mathbb{C} \) be a subring, denote \( M_k(\Gamma, A) = M_k(\Gamma) \cap A[[q]] \)
Prop. (IV.5.1.11) (Petersson Inner Product). Let \( f \in M_k(\Gamma, \chi) \) and \( g \in S_k(\Gamma, \chi) \), then we see \( \overline{fgy^k} \) is invariant under the action of \( \Gamma \), thus we can define

\[
(f, g) = \frac{1}{[SL(2, \mathbb{Z}) : \{\pm I\}]} \int_{\H} \overline{fgy^k} \frac{dx dy}{y^2},
\]

which is finite, and restricts to an inner product on \( S_k(\Gamma, \chi) \). Moreover, this inner form is invariant of the \( \Gamma \) chosen.

Proof: To show it is finite, notice that \( \Gamma \setminus \H \) is a finite translations of the fundamental domain of \( \Gamma(1) \), and for \( \alpha \in SL(2, \mathbb{Z}) \),

\[
\int_{\alpha(D)} fgy^k \frac{dx dy}{y^2} = \int_D f \circ \alpha \overline{f \circ \alpha y}(\alpha(z))^k \frac{dx dy}{y^2} = \int_D f[\alpha]_k g[\alpha]_k \overline{f[y]^k} \frac{dx dy}{y^2},
\]

and \( f[\alpha]_k, g[\alpha]_k \) are modular forms for \( \Gamma' = \alpha^{-1} \Gamma \alpha, \chi'(\gamma') = \chi(\alpha \gamma \alpha^{-1}) \), thus it suffices to prove the integral is finite on the fundamental domain \( F \) for \( \Gamma(1) \) (IV.5.1.5). For this, notice \( f \) is of exponential decay for \( y \), thus the integral is bounded by \( \int_y^\infty e^{-cy} y^{k-2} dy < \infty \). \( \square \)

Prop. (IV.5.1.12) (Twisted Modular Forms). For each Dirichlet character \( \chi \mod N \), we can define a character of \( \Gamma_0(N) \):

\[
\chi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \chi(d).
\]

Then we can define

\[
M_k(N, \chi) = M_k(\Gamma_0(N), \chi) = \{ f \in M_k(\Gamma_1(N)) | f[\gamma]_k = \chi(d) f, \gamma \in \Gamma_0(N) \}(IV.5.1.9).
\]

and \( S_k(N, \chi) = S_k(\Gamma_1(N)) \cap M_k(N, \chi) \). Then we have direct sum decompositions:

\[
M_k(\Gamma_1(N)) = \bigoplus_\chi M_k(N, \chi), \quad S_k(\Gamma_1(N)) = \bigoplus_\chi S_k(N, \chi),
\]

where the summation is over all Dirichlet character \( \mod N \). Moreover, the summands are orthogonal w.r.t. the Petersson inner product (IV.5.1.11).

Proof: Cf. [Diamond, P418]. \( \square \)

2 Modular Curves

Prop. (IV.5.2.1) (Local Picture). Let \( D \) be the unit disk and \( \Delta \) be a finite group acting on \( D \) and fixing 0, then by Schwarz lemma, \( \Delta \) is a finite subgroup of \( \text{Aut}(D, 0) \cong \mathbb{R}/\mathbb{Z} \), so it is a finite cyclic group. If \(|\Delta| = m\), then \( z^m \) is invariant under \( \Delta \) and defines a function on \( \Delta \setminus D \). It is a homeomorphism from \( \Delta \setminus D \) to \( D \), thus defines a complex structure on \( \Delta \setminus D \).

Let \( X = \{ z \in \mathbb{C} | \text{Im}(z) > c \} \) and \( h \) an integer. Let \( \mathbb{Z} \) acts on \( X \) by \( nz = z + nh \). This action can extend to \( X^* = X \cup \{ \infty \} \), and we can consider the quotient space \( \mathbb{Z} \setminus X^* \). The function \( q(z) = e^{2\pi iz/h} \) is a homeomorphism from \( \mathbb{Z} \setminus X^* \) onto the open disk of radius \( e^{-2\pi c/h} \) and center 0, which defines a complex structure on \( \mathbb{Z} \setminus X^* \).

Lemma (IV.5.2.2) (Modular Curves for \( \Gamma(1) \)). Let \( \mathcal{H}^* = \mathcal{H} \cup \{ \infty \} \), the Riemann surface \( \Gamma(1) \setminus \mathcal{H}^* \) is compact and of genus 0, so isomorphic to the Riemann surface.
Proof: We first define a complex structure on $\Gamma(1)\setminus \mathcal{H}$: Let $p : \mathcal{H} \to \Gamma(1)\setminus \mathcal{H}$ be the quotient map, and let $Q$ be a point of $\mathcal{H}$ mapping to $P$. If $Q$ is not an elliptic point, then we can choose a nbhd of $Q$ that maps isomorphically to a nbhd of $P$, so we can define the complex structure near $P$.

If $Q = i$, then the map $z \mapsto \frac{z-i}{z+1}$ maps some open nbhd of $i$ to an open disk $D'$ with center 0, and the action of $S$ is transformed to the action $z \mapsto -z$. By the local picture, $f(z) = \left(\frac{z-i}{z+1}\right)^2$ is invariant under action of $S$ and defines a complex structure near $i$. Similarly, if $Q = \rho$, then $g(z) = \left(\frac{z-\rho}{z-\bar{\rho}}\right)^3$ is invariant under the action of $ST$, and defines a complex structure near $p(\rho)$.

The space $\Gamma \setminus \mathcal{H}$ we get is not compact, and it can be compactified by adding a point $\infty$, the resulting space is compact because it is a quotient space of $\mathcal{D} \cup \{\infty\}$, which is compact. And we give a complex structure on the resulting space: The function $q = e^{2\pi iz}$ is a function mapping a nbhd of $\infty$ in the fundamental domain to an open disk with center 0, and thus giving a complex structure near $\infty$.

It can be seen directly that $\Gamma(1)\setminus \mathcal{H}^*$ is homeomorphic to a sphere, and then use the fact any Riemann surface of genus $0$ is isomorphic to $S^1(IX.12.7.1)$. \hfill \square

Prop. (IV.5.2.3) (Modular Curves for $\Gamma$). For a congruence group $\Gamma$, the quotient space $\Gamma\setminus \mathcal{H}$ can be compactified to a Riemann surface $X(\Gamma)$ by adjoining f.m. points.

Proof: The quotient space $\Gamma\setminus \mathcal{H}$ can be given a complex structure exactly the same way as $\Gamma(1)\setminus \mathcal{H}$, and it can be compactified by adding $\mathbb{P}^1(\mathbb{Q})$, which has only f.m. orbits under action of $\Gamma$, by (IV.5.1.8) and the fact $\Gamma$ has finite index in $\Gamma(1)$. For the complex structure: if $h$ is the smallest positive integer that $T^h \in \Gamma$, then $q = e^{2\pi iz/h}$ is a homeomorphism of a nbhd of $\infty$ in $\Gamma\setminus \mathcal{H}$ to some open disk of 0, thus defines a complex structure near $\infty$. For any other cusps $\alpha$, let $\gamma \in \Gamma(1)$ satisfies $\alpha = \gamma(\infty)$, then $z \mapsto q(\gamma^{-1}(z))$ defines a complex structure near $\alpha$. \hfill \square

Def. (IV.5.2.4) (Notations). Let $\Gamma$ be a congruence subgroup, then we denote

$$Y(\Gamma) = \Gamma\setminus \mathcal{H}, \quad X(\Gamma) = \Gamma\setminus \mathcal{H}^*.$$

Also abbreviate $Y(\Gamma(N))$ to $Y(N)$, $X(\Gamma(N))$ to $X(N)$, and $Y(\Gamma_0(N))$ to $Y_0(N)$, and $X(\Gamma_0(N))$ to $X_0(N)$.

Prop. (IV.5.2.5) (Modular Form as Differential Forms). A holomorphic modular form $M_k(\Gamma)$ is just a holomorphic form on $X(\Gamma)$ of degree $k$, and its dimension can be calculated, see 3.

Similarly, an automorphic function for $\Gamma$ is the same as a function on $X(\Gamma)(IV.5.2.3)$. In particular, if it is holomorphic and vanishes at cusps, then it is constant.

Proof: \hfill \square

3 Dimension Formulae

Def. (IV.5.3.1) (Notations). In this subsection,

- $\Gamma$ is a congruence subgroup of $SL_2(\mathbb{Z})$.
- $g$ is the genus of $X(\Gamma)$.
- $d$ is the degree of the map $X(\Gamma) \to X(1)$ which is equal to $|PSL(2, \mathbb{Z}) : \Gamma|$.
- $\varepsilon_2$ the number of elliptic points with period 2.
- $\varepsilon_3$ the number of elliptic points with period 3.
• $\varepsilon_{\infty}$ the number of cusps.
• $\varepsilon_{\text{reg}}^\infty$ the number of regular cusps (III.2.1.7).
• $\varepsilon_{\text{irr}}^\infty$ the number of irregular cusps.

Prop. (IV.5.3.2) (Genus Formula).

\[ g = 1 + \frac{d}{12} - \frac{\varepsilon_2}{4} - \frac{\varepsilon_3}{3} - \frac{\varepsilon_{\infty}}{2}. \]

Proof: Cf. [Diamond, P68]. □

Prop. (IV.5.3.3) (Zeros and Poles of Automorphic Forms). Let $f$ be a meromorphic forms for $\Gamma$ of weight $2k$, then

\[ \frac{1}{2} \sum_{Q \text{ elliptic points with period } 2} \text{ord}_Q(f) + \frac{1}{3} \sum_{Q \text{ elliptic points with period } 3} \text{ord}_Q(f) + \sum_{Q \text{ others}} \text{ord}_Q(f) = kd/6, \]

where the sum is over a set of representatives for points in $\Gamma \backslash \mathcal{H}^*$.

Proof: Cf. [Milne, P53]. □

Prop. (IV.5.3.4) (Dimension Formulae for $k$ Even). If $k$ is even, then

\[
\dim(M_k(\Gamma)) = \begin{cases} 
\frac{(k-1)d}{12} + \left(\frac{1}{4} - \{\frac{k}{4}\}\right)\varepsilon_2 + \left(\frac{1}{3} - \{\frac{k}{3}\}\right)\varepsilon_3 + \frac{1}{2}\varepsilon_{\infty} & k \geq 2 \\
1 & k = 0 \\
0 & k < 0 
\end{cases}
\]

and

\[
\dim(S_k(\Gamma)) = \begin{cases} 
\frac{(k-1)d}{12} + \left(\frac{1}{4} - \{\frac{k}{4}\}\right)\varepsilon_2 + \left(\frac{1}{3} - \{\frac{k}{3}\}\right)\varepsilon_3 - \frac{1}{2}\varepsilon_{\infty} & k \geq 4 \\
g & k = 2 \\
0 & k \leq 0 
\end{cases}
\]

Proof: Cf. [Diamond P87]. □

Cor. (IV.5.3.5) (Modular Forms for $SL(2, \mathbb{Z})$).

\[ M(\Gamma(1)) = \mathbb{C}[E_4, E_6], \quad S(\Gamma(1)) = \Delta \cdot M(\Gamma(1)). \]

Thus for $k \geq 4$ even,

\[ \dim(S_k(\Gamma(1))) = \begin{cases} 
\left\lfloor \frac{k}{12} \right\rfloor - 1 & k \equiv 2 \mod 12 \\
\left\lfloor \frac{k}{12} \right\rfloor & \text{otherwise} 
\end{cases}. \]

\[ M_k(\Gamma(1)) = S_k(\Gamma(1)) \oplus \mathbb{C}E_k. \]

Proof: Cf. [Diamond P88]. □

Cor. (IV.5.3.6). $M(\Gamma(1)) = M(\Gamma(1), \mathbb{Z}) \otimes \mathbb{C}, \quad S(\Gamma(1)) = S(\Gamma(1), \mathbb{Z}) \otimes \mathbb{C}(IV.5.1.10)$.

Proof: By (IV.5.4.8), $E_4, E_6$ have Fourier coefficients in $\mathbb{Z}$. □

Prop. (IV.5.3.7) (Dimension Formulae for $k$ Odd). For $k$ odd,
if \( k < 0 \) or \(-I \in \Gamma\), then \( M_k(\Gamma) = S_k(\Gamma) = 0 \).

- If \( k \geq 3 \), then

\[
\dim(M_k(\Gamma)) = (k - 1)(g - 1) + \left\lfloor \frac{k}{3} \right\rfloor \varepsilon_3 + \frac{k}{2} \varepsilon_{\text{reg}} + \frac{k - 1}{2} \varepsilon_{\text{irr}}
\]

\[
\dim(S_k(\Gamma)) = (k - 1)(g - 1) + \left\lfloor \frac{k}{3} \right\rfloor \varepsilon_3 + \left( \frac{k}{2} - 1 \right) \varepsilon_{\text{reg}} + \frac{k - 1}{2} \varepsilon_{\text{irr}}
\]

- If \( k = 1 \), then\( \dim(M_1(\Gamma)) \)

\[
\begin{cases}
\varepsilon_{\text{reg}} > 2g - 2 & \text{if } \dim(M_1(\Gamma)) = \dim(M_1(\Gamma)) - \frac{\varepsilon_{\text{reg}}}{2}
\\
\varepsilon_{\text{reg}} \leq 2g - 2 & \text{if } \dim(S_1(\Gamma)) = \dim(M_1(\Gamma)) - \frac{\varepsilon_{\text{reg}}}{2}
\end{cases}
\]

**Proof:** Cf.[Diamond P91].

---

**Explicit Dimension Formulae for \( \Gamma(N), \Gamma_1(N), \Gamma_0(N) \)**

**Prop. (IV.5.3.8) (Elliptic Points).**

**Prop. (IV.5.3.9).** The degree of the mapping \( X(N) \to X(1) \) is

\[
d = d_N = |SL(2, \mathbb{Z}) : \{\pm 1\} \Gamma(N)| = \begin{cases} 
1/2N^3 \prod_{p|N} (1 - 1/p^2) & N > 2 \\
6 & N = 2
\end{cases}
\]

**Proof:** Cf.[D-S16]P101.

**Prop. (IV.5.3.10) (Regular Cusps).** All the cusps of \( \Gamma_0(N) \) and \( \Gamma(N) \) are regular. The only irregular cusp of \( \Gamma_1(N) \) are \( s = 1/2 \) for \( N = 4 \).

**Proof:** Cf.[?]P103.

**Lemma (IV.5.3.11).** Lists of statistics for \( \Gamma_0(N), \Gamma_1(N), \Gamma(N) \).

<table>
<thead>
<tr>
<th>( \Gamma )</th>
<th>( d )</th>
<th>( \varepsilon_2 )</th>
<th>( \varepsilon_3 )</th>
<th>( \varepsilon_\infty )</th>
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<td>( \Gamma_0(N), N &gt; 2 )</td>
<td>( \frac{2d_N}{N\varphi(N)} )</td>
<td>( \prod_{p</td>
<td>N} \prod_{1 + (\frac{1}{p})}^{4/4} \frac{N}{4</td>
<td>N} )</td>
</tr>
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<td>0</td>
<td>2</td>
</tr>
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<td>2</td>
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<tr>
<td>( \Gamma_1(4) )</td>
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<td>0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>( \Gamma_1(N), N &gt; 4 )</td>
<td>( d_N/N )</td>
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<td>0</td>
<td>( \frac{1}{2} \sum_{d</td>
</tr>
<tr>
<td>( \Gamma(1) )</td>
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<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \Gamma(N), N &gt; 2 )</td>
<td>( d_N )</td>
<td>0</td>
<td>0</td>
<td>( d_N/N )</td>
</tr>
</tbody>
</table>

where

\[
d_N = \begin{cases} 
1/2 \prod_{p|N} N^3(1 - 1/p^2) & N > 2 \\
6 & N = 2
\end{cases}
\]

**Proof:** Cf.[Diamond P107].

\[\square\]
Prop. (IV.5.3.12) (List of Dimension Formulae). Lists of dimension formulae for $\Gamma_0(N), \Gamma_1(N), \Gamma(N)$.

<table>
<thead>
<tr>
<th>$\Gamma$</th>
<th>$g$</th>
<th>$\dim(M_k(\Gamma))&amp; \dim(S_k(\Gamma))$, $2k, k \geq 2$</th>
<th>$\dim(M_k(\Gamma))&amp; \dim(S_k(\Gamma))$, $2k+1, k \geq 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_0(N), N &gt; 2$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$\Gamma_1(2)(\Gamma_0(2))$</td>
<td>$0$</td>
<td>$\lfloor \frac{k}{2} \rfloor \pm 1$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\Gamma_1(3)$</td>
<td>$0$</td>
<td>$\lfloor \frac{k}{2} \rfloor \pm 1$</td>
<td>$\lfloor \frac{k}{2} \rfloor \pm 1$</td>
</tr>
<tr>
<td>$\Gamma_1(4)$</td>
<td>$0$</td>
<td>$\lfloor \frac{k}{2} \rfloor \pm 1$</td>
<td>$\lfloor \frac{k}{2} \rfloor \pm 1$</td>
</tr>
<tr>
<td>$\Gamma_1(N), N &gt; 4$</td>
<td>$1 + \frac{d_N}{12N} - \frac{1}{4} \sum_{d</td>
<td>N} \varphi(d) \varphi(N/d)$</td>
<td>$\frac{(k-1)d_N}{12N} + \frac{1}{4} \sum_{d</td>
</tr>
<tr>
<td>$\Gamma(1)$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\Gamma(2)$</td>
<td>$0$</td>
<td>$\frac{k-1}{2}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\Gamma(N), N &gt; 2$</td>
<td>$1 + \frac{d_N(N-6)}{12N}$</td>
<td>$0$</td>
<td>$\frac{(k-1)d_N}{12N} + \frac{d_N}{2N}$</td>
</tr>
</tbody>
</table>

Proof: This follows from (IV.5.3.4)-(IV.5.3.7) (IV.5.3.11) and (IV.5.3.10). □

Remark (IV.5.3.13). The only case that is not resolved is the dimensions of $\dim(M_1(\Gamma))$.

Applications


4 Eisenstein Series

Def. (IV.5.4.1) (Eisenstein Series). Let $\Gamma$ be a congruence subgroup of $\Gamma(1)$, define the space of Eisenstein series $E_k(\Gamma, \chi)$ to be the orthogonal complement of $S_k(\Gamma, \chi)$ in $M_k(\Gamma, \chi)$. Also denote $E_k(\Gamma) = E_k(\Gamma, 1)$.

Prop. (IV.5.4.2) (Dimensions of Eisenstein Series). Notation as in (IV.5.3.1), by (IV.5.3.4) and (IV.5.3.7), the dimensions of the space of Eisenstein series satisfy:

$$\dim(E_k(\Gamma)) = \begin{cases} 
\varepsilon_\infty & k \geq 4, 2|k \\
\varepsilon_\infty - 1 & k = 2 \\
1 & k = 0 \\
\varepsilon_{\infty g} & k \geq 3, 2|k + 1, -1 \notin \Gamma \\
\varepsilon_{\infty g}/2 & k = 1, -1 \notin \Gamma \\
0 & k < 0 \text{ or } 2|k + 1, -1 \in \Gamma 
\end{cases}$$

And also the dimension of $E_k(\Gamma_0(N)), E_k(\Gamma_1(N)), E_k(\Gamma(N))$ can be read off from (IV.5.3.12).

Eisenstein Series for $SL(2, \mathbb{Z})$

Def. (IV.5.4.3). In this subsection, denote $q = e^{2\pi i z}$.

Prop. (IV.5.4.4) (Weakly Modular Forms and Lattices). Let $\mathcal{L}$ be the set of lattices in $\mathbb{C}$, if $F : \mathcal{L} \to \mathbb{C}$ is a function of weight $2k$, i.e. $F(\lambda \Lambda) = \lambda^{-2k} F(\Lambda)$ for $\lambda \in \mathbb{C}^*$, then $f(z) = F(\Lambda(z, 1))$ is a weakly modular form on $\mathcal{H}$ for $\Gamma(1)$ of weight $2k$, and this is a bijection between functions of weight $2k$ on $\mathcal{L}$ and weakly modular functions on $\mathcal{H}$ for $\Gamma(1)$ of weight $2k$. 
Proof: By the hypothesis, there is a function $f$ on $H$ that for any $w_1, w_2$ with $w_1/w_2 \in H$,

$$F(\Lambda(w_1, w_2)) = w_2^{-2k}f(w_1/w_2).$$

Then the invariance of $F$ under $SL(2, \mathbb{Z})$ action implies $f$ is weakly modular of weight $2k$. \qed

Prop. (IV.5.4.5) (Eisenstein Series for $SL(2, \mathbb{Z})$). Let $k > 2$ be an even integer and $\Lambda$ a lattice of $\mathbb{C}$, define the Eisenstein series of weight $k$ to be

$$G_k(\Lambda) = \sum_{\omega \in \Lambda} \frac{1}{\omega^k},$$

and also for a complex number $z$, let $\Lambda_z$ be the lattice generated by 1 and $z$, and let $G_k(z) = G_k(\Lambda_z)$. Then $G_k(z) \in M_k(\Gamma(1))$, and

$$G_k(z) = 2\zeta(k) + 2 \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n$$

And denote

$$E_k(z) = G_k(z)/2\zeta(k) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n$$

the normalized Eisenstein series.

Proof: $G_k(z)$ is weakly modular of weight $k$ by (IV.5.4.4). For the expansion, notice by (X.2.5.7),

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{z+n} + \frac{1}{z-n} \right),$$

and by definition

$$z \cot(\pi z) = \pi i - \frac{\pi i}{1-q} = \pi i - 2\pi i \sum_{n=1}^{\infty} q^n$$

where $z \in H$.

Taking $(k-1)$-th derivative of this, we get

$$\sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1}q^n,$$

thus

$$G_k(z) = \sum_{(m,n) \neq (0,0)} \frac{1}{(nz+m)^{2k}}$$

$$= 2\zeta(2k) + 2 \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} \frac{1}{(nz+m)^{2k}}$$

$$= 2\zeta(2k) + 2 \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} a^{2k-1}q^{an}$$

$$= 2\zeta(k) + 2 \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n$$

Finally the assertion about $E_k(z)$ follows from (III.3.1.52). \qed
Cor. (IV.5.4.6). By (IV.5.4.5) and (X.2.5.5):

\[
E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n, \quad E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n
\]
\[
E_8(z) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n)q^n, \quad E_{10}(z) = 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n)q^n
\]
\[
E_{12}(z) = 1 + 65520 \sum_{n=1}^{\infty} \sigma_9(n)q^n, \quad E_{14}(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_5(n)q^n
\]

Cor. (IV.5.4.7) (Ramanujan Identities). By (IV.5.3.5), \( \dim M_8(\Gamma(1)) = \dim M_{10}(\Gamma(1)) = 1 \), thus there are equations

\[
E_2^4 = E_8, \quad E_4E_6 = E_{10}
\]

which is equivalent to the following equations:

\[
\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m)
\]
\[
11\sigma_9(n) = 21\sigma_5(n) - 10\sigma_3(n) + 1054 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_5(n-m).
\]

Prop. (IV.5.4.8) (Discriminant Function). By (IV.5.4.6), we can define the discriminant function

\[
\Delta(z) = \frac{1}{1728} (E_4^3 - E_6^2) = q - 24q^2 + 252q^3 - 1472q^4 + \tau(5)q^5 + \ldots
\]

So \( \Delta(z) \subset S_{12}(\Gamma(1)) \), and the coefficients \( \tau(n) \) are called the Ramanujan \( \tau \)-function.

Cor. (IV.5.4.9) (Modular Function). By (IV.5.3.3), the discriminant function \( \Delta(z) \) has exactly one simple zero at \( \infty \), thus we can define the modular function \( j \) on \( \mathcal{H} \) as

\[
j : \mathcal{H} \to \mathbb{C}, j(z) = \frac{E_4(z)^3}{\Delta(z)} = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \ldots
\]

which is an automorphic function for \( \Gamma(1) \), and it subjects onto \( \mathbb{C} \).

Proof: To show it surjects onto \( \mathbb{C} \), notice it induces a holomorphic map \( X(1) \cong \mathbb{P}^1 \to \mathbb{P}^1 \), and it maps \( \infty \) to \( \infty \) with no ramification, thus it has degree 1, so it is surjective.

Prop. (IV.5.4.10).

\[
\Delta(z) = \eta(z)^{24} = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.
\]

where \( \eta(z) \) the Dedekind eta function from (III.3.1.49), and \( \dim(S(SL(2,\mathbb{Z}))) = 1 \), spanned by \( \Delta \).

Proof: We show that \( \eta^{24}(z) \) is a holomorphic cusp form in \( S_{12}(\Gamma(1)) \), and it has a first order pole at \( \infty \) by the expression, and has no zero on \( \mathcal{H} \), thus it divides every cusp form in \( S_{12}(\Gamma(1)) \), and the quotient is a holomorphic modular function, thus is a constant. The assertion follows by comparing coefficients.
To show this, it suffices to show
\[ \eta(\gamma(z)) = \varepsilon(\gamma)(cz + d)^{1/2}\eta(z) \]
for every \( \gamma \in \Gamma(1) \), where \( \varepsilon(\gamma) \) is a 24-th root of unity. Because \( [\gamma]_k \) is an action (IV.5.1.9), by 4, it suffices to show for \( S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \) and \( T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \). The case for \( T \) is clear from the last expression of \( \eta(z) \). For \( S \): 

**Def. (IV.5.4.11) (Poincaré Series).** Let \( \Gamma \) be a congruence subgroup, \( T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \). Let \( h \) be the minimal positive integer that \( T^h \subset \Gamma \), and define \( \Gamma_0 \) the subgroup of \( \Gamma \) generated by \( T^h \). Then we define the Poincaré series of weight \( 2k \) and character \( n \) for \( \Gamma \) to be the series

\[ \varphi_n(z) = \sum_{\Gamma \setminus \Gamma'} \frac{e^{2\pi in\gamma(z)}}{(cz + d)^{2k}}, \]

where \( \Gamma' \) is the image of \( \Gamma \) in \( \Gamma(1)/\{\pm 1\} \).

**Prop. (IV.5.4.12).** For \( k \geq 1, n \geq 0 \), The Poincaré series converges absolutely on compact subsets of \( \mathcal{H} \), and is invariant under \( \Gamma \)-action, and it is a modular form of weight \( 2k \) for \( \Gamma \). Moreover,

- \( \varphi_0(z) \) vanishes at all finite cusps, and \( \varphi_0(\infty) = 1 \).
- for \( n \geq 1, \varphi_n(z) \) are cusp forms.

**Proof:** Cf.[Mil, P62].

**Prop. (IV.5.4.13).** The Poincaré series \( \varphi_n(z) \) of weight \( 2k \) spans \( S_{2k}(\Gamma) \).

**Proof:** Cf.[Mil17c]P62.

**General Eisenstein Series**

5 Modular Curves are Algebraic

**Prop. (IV.5.5.1) (X_0(N)).** The field \( C(X_0(N)) \) of modular functions for \( \Gamma_0(N) \) is generated by \( j(z) \) and \( j(Nz) \) over \( \mathbb{C} \), and the minimal polynomial \( F(j, Y) \) of \( j(Nz) \) over \( C(j) \) has degree \( d = [\text{PSL}(2,\mathbb{Z}) : \Gamma_0(N)][(IV.5.3.1)] \), and \( F(j, Y) \in \mathbb{Z}[j, Y] \).

When \( N > 1 \), \( F(X, Y) \) is symmetric in \( X, Y \), and when \( N = p \) is a prime,

\[ F(X, Y) \equiv X^{p+1} + Y^{p+1} - X^pY^p - XY \mod p. \]

**Proof:**

**Cor. (IV.5.5.2).** The field of functions on \( X(1) \) is \( \mathbb{C}(j) \).
6 Hecke Algebra

We only considered Hecke algebra of level 1.

Prop. (IV.5.6.1).

Prop. (IV.5.6.2).

- \( R(1) = T(1) = \text{id} \).
- If \((m, n) = 1\), then \( R(m) R(n) = R(mn) \).
- If \((m, n) = 1\), then \( T(m) \circ T(n) = T(mn) \).
- If \( p \) is a prime and \( n \geq 1 \), then \( T(p^n) \circ T(p) = T(p^{n+1}) + pR(p) \circ T(p^{n-1}) \).

Proof:

Cor. (IV.5.6.3). For any \( m, n \), \( T(m) \circ T(n) = \sum_{d|(m,n), d > 0} dR(d) \circ T(mn/d^2) \).

Proof: It suffices to prove by induction that

\[
T(p^r)T(p^s) = \sum_{i \leq \min(r,s)} p^i R(p^j)T(p^{r-s-2i}).
\]

Cor. (IV.5.6.4). If \( f \) is a cusp form for \( \Gamma(1) \) and \( T_n f(\infty) = 0 \) for all \( n \geq 1 \), then \( f = 0 \).

Prop. (IV.5.6.5) (Normalized Hecke Eigenforms). If \( f = \sum_{n=0}^\infty c_n q^n \) be a nonzero modular form of weight \( 2k \) that satisfies

\[
T(n)f = \lambda(n)f
\]

for all \( n \geq 1 \) and \( \lambda(n) \in \mathbb{C} \), then \( c_1 \neq 0 \), and if \( f \) is normalized that \( c(1) = 1 \), then \( c_n = \lambda(n) \) for all \( n \geq 1 \). In particular, \( c_n \) are all real, because \( T(n) \) is Hermitian.

Proof: Cf.[Milne, P75].

7 L-Functions

Prop. (IV.5.7.1). If \( f \) is a cuspidal modular form for \( \Gamma(1) \), then its Fourier coefficients satisfy \( |a_n| \leq O(n^{k/2}) \).

Proof: Cf.[Bump, P32].

Thm. (IV.5.7.2) (Ramanujan Conjecture). If \( f \) is a cuspidal modular form for \( \Gamma(1) \), then its Fourier coefficients satisfy \( |a_n| \leq O(n^{(k-1)/2} + \varepsilon) \).

Def. (IV.5.7.3) (L-functions associated to Modular Forms). Each \( f \in M_k(\Gamma_1(N)) \) has an associated \( L \)-functions: if \( f = \sum_{n=0}^\infty a_n q^n \), define

\[
L(s, f) = \sum_{n=1}^\infty a_n n^{-s}.
\]
Prop. (IV.5.7.4) (Hecke). Let \(a_0, a_1, \ldots\) be a sequence of complex numbers s.t. \(a_n = O(n^M)\) for some integer \(M\). Given \(\lambda > 0, k > 0, C = \pm 1\), write
\[
\varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad \Phi(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) \varphi(s), \quad f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi inz/\lambda}.
\]

Then the following conditions are equivalent:

1. The function \(\Lambda(s) = \Phi(s) + \frac{a_0}{s} + \frac{Ca_0}{k-s}\) can be analytically continued to a holomorphic function of the whole plane which is bounded on vertical strips, and it satisfies the functional equation
   \[
   \Phi(s) = C \Phi(k-s).
   \]

2. In the upper half plane, \(f\) satisfies the functional equation
   \[
   f(-1/z) = C(z/i)^k f(z).
   \]

Proof: Notice first
\[
\int_0^{\infty} e^{-2\pi n/t} t^{s} dt = (\frac{2\pi}{\lambda})^{-s} \int_0^{\infty} e^{-t^{s}/t} dt = (\frac{2\pi}{\lambda})^{-s} \Gamma(s).
\]
Thus for \(\text{Re}(s)\) large,
\[
\Phi(s) = \int_0^{\infty} (f(it) - a_0) t^{s} dt = \int_1^{\infty} (f(it) - a_0) t^{s} dt + \int_1^{\infty} (f(i t) - a_0) t^{-s} dt.
\]

If 2 holds, then \(f(i/t) = C t^k f(it)\), and
\[
\Phi(s) = \int_1^{\infty} (f(it) - a_0) t^{s} dt + \int_1^{\infty} (C t^k f(it) - a_0) t^{-s} dt
\]
\[
= \int_1^{\infty} (f(it) - a_0) t^{s} dt + C \int_1^{\infty} (f(it) - a_0) t^{k-s} dt + \int_1^{\infty} (C t^{k-s} a_0 - t^{-s} a_0) \frac{dt}{t}.
\]

The first two integral is absolutely convergent for any \(s\) and then can be extended to an analytic function of the whole plane, and the final term equals
\[
-(\frac{a_0}{s} + \frac{Ca_0}{k-s})
\]
when \(\text{Re}(s)\) is large, so it can be extended to a meromorphic function on the whole plane. And \(\Phi(s) = \Phi(k-s)\) follows easily from the form above. Also it is bounded on vertical strips because it attain maximum at the real axis.

Conversely, first notice it suffices to prove for real \(y > 0\),
\[
f(\frac{i}{y}) = C y^k f(iy)
\]
because this implies these two holomorphic functions on \(H\) coincide on the imaginary axis, so they must be equal. Notice
\[
\int_0^{\infty} (f(iy) - a_0) y^{s} \frac{dy}{y} = \Phi(s)
\]
converges absolutely for \( \text{Re}(s) \) sufficiently large, so for \( \sigma > 0 \) sufficiently large, Mellin inversion formula (IV.7.2.16) shows

\[
f(iy) - a_0 = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Phi(s)y^{-s}ds = \frac{C}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \Phi(k-s)y^{-s}ds
\]

\[
= \frac{C}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} (\Lambda(k-s) - \frac{Ca_0}{s} - \frac{a_0}{k-s})y^{-s}ds
\]

Notice \( \Phi(\sigma + it) \) decays exponentially for fixed \( \sigma = \alpha \) sufficiently large, because of the Stirling formula (X.2.8.8), and for \( \sigma = \beta \) sufficiently small, \( \Phi(\sigma) = C\Phi(k-s) \) also shows \( \Phi(\sigma + it) \) decays exponentially. And also \( \frac{Ca_0}{\sigma+i\delta} + \frac{a_0}{k-\sigma+i\delta} = O(t^{-1}) \) for \( \sigma = \alpha \) or \( \beta \), so \( \Lambda(\sigma + it) = O(t^{-1}) \) for \( \sigma \) large or small, and also the hypothesis shows \( \Lambda \) is bounded on the strip \( \alpha < \text{Re}(z) < \beta \), thus by (X.2.4.9), \( \Lambda(\sigma + it) \to 0 \) for \( t \to 0 \) and \( \sigma \) in any compact set. So we can move the integration of \( \Lambda(s) \) to the left or to the right. Then

\[
f(iy) - a_0 = \frac{C}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} (\Lambda(k-s) - \frac{Ca_0}{s} - \frac{a_0}{k-s})y^{-s}ds
\]

\[
= \frac{C}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} y^{-k}\Lambda(s)y^sds - \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} (\frac{a_0}{s} + \frac{Ca_0}{k-s})y^{-s}ds
\]

\[
= Cy^{-k} \Phi(s)y^sds + Cy^{-k} \int_{\sigma-i\infty}^{\sigma+i\infty} (\frac{a_0}{s} + \frac{Ca_0}{k-s})y^sds + \frac{1}{2\pi i} \int_{k-\sigma+i\infty}^{\sigma+i\infty} (\frac{a_0}{s} + \frac{Ca_0}{k-s})y^{-s}ds
\]

\[
= Cy^{-k}(f(iy) - a_0) + \frac{y^{-k}}{2\pi i} \int_{\gamma} (\frac{Ca_0}{s} + \frac{a_0}{k-s})y^sds
\]

\[
= Cy^{-k}(f(iy) - a_0) + \frac{y^{-k}}{2\pi i} \int_{\gamma} (\frac{Ca_0}{s} + \frac{a_0}{k-s})y^sds
\]

So we are done. \( \square \)

Def. (IV.5.7.5) (L-Functions). Let \( a_1, a_2, \ldots \) be a sequence of complex numbers such that \( a_n = O(n^M) \) for some \( M > 0 \). Let

\[
L(s) = \sum a_n n^{-s}, \quad \Lambda(s) = (2\pi)^{-s}\Gamma(s)L(s), \quad f(z) = \sum a_n e^{-2\pi inz}.
\]

More generally, let \( m > 0 \) and \( \chi \) a primitive character mod \( m \), then we define twisted L-function as

\[
L(s, \chi) = \sum a_n \chi(n) n^{-s}, \quad \Lambda(s, \chi) = (2\pi)^{-s}\Gamma(s)L(s, \chi)
\]

Prop. (IV.5.7.6) (Converse Theorem (Correspondence for \( \Gamma(1) \))). If \( f \) is a cusp form of weight \( k \) for \( \Gamma(1)(k) \) is even), then \( \Lambda(s, f) = (2\pi)^{-s}\Gamma(s)L(s, f) \) has analytic continuation to all \( s \), is bounded on vertical strips, and satisfies a functional equation

\[
\Lambda(s, f) = (-1)^{k/2}\Lambda(k-s, f).
\]

Conversely, let \( a_0, a_1, \ldots \) be a sequence of complex numbers that \( a_n = O(n^M) \) for some \( M > 0 \). Let \( f(s), L(s), \Lambda(s) \) be defined in (IV.5.7.5) and \( \Lambda(s) \) has analytic continuation to all \( s \) and satisfies the above functional equation then \( f(z) \) is an element of \( S_k(\Gamma(1)) \).

Moreover, in this case, \( f \) is a normalized Hecke eigenform (IV.5.6.5) iff \( L(s, f) \) has a Euler product formula

\[
L(s) = \prod_p (1 - a_p p^{-s} + p^{k-1+2s})^{-1}
\]

where \( 1 - a_s X + p^{k-1} X^2 = (1 - aX)(1 - \bar{a}X) \), where \( |a| = p^{k-1} \).
Proof: The first part is a direct consequence of (IV.5.7.4). Why $a_n$ bounded?

If $f$ is a normalized Hecke eigenform, then by (IV.5.6.5) and (IV.5.6.2),

$$
\begin{align*}
&c_mc_n = c_{mn}, \\
&c_pc_p = c_{p^{n+1}} + p^{2k-1}c_{p^{k-1}}
\end{align*}
$$

Prop. (IV.5.7.7) (Functional Equation associated to $S_k(N, \psi)$). Notation as in (IV.5.7.5). If $f(z) \in S_k(N, \psi)(IV.5.1.9)$. Denote $w_N = \begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix}$, then $w_N$ stabilizes $\Gamma_0(N)$ because

$$
\begin{bmatrix} a & b \\ c & d \end{bmatrix} w_N^{-1} = \begin{bmatrix} d & c/N \\ bN & a \end{bmatrix}
$$

Also notice $\psi(a_\gamma) = \overline{\psi(d_\gamma)}$, so

$$
\begin{align*}
f[w_N]k[\gamma]k &= f[w_N \gamma w_N^{-1}]k[w_N]k = \overline{\psi(d)}f[w_N].
\end{align*}
$$

So $g = f[w_N]k \in S_k(N, \overline{\psi})$.

Now if

$$
f(z) = \sum a_ne^{2\pi inz/N}, \quad g(z) = \sum b_ne^{2\pi inz/N},
$$

and $\chi$ is a primitive character mod $D$, define $L(s, f, \chi)$, $L(s, g, \chi)$, $\Lambda(s, f, \chi), \Lambda(s, g, \chi)$ as in (IV.5.7.5), then $\Lambda(s, f, \chi)$ extends to an analytic function for all $s \in \mathbb{C}$, and there are functional equations

$$
\Lambda(s, f, \chi) = i^k \chi(N) \psi(D) \frac{\tau(\chi)^2}{D} (D^2 N)^{-s+k/2} \Lambda(k-s, g, \overline{\chi}), \quad (IV.1)
$$

where $\tau(\chi)$ is the Gauss sum of $\chi(III.3.1.42)$.

In particular, if $D = 1$, then

$$
\Lambda(s, f) = i^k N^{-s+k/2} \Lambda(k-s, g). \quad (IV.2)
$$

Proof: Let

$$
f_{\chi}(z) = \sum \chi(n) a_n q^n, \quad g_{\overline{\chi}}(z) = \sum \overline{\chi(n)} b_n q^n.
$$

Use (III.3.1.46) on $f_{\chi}(z)$, we get

$$
f_{\chi} = \frac{\chi(-1)\tau(\chi)}{D} \sum_{m \in (\mathbb{Z}/D\mathbb{Z})^*} \overline{\chi(m)} f_{\chi}[D \begin{bmatrix} m \\ D \end{bmatrix}]_k
$$

Now

$$
f_{\chi}[D^{2N} -1]_k = f_{\chi}[D^{-1/DN} -1]_k = \frac{\chi(-1)\tau(\chi)}{D} \sum_{m \in (\mathbb{Z}/D\mathbb{Z})^*} \overline{\chi(m)} g[w_N^{-1}]_k[D \begin{bmatrix} m \\ D \end{bmatrix}]_k[D^{-1/DN} -1]_k
$$

$$
= \frac{\chi(-1)\tau(\chi)}{D} \sum_{m \in (\mathbb{Z}/D\mathbb{Z})^*} \overline{\chi(m)} g[D \begin{bmatrix} m \\ -Nm \end{bmatrix}]_l[D \begin{bmatrix} r \\ s \end{bmatrix}]_k
$$
where \((r, s)\) are integers chosen that \(Ds - rNm = 1\). Thus \(\overline{\chi(m)} = \chi(-N)\chi(r)\), and because \(g \in M_k(N, \overline{\psi})\),

\[
 f_\chi\left[D^2N -1 \right]_k = \frac{\chi(N)\tau(\chi)}{D} \sum_{r \in \mathbb{Z}/D\mathbb{Z}} \chi(r)\psi(D)g\left[D \begin{bmatrix} r \\ D \end{bmatrix}\right]_k.
\]

Compare this with the formula

\[
 g_\chi = \frac{\chi(-1)\tau(\overline{\chi})}{D} \sum_{m \in \mathbb{Z}/D\mathbb{Z}} \chi(m)g\left[D \begin{bmatrix} m \\ D \end{bmatrix}\right]_k,
\]

we get

\[
 f_\chi\left[D^2N -1 \right]_k = \chi(-N)\psi(D)\frac{\tau(\chi)}{\tau(\overline{\chi})} g_\chi = \chi(N)\psi(D)\frac{\tau(\chi)^2}{D} g_\chi(III.3.143)(III.3.145). \tag{IV.3}
\]

Now similar to the proof of (IV.5.7.4),

\[
 \Lambda(s, f, \chi) = \int_0^\infty f_\chi(iy)y^sdy.
\]

So when \(\text{Re}(s)\) is large,

\[
 \chi(N)\psi(D)\frac{\tau(\chi)^2}{D} \Lambda(s, g, \overline{\chi}) = \chi(N)\psi(D)\frac{\tau(\chi)^2}{D} \int_0^\infty g_\chi(iy)y^sdy
\]

\[
 = \int_0^\sqrt{DN} (D^2N)^{-k/2}(iy)^{-k} f_\chi\left(-\frac{1}{D^2Ni}y\right)y^sdy + \chi(N)\psi(D)\frac{\tau(\chi)^2}{D} \int_0^\sqrt{DN} g_\chi(iy)y^sdy
\]

\[
 = \int_0^\sqrt{DN} (D^2N)^{k/2}t^k i^{-k} f_\chi(it)(D^2Nt)^{-s}dt + \chi(N)\psi(D)\frac{\tau(\chi)^2}{D} \int_0^\sqrt{DN} g_\chi(iy)y^sdy
\]

\[
 = i^{-k}(D^2N)^{k/2-s} \int_0^\sqrt{DN} f_\chi(it)t^{k-s}dt + \chi(N)\psi(D)\frac{\tau(\chi)^2}{D} \int_0^\sqrt{DN} g_\chi(iy)y^sdy
\]

Both integral are absolutely convergent for any \(s\). And similarly when \(\text{Re}(s)\) is small,

\[
 i^{-k}(D^2N)^{k/2-s} \Lambda(k-s, f, \chi) = i^{-k}(D^2N)^{k/2-s} \int_0^\sqrt{DN} f_\chi(it)t^{k-s}dt
\]

\[
 = i^{-k}(D^2N)^{k/2-s} \int_0^\sqrt{DN} f_\chi(it)t^{k-s}dt + \chi(N)\psi(D)\frac{\tau(\chi)^2}{D} \int_0^\sqrt{DN} g_\chi(iy)y^sdy
\]

Thus we get the desired result. \(\square\)

**Cor.** (IV.5.7.8). As \(f[w_N^2] = (-1)^k f\), if \(f \in S_k(\Gamma_1(N))\) satisfies \(f[w_N] = c f\), \(c = \varepsilon i^k\), \(\varepsilon = \pm 1\), then

\[
 N^{s/2} \Lambda(s, f) = \varepsilon (-1)^k (N^{(k-s)/2} \Lambda(k-s, f)).
\]

so \(\text{ord}_{1/2}(L(s, f))\) is even if \(\varepsilon = (-1)^k\) and odd if \(\varepsilon = (-1)^{k+1}\).

In particular, if \(N = 1\) and \(k \equiv 2 \mod 4\), \(L(1/2, f) = 0\).

It is conjectured that \(L(1/2, f) \neq 0\) if \(4|k\), Cf.[Modular Forms, Ribet] P148.
Prop. (IV.5.7.9) (Converse Theorem, Weil). Let \( N > 0 \) and \( \psi \) a character mod \( N \). Suppose \( a_n, b_n \) are two sequence of complex numbers that \( |a_n|, |b_n| = O(n^M) \) for some positive integer \( M \). If \((D, N) = 1 \) and \( \chi \) is a primitive character mod \( D \), let
\[
L_1(s, \chi) = \sum \chi(n)a_n n^{-s}, \quad L_2(s, \chi) = \sum \overline{\chi(n)}b_n n^{-s}.
\]
and \( \Lambda_i(s, \chi) = (2\pi)^{-s}\Gamma(s)L_i(s, \chi_i) \).

Now if for \( D \) equals to a.e. \( p \) and any primitive character mod \( D \), \( \Lambda_i(s, \chi) \) has analytic continuation to all \( s \), are bounded on vertical strips, and satisfy the functional equation
\[
\Lambda_1(s, \chi) = i^k\chi(N)\psi(D)\frac{\tau(\chi)^2}{D}(D^2N)^{-s+k/2}\Lambda_2(k-s, \chi),
\]
where \( \tau(\chi) \) is the Gauss sum of \( \chi(\text{III.3.142}) \), then \( f(z) = \sum a_n e^{2\pi inz} \) is a modular form in \( M_k(\Gamma_0(N), \psi) \).

Proof: Let
\[
f_{\chi}(z) = \sum \chi(n)a_n q^n, \quad g_{\chi}(z) = \sum \overline{\chi(n)}b_n q^n.
\]
We first show equation IV.3 holds. As in the proof of (IV.5.7.4), it suffices to show the functional equation it is true on the positive imaginary axis. If \( \sigma = \text{Re}(s) \) is sufficiently large, then
\[
\int_0^\infty f_{\chi}(iy)y^{s} \frac{dy}{y} = \Lambda_1(s, \chi)
\]
so by Mellin inversion formula
\[
f_{\chi}(iy) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Lambda_1(s, \chi)y^{-s} ds = i^k\chi(N)\psi(-D)\frac{\tau(\chi)^2}{D} \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} (D^2N)^{-s+k/2} \Lambda_2(k-s, \chi) ds
\]

Same argument of Phragmén-Lindelöf principle as in proof of (IV.5.7.4) shows \( \Lambda_2(\sigma + it, \chi) \) converges to 0 for \( t \to \infty \), uniformly on any compact subset, so we can move the integral horizontally and make a change of variable \( s \mapsto k-s \) to get
\[
f_{\chi}(iy) = i^k\chi(N)\psi(-D)\frac{\tau(\chi)^2}{D}(D^2N)^{-k/2}y^{-k} \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Lambda_2(k-s, \chi)(D^2Ny)^s ds
\]
which is equivalent to IV.3.

The rest is to manipulate \( 2 \times 2 \) matrices and use (III.2.1.13) to show that \( g \in M_k(N, \overline{\psi}) \), Cf.[Bump, P62].

Remark (IV.5.7.10). For \( \Gamma(1) \), the function only requires one functional equation, and we used the fact \( \Gamma(1) \) is generated by \( S \) and \( T \). But \( \Gamma_0(N) \) is not generated by two elements, so we must assume functional equations for the twists \( L(s, f, \chi) \) also.

Prop. (IV.5.7.11) (Dirichlet Series for \( \Delta \) Function). The Dirichlet series of \( L(\Delta, s) \) for \( \Delta(IV.5.4.8) \) has an Euler product expansion of the form
Rankin-Selberg Method

**Def. (IV.5.7.12) (Rankin-Selberg Method).** The converse theorems like (IV.5.7.4), shows that a possible method of proving the existence of an automorphic form is to prove by any method the functional equations of sufficiently many of the the $L$-series attached to it. One of the most powerful methods of doing this is the **Rankin-Selberg method**, which seeks to represent an $L$-function as an integral of one or more automorphic forms against an Eisenstein series, itself a type of automorphic form.

8 Eichler-Shimura Relation

9 Modularity

**Prop. (IV.5.9.1) (Elliptic Curves and Modular Curves).** Modular curves $Y_0(N), Y_1(N), Y(N)$ parametrizes elliptic curves over $\mathbb{C}$ with additional structures.

**Proof:**

**Cor. (IV.5.9.2) ($j$-Invariant).** Take $N = 1$, then elliptic curves over $\mathbb{C}$ corresponds to an orbit $SL(2, \mathbb{Z}) \tau \subset Y(1)$, thus we can associate it the value $j(\tau)$, denoted by $j(E)$, called the $j$-invariant of $E$.

**Prop. (IV.5.9.3).** If $f = \sum_{n \geq 1} a_n q^n$ is an eigenform in $S_k(\Gamma_1(N))$, then the coefficients

**Thm. (IV.5.9.4) (Modularity Theorem).** An elliptic curve over $\mathbb{C}$ with rational $j$-invariant arises from a modular form.

**Conjecture (IV.5.9.5) (Taniyama-Shimura-Weil).** For any elliptic curve $E$ over $\mathbb{Q}$, there exists a surjective map $X_0(N) \rightarrow E$, where $N$ is the conductor of $E$.

**Conjecture (IV.5.9.6) (Taniyama-Shimura-Weil).** For any elliptic curve $E$ over $\mathbb{Q}$, there $L(E, s) = L(f, s)$ for some normalized eigenform on $\Gamma_0(N)$ of weight 2, where $N$ is the conductor of $E$.

10 Modular Forms Mod $p$

**Def. (IV.5.10.1) (Modular Forms Mod $p$).** Define $M_k(\Gamma, \mathbb{F}_p) = M_k(\Gamma, \mathbb{Z}) \otimes \mathbb{F}_p$, called the space of modular forms mod $p$ of weight $k$.

**Prop. (IV.5.10.2) (Serre’s Equality).** There is an isomorphism $M_{p+1}(SL(2, \mathbb{Z}), \mathbb{F}_p) \cong M_2(\Gamma_0(p), \mathbb{F}_p)$.

**Proof:** Because $E_k(z)1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$, and by Kummer’s congruence (VIII.1.7.1) $\text{ord}_p(B_{p-1}) = -1$, thus $E_{p-1}$ mod 1 mod $p$. Then multiplying by $E_{p-1} : M_2(\Gamma_0(p), \mathbb{F}_p) \rightarrow M_{p+1}(\Gamma_0(p), \mathbb{F}_p)$ raises the level by $p - 1$. Then we compose with the natural averaging map $M_{p+1}(\Gamma_0(p), \mathbb{F}_p) \rightarrow M_{p+1}(SL(2, \mathbb{Z}), \mathbb{F}_p)$, which is dual to the natural inclusion $M_{p+1}(SL(2, \mathbb{Z}), \mathbb{F}_p) \rightarrow M_{p+1}(\Gamma_0(p), \mathbb{F}_p)$.

Why isomorphism? □
11 Hilbert Modular Forms

12 Fermat’s Last Theorem

Thm. (IV.5.12.1) (Fermat’s Last Theorem). If \( l \geq 5 \), then there are no integral solution to the equation \( a^l + b^l = c^l \).

Proof:

Step 1: Frey’s Curve: \( y^2 = x(x - a^l)(x + b^l) \). It defines an elliptic curve, has semistable reduction with discriminant \( \Delta = 16(abc)^l \).

The Galois representation: \( \rho_{E,l} : G(\mathbb{Q}) \to Aut(E[l]) \cong GL_2(\mathbb{F}_l) \) is unramified.

Step 2: Taniyama-Shimura-Weil Conjecture: \( E \) is associated with a cuspidal modular eigenform \( f_E(q) = \sum a_n q^n, a_n \in \mathbb{Q} \) of level \( \Gamma_0(N) \) and weight 2.

This is defined by \( a_1 = 1 \), and \( a_p = p + 1 - |E(\mathbb{F}_p)| \) for \( p \) not divisible by \( N \).

By Eichler-Shimura, weight 2 cusp eigenform over \( \mathbb{Q} \) can always associated to a Galois representation \( \rho_f : G(\mathbb{Q}) \to GL_2(\mathbb{Q}_l) \). Weil did the converse.

Step 3: Serre+Ribet’s Level-Lowering: there should be a cuspidal modular eigenform \( f' \) of level \( \Gamma_0(2) \) and weight 2. s.t. \( a_n(f') \equiv a_n(f) \mod l, \forall (n,N) = 1 \).

Step 4: But there is no cuspidal form of level \( \Gamma_0(2) \) and weight 2, because there \( X_0(2) = \mathbb{P}^1 \), and \( H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) = 0 \).

□

Prop. (IV.5.12.2) (Modular Lifting Theorem). The main tool.

Prop. (IV.5.12.3) (Weil’s Theorem). modular deformation ring \( T \) is isomorphic to the Galois deformation ring \( R_\rho \).
IV.6 Shimura Varieties

Main references are [Lan20], [Mil17b], [Mil11].

Def. (IV.6.0.1) (Special Subvarieties).

Conjecture (IV.6.0.2) (André-Oort). Let \( S \) be a Shimura variety. Let \( V \subset S \) be a subvariety, then there are only f.m. maximal special subvarieties contained in \( V \).

1 Locally Symmetric Varieties

Prop. (IV.6.1.1).

2 Shimura Varieties of PEL Type

Def. (IV.6.2.1).
IV.7 Quadratic Forms over Fields

Basic references are [Quadratic Forms over Fields Y.T.Lam], [Quadratic Forms Clark] and [Algebraic and Geometric Theory of Quadratic Forms].

All fields $K$ in this section has $\text{char} \neq 2$.

1 Quadratic Forms

This subsection should be regarded as a continuation of 7. In fact, most materials in this subsection are trivial facts.

**Def. (IV.7.1.1).** Given a field $K$ of $\text{char} \neq 2$, a **quadratic form** over $K$ is a bilinear form on $K^n$ for some $K$. It is represented by a symmetric matrix.

The reason that $\text{char} K \neq 2$ is because only in this case, a quadratic form $q$ is equivalent to a symmetric bilinear form $B$, and I will use this equivalence freely.

The determinant $\det$ is a function from the set of quadratic forms to $K^\ast \rtimes (K^\ast)^2$ that is invariant under congruence.

**Def. (IV.7.1.2).** A field is called **quadratically closed** iff $K^2 = K$, or equivalently $K$ has no quadratic extensions.

**Def. (IV.7.1.3).** The category of **quadratic spaces** is a category with objects as finite dimensional spaces with a quadratic form, and its morphisms are isometric embeddings.

**Def. (IV.7.1.4).** A quadratic form is called **universal** if it represents every element of $K^\ast$.

**Non-Degeneracy**

**Def. (IV.7.1.5) (Non-degeneracy).** A quadratic space is called **non-degenerate** if $v \mapsto B(v, \cdot)$ is an isomorphism from $V$ to $V^\ast$. Notice if $\dim V = \infty$, this cannot happen, because $\dim V^\ast > \dim V$ (I.1.3.3). And in case $\dim V < \infty$, $\dim V = \dim V^\ast$, so it suffices to show $v \mapsto B(v, \cdot)$ is injective, i.e. if $v \neq 0$, then there is a $w$ that $B(v, w) \neq 0$.

**Prop. (IV.7.1.6) (Radical Splitting).** The **radical** of a quadratic space is defined to be $\text{rad}(V) = V^\perp$. Then for any quadratic form $V$, there is an orthogonal decomposition $V = \text{rad}(V) \oplus W$, where $W$ is a non-degenerate form.

**Proof:** In fact, by the definition, any complement space of $\text{rad}(V)$ in $V$ can be chosen as the orthogonal complement $W$. □

**Prop. (IV.7.1.7).** If $W$ is a non-degenerate sub-quadratic space of $V$, then $W \oplus W^\perp = V$.

**Proof:** Since $W$ is non-degenerate, $W \cap W^\perp = 0$. and for any $v \in V$, $B(v, \cdot) \in W^\ast$, so by degeneracy, there is a $w \in W$ that $B(v, \cdot) = B(w, \cdot)$, then $z = v - w \in W^\perp$ and $v = w + z$. □

**Prop. (IV.7.1.8) (Perp and Non-Degeneracy).** If $V$ is a non-degenerate quadratic space, then for any non-degenerate subspace $W$, $\dim W + \dim W^\perp = \dim V$, and $(W^\perp)^\perp = W$.

**Proof:** The first is immediate from the fact $\dim \text{Ker} + \dim \text{Coker} = \dim V$. The second is by dimensional reason. □

**Cor. (IV.7.1.9).** A subspace $W$ of a non-degenerate quadratic space $V$ is a non-degenerate quadratic space iff $W \cap W^\perp = 0$. 

Diagonalizability

Prop. (IV.7.1.10) (Quadratic Form Representable). Any quadratic forms over \( K \) of char \( \neq 2 \) is diagonalizable, and if \( \alpha \in K^\ast \) is represented by \( K \), then it is diagonalizable to a matrix with first entry \( \alpha \).

Proof: Use (I.1.7.4), since in this case, a quadratic form is equivalent to a symmetric form. And if \( \alpha = B(v,v) \), then we can choose \( v \) in the first place in the proof of (I.1.7.4).

Cor. (IV.7.1.11). Over a quadratically closed field \( K \) of char \( \neq 2 \), any non-degenerate quadratic form is congruent to \( x_1^2 + \ldots + x_n^2 \).

Proof: Because in this case, we can make \( \sum a_ix_i^2 \) into \( \sum (\sqrt{a_i}x_i)^2 \).

Def. (IV.7.1.12). We will use the notation \( \langle \alpha_1, \ldots, \alpha_n \rangle \) for the diagonal quadratic form \( \sum \alpha_i x_i^2 \).

Isotropic and Hyperbolic Spaces

Def. (IV.7.1.13) (Isotropic). Given a non-degenerate quadratic space \( V \), a vector \( v \) is called isotropic if \( B(v,v) = 0 \). \( V \) itself called isotropic if it is non-degenerate and there exists an isotropic vector, otherwise it is called anisotropic.

Def. (IV.7.1.14) (Hyperbolic). The hyperbolic plane \( \mathbb{H} \) is the 2-dimensional space with quadratic form \( H(x,y) = xy \), which is congruent to \( \frac{1}{2}(x^2 - y^2) \).

A quadratic space is called hyperbolic if it is isomorphic to a direct sum of hyperbolic planes.

Lemma (IV.7.1.15). If \( V \) is a non-degenerate isotropic space, then there is an isometric imbedding of the hyperbolic plane into \( V \).

Proof: There is a \( u \in V \) that \( B(u,u) = 0 \). By non-degeneracy, there is a \( w \) that \( B(u,w) \neq 0 \). We may assume \( B(u,w) = 1 \). Now I claim there is an \( \alpha \) that \( q(\alpha u + v) = 0 \): in fact, \( q(\alpha u + v) = 2\alpha B(u,w) + q(w) \). Now \( v = \alpha u + w \), then \( q(u) = q(v) = 0 \), and \( B(u,v) = 1 \), so it is isomorphic to \( \mathbb{H} \).

Prop. (IV.7.1.16) (Isotropic Complement). If \( V \) is a non-degenerate quadratic space, and \( U \subset V \) is an isotropic space with basis \( u_1, \ldots, u_m \), then there is another isotropic space \( V \) with basis \( v_1, \ldots, v_m \) that \( B(u_i, v_j) = \delta_{ij} \).

Proof: Use induction on \( m \). The \( m = 1 \) case is lemma (IV.7.1.15) above. If this is true for \( n < m \), let \( W = \{ u_2, \ldots, u_m \} \), then \( W^\perp \subset \{ u_1 \}^\perp \), then \( u_1 \in W \) by (IV.7.1.8), contradiction, so there is a \( v \in W^\perp \) that \( B(u_1,v) \neq 0 \), so by the same proof as (IV.7.1.15), there is a \( \alpha u_1 + v \) that is isotropic, and a \( \mathbb{H} \subset W^\perp \), so by (IV.7.1.8), \( W \subset \mathbb{H}^\perp \), so by induction, we can find in \( \mathbb{H}^\perp \) elements \( v_2, \ldots, v_m \) that satisfies the requirement.

Cor. (IV.7.1.17). \( \langle a, -a \rangle \cong \mathbb{H} \), because it is isotropic, and it has dimension 2.

Cor. (IV.7.1.18) (Isotropic Form is Universal). A non-degenerate isotropic space is universal, because hyperbolic plane does.

Cor. (IV.7.1.19). A maximal totally isotropic space in a non-degenerate quadratic space \( V \) has dimension at most \( \frac{1}{2} \dim V \), and equality holds if \( V \) is hyperbolic.
Prop. (IV.7.1.20) (First Representation Theorem). If $q$ is a non-degenerate quadratic form, then $q$ represents $\alpha \in K^*$ iff $q \oplus \langle -\alpha \rangle$ is isotropic.

Proof: If $q$ represent $\alpha$, then by (IV.7.1.10) shows that $q$ is equivalent to $\langle \alpha, \alpha_1, \ldots, \alpha_n \rangle$, so $q \oplus \langle -\alpha \rangle$ contains a $\langle \alpha, -\alpha \rangle$ which is isomorphic to $\mathbb{H}$ by (IV.7.1.15).

Conversely, if $q \oplus \langle -\alpha \rangle$ is isotropic, then there is a $-\alpha x_0^2 + \sum \alpha_i x_i^2 = 0$. If $x_0 \neq 0$, then $q$ represent $\alpha$, and if $x_0 = 0$, then $q$ is isotropic, thus represent any element (IV.7.1.18).

Cor. (IV.7.1.21). The following are equivalent:
- Any $n$-quadratic form over $K$ is universal.
- Any $(n+1)$-quadratic form over $K$ is isotropic.

Prop. (IV.7.1.22) (Isotropy Criterion). For two non-degenerate forms $f, g$ over $K$, $h = \langle f, -g \rangle$ is isotropic iff there is an $\alpha \in K^*$ that is represented by both $f$ and $g$.

Proof: Easy, notice to use isotropic form is universal (IV.7.1.18).

2 Witt Theory

Prop. (IV.7.2.1) (Witt Cancellation Theorem). If $U_1, U_2, V_1, V_2$ are quadratic spaces and $V_1 \cong V_2$, $V_1 \oplus U_1 \cong V_2 \oplus U_2$, then $U_1 \cong U_2$.

Proof: We may identify $V_1 = V_2 = V$, and $W = U_1 \oplus V = U_2 \oplus V$.

First if $V$ is totally isotropic and $U_1$ is non-degenerate, then there is a matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ that

\[
M^t \begin{bmatrix} 0 & 0 \\ 0 & B_2 \end{bmatrix} M = \begin{bmatrix} 0 & 0 \\ 0 & B_1 \end{bmatrix}
\]

So $B_1 = D^t B_2 D$. As $B_1$ is non-singular, so is $D$, thus $U_1 \cong U_2$.

Now if $V$ is isotropic but $U_1, U_2$ are not non-degenerate, then we may assume in their diagonalization, $U_1$ has less 0s, it has $r$ 0s, then we can extract from both $U_i$ a zero part, thus reducing to the above case.

Now if dim $V = 1$, $V = \langle a \rangle$, if $a = 0$, then we are done by the above argument, and if $a = 0$, then find $q(x) = a$, then by (IV.7.2.13), we can find a $\tau \in O(W)$ that $\tau(V_1) = V_2$, so now $U_1, U_2$ as the orthogonal complement of $V_1, V_2$, they are isometric under the map $\tau$.

So now in general, we can cancel $V$ out by moving its diagonal part once a time.

Cor. (IV.7.2.2). If $X$ is a quadratic space and $V_1, V_2$ are non-degenerate subspaces of $X$, then any isometry $V_1 \cong V_2$ extends to an isometry of $X$.

Proof: $V_i \oplus V_i^\perp = X$ by (IV.7.1.7).

Cor. (IV.7.2.3) (Witt’s Extension Theorem). If $X$ is a non-degenerate quadratic space and $f : W_1 \rightarrow W_2$ is an isometry of two subspaces of $X$, then $f$ extends to an isometry of $X$.

Notice this also holds for symplectic spaces $X$, by the same method of proof.

Proof: If $W_1$ is non-degenerate, then so does $W_2$, and we can use (IV.7.2.2). By (IV.7.1.6), we can write $W_i = U_i \oplus V_i$ where $U_i$ is totally isotropic and $V_i$ is non-degenerate. Now $X, V_i$ are non-degenerate, $V_i^\perp$ is non-degenerate also, so there is an isotropic complement $U_i^\prime \subset V_i^\perp$ (IV.7.1.16).
Let $T_i = \langle U_i, U_i' \rangle V_i$, then $T_i$ is non-degenerate and $W \subset T$. As $U_i$ is the radical of $W_i$, $U_1 \cong U_2$, and then $(U_1, U_1') \cong (U_2, U_2')$. By Witt cancellation, $W_1 \cong W_2$. So we reduced to the non-degenerate case. \hfill \square

**Cor. (IV.7.2.4)** If $V$ is a non-degenerate quadratic space, then the group of isometries of $X$ acts transitively on the set of all totally isotropic subspaces of a fixed dimension $d$.

**Prop. (IV.7.2.5) (Witt’s Decomposition Theorem).** For any quadratic space $V$, there is an orthogonal decomposition

$$V \cong \text{rad}(V) \oplus \bigoplus_{i=1}^{k} \mathbb{H} \oplus V'$$

where $V'$ is anisotropic (IV.7.1.13). Moreover the number $k = I(V)$ which is called the **Witt index** of $V$ and the isometry class of $V' = w(V)$ which is called the **non-isotropic kernel** is independent of the decomposition.

**Proof:** The existence of the decomposition follows from (IV.7.1.6) and an easy induction using (IV.7.1.15). The uniqueness is an easy corollary of (IV.7.1.15) and Witt’s cancellation theorem. \hfill \square

**Cor. (IV.7.2.6)** The Witt index equals the maximal dimension of a maximal totally isotropic subspace of $W$, by (IV.7.1.16).

**Remark (IV.7.2.7)**. This is a good reason that we will only consider non-degenerate quadratic forms from now on.

**Cor. (IV.7.2.8) (Sylvester’s Law of Nullity).** Let $q_{r,s} = [r](1) \oplus [s](1)$, then any non-degenerate quadratic form $q$ over $\mathbb{R}$ is congruent to exactly one of $q_{r,s}$, and $r - s$ is called the **signature** of $q$.

**Def. (IV.7.2.9)**. Two quadratic forms $q_1 = \langle a_1, \ldots, a_n \rangle$ and $q_2 = \langle b_1, \ldots, b_n \rangle$ are called **simply equivalent** iff there are two indices that $\langle a_i, a_j \rangle \cong \langle b_i, b_j \rangle$. Two quadratic forms are called **chain equivalent** iff there is a chain of simply equivalence between them.

**Prop. (IV.7.2.10) (Witt’s Chain equivalence Theorem).** Two diagonal quadratic forms over $K$ are equivalent iff they are chain equivalent.

**Proof:** Chain equivalent is clearly equivalent. Conversely, by Witt’s decomposition theorem, it is easy to reduce to the non-degenerate case.

Now if $q = \langle \alpha_1, \ldots, \alpha_n \rangle \cong q' = \langle \beta_1, \ldots, \beta_n \rangle$, any form $q = \langle \gamma_1, \ldots, \gamma_n \rangle$ that is chain equivalent to $q$ is equivalent to $q'$, so $\beta_1$ is represented by it, choose a form that there is a minimal $l$ that $\beta_1$ is represented by $\langle \gamma_1, \ldots, \gamma_l \rangle$, we prove that $l = 1$:

- If the minimal $l$ is not 1, then $d = \gamma_1 a_1^2 + \gamma_2 a_2^2 \neq 0$ (otherwise $l$ can be smaller), so $\langle \gamma_1, \gamma_2 \rangle \cong \langle d, \gamma_1 \gamma_2 d \rangle$ by (IV.7.1.10) and invariance of det. so $q \cong \langle d, \gamma_3, \ldots, \gamma_n, d \gamma_1 \gamma_2 \rangle$ (notice permutation is chain equivalence), and this is smaller, contradiction.

Now $l = 1$, so we may assume $\alpha_1 = \beta_1$, and then Witt’s cancellation (IV.7.2.1) shows that $\langle \alpha_2, \ldots, \alpha_n \rangle \cong \langle \beta_2, \ldots, \beta_n \rangle$, so we win by induction. \hfill \square
Orthogonal Group

Prop. (IV.7.2.11). The orthogonal group of a quadratic form $q$ is the set of matrixes $M$ that $g(Mx) = q(x)$. And it is clear $\det M = \pm$, so we can also define $O^+(V)$ and $O^-(V)$.

Def. (IV.7.2.12). A hyperplane reflection for a non-isotropic vector $v$ is defined by $x \mapsto x - \frac{2B(x,v)}{q(v)}v$, it is an element in $O(V)$.

Prop. (IV.7.2.13). If $x, y$ are two non-isotropic vectors that $q(x) = q(y)$, then there is a $\tau \in O(V)$ that $\tau(x) = y$.

Proof: First notice $q(x + y) + q(x - y) = 2q(x) + 2q(y) = 4q(x) \neq 0$, so one of $x + y, x - y$ is non-isotropic. And it can be easily calculated that $\tau_{x-y}(x) = y$ or $-\tau_{x+y}(x) = y$. □

Prop. (IV.7.2.14) (Cartan-Dieudonné). Let $V$ be a non-degenerate quadratic form of dimension $n$, then every element of the orthogonal group $O(V)$ can be represented as a product of $n$ reflections.

Proof: Cf.[Quadratic Forms Clark P22]. □

3 Witt Ring

Def. (IV.7.3.1) (Witt Ring). The Witt ring $W(K)$ of $K$ is a commutative ring whose elements are equivalent classes of non-isotropic quadratic forms over $K$, and the addition is defined by $[q_1] + [q_2] = w[q_1 \oplus q_2]$ and multiplication is defined by $[q_1] \otimes [q_2] = w[q_1 \otimes q_2]$.

Def. (IV.7.3.2) (Grothendieck-Witt Ring). There is another ring, the Grothendieck-Witt ring $\hat{W}(K)$ which is defined as the ring generated by the semiring of all non-degenerate quadratic forms over $K$.

Prop. (IV.7.3.3). The subgroup $[\mathbb{H}]$ generated by the hyperbolic plane is an ideal of $\hat{W}(K)$. And $\hat{W}(K)/([\mathbb{H}]) \cong W(K)$.

Proof: $[\mathbb{H}]$ is an ideal because $[H] \cdot [(a_1, \ldots, a_n)] \cong w(\oplus(a_i, -a_i))$ which is 0, by(IV.7.1.17). The last proposition is easy. □

4 over Local and Global Fields
Chapter V

Algebraic Geometry

V.1 Sites, Sheaves and Stacks

References are [Sta], [Algebraic Spaces and Stacks Olsson], [Vis08] and [Fibered Category to Algebraic Stacks Lamb].

1 Sites

Def. (V.1.1.1) (Sites). A site is given by a category $\mathcal{C}$ and a set $\text{Cov}(\mathcal{C})$ of families of morphisms with fixed target, called the coverings of $\mathcal{C}$ that:

- An isomorphism is a covering.
- Coverings of covering is a covering.
- Base change of a covering is a covering.

Sometimes a site is wrongly called a topology, the difference is that the morphism of site is reverse of a morphism of topology.

Def. (V.1.1.2) (Discrete Topology). A discrete topology or chaotic topology is a site that the only coverings are identities. In this way, we can regard any category as a site.

Def. (V.1.1.3) (Comma Topology). For a site $\mathcal{C}$ and an object $S$, we have the comma category $\mathcal{C} \downarrow S$ (II.1.1.10), and we can define a topology on it where the coverings are coverings of $\mathcal{C}$ that is compatible over $S$.

Def. (V.1.1.4) (Continuous Functor). A continuous functor between sites $\mathcal{C} \to \mathcal{D}$ is a functor that preserves covering and any base change by morphisms in a covering.

A morphism of sites $\mathcal{C} \to \mathcal{D}$ is given by a continuous functor $u: \mathcal{D} \to \mathcal{C}$ that $u_s$(V.1.2.10) is exact.

This exact condition is easy to be satisfied, by(V.1.2.13).

Def. (V.1.1.5) (Cocontinuous functors). A cocontinuous functor between sites $u: \mathcal{C} \to \mathcal{D}$ is a functor that for any $U \in \mathcal{C}$ and any covering $\{V_i \to u(U)\}$ in $\mathcal{D}$, there is a covering $\{U_i \to U\} \in \mathcal{C}$ that refines $\{V_i \to u(U)\}$ after the functor $u$. 
Topologies and Sieves

Def. (V.1.1.6) (Sieves). For a covering \( U = \{U_i \to U\} \) in a category \( \mathcal{C} \), define a subfunctor \( h_U \subset h_U \), where for each \( X \) \( h_U(X) \) consists of elements in \( \text{Hom}(X, U) \) that factor through some \( U_i \to U \).

A sieve \( S \) on \( U \) is a subfunctor of \( h_U \). Notice that any sieve must be of the form \( h_U \), by choosing \( \mathcal{U} \) to consist of all arrows in \( \{S(T)\}_{T \in \mathcal{C}} \).

Def. (V.1.1.7). If \( T \) is a Grothendieck topology on a category \( \mathcal{C} \), then a sieve \( S \subset h_U \) over \( U \) is said to belong to \( T \) or just a sieve of the site \( \mathcal{C} \) if there exists a covering \( \mathcal{U} \) of \( U \) that \( h_U \subset S \).

G-Spaces

Def. (V.1.1.8) (G-Spaces). A G-space is a set \( X \) with a family of subsets of \( X \) that they form a site w.r.t inclusions and that covering are all set-theoretic coverings (but not necessarily conversely). These subsets are called admissible opens of \( X \) and covers are called admissible covers. (In other words, a G-topological space is a "topological space without unions"). Morphisms of G-spaces is simultaneously a continuous map and a morphism of sites.

Def. (V.1.1.9) (Completeness). The completeness of a G-topological space \( X \):

- G0: \( \emptyset \) and \( X \) are admissible open.
- G1: Let \( \{U_i \to U\} \) be an admissible cover, then a subset \( V \subset U \) is admissible if \( V \cap U_i \) are all admissible.
- G2: Let \( \{U_i \to U\} \) be a cover of admissible opens for \( U \) admissible, then the cover is admissible if it has an admissible cover as a refinement.

Lemma (V.1.1.10) (Admissible is Local). If \( G_2 \) is satisfied for a G-topological space \( X \), then for an admissible covering \( \{X_i \to X\} \) and another covering \( \{U_i \to X\} \) between admissible opens, it is admissible iff \( U_i \cap X_j \) is an admissible covering for \( X_j \) for each \( j \). (By composition, \( \{U_i \cap X_i \to X\} \) is admissible, and it refines \( \{U_i \to X\} \).

Prop. (V.1.1.11) (Glue of Complete G-topological spaces). For sets \( \cup X_i = X \), if there are Grothendieck category \( \mathcal{I} \) on \( X_i \) making \( X_i \) into a G-topological space, and they all satisfies the completeness conditions \( G_0, G_1, G_2 \) of (V.1.1.9). Assume that \( X_i \cap X_j \) is \( \mathcal{I} \)-open in \( X_i \) and \( \mathcal{I}, \mathcal{I}_j \) restrict to the same topology on \( X_i \cap X_j \), then there is a unique Grothendieck category \( \mathcal{I} \) on \( X \) making \( X \) a G-topological space that:

- \( X_i \) is \( \mathcal{I} \)-open and \( \mathcal{I} \) restricts to \( \mathcal{I}_i \) on \( X_i \),
- \( \mathcal{I} \) satisfies the completeness conditions \( G_0, G_1, G_2 \).
- \( X_i \) is a \( \mathcal{I} \)-covering of \( X \).

Proof: By (V.1.1.9) and (V.1.1.10), the uniqueness is straightforward, for the existence,

- check Grothendieck first: Composition, base change.
- check condition 1: by hypothesis, and (V.1.1.9) applied to \( X_i \cap X_j \to X_i \) (this is admissible because \( \text{id}_{X_i} \) refined it).
- check condition 2: \( G_0 \) obvious, \( G_1 \) by if \( V \cap U_i \cap X_i \) admissible, then \( V \cap X_i \) admissible by admissibility of \( U_i \to U \), then \( V \) is admissible, \( G_2 \): obvious.
- check condition 3: because \( X_i \cap X_j \to X_i \) is admissible.
Def. (V.1.1.12). A $G$-topological space is called connected iff there isn’t two nonempty admissible open subset $X_1, X_2$ that $X_1 \cap X_2 = \emptyset$ and $\{X_1, X_2 \to X\}$ is an admissible cover.

Def. (V.1.1.13). An object $U$ in a site is called quasi-compact if for each covering of $U$, f.m. of them still forms a covering of $U$. The topology $T$ is called Noetherian if each object of $T$ is quasi-compact.

Given a site $T$, we can define a new site $T^f$ whose coverings are coverings of $T$ that are finite. Then this is truly a site and it is Noetherian.

**G-torsor**

Def. (V.1.1.14) (Torsors). Let $C$ be a site and $G$ a sheaf of groups over $C$, then a pseudo $G$-torsor is a sheaf of sets $F$ over $C$ endowed with an action $G \times F \to F$ that $G \times F \to F \times F : (g, f) \mapsto (gf, f)$ is an isomorphism.

A pseudo $G$-torsor is called a $G$-torsor if for any $U \in C$, there is a covering $\{U_i \to U\}$ that $F(U_i)$ is non-empty for any $i$.

Prop. (V.1.1.15). A $G$-torsor on a site is trivial iff $\Gamma(C, F) \neq 0$ (V.6.1.1).

Proof: This is because the transitive action of $G$ on the global section induces an isomorphism $G \to F$.

Prop. (V.1.1.16). If $C$ is a subcanonical site, and $G$ a sheaf of groups over $C$, then a sheaf of sets $F$ together with an action $\alpha : G \times F \to F$ is a $G$-torsor iff for any $U \in C$, there is a covering $\{U_i \to U\}$ that the restrictions to $U_i$ are trivial torsors, i.e. $\alpha|_{C/U_i} = \pi_2 : G|_{U_i} \times F|_{U_i} \to F|_{U_i}$.

Proof: If $F$ is a $G$-torsor, then the restrictions of the torsor on $U_i$ are trivial because they have global sections in $\Gamma(C/U_i, F) = F(U_i)$, by (V.1.1.15). Conversely, if there is a covering $\{U_i \to U\}$ that the restrictions to $U_i$ are trivial torsors, then the map $G \times F \to F \times F : (g, f) \mapsto (gf, f)$ are isomorphisms when restricted to $U_i$, which means $G(U) \times F(U) \to F(U) \times F(U)$ is an isomorphism for any $U$, because they are sheaves, so it is an isomorphism, and $F$ is a $G$-torsor.

Cor. (V.1.1.17) (Representable $G$-Torsor). If $C$ is a site and $G$ is a group object in $C$, then

- $X \to Y$ is a $G$-torsor in the category $C/Y$ iff $X \to Y$ is a $G$-equivariant map(where the action of $G$ on $Y$ is trivial), $G \times X \to X \times_Y X : (g, x) \mapsto (gx, x)$ is an isomorphism, and $\{X \to Y\}$ is refined by a covering of $Y$.

- If $C$ is a subcanonical site, then $X \to Y$ is a $G$-torsor in the category $C/Y$ iff $X \to Y$ is a $G$-equivariant map(where the action of $G$ on $Y$ is trivial), and there exists a covering $\{Y_i \to Y\}$ that each $Y_i \times_Y X \to Y_i$ is a trivial torsor, i.e. $G$-equivariantly isomorphic to $G \times Y_i \to Y_i$.

Cor. (V.1.1.18). If $C$ is a site and $G$ is a group object in $C$, $X \to Y$ is a $G$-torsor in the category $C/Y$, then the map $G \times G \times X \to X \times_Y X \times_Y X : (g, h, x) \mapsto (ghx, hx, x)$ is an isomorphism.

**Preshaves**

Def. (V.1.1.19) (Preshaves). On a site $C$, a presheaf is a functor $C^{op} \to Set$. The category of presheaves on $C$ is denoted by $PSh(C)$. For any $U \in C$, $\Gamma(U, -)$ is the functor $PSh(C) \to Set : F \mapsto F(U)$.
Prop. (V.1.1.20). The category of presheaves on a site admits all limits and colimits, and it commutes with $\Gamma(U, -)$ for any $U \in \mathcal{C}$.

Proof: This is obvious. □

Def. (V.1.1.21) (Points). A point of a site is a Cf.[Sta]00Y3.

2 Sheaves and Topoi

Def. (V.1.2.1) (Sheaf). Let $C$ be a site, then a presheaf $\mathcal{F}$ on $C$ is called a sheaf iff

\[ \text{the category of sheaves on a site } \text{is denoted by } Sh(C). \]

Def. (V.1.2.2) (Effective Epimorphisms). An epimorphism $\{U_i \to V\}$ in a category is called a family of effective epimorphisms if

\[ \text{Hom}(V, Z) \to \prod \text{Hom}(U_i, Z) \Rightarrow \prod \text{Hom}(U_i \times_U U_j, Z) \]

is exact for each $Z$. Similarly for a family of universal effective epimorphisms.

Prop. (V.1.2.3) (Subcanonical Site). The set of all families of universal effective epimorphisms in a category forms a Grothendieck topology, called the canonical topology. It is the finest topology that all representable presheaves are sheaves.

Topologies that are coarser than the canonical topology are called subcanonical topology. Equivalently, a subcanonical topology is a topology that every representable presheaf is a sheaf.

Proof: We only need to verify that family of universal effective epimorphisms is closed under composition. For this, first prove epimorphism, then use epimorphism to prove effectiveness. Universal follows routinely. Cf.[Tamme]. □

Prop. (V.1.2.4). For a subcanonical topology on $C$, its restriction on a localizing category $C/S$ is subcanonical.

Proof: The only nontrivial part is that the glued morphism is a morphism over $S$. For this, consider its composition that maps to $S$, then the uniqueness of the exact sequence (V.1.2.2) will show that it is truly a $S$-morphism. □

Prop. (V.1.2.5). Let $C$ be a subcanonical site, and $f : X \to Y$ is an arrow in $C/S$, suppose there is a covering $\{S_i \to S\}$ that the pullback of $f$ to $C/S_i$ are all isomorphisms, then $f$ is an isomorphism.

Proof: This follows from (V.1.4.3) and (V.1.4.6). □

Prop. (V.1.2.6) (Sheafification). The operator $F^+$ is the presheaf that

\[ F^+(U) = \varprojlim \ker \prod F(U_i) \Rightarrow \prod F(U_i \times_U U_j) = \check{H}^0(U, F) \]

It is a separated presheaf, i.e. $0 \to F(U) \to \prod_i F(U_i)$ and when $F$ is separated, $F \to F^+$ is injective and $F^+$ is a sheaf. (The problem of separated is that the cover may not be identical in $U_i \times_U U_j$ but only on a cover of it). $F^+$ is left exact.

The sheafification $F^{++}$ is exact and it is left adjoint to the forgetful functor.

So the forgetful functor is left exact and it preserves injectives. Thus the sheaf cokernel is the sheafification of the presheaf kernel, the sheaf kernel is the presheaf kernel.
Proof: The separatedness is simple. For sheaf condition, an element of $F^+(U_i)$ is represented by a covering $\{V_{ij} \to U_i\}$, and there restriction to $U_i \times_U U_j$ coincide by separatedness hence the covering $\{V_{ij} \to U\}$ is an element of $F^+(U)$.

Sh is left exact because $(-)^+$ is left exact from $PAb$ to $PAb$ by (V.6.2.4) checked on every element $U$. It is right exact trivially, hence it is exact. □

Def. (V.1.2.7) (Constant Sheaf). The constant sheaf $\mathcal{S}$ for a set $S$ is the sheafification of the constant presheaf $U \mapsto S$.

Transfer of Sheaves under Morphisms of Sites

Def. (V.1.2.8) (Functoriality of Presheaves). Given a continuous functor of sites $u : T \to T'$, which should be regarded as an inverse map, there are maps

$$u^p F'(U) = F'(u(U)) : \mathcal{P}' \to \mathcal{P}, \quad u_p(F)(U') = \lim_{U_i|U' \to u(U_i)} F(U_i) : \mathcal{P} \to \mathcal{P}'$$

Then $u_p$ is left adjoint to $u^p$.

We can also define a functor

$$p^u : pF(U') = \lim_{U_i|U' \to u(U_i)} F(U_i).$$

Then this functor is right adjoint to $u^p$, by duality.

Proof: A map $f \in \text{Mor}(u_p(F), G)$ is represented by compatible maps

$$\lim_{U_i:U' \to u(U_i)} F(U_i) \to G(U')$$

, and this is represented by compatible maps $F(U_i) \to G(U)$ which is indexed over $\prod_{U' \in \mathcal{C}'} I_{U'U}$. Now this is equivalent to compatible maps $F(U_i) \to G(u(U_i))$, which is a map $g \in \text{Mor}(F, p^u(G))$. □

Cor. (V.1.2.9). $u^p$ is exact.

Def. (V.1.2.10) (Functoriality of Sheaves).

- Given a continuous functor $u : \mathcal{C} \to \mathcal{C}'$ between sites, there are maps

$$u_s = \# \circ u_p \circ \iota : \mathcal{S} \to \mathcal{S}', \quad u^s = u^p \circ \iota : \mathcal{S}' \to \mathcal{S}. \quad u_s \text{ is left adjoint to } u^s, \text{ by adjointness of } u_p, u^p \text{ and } \#, \iota.$$

- Given a cocontinuous functor $u : \mathcal{C} \to \mathcal{C}'$ between sites, there are maps

$$u^s = \# \circ p^u \circ \iota : \mathcal{S}' \to \mathcal{S}, \quad s_u = p^u \circ \iota : \mathcal{S} \to \mathcal{S}'.$$

$u^s$ is left adjoint to $s_u$, by adjointness of $u^p, p^u$ and $\#, \iota$. Moreover, $u^s$ is exact.

Proof: 1: Notice if $\mathcal{F}$ is a sheaf, then $u^s \mathcal{F}$ is also a sheaf, by continuity(V.1.1.4).

2: $p^u \mathcal{F}$ is a sheaf by [Sta][0XK]. $u^p$ is clearly right exact, and it is left exact because $\iota, \# \text{ do, and } u^p$ is exact by (V.1.2.9). □
Cor. (V.1.2.11). When \( u \) is continuous, \((u_p(G))^\sharp \cong (u_p(G^\sharp))^\sharp\) for any presheaf \( G \) on \( T \).
When \( u \) is cocontinuous, \((u^pG)^\sharp \cong (u^p(G^\sharp))^\sharp\) for any presheaf \( G \) on \( T \).

\[ \text{Proof: Use Yoneda lemma.} \]

Prop. (V.1.2.12). For \( Z \in T, u_p h_Z = h_u(Z) \).

\[ \text{Proof: Use the adjointness of } u \text{ over a site of categories. In particular, if } \]
\( I \text{ condition hold, because by the definition of sheafification functor,}\)
\( \text{topology, it suffice to show that for any covering, there is a refinement covering of it that sheaf}\)
\( \text{and } \sharp \text{ so we conclude.}\)

\[ \text{The last assertion is clear.} \]

Prop. (V.1.2.13) (When is } \mathcal{U},f \text{ Exact). If } f : \mathcal{C} \to \mathcal{C}' \text{ is a continuous functor between sites that } \mathcal{I}_{U'} \text{ is cofiltered for any } U' \in \mathcal{C}', \text{ where } \mathcal{I}_{U'} \text{ is the category of all } (U, \varphi) \text{ that } U' \to u(U), \text{ then } u_\ast \text{ is exact.}\)

In particular, this is the case when \( \mathcal{C}, \mathcal{C}' \) both have weakly final objects and finite fiber products and \( u \) preserves them. Notice the condition of weakly final objects can be released if we can show \( \mathcal{I}_{U'} \) is nonempty for any \( U' \).

\[ \text{The last assertion is clear.} \]

Prop. (V.1.2.14). For a topology \( T \) and an object \( Z \) of \( T \), there is a category \( T/Z \) as objects over \( T \), and \( i : T/Z \to T \) is continuous. Then \( i^\ast \) is exact.

\[ \text{Proof: } R^q i^\ast(F) = (i^p(H^q(F))^\ast)(V.6.1.5), \text{ and } (H^q(F))^\ast = 0(V.6.2.12), \text{ so it suffices to show } i^p \text{ and } \ast \text{ commutes. But } i^\ast \text{ and } + \text{ commutes obviously.} \]

Prop. (V.1.2.15) (Sheaf Condition is Local). To check sheaf condition for presheaf \( w.r.t. \) a topology, it suffice to show that for any covering, there is a refinement covering of it that sheaf condition hold, because by the definition of sheafification functor, \( F^+ = F \), so \( F \) is a sheaf.

Cor. (V.1.2.16). For two topology on a same category that \( \mathcal{I}' \) is finer than \( \mathcal{I} \), then any \( \mathcal{I}' \)-sheaf is a \( \mathcal{I} \)-sheaf. And if any covering in \( \mathcal{I}' \) can be refined by a covering in \( \mathcal{I} \), then \( \mathcal{S} \to \mathcal{S}' \) is an equivalence of categories. In particular, if \( T \) is Noetherian, \( \mathcal{S}(T) \) and \( \mathcal{S}(T') (V.1.1.13) \) are equivalent.

**Topoi**

**Def. (V.1.2.17) (Topoi).** A topos is a category that is equivalent to the category of sheaves over a site \( \mathcal{C} \). A morphism of topoi \( f : Sh(\mathcal{D}) \to Sh(\mathcal{C}) \) consists of a pair of adjoint functors \( f_* : Sh(\mathcal{C}) \to Sh(\mathcal{D}) \) and \( f^\ast : Sh(\mathcal{D}) \to Sh(\mathcal{C}) \) that \( f_* \) is right adjoint to \( f^\ast \) and \( f^\ast \) is exact. Compositions of morphisms of topoi are defined routinely.

**Prop. (V.1.2.18) (Sites and Topoi).** A morphism of sites \( f : \mathcal{D} \to \mathcal{C} \) consists of a continuous functor \( u : C \to D \) that \( u_\ast \) is exact, so it induces a functor of topoi \( f : Sh(\mathcal{C}) \to Sh(\mathcal{D}) \), if we define \( f^\ast = u_\ast, f_* = u^\ast \), by (V.1.2.10).
Prop. (V.1.2.19) (Cocontinuous Maps and Topoi). Let $u : \mathcal{C} \to \mathcal{D}$ be a cocontinuous functor between sites, then this defines a morphism $g$ of topoi $\text{Sh}(\mathcal{C}) \to \text{Sh}(\mathcal{D})$, by letting $g_* = s u$ and $g^{-1} = u^*$, by (V.1.2.10).

Prop. (V.1.2.20) (Special Cocontinuous Functor). A functor $u : \mathcal{C} \to \mathcal{D}$ between sites is called a special cocontinuous functor if:

- $u$ is continuous and cocontinuous,
- Given any $a, b : U' \to U \in \mathcal{C}$, there is a covering $\{f_i : U'_i \to U'\}$ that $af_i = bf_i$,
- Given any $U', U \in \mathcal{C}$ and a morphism $c : u(U') \to u(U) \in \mathcal{D}$, there exists a covering $\{f_i : U' \to U\} \in \mathcal{C}$ and morphisms $c_i : U'_i \to U$ that $u(c_i) = c \circ u(f)$.
- Given any $V \in \mathcal{D}$, there is a covering of the form $\{u(U_i) \to V\}$ in $\mathcal{D}$.

Then for a cocontinuous functor $u$, $(u^*, s u)$ defines an equivalence of topoi $\text{Sh}(\mathcal{C}) \cong \text{Sh}(\mathcal{D})$.

Proof: Cf. [Sta]03A0 \qed

Cor. (V.1.2.21) (Comparing Topologies). Let $T'$ be a fully subcategory of $T$, $i : T' \to T$ is a continuous functor of sites, if each object $U$ of $T'$ and a covering $\{U_i \to U\}$ in $T$ has a refinement $\{U'_i \to U\}$ in $T'$, and each object $U$ of $T$ has a covering $\{U_i \to U\}$ with objects $U_i \in T'$, then $i^*, i_*$ forms an equivalence between sheaves on $T$ and sheaves on $T'$. $i_* G$ is called the extension of sheaf.

Cor. (V.1.2.22) (Extending Sheaf). Let $T'$ be a fully subcategory of $T$, $i : T' \to T$ is a continuous and cocontinuous functor, then $G \cong i^* i_* G$ for any sheaf $G$ on $T'$ (This is implicit in the proof of the last proposition).

In particular, this applies to the case $T' = T / Z \to T$, the localization category, in with case $i^* F(Z') = F(Z')$ is called the restriction sheaf.

Prop. (V.1.2.23) (Localizing at Sheaves). Let $\mathcal{C}$ be a site and $\mathcal{F}_i$ be a set of topos on $\mathcal{C}$, then there is an equivalence $\text{Sh}(\mathcal{C}) \cong \text{Sh}(\mathcal{C}')$ induced by a special cocontinuous functor (V.1.2.20) that

- $\mathcal{C}'$ has the subcanonical topology,
- A family of morphisms $\{V_i \to V\}$ are a covering of $\mathcal{C}'$ iff $\bigsqcup h_{V_i} \to h_V$ is surjective.
- $\mathcal{C}'$ has fiber products and a final object.
- Every subsheaf of a representable sheaf is representable,
- Each $g_* \mathcal{F}_i$ is a representable sheaf.

Proof: Cf. [Sta]03CI. \qed

3 Sites over Schemes

Prop. (V.1.3.1). Fiber products exist in the category of schemes, by (V.2.7.15).

Zariski Topology

Def. (V.1.3.2). The Zariski topology has the the covering of a scheme $T$ as classes of open immersions $\{T_i \to T\}$ that their images cover $T$.

The big Zariski site $\text{Sch}_{\text{Zar}} / S$ has the objects as all schemes over $S$.

The small Zariski site $\text{S}_{\text{Zar}}$ has the objects as all open subschemes over $S$. 
The **restricted Zariski site** \( S_{\text{Zarfp}} \) has the objects as all schemes that are qcqs open subschemes of \( S \).

The **big affine Zariski site** \( \text{Aff}_{\text{Zar}}/S \) has the objects as all schemes affine over \( S \). These are all topologies because open immersions satisfies base change trick (V.4.4.44).

In particular when the cover has only one element and is affine, the descent datum is equivalent to compatible isomorphisms

\[
\varphi_{13} : N \otimes_A B \otimes_A B \xrightarrow{\varphi_{12}} B \otimes_A M \otimes_B B \xrightarrow{\varphi_{21}} B \otimes_A B \otimes_A M.
\]

**Prop. (V.1.3.3).** A sheaf w.r.t the small Zariski topology is equivalent to a sheaf on \( S \), trivially, so the sheaf cohomology on \( \text{Aff}_{\text{Zar}}/S \) is equivalent to usual sheaf cohomology on \( S \).

**Prop. (V.1.3.4).** If \( X \) is qs, then \( \widetilde{X_{\text{Zar}}} \to \widetilde{X_{\text{Zarfp}}} \) is an equivalence by \( i_\ast \) and \( i^\ast \), the same proof as (V.1.3.9).

### Étale Topology

**Def. (V.1.3.5) (Étale Topology).** The **étale topology** has the covering of a scheme \( T \) as classes of étale morphisms that their images cover \( T \).

The **big étale site** \( \text{Sch}_{\text{étale}}/S \) has the objects as all schemes over \( S \).

The **small étale site** \( S_{\text{étale}} \) has the objects as all schemes that are étale over \( S \).

The **restricted étale site** \( S_{\text{étfp}} \) has the objects as all schemes that are étale and qcqs over \( S \).

The **big affine étale site** \( \text{Aff}_{\text{ét}}/S \) has the objects as all schemes affine over \( S \).

These are truly topologies because étale is stable under base change and composition.

**Prop. (V.1.3.6).** Zariski covering is étale, because open immersions are étale.

**Prop. (V.1.3.7).** An étale covering of a qc scheme can be refined a finite affine étale covering, this is because étale map are open (V.5.5.3). Thus so does all above coverings.

**Prop. (V.1.3.8).** The restricted étale site of a qc scheme \( X \) is Noetherian, because étale map is open, and any object in \( X_{\text{étfp}} \) is qc.

**Prop. (V.1.3.9).** If \( X \) is qs, then \( \widetilde{X_{\text{ét}}} \to \widetilde{X_{\text{étfp}}} \) is an equivalence by \( i_\ast \) and \( i^\ast \).

**Proof:** Want to use (V.1.2.21), one condition is satisfied by (V.1.3.7), so it suffice to check any étale scheme \( X'/X \) has a covering of qcqs étale schemes over \( X \). For any point \( p \in X' \), there is an affine nbhd \( U' \) that maps to an affine nbhd \( U' \) of \( X \) and the ring map is f.p., so \( U' \to U \) is étale and f.p, and \( U \to X \) is open immersion and qs, it is qc because \( X \) is qs and \( U \) is qc (V.4.4.26).

**Prop. (V.1.3.10) (Cohomology Big and Small Sites).** The inclusion of small sites to the big sites has no infection on the sheaf cohomology, by (V.6.1.6). This is applicable to all topologies \( \tau \) considered here.

**Prop. (V.1.3.11) (Topological Invariance of Étale Site).** If \( S' \to S \) is universally homeomorphism, then \( f^\ast, f_* : S'_{\text{ét}} \to S_{\text{ét}} \) induces an equivalence of categories. Especially for the case of reduced structure.

**Proof:** Cf. [Étale Cohomology Conrad P18].

□
**Smooth Topology**

This topology will be shown to be identical to the Étale topology, so it is not so important.

**Syntomic Topology**

**Def. (V.1.3.12).** The **syntomic topology** has the covering of a scheme \( T \) as classes of syntomic morphisms that their images cover \( T \).

**fpff Topology**

**Def. (V.1.3.13).** The **fpff topology** has the covering of a scheme \( T \) as classes of flat locally of finite presentation morphisms that their images cover \( T \). (f.f.+locally of f.p.).

The **big Zariski site** \( \text{Sch}_{fpff}/S \) has the objects as all schemes over \( S \).

The **big affine Zariski site** \( \text{Af}_{fpff}/S \) has the objects as all schemes affine over \( S \).

They are all topologies because flatness and finite presentation satisfies base change trick by(V.5.1.2) and(V.5.8.3).

**Prop. (V.1.3.14) (Syntomic Covering is fpff).** A syntomic covering is fpff by definition(I.7.4.17).

**Prop. (V.1.3.15).** A fpff covering of an affine scheme can be refined a finite affine fpff covering, because fpff map are open(V.5.1.8).

**fpqc Topology**

**Def. (V.1.3.16).** The **fpqc topology** has the covering of a scheme \( T \) as classes of flat morphisms s.t. their images cover \( T \) and for any affine open \( U \subset T \), the restriction on \( T \) can be refined by a finite affine cover of open affine subschemes of the covering(f.f.+qc). It is a topology by(V.5.1.2) and(V.4.4.24).

When the covering consists of affine schemes, it is called **standard fpqc covering**.

**Prop. (V.1.3.17) (fpff is fpqc).** Fpff coverings are fpqc.

**Proof:** Use(V.5.1.8), we see that fpff covering consists of open morphisms, thus it is qc because affine scheme is quasi-compact. \( \square \)

**Prop. (V.1.3.18).** A covering consisting of flat morphisms refined by a fpqc covering is a fpqc covering.

Hence being fpqc is local on the target, because a Zariski cover is a fpqc covering.

If \( U \) is a covering consisting of flat morphisms that there is a fpqc covering \( V \) that \( U \times V \to V \) is a fpqc covering, then \( U \) is fpqc, because \( U \times V \) does and it refines \( U \).

**Remark (V.1.3.19).** Defining fpqc sites has inescapable set-theoretic difficulties, thus we don’t consider fpqc sites and fpqc cohomologies. Cf.[Sta]0BBK.

**Lemma (V.1.3.20) (Checking Sheaf Condition).** A presheaf is a sheaf w.r.t the fpqc topology iff it is a sheaf w.r.t the Zariski topology and satisfies sheaf property w.r.t the single covering \( V \to U \) f.f. between affine schemes.

Notice this is a special case of(V.1.5.7).
Proof: For any covering \{X_i \to X\}, choose an affine open refinement \{U_i \to X\}, then by definition, the pullback cover on U_i can use refined by a finite affine cover \(U_{ik} \to U_i\), so the composition covering of \(U_{ij} \to U_i\) and \(U_i \to X\) refines \(X_i \to X\). And sheaf condition for \(\{U_{ij} \to U_i\}\) is the same as sheaf condition for \(\{\prod U_{ij} \to U_i\}\). Thus the result follows from (V.1.2.15). □

Prop. (V.1.3.21) (fpqc Site is Subcanonical). The coverings in \(X_{fpqc}\) are families of universal effective epimorphisms, in the category of \(X\)-schemes.

Proof: By (V.1.3.20), it suffices to show that any representable presheaf is a sheaf w.r.t Zariski topology and f.f. affine morphisms. The Zariski case follows from (V.1.5.3), for the second, \(\text{Spec } B \to \text{Spec } Z\), for any scheme \(X\), the morphism corresponds to \(0 \to \text{Hom}(R, A) \to \text{Hom}(R, B) \to \text{Hom}(R, B \otimes_A B)\), but this follows immediately from (I.7.2.2), with \(M = A\). □

Cor. (V.1.3.22). For \(f : Y \to X\) a morphism of schemes, if \(Z \in X_\tau\) for the above topologies \(\tau\), then \(f^*(\text{Hom}_X(-, Z)) \cong \text{Hom}(-, Z \otimes_X Y)\), in other words, the inverse sheaf of a representable sheaf is representable.

Proof: By definition, \(f^*(\text{Hom}_X(-, Z))\) is the sheaf associated to the presheaf \(f_p(\text{Hom}_X(-, Z))\), which by (V.1.2.12) is just the presheaf represented by \(Z \otimes_X Y\), but by the proposition, it is already a sheaf. □

Prop. (V.1.3.23) (Coherent Sheaves on \((\text{Sch}/X)_{fpqc}\)). Let \(M\) be a \(\mathbb{Q}\)co sheaf of \(\mathcal{O}_X\)-modules, then the functor \(X' \to \Gamma(X', M \otimes_{\mathcal{O}_X} \mathcal{O}_{X'})\) is an Abelian sheaf on \((\text{Sch}/X)_{fpqc}\), by (V.1.5.13).

Prop. (V.1.3.24). For any \(\mathbb{Q}\)co sheaf \(F\) on a separated scheme \(X\). If \(\mathcal{T}\) is a Grothendieck topology on \(\text{Sch}/S\) containing the Zariski topology and every cover is refined by a fpqc cover by a finite collection of affine schemes, then \(H^p(\mathcal{T}, X, F) = H^p(X, F)\). Same as the proof of (V.6.7.2), with the Zariski-Poincare lemma replaced by the fpqc-Poincare lemma.

**PH-Covering**

Def. (V.1.3.25) (Standard PH-Covering).

Def. (V.1.3.26) (PH-Topology).

Prop. (V.1.3.27) (Zariski Covering is PH-Covering). A Zariski covering is a PH-covering.

Proof: Cf.[[Sta]0DBH]. □

Prop. (V.1.3.28). A proper surjective morphism is a ph-covering.

Proof: Cf.[[Sta]0DES]. □

**V-Topology**

Def. (V.1.3.29) (Standard V-Covering). A finite family of morphisms morphism \(T_i \to X\) of affine schemes is a covering in the **standard v-topology** if for any morphism \(\text{Spec } V \to X\) where \(V\) is a valuation ring, there is an extension of valuation rings (I.9.2.11) \(V \to W\) and a morphism \(\text{Spec } W \to \text{Spec } V \times_X T_i\) for some \(i\).
Def. (V.1.3.30) (V-Topology). A family of morphisms \( \{ T_i \to T \} \) is called a \textbf{v-covering} in the \textbf{v-topology} if for any open affine subscheme \( U \) of \( T \), the base change is refined by a standard v-covering of \( U \).

The v-coverings form a topology, by\([\text{Sta}0\text{ETJ}]\).

Lemma (V.1.3.31). A standard fpqc covering is a standard v-covering.

Proof: Cf.[\text{Sta}0\text{22E}] \hfill \Box

Prop. (V.1.3.32) (fpqc Covering is v-Covering). A fpqc covering is a v-covering.

Proof: This follows immediately from(V.1.3.31) \hfill \Box

Prop. (V.1.3.33). A standard ph-covering is a standard v-covering.

Proof: Cf.[\text{Sta}0\text{ETD}] \hfill \Box

Prop. (V.1.3.34) (ph-Covering is V-Covering). A ph-covering is a v-covering.

Proof: This follows immediately from(V.1.3.33) \hfill \Box

Cor. (V.1.3.35). A proper and surjective map is a v-covering, by(V.1.3.34) and(V.1.3.28).

\textbf{Arc-Topology}

Def. (V.1.3.36) (Arc-Topology). A finite family of morphisms \( \{ T_i \to X \} \) of schemes is a covering in the \textbf{arc-topology} if for any morphism \( \text{Spec} \, V \to X \) where \( V \) is a rank1-valuation ring, there is a rank1-valuation ring \( W \) and a morphism \( \text{Spec} \, W \to \text{Spec} \, V \times_X T_i \) for some \( i \) that \( V \to W \) is f.f..

Prop. (V.1.3.37). V-coverings are arc coverings.

Proof: This is by the definition and(I.9.2.11) \hfill \Box

4 \textbf{Stacks}

Def. (V.1.4.1) (Stacks). Let \( F \to C \) be a fibered category on a site \( C \). Then \( F \) is called a \textbf{prestack} over \( C \) if for each covering \( \{ U_i \to U \} \) in \( C \), the functor \( \text{Hom}(h_{U_i}, F) \to \text{Hom}(h_U, F)(\text{V.1.1.6}) \) is fully faithful. It is called a \textbf{stack} over \( C \) if it is moreover an equivalence of categories.

Cor. (V.1.4.2) (Stacks and Sheaves). A stack fibered in sets over a site \( C \) is equivalent to a sheaf on \( C \), by(II.1.7.25).

Cor. (V.1.4.3). A site is subcanonical iff any representable fibered category \( h_U \to C \) is a stack.

Def. (V.1.4.4) (Category of Descent Datum). Let \( F \to C \) be a fibered category on a site \( C \), \( U \in C \) and \( U \) is a covering of \( U \), then we can define the \textbf{category of descent datum} \( F_{\text{desc}}(U) \) to be the category of tuples \( (\xi_i, \xi_{ij}, \xi_{ijk}) \) where \( \xi_{ij} \in F(U_{ij}) \) with Cartesian morphisms between them that are commutative. A morphism in \( F_{\text{desc}}(U) \) is a family of morphisms \( (\varphi_i, \varphi_{ij}, \varphi_{ijk}) \) commuting with the Cartesian morphisms.

Then there is an equivalence of categories \( \text{Hom}(h_U, F) \cong F_{\text{desc}}(U) \).
Proof: For any $F : h_U \to \mathcal{F}$, $U_i \to U \in h_U(U_i)$, applying $F$ to the arrows $\xi_{ij} \to \xi_i \in h_U$, we get an element in $\mathcal{F}_{\text{desc}}(U)$. Also a natural transformation $F \to G$ maps to a morphism in $\mathcal{F}_{\text{desc}}(U)$, so we get a functor $T : \text{Hom}(h_U, \mathcal{F}) \to \mathcal{F}_{\text{desc}}(U)$.

Conversely, choose an arbitrary choice of pullbacks (that coincide with $\xi_{ij} \to \xi_i$), for any arrow $f : T \to U \in h_U(T)$, we choose a $U_i$ that $T \to U$ factors as $T \to U_i \to U$ (also for $f = \text{id}_{U_i}$, choose $U_i$), then define $F(f)$ as the pullback of $\xi_i$ along $T \to U_i$. For any morphisms $T' \to T \to U \in h_U$ and their choice of $U_i, U_j$ that $T' \to U$ factors through $U_j \to U$ and $T \to U$ factors as $U_i \to U$, then $T' \to U_j$ factors through $U_{ij}$, i.e. we have a commutative diagram

$$
\begin{array}{ccc}
T' & \to & U_{ij} \\
\downarrow & & \downarrow \\
T & \to & U_i \\
\end{array}
$$

Then by Cartesian properties, we get a unique map $F(T') \to F(T)$, which is Cartesian by (II.1.7.8). It can be shown these maps make $F$ a functor, using Cartesian property and the existence of $\xi_{ijk}$. And also for any map of descent datum, we get a natural transformation in $\text{Hom}(h_U, \mathcal{F})$. Thus we get a functor $S : \mathcal{F}_{\text{desc}}(U) \to \text{Hom}(h_U, \mathcal{F})$.

The construction of $S$ shows $T$ is essentially surjective and full, and also faithful, so $(S, T)$ is an equivalence of categories. □

Prop. (V.1.4.5) (Equivalence Categories and Stacks). Let $\mathcal{F} \to \mathcal{G}$ be an equivalence of fibered categories over a site $\mathcal{C}$, then $\mathcal{F}$ is a prestack/stack iff $\mathcal{G}$ is.

Proof: There is a strict commutative diagram of categories

$$
\begin{array}{ccc}
\text{Hom}(h_U, \mathcal{F}) & \to & \text{Hom}(h_U, \mathcal{F}) \\
\downarrow & & \downarrow \\
\text{Hom}(h_U, \mathcal{G}) & \to & \text{Hom}(h_U, \mathcal{G})
\end{array}
$$

that the vertical arrows are equivalences of categories, then we are done. □

Prop. (V.1.4.6) (Prestack and Hom Functor). Let $\mathcal{F}$ be a fibered category over a site $\mathcal{C}$, then $\mathcal{F}$ is a prestack iff for any object $S$ of $\mathcal{C}$ and two objects $\xi, \eta \in \mathcal{F}(S)$, the presheaf $\text{Hom}_S(\xi, \eta) : (\mathcal{C}/S)^{\text{op}} \to \text{(Set)}(\text{II.1.7.17})$ is a sheaf in the comma site $\mathcal{C}/S$ (V.1.1.3).

Proof: By (V.1.4.5), it suffices to show $\text{Hom}_S(\xi, \eta)$ is a stack in the comma topology $\mathcal{C}/S$. Then it can be shown that $\text{Hom}(h_U, \text{Hom}_S(\xi, \eta)) \to \text{Hom}(h_U, \text{Hom}_S(\xi, \eta))$ is an equivalence of categories iff $\text{Hom}(h_U, \mathcal{F}) \to \text{Hom}(h_U, \mathcal{F})$ is fully faithful. □

Lemma (V.1.4.7). Let $\mathcal{F}$ be a prestack over a site $\mathcal{C}$, $S, S'$ be sieves belonging to the topology of $\mathcal{C}$ that $S' \subset S$, then the restriction functor

$$
\text{Hom}_\mathcal{C}(S, \mathcal{F}) \to \text{Hom}_\mathcal{C}(S', \mathcal{F})
$$

is faithful.

Proof: it suffices to show for $S' = h_U$ for some covering $U$. Let $F, G \in \text{Hom}_\mathcal{C}(S', \mathcal{F})$ and $\varphi, \psi$ be two natural transformations from $F$ to $G$ that induce the same natural transformation from the
restriction of $F$ to $h_{U}$ to the restriction of $G$ to $h_{U}$, then $\varphi = \psi$. For this, just notice there are commutative diagrams

$$F(T \times_{U} U_{i}/U) \xrightarrow{\varphi_{T \times_{U} U_{i}/U}} G(T \times_{U} U_{i}/U)$$

$$\downarrow \quad \downarrow$$

$$F(T/U) \xrightarrow{\varphi_{T/U}} G(T/U)$$

where the vertical arrows are Cartesian, and the hypothesis implies $\varphi_{T \times_{U} U_{i}/U} = \psi_{T \times_{U} U_{i}/U}$. Then we deduce $\varphi_{T/U} = \psi_{T/U}$ as $\text{Hom}_{T}(F(T/U), G(T/U))$ is a sheaf, as $F$ is a presheaf (V.1.4.6).

\[\square\]

**Prop. (V.1.4.8) (Stack and Sieves).** A prestack $F \to C$ is stack iff for any $U \in C$ and any sieve $S$ on $U$ belonging to $T$, $\text{Hom}(h_{U}, F) \to \text{Hom}(S, F)$ is an equivalence of categories.

**Proof:** Let $S$ be a sieve on $U$ belong to $C$, choose a covering $U$ of $U$ that $h_{U} \subset S \subset h_{U}$, then there is a factorization

$$\text{Hom}(h_{U}, F) \to \text{Hom}(S, F) \to \text{Hom}(h_{U}, F).$$

Then we are done by (V.1.4.7).

\[\square\]

**Cor. (V.1.4.9).** Let $T, T'$ be two topologies on a category $C$ that $T'$ is subordinate to $T$ and $F \to C$ is a fibered category, then if $F$ is a prestack/stack relative to $T$, it is also true for $T'$.

**Prop. (V.1.4.10) (2-Fiber Products of Stacks).** There is a natural 2-category of stacks over $C$ defined as a sub-2-category of the 2-category of fibered-categories over $C$, and the $(2, 1)$-category of stacks over $C$ has 2-fibered products, which coincides with that of (II.1.7.13).

**Proof:** Let $\mathcal{X} \to S, \mathcal{Y} \to S$ be morphisms of stacks over $C$, then the category $\mathcal{X} \times_{S} \mathcal{Y}$ is a fibered category over $C$, by (II.1.7.13). It remains to show that the morphism presheaves sheaves and descent datum are effective.

For this, Cf.[Sta]026G.

\[\square\]

**Prop. (V.1.4.11) (Associated Stack in Groupoids).** Let $C$ be a site and $F$ be a prestack, and $F_{\text{cart}}$ is the associated category fibered in groupoids (II.1.7.22), then $F_{\text{cart}}$ is also a prestack. And in this case, $F$ is a stack if $F_{\text{cart}}$ is a stack.

**Proof:** The categories $F(U)$ and $F_{\text{cart}}(U)$ have the same isomorphism classes of objects, as isomorphisms are Cartesian, so it suffices to show $F$ is a prestack iff $F_{\text{cart}}$ is a prestack. For this, use (V.1.4.6) and consider $\xi, \eta \in F(U)$ and a covering $\{U_{i} \to U\}$, if there are arrows $\alpha_{i} : \xi_{i} \to \eta_{i}$ that are compatible, then there is a unique arrow $\alpha : \xi \to \eta$ restricting to $\alpha_{i}$, thus it suffices to show $\alpha$ is Cartesian. But since $F_{\text{cart}}$ is a category fibered in groupoids, each $\alpha_{i}$ is invertible, and their inverses glue together to a morphism $\beta : \eta \to \xi$ which is the inverse of $\alpha$, so $\alpha$ is also an isomorphism thus Cartesian.

\[\square\]

**Prop. (V.1.4.12) (2-Fibered Products of Stacks Fibered in Groupoids).** Let $C$ be a category, the 2-category of stacks fibered in groupoids over $C$ (automatically a $(2, 1)$-category) has 2-fiber products, and coincide with that of (V.1.4.10).

**Proof:** This is clear from (V.1.4.10) and (II.1.7.23).
Prop. (V.1.4.13) (Equivalence of Stacks). Let $F : S_1 \to S_2$ be a morphism of stacks over a site $C$. If $F$ is fully faithful, then $F$ is an equivalence iff for any $x \in S_{2,U}$, there exists a covering $\{f_i : U_i \to U\}$ s.t. $f_i^* x$ is in the essential image of the functor $F : S_{1,U} \to S_{2,U}$.

Proof: Easy. □

Prop. (V.1.4.14) (Subcanonical Site Prestack). If $C$ is a subcanonical site (V.1.2.3) and $\mathcal{P}$ is a class of arrows in $C$ stable under base change, then the corresponding fibered category $\mathcal{P} \to C$ is a prestack.

Proof: By (V.1.4.6), we need to prove for any covering $\{U_i \to U\}$ and arrows $X \to U, Y \to U$, $X_i = U_i \times_U X$, and $X_{ij} = U_{ij} \times_U X$ and analogous for $Y$, if there are arrows $f_i : X_i \to Y_i$ over $U_i$ that the arrows $X_{ij} \to Y_{ij}$ induced by $f_i$ and $f_j$ coincide, then there is a unique arrow $f : X \to Y$ over $U$ that induces $f_i$.

Notice that the composite $X_i \xrightarrow{f_i} Y_i \to Y$ gives sections $g_i \in h_Y(X_i)$, and the pullback of $g_i, g_j$ to $X_{ij}$ coincide by hypothesis. Now $h_Y$ is a sheaf by hypothesis, so there is an arrow $f : X \to Y$ that pulls back to $f_i$ on $X_i$. Finally $f$ is compatible over $U$ because $(Y \to U) \circ f$ and $(X \to U)$ coincide when composed with $X_i \to X$, and $h_Y$ is a sheaf. □

Prop. (V.1.4.15) (Category of Sheaves is a Stack). Let $C$ be a site, we denote $(\text{Sh}/C)(X) = \text{Sh}(C/X)$, then $\text{Sh}/C$ is a stack over $C$.

Proof: To show $\text{Sh}/C$ is a prestack, by (V.1.4.6), it suffices to show for any $F,G \in \text{Sh}(C/X)$, $\text{Hom}_X(F,G)$ is a sheaf. For this, let $\{U_i \to U\}$ be a covering, and $\varphi_i : F_{U_i} \to G_{U_i}$ be morphisms of sheaves that their restrictions to $F_{U_{ij}} \to G_{U_{ij}}$ are compatible, then for any $T \to U$, there are commutative diagrams

\[
\begin{array}{ccc}
F(T) & \to & \prod_i F_i(T_i) \\
\downarrow & & \downarrow \\
G(T) & \to & \prod_i G_i(T_i)
\end{array}
\]

where $\varphi_T : F(T) \to G(T)$ is the unique function of sets that makes the diagram commutative. And it can be shown that these $\varphi_T$ defines a natural transformation $F_U \to G_U$.

Now for any covering $\{U_i \to U\}$ and a descent datum $(F_i, F_{ij})$, we need to show it is effective. We define a function $F$ on $C/U$: $F(T) = \text{equal}(\prod_i F_i(T_i) \to \prod_{ij} F(T_{ij}))$, then it can be shown that this is a sheaf by spectral sequence.

Then it suffices to check $F_{U_i} = F_k$. For any $T \to U_k$, $T_i$ maps to $U_{ik}$, so $F_i(T_i) = F_k(T_i)$, thus for any $s \in F_k(T)$, we can produce an element $(s_{T_i}) \in \prod_i F_i(T_i)$ that satisfies compatibility conditions, which gives us an element of $F(T)$. It can be shown this is a natural transformation $F_k \to F_{U_k}$, and it is an isomorphism of sheaves. □

Prop. (V.1.4.16) (Descent Along Torsors). Let $C$ be a site, $G$ a group object and $X \to Y$ a $G$-torsor, $\mathcal{F} \to C$ a stack. Then there exists a canonical equivalence of categories between $\mathcal{F}(Y)$ and the category of $G$-equivariant objects $\mathcal{F}^G(X)$ (II.1.7.15).

Proof: By (V.1.1.17) $X \to Y$ is refined by a covering, thus by (V.1.4.8), $\mathcal{F}(Y)$ is equivalent to $\mathcal{F}(X \to Y)$. And we check $\mathcal{F}(X \to Y) \cong \mathcal{F}^G(X)$. Then by (V.1.1.18), $\mathcal{F}(X \to Y)$ consists of elements $\xi \in \mathcal{F}(X), \eta \in \mathcal{F}(G \times X)$ and Cartesian arrows $\varphi_1, \varphi_2$ over $\alpha, \pi_2$. Cf. [Vis08]P106. □
Stackifications

Prop. (V.1.4.17) (Stackification). Let \( C \) be a site and \( p : \mathcal{F} \to C \) a fibered category over \( C \), then there exists a stack \( \mathcal{F}' : \mathcal{F}' \to C \) and a morphism \( G : \mathcal{F} \to \mathcal{F}' \) of fibered categories over \( C \) s.t. for any stack \( \mathcal{X} \to C \), a morphism \( F : \mathcal{F} \to \mathcal{X} \) of fibered categories over \( C \) factors through \( G \) 2-commutatively and uniquely up to 2-isomorphisms. In other words, there is a canonical equivalence of categories:

\[
\text{Mor}_{\text{Fib}/C}(\mathcal{F}, \mathcal{X}) \cong \text{Mor}_{\text{Sta}/C}(\mathcal{F}', \mathcal{X}).
\]

In particular, such a stack \( \mathcal{F}' \) is determined up to unique 2-isomorphisms, and is called the stackification of \( \mathcal{F} \).

Proof: By (II.1.7.18), we may assume \( \mathcal{F} \) is split, thus \( \mathcal{F} \) corresponds to the functor \( C^{\text{op}} \to \text{Cat} : U \mapsto \text{Hom}(h_U, \mathcal{F}) \). Then we define a functor

\[
\mathcal{F}' : C^{\text{op}} \to \text{Cat} : U \mapsto \varprojlim_{U \in \text{Cov}(U)} \text{Hom}(h_U, \mathcal{F}).
\]

then there is a natural map of fibered categories \( \mathcal{F} \to \mathcal{F}' \). For any stack \( \mathcal{X} \to C \), because \( \text{Hom}(h_U, \mathcal{X}) \to \text{Hom}(h_U, \mathcal{X}) \) is an isomorphism for any covering \( U \) of \( U \), we get the desired equivalence of categories.

\[\square\]

Cor. (V.1.4.18). Let \( G : \mathcal{S} \to \mathcal{S}' \) be the stackification of a fibered category over \( C \), then

- For any \( U \in C \) and \( x, y \in \mathcal{S}_U \), the map

\[
\text{Hom}(x, y) \to \text{Hom}(G(x), G(y))
\]

identifies the RHS as the shification of the LHS.

- For any \( U \in C \) and \( x \in \mathcal{S}'_U \), there exists a covering \( \{U_i \to U\} \) that for any \( i \), \( x|_{U_i} \) is in the essential image of \( G_U : \mathcal{S}_U \to \mathcal{S}'_U \).

Proof: Cf.[Sta]0435. \[\square\]

Prop. (V.1.4.19) (Stackifications Commute with 2-Fibered Products). Stackifications commute with 2-fibered products.

Proof: Cf.[Sta]04Y1. \[\square\]

Prop. (V.1.4.20). If \( \mathcal{F}, \mathcal{G} \) are prestacks over a topological space \( X \), if there is a morphism \( \eta : \mathcal{F} \to \mathcal{G} \) that satisfies:

- \( \mathcal{F} \) is a stack and \( \mathcal{G} \) is a prestack.
- \( \eta \) induces isomorphisms on stalks.
- \( \eta(U) : \mathcal{F}(U) \to \mathcal{G}(U) \) is fully faithful.

Then \( \eta \) is an equivalence of prestacks. In particular, \( \mathcal{G} \) is also a stack.

Proof: Let \( \mathcal{H} \) be the stackification of \( \mathcal{G} \), then \( \mathcal{F} \to \mathcal{G} \to \mathcal{H} \) is a morphism of stacks that is isomorphism on the stalk, so it is an isomorphism? But \( \mathcal{G} \) is separated, so for any open \( U \), \( \mathcal{F}(U) \to \mathcal{G}(U), \mathcal{G}(U) \to \mathcal{H}(U) \) is fully faithful, and their composition is an equivalence, thus both of them are equivalences. \[\square\]
Gerbes

**Def. (V.1.4.21) (Gerbe).** Let $\mathcal{C}$ be a site, a **gerbe** over $\mathcal{C}$ is a stack in groupoids over $\mathcal{C}$ s.t.

- For any $U \in \mathcal{C}$, there is a covering $\{U_i \to U\}$ that $S_{U_i}$ is nonempty for any $i$.
- For any $U \in \mathcal{C}$ and $x, y \in S_U$, there exists a covering $\{U_i \to U\}$ that $x|_{U_i} = y|_{U_i}$ for any $i$.

**Prop. (V.1.4.22).** Let $p : S \to \mathcal{C}$ be a gerbe over a site $\mathcal{C}$, assume that for all $U \in \mathcal{C}$ and $x \in S_U$, the sheaf of groups $Aut(x)$ on $\mathcal{C} \triangleleft U$ is Abelian, then there exists a sheaf of Abelian groups over $\mathcal{C}$ and for any $x \in S_U$ an isomorphism $G|_U \to Aut(x)$ that for any morphism $\varphi : x \to y \in S_U$, the diagram

$$
\begin{array}{ccc}
G|_U & \longrightarrow & G|_U \\
\downarrow & & \downarrow \\
Aut(x) & \longrightarrow & Aut(y)
\end{array}
$$

is commutative.

**Proof:** It can be checked by using the fact $Aut(x)$ is Abelian that there are canonical morphisms $Aut(x) \to Aut(y)$ induced by any morphism $\varphi : x \to y$.

If there is no morphism from $x$ to $y$, then we can use the condition of gerbe to obtain morphisms $Aut(x)|_{U_i} \to Aut(y)|_{U_i}$ locally, and then glue together. Similarly, if $S_U$ is empty, then we can restrict to a covering and then glue.

Finally, notice this gives an Abelian sheaf $G = Aut$ on $\mathcal{C}$.

Bands

5 Descent for Schemes

Main References are [Sta]Chap34, 10.158.

General Principal

**Prop. (V.1.5.1).** A property of schemes is called **local** in a topology if for any covering $\{U_i \to S\}$, $S$ has $P$ iff $U_i$ has $P$. A property of morphisms is called **local** in a topology if for any covering $\{U_i \to S\}$, $X \to S$ has $P$ iff $X \times_S U_i \to U_i$ has $P$.

**Prop. (V.1.5.2) (Twists and Čech Cohomology).** Let $\xi$ be an object of a stack $\mathcal{F}$ over a site $\mathcal{C}$ lying over an object $U$ of $\mathcal{C}$, we call an object $\xi' \in \mathcal{F}(U)$ a **twist** of $\xi$ if there is some covering $\{U_i \to U\}$ that the pullback of $\xi$ and $\xi'$ to $U_i \to U$ are isomorphic.

Then there is a natural bijection between $\mathcal{F}(U)$-isomorphism classes of twists of $\xi$ with $\check{H}^1(U, Aut(\xi))$, where $\xi$ is seen as a presheaf so $Aut E$ is also a presheaf.

**Proof:** Cf.[Appendix of Lamb].

**Zariski Descent**

**Lemma (V.1.5.3) (Zariski Descent of Qco Sheaves).** The fibered category $X \mapsto (QCo/X)$ is a stack in the Zariski topology $Sch_{Zar}/S$. 
Prop. (V.1.5.4) (Zariski Descent of Schemes). The fibered category $X \mapsto Sch/X$ is a stack in the Zariski topology $Sch_{Zar}/S$.

Proof: Firstly it is a prestack by (V.1.4.14) and (V.1.3.21).

To show any descent datum of effective, let $\{U_i \to U\}$ be a Zariski covering, and $X_i \to U_i$ be schemes with descent datum $\varphi_{ij} : X_i \times_{U_i} U_{ij} \cong X_j \times_{U_j} U_{ij}$, then we define $X = \bigsqcup X_i/ \sim$, where $x_i \sim x_j$ if $x_i \in U_{ij}, x_j \in U_{ji}$ and $\varphi_{ij}(x_i) = \varphi_{ji}(x_j)$. It can be shown this is an equivalence relation. Denote $\varphi_i : X_i \to X$ the natural map, and $U_i = \varphi_i(X_i)$. Define the topology on $X$ as the quotient topology. In particular, $\varphi_i$ is a homeomorphism onto the image. Then we can use (V.1.4.15) to glue the sheaves of rings $(\varphi_i)_* (O_{X_i})$ to a sheaf of rings $O_X$ on $X$ that $\varphi_i^* O_X = O_{X_i}$. Also we have a map $f : X \to U$ by set-theoretical and topological consideration. For the structure map $f^{-1}(O_U) \to O_X$, use (V.1.4.15).

Cor. (V.1.5.5). If $\mathcal{P}$ is a subclass of arrows of schemes that is stable under base change and local on the target, then $\mathcal{P}/S$ is a stack over $(Sch/S)_{Zar}$.

Cor. (V.1.5.6) (Zariski Descent of Schemes with a Qco Sheaf). The fibered category $X \mapsto \{\text{Schemes over } X \text{ with a Qco sheaf } \mathcal{F}\}$ is a stack in the Zariski topology $Sch_{Zar}/S$.

Proof: This is a combination of (V.1.5.4) and (V.1.5.3).

fpqc Descent

Prop. (V.1.5.7) (Reduction to Zariski). Let $S$ be a scheme and $\mathcal{F}$ be a fibered category over the category $(Sch/S)$. Suppose that

- $\mathcal{F}$ is a stack w.r.t. the Zariski topology.
- When $V \to U$ is a f.f. morphism of affine $S$-schemes, $\mathcal{F}(U) \to \mathcal{F}(V \to U)$ is an equivalence of categories.

then $\mathcal{F}$ is a stack w.r.t. the fpqc topology.

Proof: Firstly, to show $\mathcal{F}$ is a prestack, using (V.1.4.6), it suffices to show for an $S$-scheme $T \to S$ and objects $\xi, \eta \in F(T)$, the functor

$$\text{Hom}_T(\xi, \eta) : (Sch/T)^{op} \to \text{Set}$$

is a sheaf. But then we can use (V.1.3.20) to achieve this.

Next, according to (II.1.7.18) and (V.1.4.5), we may assume $\mathcal{F}$ is splitting.

Notice that $\mathcal{F}(\emptyset)$ is equivalent to the category pt. This is because $\mathcal{F}(\emptyset)$ is equivalent to $\mathcal{F}(\mathcal{U})$, where $\mathcal{U}$ is the null Zariski covering of $\emptyset$ (with no mapping or objects at all!). Then for any disjoint union of open subschemes $U = \bigsqcup U_i$, there is a natural isomorphism of categories $\mathcal{F}(U) \cong \prod_i \mathcal{F}(U_i)$.

Thus for any covering $\mathcal{U} = \{U_i \to X\}$, to show $\mathcal{F}(X) \to \mathcal{F}(\bigsqcup U_i \to X)$ is an equivalence of categories, it suffices to show that $\mathcal{F}(X) \to \mathcal{F}(\prod U_i \to X)$ is an equivalence of categories: this is because $\prod U_i \times_X \prod U_i \cong \prod U_i \times_X U_j$, so $\mathcal{F}(\prod U_i \to X) \to \mathcal{F}(\mathcal{U})$ is an equivalence of categories.

If $\mathcal{U} = \{V \to U\}$ is a covering of $F$ with a single morphism that is qc and $U$ is affine, then by the proof of (V.1.4.8), we can choose a finite affine cover of $V$ reduce to the case of covering of f.m. affine maps, then we finish by the case above.
If \( U = \{ V \to U \} \) is a covering of \( F \) with a single morphism and \( U \) is affine, then we can find a Zariski covering \( \{ V_i \to V \} \) that each \( V_i \) is qc. and surjects onto \( U \). Then there are maps of categories \( \mathcal{F}(U) \to \mathcal{F}(\{ V \to U \}) \) where \( \mathcal{F}(U) \to \mathcal{F}(\{ V_i \to V \}) \) is an equivalence and \( \mathcal{F}(U) \to \mathcal{F}(\{ V \to U \}) \) is fully faithful. Thus to show \( \mathcal{F}(\{ V \to U \}) \to \mathcal{F}(\{ V_i \to V \}) \) is an equivalence, it suffices to show \( \mathcal{F}(\{ V \to U \}) \to \mathcal{F}(\{ V \to U \}) \) is faithful, which is true by (V.1.4.7).

For general case, Cf.[Vis08]P88.

**Prop. (V.1.5.8).** \( X \to \text{Sch}/X \) is a prestack over \( (\text{Sch}/S)_{fpqc} \), by (V.1.3.21) and (V.1.4.14).

**Def. (V.1.5.9).** For any ring map \( A \to B \), we can define a category \( \text{Mod}_{A \to B} \) as follows: its objects are pairs \( (N, \psi) \), where \( N \) is a \( B \)-module and \( \psi : N \otimes_A B \to B \otimes_A N \) is an isomorphism of \( B \otimes_A B \)-modules that

\[
\psi_{12} \circ \psi_{01} = \psi_{02} : N \otimes_A B \otimes_A B \to B \otimes_A B \otimes_A N
\]

where \( \psi_{ij} \) are interchanging the \( i, j \)-part using \( \psi \). The arrows in \( \text{Mod}_{A \to B} \) are maps of \( N \) that is compatible with \( \psi \).

**Lemma (V.1.5.10) (Affine Fpqc Descent).** We have a functor \( F : \text{Mod}_A \to \text{Mod}_{A \to B} \), where \( M \) is mapped to \( B \otimes_A M \), and \( \psi_M \) being

\[
\psi_M : (B \otimes_A M) \to B \otimes_A (B \otimes_A M) : b \otimes m \otimes b' \mapsto b \otimes b \otimes m.
\]

Then when \( A \to B \) is f.f., this is an equivalence of categories.

**Proof:** We construct an inverse \( T \) that maps \( (N, \psi) \) to \( \{ n \mid \psi(n \otimes 1) = 1 \otimes n \} \). Then \( TF \cong \text{id} \) because of the first exactness of (I.7.2.2).

And for \( FT \), let \( F((N, \psi)) = M \). Notice if \( \psi(n \otimes 1) = \sum_i b_i \otimes n_i \), then

\[
\sum_i b_i \otimes 1 \otimes n_i = \otimes_i b_i \otimes \psi(n \otimes 1).
\]

so \( \psi(n \otimes 1) \in \text{Ker}(\text{id}_B \otimes (n \mapsto \psi(n \otimes 1) - 1 \otimes n)) = B \otimes M \) because \( B/A \) is flat.

So we defined a map \( N \xrightarrow{\Psi} B \otimes_A M \), and the composition \( B \otimes_A M \xrightarrow{\psi} N \xrightarrow{\Psi} B \otimes_A M \) is the identity, because

\[
\psi(bm \otimes 1) = (b \otimes 1)\psi(m \otimes 1) = (b \otimes 1)(1 \otimes m) = b \otimes m.
\]

This shows \( \Psi \) is surjective. And \( \Psi \) is injective because \( n \mapsto n \otimes 1 \) is injective as \( B/A \) is f.f., and \( \psi \) is an isomorphism. So \( \Psi \) is an isomorphism of \( N \cong FT(N) \).

And for \( \psi \), because \( B \otimes_A M \xrightarrow{\psi} N \) is an isomorphism, we check

\[
\psi((bm) \otimes b') = (b \otimes b')\psi(m \otimes 1) = (b \otimes b')(1 \otimes m) = b \otimes (bm).
\]

**Remark (V.1.5.11).** In fact, a descent datum is always effective iff \( A \to B \) is universally injective. Cf.[Sta]. And f.f. extension is u.i.(I.7.1.23).

**Prop. (V.1.5.12) (fpqc Descent for Qco Sheaves).** Let \( S \) be a scheme, then the fibered category \( \text{Qco}/\text{Sch} \) is a stack over \( (\text{Sch}/S) \) w.r.t. the fpqc topology.
Proof: We use \((V.1.5.7)\), the first condition is satisfied by \((V.1.5.3)\), and for the second condition, let \(\text{Spec } B \to \text{Spec } A\) be a f.f. morphism of affine schemes, then \(QCoh(\text{Spec } B \to \text{Spec } A)\) is just equivalent to \(\text{Mod}_{A \to B}\), and \(QCoh(\text{Spec } A)\) is equivalent to \(\text{Mod}_A\), so the conclusion follows from \((V.1.5.10)\). □

Cor. \((V.1.5.13)\). For any \(Qco\) sheaf \(\mathcal{F}\) on \(S\), the functor \((\text{Sch} \downarrow S)^{op} \to \text{Ab}: T \to \Gamma(T, f^* \mathcal{F})\) is a sheaf in the fpqc topology, hence also in the fppf, étale Zariski topology.

Proof: Because this functor is just \(\text{Hom}_S(\mathcal{O}_S, \mathcal{F})\), and it is a sheaf \((V.1.4.6)\). □

Prop. \((V.1.5.14)\) (Descending Affine Morphisms). For a scheme \(S\), let \(P\) be the class of affine arrows in \(\text{Sch} \downarrow S\) that denote by \(\text{Aff}_S\) the resulting fibered category, then it is a stack over \(\text{Sch} \downarrow S\) in the fpqc topology.

Proof: Firstly \(\text{Aff}_S\) is a stack over \((\text{Sch} \downarrow S)_{\text{Zar}}\), and it satisfies the affine fpqc descent condition of \((V.1.5.7)\) (Notice the \(\{n|\psi(n \otimes 1) = 1 \otimes n\}\) is now a ring, because \(\psi\) is a ring homomorphism), so we are done by \((V.1.5.7)\). □

Cor. \((V.1.5.15)\). If \(P\) is a subclass of affine arrows stable under base change and fpqc local on the target, then \(P \downarrow S\) is a stack over \((\text{Sch} \downarrow S)_{\text{fpqc}}\).

Prop. \((V.1.5.16)\) (Descent via Ample Invertible Sheaves). Let \(S\) be a scheme and \(\mathcal{F}\) be a class of flat proper morphisms of f.p. in \(\text{Sch} \downarrow S\) that is local in the fpqc topology. Assume that for each object \(\xi: X \to U\) of \(\mathcal{F}\), we have an invertible sheaf \(\mathcal{L}_\xi\) on \(X\) that is ample relative to \(X \to U\), and for an arrow \(f: (X, \xi) \to (Y, \eta V)\), we have an isomorphism \(\rho_f: f^* \mathcal{L}_\eta \cong \mathcal{L}_\xi\) that satisfies \(\rho_{gf} = \rho_f \circ f^* \rho_g\), then \(\mathcal{F}\) is a stack in the fpqc topology.

Proof: Cf.\([Vistoli, P96]\). □

Étale Descent

Prop. \((V.1.5.17)\) (Galois Descent). Let \(L/K\) be a finite separable field extension with Galois group \(G\), then \(\text{Spec } L \to \text{Spec } K\) is a \(G\)-torsor in the étale topology. So Galois descent is a special case of étale descent along torsors \((V.1.4.16)\).

Notice this is also true for arbitrary finite separable field extensions with continuity condition added, because we can take direct limits of categories over its finite normal subextensions.

Proof: Consider the locally constant group scheme \(G = \text{Spec}(\prod_{g \in G} K)\), let \(X = \text{Spec } L, Y = \text{Spec } K\), then \(\{X \to Y\}\) is an étale cover, and the action \(G \times X \to X\) is given by

\[
L \to \prod_{g \in G} L : x \mapsto \prod_{g \in G} (g(x)).
\]

Thus \(X \to Y\) is a \(G\)-equivariant map, and there is an isomorphism \(G \times X \cong X \times_Y X : (g, x) \mapsto (gx, x)\) that corresponds to the isomorphism

\[
L \otimes_K L \cong \prod_{g \in G} L : (a, b) \mapsto \prod_{g \in G} (g(a)b)
\]

□
Cor. (V.1.5.18) (Galois Descent of Closed Immersions). Let $X$ be a scheme over a field $k$, and $K/k$ be a Galois field extension with Galois group $\Gamma$, $X' = X \otimes_k K$. Then the category of closed subschemes of $X$ is equivalent to the category of closed subschemes that is base change from some $X_{k'}$ where $k'/k$ is finite that is stable under the action of $\Gamma$. This weird finiteness condition can be removed when $X$ is algebraic over $k$.

Proof: This is because the class of closed immersions is a stack (V.1.5.15).

Remark (V.1.5.19). When $Y$ is a subvariety of $X$ and $K = k^s$, to check $Y$ is stable under action of $\Gamma$, it suffices to check that the geometric points is closed under action of $\Gamma$. This is because the geometric points are dense in $Y$ (V.8.1.10).

Cor. (V.1.5.20) (Galois Descent for Ideals of Algebras). Let $A$ be a scheme over a field $k$, and $K/k$ be a Galois field extension with Galois group $\Gamma$, then the category of ideals of $A$ is equivalent to the category of ideals of $A_K$ that is base change from some $A_{k'}$ where $k'/k$ is finite that is stable under the action of $\Gamma$. This weird finiteness condition can be removed when $A$ is f.g. over $k$.

Cor. (V.1.5.21) (Galois Descent of Morphisms). Let $X,Y$ be schemes over $k$ and $K/k$ a Galois field extension with Galois group $\Gamma$, $X' = X \otimes_k K, Y' = Y \otimes_k K$. If $Y$ is separated, then a morphism $\varphi' : X' \to Y'$ arises from a morphism $X \to Y$ iff its graph $\Gamma \varphi' \subset X' \times_k Y'$ is stable under action of $\Gamma$. In this case $\varphi$ is unique.

And when $X,Y$ are varieties and $K = k^s$, then it suffices to check the map

$$\varphi'(k^s) : X'(k^s) \to Y'(k^s)$$

commutes with action of $\Gamma$.

Cor. (V.1.5.22) (Galois Descent for Qco Sheaves). Let $K/k$ be a Galois extension with Galois group $\Gamma$ and $X$ be a scheme over $k$ with $X' = X \otimes_k K$, then $Qch/X \to (Qch/X')^\Gamma$ is an equivalence of categories.

Proof: This is because $Qco/Sch$ is a fpqc stack (V.1.5.12).

Cor. (V.1.5.23) (Galois Descent for Vector Spaces). Let $K/k$ be a Galois extension with Galois group $\Gamma$. Then the functor $V \mapsto K \otimes_k V$ induces an equivalence between the category of vector spaces over $k$ and the category of vector spaces over $K$ together with a continuous semi-linear action of $\Gamma$.

Prop. (V.1.5.24) (Failure of Étale Descent for proper smooth morphisms). Cf.[?]P107.

Properties of Morphisms Local in the Fpqc Topology

Prop. (V.1.5.25) (Morphisms local in Fpqc Topology). The following properties of morphisms are local on the target w.r.t. the fpqc topology.

1. quasi-compact.
2. (quasi-)separated.
3. Universally closed.
4. universally open.
5. universally submersive.
6. surjective.
7. universally injective.
8. universally homeomorphism.
9. (locally)of f.t.
10. (locally)of f.p.
11. properness.
12. flatness.
13. (closed/open)immersion.
14. isomorphism/monomorphism.
15. (quasi-)affine.
16. quasi-compact immersion.
17. integral
18. (locally)(quasi-)finite.
19. syntomic.
20. smooth, unramified, étale.
21. finite locally free.

Proof: Cf. [Sta]34.20.
V.2 Ringed Topoi, Ringed Sites, Ringed $G$-Spaces and Schemes

1 Ringed Topoi, Ringed Sites

Def. (V.2.1.1) (Ringed Topoi). A ringed topos is a pair $(\mathcal{C}, \mathcal{O})$ where $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$, called the structure sheaf. A morphism of ringed topoi $(f,f^\#) : (\mathcal{C}, \mathcal{O}) \to (\mathcal{C}', \mathcal{O}')$ consists of a morphism of topoi (V.1.2.17) $f : \mathcal{C} \to \mathcal{C}'$ and a map of sheaves of rings $f^\# : f^{-1}\mathcal{O}' \to \mathcal{O}$. A composition of morphisms of topoi is defined to be $(g,g^\#) \circ (f,f^\#) = (g \circ f, f^\# \circ f^{-1}(g^\#))$.

Def. (V.2.1.2) (Ringed Sites). A ringed site is a pair $(\mathcal{C}, \mathcal{O})$ where $\mathcal{C}$ is a site and $\mathcal{O}$ is a sheaf of rings on $\mathcal{C}$, called the structure sheaf. A morphism of ringed sites is a morphism of sites that the induced morphism of topoi (V.1.2.18) is a morphism of ringed topoi (V.2.1.1).

A site is naturally a ringed site where $\mathcal{O} = \mathbb{Z}$ the constant sheaf (V.1.2.7). So we only consider ringed sites afterwards, then a morphism of ringed sites is naturally a morphism of ringed sites. So whenever we say $\mathcal{C}$ is a site, it is understood as a ringed site $(\mathcal{C}, \mathbb{Z})$.

Prop. (V.2.1.3) (Multiplicative Structure Sheaf). Given a ringed topos $(\mathcal{C}, \mathcal{O})$, the presheaf $U \mapsto \mathcal{O}^\ast(U)$ is a sheaf of groups, called the multiplicative structure sheaf $\mathcal{O}^\ast$.

Proof: This comes from the sheaf property of $\mathcal{O}$ and the fact the inverse of an element is unique. □

Def. (V.2.1.4) (Local Ringed Site). A ringed site $(\mathcal{C}, \mathcal{O})$ is called a local ringed site if

$$\mathcal{O}^\sharp \to \text{Equalizer}(0, 1 : \text{pt} \to \mathcal{O})$$

is an isomorphism of sheaves, and for any $U \in \mathcal{C}$ and $f \in \mathcal{O}(U)$, there exists a covering $\{U_i \to U\}$ s.t. for any $j$, either $f|_{U_i}$ is invertible or $(1 - f)|_{U_i}$ is invertible.

Prop. (V.2.1.5) (Characterizing Local Ringed Sites). Let $(\mathcal{C}, \mathcal{O})$ be a ringed site, the following are equivalent:

- $(\mathcal{C}, \mathcal{O})$ is a local ringed site.
- For any $U \in \mathcal{C}$, $f_1, \ldots, f_n \in \mathcal{O}(U)$ that $(f_1, \ldots, f_n) = (1)$, there is a covering $\{U_i \to U\}$ that for each $j$, there exists an $i$ that $f_i$ is invertible on $U_j$.
- The map of sheaves of sets:

$$\mathcal{O} \otimes \mathcal{O} \coprod (\mathcal{O} \times \mathcal{O}) \to \mathcal{O} \times \mathcal{O}$$

which maps $(f, a)$ in the first component to $(f, af)$ and $(f, b)$ in the second component to $(f, b(1 - f))$ is surjective.

Proof: Cf.[Sta]04ES. □

Def. (V.2.1.6) (Local Ringed Topoi). If $f : \mathcal{C}' \to \mathcal{C}$ is a morphism of topoi and $(\mathcal{C}, \mathcal{O})$ is a local ringed site, then $(\mathcal{C}', f^{-1}\mathcal{O})$ is also a local ringed site. In particular, being a local ringed site is an intrinsic property, so we can define local ringed topos to be ringed topos that the underlying ringed sites are local ringed.
Proof: Because $f^{-1}$ is exact (V.1.2.17) and commutes with products and equalizers, it maps the isomorphism
$$\mathcal{O}^* \to \text{Equalizer}(0, 1 : \text{pt} \to \mathcal{O})$$
to the corresponding isomorphism of $\mathcal{C}'$, and also the sejection of
$$(\mathcal{O} \otimes \mathcal{O}) \coprod (\mathcal{O} \times \mathcal{O}) \to \mathcal{O} \times \mathcal{O}$$
in (V.2.1.5) to that of $\mathcal{C}'$, thus $(\text{Sh}(\mathcal{C}'), f^{-1}\mathcal{O})$ is also a local ringed site. □

Def. (V.2.1.7) (Morphisms of Local Ringed Topoi). A morphism of local ringed topoi is a morphism of ringed topoi $(f, f^\#) : (\text{Sh}(\mathcal{C}), \mathcal{O}_C) \to (\text{Sh}(\mathcal{D}), \mathcal{O}_D)$ that the diagram of sheaves
$$\begin{array}{ccc}
f^{-1}(\mathcal{O}_D^*) & \xrightarrow{f^\#} & \mathcal{O}_C^* \\
\downarrow & & \downarrow \\
f^{-1}(\mathcal{O}_D) & \xrightarrow{f^\#} & \mathcal{O}_C
\end{array}$$
is Cartesian, where $\mathcal{O}_C^*$ is the multiplicative structure sheaf (V.2.1.3). Morphisms of local ringed topoi are stable under compositions.

Ringed Spaces

Def. (V.2.1.8) (Ringed Spaces). A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space $X$ and a sheaf of rings $\mathcal{O}_X$ on $X$. A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a pair $(f, f^\#)$ where $f$ is a continuous map $X \to Y$ and $f^\#$, which induces a map of sites $(f^{-1}, f^\#) : X_{\text{Zar}} \to Y_{\text{Zar}}$ (V.2.6.7), and $f^\#$ is a map $f^\# : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ (V.2.6.2), such that $(f, f^\#)$ is a map of ringed topoi (V.2.1.1).

Def. (V.2.1.9) (Local Ringed Space). A local ringed space is a topological space $X$ with a sheaf of rings $\mathcal{O}_X$ that $(X, \mathcal{O}_X)$ forms a local ringed site (V.2.1.4). A morphism of local ringed space is a morphism of ringed spaces that the corresponding morphism of ringed topoi $(\text{Sh}(\mathcal{C}), \mathcal{O}_C) \to (\text{Sh}(\mathcal{D}), \mathcal{O}_D)$ is a morphism of local ringed topos (V.2.1.7).

Prop. (V.2.1.10). A ringed space $(X, \mathcal{O}_X)$ is a locally ringed space iff any stalks $\mathcal{O}_{X,x}$ is either 0 or a local rings.

Proof: Cf. [Sta]04ET. □

2 Modules on Ringed Topoi

Main References are [Sta]Chap 18.

Def. (V.2.2.1) (Modules on Ringed Topoi). Let $(\text{Sh}(\mathcal{C}), \mathcal{O})$ be a ringed topos, then a sheaf of $\mathcal{O}$-modules is a presheaf of $\mathcal{O}$-modules that the underlying presheaf of Abelian groups is a sheaf.

Def. (V.2.2.2) (Support). Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{F}$ a $\mathcal{O}_X$-module, then the support of $\mathcal{F}$ is the set of points $x$ that $\mathcal{F}_x \neq 0$. It is denoted by $\text{Supp}(\mathcal{F})$. For a section $s \in \Gamma(X, \mathcal{F})$, $\text{Supp}(s)$ is defined to be the set of points $x$ that $s_x \neq 0 \in \mathcal{F}_x$.

Prop. (V.2.2.3). Glueing sheaves is available for ringed spaces, similar to (V.1.5.3).
Prop. (V.2.2.4). Glueing ringed spaces is available.

Proof: □

Def. (V.2.2.5) (Local Properties of Modules). On a ringed site \((\mathcal{C}, \mathcal{O})\), an \(\mathcal{O}\)-module \(\mathcal{F}\) is called locally has property \(P\) if for any object \(U \in \mathcal{C}\), there is a covering \(\{U_i \to U\}\) that \(\mathcal{F}|_{U_i}\) has property \(P\).

Def. (V.2.2.6) (Intrinsic Notions of Modules). Cf.\([Sta]03DG\).

Def. (V.2.2.7) (Tensor Products Sheaf). Let \((\mathcal{C}, \mathcal{O})\) be a ringed site and \(\mathcal{F}, \mathcal{G}\) be \(\mathcal{O}\)-modules, then the tensor product \(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}\) is defined to be the shifification of the presheaf \(U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{G}(U)\).

The tensor product is an intrinsic notion\((V.2.2.6)\), so it can be defined on any ringed topoi.

Prop. (V.2.2.8) (Base Change). Let \((\mathcal{C}, \mathcal{O})\) be a ringed site and \(\mathcal{O}_2\) be a sheaf \(\mathcal{O}_1\)-algebras, \(\mathcal{G}\) be a sheaf of \(\mathcal{O}_1\)-module and \(\mathcal{F}\) a sheaf of \(\mathcal{O}_2\)-module, then
\[
\text{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \text{Hom}_{\mathcal{O}_2}(\mathcal{G} \otimes_{\mathcal{O}_1} \mathcal{O}_2, \mathcal{F}).
\]

Proof: This can be seen from the definition of tensor product and the fact shifification doesn’t bother because \(\mathcal{F}\) is a sheaf. □

Def. (V.2.2.9) (Transfer of Modules on Ringed Topoi). Let \((f, f^\sharp)\) be a morphism of ringed topoi \((\text{Sh}(\mathcal{C}), \mathcal{O}) \to (\text{Sh}(\mathcal{C}'), \mathcal{O}')\), then there are functor:

- the pushforward: \(f_* : \text{Mod}(\mathcal{O}) \to \text{Mod}(\mathcal{O}') : f_* \mathcal{F} = f_* \mathcal{F}\) as a \(\mathcal{O}'\)-module via \(\mathcal{O}' \to f_* \mathcal{O}\).
- the pullback \(f^* : \text{Mod}(\mathcal{O}') \to \text{Mod}(\mathcal{O}) : f^* \mathcal{G} = \mathcal{O} \otimes_{f^{-1} \mathcal{O}' \mathcal{O}} f^{-1} \mathcal{G}\) via \(f^{-1} \mathcal{O}' \to \mathcal{O}\).

Then \(f^*\) is left adjoint to \(f_*\), by the adjointness of \(f^{-1}\) and \(f_* (V.1.2.17)\).

Prop. (V.2.2.10) (Tensor Products and Pullbacks). Tensor products commute with pullbacks, in particular with taking stalks. So tensoring with a locally free sheaf is exact.

Proof: By\((V.2.3.2)\) and\((V.2.3.4)\), We have
\[
\text{Hom}(f^* \mathcal{F} \otimes f^* \mathcal{G}, \mathcal{H}) = \text{Hom}(\mathcal{F}, f_* \text{Hom}(f^* \mathcal{G}, \mathcal{H})) = \text{Hom}(\mathcal{F}, \text{Hom}(\mathcal{G}, f_* \mathcal{H})) = \text{Hom}(f^*(\mathcal{F} \otimes \mathcal{G}), \mathcal{H}).
\]

Prop. (V.2.2.11) (Properties of Tensor Products). Let \((\mathcal{C}, \mathcal{O})\) be a ringed site and \(\mathcal{F}, \mathcal{G}\) be sheaves of \(\mathcal{O}\)-modules, then

1. If \(\mathcal{F}, \mathcal{G}\) are locally free, then so does \(\mathcal{F} \otimes \mathcal{G}\).
2. If \(\mathcal{F}, \mathcal{G}\) are locally finite free, then so does \(\mathcal{F} \otimes \mathcal{G}\).
3. If \(\mathcal{F}, \mathcal{G}\) are locally generated by sections, then so does \(\mathcal{F} \otimes \mathcal{G}\).
4. If \(\mathcal{F}, \mathcal{G}\) are of f.t., then so does \(\mathcal{F} \otimes \mathcal{G}\).
5. If \(\mathcal{F}, \mathcal{G}\) are quasi-coherent., then so does \(\mathcal{F} \otimes \mathcal{G}\).
6. If \(\mathcal{F}, \mathcal{G}\) are of f.p., then so does \(\mathcal{F} \otimes \mathcal{G}\).
7. If \(\mathcal{F}\) is of f.p. and \(\mathcal{G}\) is coherent, then \(\mathcal{F} \otimes \mathcal{G}\) is coherent. In particular, if \(\mathcal{F}, \mathcal{G}\) are coherent, then so does \(\mathcal{F} \otimes \mathcal{G}\)(V.2.2.24).

Proof: Cf.\([Sta]03L6\). □
**Flat Modules**

**Def. (V.2.2.12) (Flat Modules and Flat Morphisms).** Let $\mathcal{C}$ be a site and $\mathcal{O}$ a presheaf of rings, then a presheaf $\mathcal{F}$ of $\mathcal{O}$-modules is called flat if the functor $P\text{Mod}(\mathcal{O}) \to P\text{Mod}(\mathcal{O}) : \mathcal{G} \mapsto \mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$ is exact.

Let $\mathcal{C}$ be a ringed site, and $\mathcal{F}$ a sheaf of $\mathcal{O}$-modules, then it is called flat if the functor $\text{Mod}(\mathcal{O}) \to \text{Mod}(\mathcal{O}) : \mathcal{G} \mapsto \mathcal{G} \otimes_{\mathcal{O}} \mathcal{F}$ is exact.

A morphism $(f, f^\flat)$ is called a flat morphism if the ring map $f^\flat : f^{-1}\mathcal{O}' \to \mathcal{O}$ is flat, or equivalently, the pullback functor $f^\ast$ is exact.

If $(f, f^\flat)$ is a morphism of ringed topoi $(\text{Sh}(\mathcal{C}), \mathcal{O}) \to (\text{Sh}(\mathcal{C}'), \mathcal{O}')$, and $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules, then $\mathcal{F}$ is flat over $(\text{Sh}(\mathcal{C}'), \mathcal{O}')$ if $\mathcal{F}$ is flat over $f^{-1}\mathcal{O}'$.

**Prop. (V.2.2.13).** Let $\mathcal{C}$ be a site and $\mathcal{O}$ a presheaf of rings with shifification $\mathcal{O}^\sharp$.

- If $\mathcal{F}$ is a presheaf of $\mathcal{O}$-modules that each $\mathcal{F}(U)$ is flat $\mathcal{O}(U)$-modules, then $\mathcal{F}$ is flat.
- If $\mathcal{F}$ is a flat presheaf of $\mathcal{O}$-modules, then $\mathcal{F}^\sharp$ is a flat $\mathcal{O}^\sharp$-modules.
- A filtered colimits of flat presheaves of modules is flat. A direct sum of flat presheaves of modules is flat.
- A filtered colimits of flat sheaves of modules is flat. A direct sum of flat sheaves of modules is flat.

**Prop. (V.2.2.14) (Flatness is Stalkwise).** Let $(X, \mathcal{O}_X)$ be a ringed space, then an $\mathcal{O}_X$-module $\mathcal{F}$ is flat iff the stalks $\mathcal{F}_x$ are all flat $\mathcal{O}_{X,x}$-modules.

**Proof:** Cf. [Sta]05NE. □

**Prop. (V.2.2.15) (Flat Morphism and Support).** Let $f : (X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ be a flat morphism of local ringed spaces, $\mathcal{F}$ an $\mathcal{O}_{X'}$-module, then $\text{Supp}(f^\ast(\mathcal{F})) = f^{-1}(\text{Supp}(\mathcal{F}))$.

**Proof:** Use the fact flat ring map of local rings is faithfully flat. □

**Modules of Finite Type & Finite Presentation**

**Def. (V.2.2.16) (Finite Type).** An $\mathcal{O}$-module is called finite type iff locally a quotient of a finite free sheaf.

**Prop. (V.2.2.17) (Extension of F.T. Sheaves).** if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of $\mathcal{O}$-modules and $\mathcal{F}_1, \mathcal{F}_3$ are of f.t., then $\mathcal{F}_2$ is of f.t.

**Proof:** For any $U \in \mathcal{C}$, choose a covering $\{U_i \to U\}$ that $\mathcal{F}_3(U_i)$ is generated by f.m. sections, then by passing to a covering, we may assume these sections come from $\mathcal{F}_2$. Pass to another covering that $\mathcal{F}_1$ is generated by f.m. sections, then on this covering, $\mathcal{F}_2$ is generated by f.m. sections. □

**Def. (V.2.2.18) (Finite Presentation).** A sheaf of modules $\mathcal{F}$ is called of finite presentation iff locally it is a cokernel of two finite free modules. The pullback of a f.p. sheaf is f.p, by the left adjointness of $f^\ast$.

**Prop. (V.2.2.19) (FP-FT-FT).** If $f : \mathcal{G} \to \mathcal{F}$ is a surjection of $\mathcal{O}$-modules, $\mathcal{F}$ is of f.p. and $\mathcal{G}$ is of f.t., then the kernel is of finite type.
Proof: We first show for \( G = \mathcal{O} \): By pass to covering, we can construct a diagram

\[
\begin{array}{c}
\mathcal{O}_{U_{ij}}^n \longrightarrow \mathcal{O}_{U_{ij}}^n \longrightarrow \mathcal{F}|_{U_{ij}} \longrightarrow 0 \\
\downarrow \alpha \quad \quad \downarrow \quad \downarrow \\
0 \longrightarrow \ker(f)|_{U_{ij}} \longrightarrow \mathcal{G}|_{U_{ij}} \longrightarrow \mathcal{F}|_{U_{ij}} \longrightarrow 0
\end{array}
\]

and then use snake lemma. The image and cokernel of \( \alpha \) are all of f.t., then \( \ker(f)|_{U_{ij}} \) is of f.t. by (V.2.2.17).

For general \( G \), locally choose a surjection \( \varphi : \mathcal{O}_{U_i}^n \rightarrow G|_{U_i} \), then \( \ker(f|_{U_i}) = \varphi(\ker(\varphi \circ f|_{U_i})) \), which is of f.t.. \( \square \)

**Prop. (V.2.2.20).** Pullbacks of a module of finite type is of finite type. Pullback of a module of finite presentation is of finite presentation.

Finite type and finite presentation are local on the target.

Proof: This is because pullback is a left adjoint thus right exact. They are local on the target because they are defined locally. \( \square \)

**Prop. (V.2.2.21).** If \( f : \mathcal{G} \rightarrow \mathcal{F} \) is surjective at the stalk at a point \( x \) and \( \mathcal{F} \) is of f.t., then it is surjective on a nbhd of \( x \). Thus the support of a f.t. sheaf is closed (look at \( 0 \rightarrow \mathcal{F} \)).

Proof: Choose a nbhd of \( x \) that \( \mathcal{F}(U) \) is generated by \( s_1, \ldots, s_n \in \mathcal{F}(U) \), because \( f \) is surjective at the stalk of \( x \), after shrinking \( U \), we may assume \( s_i = f(t_i) \) for \( t_i \in \mathcal{G}(U) \), so \( f \) is surjective on \( U \). \( \square \)

**Quasi-Coherent Sheaves**

**Def. (V.2.2.22) (Quasi-Coherent Sheaf).** An \( \mathcal{O} \)-module \( \mathcal{F} \) on a ringed site is called \( \text{quasi-coherent} \) iff locally it is a cokernel of two free modules. A locally f.p. sheaf of modules is \( \text{Qco} \).

**Prop. (V.2.2.23) (Associated Qco Sheaves).** And for a ringed space \( (X, \mathcal{O}_X) \) and a \( R = \Gamma(X, \mathcal{O}_X) \)-module \( M \), we have a coherent sheaf \( \mathcal{F}_M \) on \( X \), defined as \( \pi^*(M) \), where \( M \) is seen as a qco sheaf on \( (\text{pt}, R) \). It is the sheaf associated to the presheaf \( U \mapsto \mathcal{O}_X(U) \otimes M \).

This construction is a functor from the category of \( R \)-module to the category of Qco \( \mathcal{O}_X \)-modules, and it commutes with colimits because \( \pi^* \) does. And it is left adjoint to \( \Gamma \) by (V.2.2.9):

\[
\text{Hom}_A(M, \Gamma(X, \mathcal{G})) \cong \text{Hom}_{\mathcal{O}_X}(\mathcal{F}_M, \mathcal{G})
\]

**Def. (V.2.2.24) (Coherent Sheaves).** A \( \text{coherent sheaf} \) is a \( \mathcal{O} \)-module \( \mathcal{F} \) that is of f.t. and for any object \( U \) and for any set of elements of \( \Gamma(U, \mathcal{F}) \), the kernel of \( \oplus \mathcal{O}_U \rightarrow \mathcal{F}|_U \) is of f.t..

A coherent sheaf is of f.p., by base change to a smaller covering, and it is \( \text{Qco} \).

**Prop. (V.2.2.25) (Properties of Coherent Sheaves).** Any f.t. subsheaf of a coherent sheaf is coherent, by definition. Any kernel of a morphism from a f.t. sheaf to a coherent sheaf is of f.t..

The category of coherent sheaves is a weak Serre subcategory of \( \mathcal{O}_X \)-modules. In particular, if \( \mathcal{O}_X \) is coherent, then a sheaf is coherent iff it is f.p.
Prop. (V.2.2.26). The pullback of a (quasi-)coherent module is a (quasi-)coherent, because \( f^* \) is a left adjoint.

Prop. (V.2.2.27). (Quasi-)Coherence is local on the target.

Proof: This is because they are defined locally. □

Prop. (V.2.2.28). Let \( (X, \mathcal{O}_X) \) be a ringed space and \( x \in X \).

- Let \( f : \mathcal{G} \to \mathcal{F} \) be a map of \( \mathcal{O}_X \)-modules. If \( \mathcal{G} \) is of f.t. and \( \mathcal{F} \) is coherent, and \( f \) is injective at the stalk of \( x \), then there exists a nbhd \( U \) of \( x \) that \( f|_U \) is injective.

- Let \( f : \mathcal{G} \to \mathcal{F} \) be a map of coherent \( \mathcal{O}_X \)-modules that is surjective at the stalk of \( x \), then there exists a nbhd \( U \) of \( x \) that \( f|_U \) is surjective.

- Let \( f : \mathcal{G} \to \mathcal{F} \) be a map of coherent \( \mathcal{O}_X \)-modules that is isomorphism at the stalk of \( x \), then there exists a nbhd \( U \) of \( x \) that \( f|_U \) is an isomorphism.

Proof: 1: Consider the kernel of \( f \), then it is of f.t. by definition(V.2.2.24). Then \( \text{Ker}(f)_x = 0 \), so there is a nbhd \( U \) of \( x \) that \( \text{Ker}(f)|_U = 0 \), by(V.2.2.21), which means \( f|_U \) is injective.

2: this is immediate from(V.2.2.21).

3 follows from 1 and 2. □
3 Construction of Sheaves

Internal Hom

Def. (V.2.3.1) (Internal Hom). Let \( C \) is a category and \( \mathcal{O} \) is a presheaf of rings, \( \mathcal{F}, \mathcal{G} \) be presheaves of \( \mathcal{O} \)-modules, then \( U \mapsto \text{Hom}_{\mathcal{O}|_U}(\mathcal{F}|_U, \mathcal{G}|_U) \) defines a presheaf \( \text{Hom}(\mathcal{F}, \mathcal{G}) \) of \( \mathcal{O} \)-modules, and there is a natural evaluation map

\[
\mathcal{F} \otimes_{\mathcal{O}} \text{Hom}(\mathcal{F}, \mathcal{G}) \to \mathcal{G}.
\]

Now if \( C \) is a site and \( \mathcal{O} \) is a sheaf of rings, \( \mathcal{F}, \mathcal{G} \) be sheaves of \( \mathcal{O} \)-modules, then \( \text{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \) is a sheaf of \( \mathcal{O} \)-modules.

Proof: We need to check the sheaf properties. Let \( \{U_i \to U\} \) be a covering of \( C \), for any morphism \( V \to U \), let \( V_i = U_i \times_U V \), then \( (U_i \times_U U_j) \times_U V = V_i \times_V V_j \). Now if there are \( \varphi_i \in \text{Hom}_{\mathcal{O}|_{U_i}}(\mathcal{F}|_{U_i}, \mathcal{G}|_{U_i}) \) that there restriction in \( \text{Hom}_{\mathcal{O}|_{U_i \times_U U_j}}(\mathcal{F}|_{U_i \times_U U_j}, \mathcal{G}|_{U_i \times_U U_j}) \) are compatible, then we can define \( \varphi(V) : \mathcal{F}(V) \to \mathcal{G}(V) \) from the diagram:

\[
\begin{array}{ccc}
\mathcal{F}(V) & \to & \prod \mathcal{F}(V_i) \\
\downarrow \varphi & & \downarrow \prod \varphi_i \\
\mathcal{G}(V) & \to & \prod \mathcal{G}(V_i)
\end{array}
\]

and it can be verified that \( \varphi(V) \) is functorial in \( V \), thus defines an element in \( \text{Hom}_{\mathcal{O}|_U}(\mathcal{F}|_U, \mathcal{G}|_U) \). The verification of uniqueness is omitted. \( \square \)

Prop. (V.2.3.2) (Tensoring and Inner Hom). If \( C \) is a site, \( \mathcal{O} \) is a sheaf of rings and \( \mathcal{F}, \mathcal{G}, \mathcal{H} \) are sheaves of \( \mathcal{O} \)-modules, then there is a canonical isomorphism of sheaves:

\[
\text{Hom}_{\mathcal{O}}(\mathcal{H} \otimes_{\mathcal{O}} \mathcal{F}, \mathcal{G}) \cong \text{Hom}_{\mathcal{O}}(\mathcal{F}, \text{Hom}_{\mathcal{O}}(\mathcal{H}, \mathcal{G})).
\]

In particular, taking limit over \( C \), we see \( - \otimes_{\mathcal{O}} \mathcal{H} \) is left adjoint to \( \text{Hom}_{\mathcal{O}}(\mathcal{F}, -) \):

\[
\text{Hom}_{\mathcal{O}}(\mathcal{H} \otimes_{\mathcal{O}} \mathcal{F}, \mathcal{G}) \cong \text{Hom}_{\mathcal{O}}(\mathcal{F}, \text{Hom}_{\mathcal{O}}(\mathcal{H}, \mathcal{G})).
\]

In particular, the monoidal category of \( \mathcal{O}_X \)-modules is closed (V.2.2.12).

Proof: Omitted (Recall the definition of tensor product sheaf (V.2.2.7)). \( \square \)

Prop. (V.2.3.3).

\[
\text{Hom}(\varinjlim A_i, B) \cong \varprojlim \text{Hom}(A_i, B) \quad \text{Hom}(A, \varprojlim B_i) \cong \varinjlim \text{Hom}(A, B_i)
\]

Proof: This is immediate from (II.1.4.8). \( \square \)

Prop. (V.2.3.4). \( f^* \) is left adjoint to \( f_* \) by (V.2.2.9): \( \text{Hom}_{\mathcal{O}_X}(f^* \mathcal{G}, \mathcal{F}) \cong \text{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_* \mathcal{F}) \). But in fact

\[
f_*(\text{Hom}_{\mathcal{O}}(f^* \mathcal{G}, \mathcal{F})) \cong \text{Hom}_{\mathcal{O}}(\mathcal{G}, f_* \mathcal{G}).
\]

by checking on every open subset \( U \subset Y \).

Remark (V.2.3.5). The \( f^* \) may not be exact. \( f^{-1} \) is exact, but we tensored with \( \mathcal{O}_X \), it is exact when \( f \) is flat (V.2.2.12).
**Prop. (V.2.3.6).** Let $f^*: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces, and $\mathcal{F}, \mathcal{G}$ are $\mathcal{O}_Y$-modules. If $\mathcal{F}$ is f.p. and $f$ is flat, then the canonical map
\[ f^* \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{F}, f^*\mathcal{G}) \]
is an isomorphism.

**Proof:**
\[ □ \]

**Prop. (V.2.3.7).** Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. If $\mathcal{F}$ is f.p. or locally free, then the canonical map
\[ \mathcal{H}om(\mathcal{F}, \mathcal{G})_x \to \mathcal{H}om_{\mathcal{O}_X,x}(\mathcal{F}_x, \mathcal{G}_x) \]
is an isomorphism.

**Proof:** Choose a presentation of $\mathcal{F}$. This follows from the exactness of taking stalks and (V.2.3.3).
\[ □ \]

**Prop. (V.2.3.8).** Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. If $\mathcal{F}$ is f.p. and $\mathcal{G}$ is coherent, then $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is also coherent. In particular, this applies to $\mathcal{F}, \mathcal{G}$ both coherent.

**Proof:** This follows from (V.2.3.3) and (V.2.2.25).
\[ □ \]

**Tensor Sheaves**

**Def. (V.2.3.9) (Tensor Sheaves).** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site and $\mathcal{F}$ an $\mathcal{O}$-module, then we define
- $T(\mathcal{F})$ to be the sheafification of the presheaf $U \mapsto T_{\mathcal{O}_X(U)}(\mathcal{F}(U))$(I.5.1.16).
- $\wedge \mathcal{F}$ to be the sheafification of the presheaf $U \mapsto \wedge_{\mathcal{O}_X(U)}(\mathcal{F}(U))$(I.5.1.16).
- $\text{Sym}(\mathcal{F})$ to be the sheafification of the presheaf $U \mapsto \text{Sym}_{\mathcal{O}_X(U)}(\mathcal{F}(U))$(I.5.1.16).

**Cor. (V.2.3.10).** Over a ringed space $(X, \mathcal{O}_X)$, the construction of $T(\mathcal{F}), \wedge(\mathcal{F})$ and $\text{Sym}(\mathcal{F})$ commutes with taking stalks, because the construction of tensor algebras and shifification are both left adjoints(I.5.1.17). Also they commutes with pullbacks, because they satisfy the same universal properties.

**Prop. (V.2.3.11).** Let $\mathcal{F}$ be an $(\mathcal{C}, \mathcal{O})$-module, then the following properties are preserved under the construction of $T(\mathcal{F}), \wedge(\mathcal{F})$ and $\text{Sym}(\mathcal{F})$:
- Locally generated by sections.
- Finite Type.
- Finite Presented.
- Coherent.
- Quasi-coherent.
- Locally free.

**Proof:** Cf.[Sta]01CL.
4 Sheaf of Differentials

Prop. (V.2.4.1). If $\mathcal{C}$ is a site and $\mathcal{O}_1 \to \mathcal{O}_2$ is a homomorphism of sheaves of rings and $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules, then an $\mathcal{O}_1$-derivation from $\mathcal{O}_2$ to $\mathcal{F}$ is a map that for any $U \in \mathcal{C}$, the map $\mathcal{O}_2(U) \to \mathcal{F}(U)$ is an $\mathcal{O}_1(U)$-derivation (I.7.3.1).

Prop. (V.2.4.2) (Sheaf of Differentials). Let $\mathcal{C}$ be a site and $\mathcal{O}_1 \to \mathcal{O}_2$ is a homomorphism of sheaves of rings and $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules, then the functor $\text{Mod}(\mathcal{O}_2) \to \text{Ab} : \mathcal{F} \to \text{Der}_{\mathcal{O}_1}(\mathcal{O}_2, \mathcal{F})$ is representable by a sheaf of modules $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$, called the sheaf of differentials, and the map $d : \mathcal{O}_2 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is called the universal derivation.

Proof: The construction is similar to that of (I.7.3.4): if for any sheaf $\mathcal{F}$ we denote $\mathcal{O}_2[\mathcal{F}]$ generated by shifification of the presheaf $U \mapsto \mathcal{O}_2(U)[\mathcal{F}(U)]$, then $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the cokernel of $\mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_2 \times \mathcal{O}_2] \oplus \mathcal{O}_2[\mathcal{O}_1] \to \mathcal{O}_2[\mathcal{O}_2]$. □

Prop. (V.2.4.3) (Localness of Sheaf of Differentials). If $\mathcal{O}_1 \to \mathcal{O}_2$ is a homomorphism of presheaves of rings, then $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ is the sheafification of the presheaf $U \mapsto \Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)}$.

Proof: This is because the construction of $\Omega_{\mathcal{O}_2/\mathcal{O}_1}$ (I.7.3.4) for all $U$ gives an exact sequence of presheaves, and the shifification of which is just the construction in (V.2.4.2), so we are done because shifification is exact (V.1.2.6). □

Prop. (V.2.4.4) (Change of Sites). Let $f : Sh(\mathcal{D}) \to Sh(\mathcal{C})$ be a morphism of topoi and $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ a homomorphism of rings on $\mathcal{C}$, then there is a canonical isomorphism $f^{-1}\Omega_{\mathcal{O}_2/\mathcal{O}_1} \cong \Omega_{f^{-1}\mathcal{O}_2/f^{-1}\mathcal{O}_1}$ compatible with the universal derivations.

Proof: This follows from the construction (V.2.4.2) and the fact $f^{-1}$ is exact (V.2.4.2). □

Prop. (V.2.4.5) (Functoriality of $\Omega$). Let $\varphi : (\mathcal{O}_1 \to \mathcal{O}_2) \to (\mathcal{O}_1' \to \mathcal{O}_2')$ be a commutative diagram of sheaves of rings over a site $\mathcal{C}$, then the map $\mathcal{O}_2' \to \mathcal{O}_2$ composed with the derivative $\mathcal{O}_2' \to \Omega_{\mathcal{O}_2'/\mathcal{O}_1'}$ is an $\mathcal{O}_1'$-derivative, thus we obtain a map of $\mathcal{O}_2$-modules $\Omega_{\mathcal{O}_2'/\mathcal{O}_1'} \to \Omega_{\mathcal{O}_2'/\mathcal{O}_1'}$, or equivalently a map of $\mathcal{O}_2'$-modules $\Omega_{\mathcal{O}_2'/\mathcal{O}_1'} \otimes_{\mathcal{O}_2} \mathcal{O}_2' \to \Omega_{\mathcal{O}_2'/\mathcal{O}_1'}$. Thus $\Omega_{\mathcal{O}_2'/\mathcal{O}_1'}$ is a functor of arrows.

Moreover, if $\mathcal{O}_2' = \mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{O}_1'$, then this map is an isomorphism, by (V.2.4.3) and (I.7.3.6).

Prop. (V.2.4.6). Let $\mathcal{O}_1 \to \mathcal{O}_2 \to \mathcal{O}_2'$ be a map of sheaves of rings that $\mathcal{O}_2 \to \mathcal{O}_2'$ is surjective with kernel $\mathcal{I} \subset \mathcal{O}_2$, then there is a canonical exact sequence of $\mathcal{O}_2'$-modules

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathcal{O}_2/\mathcal{O}_1} \otimes_{\mathcal{O}_2} \mathcal{O}_2' \to \Omega_{\mathcal{O}_2'/\mathcal{O}_1} \to 0,$$

where the first map is characterized by mapping local sections $f$ of $\mathcal{I}$ to $df \otimes 1$.

Proof: The first map is well-defined if $d(\mathcal{I}^2) = 0$. To show the exactness, let $\mathcal{O}_2'' \subset \mathcal{O}_2'$ to be the presheaf of $\mathcal{O}_1$-algebras that $\mathcal{O}_2''(U)$ the image of $\mathcal{O}_2(U) \to \mathcal{O}_2'(U)$. Then there is an exact sequence

$$\mathcal{I}(U)/\mathcal{I}(U)^2 \to \Omega_{\mathcal{O}_2(U)/\mathcal{O}_1(U)} \otimes_{\mathcal{O}_2(U)} \mathcal{O}_2''(U) \to \Omega_{\mathcal{O}_2''(U)/\mathcal{O}_1(U)} \to 0$$

by (I.7.3.8). Now shifification of these presheaves gives use the desired result by (V.2.4.3). □

Def. (V.2.4.7) (Sheaf of Differentials). Let $(X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ be a morphism of ringed sites,

- Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, an $S$-derivation from $\mathcal{O}_X$ to $\mathcal{F}$ is a derivation over $f^{-1}\mathcal{O}_S$. The set of $S$-derivations is denoted by $\text{Ders}_S(\mathcal{O}_X, \mathcal{F})$.

- the sheaf of differentials $\Omega_{X/S}$ is defined to be a sheaf of modules $\Omega_{\mathcal{O}_X/f^{-1}\mathcal{O}_S}$ (V.2.4.2), with a universal derivation $d_{X/S} : \mathcal{O}_X \to \Omega_{X/S}$.
5 Locally Free sheaves

Prop. (V.2.5.1). Pullback of (finite)locally free sheaves are locally free.

Prop. (V.2.5.2) (Properties of Locally Free Sheaves). For a finite locally free sheaf \( \mathcal{E} \) on a ringed site \((\mathcal{C}, \mathcal{O})\), denote \( \mathcal{E}^\vee = \text{Hom}_\mathcal{O}(\mathcal{E}, \mathcal{O}) \), then:

- \( \mathcal{E}^\vee \cong \mathcal{E} \).
- \( \text{Hom}_\mathcal{O}(\mathcal{E}, \mathcal{F}) \cong \mathcal{E}^\vee \otimes \mathcal{O} \mathcal{F} \).
- \( \text{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{G}) \otimes \mathcal{H} \cong \text{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{G} \otimes \mathcal{H}) \) if \( \mathcal{F} \) or \( \mathcal{H} \) is finite locally free.
- \( \text{Hom}_\mathcal{O}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G}) \cong \text{Hom}_\mathcal{O}(\mathcal{F}, \mathcal{E}^\vee \otimes \mathcal{G}) \), by the first and (V.2.3.2).

Proof: We define the map, and verify locally, which is by \( 1 \). □

Invertible Sheaves

Def. (V.2.5.3) (Invertible Sheaf). An invertible sheaf \( \mathcal{L} \) on a ringed site \((\mathcal{C}, \mathcal{O})\) is an invertible object in the symmetric monoidal category \( \text{Mod}_\mathcal{O}(\text{II.1.4.23}) \).

Prop. (V.2.5.4). Let \((\mathcal{C}, \mathcal{O})\) be a ringed site and \( \mathcal{L} \) an \( \mathcal{O} \)-module, the following are equivalent:

- \( \mathcal{L} \) is an invertible sheaf.
- There exists some \( \mathcal{O} \)-module \( \mathcal{N} \) that \( \mathcal{L} \otimes \mathcal{O} \mathcal{N} \cong \mathcal{O} \).

And in this case, \( \mathcal{L} \) is flat and of finite presentation, and \( \mathcal{N} \cong \text{Hom}_\mathcal{O}(\mathcal{L}, \mathcal{O}) \).

Proof: \( \mathcal{L} \) is flat because tensoring \( \mathcal{L} \) is an equivalence thus exact. Let \( \psi : \mathcal{L} \otimes \mathcal{O} \mathcal{N} \cong \mathcal{O} \) the isomorphism, \( U \) an element of \( \mathcal{C} \), then by construction of \( \otimes \), after localization, we may assume there exists sections \( x_i \in \mathcal{L}(U), y_i \in \mathcal{N}(U) \) that \( \psi(x_i \otimes y_i) = 1 \). Then there is an automorphism of \( \mathcal{L}|_U : x \mapsto \sum \psi(x \otimes y_i)x_i \). This automorphism factors through

\[
\mathcal{L}|_U \to \mathcal{O}^\oplus_n \to \mathcal{L}|_U,
\]

thus \( \mathcal{L}|_U \) is a direct summand of a finite free \( \mathcal{O}_U \)-module, thus \( \mathcal{L} \) is of finite presentation.

Assume \( \mathcal{L} \) is invertible, consider the evaluation map

\[
\mathcal{L} \otimes \mathcal{O} \text{Hom}(\mathcal{L}, \mathcal{O}) \to \mathcal{O},
\]

and by (V.2.3.2),

\[
\text{Hom}(\mathcal{O}, \mathcal{O}) = \text{Hom}_\mathcal{O}(\mathcal{L} \otimes \mathcal{N}, \mathcal{O}) \to \text{Hom}_\mathcal{O}(\mathcal{N}, \text{Hom}_\mathcal{O}(\mathcal{L}, \mathcal{O})).
\]

The image of \( 1 \) gives a morphism \( \mathcal{N} \to \text{Hom}_\mathcal{O}(\mathcal{L}, \mathcal{O}) \). Tensoring \( \mathcal{L} \) gives the inverse of the evaluation map. □

Cor. (V.2.5.5). The pullback of an invertible sheaf is an invertible sheaf, because tensoring commutes with pullbacks (V.2.2.10).

Def. (V.2.5.6) (Picard Groups). For any ringed site \((\mathcal{C}, \mathcal{O})\), there is a set of invertible modules over \( \mathcal{C} \) that any invertible module is isomorphic to exactly one of them. Then this set forms an Abelian group, called the Picard group \( \text{Pic}(\mathcal{C}) \).
Prop. (V.2.5.7) (Invertible Sheaves and Locally Free Sheaves of Rank 1). If $(X, \mathcal{O}_X)$ is a ringed space, then any locally free $\mathcal{O}_X$-module of rank 1 is invertible. And when $(X, \mathcal{O}_X)$ is a local ringed space, the converse holds as well.

Proof: Assume $\mathcal{L}$ is locally free of rank 1 and consider the evaluation map (V.2.3.1)

$$\mathcal{L} \otimes_{\mathcal{O}} \text{Hom}(\mathcal{L}, \mathcal{O}) \to \mathcal{O}.$$ 

This map is an isomorphism when restricting to any trivializing covering of $\mathcal{L}$, so it is an isomorphism. Thus $\mathcal{L}$ is invertible by (V.2.5.4).

Assume $(\mathcal{S}, \mathcal{O})$ is a local ringed topos and $\mathcal{L}$ is invertible, the proof of (V.2.5.4) shows there exists a covering $\{U_i \to U\}$ that $\mathcal{L}|_{U_i}$ is a direct summand of a finite free $\mathcal{O}_{U_i}$-module. Replacing $U_i$ by $U$, let $\pi$ be the projection of $\mathcal{O}_U'$ onto $\mathcal{L}|_U$ which corresponds to a matrix with entries in $\mathcal{O}(U)$. The image of $\pi$ acting on $\mathcal{O}(U)^*$ is a finite free $\mathcal{O}(U)$-module $M$, thus there are $f_1, \ldots, f_\ell$ generating unit ideal of $\mathcal{O}(U)$ such that $M_{f_i}$ is finite free. Now by definition of local ringed topos (V.2.1.5), after replacing $U$ by a covering, we may assume $M$ is finite free, which means $\mathcal{L}|_U$ is free summand of $\mathcal{O}_U'$. But $\mathcal{L}$ is invertible, thus rank of $\mathcal{L}$ is 1. 

\[ \square \]

6 Sheaves on Spaces

Sheaves on Topological Spaces

Remark (V.2.6.1). A topological space can be regarded as a ringed space by assigning the locally constant sheaf $\mathbb{Z}$ as the structure sheaf.

Def. (V.2.6.2) (Grothendieck’s Six Operators). Let $f : X \to Y$ be a continuous map of topological spaces, then the inverse image defines a continuous map between sites $Y_{\text{Zar}} \to X_{\text{Zar}}$, so by (V.1.2.8) and (V.1.2.10) we can define

- the pushforward $f\_F$, $f\_F(U) = F(f^{-1}(U))$ sends presheaf to presheaf.
- the direct image $f_*\mathcal{F}$, $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ sends sheaf to sheaf.
- the inverse image $f\_\mathcal{G}$, $f\_\mathcal{G}(U) = \lim_{\to f(U) \subset V} \mathcal{G}(V)$ that sends presheaf to presheaf.
- the inverse image $f^{-1}\mathcal{G} = f_*\mathcal{G}$ that sends sheaf to sheaf.
- For a morphism of locally compact spaces, we can define a proper direct image:

$$f_!(\mathcal{F})(U) = \{ s \in \Gamma(f^{-1}(U), \mathcal{F}) | \text{Supp}(s) \to U \text{ proper} \}.$$ 

This is a subsheaf of $f_*\mathcal{F}$ and it is left exact. we denote $\Gamma_c(X, \mathcal{F})$ as the group $f_!(\mathcal{F})$ where $f : X \to \text{pt}$. And the stalk $f_!(\mathcal{F})_y = \Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$ Cf.[Gelfand P224 P225].
- the proper inverse image (special case) $i^!$ for a closed immersion $Z \subset X$ defined by

$$i^!(\mathcal{F})(U = V \cap Z) = \{ s \in \Gamma(V, \mathcal{F}) | \text{Supp}(s) \in Z \}.$$ 

- sends Abelian sheaves to Abelian sheaves.
- the internal tensor product.
- the internal Hom.
Proof: Check that \( f_1 \) is a sheaf: it is separated clearly, it suffices to show that for a covering \( \cup U_i = W \) and \( \xi \in F(f^{-1}(U_i)) \), the section \( \xi \in F(f^{-1}(W)) \) they generated by sheaf property of \( F \) is in \( f_1F(W) \). For a compact subset \( K \), there is a finite cover \( \cup_j U_j \) of it, thus \( K - \cup_{i \neq j} U_j \) is compact in \( U_j \), thus its inverse image is compact in \( \text{Supp}(\xi) \). there are f.m. \( U_j \), thus the inverse image of \( K \) is compact in \( \text{Supp}(\xi) \). \( \square \)

Def. (V.2.6.3) (Stalks). The convenience of rings spaces compared to the case of sites is the it has stalk functors: For any presheaf \( F \) on \( X \) and a point \( i : x \to X \), define the \textit{stalk} \( F_x = i_x(F) \) (V.2.6.2).

Prop. (V.2.6.4) (Stalks Commutes with Shiffification). Taking stalks commutes with shiffification.

Proof: Cf.[Sta]007Z. \( \square \)

Prop. (V.2.6.5). If a sheaf on a ringed space has only one non-vanishing stalk, then it is a skyscraper sheaf. (Because the restriction map to that point for every open set is an isomorphism).

Prop. (V.2.6.6) (Stalks). Taking stalks is a left adjoint to the skyscraper sheaf from Presheaves to Sets, thus it preserves cokernel. Moreover, for a map of sheaves \( F \to G \) on \( X \),
- \( \varphi \) is a monomorphism iff \( \varphi_x \) is injective for all \( x \in X \).
- \( \varphi \) is an epimorphism iff \( \varphi_x \) is surjective for all \( x \in X \).
- \( \varphi \) is an isomorphism iff \( \varphi_x \) is surjective for all \( x \in X \).

Proof: 1: If \( \varphi \) is a monomorphism, then \( \varphi_x \) is clearly. Conversely, if \( \varphi_x \) are all injective, if \( s \in F(U) \) mapsto \( 0 \in (U) \), then \( s_x \) mapsto \( 0 \in G_x \) for all \( x \), thus \( s_x = 0 \) for all \( x \), thus \( s = 0 \).
2: If \( \varphi \) is an epimorphism, then \( \varphi_x \) is surjective by definition. The converse is also true.
3: If \( \varphi_x \) is isomorphism for all \( x \), then \( \varphi \) is monomorphism by 1, and for any \( t \in G(U) \), \( t \) is locally coming from some section of \( s \), and these sections are compatible on their intersections because of monomorphism, so they glue together to a section \( s \in F(U) \) that \( \varphi(U)(s) = t \). \( \square \)

Prop. (V.2.6.7) (Topological Spaces and Sites). Let \( f : X \to Y \) be a continuous map of topological spaces, then \( f \) induces a map of sites \( X_{\text{Zar}} \to Y_{\text{Zar}} \) because \( f^{-1} \) is exact (V.1.2.13), thus induces a map of topoi \( f : Sh(X) \to Sh(Y) \) (V.1.2.18).

Cor. (V.2.6.8). Let \( f : X \to Y \) be a continuous map of topological spaces,
- Let \( G \) be a presheaf on \( Y \), then there is a canonical bijection of stalks \( (f_!(G))_x = G_{f(x)} \). If \( G \) is a sheaf on \( Y \), then there is a canonical bijection of stalks \( (f^{-1}(G))_x = G_{f(x)} \).
- \( f^{-1} \) is left adjoint to \( f_! \).
- \( f_! \) is left exact when \( X,Y \) are locally compact. And \( j_! \) is left adjoint to the functor \( f^{-1} \) for an inclusion of open subset \( j : U \subset X \).
- \( i^! \) is right adjoint to \( i_* \) for a closed immersion \( i : Z \to X \), in particular \( i_* \) is exact when \( i \) is a closed immersion.

Proof: 1: This is because \((\_)_p \) commutes with composition (V.2.6.3), and also shiffification commutes with \((\_)_p \) (V.2.6.4).
2: This is immediate.
3: 4: The adjointness follows form the fact that any section under a homomorphism \( i_*G \to F \) has support contained in \( Z \). \( \square \)
Prop. (V.2.6.9). Let \( i : Z \to X \) be a closed immersion, then the functor \( i_* : Ab(Z) \to Ab(X) \) is exact, fully faithful, with the essential image those sheaves with support in \( Z \).

Prop. (V.2.6.10) (Canonical Exact Sequence).

\[
0 \to j_*j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0
\]

Proof: Cf.[Sta]02UT. \( \square \)

Prop. (V.2.6.11). On a topological space \( X \), for a qc open subset \( U \), \((\oplus \mathcal{F}_i)(U) = \oplus \mathcal{F}_i(U)\). This uses the compactness of \( U \).

Morphisms of Local Ringed Spaces

Def. (V.2.6.12) (Open Immersion of Ringed Spaces). A morphism \( f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \) of ringed spaces is called an open immersion if \( f \) is a homeomorphism of \( X \) onto an open subset of \( Y \), and the map \( f_* : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X \) is an isomorphism.

Prop. (V.2.6.13). Let \( (X, \mathcal{O}_X) \) be a ringed space, \( U \subset X \) an open subset, and \( \mathcal{O}_U = \mathcal{O}_X|_U \) is a sheaf of rings on \( U \), then \( (U, \mathcal{O}_U) \to (X, \mathcal{O}_X) \) is an open immersion, and \( (U, \mathcal{O}_U) \) is called the open subspace associated to \( U \).

Prop. (V.2.6.14) (Universal Property of Open Immersions). Let \( f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \) be an open immersion of ringed spaces, then it has the universal property that any morphism of ringed spaces \( (T, \mathcal{O}_T) \to (Y, \mathcal{O}_Y) \) that factors set-theoretically through \( f(X) \) factors uniquely through \( (X, \mathcal{O}_X) \).

Def. (V.2.6.15) (Closed Immersion). Let \( i : Z \to X \) be a morphism of local ringed spaces, then \( i \) is called a closed immersion if:

- \( i \) is a homeomorphism of \( Z \) onto a closed subspace of \( X \).
- the map \( \mathcal{O}_X \to i_*\mathcal{O}_Z \) corresponding to \( f^* \) is surjective with kernel \( \mathcal{I} \).
- the \( \mathcal{O}_X \)-module \( \mathcal{I} \) is locally generated by sections.

And for a closed immersion, \( \mathcal{I} \) is called the ideal sheaf of \( i \).

Def. (V.2.6.16) (Closed Immersion Defined by Ideals). Let \( (X, \mathcal{O}_X) \) be a local ringed space, and \( \mathcal{I} \subset \mathcal{O}_X \) a sheaf of ideals on \( X \) locally generated by sections , Let \( Z \) be the support of the sheaf of rings \( \mathcal{O}_X/\mathcal{I} \). \( Z \) is closed in \( X \) because it is the support of 1. by(V.2.6.9), there is a unique sheaf of rings \( \mathcal{O}_Z \) on \( Z \) that \( i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I} \). For any \( z \in Z \), the stalk \( \mathcal{O}_z = \mathcal{O}_{X,z}/\mathcal{I}_z \) is a quotient of a local ring and is non-zero, thus a local ring. Then \( (Z, \mathcal{O}_Z) \) is a local ringed space and \( i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X) \) is a closed immersion, called the closed immersion defined \( \mathcal{I} \).

Prop. (V.2.6.17) (Closed Immersions are Equivalent to Ideals). Let \( f : X \to Y \) be a closed immersion of local ringed spaces with ideal sheaf \( \mathcal{I} \). Let \( i : Z \to X \) be the closed immersion defined by \( \mathcal{I} \)(V.2.6.16), then \( f \) is isomorphic to \( i \).

Proof: Because \( f_*\mathcal{O}_Z \cong \mathcal{O}_X/\mathcal{I} \) on \( X \). \( \square \)

Prop. (V.2.6.18). For a closed immersion of ringed spaces \( f, f_* \) on \( \mathcal{O}_X \)-mod is fully faithful, with image those modules annihilated by \( \mathcal{I} \), where \( \mathcal{I} \) is the structural kernel.

Proof: Cf.[Sta]08KS. \( \square \)
7 Spec and Schemes

Def. (V.2.7.1) (Spec). Given a commutative ring $A$, the spectrum of $A$ Spec $A$ is a locally ringed space whose underlying space is the set of primes of $A$, and the topology is generated by the standard open subsets $D(f) = \{ p | f \notin p \}$.

To define the structure sheaf $\mathcal{O}_X$, we first define a sheaf on the site of standard open subsets, which takes value $Rf$ on $D(f)$. This is truly a sheaf by (I.7.2.3), and then we can use (V.1.2.21) to extend this sheaf to a sheaf on Spec $A$, called the structure sheaf $\mathcal{O}_X$.

Def. (V.2.7.2) (Schemes). The category of schemes is a fully faithful category of the category of local ringed spaces (V.2.1.9) that locally isomorphic to Spec $A$.

Lemma (V.2.7.3). If $X$ is a local ringed space, $x \in X$, and $Y = \text{Spec} A$ an affine scheme, $f : X \to Y$ is a morphism, consider the ring map $\Gamma(X, \mathcal{O}_X) \xrightarrow{f^*} \Gamma(Y, \mathcal{O}_Y) \to \mathcal{O}_{X,x}$, and consider the inverse image $p$ of $m_x$, which corresponds to $y \in Y$, then $f(x) = y$.

Proof: There are commutative diagrams

$$
\begin{align*}
\Gamma(Y, \mathcal{O}_Y) &\to \mathcal{O}_{Y,f(x)} \\
\downarrow & \downarrow \\
\Gamma(X, \mathcal{O}_X) &\to \mathcal{O}_{X,x}
\end{align*}
$$

and the map of local rings is a local ring map. So the inverse image of $m_x$ is just $m_{f(x)}$, so $m_{f(x)} = m_y$. \qed

Prop. (V.2.7.4). Let $X$ be a local ringed spaces and $Y = \text{Spec} A$ an affine scheme, then the map $\text{Hom}(X, \text{Spec} A) \to \text{Hom}(A, \Gamma(X, \mathcal{O}_X))$ is an isomorphism.

Proof: The inverse map is constructed as follows: for any $\varphi : A \to \Gamma(X, \mathcal{O}_X)$ and $x \in X$, define $\Phi(x)$ to be the point corresponding to the inverse image of $m_x$ in $A \to \Gamma(X, \mathcal{O}_X) \to \mathcal{O}_{X,x}$. In this way, $\Phi^{-1}(D(f))$ is just $D(\varphi(f)) \subset X$ which is open, thus $\Phi$ is continuous. Now we want to construct a sheaf homomorphism $f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$, and it suffices to construct compatible maps on the affine open basis $D(f)$, by (V.1.4.15). Now $\Gamma(D(f), \mathcal{O}_Y) = A_f$, and because $f$ is invertible on $D(\varphi(f))$, there is by universal property a unique map $A_f \to D(\varphi(f))$ extending $\varphi$. Then by universal property these maps are compatible. Notice the construction here also shows the homomorphism is determined by the map set-theoretically.

Finally, we need to show this induces a local ring map on the stalks, and this is quite obvious from the definition.

Then we show these two maps are inverse to each other: It suffices to show any ring map $A \to \Gamma(X, \mathcal{O}_X)$ comes uniquely from a map $X \to Y$: the uniqueness is proven by (V.2.7.3), and the sheaf homomorphism is determined by the set-theoretical map by the above argument. \qed

Cor. (V.2.7.5) (Adjointness of Spec and $\Gamma$). The Spec operator from $\text{CRing}^{op}$ to Scheme is right adjoint to $X \to \Gamma(X, \mathcal{O}_X)$,

$$\text{Hom}_{\text{Sch}}(X, \text{Spec}(A)) \cong \text{Hom}_{\text{CRing}}(A, \Gamma(X, \mathcal{O}_X)).$$

Notice the category of schemes is a full subcategory of the category of locally ringed spaces.
Cor. (V.2.7.6). The category of affine schemes is equivalent to the opposite of the category of rings.

Prop. (V.2.7.7) (Points of Schemes). Let $X$ be a scheme and $R$ a local ring, then there is a natural bijection between morphisms $\text{Spec } R \to X$ and pairs $(p, \varphi)$ where $p \in X$ is a point and $\varphi : \mathcal{O}_{X, x} \to R$ is a local ring map.

Proof: Consider where the closed point of $\text{Spec } R$ is mapped to and choose an affine open nbhd of that point, then we reduce to the affine case, which is by (V.2.7.5).

Cor. (V.2.7.8). If $f : Y \to X$ is a morphism of schemes that $f(y) = x$, then we have a commutative diagram
\[
\begin{array}{ccc}
\mathcal{O}_{Y, y} & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\mathcal{O}_{X, x} & \longrightarrow & X
\end{array}
\]

Prop. (V.2.7.11) (Scheme is Sober). The underlying space of a scheme is sober.

Proof: Firstly this is true for affine schemes, by (IX.1.15.13). Then notice for any affine open subscheme $U$, the generic point for $Z \cap U$ is the generic point for $Z$.

Construction of Schemes

Prop. (V.2.7.12) (Global Spec). There is an $S$-scheme $f : \text{Spec}_S \mathcal{A} \to S$ for every Qco sheaf of $\mathcal{O}_S$-algebras $\mathcal{A}$ on $S$ that for any affine open subscheme $U \subset X$, $f^{-1}(U) \cong \text{Spec } \mathcal{A}(U)$ over $U$.

This construction is right adjoint to the direct image map:
\[
\text{Hom}_{\text{Alg}_{\mathcal{O}_S}}(\mathcal{A}, \pi_* \mathcal{O}_X) \cong \text{Hom}_{\text{Sch/}S}(X, \text{Spec}_S \mathcal{A}).
\]

and defines an equivalence of affine morphisms over $S$ and Qco $\mathcal{O}_S$-algebras. Moreover, this defines an equivalence of the category of $\mathcal{A}$-modules and the category of $\mathcal{O}_{\text{Spec } \mathcal{A}}$-modules.

Proof: Choose an affine open covering $\{U_i \to X\}$ of $X$, and consider the schemes $\text{Spec } \mathcal{A}(U_i) \to U_i$ over $U_i$, then their restrictions to $U_{ij}$ are compatible, because this is true after further restriction to an affine open covering of $U_{ij}$, we can use (V.1.5.4). Then we can use (V.1.5.4) to get an $S$-scheme $f : \text{Spec}_S \mathcal{A} \to S$ that $f^{-1}(U_i) \cong \text{Spec } \mathcal{A}(U_i)$.

Now we check that for any affine open subset $U \subset X$, $f^{-1}(U) \cong \text{Spec } \mathcal{A}(U)$ over $U$. But this is true after base change to $U_i \cap U$ for any $i$, so it is true, by (V.1.5.4).

To show the adjointness condition, by (V.1.5.4), it suffices to show canonical (thus compatible) isomorphism for $S$ affine. In this case, this reduces to (V.2.7.5).

Cor. (V.2.7.13). Let $S$ be a scheme and $\mathcal{A}$ is a Qco sheaf of $\mathcal{O}_S$-algebras, then

- For any morphism $g : S' \to S$, $S' \times_S \text{Spec}_S(\mathcal{A}) \cong \text{Spec}_{S'}(g^* \mathcal{A})$. 


V.2. RINGED TOPOI, RINGED SITES, RINGED G-SPACES AND SCHEMES

- The natural map \( A \to \pi_* \mathcal{O}_{\text{Spec}_S(A)} \) is an isomorphism of \( \mathcal{O}_S \)-algebras.

Proof: 1: It can be checked that \( S' \times_S \text{Spec}_S(A) \) and \( \text{Spec}_{S'}(g^*A) \) satisfy the same universal property.

2: It suffices to check on affine opens, then it is trivial.

Lemma (V.2.7.14) (Affine Case). Fiber product of affine schemes is also affine that corresponds to the tensor product of their corresponding rings, by (V.2.7.5).

Prop. (V.2.7.15) (Finite Limits of Schemes). Fiber products exist in the category of schemes, and there is a final object \( \text{Spec} \mathbb{Z} \), so arbitrary limits exist in the category of schemes (II.1.1.35).

In fact, finite limits exists in the category of ringed spaces, and finite limits of schemes coincide with finite limits as ringed spaces.

Proof: Let \( f : X \to S, g : Y \to S \), and \( U_i \) is an affine open covering of \( S \), \( V_j \) is an affine open covering of \( f^{-1}(U_i) \) and \( W_k \) is an affine open covering of \( g^{-1}(U_i) \), then we can check \( h_{V_{ij} \times_{h_{U_i}} h_{W_{ik}}} \) is a covering of \( h_X \times_{h_S} h_Y \) by representable open subfunctors (V.3.5.7), by (V.2.7.14), thus it is representable.

Cor. (V.2.7.16) (Open Subschemes). Let \( X \to S, Y \to S \), and \( V \subset X, W \subset Y \) be open subschemes mapping into open subscheme \( U \subset S \), then there is a natural open immersion \( V \times_U W \to X \times_S Y \) with image \( \pi_1^{-1}V \cap \pi_2^{-1}(W) \).

Proof: There is a natural map \( V \times_U W \to X \times_S Y \) by Yoneda lemma, and this map has the universal property that any map \( f : (T, \mathcal{O}_T) \to X \times_S Y \) that \( \pi_1 \circ f \) has image in \( V \) and \( \pi_2 \circ f \) has image in \( W \) factors uniquely through \( V \times_U W \). But so does the open immersion \( \pi_1^{-1}V \cap \pi_2^{-1}(W) \to X \times_S Y \) (V.2.6.14), so they are equal.

Cor. (V.2.7.17). Let \( f : X \to S, g : Y \to S \), and \( U_i \) is an affine open covering of \( S \), \( V_{ij} \) is an affine open covering of \( f^{-1}(U_i) \) and \( W_{ik} \) is an affine open covering of \( g^{-1}(U_i) \), then

\[
X \times_S Y = \bigcup_i \bigcup_{j,k} V_{ij} \times_{U_i} W_{ik}
\]
is an affine open covering of \( X \times_S Y \).

Also, the structure sheaf of \( X \otimes_S Y \) is given by \( \mathcal{O}_{X \times_S Y} = \pi_1^{-1} \mathcal{O}_X \otimes_{\pi_2^{-1} \mathcal{O}_S} \pi_2^{-1} \mathcal{O}_Y \).

Cor. (V.2.7.18). The equalizer of two morphisms from \( X \) to \( Y \) exists, it is a locally closed subscheme of \( X \), and it is a closed subscheme of \( X \) if \( Y \) is separated.

Proof: because it is the base change of \( \Delta : Y \to Y \times Y \) (II.1.1.35), then use (V.4.4.59).

Remark (V.2.7.19) (Infinite Product of Schemes Doesn’t Exists). WARNING: infinite products of schemes may not exists, Cf. [Sta] 0CNH. Intuitively, if you want to glue affine products together, you will notice you can identify only those products that is equal a.e..

8 Rational Maps

Def. (V.2.8.1) (Rational Maps). Let \( X, Y \) be schemes, a rational map \( f : X \to Y \) is an equivalence class of maps \( U \to Y \) where \( U \) is an open subset of \( X \). A rational function on \( X \) is a rational map \( X \to \mathbb{A}^1 \). It has a ring structure. The ring of rational functions is denoted by \( R(X) \).

Prop. (V.2.8.2). If \( X \) is a scheme with f.m. generic points \( \eta_i \), then

\[
R(X) = \prod \mathcal{O}_{X, \eta_i}.
\]

Proof: Cf. [Sta] 01RV.
9  Associated Points

Main References are [Sta]Chap30.

Def. (V.2.9.1). For a scheme $X$ and a Qco sheaf $\mathcal{F}$ on $X$, a point is called associated to $\mathcal{F}$ iff $m_x$ is associated to $\mathcal{F}_x$, which is equivalent to $m_x$ are all zero-divisors in $M$ by (I.5.4.10). When $\mathcal{F} = \mathcal{O}_X$, $x$ is called an associated point of $X$.

Prop. (V.2.9.2). If $X$ is locally Noetherian, then an associated prime is equivalent to it is an associated prime of $\Gamma(X, \mathcal{O}_X)$ of $\Gamma(U, \mathcal{F})$ for a nbhd $U$ of $x$.

Proof: Cf.[Sta]02OK.

Prop. (V.2.9.3). Same results of associated points are parallel to the discussion of associated primes:

- relations of $\text{Ass}(\mathcal{F})$ w.r.t exact sequences (I.5.4.7).
- $\text{Ass}(\mathcal{F}) \subset \text{Supp}(\mathcal{F})$ (I.5.4.8).
- When $X$ is locally Noetherian and $\mathcal{F}$ is coherent, for a quasi-compact open set $U$ of $X$, the number of associated points in $U$ is finite (I.5.4.8).
- When $X$ is locally Noetherian, $\mathcal{F} = 0$ iff $\text{Ass}(\mathcal{F})$ is empty (I.5.4.8).
- When $X$ is locally Noetherian, If $\text{Ass}(\mathcal{F}) \subset$ an open subset $U$, then $\Gamma(X, \mathcal{F}) \rightarrow \Gamma(U, \mathcal{F})$ is injective (I.5.4.13).
- If $X$ is locally Noetherian, then the minimal elements(under specialization) of $\text{Supp}(\mathcal{F})$ are associated points of $\mathcal{F}$. in particular, any generic point of an irreducible component of $X$ is an associated points of $X$.
- If $X$ is locally Noetherian, then if a map $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ that is injective at all the stalks of $\text{Ass}(\mathcal{F})$, then $\varphi$ is injective.
V.3 Quasi-Coherent Sheaves on Schemes

1 (Quasi-)Coherent Sheaves

Lemma (V.3.1.1) (Associated Qco Sheaves on Affine Scheme). On an affine scheme Spec $A$, there is a sheaf $\tilde{M}$, that is $M_f$ on Spec $A_f$. To check it is a sheaf, we only need to check to affine coverings, and this is by (I.7.2.2).

Prop. (V.3.1.2) (Qco Sheaves on Affine Schemes). For any $A$-module $M$, there is a sheaf of modules $F_M$ on $X = \text{Spec } A$ by (V.2.2.23). This is left adjoint to $\Gamma$ and defines a functor from the category of $A$-modules to the category of $O_{\text{Spec } A}$-modules.

And in fact, $F_M$ is naturally isomorphic to $\tilde{M}$, and $M \mapsto \tilde{M}$ is an equivalence to the category of quasi-coherent sheaves over Spec $A$. In particular,

$$\text{Hom}_A(M, \Gamma(X, G)) \cong \text{Hom}_{O_X} (\tilde{M}, G),$$

so $M \mapsto \tilde{M}$ commutes with colimits and is exact, and commutes with pullbacks.

When $A$ is Noetherian, this also induces an equivalence between finite $A$-modules and coherent sheaves over Spec $A$, because finiteness for modules is local (V.4.1.5).

Proof: By universal property of $F_M$ (V.2.2.23), there is a natural map $F_M \to \tilde{M}$ corresponding to the ring map $M \mapsto \Gamma(\text{Spec } A, \tilde{M}) = M$. The induced maps on the stalk at a point $x$ is $M \otimes_A O_{X,x} \to M_p$, which is isomorphism, so $F_M \cong \tilde{M}$.

From the universal property of $\tilde{M} = F_M$, $\text{Hom}(\tilde{M}, N) = \text{Hom}(M, N)$, thus $\sim$ is fully faithful, to show it is an equivalence, it suffices to show for any Qco sheaf $F$ on Spec $A$, the natural map $\Gamma(X, F) \to F$ is an isomorphism: \footnote{Cf. [Sta]01IA.}

If $0 \to M_1 \to M \to M_2 \to 0$ is exact, then $0 \to \tilde{M}_1 \to \tilde{M} \to \tilde{M}_2 \to 0$ is exact, because localization is exact.

Prop. (V.3.1.3) (properties of (Quasi-)Coherent Sheaves on Schemes).

- $(Q)co(X)$ form weak Serre subcategories of Mod$_{O_X}$.
- Any colimit of Qco sheaves is a Qco sheaf, because localization is exact.
- Tensor product of two (Q)co sheaf is (Q)co, and locally free if they are locally free. More explicitly, $M \otimes_A N \cong \tilde{M} \otimes_{O_X} \tilde{N}$ as tensor product commutes with $\pi^*$.
- If $F$ is Qco, then so does $T(F), Sym(F)$ and $\wedge(F)$, by (V.2.3.11).
- Given two Qco sheaves $F, G$ that $F$ is f.p., then $\text{Hom}(F, G)$ is Qco, by (V.2.3.8).
- pullback of (qco)coherent sheaves are (qco)coherent, by (V.2.2.26). More explicitly, if Spec $B \subset Y$ is mapped into Spec $A \subset X$, then $f^*(F)(\text{Spec } B) = F(\text{Spec } A) \otimes_A B$, using the fact $\tilde{M} = F_M = \pi^* M$.

Proof: 1: Coherent case is by (V.2.2.25). For Qco, it follows from (V.3.1.2) that the kernel and cokernel of $\varphi : \tilde{M} \to \tilde{N}$ is just Ker $\varphi$ and Coker $\varphi$ which is Qco, for the extension of Qco, use (V.6.7.3) that the global section is exact, so there is a morphism of exact sequences $\Gamma(X, F_i)(U) \to F_i(U)$, and five lemma gives the result.
The last two are Qco because two are maps between affine schemes, so the first is Qco. □

**Proof:** The question is local so we let Y be affine, and then X is qcqs, so we cover it with affine opens $U_i$ and their intersections are $U_{ijk}$. Then we see by sheaf property

$$0 \to f_*\mathcal{F} \to \bigoplus_i f_*(\mathcal{F}|_{U_i}) \to \bigoplus_{ijk} f_*(\mathcal{F}|_{U_{ijk}})$$

The last two are Qco because two are maps between affine schemes, so the first is Qco. □

**Prop. (V.3.1.5) (Qco Sheaves on Qcqs Schemes).** For a qcqs scheme $X$ and $s \in \Gamma(X, \mathcal{O}_X)$, and a Qco module $F$, $(\Gamma(X, \mathcal{F}))_s \cong \Gamma(X_s, \mathcal{F})$.

**Proof:** The canonical map $f : X \to \text{Spec} \Gamma(X, \mathcal{O}_X)(\text{V.2.7.5})$ is qcqs, so $f_*\mathcal{F}$ is Qco on Spec $\Gamma(X, \mathcal{O}_X)$ by (V.3.1.4). Then the result follows from the fact $f^{-1}(\text{Spec} \Gamma(X, \mathcal{O}_X)_s)) = X_s$ and the definition of $f_*$. □

**Lemma (V.3.1.6).** Let $i : U \to X$ be a quasi-compact open immersion of schemes, $F$ a Qco $\mathcal{O}_X$-module, and $G \subset F|_U$ a Qco $\mathcal{O}_U$-submodule, then there exists a Qco $\mathcal{O}_X$-submodule $G' \subset F$ that $G'|_U = G$.

**Proof:** immersion is separated (V.4.4.69), so $i_*G$ is a Qco $\mathcal{O}_X$-sheaf by (V.3.1.4), and it is a submodule of $i_*i^*F$, so the kernel

$$\mathcal{H} = \text{Ker}(\mathcal{F} \oplus i_*G \to i_*i^*\mathcal{F})$$

is also Qco by (V.3.1.3), and $\mathcal{H} \subset \mathcal{F}$, $\mathcal{H}|_U = G$. □

**Prop. (V.3.1.7) (Extending Qco Sheaves).** Let $X$ be a qcqs scheme, and $U \subset X$ a qc open subset, $F$ a Qco $\mathcal{O}_X$-module, and $G \subset F|_U$ a Qco $\mathcal{O}_U$-submodule of f.t., then there exists a Qco $\mathcal{O}_X$-submodule $G' \subset F$ of finite type that $G'|_U = G$.

**Proof:** Let $n$ be the minimal number of affine open subsets $U_i$ that $X = U \cup \bigcup U_i$, by induction on $n$, it suffices to prove for $n = 1$. Thus we may assume $X = U \cup V$ where $U, V$ are affine opens. Now $U \cap V$ is qc because $X$ is qs. Then we can change $(X, U)$ to $(V, U \cap V)$, because we can glue the resulting sheaf. Then we reduce to the case $X$ is affine.

Let $X = \text{Spec} R$ and $\mathcal{F} = \mathcal{M}$, then by (V.3.1.6), there exists a Qco sheaf $\widetilde{N}$ that $\widetilde{N}|_U = G$. By hypothesis we can cover $U$ by f.m. open affine $D(f_i)$ that $N_{f_i}$ is f.g., by element $x_{ij}/f_{ji}^{n_i}$. Let $N'$ be the submodule of $N$ generated by these elements $x_{ij}$, then $\widetilde{N}'$ meets our requirement. □

**Cor. (V.3.1.8) (Qco Sheaf is a Direct Union of Qco Sheaves of F.T.).** Let $X$ be a qcqs scheme, then any Qco sheaf $\mathcal{F}$ on $X$ is a direct colimit of its Qco subsheaves of f.t.

**Proof:** It is a direct colimit because the sum of two Qco sheaves of f.t. is also Qco of f.t.. Now for any affine open $U \subset X$ and $s \in \mathcal{O}_X(U)$, $s$ generates a Qco $\mathcal{O}_U$-submodule of f.t. of $\mathcal{F}|_U$, and by (V.3.1.7) this extends to a Qco $\mathcal{O}_X$-submodule of $\mathcal{F}$. Then we see that the direct colimit of Qco subsheaves of f.t. of $\mathcal{F}$ contains elements of $\mathcal{F}(U)$ for any affine open subset $U$, thus it is just $\mathcal{F}$. □

**Prop. (V.3.1.9) (Locally Free Sheaves).** Suppose $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves on a scheme $X$, then

- If $\mathcal{F}'$, $\mathcal{F}''$ are both locally free, then so is $\mathcal{F}$.
- If $\mathcal{F}, \mathcal{F}'$ are both locally free of finite rank, then so is $\mathcal{F}'$. 

...
Proof: Firstly of all in each case all sheaves are Qco by (V.3.1.3).

1: on an affine open \( U = \text{Spec} A \subset X \), the exact sequence is induced by \( 0 \to A' \to \Gamma(U, \mathcal{F}) \to A^J \to 0 \), so it splits and \( \Gamma(U, \mathcal{F}) \cong A^{I+J} \) is free.

2: on an affine open \( U = \text{Spec} A \subset X \), the exact sequence is induced by \( 0 \to \Gamma(U, \mathcal{F}') \to A^m \to A^n \to 0 \), where the map \( A^m \to A^n \) is represented by a \( n \times m \) matrix \( M \). Then \( U \) is covered by open subsets \( U_i \) that some \( n \times n \) minor of \( M \) is invertible. Then after a change of coordinates on each subset, \( \mathcal{F}(U_i) \cong A^{m-n} \).

□

Prop. (V.3.1.10). For a exact sequence of locally free sheaves: \( 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \), there is a filtration of \( \text{Sym}^r \mathcal{F} \):

\[
0 = G^{r+1} \subset G^r \subset \ldots \subset G^0 = \text{Sym}^r \mathcal{F}
\]

that

\[
G^p / G^{p+1} \cong \text{Sym}^p \mathcal{F}' \otimes \text{Sym}^{r-p} \mathcal{F}''.
\]

Proof: On any affine open subset, choose a splitting of the exact sequence, then use coordinates.

□

Prop. (V.3.1.11). For a exact sequence of locally free sheaves: \( 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \), there is a filtration of \( \wedge^r \mathcal{F} \):

\[
0 = G^{r+1} \subset G^r \subset \ldots \subset G^0 = \wedge^r \mathcal{F}
\]

that

\[
G^p / G^{p+1} \cong \wedge^p \mathcal{F}' \otimes \wedge^{r-p} \mathcal{F}''.
\]

In particular,

\[
\wedge \mathcal{F}' \otimes \wedge \mathcal{F}'' \cong \wedge \mathcal{F}.
\]

and when \( \mathcal{F}'' \cong L \) is a line bundle, there is an exact sequence

\[
0 \to \wedge^r (\mathcal{F}') \to \wedge^r (\mathcal{F}) \to \wedge^{r-1} (\mathcal{F}') \otimes L \to 0
\]

Proof: On any affine open subset, choose a splitting of the exact sequence, then use coordinates.

□

Prop. (V.3.1.12) (Perfect Pairing Wedge Product Sheaf). Let \( \mathcal{F} \) be a locally free sheaf of rank \( n \), then there is a perfect pairing \( \wedge^r \mathcal{F} \otimes \wedge^{n-r} \mathcal{F} \to \wedge \mathcal{F} \) which is a perfect pairing, i.e. it induces an isomorphism \( \wedge^r \mathcal{F} \cong (\wedge^{n-r} \mathcal{F})' \otimes \wedge \mathcal{F} \).

Proof: The map is natural, and the isomorphism can be seen at the level of stalls, by (V.2.3.10).

□

Def. (V.3.1.13). ?

- for a closed immersion \( Y \to X \), there is \( i^! : \text{Qco}(X) \to \text{Qco}(Y) \) that is right adjoint to \( i_* : i^! \mathcal{G} = i^*(\mathcal{H}_Z(\mathcal{G}))' \), where \( \mathcal{H}_Z(\mathcal{G}) \) is the sheaf of sections annihilated by \( \mathcal{I} \) and \( \mathcal{F}' \) is the maximal Qco sheaf of \( \mathcal{F} \).

- For \( f \) proper between locally Noetherian scheme, there is a inverse sheaf \( f^\dagger \mathcal{G} = \text{Hom}_Y(f_* \mathcal{O}_X, \mathcal{G}) \), which maps \( \text{Qco}(Y) \) to \( \text{Qco}(X) \) by (V.3.1.15) and (V.6.7.11). When \( f \) is affine, in particular when it is finite, then \( f^\dagger \) is right adjoint to \( f_* \) on \( \text{Qco}(V.6.7.12) \).
Coherent Sheaves

Def. (V.3.1.14) (Coherent Sheaves on Schemes). When \( \mathcal{O}_X \) is coherent over itself, coherence is equivalent to f.p.(V.2.2.25). In particular when \( X \) is a locally Noetherian scheme, a coherent sheaf is equivalent to a Qco module of f.t.. When talking about coherent sheaves over schemes, I often assume the scheme is locally Noetherian.

(Quasi-)coherent is an affine local by (V.2.2.27). \( \text{Qco} (X) \) forms an Abelian category, by (V.2.2.25).

Prop. (V.3.1.15) (Proper Pushforward). If \( f \) is proper, \( f_* \) maps coherent sheaf to a coherent sheaf. (directly from (V.6.7.30)).

Prop. (V.3.1.16) (Artin-Rees). Let \( X \) be a Noetherian scheme, \( \mathcal{F} \) a coherent \( \mathcal{O}_X \)-module, \( \mathcal{G} \) a Qco subsheaf of \( \mathcal{F} \), \( I \subset \mathcal{O}_X \) a Qco sheaf of ideals, then there exists some \( c > 0 \) that for all \( n \geq c \)

\[
I^{n-c}(I \mathcal{F} \cap \mathcal{G}) = I^n \mathcal{F} \cap \mathcal{G}.
\]

Proof: Cover \( X \) by f.m. affine open subsets, then this follows from the affine case (I.5.6.9).

Cor. (V.3.1.17) (Vanish Analytically). Let \( X \) be a Noetherian scheme, \( \mathcal{F} \) a coherent \( \mathcal{O}_X \)-module, for any element \( f \in \bigcap_n m_x^n \mathcal{F} \), \( f \) vanishes at a nbhd of \( x \).

Proof: This follows from the intersection theorem (I.5.6.10).

Prop. (V.3.1.18) (Deligne). On a Noetherian scheme \( X \), let \( \mathcal{F} \) be a Qco sheaf, \( \mathcal{G} \) be a coherent sheaf and \( \mathcal{I} \) be a Qco sheaf of ideals correspongding to \( Z, U = X - Z \), then we have

\[
\lim_n \text{Hom}_{\mathcal{O}_X}(\mathcal{I}^n \mathcal{G}, \mathcal{F}) \cong \text{Hom}_{\mathcal{O}_U}(\mathcal{G}|_U, \mathcal{F}|_U).
\]

In particular,

\[
\lim_n \text{Hom}_{\mathcal{O}_X}(\mathcal{I}^n, \mathcal{F}) \cong \Gamma(U, \mathcal{F}).
\]

Proof: Cf.[Sta]01YB.

Prop. (V.3.1.19) (Kleinmann). If \( X \) is a Noetherian integral separated locally factorial scheme, then every coherent sheaf on \( X \) is a quotient of a finite locally free sheaf.

Proof: Cf.[Hartshorne P238].

Prop. (V.3.1.20) (Support of Modules). The support(V.2.2.2) of a sheaf of modules of f.t over a scheme is closed(V.2.2.21), e.g. coherent sheaf.

This have many consequences applied to kernel and cokernel, for example, a coherent sheaf is locally free iff all its stalk is free (choose a presentation and see kernel and cokernel).

For a flat morphism \( f \), \( \text{Supp}(f^*(\mathcal{F})) = f^{-1}(\text{Supp} \mathcal{F}) \), by (V.2.2.15).

Proof: because for a set of generators \( x_i \) of \( M \), \( \text{Ann}(\mathcal{F}) = \bigcup \text{Ann}(x_i) \), and \( \text{Ann}(x_i) \) is closed.

Cor. (V.3.1.21) (Semicontinuity). For a Qco sheaf \( \mathcal{F} \) of f.t., \( \varphi(y) = \dim_{k(y)}(\mathcal{F} \otimes k(y)) \) is an upper semicontinuous function on the scheme.

Proof: By Nakayama, \( \varphi(y) \) is equal to the minimal number of generators of the \( \mathcal{O}_y \)-module \( \mathcal{F}_y \). But these generators extends to a nbhd of \( y \), so \( \varphi \leq n \) on this nbhd.
Prop. (V.3.1.22) (Maximal Qco Submodule). For $X$ a scheme and any $\mathcal{O}_X$-module $\mathcal{F}$, there is a Qco submodule of $\mathcal{F}$ maximal among all Qco submodules of $\mathcal{F}$. This is because direct colimit of Qco sheaves are Qco(V.3.1.3).

Prop. (V.3.1.23). A f.t. Qco sheaf on a scheme has a minimal closed scheme structure on its support, it is generated locally by the Qco ideal $\text{Ann}_A(M)$ (V.4.4.42). And there is a f.t. Qco sheaf $\mathcal{G}$ that $i_*(\mathcal{G}) = \mathcal{F}$.

Proof: Cf.[Sta]01QY.

Devisage of Coherent Sheaves

Lemma (V.3.1.24). Let $\mathcal{F}$ be a coherent sheaf on a Noetherian scheme $X$, let $I$ be a sheaf of ideals that correspond to $Z$, then $\text{Supp}(\mathcal{F}) \subset Z$ iff $I^n\mathcal{F} = 0$ for some $n$. (This follows easily from Noetherian and(I.5.4.5)).

Lemma (V.3.1.25). If we have a coherent sheaf $\mathcal{F}$ on a Noetherian scheme $X$, that $\text{Supp}(\mathcal{F}) = Z_1 \cup Z_2$, then we have an exact sequence of coherent sheaves $0 \to \mathcal{G}_1 \to \mathcal{F} \to \mathcal{G}_2 \to 0$ that $\text{Supp}(\mathcal{G}_i) \subset Z_i$.

Proof: Let $I$ be the reduced ideal sheaf of $Z_1$, we use the exact sequence $0 \to I^n\mathcal{F} \to \mathcal{F} \to \text{Coker} \to 0$, by(V.3.1.24), we can choose $n$ that $\text{Supp}(I^n\mathcal{F}) \subset Z_2$, thus the result.

Prop. (V.3.1.26). Let $\mathcal{F}$ be a coherent sheaf on a Noetherian scheme $X$, then there is a filtration of coherent sheaves that the quotients are pushforward of ideal sheaves on integral subschemes of $X$. This is analogous to the filtration in the module case.

Proof: We consider the set of these counterexamples and their $\text{Supp}$, then use Noetherian induction, the minimal one if not irreducible, then from(V.3.1.25) we find a filtration for it. Then let the ideal of sheaf be $\mathcal{I}$, then $I^n\mathcal{F} = 0$, then we should use [[Sta]01YE] to finish to proof. Cf.[[Sta]01YF].

Prop. (V.3.1.27). Let $P$ be a property of coherent sheaves on $X$ Noetherian that
• for an exact sequence of sheaves: $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$, if $\mathcal{F}_1$ has $P$, then $\mathcal{F}$ has $P$.
• If $\mathcal{F}^\oplus$ has $P$, then $\mathcal{F}$ has $P$.
• For every integral closed subscheme $Z$ of $X$ with generic point $\xi$, there is a coherent sheaf $\mathcal{G}$ that
  1. $\text{Supp}\mathcal{G} \subset Z$.
  2. $\mathcal{G}_\xi$ is annihilated by $m_\xi$.
  3. For every sheaf of ideal $\mathcal{I}$ on $X$ that $\mathcal{I}_\xi = \mathcal{O}_{X,\xi}$, there is a sheaf $\mathcal{G}' \subset \mathcal{I}\mathcal{G}$ that $\mathcal{G}'_\xi = \mathcal{G}_\xi$ and has $P$.
Then we have $P$ holds for every coherent sheaf on $X$.

Proof: Use Noetherian induction, the minimal counterexample should have $\text{Supp}$ irreducible by(V.3.1.25) and then we use[[Sta]01YL]. Note this has nothing to do with reducedness.
2 Projective Spaces

Def. (V.3.2.1) (Projective Scheme). For a graded ring $S$, we have a scheme $\text{Proj}(S)$ that consists of homogenous primes of $S$ minus $S_+$ and the affine cover is $D(f) = \{ p | f \not\in p \}$, and $\mathcal{O}(D(f)) = \text{Spec} S(f)$, where $S(T)$ is the degree zero part of $T^{-1}S$. It has $\mathcal{O}_p = S(p)$.

Proof: Define the sheaf using stalks, then we only have to check that $\text{Spec} S(f) \cong \text{homogenous } p \in S_f$ by natural intersection of ideals $\varphi$ and $S(p) \cong (S(f))_{\varphi(p)}$ for $p \in D(f)$.

We check that for $S(f) \subset S_f$, $p \to p \cap S(f)$ and $p' \to pS$ is natural and inverse to each other. $S(f) \to S(p)$ maps $\varphi(p)$ to invertible, and any $x/a \in S(p)$ can be written as $x \deg f / f \deg a$. □

Prop. (V.3.2.2) (Representing Functor of Projective Schemes). Let $S$ be a graded ring generated by $S_1$ over $S_0$, then $\text{Proj}(S)$ represents the functor that maps a scheme $Y$ to the set of pairs $(\mathcal{L}, \psi)$, where $\mathcal{L}$ is an invertible sheaf on $Y$, and $\psi : S \to \Gamma(Y, \mathcal{L})$ is a graded ring homomorphism that $\mathcal{L}$ is generated by the global sections $\psi(S_1)$, up to strict equivalences.

Proof: Cf.[Sta]01NA. □

Def. (V.3.2.3) (Projective Space). Let $A$ be a ring, the projective space $\mathbb{P}^n$ is defined to be $\text{Proj}(A[T_0, \ldots, T_n])$ with $\deg(T_i) = 1$. It represents the functor that maps a scheme $T$ to the pairs $(\mathcal{L}, (s_0, \ldots, s_n))$, where $\mathcal{L}$ is an invertible sheaf on $T$, and $s_0, \ldots, s_n \in \Gamma(T, \mathcal{L})$ that generate $\mathcal{L}$. For any scheme $S$, $\mathbb{P}^n_S \times_Z S$ is called the projective space over $S$.

Prop. (V.3.2.4) (Segre Embedding). Let $S$ be a scheme, there is a natural closed immersion

$$\mathbb{P}^m_S \times_S \mathbb{P}^n_S \to \mathbb{P}^{m+n}_S$$

called the Segre embedding.

Proof: It suffices to write for $S = \mathbb{Z}$, and in this case, it suffices to write down an invertible sheaf on $\mathbb{P}_S^m \times_S \mathbb{P}_S^n$ with $(n + 1)(m + 1)$ global sections that generate it. Then we take the invertible sheaf $\mathcal{L} = \pi_1^* \mathcal{O}_{\mathbb{P}^m}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^n}(1)$ and the sections $X_i Y_j$, where $(X_0, \ldots, X_m)$ generate $\mathcal{O}_{\mathbb{P}^m}(1)$ and $(Y_0, \ldots, Y_n)$ generate $\mathcal{O}_{\mathbb{P}^n}(1)$.

It is a closed immersion by[Sta]01WD. □

Prop. (V.3.2.5) (Venerose Embedding).

Prop. (V.3.2.6) (Projection Map).

Prop. (V.3.2.7). If $X \subset \mathbb{P}_k^n$ is a closed subvariety disjoint from a $d$-dimensional subspace $L \subset \mathbb{P}_k^n$, then the projection $\pi : X \to \mathbb{P}_k^{n-d-1}$ with center $L$ induces a finite map $X \to \pi(X)$.

Proof: Cf.[Shafarovich]1, P63]. □

Cor. (V.3.2.8). If $F_0, \ldots, F_s$ are forms of degree $m > 0$ on $\mathbb{P}_k^n$ having no common zero on a closed subvariety $X \subset \mathbb{P}_k^n$, then $\varphi(x) = [F_0(x), \ldots, F_s(x)]$ defines a finite map $\varphi : X \to \varphi(X)$.

Proof: Let $v_m : \mathbb{P}_k^n \to \mathbb{P}_k^N$ be the Veronese embedding of degree $m$, then $X \to v_m(X)$ is an isomorphism, and $\varphi$ is a composition of $v_m$ and a projection $\mathbb{P}_k^N \to \mathbb{P}_k^s$, thus it is a finite map, by(V.3.2.7). □
Prop. (V.3.2.9) (Tensor and Proj). For two graded ring with the same $S_0 = A$, Proj$(S \times_A T) \cong X \times_A Y$, where $(S \times_A T)_n = S_n \times_A T_n$

Proof:

Prop. (V.3.2.10). For a graded $S$-module, there is a Qco-sheaf $\tilde{M}$ on Proj$S$, that $\tilde{M}_p = M(p)$ and $\tilde{M}|_{D^+(f)} \cong \tilde{M}(f)$, the construction is as in (V.3.2.1).

Def. (V.3.2.11) (Relative Proj). The relative Proj $S$ over locally Noetherian $Y$ of a Qco graded $\mathcal{O}_Y$-algebra $S$ f.g. over $S_0$ by coherent $S_1$ is the gluing of locally Proj $S$. Proj $S \to Y$ is locally projective thus proper. It is equipped with invertible sheaf $\mathcal{O}(1)$ by glueing.

Prop. (V.3.2.12) (Closed Subscheme of Projective Scheme). The closed scheme of $X = \mathbb{P}^n_A$ corresponds to the saturated homogenous ideal $I_Y$, (i.e. for any $s$, if there is an $n$ that for any $i, x^n_i s \in I_Y$, then $s \in I_Y$).

So projective scheme over Spec $S_0$ corresponds to Proj $S$, where $S$ are f.g. over $S_0$ by $S_1$ saturated in the sense above.

Proof: A closed immersion is proper, thus the kernel $I_Y$ of the structural map is a Qco (V.3.1.3), so it must be an ideal on every affine open, because Qco is affine local. Then we should use (V.3.3.3), $\Gamma_s(I_Y)$ will suffice. Cf. [Hartshorne Ex2.5.10].

Prop. (V.3.2.13). The global section of a projective space Proj $S \to$ Spec $S_0$ is just $S_0$, this is by (V.3.3.3).

Prop. (V.3.2.14) (Grassmannian). The functor $G(n, k)$ is defined to be $G(n, k)(T) = \text{the isomorphism classes of surjections } \mathcal{O}_T^n \to Q$, where $Q$ is a locally free $\mathcal{O}_T$-modules of dimension $n - k$. It is a functor by pulling back surjections. Then this functor is representably by a scheme, called the Grassmannian over $\mathbb{Z}$. For any scheme $S$, $G(n, k) \otimes_{\mathbb{Z}} S$ is called the Grassmannian over $S$.

Proof: Cf. [Sta]098T.

Prop. (V.3.2.15). There is a canonical isomorphism $G(n, n + 1) \cong \mathbb{P}^n$ identifying sections with surjections from $\mathcal{O}_X^n$.

Prop. (V.3.2.16). A quasi-projective scheme $X$ over a field $k$ of dimension $r$ can be covered by $r + 1$ open affine subsets. This is because there are $r$ hyperplane that intersect $X$ non-empty. This can happen by choosing a hyperplane non-intersecting the generic point of $X$, otherwise we choose many hyperplane, then their intersection is empty.

Invertible Sheaves on Proj

Def. (V.3.2.17). Define $\mathcal{O}_{\mathbb{P}_k^n}(1) = \mathbb{Z}[X_0, \ldots, X_n](1)$, this is an invertible sheaf. The invertible Serre twisting sheaf $\mathcal{O}(1)$ on $\mathbb{P}_k^n$ is the pullback of that of $\mathbb{P}_Y^n$ and an invertible Serre twisting sheaf of the relative $X = \text{Proj} S$ over $Y$ is locally the pullback of that of $\mathbb{P}_Y^n$. Giving a Serre twisting sheaf of $X$ over $Y$, the Serre twisting sheaf of $\mathcal{F}$ over $X$ is the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Prop. (V.3.2.18). For $X$ projective over Spec$(A)$, (i.e. $X = \text{Proj}(S)(V.3.2.12)$), $\tilde{M} \otimes_S N \cong \tilde{M} \otimes_{\mathcal{O}_X} N$ and many other properties involving the Serre twisting, all this boil down to the fact that $(M \otimes_S N)(f) = M(f) \otimes_{\mathcal{O}(f)} N(f)$ for $f \in S_1$.

and by virtue of (V.3.2.10), when $X = \text{Proj}(S)$ projective, we have:
• \( \widetilde{M}(n) \cong \widehat{M}(n) \).

For a graded ring map \( S \rightarrow T \), we have the corresponding \( \text{Proj} \) map \( f : U \rightarrow T \) that \( f^*(\widetilde{M}) \cong (M \otimes_S T)_U \) and \( f_*(\widehat{N})_U \cong \widehat{N}_S \). That’s to say, \( f^*(\widetilde{M}(n)) = f^*(\widetilde{M})(n) \) and \( f_*(\widehat{M}(n)) = f_*(\widehat{M})(n) \).

**Cor.** (V.3.2.19). \( \mathcal{O}_X(n) \otimes \mathcal{O}_Y(m) \cong \mathcal{O}_X(n + m) \) for any scheme \( X \) projective over \( Y \).

**Prop.** (V.3.2.20) (Twisting of \( \text{Proj} \)). With notation as in (V.3.2.11), Let \( S' = S \ast L : S'_d = S_d \otimes L^d \), then \( \varphi : \text{Proj} S' \rightarrow \text{Proj} S \) is an isomorphism that induces

\[ \mathcal{O}'(1) \cong \varphi^* \mathcal{O}(1) \otimes \pi'^* \mathcal{L}. \]

**Prop.** (V.3.2.21). If \( Y \) is Noetherian and admits an ample invertible sheaf, then by definition, we have \( S_1 \otimes \mathcal{L}^n \) is base point free for some \( n \), thus we have a morphism \( \text{Proj} S \ast \mathcal{L}^n \rightarrow \mathbb{P}_Y^n \), so \( P = \text{Proj} S \) is \( H \)-quasi-projective with \( \mathcal{O}(1) = \mathcal{O}_P(1) \otimes \pi^* \mathcal{L}^n \).

**Prop.** (V.3.2.22) (Functorial Definition of a Projective Spaces). The projective space \( \mathbb{P}(\mathcal{E}) \) represents the following functor:

\[ \text{Sch}/S \rightarrow \text{Set} : (f : X \rightarrow S) \mapsto \{ \text{invertible quotients of } f^* \mathcal{E} \} \]

**Proof:**

\[ \square \]

**Relative \( \text{Proj} \)**

**Def.** (V.3.2.23) (Vector Bundles). Let \( S \) be a scheme, \( \pi : V \rightarrow S \) is called a vector bundle if it is affine, and \( \pi_* \mathcal{O}_V \) endowed with the structure of a graded \( \mathcal{O}_S \)-algebra structure \( \pi_* \mathcal{O}_V = \oplus \mathcal{E}_n \), where \( \mathcal{E}_0 = \mathcal{O}_S \), and \( \text{Sym}^n \mathcal{E}_1 \rightarrow \mathcal{E}_n \) is an isomorphism for any \( n \).

A morphism of vector bundles is a map \( E' \rightarrow E \) over \( S \) that the associated map \( f_* : \pi_* \mathcal{O}_V \rightarrow \pi'_* \mathcal{O}_V' \) is compatible with grading.

When \( \mathcal{E} \) is \( \mathcal{Qco} \), we can define the associated vector bundle \( V(\mathcal{E}) \) as \( \text{Spec}_{\mathcal{S}} \text{Sym}(\mathcal{E})(\text{V.2.3.9})(\text{V.2.7.12}) \). In this way, the category of vector bundles over \( S \) is anti-equivalent to the category of \( \mathcal{Qco} \) \( \mathcal{O}_S \)-algebras.

When \( \mathcal{E} \) is \( \mathcal{Qco} \), we can define the associated projective space bundle \( \mathbb{P}(\mathcal{E}) \) as \( \text{Proj}_{\mathcal{S}} \text{Sym}(\mathcal{E})(\text{V.2.3.9})(\text{V.2.7.12}) \). It is equipped with a Serre twisting sheaf \( \mathcal{O}(1) \), which is the glue of locally the Serre sheaf in projective space. There is a surjective morphism \( \pi^* (\mathcal{E}) \rightarrow \mathcal{O}(1) \) (local check).

**Prop.** (V.3.2.24). Let \( g : Y \rightarrow X \) by a scheme over \( X \), a morphism \( Y \rightarrow \mathbb{P}(\mathcal{E}) \) over \( X \) is equivalent to an invertible sheaf \( \mathcal{L} \) on \( Y \) and a surjective map \( g^* \mathcal{E} \rightarrow \mathcal{L} \).

In particular, giving a morphism \( X \rightarrow \mathbb{P}_A^n \) is equivalent to a base point free invertible sheaf with \( n \) generators on \( X \).

**Proof:** If there is a morphism, it will pullback \( \pi^*(\mathcal{E}) \rightarrow \mathcal{O}(1) \) into \( g^* \mathcal{E} \rightarrow \mathcal{L} \). For the converse, construct locally and glue, we have the natural morphisms \( A[x_1/x_1, \ldots, x_n/x_1] \rightarrow \mathcal{O}_{X_{s_i}} : x_j/x_i \rightarrow s_j/s_i \) in a homogenous sense. It is natural hence glue together. For the module, maps \( x_i \rightarrow s_i \). \( \square \)

**Cor.** (V.3.2.25). All automorphisms of \( \mathbb{P}_k^n \) is linear.

**Proof:** The Picard group of \( \mathbb{P}_k^n \) is \( \mathbb{Z} \) and is generated by \( \mathcal{O}(1)(\text{V.7.1.20}) \), so the automorphism will map \( \mathcal{O}(1) \) to \( \mathcal{O}(\pm 1) \) and \( \mathcal{O}(-1) \) has no global section(V.3.3.2). And the globals section is \( n \)-dimensional and determines the morphism by the prop. \( \square \)
3 Invertible Sheaves

General invertible sheaves on a ringed site is treated in §5.

Prop. (V.3.3.1). Giving a morphism \( \mathcal{X} \to \mathbb{P}_A^n \) is essentially equivalent to a base point free invertible sheaf with \( n \) generators on \( \mathcal{X} \). This follows from (V.3.2.24).

Prop. (V.3.3.2) (Global Sections). Let \( \mathcal{L} \) be an invertible sheaf over qcqs scheme \( \mathcal{X} \), for a \( \text{Qco} \) module \( \mathcal{F} \) let the global section functor \( \Gamma_*(\mathcal{F}) = \oplus \Gamma(\mathcal{X}, \mathcal{F} \otimes \mathcal{L}^n) \), then

\[
\Gamma_*(\mathcal{F})_f \cong \mathcal{F}(X_f).
\]

where \( s \in \Gamma(\mathcal{X}, \mathcal{L}) \). In particular that if there is a section \( f \) of \( \mathcal{F} \) on \( \mathcal{X} \), then for some \( n \), \( f \otimes s^n \) is a global section of \( \mathcal{F} \otimes \mathcal{L}^n \).

Proof: This is nearly the same as the proof that \( (\text{Spec } A)_f = \text{Spec } A_f \), Cf.[Sta]01PW. \( \square \)

Cor. (V.3.3.3). when \( \mathcal{X} = \text{Proj } S \) projective over \( \text{Spec } S_0 \) and \( \mathcal{F} \) \( \text{Qco} \), \( \Gamma_*(\mathcal{F}) \cong \mathcal{F} \), where \( \Gamma_*(\mathcal{F}) = \oplus_{n \in \mathbb{Z}} \Gamma(\mathcal{X}, \mathcal{F}(n)) \), which is a graded \( S \)-module. In particular, \( \Gamma_* \) for projective space \( \mathbb{P}_A^n \) equals \( A[x_1, \ldots, x_n] \).

Ample Invertible Sheaves

Def. (V.3.3.4) (Ample Invertible Sheaves). On a quasi-compact scheme \( \mathcal{X} \), an invertible sheaf \( \mathcal{L} \) is called ample iff there is some \( n \geq 1 \) and sections \( s_i \in \Gamma(\mathcal{X}, \mathcal{L}^n) \) that \( X_{s_i} \) is an affine cover of \( \mathcal{X} \). In particular, an ample invertible sheaf is generated by global sections.

For a qc morphism \( f : \mathcal{X} \to \mathcal{Y} \), an invertible sheaf on \( \mathcal{X} \) is called \( f \)-ample iff it is ample restricted to every open subscheme \( f^{-1}(V) \), where \( V \) are affine open in \( \mathcal{Y} \). In particular, an invertible sheaf \( \mathcal{L} \) on a quasi-compact scheme \( \mathcal{X} \) is ample iff it is \( f \)-ample where \( f : \mathcal{X} \to \text{Spec } Z \).

Prop. (V.3.3.5). An invertible sheaf \( \mathcal{L} \) is \( (f-) \)ample iff \( \mathcal{L}^m \) is \( (f-) \)ample.

Prop. (V.3.3.6) (Ample Implies Separatedness). When there is a \( f \)-ample sheaf for \( f : \mathcal{X} \to \mathcal{Y} \) \( \text{qc} \), then \( f \) is separated. In particular, if there is an ample line bundle over a \( \text{qc} \) scheme \( \mathcal{X} \), then \( \mathcal{X} \) is separated.

Proof: [Sta]09MP. \( \square \)

Prop. (V.3.3.7). Let \( \mathcal{X} \) be a \( \text{qc} \) scheme and \( \mathcal{L} \) be an invertible sheaf on \( \mathcal{X} \), \( S = \Gamma_*(\mathcal{X}, \mathcal{L}) \), then the following are equivalent:

1. \( \mathcal{L} \) is ample.
2. The open subsets \( X_s \), where \( s \in \Gamma_*(\mathcal{X}, \mathcal{L}) \) homogeneous, cover \( \mathcal{X} \), and the associated morphism \( \mathcal{X} \to \text{Proj } S \) is an open immersion.
3. The open subsets \( X_s \) where \( s \in \Gamma_*(\mathcal{X}, \mathcal{L}) \) homogeneous, form a topological basis for \( \mathcal{X} \).
4. The open subsets \( X_s \) that is affine and where \( s \in \Gamma_*(\mathcal{X}, \mathcal{L}) \) homogeneous, form a topological basis for \( \mathcal{X} \).
5. For any \( \text{Qco} \) sheaf \( \mathcal{F} \) on \( \mathcal{X} \), the sum of images of the canonical maps \( \Gamma(\mathcal{X}, \mathcal{F} \times \mathcal{L}^n) \otimes \mathcal{L}^{\otimes -n} \to \mathcal{F} \) is surjective.
6. $X$ is quasi-separated, and for any Qco sheaf $\mathcal{F}$ on $X$ of f.t., $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections for $n$ sufficiently large.

**Proof:** Cf.[Sta]01Q3. \hfill \square

**Cor.** (V.3.3.8). The pullback of an ample invertible sheaf along a qc immersion is an ample invertible sheaf.

**Cor.** (V.3.3.9). Let $S$ be a quasi-separated scheme and $X, Y$ be schemes over $S$. If $\mathcal{L}$ is an ample invertible sheaf over $X$ and $\mathcal{N}$ an ample invertible sheaf over $Y$, then $M = \pi_1^* \mathcal{L} \otimes_{\mathcal{O}_{X \times_S Y}} \pi_2^* \mathcal{N}$ is ample over $X \times_S Y$.

**Proof:** Because $X \times_S Y \to X \times Y$ is a qc immersion, by (V.3.3.8), it suffices to show for $S = \text{Spec } Z$. Then if $X_s$ is an affine nbhd of $x$ and $Y_t$ is an affine nbhd of $y$, then $(X \times Y)_{\pi_1^* \otimes \pi_1^*}$ is an affine nbhd of $x \times y$. \hfill \square

**Cor.** (V.3.3.10) (Tensor Product of Ample Invertible Sheaves is Ample). If $M$ is an invertible sheaf generated by global sections and $\mathcal{L}$ is an ample invertible sheaf, then $\mathcal{L} \otimes M$ is ample. In particular if $\mathcal{L}, \mathcal{M}$ are ample invertible sheaves, then $\mathcal{L} \otimes \mathcal{M}$ is ample.

**Proof:** For any $x \in X$ and $U$ a nbhd of $x$, choose $s \in \Gamma(X, \mathcal{L}^n)$ that $x \in X_s \subset U$, and choose $t \in \Gamma(X, \mathcal{M})$ that $t_x \neq 0$, then $x \in X_{s \otimes M} \subset U$, thus $X_r$ form a basis for $X$ where $r \in \Gamma(X, (\mathcal{L} \otimes \mathcal{M})^n)$, so $\mathcal{L} \otimes \mathcal{M}$ is ample. \hfill \square

**Cor.** (V.3.3.11). If $\mathcal{L}$ is ample and $\mathcal{M}$ is an invertible sheaf, then $\mathcal{M} \otimes \mathcal{L}^n$ is ample for $n$ sufficiently large.

**Proof:** This is because $\mathcal{L} \otimes \mathcal{M}^n$ is generated by global sections for $n$ sufficiently large, thus $\mathcal{L} \otimes \mathcal{M}^{n+1}$ is ample, by (V.3.3.10). \hfill \square

**Lemma** (V.3.3.12). For an invertible sheaf $\mathcal{L}$ on a qc scheme $X$, if for each Qco sheaf of ideals $\mathcal{I} \in \mathcal{O}_X$, there is a $n$ that $H^1(X, \mathcal{I} \otimes \mathcal{L}^n) = 0$, then $\mathcal{L}$ is ample.

**Proof:** For any closed pt $P$, choose an open affine nbhd $U$ that $\mathcal{L}$ is trivial, let $Y = X - U$, by the exact sequence $0 \to \mathcal{I}_{Y \cup \{p\}} \to \mathcal{I}_Y \to k(P) \to 0$, we have

$$0 \to \mathcal{I}_{Y \cup \{p\}} \otimes \mathcal{L}^n \to \mathcal{I}_Y \mathcal{L}^n \to k(P) \otimes \mathcal{L}^n \to 0.$$

Thus by assumption we have a surjective map $\Gamma(X, \mathcal{I}_Y \otimes \mathcal{L}^n) \to \Gamma(X, k(P) \otimes \mathcal{L}^n)$. Now $k(P) \otimes \mathcal{L}^n$ is $A/m_P$, so we let $s \in \Gamma(X, \mathcal{I}_Y \otimes \mathcal{L}^n)$ maps to a section in $\Gamma(X, k(P) \otimes \mathcal{L}^n)$ that restricts to $1 \in A/m_P$, then $P \in \text{Supp } s \subset U$ is affine. So we find an affine $X_s$ for every closed pt of $X$, these will cover $X$. \hfill \square

**Prop.** (V.3.3.13) (Serre’s Cohomological Criterion of Ampleness). If $X$ is proper over a Noetherian affine scheme, $\mathcal{L}$ is an invertible sheaf, then the following is equivalent.

- $\mathcal{L}$ is ample
- For each coherent sheaf $\mathcal{F}$, $H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$ for $n$ large enough.
- For each Qco sheaf of ideals $\mathcal{I} \in \mathcal{O}_X$, there is a $n$ that $H^1(X, \mathcal{I} \otimes \mathcal{L}^n) = 0$ (Notice in this case $H$-ample $\iff$ ample).
**V.3. QUASI-COHERENT SHEAVES ON SCHEMES**

**Proof:** 1 → 2: Because \( \mathcal{L}^m \) is \( H \)-very ample for some \( m \), thus \( X \) is projective, then we use Serre theorem (V.6.7.32).

3 → 1: (V.3.3.12). \( \square \)

**Prop. (V.3.3.14)**. \( f : X \to Y \), let \( \mathcal{L} \) be \( f \)-ample on \( X \) and \( \mathcal{M} \) ample on \( Y \), then \( \mathcal{L} \otimes f^* \mathcal{M}^n \) is ample for \( n \) large.

**Proof:** Cf.[Sta]0892? \( \square \)

**Cor. (V.3.3.15)**. If \( f : X \to Y \) is quasi-affine, then the pullback of an ample invertible sheaf is ample, by (V.4.4.19) and (V.3.3.5).

**Prop. (V.3.3.16) (Pullback of Ampleness).** If \( f : Y \to X \) is finite and surjective morphism between schemes proper over a Noetherian affine scheme, then for an invertible sheaf \( \mathcal{L} \) on \( X \), \( \mathcal{L} \) is ample iff \( f^* \mathcal{L} \) is ample.

**Proof:** One direction follows from (V.3.3.15), For the other we use Serre criterion (V.3.3.13) and devissage (V.3.1.27). We only verify 3: By (V.4.4.37), there exists such coherent sheaf \( f_* \mathcal{F} \) for any integral subscheme, and for any Qco sheaf of ideals \( \mathcal{I} \), \( f_* \mathcal{F} = f_*(f^{-1} \mathcal{I} \mathcal{F}) \) because \( f \) is affine, thus

\[
H^p(X, \mathcal{I} f_* \mathcal{F}) = H^p(X, f_*(f^{-1} \mathcal{I} \mathcal{F} \otimes \mathcal{L}^n)) = H^p(Y, f^{-1} \mathcal{I} \mathcal{F} \otimes \mathcal{L}^n)
\]

by projection formula, and \( f \) is affine. This vanish for \( n \) large. \( \square \)

**Prop. (V.3.3.17)**. If \( i : Z \to X \) is a closed immersion that induce homeomorphism on topology between Noetherian schemes, then \( \mathcal{L} \) is ample iff \( i^* \mathcal{L} \) is ample.

In particular, this applies to \( X_{red} \to X \).

**Proof:** Cf.[Sta]09MS. \( \square \)

**Prop. (V.3.3.18)**. Let \( X \) be a scheme. Let \( \mathcal{L} \) be an ample invertible \( \mathcal{O}_X \)-module. Let \( n_0 \) be an integer. If \( H^p(X, \mathcal{L}^{-n}) = 0 \) for \( n \geq n_0 \) and \( p > 0 \), then \( X \) is affine.

**Proof:** Cf.[Sta]0EBD. \( \square \)

**Very Ample Invertible Sheaves**

**Def. (V.3.3.19) (Very Ampleness).** Let \( f : X \to S \) be a morphism, a \( f \)-very ample invertible sheaf on \( X \) is the pullback of \( \mathcal{O}(1) \) along some immersion \( X \to \text{Proj}(\mathcal{E}) \) for some Qco module \( \mathcal{E} \) over \( Y \), Cf.(V.3.2.17). It is called \( H \)-very ample iff \( \mathcal{E} \) is trivial. Notice when \( X \) is proper, this immersion must be closed by (V.4.5.3).

When \( S \) is affine and \( f : X \to S \) is of f.t., \( f \)-very ample is equivalent to \( H \)-very ample.

**Proof:** Cf.[Sta]02NP. \( \square \)

**Prop. (V.3.3.20) (Tensor Product of Very Ample Line Bundles).** Let \( f : X \to \text{Spec} A \) be a morphism. If \( \mathcal{L} \) is \( H \)-very ample and \( \mathcal{M} \) is generated by global sections, then \( \mathcal{L} \otimes \mathcal{M} \) is \( H \)-very ample. In particular, the tensor product of two \( H \)-very ample invertible sheaves is \( H \)-very ample.

**Proof:** The hypothesis means \( \mathcal{L} = \varphi^* \mathcal{O}(1) \), where \( \varphi : X \to \mathbb{P}_A^n \) is an immersion, and \( \mathcal{M} = \psi^* \mathcal{O}(1) \), where \( \psi : X \to \mathbb{P}_k^n \) is a morphism. Then the product \( T : X \to \mathbb{P}_k^n \times \mathbb{P}_k^n \) is also an immersion, by base change trick (V.4.4.2), as \( \mathbb{P}_k^n \times \mathbb{P}_k^n \to \mathbb{P}_k^n \) is separated. Then \( S \circ T : X \to \mathbb{P}^{mn+m+n} \), where \( S : \mathbb{P}_k^n \times \mathbb{P}_k^n \to \mathbb{P}^{mn+m+n} \) is the Segre embedding, is also an immersion, and \((ST)^* \mathcal{O}(1) \cong \mathcal{L} \otimes \mathcal{M} \), thus it is \( H \)-very ample. \( \square \)
Prop. (V.3.3.21) (Ample and $H$-Very Ample). If $f : X \to S$ is locally of f.t. and $\mathcal{L}$ is an ample invertible sheaf on $X$, then $\mathcal{L}^m$ is $H$-very ample for $m$ sufficiently large.

Proof: Choose an affine open cover $\{V_i\}$ of $S$. By (V.3.3.7), there are f.m. affine opens $X_{s_i}$ that cover $X$ refining a inverse image of $\{V_i\}$. Now $\mathcal{O}_X(X_{s_i})$ is f.g. over $\mathcal{O}_S(V_i)$, so we can find f.m. $f_{ij} \in \mathcal{O}_X(X_{s_i})$ that generates it over $\mathcal{O}_S(V_i)$. By (V.3.3.2), we can write each $f_{ij} = s_{ij}/s_i^{e_{ij}}$ for some $a_{ij}$ homogenous. We can multiply by a factor to make all the $s_i^{e_{ij}}$ the same degree $N$, and $f_{ij} = s_{ij}s_i^{N/\deg(s_i)-e_{ij}}$, then all the elements $s_i$, $s_{ij}$ generates the invertible sheaf $\mathcal{L}$, thus inducing a map $j : X \to \mathbb{P}_k^m$. This map is an immersion, because $j^{-1}(D(T_i)) = X_{s_i}$ and the function $T_{ij}/T_i$ on $D(T_i)$ pulls back to $s_{ij}/s_j$. Thus $j$ is locally a closed immersion, thus an immersion.

Now $\mathcal{L}^{\otimes d_i}$ is $H$-very ample for some $d_i$, in particular it is separated, and by (V.3.3.7) there is some $d_2 \mathcal{L}^{\otimes d}$ is generated by global sections for $d \geq d_2$, then by (V.3.3.20), $\mathcal{L}^{\otimes d}$ is $H$-very ample for $d \geq d_1 + d_2$. □

Prop. (V.3.3.22) (f-Very Ample Implies f-Ample). If $f : X \to S$ is qc, then f-very ample implies f-ample.

Proof: Cf.[Sta]01VN. □

Cor. (V.3.3.23) (Serre). If $f : X \to S$ is of f.t. and $S$ is affine, $\mathcal{L}$ is an invertible sheaf on $X$, then the following are equivalent:

- $\mathcal{L}$ is ample.
- $\mathcal{L}$ is f-ample.
- $\mathcal{L}^{\otimes n}$ is $(H)$-f-very ample for some (all large)$n$.

Proof: This follows from (V.3.3.19) (V.3.3.21) and (V.3.3.22). □

Cor. (V.3.3.24). If $f : X \to S$ is of f.t. and $S$ is quasi-compact, $\mathcal{L}$ is an invertible sheaf on $X$, then the following are equivalent:

- $\mathcal{L}$ is f-ample.
- $\mathcal{L}^{\otimes n}$ is $(H)$-f-very ample for some (all large)$n$.

Proof: Cf.[Sta]01VU. □

Prop. (V.3.3.25). When $X$ is Noetherian and has an $H$-ample invertible sheaf, any coherent sheaf is a quotient of a a finite direct sum of $\mathcal{O}(-n)$.

Proof: This is because $X$ is qc and $\mathcal{F}(n)$ is globally generated for some $n$. So for any pt $p$ we find a.m. section that generate the stalk, then by coherence, there is a nbhd that generate the stalk, and the compactness shows that there is f.m that generate the stalk, thus $\mathcal{O}_X^N \to \mathcal{F}(n)$ surjective, then we tensor it with $\mathcal{O}_X(-n)$. □

4 Sheaf of Differentials

Prop. (V.3.4.1) (Differentials on Schemes). Consider a morphism of schemes $X \to Y$, we define the sheaf of differentials $\Omega_{X/Y}$ together with an $S$-derivative $\mathcal{O}_X \to \Omega_{X/S}$ as for ringed sites (V.2.4.7). Then it is a Qco sheaf by (V.2.4.5). In fact, If $U = \text{Spec } A$ is mapped into $\text{Spec } B \subset S$, then $\Omega_{X/S}(U) \cong \Omega_{A/B}$.

Thus when $X \to Y$ is locally of f.t.(or locally of f.p.), then $\Omega_{X/S}$ is an $\mathcal{O}_X$-module of f.t.(or of f.p.).
Prop. (V.3.4.2) (Base change and Differentials). Let \( X' \xrightarrow{f} X \)
\( \xrightarrow{g'} S' \xrightarrow{g} S \)
be a commutative diagram
of schemes, then there is a canonical map
\[ f^* \Omega_X/S \to \Omega_{X'}/S' \]
which is an isomorphism if the diagram is a fiber product square.

In particular, the stalk of \( \Omega_X/S \)
at \( p \) is \( \Omega_{X'/S'} 
(\mathfrak{p}) \).

\[ \text{Proof:} \quad \text{Such a diagram gives a diagram} \]
\[ (f^{-1} \mathcal{O}_S \to f^{-1} \mathcal{O}_X) \to ((g')^{-1} \mathcal{O}_{S'}) \to \mathcal{O}_{X'}) \]
of sheaves of rings on \( X'_{\text{Zar}} \), thus the conclusion follows from (V.2.4.5) and (V.2.7.17).

Prop. (V.3.4.3). Let \( X, Y \) be schemes over another scheme \( S \), then
\[ \pi_1^* \Omega_X/S \oplus \pi_2^* \Omega_Y/S \cong \Omega_{X \times_S Y}/S, \]
where the maps are given by (V.3.4.2).

\[ \text{Proof:} \quad \text{It suffices to check on affine subschemes, so we may assume} \]
\( X, Y, S \) are affine, thus the map is given by
\[ \mathcal{O}_A/S \oplus \mathcal{O}_B \oplus \mathcal{O}_A \otimes \mathcal{O}_B \to \mathcal{O}_A \otimes \mathcal{O}_B \]
which is an isomorphism by (I.7.3.6).

Prop. (V.3.4.4). Let \( X, Y \) be schemes over another scheme \( S \), then \( \Omega_{X \times_S Y}/S \cong \pi_1^* \Omega_X/S \otimes \pi_2^* \Omega_Y/S \).

Prop. (V.3.4.5) (Jacobi-Zariski Sequence). Let \( f : X \to Y \) and \( g : Y \to Z \), then there is an
exact sequence of sheaves on \( X \):
\[ f^* \Omega_Z \to \Omega_X \to 0. \]

Where the maps come from (V.3.4.2).

\[ \text{Proof:} \quad \text{Immediate from (I.7.3.8).} \]

Prop. (V.3.4.6). The stalk of the differential sheaf \( \Omega_{X/k} \) at a rational point \( x \) of a scheme over a field \( k \) is just the Zariski cotangent space \( m_x/m_x^2 \).

\[ \text{Proof:} \quad \text{Using the Jacobi exact sequence} \]
\[ (\text{I.7.3.8}) \text{ on an affine nbhd Spec } \mathcal{A} \text{ of } x \text{ for } \mathcal{A} \text{ and } m_x. \]

Then we verified that there is a right inverse of \( A/m^2 \to k(x) = x \), then it follows that \( m_x/m_x^2 \cong \\
\Omega_{A/k} \otimes_A k(x) = \Omega_{A_{m_x}/k} \) which is the stalk of \( \Omega_{X/k} \) by (V.3.4.1).

Prop. (V.3.4.7). If \( X = \mathbb{P}^n_A \) over \( Y = \text{Spec } A \), then there is an exact sequence
\[ 0 \to \Omega_X \to (\mathcal{O}_X(-1))^{n+1} \to \mathcal{O}_X \to 0. \]

This is because locally the kernel is generated by \( (e_j - (x_j/x_i)e_i)/e_i = d((x_j/x_i)) \).

When \( A \) is a field \( k \), this sequence is locally free by (V.8.1.17), so taking dual we get:
\[ 0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{n+1} \to T_X \to 0. \]

Taking highest exterior power we get \( \omega_X \cong \mathcal{O}_X(-n-1) \).
Conormal Sheaves

Def. (V.3.4.8) (Conormal Sheaf of an Immersion). Let \( i : Z \to X \) be a closed immersion with corresponding sheaf of ideals \( \mathcal{I} \). Consider the \( \mathbb{Q}_{\text{co}} \) sheaf \( \mathcal{I}/\mathcal{I}^2 \), which is annihilated by \( \mathcal{I} \), thus corresponds to a sheaf on \( Z \) by (V.2.6.18), called the conormal sheaf \( \mathcal{C}_{Z/X} \) of \( Z \).

More generally, if \( i \) is any immersion, we can define the conormal sheaf as the conormal sheaf of the closed immersion \( i : Z \to X \setminus \partial Z \). And also the normal sheaf \( \mathcal{N}_{Z/X} \) is defined to be \( \mathcal{N}_{Z/X} = \text{Hom}_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z) \).

Prop. (V.3.4.9) (Pullback of Conormal Sheaf). Let \( \begin{array}{ccc} Z' & \xrightarrow{i'} & X \\ \downarrow f & & \downarrow g \\ Z & \xrightarrow{i} & X' \end{array} \) be a fiber product square where \( i, i' \) are immersions, then the canonical map \( f^*\mathcal{C}_{Z'/X'} \to \mathcal{C}_{Z/X} \)

is surjective, and if \( g \) is flat, it is an isomorphism.

Proof: Change \( X' \) to \( X' \setminus \partial Z' \) and \( X \) to \( X \setminus (g^{-1}\partial Z' \cup \partial Z) \), we may assume \( i \) is a closed immersion. Then we may localize to the case \( X' \) and \( X \) is affine. Then we notice if \( R' \to R \) is a ring map and \( I' \subset R' \) is an ideal, with \( I = I'R \), then \( (I'/I'^2) \otimes_{R'} R \to I/I^2 \) is surjective, and if \( R/R' \) is flat, then \( I \cong I' \otimes_{R'} R \), and the map is an isomorphism. \( \square \)

Prop. (V.3.4.10). Let \( \begin{array}{ccc} Z & \xrightarrow{i} & Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{j} & Y \end{array} \) be immersions of schemes, then there is a canonical exact sequence

\[ i^*\mathcal{C}_{Y/X} \to \mathcal{C}_{Z/X} \to \mathcal{C}_{Z/Y} \to 0 \]

Proof: By changing \( Y \) to \( Y \setminus \partial Z \) and \( X \) to \( X \setminus (\partial(Y \setminus Z)) \), we can assume the immersions are closed immersion. Now by restricting to affine subsets, it suffices to show that for surjective ring maps \( C \to B \to A \), if \( I = \text{Ker}(B \to A) \), \( J = \text{Ker}(C \to A) \), \( K = \text{Ker}(C \to B) \), then there is an exact sequence

\[ K/K^2 \otimes_B A \to J/J^2 \to I/I^2 \to 0. \]

But this follows from the observation \( K = \text{Ker}(J \to I) \). \( \square \)

Prop. (V.3.4.11) (Conormal Sheaf of the Diagonal). Let \( f : X \to S \) be a morphism, then there is a canonical isomorphism between \( \Omega_{X/S} \) and the conormal sheaf of the diagonal \( \Delta : X \to X \otimes_S X \).

Proof: Cf. [Sta]08S2. \( \square \)

Cor. (V.3.4.12). If \( f : X \to S \) is a monomorphism, e.g. an immersion, then \( \Omega_{X/S} = 0 \).

Prop. (V.3.4.13). If \( f : Z \to X \) is an immersion of schemes over \( S \), then there is an exact sequence of sheaves on \( Z \):

\[ \mathcal{C}_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0. \]

Proof: Replace \( X \) be \( X \setminus \partial Z \), we can assume \( f \) is a closed immersion. This follows immediate from (V.2.4.6). \( \square \)
Prop. (V.3.4.14). If \( i : Z \to X \) is an immersion over \( S \) that locally has a left inverse, then the canonical sequence
\[
0 \to C_{Z/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0.
\]
is locally split exact. In particular, if \( s : S \to X \) is a section of the structure morphism \( X \to S \), then the map \( C_{S/X} \to s^*\Omega_{X/S} \) is an isomorphism.

Proof: Cf.

Prop. (V.3.4.15). Let \( Z \xrightarrow{i} X \xrightarrow{j} Y \) be a commutative diagram where \( i, j \) are immersions, then there is a canonical exact sequence
\[
C_{Z/Y} \to C_{Z/X} \to i^*\Omega_{X/Y} \to 0,
\]
where the first arrows comes from (V.3.4.9) and the second map comes from (V.3.4.13).

Proof: By replacing \( Y \) by \( Y \setminus \partial Z \) and \( X \) by \( X \setminus (\partial i(Z) \cup \partial j(Z)) \), then we may assume \( i, j \) are closed immersion. Then we check locally, the exactness follows from (XI.2.1.5).

5 Others

Def. (V.3.5.1) (Frobenius). For a scheme over \( \mathbb{F}_p \), the absolute Frobenius is the unique morphism of schemes \( \text{Frob}_{p,X} : X \to X \) that is \( x \to x^p \) on the sections. The Frobenius is functional in \( X \).

For a scheme \( X \) over a scheme \( S \) over \( \mathbb{F}_p \), consider the base change \( X^{(p,S)} \to X \) of \( \text{Frob}_{p,S} \) by \( X \to S \), the functionality of Frobenius and the universal property of base change gives us a morphism \( F_{X/S} : X \to X^{(p,S)} \), which is called the relative Frobenius of \( X \) over \( S \).

For a scheme \( X \) over \( \mathbb{F}_q \), \( q = p^n \), then geometric Frobenius is defined to be \( \pi_X = \text{Frob}_{X}^n \), as a morphism of schemes over \( \mathbb{F}_q \). More generally, if \( X \) is a scheme over a scheme \( S \) over \( \mathbb{F}_q \), and there is a scheme \( X_0 \) over \( \mathbb{F}_q \) that \( X_0 \otimes_{\mathbb{F}_q} S = X \), then the geometric Frobenius is defined to be the extension of scalers of the geometric Frobenius \( \pi_{X_0} \) to \( S \), i.e, the map \( \pi_X : X \to X \) induced by \( X \to X_0 \xrightarrow{\pi_{X_0}} X_0 \).

Remark (V.3.5.2). If \( S = \text{Spec} R \) and \( X = \text{Spec} R[X_1, \ldots, X_n]/I \), then \( X^{(p)} = \text{Spec} R[X_1, \ldots, X_n]/I^{(p)} \), and \( F_{X/S} \) is given by \( r \to r, X_i \to X_i^p \).

Cor. (V.3.5.3). Frobenius is the relative Frobenius over \( \mathbb{F}_p \). Relative Frobenius is universal homeomorphism, because Frobenius is universal homeomorphism (I.5.3.22) and use definition, in particular, it is integral.

The relative Frobenius has nice functoriality properties. It is functorial for schemes over \( S \), and for \( X \to S \) and \( T \to S \), \( \text{Frob}_{X/T} \) is the base change of \( \text{Frob}_{X/S} \) by \( (X_T)^{(p,T)} \to X^{(p,S)} \). The composition \( \text{Frob}_{X/S}^n \) has two definitions, they are equal, Cf.

Prop. (V.3.5.4). If \( X \) is a scheme over a field \( k \) of char \( p \), then \( X^{(p)} \) is reduced if \( X \) is geometrically reduced. This follows from (V.4.3.2).
Schemes as Functors

Def. (V.3.5.5) (Open Subfunctors). A subfunctor $H \subset F$ of functors on the category of schemes is called an open immersion if for any $h_X \to F$ where $X$ is a scheme, the pullback $h_X \times_F H$ is an open subscheme of $X$.

Def. (V.3.5.6) (Covering Subfunctors). A covering of a functor $F$ on the category of schemes is a family of open subfunctors $F_i$ that for any $h_X \to F$ where $X$ is a scheme, $h_X \times_F F_i$ is an open covering of $X$.

Prop. (V.3.5.7) (A Representability Criterion). Let $F$ be a Zariski sheaf on $Sch_{\text{Zar}}$, and there is a covering $F = \cup F_i$ by open subfunctors (V.3.5.5)(V.3.5.6) that are representable by schemes, then $F$ is representable by a scheme.

Proof: Let $(X_i, \xi_i)$ represents $F_i$, where $\xi_i \in F_i(X_i)$. Because $F_j \subset F$ is representable by open immersion, there are open subsets $U_{ij} \subset X_i$ that $T \to X_i$ factors through $U_{ij}$ if $\xi_i|_T \in F_i(T)$. In particular, $\xi_i|U_{ij} \subset F_j(U_{ij})$, and therefore there is a canonical map $\varphi_{ij} : U_{ij} \to X_j$ that $\varphi_{ij}^*\xi_j = \xi_i|U_{ij}$. By definition of $U_{ij}$ this map factors through $U_{ij}$.

For the rest, Cf.[Sta]01JJ.

Prop. (V.3.5.8) (Strong Yoneda Lemma). The map $X \mapsto \tilde{X} : \text{Spec} R \mapsto \text{Hom}(\text{Spec} R, X)$ is a fully faithful embedding of the category $Sch_S$ into the category of presheaves on $Aff_S$.

Proof: It is faithful because any a map $Y \to X$ is determined by an affine covering of $Y$. To show it is full, notice they are Zariski sheaves on $Aff_S$, and a covering of $Y$ glue together to give a map $Y \to X$.

Def. (V.3.5.9) (Closed Subfunctors). Let $Z$ be a subfunctor of a functor $X$ on $Alg_k$. We say $Z$ is a closed subfunctor of $X$ if for any map of functors $f : h^A \to X$, the subfunctor $Z \otimes_X h^A$ of $h^A$ is represented by a quotient of $A$.

Prop. (V.3.5.10) (Closed Subfunctor of Schemes). Let $X$ be a scheme over $k$, then the closed subfunctors of $X$ are exactly of the form $Z$ for a closed subscheme $Z$ of $X$.

Proof: If $Z$ is a closed subscheme of $X$, then for any $f : h^A \to X$, $f^{-1}(Z)$ is the pullback of $Z$ along $Spec A \to X$, so it is a closed subscheme of $Spec A$, so represented by some quotient of $A$. Conversely, if $Z$ is a closed subfunctor of $X$, then for each affine open subset $U$ of $X$, $Z \cap h_U$ is represented by a quotient of $O(U)$ by some ideal $I(U)$. Because of the uniqueness, $I(U)$ and $I(U')$ coincides on the intersection $U \cap U'$, thus $U \mapsto I(U)$ defines a sheaf of ideals $I$ on $X$.

Now $Z = h_{Z'}$, where $Z'$ is the closed subscheme of $X$ defined by $I$, because for any $Spec R \to X$, the pullback of $Z$ and $Z'$ to $R$ are the same, because they are all closed subschemes of $Spec R$ and they are equal on an open covering of $Spec R$(The pullback of the open coverings of $X$). Now if $Spec R \to X$ is represented by an element $\alpha \in X(R)$, $Z \times_X h^R(R)$ is the set $\{\varphi \in \text{Hom}(R, R)|X(\varphi)(\alpha) \in Z(R)\}$. So $id_R \in Z \times_X h^R(R) \iff \alpha \in Z(R)/R.$ From this we see that $Z(R) = Z'(R)$ for any $R$.

Prop. (V.3.5.11). The pullback of a closed subfunctor is also a closed subfunctor. The intersection of closed subfunctors is a closed subfunctor.

Lemma (V.3.5.12). Let $B$ be a $k$-algebra and $X$ a functor, define $X_*$ to be the functor that $X_*(R) = X(R \otimes_k B)$. Then if $Z$ is a closed subfunctor of $X$, $Z_*$ is also a closed subfunctor of $X_*$. 
Proof: Let $A$ be a $k$-algebra, and $\alpha \in X_*(A)$. To prove $Z_*$ is closed in $X_*$, we need to show there exists an ideal $a \subset A$ that for any homomorphism $\varphi : A \to R$,

$$X_*(\varphi)(\alpha) \in Z_*(R) \iff \varphi(a) = 0.$$  

Because $Z$ is closed in $X$, there exists an ideal $b$ of $A \otimes_k B$ that for any $\varphi : A \to R$,

$$X(\varphi \otimes B)(\alpha) \in Z_*(R) \iff (\varphi \otimes B)(b) = 0.$$  

Now by (I.5.1.21), there is an ideal $a \subset A$ that an ideal $I$ of $A$ satisfies $b \subset I \otimes B \iff a \subset I$, thus we are done.  

□

Prop. (V.3.5.13). Let $Z$ be a closed subfunctor of a functor $X$ on $\text{Alg}_k$. If $Y$ is an scheme, then $\mathcal{H}om(Y, Z)$ is a closed subfunctor of $\mathcal{H}om(Y, X)$.

Proof: If $Y = h^B$, then $\mathcal{H}om(Y, X)(R) = X(B \otimes R)$, thus the conclusion follows from (V.3.5.12). For a general $Y$, let $Y_i$ be an affine open covering of $Y$, then there are maps $\rho_i : \mathcal{H}om(Y, X) \to \mathcal{H}om(Y_i, X)$. Now $\mathcal{H}om(Y_i, Z)$ is closed subfunctor of $\mathcal{H}om(Y_i, X)$, thus we are done if we can show that $\mathcal{H}om(Y, Z) = \cap \rho_i^{-1}(\mathcal{H}om(Y_i, Z))$. But this is equivalent to any map $Y_R \to X_R$ that maps $(Y_i)_R$ into $Z_R$ maps $Y_R$ into $Z_R$, which is clear. □

Def. (V.3.5.14) (Fat Subfunctor). Let $F$ be a sheaf of sets on $\text{Alg}_k$ w.r.t. the fppf topology, then a subfunctor $D$ of $F$ is called a fat subfunctor if the shifification of $D$ w.r.t. the fppf topology is just $F$.

Geometry of schemes

Def. (V.3.5.15) (Tangent Spaces). Let $X$ be a scheme and $x \in X$. For $A = k(x)[\varepsilon]/\varepsilon^2$, an $A$-point of $X$ corresponds to a point $x$ and an element in the dual of the $k(x)$-space $m_x/m_x^2$, i.e. the Zariski tangent space. (notice the local map).

Prop. (V.3.5.16) (Smoothness and Tangent Spaces). By (V.5.3.11) and the definition of regular local ring, if $X$ is a locally algebraic scheme over a field, then $X$ is smooth at a point $x$ iff for any base change of fields, the dimension of tangent space of $X_K$ at a pt $x'$ over $x_K$ equals the dimension of $O_{X_K,x'}$.
V.4 Properties of Schemes

Main References are [Har77], [Sta] and [Hartshorne Solution 田翊].

1 Basic Scheme Properties

Affine Local Properties of Schemes

Lemma (V.4.1.1) (Nike’s Trick). In a scheme $X$ and $x \subset \text{Spec } A \cap \text{Spec } B$, $x$ has an open nbhd in $\text{Spec } A \cap \text{Spec } B$ that are distinguished in both $\text{Spec } A$ and $\text{Spec } B$.

Proof: Choose a nbhd of $x$ that is distinguished in $\text{Spec } A$ that is contained in $\text{Spec } A \cap \text{Spec } B$, then because distinguished of distinguished is distinguished, we may assume $i : \text{Spec } A \subset \text{Spec } B$. Now let $f \in B$ be an element that $D(f) \subset \text{Spec } A$, then I claim $D(i^\#(f)) = D(f)$, this will finish the proof, but this is equivalent to $i^{-1}(\text{Spec } B) = \text{Spec } A_{i^\#(f)}$, which is true for ideal-theoretical reason.

Prop. (V.4.1.2) (Affine Communication Theorem). A property $P$ of affine open subsets is called affine local if: $\text{Spec } (A)$ has $P \Rightarrow$ all $\text{Spec } (A_f)$ has $P$, and any cover of $\text{Spec } (A_f_i)$ has $P \Rightarrow \text{Spec } (A) has P$. Notice a stalk-wise property is obviously affine-local.

Now if we call $X$ has $\tilde{P}$ if $X = \bigcup_i \text{Spec } A_i$ that $A_i$ has $P$. Then the following is equivalent:

- all open affine subscheme of $X$ has $P$.
- all open subscheme of $X$ has $\tilde{P}$.
- $X$ has a cover of open subschemes that has $\tilde{P}$.
- $X$ has $\tilde{P}$.

Proof: $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$ is obvious, only need to prove $4 \rightarrow 1$: if $X = \bigcup \text{Spec } A_i$, for any open affine subscheme of $X$, by(V.4.1.1), it can be covered by distinguished opens that are also distinguished in some $\text{Spec } A_i$, so by hypothesis it has $P$.

Remark (V.4.1.3). When proving locality of morphism properties using affine communication theorem, one usually resort to 1.

Prop. (V.4.1.4) (List of Stalkwise Properties). All properties defined by a ring-theoretic property that is stalkwise. (I.5.1.55)

Prop. (V.4.1.5) (List of properties affine local on the target). (All the property besides the $H$-projectiveness is local on the target).

1. Because affineness is local on the target(V.1.5.25), all properties defined by a ring-theoretic property local on the target(I.5.1.57).
2. All properties that is stalkwise.
3. All properties that satisfies faithfully flat descent.(V.1.5.25)
4. Locally projective morphism.

Prop. (V.4.1.6) (List of properties affine local on the source). (not complete)

1. All properties defined to be local ring map property local on the source(I.5.1.57)
2. Openness.

Proof:  
1. Trivial.  
2. Trivial.  

Irreducible

Def. (V.4.1.7). A scheme is called irreducible if its underlying topological space is irreducible.

Prop. (V.4.1.8) (Nearly Affine Local). For a scheme, the following are equivalent:  
1. It is irreducible.  
2. There is an affine cover $U_i$ of $X$ that $U_i$ are all irreducible and $U_i \cap U_j \neq \emptyset$.  
3. Every affine open subset of $X$ is irreducible.

Proof: A scheme is sober (V.2.7.11), if $X$ is irreducible, then $X$ has a unique generic pt $\eta$ that $\overline{\{\eta\}} = X$, then 2, 3 all holds. If 2 holds, then for a decomposition $X = Z_1 \cup Z_2$, any $U_i$ belongs to $Z_1$ or $Z_2$, so it is easy to see $Z_1 = X$ or $Z_2 = X$. If 3 holds, then choose an affine cover $U_i$ of $X$, then $U_i \cap U_j \neq 0$, otherwise $U_i \coprod U_j$ is affine and not irreducible, contradiction, so 2 holds.

Cor. (V.4.1.9). The fiber product of irreducible schemes is irreducible, because.

Reducedness

Def. (V.4.1.10) (Reduced Schemes). A scheme is called reduced if $\mathcal{O}_X(U)$ is reduced for every open set $U$. Reduced is a stalk-wise property (V.4.1.5), it suffices to check reducedness at closed pts.

Prop. (V.4.1.11). For a reduced scheme $X$, $\Gamma(X, \mathcal{O}_X) \to \prod_{x \in X} k(x)$ is injective.

Prop. (V.4.1.12) (Reduction). There is a $X_{red} \to X$ associated to every scheme, it is $\text{Spec}(\mathcal{O}_X/\mathcal{N})$ where $\mathcal{N}$ is the sheaf of nilpotent elements. This construction is right adjoint to the forgetful functor by the adjoint property of $\text{Spec}$ (V.2.7.12). $X_{red} \to X$ is an closed immersion.

It’s useful to change to $X_{red}$ when the proposition only involve topology because $X_{red}$ has the same topology as $X$. A map can induce a map on their reduced structure.

Prop. (V.4.1.13) (Induced Reduced Scheme Structure). Let $Z$ be a locally closed subset of a scheme $X$, There is a unique reduced subscheme $Z_{red}$ of $X$ with underlying topological space $Z$, called the induced reduced scheme structure of $Z$. It has the universal property that any morphism from a reduced scheme $Y$ to $X$ that has image in $Z$ factors through this subscheme (By virtue of reducedness).

In particular, there is a closed subscheme structure $X_{red}$ of $X$, called the underlying reduced subscheme of $X$.

Proof: The uniqueness is clear by the universal property. The existence is clear when $X$ is affine and $Z$ is closed in $X$. Then we can use the uniqueness to glue them to a global subscheme structure.
Integral

**Def. (V.4.1.14) (Integral Schemes).** A scheme $X$ is called integral if $O_X(U)$ are all integral. This is equivalent to reduced and irreducible. So a scheme is integral iff there is an integral open affine cover that are pairwise-intersect (V.4.1.7).

**Proof:** If $X$ is irreducible and reduced, then so does any affine subscheme $\text{Spec } R$, so $R$ is integral as $(0)$ is the generic prime, because it has only one minimal prime consisting of nilpotent elements. Conversely, if $X$ is reduced, then any affine subscheme $\text{Spec } R$ is integral so reduced, and is irreducible by the presence of prime $(0)$. □

**Cor. (V.4.1.15).** The projective space over an integral scheme is integral. (Check the affine covers are dense). The projective space $P^n_{\mathbb{Z}}$ is integral.

**Prop. (V.4.1.16) (Integral is Almost Stalkwise).** Let $X$ be a non-empty and connected scheme, then $X$ is integral iff all the

**Def. (V.4.1.17) (Function Field).** Let $X$ be an integral scheme with generic point $\eta$, then $R(X) \cong O_{X,\eta}$ is a field (V.2.8.2), called the function field of $X$, denoted by $K(A)$. Then any rational function on $X$ is defined on an open dense subset of $X$.

**Prop. (V.4.1.18).** If $X$ is an integral scheme and $Z_1, Z_2$ are closed subschemes of $X$ with generic points $\eta_1, \eta_2$, then $O_{X,\eta_1} \nsubseteq O_{X,\eta_2}$. In particular, if $Z = \{x\}$ consists of a closed point, then there is a rational function defined near $x$ that is not in $O_{X,\eta_2}$.

**Proof:** [Sta]02NF. □

Noetherian

**Def. (V.4.1.19) (Noetherian Scheme).** A scheme is called locally Noetherian if it can be covered by open affine schemes of noetherian rings. It is called Noetherian if moreover it is quasi-compact.

(Locally)Noetherian is affine local (V.4.1.5).

**Prop. (V.4.1.20) (Noetherian Scheme is Noetherian).** The underlying space of a Noetherian scheme is a Noetherian space.

**Proof:** By (IX.1.14.3), we are reduced to the affine case. Now it is clear from the definition. □

**Prop. (V.4.1.21).** Any locally closed subscheme of a (locally)Noetherian scheme is (locally)Noetherian. In particular, an subset of a Noetherian scheme is quasi-compact.

**Proof:** This is because any localization and quotients of a Noetherian ring is Noetherian (I.5.1.34), and any subset of a Noetherian space is quasi-compact (IX.1.14.2) (V.4.1.20). □

**Prop. (V.4.1.22) (Finitely Many Irreducible Components).** For a closed subscheme in a locally Noetherian space, the collection of its irreducible components is locally finite in $X$, because a Noetherian space has f.m. irreducible components (IX.1.14.4).

**Prop. (V.4.1.23).** Let $k'/k$ be a f.g. field extension, then a scheme $X$ over $k$ is locally Noetherian iff $X_{k'}$ is locally Noetherian.

**Proof:** Locally Noetherian is affine local, so the problem is totally ring-theoretic. If $X_{k'}$ is Noetherian, then so does $X$ by ff descent (I.7.2.1). If $X$ is Noetherian, then so does $X_{k'}$ by (I.5.1.38). □
Jacobson

**Def. (V.4.1.24).** An scheme is called Jacobson iff its underlying topological space is Jacobson (IX.1.14.19). In particular, an affine scheme Spec $R$ is Jacobson iff $R$ is Jacobson (I.5.9.4).

So by (IX.1.14.20), being Jacobson is a local property.

**Prop. (V.4.1.25) (Locally Algebraic Scheme is Jacobson).** For a scheme locally algebraic over a field $k$, the set of closed points $X_0$ is dense in every closed subset of $X$, because it is a Jacobson space by (IX.1.14.20) and (I.5.9.9). Equivalently, every locally closed subset of $X$ contains a closed point.

Moreover, the residue field of a closed point is finite over $k$ by (I.5.9.9), and the converse is also true. In particular, by (V.2.7.9), the closed points of $X$ are just the geometric points.

**Proof:** For the converse, because $k \subset A/p_x \subset k(x)$ are finite hence integral extensions, by (I.5.5.3) $A/p_x$ is a field, thus $x$ is a closed point.

**Cor. (V.4.1.26) (Algebraic Scheme Preserves Closed Points).** A morphism between algebraic schemes over a field $k$ maps closed points to closed points.

**Remark (V.4.1.27).** When $X$ is geo.reduced, a stronger statement shows the set of separable closed points of $X$ is dense in $X$, Cf. (V.4.3.3).

**Cor. (V.4.1.28) (Check Surjectiveness on Geometric Points).** If a morphism of algebraic schemes over a field $k$ is surjective on geometric points, then it is surjective.

**Proof:** By Chevalley theorem (V.5.8.6), the image is a constructible set, thus the supplement set is also constructible. Now if it is not surjective, then there is an open closed subset $U \cap Z$ not in the image. But this set contains a closed point, which is a contradiction by (V.4.1.25).

Cohen-Macaulay

**Def. (V.4.1.29).** A scheme is called C.M. iff all its stalks is C.M. local.

2 Normal & Regular

**Def. (V.4.2.1) (Normal & Regular Schemes).** A scheme is called normal if all its stalks are normal domains (I.6.5.1), or equivalently all its affine sections are normal rings. In particular, a normal scheme is reduced.

A locally Noetherian scheme is called regular iff all its stalk are regular local rings (I.6.5.13), i.e. all affine opens are regular rings. Regular only have to be checked at close pt by (I.6.5.13).

**Prop. (V.4.2.2) (Normalization).** For an integral scheme $X$, there is a $X_{nom} \to X$ which is Spec$(\mathcal{O}_{X,nom})$, any dominant morphism $f$ from a normal integral scheme to $X$ will factor through $X_{nom}$. (Use the adjointness for Spec and notice $f$ maps generic to generic).

**Proof:**

**Prop. (V.4.2.3).** Let $X$ be a locally Noetherian scheme, then $X$ is normal iff it is a disjoint union of integral normal schemes.
Proof: Cf.[Sta]033N.

Cor. (V.4.2.4). A normal scheme is integral iff it is connected.

Prop. (V.4.2.5). If \( X \) is an integral normal scheme, then \( \Gamma(X, \mathcal{O}_X) \) is a normal ring.
Proof: Cf.[Sta]0358.

Prop. (V.4.2.6) (Regular and Normal). Regular scheme is C.M and locally factorial, hence normal, by(I.6.5.17) and(I.6.5.16). A Normal scheme is regular in codimension 1, by(I.6.5.25).

Prop. (V.4.2.7). For a locally Noetherian scheme of dimension \( \leq 1 \), normal is equivalent to regular.
Proof: This is because for a Noetherian local domain of dim 1, principal \( \iff \) normal \( \iff \) regular \( \iff \) DVR.

Dedekind Scheme

Def. (V.4.2.8) (Dedekind Scheme). A Dedekind scheme is an integral Noetherian normal scheme of dimension 1.

Prop. (V.4.2.9). Let \( X \) be a Dedekind scheme and \( x \in X \) is a closed pt, let \( \hat{X} = \text{Spec}(\hat{O}_{X,x}) \to X \) be the completion of \( X \) at \( x \), then there is a pullback of categories:

\[
\begin{array}{ccc}
\text{Bun}_X & \to & \text{Bun}_{X-\{x\}} \\
\downarrow & & \downarrow \\
\text{Bun}_{\hat{X}} & \to & \text{Bun}_{\hat{X}-\{x\}}
\end{array}
\]

Proof: We may study locally near \( x \), then we can assume that \( X \) is affine. Now shrink \( X \) even more, we can assume that \( x \) is defined by a single \( f \in A(\text{localized at the maximal ideal defined by } x) \), then we finish by(I.7.2.12).

3 Geometrical properties

Def. (V.4.3.1).
- A scheme \( X \) is called geometrically integral/reduced/separated/irreducible... over a field \( k \) iff for any field extension \( k'/k \), \( X_{k'} \) is integral/reduced/separated/....
- A locally Noetherian scheme is called geometrically regular iff for any f.g. field extension \( K/k \), \( X_K \) is regular. It is stalkwise by(V.4.3.19).

Geo.reducedness

Prop. (V.4.3.2) (Geo.Reduced). For a scheme \( X \) over a field \( k \), the following are equivalent:
1. \( X \) is geometrically reduced.
2. For every reduced \( k \)-scheme \( Y \), the product \( X \otimes_k Y \) is reduced.
3. All stalks are geometrically reduced ring.
4. \( X \) is reduced and for every maximal point \( \eta \) of \( X \), the residue field \( k(\eta) \) is separable over \( k \).
5. $X_{k_{sep}}$ is reduced.
6. $X_K$ is reduced for every finite purely inseparable field extension $K/k$.
7. $X_{k_{s1/p}}$ is reduced.

Proof: As reduced is local, these all follows from (I.6.6.2).

Prop. (V.4.3.3) (Geo.Reduced and Geometric Points). Let $X$ be a locally algebraic geo.reduced scheme over a field $k$, then the set of closed points with finite separable field extensions $k(x)/x$ is dense in $X$.

Proof: Combine (II.1.1.40) and (V.5.3.15).

Def. (V.4.3.4) (Density of Points). Let $X$ be an algebraic scheme over a field $k$ and $k'/k$ be a field extension, then a subset $S \subset X(k')$ is said to be schematically dense in $X$ if the only points closed subscheme $Z \subset X$ over $k$ that $S \subset Z(k')$ is $X$ itself.

Prop. (V.4.3.5) (Schematically Dense Subset). Let $X$ be an algebraic scheme over a field $k$, $S \subset X(k)$ be a subset. Then the following are equivalent:
1. $S$ is schematically dense in $X$.
2. $X$ is reduced and $S$ is dense in $|X|$.
3. The family of homomorphisms $O_X \to k : f \mapsto f(s)$ is jointly injective.

Proof: 1 $\Rightarrow$ 2: Let $\mathcal{S}$ be the induced reduced structure of the closure $S$ in $X$ (V.4.1.13), then $\mathcal{S} = X$, so $X$ is reduced with $X = \mathcal{S}$.

2 $\Rightarrow$ 3: Cf. [Milne Algebraic Groups, P10].

Cor. (V.4.3.6). A schematically dense subset remains schematically dense after field base changes.

Cor. (V.4.3.7). The schematic closure of a subset commutes with base change.

Proof: Use the third definition in (V.4.3.5), notice that the valuation maps are also jointly injective because $k'/k$ is flat.

Cor. (V.4.3.8). If $X$ admits a schematically dense subset $S$, then $X$ is geo.reduced.

Prop. (V.4.3.9). If $X(k')$ is dense in $X$, then $X$ is reduced. Conversely, if $X(k')$ is dense in $|X_{k'}|$ and $X$ is geo.reduced, then $X(k')$ is dense in $X$.

Proof: $X$ is reduced because $X_{red}(k') = X(k')$. Conversely if $Z \subset X$ is a closed subscheme that $Z(k') = X(k')$, then $|Z_{k'}| = |X_{k'}|$ by condition, and then $Z_{k'} = X_{k'}$ as $X_{k'}$ is reduced. Thus $Z = X$ by flatness.

Cor. (V.4.3.10) (Geometric Points Schematically Dense). If $X$ is locally algebraic and geo.reduced, then $X(k')$ is schematically dense in $X$ for any separably closed field $k'$ containing $k$.

Proof: $X(k')$ is dense in $|X_{k'}|$ by (V.4.3.3), thus it is schematically dense in $X$ by the proposition.

Cor. (V.4.3.11). If $Z, Z'$ are closed subvarieties of a locally algebraic algebraic scheme $X$ over $k$ that $Z(k') = Z'(k') \subset X(k')$ for some separably closed field $k'$ containing $k$, then $Z = Z'$. In other words, a closed subvariety of $X$ is determined by the subset $Z(k^s) \subset X(k^s)$.

Proof: The closed subscheme $Z \cap Z'$ satisfies $Z \cap Z'(k') = Z(k')$, so $Z \cap Z' = Z$ by (V.4.3.10). Similarly $Z \cap Z' = Z'$. 

□
Geo.Connected and Geo.Irreducible

**Prop. (V.4.3.12) (Geo.Connectedness).** For a scheme $X$ over a field $k$, the following are equivalent:

- For every connected $k$-scheme $Y$, the product $X \otimes_k Y$ is connected.
- $X$ is geometrically connected.
- $X_{K'}$ is connected.
- $X_K$ is connected for any finite separable extension $K/k$.

**Proof:** Cf. [Sta]0385, 0389. □

**Prop. (V.4.3.13) (Invariance of Base Change).** Let $X$ be a scheme over a field $k$ and $k'/k$ a field extension, then $X$ is geo.connected iff $X_{k'}$ is geo.connected.

**Proof:** Cf.[Sta]054N. □

**Prop. (V.4.3.14).** Let $T \to X$ be a map of schemes over a field $k$, if $T$ is geo.connected and $X$ is connected, then $X$ is geo.connected.

**Proof:** Cf.[Sta]056R. □

**Cor. (V.4.3.15) (Connected with a Rational Point).** Let $X$ be a scheme over a field $k$. Assume $X$ is connected and has a point $x$ that $k$ is alg.closed in $k(x)$, then $X$ is geo.connected. In particular, if $X$ is connected and has a rational point, then $X$ is geo.connected.

**Proof:** Cf.[Sta]04KV. □

**Prop. (V.4.3.16) (Geometrically Irreducible).** For a scheme $X$ over a field $k$, the following are equivalent:

- For every irreducible $k$-scheme $Y$, the product $X \otimes_k Y$ is irreducible.
- $X$ is geometrically irreducible.
- $X_{K'}$ is irreducible.
- $X$ is irreducible and if $\eta$ is the generic pt of $X$, then $k$ is separably closed in $k(\eta)$.
- $X_K$ is irreducible for any finite separable extension $K/k$.

**Proof:** Cf.[Gortz P136]. □

Geo.Integral

**Cor. (V.4.3.17) (geometrically Integral).** For a scheme $X$ over a field $k$, the following are equivalent:

- For every integral $k$-scheme $Y$, the product $X \otimes_k Y$ is integral.
- $X$ is geometrically integral.
- $X_{K'}$ is irreducible for any finite extension $K/k$.
- $X_K$ is integral.
- $X$ is integral and if $\eta$ is the generic pt of $X$, then $k$ is alg.closed in $k(\eta)$ and $k(\eta)/k$ is separable.

**Proof:** $1 \to 2 \to 3 \to 4$ is easy, Cf.[Gortz P136]. □
Geometrically Regular

Def. (V.4.3.18) (Geometrically Regular). Let $X$ be a locally Noetherian ring over a field $k$. Then $X$ is called geometrically regular iff $X_{k'}$ is regular for every f.g. field extension $k'/k$.

Prop. (V.4.3.19). Let $X$ be a locally Noetherian scheme over a field $k$, then $X$ is geometrically regular iff the local ring $\mathcal{O}_{X,x}$ is geometrically regular over $k$. And it suffice to check for finite purely inseparable field extensions $k'/k$.

Proof: For a finite purely inseparable field extension, $\mathcal{O}_{X,x} \otimes_k k'$ is also a local ring because their spectra are the same (I.5.3.22), so $\mathcal{O}_{X,x}$ is geometrically regular by (I.6.6.6).

Conversely, if $\mathcal{O}_{X,x}$ is regular, then for any field extension $k'/k$, stalks of $X_{k'}$ are localization of $\mathcal{O}_{X,x} \otimes_k k'$, so it is regular by (I.6.5.13).

Cor. (V.4.3.20). A geometrically regular ring is geometrically reduced, by (I.6.5.16).

Prop. (V.4.3.21) (Partially Invariance Under Base Change). If $k'/k$ is a f.g. field extension, then $X_{k'}$ is geometrically regular over $k'$ iff $X_k$ is geometrically regular over $k$.

Proof: One direction is trivial, for the other, Cf. [Sta]038W.

Prop. (V.4.3.22).

4 Basic Morphism Properties

Main references are [Sta]02WE.

Base Change Trick

Prop. (V.4.4.1) (Base Change Trick). If a property $P$ of morphisms satisfy:

- Closed immersion has $P$.
- Stable under base change and composition.

Then

- it is stable under product.
- $g \circ f : X \to Y \to Z$ has $P + g$ separated $\Rightarrow f$ has $P$.
- it is stable under $f_{\text{red}}$. (Notice $X_{\text{red}} \to X$ is closed immersion).

Proof: For the product, we may assume one of them is identity and use composition, but then the product is just base change, so it has $P$.

For the second, factorize $f : X \to X \times_Z Y \to Y$, the first is base change of $\Delta : Y \to Y \times_Z Y$, so it satisfies $P$ because $g$ is separable, and the second map is a base change of $X \to Z$, so it satisfies $P$, so $f$ satisfies $P$.

$X_{\text{red}} \to X \to Y = X_{\text{red}} \to Y_{\text{red}} \to Y$ has $P$ because $X_{\text{red}} \to X$ is closed immersion, and $Y_{\text{red}} \to Y$ is separable because closed immersion is separable (checked directly), so by what has been proved, $X_{\text{red}} \to Y_{\text{red}}$ has $P$.

Prop. (V.4.4.2). Lists of properties satisfying the base change trick (V.4.4.1) (not complete):

1. Universal closed/universal injective morphisms.
2. Affineness.
3. Morphisms (locally) of finite type.
4. Finite Morphisms.
5. Morphisms (locally) of finite presentation.
7. (Closed/Open) Immersions.
8. Quasi-compactness.
9. (Quasi-) Separatedness.
11. Unramified.

Proof:
1. Trivial.
2. Because affineness is local on the target (V.4.1.5), this follows from (V.2.7.17) and (V.2.7.14).
3. Trivial.
4. Trivial.
5. By (I.6.7.9).
6. Affine morphism is quasi-compact: because quasi-compactness is local on the target, we can reduced to the affine case, thus it is quasi-compact, by (V.4.4.25). To show a base change of quasi-compact morphism is quasi-compact, because quasi-compactness is local on the target (V.4.1.5), then we can choose a cover by affine opens that the image is contained in an affine open, thus it reduces to show a map between affine schemes is quasi-compact, which is (V.4.4.25).
7. For closed immersions, use (V.2.7.16) and check locally, for open immersions, use (V.2.7.16).
8. It suffices to show an affine map is quasi-compact.
9. Closed immersion is separated is checked directly. Composition: For \( X \to Y \to Z \), the diagonal map decomposes as \( X \to X \times_Y X \to X \times_Z X \), the second one is closed immersion (or quasi-compact) by (V.4.4.61), so this follows from that of closed immersion and qc. Base change: The diagonal commutes with base change (II.1.1.36), so this follows from that of closed immersion and qc.
10. Because universally closed, f.t. and separatedness both do (V.4.4.2).
11. □

Injectivity and Monomorphisms

Def. (V.4.4.3) (Injectivity and Monomorphisms). A morphism of schemes is called injective if it is injective topologically. A morphism of schemes is called a monomorphism if it is a monomorphism in the category of schemes.

Prop. (V.4.4.4) (Universally Injective). For a morphism of schemes, the following are equivalent:
• It is universally injective.
• It is injective and the residue field extension are all purely inseparable.
• The diagonal map is surjective.
• For any field \( K \), \( \text{Hom}(\text{Spec} \ K, X) \to \text{Hom}(\text{Spec} \ K, S) \) is injective.

_Proof:_ Cf.\[Sta\]01S4.

Prop. (V.4.4.5) (Monomorphism and Injectivity). A morphism of schemes \( j : X \to Y \) is a monomorphism if \( j \) is an injective map and for any \( x \in X \), \( \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x} \) is surjective.

_Proof:_ First check topologically, then check the local ring map.

Cor. (V.4.4.6). For any scheme \( X \) and a point \( x \), \( \text{Spec} \mathcal{O}_{X,x} \to X \) is a monomorphism.

Closed Map

Prop. (V.4.4.7) (Universal Closed). Universal closedness is local on the basis and satisfies the base change trick(V.4.4.2).

Prop. (V.4.4.8). If \( g \) is surjective, then \( f \circ g \) is universally closed iff \( f \) is universally closed (because surjective is S.u.B).

Prop. (V.4.4.9) (Closed Map and Specialization). The image of a quasi-compact morphism is closed iff it is stable under specialization. And it is a closed map iff specializations lift along \( f \).

_Proof:_ For the first, the question is local, so reduce to \( Y \) affine, and then \( X \) is qc = \( \bigcup U_i \), then we can replace \( X \) by an affine \( \coprod U_i \), then reduce to the affine case(I.5.3.13).

For the second, for any closed subset of \( X \) with its induced reduced structure, the restriction to it is still qc and specialization lifts, so we prove the image is closed. Now the image is stable under specialization, so the result follows from the first assertion.

Affine Map

Prop. (V.4.4.10). \( X \) is affine if there is a finite set of elements \( f_i \in \Gamma(X, \mathcal{O}_X) \) that generate the unit ideal and \( X_{f_i} \) is affine.

_Proof:_ First prove that \( X_{f_i} \cap X_{f_j} = X_{f_if_j} \) is affine because affine intersect \( X_{f_i} \) is affine. Second, prove \( \Gamma(X_f, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)_f \), finally glue them to get a map \( X \to \text{Spec}(A) \) and use the fact isomorphism is local on the target(V.4.1.5). \( X \) is affine scheme if \( X \to \text{Spec}(\Gamma(X)) \) is affine.

Cor. (V.4.4.11). Affineness is affine local on the target, and it satisfies the base change trick(V.4.4.2).

Prop. (V.4.4.12) (Serre Criterion of Affiness). For a qc scheme \( (X, \mathcal{O}_X) \), it is isomorphic to an affine scheme as a ringed space \( \iff X \) is (Co)\(h\)-acyclic \( \iff H^1(X, \mathcal{I}) = 0 \) for every Qco sheaf of ideals \( \mathcal{I} \).
Proof: The case of affine scheme is proven by (V.6.7.1) and (V.6.7.2). The converse: For every point \( p \), choose an open affine nbhd \( U \), let \( Y = X - U \), by the exact sequence

\[
0 \to \mathcal{I}_{Y \cup \{p\}} \to \mathcal{I}_Y \to k(P) \to 0,
\]
we have a surjective map \( \Gamma(X, \mathcal{I}_Y) \to \Gamma(X, k(P)) \) thus there is a \( f \in A = \Gamma(X, \mathcal{O}_X) \) that \( P \in X_f \subset U \) is affine. So using (V.4.4.10), we only have to show that for f.m \( f_i \), they generate \( \Gamma(X, \mathcal{O}_X) \). This is by considering the kernel \( F \) of \( \mathcal{O}_X \to \mathcal{O}_X : (a_1, \ldots, a_r) \to \sum f_ia_i \), and there is a filtration on \( F \), the quotient of which are all coherent sheaves because kernel and cokernel are Qco, and there by induction and hypothesis, \( H^1(X,F) = 0 \), thus the result. □

Cor. (V.4.4.13). If \( X \) is qcqs, then if \( H^1(X,\mathcal{I}) = 0 \) for every Qco sheaf of ideals \( \mathcal{I} \) of f.t., then \( X \) is an affine scheme. (Because by (V.6.1.10), it we can use colimit to show that \( H^1(X,\mathcal{I}) = 0 \) for Qco sheaf of ideals).

Cor. (V.4.4.14). For a Noetherian scheme \( X \), \( X \) is affine iff \( X_{red} \) is affine.

Proof: The canonical exact sequence (V.6.6.2) reads: \( 0 \to \mathcal{N}\mathcal{F} \to \mathcal{F} \to i^*\mathcal{F} \to 0 \), so iff \( X_{red} \) is affine, then we have \( H^i(F) \cong H^i(N\mathcal{F}) \), and notice \( N^k = 0 \) for some \( k \). □

Cor. (V.4.4.15). For a Noetherian reduced scheme \( X \), \( X \) is affine iff each irreducible component is affine. (The same as the above, notice that \( \prod p_i = 0 \), for the minimal primes of \( A \).) (The reducedness can be dropped by the last proposition).

Lemma (V.4.4.16). If a morphism \( X \to Y \) is a homeomorphism onto a closed subset of \( Y \), then \( f \) is affine.

Proof: Cf.[Sta]04DE. □

Quasi-affine

Def. (V.4.4.17) (Quasi-Affine Morphism). A scheme is called quasi-affine iff it is isomorphic to a qc open subscheme of an affine scheme. A morphism is called quasi-affine iff the inverse of any affine scheme is quasi-affine.

Prop. (V.4.4.18). Quasi-affine is local on the target and satisfies the base change trick.

Proof: Cf.[Sta]01SN. □

Prop. (V.4.4.19). A morphism \( f \) is quasi-affine iff \( \mathcal{O}_X \) is \( f \)-ample. In particular, A scheme is quasi-affine iff \( \mathcal{O}_X \) is ample.

Proof: Cf.[Sta]01QE. □

Prop. (V.4.4.20). If \( f : Y \to X \) is a quasi-finite morphism of schemes, and \( T \subset Y \) is nowhere dense, then \( f(T) \subset X \) is also nowhere dense.

Proof: Cf.[Sta]03J2. □
**Dominant**

**Prop. (V.4.4.21).** Let \( f : X \to S \) be a map of schemes.

- If every generic point of irreducible components of \( S \) is in the image of \( f \), then \( f \) is dominant.
- If \( f \) is quasi-compact, then the converse is also true. More precisely, if a generic point \( \eta \) is not in the image, then it is not in the closure of the image.
- If \( X \) has only f.m. irreducible component, then the converse is also true.

**Proof:** Cf.[Sta]Chap28.8.

**Def. (V.4.4.22) (Dominant Rational Maps).** A rational map between irreducible schemes is called a **dominant map** if it maps the generic point to the generic point.

**Prop. (V.4.4.23) (Dominant Map between Integral Schemes).** If \( f : X \to S \) is a map between integral schemes, then the following are equivalent:

- \( f \) is dominant.
- \( f(\eta_X) = \eta_Y \).
- for some(all) affine open subset \( U \subset X, V \subset Y \) with \( f(U) \subset V \), the ring map \( \mathcal{O}_Y(V) \to \mathcal{O}_X(U) \) is injective.
- for some(all) \( x \in X \), the local ring map \( \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x} \) is injective.

**Proof:** Cf.[Sta]0CC1.

**Quasi-Compact**

**Prop. (V.4.4.24) (Quasi-Compact Morphism).** A morphism \( f : X \to S \) of schemes is quasi-compact iff the inverse image of any quasi-compact open subsets is quasi-compact, because affine opens form a basis of \( X \).

Quasi-compactness is local on the target and satisfies the base change trick(V.4.4.2).

**Prop. (V.4.4.25).** A map between affine schemes is quasi-compact.

**Proof:** Because quasi-compactness is local on the target(V.4.1.5), it suffices to show the inverse image of a distinguished open subset is quasi-compact, and this is true.

**Prop. (V.4.4.26).** Let \( f : X \to Y, g : Y \to Z \). If \( g \circ f \) is quasi-compact and \( g \) is qs, then \( f \) is qc.

**Proof:** Factor it through \( X \to X \times_Z Y \to Y \). The second map is a base change of \( X \to Z \) hence qc, the first map is a section of \( X \times_Z Y \to X \), which is a base change of \( Y \to Z \), hence qs, so by(VIII.4.3.19), the first map is also qc.

**Prop. (V.4.4.27).** For a field extension \( k'/k \), a scheme \( X \) over \( k \) is qc iff \( X_{k'} \) is qc.

**Proof:** One direction is trivial, the other is by fpqc descent(V.1.5.25).

**Prop. (V.4.4.28).** Any map between Noetherian schemes is quasi-compact, by(V.4.1.21).
Finite Type

Def. (V.4.4.29) (Morphisms of Finite Type). A morphism \( f : X \to S \) is called of \textbf{locally finite type} if for there exists an affine open cover \( \{ \text{Spec}(B_i) \} \) of \( S \) that \( f^{-1}(U_i) \) has an affine open cover of spec of finite generated \( B_i \)-algebras. It is called \textbf{finite type} if moreover it is quasi-compact.

A scheme over a field \( k \) is called \textbf{(locally)algebraic} iff it is (locally) of finite type over \( \text{Spec} \ k \).

(Locally)Finite type is affine local on the target and on the source, and satisfies the base change trick (V.4.4.2).

Prop. (V.4.4.30) (Algebraic Schemes of Dimension 0). Let \( X \) be a locally algebraic curves over a field \( k \) of dimension 0, then \( X \) is a disjoint union of f.d. local Artinian \( k \)-algebras.

Proof: Cf.[Sta]06LH.

Prop. (V.4.4.31) (Dimensions for Locally Algebraic \( k \)-Schemes). Let \( k \) be a field and \( X \) is a locally algebraic \( k \)-scheme.

if \( X \) is irreducible, then \( \dim X = \dim U \) for any nonempty open \( U \subset X \).

Proof: It suffices to show any two affine open subsets \( U, U' \) of \( X \) have the same dimension, then use(IX.1.14.24). Now \( U \cap U' \neq \emptyset \) as \( X \) is irreducible, and then it contains a maximal pt \( x \) by Hilbert’s Nullstellensatz, and then \( \dim U = \dim (O_{X,x}) = \dim U' \) because f.g. algebras over a field is catenary.

Integral & Finite Map

Def. (V.4.4.32) (Finite and Integral Map). A morphism \( f : X \to S \) is called \textbf{integral} if it is affine and the inverse image an affine cover is integral ring extension.

Integral is affine local on the target and satisfies the base change trick (V.4.4.2).

A morphism \( f : X \to S \) is called \textbf{finite} if it is affine and the inverse image of an affine cover is finite module.

Finiteness is affine local on the target and satisfies the base change trick (V.4.4.2).

A morphism \( f : X \to S \) is called \textbf{(locally) quasi-finite} if it is (locally) of f.t. and the inverse of a point is a discrete set.

Prop. (V.4.4.33) (Integral Morphism is Closed). Specialization lifts along an integral morphism. In particular, an integral morphism is closed, by (V.4.4.9).

Proof: If \( f(x) = y, y \to y' \), then we can choose an open affine \( U \) containing \( y' \), which also contains \( y \). So we can choose an open affine containing \( x \) mapping into \( U \), then it reduces to the affine case(I.5.3.12).

Prop. (V.4.4.34) (Chevalley). If \( f : Y \to X \) is integral surjective, \( Y \) is affine, then \( X \) is affine.

Proof: Cf.[Sta]05YU.

Lemma (V.4.4.35). If \( f : Y \to X \) is finite surjective, \( Y \) is affine, then \( X \) is affine.

Proof:

Prop. (V.4.4.36) (Integral and Affine u.c.). Integral map is equivalent to u.c. and affine.


Proof: Integral is stable under base change. If it is integral, then it is closed by (I.5.5.5). Conversely, Cf. [[Sta]01WM]. □

Lemma (V.4.4.37). For \( f : Y \to X \) finite surjective and \( X \) locally Noetherian, for every integral subscheme \( Z \) of \( X \) with generic point \( \xi \), there is a coherent sheaf \( \mathcal{F} \) on \( Y \) that the support of \( f_*\mathcal{F} \) is \( Z \) and \( (f_*\mathcal{F})_\xi \) is annihilated by \( m_\xi \).

Proof: We consider an inverse image of \( \xi = \xi' \), and let \( Z' = \{ \xi' \} \) with the induced reduced structure, then let \( \mathcal{F} = i_*\mathcal{O}_{Z'} \) on \( Y \), \( \mathcal{F} \) is coherent, then we need to show that \( (f_*\mathcal{F})_\xi \) is annihilated by \( m_\xi \). This is because it factors through \( Z \). Cf. [[Sta]01YO]. □

Prop. (V.4.4.38) (Chevalley). Finite \( \iff \) quasi-finite+proper.

Proof: The fiber of \( f : X \to S \) is \( \text{Spec}(k(y) \otimes_A B) \), which is Artinian (I.5.1.51), so it has finitely many primes. Finite morphism is proper because it is integral (V.4.4.36).

For the converse, one should use Zariski’s Main Theorem. □

Prop. (V.4.4.39). An integral morphism of f.t. is finite, trivially.

Immersions

Def. (V.4.4.40) (Immersions). A closed immersion of schemes is a closed immersion of local ringed spaces (V.2.6.15). A closed subscheme of a scheme \( X \) is a closed sub-ringed space (V.2.6.15) that is also a scheme.

An open immersion of schemes is an open immersion of local ringed spaces (V.2.6.15). An open subscheme of a scheme \( X \) is an open subspace (V.2.6.15) that is also a scheme.

An immersion is a morphism that is a closed immersion of an open immersion.

Lemma (V.4.4.41) (Closed Immersion for Schemes). Let \( f : Y \to X \) be a morphism of schemes that induces a homeomorphism of \( Y \) onto a closed subset of \( X \), and \( f^\#: \mathcal{O}_X \to f_*\mathcal{O}_Y \) is surjective, then it is a closed immersion (V.2.6.15).

Proof: It suffices to show that the kernel of \( f^\# \) is Qco. For this, notice that \( f \) is quasi-compact, and it is a monomorphism by (V.4.4.5), in particular separated by (V.4.4.69). Then (V.3.1.4) shows \( f_*\mathcal{O}_Y \) is Qco. Then the kernel is Qco by (V.3.1.3). □

Prop. (V.4.4.42) (Closed Subschemes of Affine Schemes). The closed sub-ringed spaces of \( X = \text{Spec } A \) are all closed subschemes, and they corresponds to ideals \( I \subset A \):

- If \( I \subset A \) is an ideal, then the morphism \( Z = \text{Spec } A/I \to \text{Spec } A \) is a homeomorphism of \( Z \) onto a closed subspace \( V(I) \) of \( X \), and also the stalk map at a point \( \mathfrak{p} \subset Z \) is \( R_{\mathfrak{p}} \to (R/I)_{\mathfrak{p}/I} = R_{\mathfrak{p}}/IR_{\mathfrak{p}} \), which is surjective. So this is a closed immersion by (V.4.4.41).

- By (V.2.6.17), for any closed subscheme \( Z \) of \( X \) with sheaf of ideals \( \mathcal{I} \), \( Z \) is isomorphic to the closed subscheme of \( X \) defined by \( \mathcal{I} \). Now \( \mathcal{I} \) is locally generated by sections, so the quotient sheaf \( \mathcal{O}_X/\mathcal{I} \) is a Qco sheaf, so it is of the form \( \bar{S} \) for some \( A \)-module \( S \), by (V.3.1.2). Then \( \mathcal{I} \), as the kernel of \( \mathcal{O}_X \to \bar{S} \), is also Qco (V.3.1.3), so it equals \( \bar{I} \) for some ideal \( I \subset A \). Thus \( S = R/I \), and we are done.
Prop. (V.4.4.43) (Closed Subscheme of Schemes). The closed sub-ringed spaces of a scheme $X$ are all closed subschemes, and they corresponds to Qco $\mathcal{O}_X$-sheaves of ideals via the ideal sheaf (V.2.6.15):

Proof: Let $i : Y \to X$ be a closed immersion, for any $x \in X$, choose an open affine nbhd $U$ of $x \in X$, then $i : i^{-1}(U) \to U$ is also a closed immersion, so it corresponds to $\text{Spec} A/I \to \text{Spec} A$ for some ideal $I$ by (V.4.4.42). So $Z$ is a scheme, and the ideal sheaf $\mathcal{I}$ is Qco.

Prop. (V.4.4.44). Closed immersion satisfies the base change trick (V.4.4.2). Open immersion are stable under base change and composition. Immersions are stable under base change and composition.

Proof: For immersion, shrink the open subset.

Remark (V.4.4.45). A open immersion followed by a closed immersion can be written as a closed immersion followed by an open immersion, but not reversely. The reverse happens if the immersion is quasi-compact or the source is reduced (use the reduced induced structure).

Def. (V.4.4.46) (Scheme-Theoretical Image). For a morphism $f : X \to Y$, there is a closed scheme called scheme-theoretic image that is the smallest closed subscheme of $Y$ that $f$ factors through $Z$.

For an immersion of schemes, the scheme-theoretic image of the immersion is called the scheme-theoretic closure.

Proof: Consider the kernel of the structural map, and the kernel contains a maximal Qco sheaf of ideals $\mathcal{I}$ by (V.3.1.22).

Prop. (V.4.4.47). Let $f : X \to Y$ be a qc morphism of schemes and $Z$ be the scheme-theoretical image, then

- the kernel $\mathcal{I} = \ker(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ is Qco, thus $Z$ is the closed subscheme determined by $\mathcal{I}$.
- For any open subscheme $U \subset Y$, the scheme-theoretical image of $f|_{f^{-1}(U)}$ is equal to $Z \cap U$.
- $f(X)$ is dense in $Z$.

Proof: 1: As being Qco is local, it suffices to show for $Y$ affine. Then $X = \bigcup U_i$ is a finite union of affine schemes. Now take $X' = \bigsqcup U_i$, then there are maps $X' \xrightarrow{f'} X \to Y$. Then $\mathcal{O}_X$ is a subsheaf of $f'_*\mathcal{O}_{X'}$. So $\mathcal{I} = \ker(\mathcal{O}_Y \to f'_*\mathcal{O}_{X'})$. Now $f \circ f'$ is qcs, thus by (V.3.1.4), $f_*f'_*\mathcal{O}_{X'}$ is Qco, thus also $\mathcal{I}$ is Qco.

2 follows from 1 as the formation of $\mathcal{I}$ commutes with restriction to open subschemes.

3 follows from 2 as the scheme-theoretical image of empty set is empty.

Prop. (V.4.4.48). If $f : X \to Y$ is a morphism and $X$ is reduced, then the scheme-theoretical image of $f$ is the induced-reduced structure (V.4.1.13) of $\overline{f(X)} \subset Y$.

Proof: This is clear.

Def. (V.4.4.49) (Scheme-Theoretically Dense). An open subscheme $U \subset X$ is called scheme-theoretically dense if for any open subscheme $V$ of $X$, the scheme-theoretical closure of $U \cap V$ in $V$ is equal to $V$.

Prop. (V.4.4.50). If the inclusion $U \to X$ is qc, then $U$ is scheme-theoretically dense in $X$ if the scheme-theoretical closure of $U$ is $X$, by (V.4.4.47).
Prop. (V.4.4.51). Let \( j : U \to X \) be an open immersion of schemes, then \( U \) is scheme-theoretically dense in \( X \) iff \( \mathcal{O}_X \to j_*\mathcal{O}_U \) is injective.

\textit{Proof:} If it is not injective, then we can find an open subscheme \( V \) of \( X \) that the kernel is non-zero, thus it contains a non-zero \( \text{Qco} \) ideal sheaf, which means the scheme-theoretical closure of \( U \cap V \) is not \( V \). \hfill \Box

Cor. (V.4.4.52). If \( U, V \) are open subschemes of \( X \) scheme-theoretically dense in \( X \), then \( U \cap V \) is also scheme-theoretically dense in \( X \).

\textit{Proof:} \( \mathcal{O}_X \to \mathcal{O}_X(U) \to \mathcal{O}_X(U \cap V) \) is injective. \hfill \Box

Prop. (V.4.4.53) (Scheme-Theoretical Image of Immersions). Let \( f : Z \to X \) be an immersion and either \( f \) is qc or \( Z \) is reduced. Let \( \overline{Z} \) be the scheme-theoretically image of \( f \), then the morphism \( Z \to \overline{Z} \) is an open immersion that identifies \( Z \) with a scheme-theoretical dense open subscheme of \( \overline{Z} \), and also \( Z \) is dense in \( \overline{Z} \).

\textit{Proof:} Cf.\([\text{Sta}]01RG\). \hfill \Box

Prop. (V.4.4.54) (Immersions are Monomorphisms). An immersion \( f \) is a monomorphism.

\textit{Proof:} It is easy to check (V.4.4.5) for both open and closed immersions. \hfill \Box

Prop. (V.4.4.55) (Immersions are Monomorphisms). An immersion is a monomorphism.

\textit{Proof:} It is easy to check (V.4.4.5) for both open and closed immersions. \hfill \Box

Prop. (V.4.4.56) (Equivalent Definitions of Closed Immersion). The following are equivalent for a morphism \( f \):

- \( f \) is a closed immersion.
- \( f \) is a proper monomorphism.
- \( f \) is proper, unramified and u.i..
- \( f \) is a u.c., unramified monomorphism.
- \( f \) is u.c., unramified and u.i..
- \( f \) is u.c., locally of f.t. and a monomorphism.
- \( f \) is u.c., u.i., locally of f.t. and formally unramified.

\textit{Proof:} \( 4 - 7 \) are equivalent by (V.5.4.11). For the rest, Cf. \([\text{Sta}], 04XV\). \hfill \Box
Universal Homeomorphism

**Prop. (V.4.4.57).** A morphism is a universally homeomorphism iff it is integral, surjective and universally injective.

**Proof:** A universally homeomorphism is affine by(V.4.4.16). It is clearly u.c, so it is integral by(V.4.4.36). Conversely, it is integral hence u.c, and universally bijective, so it is universally homeomorphism. □

**Cor. (V.4.4.58).** The reduction \( X_{\text{red}} \to X \) is a universal homeomorphism, as closed immersion is u.c.

Separatedness

**Def. (V.4.4.59) (Separatedness).** A map \( f : X \to Y \) is called **separated** if the diagonal \( \Delta : X \to X \times_Y X \) is a closed immersion. It is called **quasi-separated** if the diagonal is quasi-compact.

In fact \( \Delta \) is always an immersion because maps between affine scheme is separated so \( \Delta(X) \) is closed in \( \bigcup U_{ij} \otimes_{V_i} U_{ij} \) where \( U, V \) are affine open, hence it suffice to check the image is closed.

**Prop. (V.4.4.60).** (Quasi-)Separateness is local on the target because closed immersion and quasi-compact is local on the target(V.4.1.5).

(Quasi-)Separatedness satisfies base change trick by(V.4.4.2).

**Prop. (V.4.4.61).** By(II.1.1.38), for \( X \to S \) and \( Y \to S \), the map \( X = X \times_Y Y \to X \times_S Y \) is an immersion. It is closed immersion if \( Y \to S \) is separated, and it is qc if \( Y \to S \) is quasi-separated.

**Cor. (V.4.4.62).** for \( X \to Y \) is a morphism of schemes over \( S \), the map \( X = X \times_Y Y \to X \times_S Y \) is an immersion. It is closed immersion if \( Y \to S \) is separated, and it is qc if \( Y \to S \) is quasi-separated.

**Cor. (V.4.4.63).** If \( s : S \to X \) is a section of \( f : X \to S \), the above proposition applies to this case, because \( S = S \times_X X \to S \times_S X = X \).

**Prop. (V.4.4.64) (Characterization of Separatedness).** A morphism is quasi-separated iff for any two affine open that mapped to an affine open, their intersection is quasi-compact. This is because quasi-compact is local on the target.

A morphism is separated iff for any two affine open that mapped to an affine open, their intersection is affine and \( O(U) \otimes_{O(W)} O(V) \to O(U \cap V) \) is surjective. This is because closed immersion is local on the target(V.4.1.5).

**Cor. (V.4.4.65).** A locally Noetherian scheme is quasi-separated.

**Cor. (V.4.4.66).** If \( g \circ f \) is (quasi-)separated, then so is \( f \).

**Cor. (V.4.4.67).** If \( X \) is (quasi-)separated, then \( X \to Y \) is (quasi-)separated.

**Prop. (V.4.4.68) (Injective Maps are Separated).** Injective maps of schemes are separated.

**Proof:** Let \( f : X \to Y \) be an injective map. Firstly \( X \times_Y X \) is a union of affine subschemes of the form \( U \otimes_V U \) where \( U, V \) are affine and \( f(U) \subset V \): let \( z \in X \times_Y X \), then \( \pi_1(z) = \pi_2(z) \), because they map to the same point in \( Y \), thus we can choose affine nbdls \( U, V \) of \( \pi_1(z) \) and \( f(\pi_1(z)) \). Now for each of these \( U \otimes_V U \), \( \Delta_U \to U \times_V U \) is closed immersion, thus \( \Delta_X \) is also closed immersion(V.4.1.5). □
Prop. (V.4.4.69). monomorphism is separated because the diagonal map is isomorphism (II.1.1.39), so immersions are separated as they are monomorphisms (V.4.4.55).

Prop. (V.4.4.70). (Quasi-)Affine morphism is separated (Check closed immersion directly).

Prop. (V.4.4.71) (Scheme-Theoretic Equalizer). If $X, Y$ are schemes over $S$ and $a, b : X \to S$ are morphisms, then there is a largest locally closed subscheme $Z$ of $X$ that $a|_Z = b|_Z$. And if $Y/S$ is separated, $Z$ is a closed subscheme of $X$.

Proof: By definition, $Z$ should be the fibered product:

\[
\begin{array}{ccc}
Z & \longrightarrow & X \\
\downarrow & & \downarrow^{(a,b)} \\
Y & \xrightarrow{\Delta_Y} & Y \times_S Y
\end{array}
\]

then the theorem follows from the definition and base change trick of locally closed morphisms. □

5 Proper & Projective

Prop. (V.4.5.1) (Proper). A morphism that is separated, of finite type and universally closed is called proper.

Properness is local on the target, because all these three properties do (V.4.1.5). Properness satisfies the base change trick (V.4.4.2).

Prop. (V.4.5.2). The class of proper morphisms satisfies the base change trick (V.4.4.1), by valuation criterion (V.4.5.7) and fibered products tricks.

Proof: Closed immersion is proper because it is f.t. and is affine so separated (V.4.4.59), and it is universally closed because immersions are stable under base change (V.4.4.44). □

Prop. (V.4.5.3) (Image of Proper Map). If $f : X \to Y$ is morphism between separated schemes f.t over $S$, then if $X$ is proper, then $f$ is proper (by base change trick) thus the image is closed, and is proper in its scheme-theoretic structure (V.4.4.8).

Proof: Notice proper is qc and use (V.4.4.46). □

Cor. (V.4.5.4) (Connected Proper to Affine Constant). A morphism from a connected proper scheme to an Noetherian affine scheme Spec $A$ is constant.

Proof: Because the image is proper and use (V.6.7.31), so the global section $A$ of its image is a finite module over Spec $k$ thus Artinian so has finitely many point (I.5.1.51). So it is discrete. But $X$ is connected, thus it is constant. □

Prop. (V.4.5.5). Finite morphism is proper, by (V.4.4.36) (V.4.4.70).

Valuation Criteria

Lemma (V.4.5.6) (Valuation Criteria Lemma). If $X$ is a scheme and $x \to y$ is a specialization of pts, then for any field extension $K/k(x)$, there is a valuation ring $A \subset K$ and a morphisms $\text{Spec} A \to X$ that maps the generic pt $\eta$ to $x$ and the unique closed pt to $y$. 

Proof: There is a morphism $O_{X,y} \to k(x) \to K$, so there is a valuation ring $A$ with field of fractions $K$ that dominate the image of $O_{X,y}$, which is also a local ring (I.5.1.8), by (I.9.2.1). Then this is what we desire.

Prop. (V.4.5.7) (Valuation Criteria). The valuation criterion for $\text{Spec} K \to \text{Spec} R$ where $R$ is a valuation ring with field of fractions $K$: Given a morphism $X \to S$,

1. If it is qc, then it is universally closed iff there is at least one lifting.
2. It is separated iff it is quasi-separated and there is at most one lifting.
3. It is proper iff it is finite type, quasi-separated and lifting exists uniquely.

Proof:

1. Firstly, in this case, by (V.4.4.9), it suffices to prove that: specializations lift along any base change of $f$ iff it has has least one lifting. If specializations lift along any base change of $f$, change $S$ to $\text{Spec} A$ and $X$ to $X \times_S \text{Spec} A$. Let $x'$ be the image of $\text{Spec} K \to X$, then by hypothesis there is a specialization $x' \to x$ where $x$ maps to the closed pt of $\text{Spec} A$. Then we get a map $A \to O_{X,x} \to k(x') \to K$, which is exactly the quotient map $A \to K$. So the image of $O_{X,x}$ in $K$ dominates $A$, which means it is just $A$. Thus we get a map $O_{X,x} \to A$, which gives a map $\text{Spec} A \to X$ that commutes $\text{Spec} K$.

Conversely, if $f$ has at least one lifting, then any base change of $f$ also has at least one lifting by categorial reason. Thus it suffices to show specializations lift along $f$. Let $s' \to s$ be a specialization in $S$ and $x' \in X$ maps to $s'$, we can apply (V.4.5.6) to $k(s') \subset k(x') = K$, then we get a lifting diagram, and the image of the closed pt of $\text{Spec} A$ under the lifting is a point mapping to $s$.

2. If it is separated, then if there are two lifting, then consider their equalizer, it is a closed subscheme of $\text{Spec} A$ by (V.2.7.18), and it contains the generic pt, so it equals $\text{Spec} A$, as desired.

Conversely, if there are at most two lifting, then we want to prove the diagonal is closed. But by (V.4.4.59) and (V.4.4.54) and the valuation criterion for u.c., it suffices to prove the existence of a lifting for the diagonal (V.4.5.7). But in fact, a valuation diagram for the diagonal correspond to two lifting of a valuation criterion for $X \to S$, then they are the same, and $\text{Spec} A \to X \times_S X$ lifts along the diagonal.

3. follows from the above two.

Prop. (V.4.5.8) (Extension of Rational Maps). Let $X, Y$ be schemes over $S$, $X$ is locally Noetherian and $Y/S$ is proper. If there is a morphism from an open subset $U$ of $X$ to $Y$, and there is a point $x$ in the closure of $U$ with the stalk being a valuation ring, then the morphism can be extended to an open set containing $x$.

Proof: We can replace $X$ by an affine open nbhd of $X$. By (V.4.6.1), we assume $X$ is affine and $\Gamma(X, O_X) \subset O_{X,x}$. In particular $X$ is integral with generic pt $\xi$ with residue field $K$. Then $U$ contains $\xi$. By the valuation criterion (V.4.5.7), the morphism $\text{Spec} K \xrightarrow{\xi} U \to Y$ can be lifted to a morphism $\text{Spec} O_{X,x} \to Y$, thus lemma (V.4.6.2) shows there is a morphism on a nbhd $V$ of $X$ spreading this morphisms.
Now because \( Y \cong S \) is separated, the equalizer of these morphisms and \( f \) on their intersection is a closed subscheme by (V.4.4.71), but it contains \( \xi \), so they coincide on the intersection, so we are done. □

**Prop. (V.4.5.9) (Singularity in Codimension1).** Let \( \varphi : X \to X' \) be a rational map of a locally Noetherian scheme \( X/K \) regular in codimension1 to a proper scheme \( X'/K \) with maximal domain \( U \), then

\[
\text{codim}(X \setminus U, X) \geq 2.
\]

In particular, if \( X \) is regular curve, then \( \varphi \) is a morphism.

**Proof:** We use (V.4.5.8), noticing that the stalk at a point of codimension1 is a DVR (I.6.5.15). □

**Projective Morphism**

**Def. (V.4.5.10) (Projective Morphism).** A projective morphism \( X \to Y \) is a closed immersion \( X \to \text{Proj}(\mathcal{E}) \) for some Qco f.t. module \( \mathcal{E} \) over \( Y \).

An \( H \)-projective morphism \( X \to Y \) is a closed immersion \( X \to \mathbb{P}^n_Y \).

An \( H \)-quasi-projective morphism is a \( H \)-projective morphism composed with a quasi-compact open immersion.

A locally projective morphism \( f : X \to Y \) is a morphism \( f \) that there exists a covering \( U_i \) of \( Y \) that \( f^{-1}(U_i) \to U_i \) is projective.

Some proposition about projective is written before the language of Hartshorne so I may not have changed them to the more general projective notion yet.

**Prop. (V.4.5.11).** For a morphism \( f : X \to S \), the following are equivalent:

- \( f \) is locally projective.
- There is a covering \( U_i \) of \( S \) that \( f^{-1}(U_i) \to U_i \) is \( H \)-projective.

**Proof:** Clearly 2 implies 1, and for the converse, it suffices to show that projective morphism is locally \( H \)-projective. Locally on each affine open nbhd \( U = \text{Spec} R \), \( X_U \) is isomorphic to a closed subscheme of \( \text{Proj}(\mathcal{E}) \) for some f.t. \( \mathcal{E} \) over \( Y \). Write \( \mathcal{E} = \mathcal{M} \) for some f.t. \( R \)-module \( M \), and choose a set of generators \( x_1, \ldots, x_n \) for \( M \), which induces a surjection of graded \( R \)-algebra \( R[x_0, \ldots, x_n] \to \text{Sym}_R(M) \), then the corresponding morphism \( \text{Proj}(\mathcal{E}) \to \mathbb{P}^n \) is a closed immersion, so \( f^{-1}(U) \) is a \( H \)-projective scheme over \( U \). □

**Prop. (V.4.5.12).** \( H \)-(Quasi-)Projectiveness satisfies the base change trick (V.4.4.1). (because Segre embedding is closed). Disjoint union of f.m. projective morphisms is projective (embed into the Segre embedding).

**Cor. (V.4.5.13).** Projective morphism is locally projective and locally projective is proper.

A quasi-projective morphism is of f.t. and separated (V.4.4.69).

**Proof:** Because locally projective and proper are both local on the base (V.4.4.2), it suffices to show that \( H \)-projective morphism is proper by (V.4.5.11).

Because properness satisfies base change trick (V.4.4.2), it suffices to show \( \mathbb{P}^n_S \to S \) is proper. Also this is base change of \( \mathbb{P}^n_Z \to \mathbb{Z} \), it suffices to check this one. \( \mathbb{P}^n_\mathbb{Z} \) is clearly separated by (VIII.4.3.20) (looking at the natural affine covering), and qc. Finally we show it is u.c. using valuation criterion (V.4.4.7):
Let \( \text{Spec} \mathbb{K} \rightarrow X \rightarrow \text{Spec} \mathbb{R} \rightarrow \text{Spec} \mathbb{Z} \) be a diagram, by induction on \( n \), we may assume the image \( \xi_1 \) of \( \text{Spec} \mathbb{K} \) is not contained in any of the hypersurface \( V(x_i) \), then \( x_i \) are all invertible in \( \mathcal{O}_{\xi_1} \), and there is a morphism \( \varphi : k(\xi_1) \rightarrow \mathbb{K} \). Let \( f_{ij} \) be the image of \( x_i/x_j \) under \( \varphi \), and choose \( k \) which \( f_{k0} \) has the minimal valuation, then \( f_{ik} \in \mathcal{O}_{\xi_1} \) for any \( i \), which means the there is a map \( \mathbb{Z}[x_0/x_k, \ldots, x_n/x_k] \rightarrow R \) compatible with \( \varphi \), or equivalently a map \( \text{Spec} R \rightarrow D(x_k) \subset X \) commuting the diagram.

Prop. (V.4.5.14). Projective scheme over \( \text{Spec} A \) is of the form \( \text{Proj} S \) where \( S_0 = A \) and \( S \) is f.g over \( S_0 \) by \( S_1 \) (V.3.2.12).

Prop. (V.4.5.15). \( H \)-projectiveness is stable under base change and composition.

Proof: \[ \text{Sta}01WE. \] □

Prop. (V.4.5.16) (Chow’s Lemma). Let \( X \rightarrow S \) be separated of f.t over a Noetherian \( S \), then there is a birational, proper, surjective \( X' \rightarrow X \) that \( X' \) is quasi-projective.

\( X \) is proper iff \( X' \) can be projective. And if \( X \) is integral(irreducible,reduced), \( X' \) can be chosen to be so.

Proof: ? Basic idea: reduce the the irreducible case, and use f.t. to generate a local quasi-projectives, then the closure of the image of \( U \rightarrow X \times_S P_1 \times_S \ldots \times_S P_n \) will suffice. □

6 Technical Lemmas

Lemma (V.4.6.1). Let \( X \) be a scheme and \( x \) a point, then there exists an open affine nbhd \( U \) of \( x \) that \( \mathcal{O}_{X}(U) \rightarrow \mathcal{O}_{X,x} \) is injective, if any of the follows holds:

- \( X \) is integral.
- \( X \) is locally Noetherian.
- \( X \) is reduced with f.m. irreducible components.

Proof: This problem is clearly local hence follows from the algebra case(I.5.1.29). □

Lemma (V.4.6.2) (Spread Out Stalk Morphism). Let \( X, Y \) be schemes over \( S, s \in S \) and \( x, y \) be pts over \( S \), then:

- Let \( f, g : X \rightarrow Y \) be morphisms over \( S \) that \( f(x) = g(x) = y \) and \( f^*_x = g^*_x \), then \( f = g \) on a nbhd \( U \) of \( x \) if any of the following holds:
  (a) \( Y/S \) is locally of f.t..
  (b) \( X \) is integral.
  (c) \( X \) is locally Noetherian.
  (d) \( X \) is reduced with f.m. irreducible components.

- Let \( \varphi : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x} \) be a local ring map over \( \mathcal{O}_{S,s} \), then there is a morphism \( f \) from a nbhd \( U \) of \( x \) mapping to \( Y \) that \( f(x) = y \), and \( f^*_x = \varphi \), if any of the following holds:
  (a) \( Y/S \) is locally of f.p..
(b) $Y/S$ is locally of f.t. and $X$ is integral.
(c) $Y/S$ is locally of f.t. and $X$ is locally Noetherian.
(d) $Y/S$ is local of f.t. and $X$ is reduced with f.m. irreducible components.

Proof: Cf.\cite{Sta}0BX6.

Prop. (V.4.6.3) (Base Change of Fields is Quotient Map). For any scheme $X$ over a field $k$ and algebraic extensions $K/k$, $X_K \to X$ is a quotient map, as it is surjective\cite{V.5.1.2}, continuous and closed\cite{V.4.4.36}.
V.5 More Properties of Schemes

1 Flatness

Def. (V.5.1.1) (Flat Modules and Flat Morphisms).

Def. (V.5.1.2) (Flatness for Schemes). Flat modules and flat morphisms over schemes are defined in the same way as that of ringed spaces (V.2.2.12).

Flatness is stalkwise by (V.2.2.14), it is stable under base change, composition. A coherent \( \mathcal{O}_X \) module is flat over \( X \) iff it is locally free, by (I.7.1.8).

Thus for a Qco \( \mathcal{O}_X \)-module \( \mathcal{F} \), flatness is equivalent to: For any affine open subsets \( \text{Spec} \ A \subset X \), \( \Gamma(\text{Spec} \ A, \mathcal{F}) \) is flat over \( A \), because flatness is also stalkwise for modules (I.5.1.55). Similarly, a morphism of schemes \( f : X \to Y \) is flat iff for any affine opens \( \text{Spec} \ B \subset X, \text{Spec} \ A \subset Y \) that \( f(\text{Spec} \ B) \subset \text{Spec} \ A \), \( B \) is flat over \( A \).

Prop. (V.5.1.3) (Flat Locus is Open). For a morphism \( f : X \to S \) locally of f.p., and a Qco sheaf on \( X \) that is locally of finite presentation, the set of points that \( \mathcal{F} \) is flat over \( S \) is open.

Proof: Cf. [Sta]0399.

Prop. (V.5.1.4). For a flat morphism of ringed space, \( f^* \) is exact, because it is \( f^{-1} \) followed by tensoring with \( \mathcal{O}_X \), check on stalks.

Prop. (V.5.1.5). A finite morphism \( f : X \to S \) with \( S \) locally Noetherian is flat iff \( f_*(\mathcal{O}_X) \) is locally free.

Proof: Cf. [Sta]02KB.

Prop. (V.5.1.6) (Going-Down). Generalization lifts along a flat morphism.

Proof: We can find an affine nbhd, then choose a nbhd of the inverse image, then a generalization in an affine open is a true generalization, so it reduce to the affine case. The rest follows from going-down (I.7.1.19).

Cor. (V.5.1.7) (Flat Map and Irreducible Component). A flat morphism maps an irreducible component onto another irreducible component.

Prop. (V.5.1.8) (Flatness and Openness). A flat morphism locally of f.p. is (universally)open, hence it is qc.

And a qc f.f. morphism of schemes is submersive.

Proof: We need only consider they are both affine. Then the assertion follows from (I.7.1.27).

For the second, by (V.5.1.6), a subset whose inverse image is closed is stable under specialization (surjectiveness used), then the complement is closed by (V.4.4.9).

Prop. (V.5.1.9) (Flat Pullback of Closed Subschemes). Cf. [Sta]081I.

Prop. (V.5.1.10) (Generic Flatness). For a morphism \( f : X \to Y \) of f.t., if \( S \) is reduced and \( \mathcal{F} \) is a qco f.t. \( \mathcal{O}_X \)-module, then there exists an open subset \( U \) of \( S \) that \( X_U \to U \) is flat, of f.p., and \( \mathcal{F}_{X_U} \) is flat over \( U \) and of f.p. over \( \mathcal{O}_{X_U} \).

Proof: Cf. [Sta]P052B.
Prop. (V.5.1.11) (Flat Family and Hilbert Polynomial). For \(X/T\) projective, where \(T\) is an integral Noetherian scheme and \(X \subset \mathbb{P}^n_T\). Then for each point \(T\), \(X_t\) is a closed subscheme of \(\mathbb{P}^n_{k(t)}\), so we can consider its Hilbert Polynomial \(P_t\). Then \(X/T\) is flat iff \(P_t\) is independent of \(T\).

Proof: \(P_t(m) = \dim_{k(t)} H^0(X_t, \mathcal{O}_{X_t}(m))\) for \(m\) large by (V.6.7.20). And we may let \(X = \mathbb{P}^n_T\) and prove for any coherent sheaf \(\mathcal{F}\). Moreover, we may let \(T\) be an affine local Noetherian, because flatness is local and we only need to compare Hilbert polynomial with the generic point. Now we prove a stronger assertion: The following are equivalent:

- \(\mathcal{F}\) is flat over \(T\).
- \(H^0(X, \mathcal{F}(m))\) is a free \(A\)-module of finite rank, for \(m\) large.
- The Hilbert polynomial \(P_t\) of \(\mathcal{F}_t\) on \(X_t = \mathbb{P}^n_{k(t)}\) is independent of \(t\).

1. \(\Rightarrow\): Use the canonical cover and \(\check{\text{C}}\)ech cohomology, then we notice when \(m\) is large, \(H^0(X, \mathcal{F}(m))\) is a kernel of the \(\check{\text{C}}\)ech resolution, so it is flat. And it is also finite by (V.6.7.31). Then it is free because it is flat by (I.7.1.8).

2. \(\Leftarrow\): Let \(M = \oplus_{m\geq m_0} H^0(X, \mathcal{F}(m))\), then \(\widetilde{M} = \mathcal{F}(V.3.3.3)\), notice that the truncation doesn’t affect.

2. \(\Rightarrow\): It suffice to prove that for any \(t \in T\), when \(m\) is large,

\[
H^0(X_t, \mathcal{F}_t(m)) \cong H^0(X, \mathcal{F}(m)) \otimes_A k(t).
\]

For this, we may use (V.6.7.34) to pass to the localization and assume \(t\) is the closed pt of \(T\). Then \(A \rightarrow k(t)\) is surjective and we may let \(A^0 \rightarrow A \rightarrow k \rightarrow 0\), then by (V.6.7.33), we have \(H^0(X_t, \mathcal{F}_t(m))\) is the cokernel of \(H^0(X, \mathcal{F}(m))^q \rightarrow H^0(X, \mathcal{F}(m))\), but this cokernel is \(H^0(X, \mathcal{F}(m)) \otimes_k\) because tensoring is right-adjoint, so we are done.

3. \(\Rightarrow\): We have the rank of \(H^0(X, \mathcal{F}(m))\) at the generic and closed point of \(T\) are the same (still use \(H^0(X_t, \mathcal{F}_t(m)) \cong H^0(X, \mathcal{F}(m)) \otimes_A k(t)\)). Now (I.7.8.1) gives \(H^0(X, \mathcal{F}(m))\) is free. It is f.g. automatically.

Cor. (V.5.1.12). For a flat morphism to a connected scheme \(T\), the dimension, degree, and arithmetic genus of the fibers are independent of \(t\).

Proof: By (V.6.7.22) and (V.8.2.5).

Def. (V.5.1.13). For a surjective map of varieties \(f : X \rightarrow T\) over an alg.closed field \(k\), its fibers over closed points with induced reduced structure \(X_t\) is called a algebraic family of varieties parametrized by \(T\) if

1. \(f^{-1}(t)\) is irreducible of dimension \(\dim X - \dim T\) for every closed point \(t\).
2. If \(\zeta\) is the generic point of \(f^{-1}(t)\), then \(F^\sharp m_t\) generates the maximal ideal \(m_\zeta \subset \mathcal{O}_{\zeta, X}\).

Prop. (V.5.1.14). if \(X_t\) is an algebraic family of normal varieties over an alg.closed field \(k\) parametrized by a nonsingular curve \(T\), then it is a flat family of schemes.

Proof: By (IX.12.1.17), \(X \rightarrow T\) is flat. So what we need to do is to prove \(X_t\) is reduced so \(X_t = X_t\). Let \(A = \mathcal{O}_{x,T}\), let \(u_t\) be a uniformizer of \(\mathcal{O}_{t,T}\), then \(A/ta\) is the local ring of \(x\) on \(X_t\). By hypothesis \(X_t\) is irreducible so \(tA\) has a unique minimal prime \(p\) in \(A\), and \(t\) generate the maximal ideal of \(A_p\) by hypothesis. The local ring of \(X_t\) is \(A/p\), so \(A/p\) is normal by hypothesis. Then the result follows from (I.6.5.11).
Cor. (V.5.1.15) (Igusa). Let $X(t)$ be an algebraic family of normal varieties in $\mathbb{P}^n_k$ for $k$ alg. closed parametrized a variety $T$, then the Hilbert polynomials of $X(t)$ are independent of $t$.

Proof: Why is $X/T$ projective? Cf. [Hartshorne P265]. □

2 Dimension

Main references are [Rising Sea, Chap11].

Prop. (V.5.2.1) (Locally Algebraic Scheme is Catenary). If $X$ is a locally algebraic scheme over a field $k$ purely of dimension $n$, and $Y$ an irreducible subscheme of $X$, then $\dim Y + \text{codim}(Y, X) = \dim X$.

Proof: Choose an affine open of the generic point of $Y$, then we are reduced to the affine case (I.5.8.3) (I.5.8.6). □

Prop. (V.5.2.2). For any scheme, $\dim \mathcal{O}_x = \text{codim}(\overline{\{x\}}, X)$.

Prop. (V.5.2.3). For an integral scheme algebraic over a field $k$,

$$\dim X = \dim \mathcal{O}_{p,X} = \dim U = \text{tr.deg} \ K(X)/k$$

for any closed point $p$ and any open subscheme $U$.

Proof: Use closed point are dense (V.4.1.25) and $k$ is universal catenary to prove it is true for some $U$ and all the closed point in it, so other $U$’s because $X$ is irreducible. The last equation follows from (I.5.8.20). □

Prop. (V.5.2.4) (Finite Surjection Preserves Dimension). Let $X \to Y$ be a surjective finite morphism of algebraic integral schemes over a field $k$, then $\dim X = \dim Y$.

Proof: The hypothesis implies that for any affine open $\text{Spec} A \subset Y$, the inverse image is $\text{Spec} B$ that $A \to B$ is an injective integral ring extension, so we can use (V.5.2.3) and (I.5.8.13). □

Prop. (V.5.2.5). Let $X$ be a locally Noetherian scheme, if $U \subset X$ is an open subscheme that $U \to X$ is affine, then every irreducible complements of $X - U$ has codimension $\leq 1$. And if $U$ is dense, then equality must hold.

Proof: Cf. [[Sta]0BCU]. □

Prop. (V.5.2.6) (Local Dimension). Let $X$ be a locally algebraic scheme over a field $k$ and $x \in X$, then the local dimension $\dim_x(X)$ equals the maximal dimension of irreducible components of $X$ passing through $x$, by (I.5.8.24).

Prop. (V.5.2.7). If $f: X \to Y$ is a morphism of schemes locally of f.t, $x \in X$ and $s = f(x)$, then $\dim_x(X_s) = \dim \mathcal{O}_{X_{s,x}} + \text{tr.deg}_{k(s)} k(x)$.

Proof: This immediately reduces to the case $Y = \text{Spec} k(s)$, thus it follows from (I.5.8.25). □

Prop. (V.5.2.8) (Semicontinuity of Dimension). Let $f: X \to S$ be a morphism of schemes locally of f.t., then the function $x \mapsto \dim_x(X_{f(x)})$ is upper-semicontinuous on $X$.

Moreover, if $f$ is of f.p., then the open subsets $\{x | \dim_x(X_{f(x)}) \leq n\}$ is retrocompact.
Proof: This follows directly from (I.5.8.29).

Prop. (V.5.2.9) (Local Dimension and Base Change). Let

\[
\begin{array}{c}
X' \xrightarrow{g'} X \\
\downarrow f' \\
S' \xrightarrow{g} S
\end{array}
\]

be a fiber product diagram of schemes, and \( f \) is locally of f.t.. Suppose \( x' \in X', x = g'(x'), s' = g(s') \). Then

- \( \dim_x(X_s) = \dim_{x'}(X'_{s'}) \).
- \( \dim O_{F,x'} = \dim O_{X',x'} - \dim O_{X,x} = \text{tr} \cdot \deg_{k(s)}(k(x)) - \text{tr} \cdot \deg_{k(s')}(k(x')) \)

where \( F \) is the fiber of the morphism \( X'_{s'} \to X_s \) over \( x \). In particular, \( \dim O_{X',x'} \geq \dim O_{X,x} \) and \( \text{tr} \cdot \deg_{k(s')}(k(x')) \leq \text{tr} \cdot \deg_{k(s)}(k(x)) \).
- Given \( s', s, x \) that \( f(x) = g(s') \), there exists an \( x' \in X' \) that \( \dim O_{X',x'} = \dim O_{X,x} \) and \( \text{tr} \cdot \deg_{k(s')}(k(x')) = \text{tr} \cdot \deg_{k(s)}(k(x)) \).

Proof: It can be reduced to the case that \( S = \text{Spec } k(s), S' = \text{Spec } k(s') \) and \( X, X' \) affine. Then 1 follows from (I.5.8.28), and 2, 3 follows from (I.5.8.23).

Cor. (V.5.2.10) (Dimension and Field Extension). Let \( K/k \) be a field extension, \( X \) a locally algebraic scheme over \( k \) purely of dimension \( n \), then \( X_K \) is a scheme purely of dimension \( n \).

Remark (V.5.2.11). This proposition shows in particular local dimension behaves better than the dimension of the stalk.

Dimension and Flatness

Prop. (V.5.2.12) (Faithfully Flat Morphism). If \( f : Y \to X \) is a faithfully flat morphism, then \( \dim Y \geq \dim X \).

Proof: This is easy from (V.5.1.6).

Prop. (V.5.2.13) (Integral Flat Morphisms). If \( f : X' \to X \) is an integral flat morphism of schemes, and \( X \) is pure of dimension \( n \), then so does \( X' \). The converse holds if \( f \) is faithfully flat.

Proof: By (V.5.1.7) and (V.4.4.33), \( f \) maps an irreducible component of \( X' \) onto an irreducible component of \( X \), which then reduces to the affine case (I.5.8.14). If \( f \) is faithfully flat, then every irreducible component of \( X \) is in the image.

Cor. (V.5.2.14) (Dimension and Field Extension). If \( K/k \) is an algebraic extension, \( X \) a scheme over \( k \) purely of dimension \( n \), then \( X_K \) is a scheme purely of dimension \( n \). Compare with (V.5.2.10).

Prop. (V.5.2.15) (Dimension Extension and Flatness). Let \( f : X \to Y, g : Y \to S \) be locally of f.t., \( x \in X, y = f(x), s = g(y) \), then

\[
\dim_x(X_s) \leq \dim_x(X_y) + \dim_y(Y_s).
\]
Moreover, equality holds if \( O_{X,x} / O_{Y,y} \) is flat.

In particular, if \( S = \text{Spec} K \) and \( X,Y \) are irreducible and \( X \) is flat over \( Y \), then \( \dim X_y = \dim X - \dim Y \) for any \( y \in Y \).

**Proof:** By (V.5.2.7) and the fact transcendental degree is additive, this reduces to

\[ \dim O_{X,x} \leq \dim O_{X/y,x} + \dim O_{Y,y}. \]

We can assume \( X,Y \) is affine and \( S = \text{Spec} k(s) \), so the rest follows from (I.5.8.12).

□

**Def. (V.5.2.16) (Relative Dimension).** A morphism of schemes which is locally of f.t. is called of **relative dimension** \( n \) iff all fibers \( X_s \) are equidimensional of dimension \( n \).

**Cor. (V.5.2.17).** If \( f : X \to Y, g : Y \to Z \) are of relative dimension \( m \) and \( n \), and \( f \) is flat, then \( g \circ f \) is of relative dimension \( m + n \).

**Cor. (V.5.2.18) (Dimension Theorem).** If \( f : X \to Y \) is a dominant morphism of irreducible algebraic schemes over \( K \) that \( X \) is reduced, then there is a dense open subset \( U \) of \( Y \) that for any \( y \in Y \),

\[ \dim(X_y) = \dim X - \dim Y. \]

This is a combination of the above proposition and generic flatness (V.5.1.10).

**Prop. (V.5.2.19) (Relative Dimension and Base Change).** By (V.5.2.9), the base change of a morphism locally of f.t. of relative dimension \( n \) is still of relative dimension \( n \). In particular, for a variety over \( K \) a field, the dimension is invariant under base change of fields.

**Prop. (V.5.2.20).** For a morphism \( f : X \to Y \) between locally Noetherian schemes which is flat and locally of f.t and of relative dimension \( n \), then if \( y = f(x) \), we have \( \dim_x(X_y) = \dim_x(X) - \dim_y(Y) \).

**Proof:** Shrinking the nbhd, we may assume \( \dim_x(X) = \dim X \) and \( \dim_y(Y) = \dim Y \) and \( X,Y \) are affine. Now \( f \) is locally of f.p. and flat, so it is open (V.5.1.8). So we may assume \( f \) is surjective. Then \( \dim O_{X,a} = \dim O_{Y,b} + \dim O_{X,b,a} = \dim Y + n \) by (I.5.8.12), then taking supremum?, the result follows.

□

**Cor. (V.5.2.21).** For a morphism of schemes that is flat and of f.t., if \( Y \) is irreducible, then \( X \) is equidimensional of dimension \( Y + n \) if \( X_y \) is equidimensional of dimension \( n \) for every \( y \in Y \).

**Proof:** The proof highly relies on (V.5.2.3).

If \( X \) is equidimensional of dimension \( Y + n \), for \( Z \subset X_y \) an irreducible component, choose a closed pt \( x \) of \( Z \) not contained in any other irreducible component, then

\[ \dim_x Z = \dim_x X - \dim_y Y = \dim X - \dim \{ x \} - \dim Y + \dim \{ y \}. \]

The two closures are of the same dimension because by (I.5.9.9), their quotient field extension is finite, and use (I.5.8.13).

Conversely, for an irreducible component of \( X \), choose a closed pt \( x \) of \( Z \) not contained in any other irreducible component, then the image is also closed, by (V.4.1.26), so the result is immediate.

□
3 Smoothness

**Def. (V.5.3.1).** A morphism \( f : X \to Y \) of schemes is called **smooth** if there is an open affine cover \( \{ U_i \} \) of \( S \) and an open affine cover \( V_{ij} \) of \( f^{-1}(\{ U_i \}) \) that the ring map is smooth. A **standard smooth morphism** is the Spec map of a standard smooth ring map.

Smoothness is local on the source and target (I.7.5.15). Smoothness is stable under base change and composition (I.7.5.15).

**Lemma (V.5.3.2).** For a smooth morphism \( X \to S \), the morphism of differential \( \Omega_{X/S} \) is locally free and \( \dim_x \Omega_{X/S} = \dim_x(X_{f(x)})(\text{local dimension } \text{IX.1.14.23}) \).

**Proof:** We can assume that \( X \to S \) is standard smooth, so by the proof in (I.7.5.14), \( \Omega_{X/S} \) is free of dimension \( n - c \), and also standard smooth is relative global complete intersection (I.7.5.11), so \( U_{f(x)} \) is equidimensional of dimension \( n - c \), thus the result. \( \square \)

**Prop. (V.5.3.3) (Fiberwise and Stalkwise).** For a morphism \( X \to S \) locally of f.p., the following are equivalent:

- It is smooth at a point \( x \in X \) over \( s \in S \).
- \( \mathcal{O}_{x,X}/\mathcal{O}_{f(x),S} \) is flat and \( X_{f(x)}/k(x) \) is smooth at \( x \), by (I.7.5.20). Moreover, using ??, we even only have to check for the geometric fibers.
- \( \mathcal{O}_{x,X}/\mathcal{O}_{f(x),S} \) is flat and \( \Omega_{X/S,x} \) can be generated by \( \dim_x(X_{f(x)}) \) elements, by (V.5.3.2) and (I.7.5.25).
- \( \mathcal{O}_{x,X}/\mathcal{O}_{f(x),S} \) is flat and \( \Omega_{X/s,x} \otimes \mathcal{O}_{X,x} k(x) = \Omega_{X/S,x} \otimes \mathcal{O}_{X,x} k(x) \) can be generated by \( \dim_x(X_{f(x)}) \) elements, by Nakayama, because \( \Omega_{X/S,x} \) is of f.p. by (I.7.3.9).

**Prop. (V.5.3.4) (Smooth Morphism is Open).** Open immersion is smooth. Smooth morphism is syntomic hence flat. Smooth morphism is locally of f.p. Hence smooth morphism is universally open (V.5.1.8).

Smooth morphism is locally standard smooth (I.7.5.14).

**Prop. (V.5.3.5).** If \( X \to Y \) is smooth and a morphism \( Y \to S \), then there is an exact sequence of sheaves (V.3.4.5)(I.7.5.6):

\[
0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0
\]

**Prop. (V.5.3.6).** If \( Z \to X \to S \), \( Z/S \) is smooth and \( Z \to X \) is an immersion, then there is an exact sequence of sheaves (V.3.4.13)(I.7.5.7):

\[
0 \to \Omega_{C/X} \to i^*\Omega_{X/S} \to \Omega_{Z/S} \to 0
\]

**Prop. (V.5.3.7).** If \( X \to Y \to S \), and \( X \to Y \) is surjective, flat and locally of f.p., \( X \to S \) is smooth, then \( Y \to S \) is smooth.

**Proof:** Cf. [Sta]05B5. \( \square \)

**Prop. (V.5.3.8).** By (V.5.3.3)(V.5.3.2), A morphism is smooth of relative dimension \( n \) is equivalent to fppf+fibers equidimensional of dimension \( n \) and \( \Omega_{X/S} \) is locally free of dimension \( n \).
Smooth over Fields

Prop. (V.5.3.9) (Differential Criterion of Smoothness). Let $X$ be a scheme algebraic over a field $k$.

- If $X$ is equidimensional of dimension $n$, then $X$ is smooth over $k$ iff $\Omega_{X/S}$ is locally free of dimension $n$.
- If $\Omega_{X/k}$ is locally free, and $k$ is of char 0 or $k$ is perfect and $X$ is reduced, then $X$ is smooth over $k$.

Proof: 1: This follows from (V.5.3.3).

2: If $k$ is of characteristic 0, then this follows from (I.7.5.29).

\[\text{perfect case: [Sta]04QP.}\]

□

Cor. (V.5.3.10). Hom

Prop. (V.5.3.11) (Smooth over Field and Geo.Regular). For a scheme locally algebraic over a field $k$, $X$ is geometrically regular iff it is smooth over $k$. In particular, if $k$ is perfect, then smoothness is equivalent to regularity, by (V.4.3.19).

Proof: The question is local around $x$, so may assume $X$ is affine. Then this follows from (I.7.5.28). □

Cor. (V.5.3.12) (Hartshorne Definition). By (V.5.3.3) and (V.5.3.11), a morphism between schemes algebraic over a field $k$ is smooth of relative dimension $n$ iff $f$ is flat and every fiber of $f$ is geometrically regular of dimension $n$.

Cor. (V.5.3.13). A smooth scheme over a field $k$ is regular hence normal.

Prop. (V.5.3.14) (Smoothness and Separable Closed Points). Let $X$ be smooth over a field $k$, then the set of closed points of $X$ with finite separable residue field $k(x)/k$ is dense in $X$.

Prop. (V.5.3.15) (Generic Smoothness). Let $X$ be a locally algebraic scheme over a field $k$ that is geometrically reduced, then it contains an open dense subset that is smooth over $k$.

Proof: The problem is local, so we may assume $X$ is affine, consider its irreducible components, all their intersections can be removed, because they are nowhere dense, so we may assume $X$ is irreducible. So $X$ is integral, let $\eta$ be the generic pt, then $k(\eta)/k$ is separable, by (V.4.3.2). Then choose an affine subscheme $\text{Spec} A \subset X$, then $A$ is smooth at $(0)$ over $k$, by (I.7.5.30), then by definition, it is smooth on some dense open subscheme of $X$. □

4 Unramified

More advanced materials to learn at [[Sta]Chap40].

Def. (V.5.4.1) (Unramified Morphism). A morphism is called ($G$-)unramified iff there is an open affine cover $U_i$ and an open affine cover of $f^{-1}(U_i)$ that the induced ring map is ($G$-)unramified. Equivalently, $\Omega_{X/S} = 0$ and it is locally of f.t.(f.p.).

($G$-)unramifiedness is local on the source and target by (V.4.1.5)(V.4.1.6). ($G$-)unramifiedness is stable under base change and composition by (I.7.6.4). Moreover, unramifiedness satisfies the base change trick.
Prop. (V.5.4.2). An unramified map is locally quasi-finite.

Proof: Cf.[[Sta]02V5].

Prop. (V.5.4.3) (Fiberwise). A morphism is \((G-)\)unramified iff it is locally of \(f.t.\)\(\left(f.p.\right)\) and all the fibers \(X_s\) are disjoint unions of spectra of finite separable extensions of \(k(p)\).

Proof: By (I.7.6.7), Notice \(pS_q = qS_q\) is equivalent to every \(q\) is minimal in \(X_p\), which is equivalent to \(X_p\) is discrete.

Cor. (V.5.4.4) (Unramified over Fields). A scheme over a field \(k\) is unramified iff it is a disjoint union of spectra of finite separable extensions of \(k\), because locally of \(f.p.\) is trivially satisfied.

Prop. (V.5.4.5). A morphism \(X \rightarrow S\) is \((G-)\)unramified iff it is of \(f.t.\)\(\left(f.p.\right)\) and the diagonal is a clopen immersion of \(X\), thus all of \(X\).

Proof: If it is unramified, then it is an open immersion by (I.7.6.10). Conversely, \(\Omega_{X/S}\) is just the conormal sheaf of the diagonal map, so it is zero.

Cor. (V.5.4.6) (Sections of Unramified Morphism). Any section of an unramified morphism is an open immersion. In particular, a section of a separable unramified morphism is a clopen immersion.

Proof: This follows from the proposition and the fact \(S \rightarrow X\) is a base change of \(\Delta_{X/S}\).

Cor. (V.5.4.7). Let \(X, Y\) be schemes over \(S\), if \(f, g\) are two maps from \(X\) to \(Y\), then if \(Y/S\) is unramified and \(f, g\) are equal on a pt \(x\) of \(X\)(both on image and residue field), then there is a nbhd of \(x\) that \(f, g\) are equal.

Proof: This follows as \(\Delta_{Y/S}\) is open immersion, so the set that \(f, g\) are equal is open in \(X\).

Prop. (V.5.4.8) (Stalkwise and Fiberwise). For a morphism locally of \(f.t.\)\(\left(f.p.\right)\), the following are equivalent:

- It is \((G-)\)unramified at a point \(x\),
- The fiber \(X_{f(x)}\) is smooth over \(k(f(x))\) at \(x\).
- \(\Omega_{X_{f(x)},x} = 0\).
- \(\Omega_{X_s,x} \otimes \Omega_{X_s,x} k(x) = \Omega_{X_{f(x)},x} \otimes \Omega_{X,s,x} k(x) = 0\).
- \(m_x \mathcal{O}_{X,x} = m_x\) and \(k(x)/k(s)\) separable. (I.7.6.7).

Prop. (V.5.4.9). If \(X \rightarrow Y \rightarrow S\), \(X/S\) is unramified, then \(X/Y\) is unramified. And if \(X/S\) is \(G\)-unramified and \(Y/S\) is of \(f.t.\), then \(X/Y\) is \(G\)-unramified. (By (V.5.8.5) and (V.3.4.5)).

Cor. (V.5.4.10) (Unramified Points Base Change). If \(f\) is of \(f.t.\)\(\left(f.p.\right)\), then the set of points that \(f\) is unramified is stable under base change by the above proposition.

Prop. (V.5.4.11) (Unramified u.i. Morphism). For a morphism \(f\) of schemes, the following are equivalent:

- \(f\) is unramified and a monomorphism.
- \(f\) is unramified and universal injective.
- \(f\) is locally of \(f.t.\), formally unramified and universal injective.
- \(f\) is locally of \(f.t.\) and a monomorphism.
- \(f\) is locally of \(f.t.\) and \(X_y\) is either empty or \(X_y \rightarrow y\) is an isomorphism for all \(y \in Y\).

Proof: Cf.[[Sta], 05VH].
Noetherian Case

Prop. (V.5.4.12). Let $S$ be a Noetherian scheme, $X \to S$ a qc unramified morphism and $Y \to S$ a morphism with $Y$ Noetherian, then $\text{Mor}_S(Y,X)$ is a finite set.

Proof: Cf.[[Sta]], 0AKI. □

Ramification and Valuations

Prop. (V.5.4.13). Let $R_v$ be a DVR with fraction field $K$ and $\varphi : X \to X'$ be a morphism of schemes of f.t. over $R_v$. Let $Q \in X'(K)$ and $P \in X(K)$ with $\varphi(P) = Q$. Let $w|v$ be a valuation of $K(P)$ extending $v$. If $P$ extends to an $R_w$-valued point $\mathcal{P}$ of $X$, then using the fact $R_w \cap K = R_v$, we see $Q$ also extends to a $R_v$-valued point of $X'$.

Proof: Since the unramified point is open, $\varphi$ is also unramified at $P$, thus $K(P)/K$ is separable(V.5.4.8). For the rest, Cf.[Diophantine Geometry, P598]. □

5 Étale

More advanced materials to learn at [Sta]Chap40.

Def. (V.5.5.1). A morphism $f : X \to Y$ of schemes is called étale if there is an open affine cover $\{U_i\}$ of $S$ and an open affine cover $V_{ij}$ of $f^{-1}(\{U_i\})$ that the ring map is étale. A standard étale morphism is the Spec map of a standard étale ring map.

étale is local on the source and target(I.7.7.5). Étale is stable under base change and composition(I.7.7.5).

Prop. (V.5.5.2) (Properties of Étale).

• Étale at a point $x$ is equivalent to smooth and unramified at a $x$(I.7.7.4).
• étale at a point $x$ is equivalent to flat and $G$-unramified at that point, by(I.7.7.11). So Étale over field is equivalent to $G$-unramified, because over a field it is obviously flat.
• Étale at a point $x$ is equivalent to locally standard étale at that point(I.7.7.17).
• A morphism is étale iff it is smooth of dimension 0, by definition(V.5.3.8).
• Étale is equivalent to flat, locally of f.p. and formally unramified, by(I.7.7.11).

Cor. (V.5.5.3). Étale map is smooth, hence syntomic, flat.

Étale map is universally open because it is flat and lcoally of f.p.(V.5.1.8).

Prop. (V.5.5.4). If $X,Y$ are étale over $S$, then any map $X \to Y$ is étale,(I.7.7.13).

Prop. (V.5.5.5) (Fiberwise). A morphism of schemes is étale iff it is flat, locally of f.p., and every fiber $X_s$ is a disjoint union of spectra of finite separable field extensions of $k(s)$.

Proof: Follows from(V.5.5.2)(V.5.4.3) and(I.7.7.10). □

Cor. (V.5.5.6). A scheme is étale over a field $k$ iff it is a disjoint union of spectra of finite separable field extensions.
Prop. (V.5.5.7). If \( X \rightarrow Y \) is smooth at \( x \), then there exist a nbhd of \( x \) that it factors through \( U \xrightarrow{\pi} \text{A}_V \rightarrow V \), where \( \pi \) is étale.

Proof: Any standard smooth morphism can be factorized as an étale map over a polynomial algebra, as easily seen. \( \square \)

Prop. (V.5.5.8) (Stalkwise and Fiberwise). For a morphism locally of f.p., the following are equivalent:

- It is étale at a point \( x \).
- \( \mathcal{O}_{x,X}/\mathcal{O}_{f(x),S} \) if flat and \( X_{f(x)}/k(x) \) is smooth at \( x \).
- \( \mathcal{O}_{x,X}/\mathcal{O}_{f(x),S} \) if flat and \( X_{f(x)}/k(x) \) is unramified at \( x \).
- \( \mathcal{O}_{x,X}/\mathcal{O}_{f(x),S} \) is flat and \( \Omega_{X_{f(x)},x} = 0 \).
- \( \mathcal{O}_{x,X}/\mathcal{O}_{f(x),S} \) is flat and \( \Omega_{X_{f(x)},x} \otimes \mathcal{O}_{X,x} k(x) = 0 \).
- \( \mathcal{O}_{x,X}/\mathcal{O}_{f(x),S} \) is flat and \( m_x \mathcal{O}_{X,x} = m_x \) and \( k(x)/k(s) \) separable.

By (V.5.4.8) and (V.5.3.3).

Prop. (V.5.5.9). If \( X \rightarrow Y \rightarrow S \), and \( X \rightarrow Y \) is surjective, flat and locally of f.p., \( X \rightarrow S \) is étale, then \( Y \rightarrow S \) is étale.

Proof: Cf.\([\text{Sta}05B5]\). \( \square \)

Def. (V.5.5.10) (Étale Neighborhood). For a point \( s : \text{Spec} k \rightarrow X \), an étale nbhd of \( s \) in \( X \) is defined to be an étale map \( U \rightarrow X \) that \( s \) factors through \( U \).

Prop. (V.5.5.11). For a morphism \( f : Y \rightarrow X \) of schemes étale over field \( k \), then \( f \) is surjective iff \( Y(k_s) \rightarrow X(k_s) \) is surjective.

Proof: If \( Y \rightarrow X \) is surjective, then? \( \square \)

Étale Connected Components

Def. (V.5.5.12) (Étale Connected Components). Let \( X \) be a scheme over a field \( k \), let \( \pi_0(X) = \text{Spec}(\pi(X)) \), where \( \pi(X) \) is the largest étale subalgebra of \( \Gamma(X) \)(I.7.7.22).

Prop. (V.5.5.13). Let \( X \) be an algebraic scheme over a field \( k \), then

- for any field extension \( k'/k \), \( \pi_0(X_{k'}) = \pi_0(X)_{k'} \).
- Let \( Y \) be a schemes over a field \( k \), then \( \pi_0(X \times Y) = \pi_0(X) \times \pi_0(Y) \).

Proof: 1: Cf.\([\text{Mil17b}]P15\).
2: There is a map \( \pi(X) \times_k \pi(Y) \rightarrow \pi(X \times Y) \). Because \( \pi \) commutes with base change, we can base change to separable closure. In this case, it suffices to show if \( X,Y \) is connected then \( X \times Y \) is connected, but this follows from (V.4.3.12). \( \square \)

Prop. (V.5.5.14). Let \( X \) be an algebraic scheme over a field \( k \), then

- The mapping \( \varphi : X \rightarrow \pi_0(X) \) induces a 1 to 1 correspondence of points of \( \pi_0(X) \) and connected components of \( X \).
- For all \( x \in \pi_0(X) \), the fiber \( \varphi^{-1}(x) \) is geo.connected over \( k(x) \).
Proof: \( \pi_0(X) \) is discrete, so the inverse image of each point is a sum of connected components of \( X \). But this must be connected, because \( \pi_0(X_{k(x)}) = \pi_0(X)_{k(x)} = k(x) \). Also, this implies for the alg. closure \( \overline{k} \) of \( k(x) \), \( \pi_0(X_{\overline{k}}) = \pi_0(X)_{\overline{k}} = \overline{k} \), thus \( X_{\overline{k}(x)} \) is geo.connected. \( \square \)

Prop. (V.5.5.15) (Étale Schemes over Field). Let \( k \) be a field and \( k^s \) its separable closure. Let \( \Gamma = \text{Gal}(k^s/k) \), then the functor \( X \mapsto X(k^s) \) is an equivalence between étale schemes over \( k \) to the category of discrete \( \Gamma \)-sets.

Proof: Cf.[Sta]03QR. \( \square \)

Noetherian Case

6 Zariski’s Main Theorem

References are [Sta]Chap36.38.

Prop. (V.5.6.1) (Zariski’s Main Theorem). For a morphism \( X \to S \) that is quasi-finite and separated, if \( S \) is qcqs, Then there is a factorization \( X \to T \to S \) that \( X \to T \) is a qc open immersion and \( T \to S \) is finite.

Proof: Cf.[[Sta]05K0]. \( \square \)

7 Complete Intersection

Should be refreshed with intrinsic definition of locally complete intersection, Cf.[[Sta]].

Def. (V.5.7.1) (Locally Complete Intersection). A closed subscheme \( Y \) of a nonsingular variety \( X \) over a field \( k \) is called locally complete intersection iff \( Y \) is locally generated by \( r = \dim(Y, X) \) elements. By(I.6.4.14) \( Y \) is C.M.. In particular, by(V.8.1.17), a regular variety is always a locally complete intersection.

Def. (V.5.7.2). A variety \( Y \) of codimension \( r \) in \( \mathbb{P}^n_k \) is a strict complete intersection iff \( \mathcal{I}_Y \) can be generated by \( r \) elements. It is called a set-theoretic complete intersection iff it can be written as an intersection of \( r \) hypersurfaces.

Prop. (V.5.7.3). A local complete intersection has its ideal sheaf \( \mathcal{I} \), then \( \mathcal{I}/\mathcal{I}^2 \) locally free by(I.6.4.13).

Prop. (V.5.7.4). If \( Y \) is a complete intersection in \( \mathbb{P}^n_k \) of hypersurfaces of degree \( d_1, \ldots, d_r \), then \( \omega_Y = \mathcal{O}_Y(\sum d_i - n - 1) \).

Proof: Use the exact sequence \( 0 \to \mathcal{O}_Z(n - d) \to \mathcal{O}_Z \to i_* \mathcal{O}_Y \to 0 \) and(V.8.1.18). \( \square \)

Prop. (V.5.7.5). For a complete intersection of dimension \( q \), \( H^i(Y, \mathcal{O}_Y(n)) = 0 \) for \( 0 < i < q \). And the natural map \( \Gamma(P, \mathcal{O}_P(n)) \to H^0(Y, \mathcal{O}_Y(n)) \) is a surjection for every \( n \). In particular, \( Y \) is connected, and the arithmetic genus \( \text{p}_a(Y) = \dim H^q(Y, \mathcal{O}_Y) \).

Proof: We use induction, the case \( Y = P \) follows from(V.6.7.16), let \( Y = Z \cap H \), where \( H \) has degree \( d \), then

\( 0 \to \mathcal{O}_Z(n - d) \to \mathcal{O}_Z \to i_* \mathcal{O}_Y \to 0 \)

thus use long exact sequence. The rest is easy. \( \square \)
Cor. (V.5.7.6). If $Y$ is a nonsingular hypersurface of degree $d$ in $\mathbb{P}^n$, then $p_g(Y) = C_d^{n-1}$. If $Y$ is a non-singular curve which is an intersection of two non-singular hypersurface of degree $d, e$ in $\mathbb{P}^3_k$, then $p_g(Y) = \frac{1}{2}de(d + e - 4) + 1$.

Proof: Use the long exact sequence to reduce to $\mathbb{P}^n_k$. Cf.[Hartshorne Ex2.8.4]. □

8 More Properties of Schemes

Universal Catenary Ring

Def. (V.5.8.1). A scheme $S$ is called universally catenary iff $S$ is locally Noetherian and every scheme locally of f.t. over $S$ is catenary.

Universally catenary is a local property, this follows from (IX.1.14.28).

Prop. (V.5.8.2). A locally Noetherian scheme is universally catenary iff all its stalks are universally catenary. Cf.[[Sta]02JA].

Morphism of Finite Presentation

Def. (V.5.8.3) (Locally of Finite Presentation). A morphism between schemes $f : Y \to X$ is called of locally finite presentation iff for any point $x \in X$, there is an open affine mapped into an open affine that the ring map is of finite presentation. It is called of finite presentation iff moreover it is qcqs.

Locally of finite presentation is local on the source and target and it is stable under composition and base change but it doesn’t satisfies the base change trick by (V.4.1.6)(V.4.1.5) and (I.6.7.9).

Prop. (V.5.8.4). When the target is locally Noetherian, (locally)finite type and (locally)finite presentation is equivalent.

Prop. (V.5.8.5). For $f : Y \to X$ over $S$, if $X/S$ is locally of f.p. and $Y/S$ is locally of f.t., then $f$ is locally of f.p.. If moreover $X$ is of f.t. and $Y$ is qs, then $f$ is of f.t..

Proof: The first follows from (I.6.7.11), the second needs to check qcqs. Qc follows from (V.4.4.26). □

Prop. (V.5.8.6) (Chevalley). A qc morphism locally of f.p. maps locally constructible subset to locally constructible subset.

Proof: We prove $f(E) \cap U_i$ is constructible for every $U_i$ affine open in $X$. The inverse image of $U_i$ is qc, hence a locally constructible set is constructible (IX.1.14.9). So we reduce to the affine case (I.5.3.1). □

Cor. (V.5.8.7). As in the proposition, if the image is dense in $Y$, then it contains an open dense subscheme of $Y$.

Proof: Cf.[GAGA Serre P8] and [[Sta]005K]. □
### Finite Locally Free Morphism

**Def. (V.5.8.8) (Finite Locally Free).** A morphism \( f : X \to Y \) is called **finite locally free of rank** \( d \) iff it is affine, and \( f_* \mathcal{O}_X \) is a finite locally free \( \mathcal{O}_Y \)-module of rank \( d \).

**Cor. (V.5.8.9).** If \( f \) is finite locally free of rank \( n \), then for any locally free sheaf \( E \) of rank \( k \) on \( X \), \( f_* E \) is locally free of rank \( nk \).

**Prop. (V.5.8.10).** \( f \) is finite locally free iff it is finite, flat and of f.p.. In particular, when \( Y \) is locally Noetherian, this is equivalent to \( f \) is finite and flat.

*Proof:* Both notions are local on the target, so we reduce to the ring case, which is (I.6.1.7). \( \square \)

**Cor. (V.5.8.11).** Finite locally freeness is stable under composition and base change, and it is local on the target.

**Prop. (V.5.8.12) (Trace and Norm).** Let \( f : Y \to X \) be a finite locally free map of constant rank, then there are trace and norm maps \( \text{tr} : f_* \mathcal{O}_Y \to \mathcal{O}_X, \text{Nm} : f_* \mathcal{O}_Y \to \mathcal{O}_X \) compatible with arbitrary base change.

*Proof:* The proof is the same as that of (I.6.1.11). \( \square \)

**Prop. (V.5.8.13).** Let \( f : Y \to X \) be a finite locally free map of constant rank, and \( b \in \Gamma(Y, \mathcal{O}_Y) \), then \( f(Z(b)) = Z(\text{Nm}(b)) \).

*Proof:* We can assume \( X \) is affine, then we need to show that for a prime \( \mathfrak{p} \) with inverse images \( \mathfrak{p}_i \), \( b \in \cup \mathfrak{p}_i \) iff \( \text{Nm}(b) \in \mathfrak{p} \). We localize at \( \mathfrak{p} \), then \( \text{Nm}(b) \in \mathfrak{p} \) iff \( \text{Nm}(b) \) is non-invertible iff multiplication by \( b \) is non-invertible iff \( b \in \mathfrak{p}_i \), because \( \mathfrak{p}_i \) are all the maximal ideals of \( B_\mathfrak{p} \). \( \square \)
V.6 Cohomology on Ringed Sites

Main References are [Sta],[Har77] and [Sheaf Cohomology, Anonymous].

1 Derived Cohomology

Def. (V.6.1.1) (Setups). Let \((\mathcal{C}, \mathcal{O})\) be a ringed site, write \(K(\mathcal{O}) = K(\text{Mod}(\mathcal{O})), D(\mathcal{O}) = D(\text{Mod}(\mathcal{O}))\). The Abelian category \text{Mod}(\mathcal{O})\) contains enough injectives by (I.11.2.28) and (I.11.2.25), so we can consider right derived functor for any left exact functor.

1. The section functor \(\Gamma(U, \mathcal{F})\) is the left exact functor \(\text{Mod}(\mathcal{O}) \to \text{Mod}(\mathcal{O}(U))\) and call the derived functors \(H^i(U, \mathcal{F}) = H^i(R\Gamma(U, \mathcal{F}))\) as the \(i\)-th cohomology of \(\mathcal{F}\) at \(U\). In fact, this functor is just \(\text{Mor}_{PSh(\mathcal{C})}(h_U, \mathcal{F})\) defined in (V.6.1.2).

2. Let \(e\) be the final object in \(PSh(\mathcal{C})\), then we define the global section functor \(\Gamma(\mathcal{C}, -)\) to be the left exact functor \(\text{Mod}(\mathcal{O}) \to \text{Ab} : \mathcal{F} \mapsto \text{Mor}_{PSh(\mathcal{C})}(e, \mathcal{F}) = \lim_{\longleftarrow} \Gamma(X, \mathcal{F})\), then we define its derived functor \(R(\mathcal{C}, \mathcal{F})\), and call the derived functors \(H^i(\mathcal{C}, \mathcal{F}) = H^i(R\Gamma(\mathcal{C}, \mathcal{F}))\) the \(i\)-th cohomology group of \(\mathcal{F}\) on \(\mathcal{C}\).

3. Let \((Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{D}), \mathcal{O}')(\) be a morphism fo topoi, then \(f_*\) is a left exact functor from \(\text{Mod}(\mathcal{O})\) to \(\text{Mod}(\mathcal{O}')\)(V.1.2.17), and we call its derived functors \(R^i f_* \mathcal{F}\) the \(i\)-th higher direct image of \(\mathcal{F}\).

4. Let shification functor \(i : PSh(\mathcal{C}) \to Sh(\mathcal{C})\) is left exact, and we call the derived functors \(\mathcal{H}^p(F)\) the sheaf-cohomology presheaves of \(F\).

Def. (V.6.1.2). Let \(K\) be a presheaf of sets on \(\mathcal{C}\), then \(\mathcal{F} \mapsto \text{Mor}_{PSh(\mathcal{C})}(K, \mathcal{F})\) is a left exact functor \(\text{Mod}(\mathcal{O})\) to \(\text{Ab}\), thus we denote its derived functors as \(H^i(K, \mathcal{F})\).


Prop. (V.6.1.4) (Sheaf-Cohomology-Presheaf). The forgetful functor is right adjoint to the exact shification functor, the Grothendieck spectral sequence applies to the exact functor \(\Gamma(U, -)\) from \(PSh(\mathcal{C})\) to \(\text{Ab}\) shows its right derived functor is

\[ \mathcal{H}^p(F) = R^p i_\ast(F) : U \to H^p(U, F). \]

Prop. (V.6.1.5) (Higher Direct Image). For \(u : (\mathcal{C}', \mathcal{O}') \to (\mathcal{C}, \mathcal{O})\) a morphism of ringed sites and \(\mathcal{F}\) a sheaf on \(\mathcal{C}\), the trivial spectral sequence for \((\mathcal{Z} \circ \mathcal{U}^p) \circ \mathcal{i}\) (because \(\mathcal{Z}, \mathcal{U}^p\) are exact(V.1.2.9)) shows that \(R^p u^s \mathcal{F} = (f^p \mathcal{H}^p(\mathcal{F}))^s\). So flask sheaf thus flabby sheaf is acyclic for \(f^s\).

Prop. (V.6.1.6) (Change of Topologies). Let \(\mathcal{C}, \mathcal{C}'\) be sites and \(\mathcal{C}'\) be a fully subcategory of \(\mathcal{C}\), \(i : \mathcal{T}' \to \mathcal{T}\) is continuous and cocontinuous(V.1.1.5), then

\[ H^p(\mathcal{T}' ; U, i^s F') \cong H^p(\mathcal{T} ; U, F), \quad H^p(\mathcal{T}' ; U, F') \cong H^p(\mathcal{T} ; U, i_* F') \]

Proof: Immediate from(V.1.2.21).

Prop. (V.6.1.7) (Relative Leray Spectral Sequence). Let \((f, f^* : (Sh(\mathcal{C}), \mathcal{O}) \to (Sh(\mathcal{C}'), \mathcal{O}'), (g, g^*) : (Sh(\mathcal{C}'), \mathcal{O}') \to (Sh(\mathcal{D}), \mathcal{O}_D))\) be morphisms of ringed topoi, then for any \(\mathcal{F}^* \in K^+(\mathcal{O})\), there is a spectral sequence convergence

\[ E_2^{p,q} = R^p g_* R^q f_*(\mathcal{F}^*) \Rightarrow R^{p+q}(g \circ f)_* \mathcal{F}^*. \]
Proof: Cf. [Sta]0732. □

Cor. (V.6.1.8) (Leray Spectral Sequence). Let \((f, f^\#) : (\text{Sh}(\mathcal{C}), \mathcal{O}) \rightarrow (\text{Sh}(\mathcal{C}'), \mathcal{O}')\) be a morphism of ringed topoi, then for any \(\mathcal{F}^\bullet \in K^+(\mathcal{O})\), there is a spectral sequence convergence

\[ E_2^{p,q} = H^p(\mathcal{C}', R^q f_*(\mathcal{F}^\bullet)) \Rightarrow H^{p+q}(\mathcal{C}, \mathcal{F}^\bullet). \]

Prop. (V.6.1.9) (Projection Formula). Let \(f : X \rightarrow Y\), and \(\mathcal{E}\) be a locally free \(\mathcal{O}_Y\)-module, then we have

\[ R^i f_*(\mathcal{F} \otimes f^* \mathcal{E}) \cong R^i f_*(\mathcal{F}) \otimes \mathcal{E}. \]

Proof: It suffice to prove for \(i = 0\), because then we know that \(f^* \mathcal{E}\) and \(\mathcal{E}\) are locally free thus flat and preserves injectives (V.6.5.10) and then use Grothendieck spectral sequence.

For \(i = 0\), there is a map from the right to the left, and there stalk are both \((f_*(\mathcal{F}))_{x}\)\(^{\text{rank}}\mathcal{E}\), so they are equal. □

Prop. (V.6.1.10) (Filtered Colimits). \(H^n(U, -)\) commutes with filtered colimits if \(T\) is a Noetherian topology.

Proof: For \(n = 0\) the limit presheaf is already a sheaf, because for any finite cover, the Čech complex of the limit sheaf is the limit of Čech sheaves, and direct limit is exact.

And the limit sheaf of flask sheaves are flask, because flask need only be checked for finite covers at this case (because \(T\) and \(T^f\) have equivalent category of sheaves (V.1.1.13) and definition of flask (V.6.5.1)). Then the limit of exact Čech complexes is exact. So we can use the limit of the flask sheaf resolutions to calculate cohomology, thus the result. □

Low Dimensions


Prop. (V.6.1.11) \((H^1 \text{ and Torsors})\). Let \(\mathcal{C}\) be a site and \(\mathcal{H}\) an Abelian sheaf on \(\mathcal{C}\), then there is a canonical isomorphism of \(\mathcal{H}\)-torsors (V.1.1.14) and \(H^1(\mathcal{C}, \mathcal{H})\).

Proof: Cf. [Sta]03AJ. □

Prop. (V.6.1.12) \((H^1 \text{ and Picard Group})\). Let \((\mathcal{C}, \mathcal{O})\) be a local ringed site, then there is a canonical isomorphism of Abelian groups

\[ H^1(\mathcal{C}, \mathcal{O}^*) = \text{Pic}(\mathcal{O}). \]

Proof: Cf. [Sta]040E. □

Prop. (V.6.1.13) \((H^2 \text{ and Objects of Gerbes})\). Let \(\mathcal{C}\) be a site and \(\mathcal{S} \rightarrow \mathcal{C}\) be a gerbe whose automorphism sheaves are Abelian. Let \(\mathcal{G}\) be the sheaf defined in (V.1.4.22). If \(U\) is an object of \(\mathcal{C}\) that

- there exists a cofinal system of coverings \(\{U_i \rightarrow U\}\) that for any such covering, \(H^1(U_i, \mathcal{G}) = 0, H^1(U_i \times_U U_j, \mathcal{G}) = 0\),
- \(H^2(U, \mathcal{G}) = 0.\)

Then \(\mathcal{S}_U\) is non-empty.
Proof: By hypothesis, there is a covering \( \{ U_i \to U \} \) and \( x_i \) in \( \mathcal{S} \) lying over \( U_i \). By item 1, after refining the covering, we may assume \( H^1(U_i, \mathcal{G}) = 0 \) and \( H^1(U_{ij}, \mathcal{G}) = 0 \). Consider the sheaf

\[ \mathcal{F}_{ij} = \text{Isom}(x_i|_{U_{ij}}, x_j|_{U_{ij}}) \]

on \( C/U_{ij} \), then there is an action \( \mathcal{G}_{U_{ij}} \times \mathcal{F}_{ij} \to \mathcal{F}_{ij} \). Then \( \mathcal{F}_{ij} \) is a pseudo \( \mathcal{G}|_{U_{ij}} \)-torsor and clearly a torsor because any two objects of a gerbe is locally isomorphic.

By (V.6.1.11), these torsors are trivial, thus having a global section. In other words, there are isomorphisms \( \varphi_{ij} : x_i|_{U_{ij}} \to x_j|_{U_{ij}} \). To get an object \( x \) over \( U \), it suffices to manage the choices of \( \varphi_{ij} \) to get a descent datum. For this, use the fact \( H^2(U, \mathcal{G}) = 0 \) and \( \check{H}^2(U, \mathcal{G}) \to H^2(U, \mathcal{G}) \) is injective by Cech to Derived spectral sequence (V.6.2.11).

Base Change

Prop. (V.6.1.14) (Flat Base Change Morphism). Let

\[
\begin{array}{ccc}
(\text{Sh}(C'), \mathcal{O}_{C'}) & \xrightarrow{g'} & (\text{Sh}(C), \mathcal{O}_{C}) \\
\downarrow f' & & \downarrow f \\
(\text{Sh}(D'), \mathcal{O}_{D'}) & \xrightarrow{g} & (\text{Sh}(D), \mathcal{O}_{D})
\end{array}
\]

be a commutative diagram of ringed sites. Assume both \( g, g' \) are flat, then for any bounded below complex of \( \mathcal{O}_C \)-modules \( \mathcal{F}^\bullet \), there exists a canonical base change map

\[ g^* Rf_* \mathcal{F}^\bullet \to R(f')_* (g')^* \mathcal{F}^\bullet. \]

Proof: Cf.[Sta]0736.

Def. (V.6.1.15) (Base Change Morphism). By Leray spectral sequence (V.6.1.7), there are edge morphisms \( R^p g_*(f_* F) \to R^p (gf)_*(F) \) and \( R^p (gf)_*(F) \to g_*(R^p f_*(F)) \). So if there is a cartesian diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow v' & & \downarrow v \\
X & \xrightarrow{f} & Y
\end{array}
\]

there is a morphism \( F \to v'_* v'^* F \), and

\[ R^p f_* F \to R^p f_*(v'_* v'^* F) \to R^p (f v')_* (v'^* F) = R^p (v f')_* (v^* F) \to v_*(R^p f_*(v'^* F)). \]

Hence by adjointness a morphism

\[ v^*(R^p f_*(F)) \to R^p f'_*(v'^* F) \]

called the base change morphism.
CHAPTER V. ALGEBRAIC GEOMETRY

Others

Prop. (V.6.1.16). Let $K' \to K$ be a map of presheaves of sets on $\mathcal{C}$ whose shifification is surjective. Set $K'_p = K' \times_K \ldots \times_K K'$, then for any Abelian sheaf $\mathcal{F}$, there is a spectral sequence convergence

$$E_1^{p,q} = H^q(K'_p, \mathcal{F}) \Rightarrow H^{p+q}(K, \mathcal{F}).$$

Proof: Since shifification is exact, $(K'_p)^\sharp = (K')^\sharp_p$. Then we use (V.1.2.23) to change to a larger site $\mathcal{C}'$ where the topoi are equivalent and $K', K$ are objects in $\mathcal{C}'$ and $K' \to K$ is a covering, then we use the $E_1$ page of the Čech to sheaf spectral sequence (V.6.2.11). Notice this need modification, the modification goes back to the proof of the Grothendieck spectral sequence, where we choose the natural Čech complex resolution in place of the CE resolution, because we have (V.6.2.3). □

Cor. (V.6.1.17) (Čech-Alexander Resolution). If $\mathcal{C}$ is a site with the indiscrete topology, $X$ a weakly final object (II.1.1.13) of $\mathcal{C}$, then for any Abelian sheaf $\mathcal{F}$ on $\mathcal{C}$, the total cohomology $R\Gamma(\mathcal{C}, \mathcal{F})$ is represented by the Čech complex

$$\mathcal{F}(X) \to \mathcal{F}(X \times X) \to \mathcal{F}(X \times X \times X) \to \ldots.$$  

Proof: By (V.6.5.4), $H^q(X_p, \mathcal{F}) = 0$ for $q > 0$. The assumption says $h_X \to *$ is surjective, thus the conclusion is a special case of (V.6.1.16). □

2 Čech Cohomology

Def. (V.6.2.1) (Čech Complex and Čech Cohomology). Let $X$ be a ringed site and $\mathcal{U} : \{U_i \to U\}$ be a covering, we have a canonical complex of presheaves $\mathbb{Z}_{\mathcal{U}, \bullet}$ defined to be

$$\cdots \to \bigoplus \mathbb{Z}_{U_{i_0}i_1} \to \bigoplus \mathbb{Z}_{U_{i_0}i_1} \to \mathbb{Z}_{U_{i_0}} \to 0.$$  

And for any presheaf of $\mathcal{O}_X$-module $\mathcal{F}$, $\text{Hom}_{\mathcal{O}_X}(\mathbb{Z}_{\mathcal{U}, \bullet}, \mathcal{F})$ gives out the Čech complex $\check{\mathcal{C}}^*(\mathcal{U}, \mathcal{F})$ of $\mathcal{F}$. The cohomology $\check{H}^*(\mathcal{U}, \mathcal{F})$ of $\check{\mathcal{C}}^*(\mathcal{U}, \mathcal{F})$ is called the Čech cohomology of $\mathcal{F}$ w.r.t $\mathcal{U}$.

This complex is exact except in degree 0. This is because we have a homotopy: choose a fixed $i_0$, for a $s \in \Gamma(X, U_{i_1 \ldots i_n})$, we map it to $(hs)_{i_1 \ldots i_n} = \delta_{i,i_0}s$. In particular, an injective sheaf is Čech acyclic.

Lemma (V.6.2.2). The Čech complexes $\check{\mathcal{C}}^*(\mathcal{U}, -)$ induces a functor from $PAb(\mathcal{C})$ to $\text{Comp}^+(Ab)$, which is an exact functor.

Proof: Because in each degree this functor is a sum of functors of the form $\mathcal{F} \mapsto \mathcal{F}(U)$, which are exact functors on $PAb(\mathcal{C})$. □

Prop. (V.6.2.3) (Čech Complex as Derived Complex). Let $X$ be a site and $\mathcal{U} : \{U_i \to U\}$ be a covering, then $\check{H}^0(\mathcal{U}, -)$ is left exact, and there are quasi-isomorphisms

$$\check{\mathcal{C}}^*(\mathcal{U}, \mathcal{F}) \to R\check{H}^0(\mathcal{U}, \mathcal{F})$$

which is functorial in $\mathcal{F}$.  

Proof: Choose a functorial injective resolution of presheaves $I^\bullet$ of $F$, and consider the double complex $\tilde{\mathcal{C}}(U, I^\bullet)$. There are maps of complexes

$$\tilde{\mathcal{C}}(U, F) \to \text{Tot}(\tilde{\mathcal{C}}(U, I^\bullet)), \quad \tilde{H}^0(U, I^\bullet) \to \text{Tot}(\tilde{\mathcal{C}}(U, I^\bullet))$$

which are both quasi-isomorphism by an application of spectral sequence and the fact the columns and rows are exact in positive degrees: The columns are exact because of (V.6.2.2) and the rows are exact because $Z_{\alpha U, \bullet}$ is exact in positive degrees (V.6.2.1) and $I^p$ are injective. Then we have the desired quasi-isomorphism, and it is functorial in $F$. □

Cor. (V.6.2.4) (Čech-Cohomology). If we take colimit for coverings, $F \to \tilde{H}^0(U, F)$ is a left exact functor from presheaves to sets, the derived complex is just $\lim_{\rightarrow} \tilde{\mathcal{C}}(U, F)$, and the derived functors are just $\lim_{\rightarrow} \tilde{H}^q(U, F)$.

Proof: This is because we can take colimit of the conclusion of (V.6.2.3), because the colimit is filtered by (V.6.2.5) so exact, so the Čech complex also represents the derived complex. □

Lemma (V.6.2.5). The refinement morphism of Čech cohomologies of two coverings doesn’t depend on the refinement map chosen.

Proof: For two refinement map, there is a commutative diagram

$$\prod F(U_i) \xrightarrow{\delta^p} \prod F(U_i \times_U U_j) \xleftarrow{\Delta^1} \prod F(U'_j)$$

so it induce the same map on the kernel. □

Prop. (V.6.2.6) (Non-Abelian Čech). For a exact sequence of sheaves of groups $1 \to A \to B \to C \to 1$, where $A$ is in the center of $B$, then there is a exact sequence:

$$1 \to H^0(U, A) \to H^0(U, B) \to H^0(U, C) \to H^1(U, A) \to H^1(U, B) \to H^1(U, C) \to H^2(U, A)$$

which is by direct calculation, the last one is the Čech composed with the injection to sheaf cohomology (V.6.2.11).

Proof: The definition of non-Abelian cohomology need clarification, similar to that of (IV.3.3.3)? □

Comparison Theorems

Prop. (V.6.2.7). If two coverings are refinements of each other, then their Čech cohomology is isomorphic.

Proof: Because the refinement morphism doesn’t depend on the refinement map (V.6.2.5). □

Prop. (V.6.2.8) (Comparison Theorem for Čech Acyclicity). If there are two coverings $\mathcal{U}, \mathcal{V}$ and a presheaf $F$, then we can construct a double Čech complex with the $(p, q)$-term being $F(U_{i_1 \ldots i_p} \cap V_{j_1 \ldots j_q})$. Then the vertical and horizontal arrays calculate the Čech cohomology $\prod_j H^*(\mathcal{V}|_{V_{j_1 \ldots j_q}}, F)$, $\prod_i H^*(\mathcal{U}|_{U_{i_1 \ldots i_q}}, F)$ respectively.

So by Spectral sequence (II.3.5.8), if both higher Čech cohomology group $H^k(\mathcal{U}|_{V_{j_1 \ldots j_q}}, F)$, $H^k(\mathcal{V}|_{U_{i_1 \ldots i_q}}, F)$ vanish, i.e., they are both $F$-acyclic, then $H^*(\mathcal{U}, F) \cong H^*(\mathcal{V}, F)$. 

Cor. (V.6.2.9). If $\mathfrak{F}$ is a refinement of $\mathfrak{U}$, and $\mathfrak{F}|_{U_{i_1,\ldots,i_p}}$ are all $\mathcal{F}$ acyclic, then $H^* (\mathfrak{U}, \mathcal{F}) \cong H^* (\mathfrak{F}, \mathcal{F})$.

Proof: It suffices to prove $\mathfrak{U}|_{U_{i_1,\ldots,i_q}}$ is $\mathcal{F}$-acyclic. But $\mathfrak{U}|_{U_{i_1,\ldots,i_q}}$ and $\text{id}_{U_{i_1,\ldots,i_q}}$ are refinements of each other, so (V.6.2.7) settles the proof.

Cor. (V.6.2.10). If $\mathfrak{F}|_{U_{i_1,\ldots,i_p}}$ is $\mathcal{F}$-acyclic, then the covering $H^* (\mathfrak{U} \times \mathfrak{F}, \mathcal{F}) = H^* (\{U_i \cap V_j\}, \mathcal{F}) \cong H^* (\mathfrak{U}, \mathcal{F})$.

Proof: Because $\mathfrak{F}|_{U_{i_1,\ldots,i_p}}$ and $\mathfrak{U} \times \mathfrak{F}|_{U_{i_1,\ldots,i_p}}$ are refinement of each other, so $\mathfrak{U} \times \mathfrak{F}|_{U_{i_1,\ldots,i_p}}$ are $\mathcal{F}$-acyclic by (V.6.2.7), and $\mathfrak{U} \times \mathfrak{F}$ refines $\mathfrak{U}$, so (V.6.2.9) can be applied.

Prop. (V.6.2.11) (Čech to Sheaf). For any Abelian sheaf $\mathcal{F}$ on a site, the Grothendieck spectral sequence applied to $\Gamma(U, -) = H^0 (\{U_i \to U\}, -) \circ \iota = \check{H}^0 (U, -) \circ \iota$ gives us:

$$H^p (U, \check{H}^q (\mathcal{F})) \Rightarrow H^{p+q} (\mathcal{U}, \mathcal{F}).$$

$$\check{H}^p (U, \mathcal{F}) \Rightarrow H^{p+q} (U, \mathcal{F}).$$

Cor. (V.6.2.12). The Grothendieck spectral sequence applied to forgetful functor and exact functor shows that $\check{H}^p (\mathcal{F})^{++} = \check{H}^p (\mathcal{F})^{\sharp}$ is 0 for $p > 0$, so

$$\check{H}^p (\mathcal{F})^{\sharp} (U) = \check{H}^0 (U, \mathcal{H}^p (\mathcal{F})) = 0 \quad p > 0.$$ because $\check{H}^p (\mathcal{F})^{\sharp}$ is separated, See (V.1.2.6).

Thus the low degree of Čech to sheaf says:

$$0 \to \check{H}^1 (U, \mathcal{F}) \to H^1 (U, \mathcal{F}) \to 0 \to \check{H}^2 (U, \mathcal{F}) \to H^2 (U, \mathcal{F}).$$

Cor. (V.6.2.13) (Acyclic Covering Calculates Cohomology). If we have $H^q (U_{i_0 i_1 \ldots i_r} \setminus U, \mathcal{F}) = 0$, then $H^p (\{U_i \to U\}, \mathcal{F}) = H^p (U, \mathcal{F})$. (because $H^p (\{U_i \to U\}, \mathcal{H}^q (\mathcal{F}))$ vanish for $q > 0$).

Prop. (V.6.2.14) (Čech Acyclic Čech Comparison). If $\mathcal{C}$ is a site, $\mathfrak{G} \subset \mathcal{C}$, $\text{Cov} \subset \text{Cov}_{\mathcal{C}}$ and $\mathcal{F} \in \text{Ab}(\mathcal{C})$ that

- For any $\{U_i \to U\} \in \text{Cov}$, $U_i, U \in \mathfrak{G}$, and $U_{i_0,\ldots,i_p} \in \mathfrak{G}$.
- $\text{Cov}|_U$ is cofinal in $\text{Cov}_{\mathcal{C}}|_U$ for any $U \in \mathfrak{G}$.
- $\check{H}^q (U, \mathcal{F}) = 0$ for any $U \in \mathfrak{G}$ and $q > 0$.

Then $H^p (X, \mathcal{F}) = H^p (X, \check{H}^q (\mathcal{F})) = H^p (\{U_i \to X\}, \mathcal{F})$ for $U_i \in \mathfrak{G}$ (if such a covering exists) and $\mathcal{F}$ any Abelian sheaf on $\mathcal{C}$.

Proof: By (V.6.2.13), we only have to show that $H^q (U, \mathcal{F}) = 0$ for $U \in \mathfrak{G}$ and $q > 0$. Use induction on $q$, use Čech to sheaf: $\check{H}^p (U, \mathcal{H}^q (\mathcal{F})) \Rightarrow H^{p+q} (U, \mathcal{F})$. The case $q \neq 0$ is by condition 2, 3, and induction hypothesis. For $p = 0$, use (V.6.2.12).

3 Derived Tensor and Inner Hom

**Rtensor and Tor**

Lemma (V.6.3.1). if $P$ is a complex of $\mathcal{O}$-modules and if $\alpha, \beta : L \to M$ are homotopy equivalent maps between complexes of $\mathcal{O}$-modules, then the map $\text{Tot}_i (L \otimes \mathcal{O} P) \to \text{Tot}_i (M \otimes \mathcal{O} P)$ they induce are also homotopy equivalent.

So $\text{Tot}_i (\mathcal{O} L)$ is a functor $K(\text{Mod}_{\mathcal{O}}) \to K(\text{Mod}_{\mathcal{O}})$, and moreover an exact functor of triangulated categories.
Proof: The homotopy is easily constructed. For the second, notice the distinguished triangles in $K(\text{Mod}_\mathcal{O})$ are termwise-split short exact sequences (I.11.4.1), and they are preserved under tensoring. □

**Prop. (V.6.3.2) (Derived Tensor Product).** In the Abelian category of $\mathcal{R}$-modules, we can define a derived tensor product

$$\otimes^L : D(\text{Mod}_\mathcal{O}) \times D(\text{Mod}_\mathcal{O}) \to D(\text{Mod}_\mathcal{O})$$

that is, for complexes $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ of $\mathcal{O}$-modules, choose a $K$-flat resolution $K$ of $F$ by (V.6.5.15), then we can define $\mathcal{F} \otimes^L \mathcal{O} \mathcal{G} = \text{Tot}(K \otimes \mathcal{O} \mathcal{G})$. This is independent of the resolution chosen up to quasi-isomorphism by (V.6.5.16), so descends to a functor

$$- \otimes^L \mathcal{O} \mathcal{F} : D(\text{Mod}_\mathcal{O}) \to D(\text{Mod}_\mathcal{O}).$$

This is also functorial and descends for $\mathcal{G}$ by (V.6.5.16) and (V.6.5.13), thus we are done.

**Cor. (V.6.3.3) (Commutative Monoidal Structure).** For complexes $K, L, M$,

$$K \otimes^L L \cong L \otimes^L K, \quad (K \otimes^L L) \otimes^L M \cong K \otimes^L (L \otimes^L M).$$

Thus $D(A)$ is a commutative monoidal structure.

**Proof:** This follows from the definition and (I.11.3.11). □

**Prop. (V.6.3.4).** Let $A \to B \to C$ be ring maps, $M \in K(A)$, $N \in K(B)$ and $K \in K(C)$, then

$$(M \otimes^L_A N) \otimes^L_B K = M \otimes^L_A (N \otimes^L_B K) = (M \otimes^L_A C) \otimes^L_B (N \otimes^L_B K)$$

and

$$(M \otimes^L_A K) \otimes^L_B C \cong (M \otimes^L_A K) \otimes^L_B (N \otimes^L_B C)$$

**Proof:** For the first equation, see $K$-flat resolutions, noticing (V.6.5.14). Similarly for the second equality. The last isomorphism follows from the above, Cf. [Sta]08YU. □

**Def. (V.6.3.5) (Tor).** Let $M, N$ be $\mathcal{O}$-modules, then the torsion group $\text{Tor}_p^\mathcal{O}(M, N)$ is defined to be $H^p(M \otimes^L_N N)$.

**Prop. (V.6.3.6).** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site, and $\mathcal{F}$ is an $\mathcal{O}$-module. Then $\mathcal{F}$ is a flat $\mathcal{O}$-modules iff $\text{Tor}_1^\mathcal{O}(\mathcal{F}, \mathcal{G}) = 0$ for any $\mathcal{O}$-module $\mathcal{G}$.

**Proof:** If $\mathcal{F}$ is flat, then clearly $\text{Tor}_1^\mathcal{O}(\mathcal{F}, \mathcal{G}) = 0$ for any $\mathcal{O}$-module $\mathcal{G}$. Conversely, use the long exact sequence associated to $\mathcal{F} \otimes_{\mathcal{O}} -$ (V.6.3.1). □

**Derived Tensor and Inner Hom**

**Def. (V.6.3.7) (Hom Complex).** For two complex $P, Q$, there is a Hom complex $\text{Hom}^\bullet(P, Q)$ as

$$\text{Hom}^n(P, Q) = \prod \text{Hom}(P_i, Q_{n+i}),$$

with the differential giving by $d_n(\{f_i\})_i = \{df_i + (-1)^n f_{i+1}d\}$ and suitable signatures.

It is clear that $H^n(\text{Hom}^\bullet(P, Q)) = \text{Hom}_{\mathcal{K}(\mathcal{A})}(P, Q[n])$. 


Prop. (V.6.3.8). There is a canonical isomorphism:
\[ \text{Hom}^\bullet(K, \text{Hom}^\bullet(L, M)) = \text{Hom}^\bullet(Tot(K \otimes_R L), M). \]

Proof: Cf.[Sta]0A5Y. \[\square\]

Cor. (V.6.3.9). In the category \(D(R)\), If \(K\) is \(K\)-flat and \(I\) is \(K\)-injective, then \(\text{Hom}^\bullet(K, I)\) is \(K\)-injective.

Proof: Use definitions(V.6.5.13)(I.11.5.12). \[\square\]

Prop. (V.6.3.10). Given complexes \(K, M, L\) of \(R\)-complexes, there are canonical functorial morphisms:
\[ \begin{align*}
\text{Tot}(\text{Hom}^\bullet(L, M) \otimes_R \text{Hom}^\bullet(K, L)) & \to \text{Hom}^\bullet(K, M), \\
\text{Tot}(\text{Hom}^\bullet(L, M) \otimes_R K) & \to \text{Hom}^\bullet(\text{Hom}^\bullet(K, L), M), \\
\text{Tot}(K \otimes_R \text{Hom}^\bullet(M, L)) & \to \text{Hom}^\bullet(M, \text{Tot}(K \otimes_R L)), \\
K & \to \text{Hom}^\bullet(L, \text{Tot}(K \otimes_R L)).
\end{align*} \]

Proof: 1: [Sta]0A8I].
2:[Sta]0A60].
3:[Sta]0BYM].
4:[Sta]0A62]. \[\square\]

Prop. (V.6.3.11). In a Grothendieck Abelian category \(A\), we can define a derived Hom
\[ R\text{Hom} : (D(A))^{\text{op}} \times D(A) \to D(A) \]
that is, for complexes \(X\) and \(Y\), there is a \(K\)-injective resolution \(I\) of \(Y\) by(I.11.5.17). Thus we define \(R\text{Hom}^n(X, Y) = \text{Hom}^\bullet(X, I)\).

This does descend to \(D(A)\) because first it is independent of \(X\) chosen because of the second definition of(I.11.5.12) and homotopy induce a homotopy in the the double complex.

Also, two \(K\)-injective resolutions are quasi-isomorphic and quasi-isomorphisms induce isomorphism on \(E_1\) of the spectral sequence associated to the double complex (used injectiveness) thus on the homology of total complex by comparison.

Def. (V.6.3.12) (Ext). For any \(G, F \in K(A)\), we define
\[ \text{Ext}^n_G(F) = H^i(\text{Hom}(G, F)), \]
this is seen to be equal to \(\text{Hom}_{K(O)}(G^\bullet, Q^\bullet[n]) = \text{Hom}_{D(O)}(G^\bullet, F^\bullet[n])\) where \(Q^\bullet\) is a \(K\)-injective resolution of \(F^\bullet\), by(V.6.3.7) and the definition of \(K\)-injective(I.11.5.12). Dually for \(G^\bullet\).

Prop. (V.6.3.13) (Spectral Sequences for Ext). Let \((\mathcal{C}, \mathcal{O})\) be a ringed site and \(\mathcal{K}^\bullet\) a bounded above complex of \(\mathcal{O}\)-modules, \(F\) is an \(O\)-module, then there are two spectral sequence convergences
\[ \begin{align*}
E_2^{i,j} & = \text{Ext}^i_O(H^{-j}(\mathcal{K}^\bullet), F) \Rightarrow \text{Ext}^{i+j}_O(\mathcal{K}^\bullet, F), \\
E_1^{i,j} & = \text{Ext}^j_O(\mathcal{K}^{-i}, F) \Rightarrow \text{Ext}^{i+j}_O(\mathcal{K}^\bullet, F).
\end{align*} \]

Proof: Choose a (bounded above) injective resolution \(F^\bullet\) of \(F\), and these are the two spectral sequences associated to the double complex \(\text{Hom}_O(\mathcal{K}^\bullet, F^\bullet)\). \[\square\]
Prop. (V.6.3.14). If \( R \) is a ring and \( K, L, M \in D(R) \), then
\[
R \text{Hom}_R(K, R \text{Hom}(L, M)) = R \text{Hom}_R(K \otimes_R L, M).
\]

*Proof:* Choose a \( K \)-flat resolution \( K \) of \( L \) and a \( K \)-injective resolution \( I \) of \( M \), then
\[
R \text{Hom}^\bullet(K, I) = R \text{Hom}^\bullet(K, I) \text{ is } K \text{-injective (V.6.3.9)}, \text{ and this isomorphism is just (V.6.3.8)}. \qed
\]

Cor. (V.6.3.15). By (V.6.3.12), taking \( H^0 \), we get:
\[
\text{Hom}_{D(R)}(K, R \text{Hom}(L, M)) = \text{Hom}_{D(R)}(K \otimes_R L, M) \]
that is, derived tensor is left adjoint to inner hom.

Cor. (V.6.3.16) (\textsc{Rtensor and RHom}). For a ring map \( R \to S \) and any \( L \subset D(R), M \subset D(S) \),
there is an isomorphism
\[
R \text{Hom}_R(L, M) \cong R \text{Hom}_S(L \otimes_R S, M)
\]
*Proof:* Choose a \( K \)-flat resolution of \( L \) and a \( K \)-injective resolution of \( M \), then this is just the usual adjunction of \( \text{Tor} \) and \( \text{Hom} \). \qed

Cor. (V.6.3.17). Taking \( H^0 \), we see that the derived change of ring \( - \otimes_R S \) is left adjoint to the inclusion \( D(S) \subset D(R) \).

Prop. (V.6.3.18) (Usual Ext). \( \mathcal{A} \) is categorically equivalent to the subcategory of \( D(\mathcal{A}) \) that has only \( H^0 \) nonzero. If we define \( \text{Ext}^i_{\mathcal{A}}(X, Y) \) as \( \text{Hom}_{D(\mathcal{A})}(X[0], Y[i]) \), then it is equivalent to the \( i \)-term extension of \( Y \) by \( X \), and it is an abelian group. We have a
\[
\text{Ext}^i_{\mathcal{A}}(X, Y) \times \text{Ext}^j_{\mathcal{A}}(Y, Z) \to \text{Ext}^{i+j}_{\mathcal{A}}(X, Z)
\]
by composition or equivalently the conjunction of extensions.

*Proof:* Cf. [Gelfand P167] \qed

Cor. (V.6.3.19). The definition of \( \text{Ext}^n(X, Y) \) is equivalent to the usual definition as the derived functor of \( \text{Hom}(X, -) \). Because by (I.11.5.13) when we use a projective resolution or an injective resolution, then it is equivalent to hom in \( K(\mathcal{A})(I.11.5.12) \), which is exactly the homology group of the Hom.

4 Derivd Pullback

Def. (V.6.4.1) (Derived Pullback). Let \( (\mathcal{S}(\mathcal{C}), \mathcal{O}_\mathcal{C}) \) be a ringed topos, then for any complex \( G^\bullet \) of \( \mathcal{O}_\mathcal{C} \)-modules, there is a quasi-isomorphism \( K^\bullet \to G^\bullet \) that \( f^*K \) is \( K \)-flat for any morphism \( f : (\mathcal{S}(\mathcal{D}), \mathcal{O}_\mathcal{D}) \to (\mathcal{S}(\mathcal{C}), \mathcal{O}_\mathcal{C}) \) of ringed topoi. In fact, \( K^\bullet \) can be chosen to be a bounded above complex of flat \( \mathcal{O}_\mathcal{C} \)-modules.

Thus, for any morphism \( f : (\mathcal{S}(\mathcal{C}), \mathcal{O}) \to (\mathcal{S}(\mathcal{C}'), \mathcal{O}') \) of ringed topoi, we can define an exact functor
\[
L f^* : D(\mathcal{O}') \to D(\mathcal{O})
\]
of triangulated categories that \( L f^*K^\bullet = f^*K^\bullet \) for any \( K^\bullet \) a bounded above complex of flat \( \mathcal{O}' \)-modules.

*Proof:* \qed

Remark (V.6.4.2) (\textsc{Warning}). If \( M, N \) are two \( A \)-modules, then we can define \( M_R \otimes_R^L N \) and \( M \otimes_R^L N_R \), but there are no reason for them to be isomorphic.
5 Acyclic Sheaves

Def. (V.6.5.1) (Flask Sheaves). An Abelian sheaf on a site is called flask if it satisfies the following equivalent conditions:

- It is acyclic for the forgetful functor \( \iota \),
- It is acyclic for any \( \check{H}^0(\mathcal{U}_i \to U), - \)
- It is acyclic for all \( \Gamma(U, -) \).

Also the class of flask sheaves are adapted to \( \iota \).

An Abelian sheaf on a site is called flasque iff it is acyclic for all \( \text{Mor}(S, -) \) for any \( S \) a sheaf of sets, which is obviously flasque.

It is called flabby iff for any monomorphism \( U \to X, \mathcal{F}(X) \to \mathcal{F}(U) \) is surjective;

Proof: \( 1 \iff 3 \) is by (V.6.1.4), \( 3 \to 2 \) use Čech to sheaf1(V.6.2.11).

2 \to 1 : suffices to check (II.3.3.2) for \( \iota \), should use \( \iota \) takes injective to injective, \( \check{H}^0(\mathcal{U}_i \to U), - \) commutes with finite sum and the fact that \( \check{H}^1 = H^1 \) and long exact sequence.

Prop. (V.6.5.2). Flabby sheaf is flasque. By the way, injective sheaves on a ringed site are flabby by (I.11.2.28).

Proof: Just need to verify (II.3.3.2). Injectives are flabby, so it is sufficiently large.

For an exact sequence \( 0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \) of sheaves, if \( \mathcal{F} \) is flabby, then \( \mathcal{H} \) is just the presheaf cokernel. (It reduces to \( \check{H}^1(\mathcal{U}_i \to U), F) = 0 \), and this is done by Zorn’s lemma). Thus if \( \mathcal{F} \) is flabby, \( \mathcal{G} \) is flabby iff \( \mathcal{H} \) is flabby by five lemma).

Prop. (V.6.5.3). For a continuous functor of sites \( f : T \to T' \), if \( F' \) is a flask sheaf on \( T' \), then \( f^* F' \) is also flask.

Proof: Notice \( H^0(\mathcal{U}_i \to U), f^* F) = H^0(\{f(\mathcal{U}_i) \to f(U)\}, F) \).

Prop. (V.6.5.4). On a discrete site, all sheaves is flabby. (This is because \( \iota \) is the identity functor).

Prop. (V.6.5.5). Filtered colimits of flabby sheaves is flabby. (This is because filtered colimits is exact).

Filtered colimits of injective sheaves over a Noetherian topological space is injective. (Use Baer criterion, then notice every sub-object of \( \mathbb{Z}_U \) is finitely generated because it has only f.m. connected component (IX.1.14.4) so it maps to some \( F_\alpha \).

Cor. (V.6.5.6). For an injective Abelian presheaf \( F \) on \( T \), \( F(U) \) is injective Abelian group for every \( U \), this is because the morphism \( i : \text{pt} \to T : \text{pt} \mapsto U \) is exact (\( i_* A(V) = \bigoplus_{\text{Hom}(V,U)} A \), hence \( \check{\mathcal{D}}^\mathcal{V} \) preserves injectives.

Prop. (V.6.5.7). Let \( I \) be an injective module over a Noetherian ring \( A \), then the sheaf \( I \) on \( \text{Spec} A \) is flabby.

Proof: We have for a Qco module over \( \text{Spec} A \), \( \Gamma(U, \check{M}) \cong \varinjlim \text{Hom}(I^n, M)(V.3.1.18) \), so if we have two open set \( X - V(a) \) and \( X - V(b) \), and \( a, b \) radical, then the restriction map is induce by the inclusion \( b \subset a \), and it is surjective because \( I \) is injective and filtered colimits is exact.

Lemma (V.6.5.8). A constant sheaf on an irreducible topological space is flabby, thus flasque.

Prop. (V.6.5.9). If \( \mathcal{I} \) is an injective \( \mathcal{O}_X \)-mod, then \( \mathcal{I}|_U \) is an injective \( \mathcal{O}_U \)-mod for \( U \) open, this is because \( \mathcal{I}|_U \) is right adjoint to the exact \( j_i \).

Prop. (V.6.5.10). If \( \mathcal{I} \) is an injective \( \mathcal{O}_X \)-module, then for a coherent locally free sheaf \( \mathcal{L} \), \( \mathcal{L} \otimes \mathcal{I} \) is also injective, because tensoring with \( \mathcal{L} \) is adjoint to tensoring with \( \mathcal{L}^\mathcal{V}(V.2.5.2) \), which is exact.
Limp Sheaves

**Def. (V.6.5.11) (Limp Sheaf).** A sheaf $\mathcal{F}$ on $\mathcal{C}$ is called limp if $H^p(K, \mathcal{F})$ for every presheaf of sets $K$ and $p > 0$.

**Prop. (V.6.5.12) (Characterization of Limp Sheaves).** Let $\mathcal{C}$ be a site and $\mathcal{F}$ an Abelian sheaf, then $\mathcal{F}$ is limp iff

- $\mathcal{F}$ is flask.
- For every surjection $K' \to K$ of sheaves of sets the extended Čech complex

$$0 \to H^0(K, \mathcal{F}) \to H^0(K', \mathcal{F}) \to H^0(K' \times_K K', \mathcal{F}) \to \cdots$$

is exact.

**Proof:** Cf.[Sta]07A1.

$\square$

**K-Flat Modules**

**Def. (V.6.5.13) (K-flat).** Let $(\mathcal{C}, \mathcal{O})$ be a ringed site, a complex $K^\bullet$ of $\mathcal{O}$-modules is called K-flat if for any acyclic complex $\mathcal{F}^\bullet$ of $\mathcal{O}$-modules, the total complex $\text{Tot}(\mathcal{F}^\bullet \otimes_\mathcal{O} K^\bullet)$ is acyclic. This is equivalent to tensoring with $K^\bullet$ maps quasiiso to quasiiso, because quasiiso is equivalent to the cone is acyclic and tensoring is an exact functor of triangulated categories(V.6.3.1).

**Prop. (V.6.5.14).** If $K, K'$ are $K$-flat complexes of $\mathcal{O}$-modules,

- Tor$(K \otimes_\mathcal{O} K')$ is $K$-flat.
- $K \otimes_R R'$ is $R'$-$K$-flat.
- If $(K_1, K_2, K_3)$ is a distinguished triangle in $K(\text{Mod}_\mathcal{O})$, if two of them is $K$-flat, then the third is also $K$-flat.
- A bounded above complex of flat $\mathcal{O}$-modules is $K$-flat.
- Any filtered colimits of $K$-flat complexes are $K$-flat.

**Proof:** 1, 2: trivial.

3: use(V.6.3.1), and the long exact sequence.

4: Cf.[Sta]06YQ.

5: because we are taking termwise-colimit, and Tot and tensor all commute with filtered colimits.

$\square$

**Prop. (V.6.5.15) (K-Flat Resolutions).** Any complex of $\mathcal{O}$-modules has a $K$-flat resolution, moreover, each term is a flat $\mathcal{O}$-module.

**Proof:** Cf.[Sta]06YS).

$\square$

**Lemma (V.6.5.16).** Let $P \to Q$ be a quasi-iso of $K$-flat complexes of $\mathcal{O}$-modules, then for any complex $L$ of $\mathcal{O}$-modules, Tor$(L \otimes P) \to$ Tor$(L \otimes Q)$ is a quasi-iso.

**Proof:** Choose a $K$-flat resolution(V.6.5.15) $K$ of $L$, then notice

$$\text{Tot}(L \otimes P) \cong \text{Tot}(K \otimes P) \cong \text{Tot}(K \otimes Q) \cong \text{Tot}(L \otimes Q)$$

by definition of $K$-flatness(V.6.5.13).

$\square$
6 Cohomology on Ringed Spaces

There are three basic objects, the derived functor for \( f_* \) as an Abelian sheaf, \( f_* \) as a \( \mathcal{O}_X \)-module, \( \Gamma(U, -) \) as an Abelian sheaf. Notice that an Abelian group is just a \( \mathbb{Z} \)-module.

**Prop. (V.6.6.1) (Grothendieck).** The sheaf cohomology of an Abelian sheaf over a Noetherian topological space of dimension \( n \) vanish for \( k > n \).

**Proof:** Use (V.6.6.2) and (V.6.7.26) and long exact sequence, we can reduce to the case of \( X \) irreducible. Then we induct on dimension. Notice first any sheaf is a filtered colimits of sheaf generated by f.m sections, thus we can use (V.6.1.10) to reduce to f.m sections case. And notice \( \mathcal{F}_{\alpha'} \to \mathcal{F}_\alpha \to \mathcal{G} \), then \( G \) is generated by at most \( |\alpha| - |\alpha'| \) elements, so reduce to the one section case.

Now it is a quotient sheaf of \( \mathbb{Z} \), look at the kernel \( R \). If the kernel is \( d\mathbb{Z} \) at the generic pt, then \( R|_V \cong \mathbb{Z} \) on some nbhd, and \( R|_V/\mathbb{Z} \) supports on a lower dimension set, then we only need to consider the pushout of constant sheaf \( \mathbb{Z}_U \).

Now there is an exact sequence \( 0 \to \mathbb{Z}_U \to \mathbb{Z} \to \mathbb{Z}_Y \to 0 \) (V.6.6.2), \( \mathbb{Z} \) is flabby (V.6.5.8) so flask, and the conclusion follows by induction. \( \square \)

**Prop. (V.6.6.2) (Canonical Exact Sequence).** We have a canonical exact sequences of sheaves of modules:

\[
0 \to j_i(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0 \\
0 \to i_*i^!_\gamma \mathcal{F} \to \mathcal{F} \to j_*(\mathcal{F}|_U) \to 0
\]

(check on stalks), which is important to use reduction to calculate sheaf cohomology. The latter induces long exact sequences:

\[
0 \to H^0_Y(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}|_U) \to H^1_Y(X, \mathcal{F}) \to \cdots
\]

**Prop. (V.6.6.3).** For \( f : X \to Y \), if \( \mathcal{I} \) is an injective module on \( X \), then \( \check{H}^p(U_i \to U, f_*\mathcal{I}) = 0 \) for every open cover for an open subset \( U \) (V.6.5.1). This is because Čech cohomology is a derived functor. (Notice \( f_*\mathcal{I} \) may not be injective when \( f \) is not flat).

**Prop. (V.6.6.4).** \( H^i(X, -) \) commutes with direct limits if \( X \) is a qs ringed space.

**Proof:** Cf. [Sta]01FF. \( \square \)

**Cor. (V.6.6.5) (Mayer-Vietoris).** For \( X = U \cup V \), there is a long exact sequence

\[
0 \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \otimes H^0(V, \mathcal{F}) \to H^0(U \cap V, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \cdots
\]

derived from the Čech to sheaf1 because it has only two column, just wrap out the definition.

**Prop. (V.6.6.6).** For a subscheme of \( \mathbb{P}^2 \) defined by a \( d \)-dimensional homogenous polynomial \( f \) that \( f(1, 0, 0) \neq 0 \), then using the two open affines \( \{x_1 \neq 0\} \) and \( \{x_2 \neq 0\} \), we see that \( \dim H^0(X, \mathcal{O}_X) = 1 \), \( \dim H^1(X, \mathcal{O}_X) = \frac{1}{2}(d-1)(d-2) \).

**Proof:** We need to see that \( \sum x_0^a x_1^b / x_2^q \equiv \sum x_0^b x_2^q / x_1^a \mod f \), where \( a_i, b_j < d \). Just look at the degree of \( x_0 \). For \( H^1 \), notice \( \{x_0^{a+b} / x_1^a x_2^b\} \) where \( a + b < d \) forms a basis of \( H^1 \). \( \square \)
7 Cohomology of Schemes

Main references are [Sta]Chap29.

Zariski Cohomology

Lemma (V.6.7.1) (Zariski-Poincare). A $\mathbb{Q}$co sheaf on an affine scheme $X$ is Čech-acyclic.

Proof: Because the principal affine covers are cofinal in the ordering of covers, we only need to consider principal affine covers. Let $R \to A = \prod R_{f_i}$, then it is f.f., so we can use (I.7.2.2), just notice the higher term is $\prod i_0 \ldots i_n R_{f_0 \ldots f_n}$. □

Prop. (V.6.7.2) (Čech Derived Coh Equal when Separated). For a $\mathbb{Q}$co sheaf $\mathcal{F}$ on a separated scheme, we have $H^p(X, \mathcal{F}) = \check{H}^p(X, \mathcal{F}) = H^p(\{U_i \to X\}, \mathcal{F})$ for $U_i$ any open affine cover.

Proof: Use (V.6.2.14), the family of affine open subsets of $X$ satisfies the requirement because $X$ is separated and (V.6.7.1), thus the result. □

Cor. (V.6.7.3) (Affine $\mathbb{Q}$co Cohomology Vanish). For a $\mathbb{Q}$co sheaf on an affine scheme, $H^i(X, \mathcal{F})$ vanish for $i > 0$. For a $\mathbb{Q}$co sheaf on a qcqs scheme $X$, the sheaf cohomology vanish for $n$ large enough. (Use check to sheaf2 (V.6.2.11)).

Prop. (V.6.7.4) (Compatibility of $\mathbb{Q}$co and $\mathcal{O}_X$-mod). We have in the category of $\mathbb{Q}$co sheaves, injective objects are flabby sheaves, thus nearly calculating all derived functors are legitimate in the category of $\mathbb{Q}$co sheaves (V.6.5.1).

Proof: We use the Deligne formula (V.3.1.18) and the definition of injective, just need to consider the sheaf of ideals of the corresponding induced reduced structure. □

Prop. (V.6.7.5) (Filtered Colimits). If $X$ is qcqs, then sheaf cohomology on $X$ commutes with filtered colimits. (Follows from (V.1.3.4) the same way (VI.2.1.7) follows from (V.1.3.9)).

Prop. (V.6.7.6). In the category of $\mathbb{Q}$co($X$), we have two Ext, for $\text{Hom}_c(\mathcal{F}, -)$ and $\check{\text{Hom}}(\mathcal{F}, -)$.

We have

\[
\check{\text{Ext}}^i_X(\mathcal{F}, \mathcal{G})|_U = \check{\text{Ext}}^i(U, \mathcal{F}|_U, \mathcal{G}|_U),
\]

because both give universal $\delta$-functors for $\mathcal{G}$. In particular, we have $\check{\text{Hom}}(\mathcal{F}, -)$ is exact for $\mathcal{F}$ locally free.

Prop. (V.6.7.7). Ext and $\check{\text{Ext}}$ are universal $\delta$ functors in $\mathcal{G}$ and a $\delta$ functors in $\mathcal{F}$ using injective resolution of $\mathcal{G}$. (Notice injective are acyclic for $\check{\text{Ext}}$ because $I|_U$ is also injective).

Cor. (V.6.7.8). When $X$ is locally Noetherian and $\mathcal{F}$ is coherent, we have

\[
\check{\text{Ext}}^i(\mathcal{F}, \mathcal{G})|_x \cong \text{Ext}_c^i(\mathcal{F}_x, \mathcal{G}_x).
\]

Proof: Check local on an affine open, Use a finite locally free resolution and (V.2.3.7), notice the stalk function is exact. □

Cor. (V.6.7.9). If $X$ is locally Noetherian, suppose that every coherent sheaf is a quotient of a locally free sheaf, we can define the homological dimension $hd(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ as the minimal length of locally free resolution of $\mathcal{F}$. Then $hd(\mathcal{F}) \leq n \iff \check{\text{Ext}}^i(\mathcal{F}, \mathcal{G}) = 0$ for every $\mathcal{G}$ and every $i > n$. And $hd(\mathcal{F}) = \sup pd_{\mathcal{O}_X,x} \mathcal{F}_x$. This follows easily from (V.6.7.8).
Prop. (V.6.7.10). When \( \mathcal{L} \) is a locally free sheaf, we have:

\[
\mathcal{E}xt^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G}) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \otimes \mathcal{L}^\vee
\]

because there are maps between them (V.2.5.2), and \( \mathcal{E}xt \) is local, so check locally. In particular,

\[
\text{Ext}^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) = \text{Ext}^i(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G}) \cong \text{Ext}^i(\mathcal{F}, \mathcal{G}) \otimes \mathcal{L}^\vee.
\]

Prop. (V.6.7.11). On a Noetherian affine scheme, if \( M \) is f.g., then

\[
\mathcal{E}xt^i(\tilde{M}, \tilde{N}) \cong \mathcal{E}xt^i(M, N).
\]

So on a locally Noetherian scheme, when \( \mathcal{F} \) is coherent and \( \mathcal{G} \) Qco, \( \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \) is Qco and if moreover \( \mathcal{G} \) is coherent, then \( \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) \) is coherent (because this is true for \( \text{Ext} \) by free resolution).

Proof: Show that they are both universal effaceable. \( \square \)

Prop. (V.6.7.12). For \( f \) proper between locally Noetherian scheme, there is a inverse sheaf \( f^! \mathcal{G} = \mathcal{H}om_Y(f_*, \mathcal{O}_X, \mathcal{G}) \), which maps Qco\( (Y) \) to Qco\( (X) \) by (V.3.1.15) and (V.6.7.11). When \( f \) is affine, in particular when it is finite, then for \( \mathcal{F} \) coherent and \( \mathcal{G} \) Qco,

\[
f_* \mathcal{H}om(\mathcal{F}, f^! \mathcal{G}) \cong \mathcal{H}om_Y(f_* \mathcal{F}, \mathcal{G})
\]

and when \( X, Y \) is separated and \( X \) has the resolution property and \( f \) is flat, then

\[
\text{Ext}^i(\mathcal{F}, f^! \mathcal{G}) \cong \text{Ext}^i_Y(f_* \mathcal{F}, \mathcal{G})
\]

is also an isomorphism.

Proof: The first one is just local check, for the second one, just use Grothendieck spectral sequence and the fact \( f_* \mathcal{O}_X \) is locally free thus \( f^! \) is exact. \( \square \)

Prop. (V.6.7.13) (Künneth Formula). If \( X, Y \) are qcqs over a field \( k \) and \( \mathcal{F}, \mathcal{G} \) be Qco \( \mathcal{O}_X, \mathcal{O}_Y \)-modules, then there is a canonical isomorphism:

\[
H^n(X \times_{\text{Spec} k} Y, pr_1^* \mathcal{F} \otimes_{\mathcal{O}_X \otimes_{\text{Spec} k} Y} pr_2^* \mathcal{G}) \cong \bigoplus_{p+q=n} H^p(X, \mathcal{F}) \otimes_k H^q(Y, \mathcal{G}).
\]

Proof: Cf. [Sta]0BEF. \( \square \)

Prop. (V.6.7.14). On a locally Noetherian scheme \( X \), any Qco sheaf \( \mathcal{F} \) admits a resolution of Qco sheaves that are injective as \( \mathcal{O}_X \)-modules.

Proof: Because Qco\( (X) \) is Serre subcategory, ? \( \square \)

Lemma (V.6.7.15) (Gabber). Let \( X \) be a scheme, then there exists a cardinal \( \kappa \) that every Qco sheaf is a colimit of its \( \kappa \)-generated Qco subsheaves.

Proof: Cf. [Sta]077N. \( \square \)
Cohomology of Proper & Projective Spaces

Prop. (V.6.7.16). Let $X = \mathbb{P}^r_A$ we have:
- $H^i(X, \mathcal{O}_X(n)) = 0$ for $0 < i < r$ and all $n$.
- There is a perfect pairing
  \[ H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n - r - 1)) \rightarrow H^r(X, \mathcal{O}_X(-r - 1)) \cong A. \]
- Of course for $i > r$, the cohomology vanish because $X$ is separated. And when $n > 0$, $H^r(X, \mathcal{O}_X(n - r - 1)) = 0$.

Proof: $X$ is separated, we use Cech cohomology, the second one is easy, $(x_0x_1 \ldots x_r)^{-1}$ forms a basis of $H^r$.

For the first one, induction on $r$, Cf.[Hartshorne P225].

Prop. (V.6.7.17). Let $X = \mathbb{P}^n_k$ and $0 \leq p, q \leq n$, then $H^q(X, \Omega^p_X) = 0$ for $p \neq q$ and when $p = q$, $H^q(X, \Omega^p_X) = k$.

Proof: By (V.3.1.11) and (V.3.4.7), there is an exact sequence $0 \rightarrow \Omega^q \rightarrow \wedge^q \mathcal{O}(-1) \rightarrow \Omega^{q-1} \rightarrow 0$, and the middle has vanishing $q$-th cohomology by (V.6.7.16), thus we can induct and (V.6.7.16) gives the result.

Def. (V.6.7.18) (Euler Characteristic). Let $X$ be proper over a field $k$ and $\mathcal{F}$ be coherent, then the Euler characteristic of $\mathcal{F}$ is defined to be:

\[ \chi(\mathcal{F}) = \sum (-1)^i \dim_k H^i(X, \mathcal{F}). \]

It is definable by (V.6.7.2), and It is clearly an additive functor on $\text{Coh}(X)$.

Prop. (V.6.7.19). For a proper scheme $X$ over a field $k$ and $\mathcal{L}_i$ be invertible sheaves on $X$. Then for any coherent sheaf $\mathcal{F}$ on $X$,

\[ \chi(X, \mathcal{F} \otimes \mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_r^{n_r}) \]

is a polynomial in $(n_1, \ldots, n_r)$ of total degree at most $\dim \text{Supp} \mathcal{F}$.

Proof: Cf.[Sta]0BEM.

Cor. (V.6.7.20) (Hilbert Polynomial). For a projective scheme over a field $k$ and a coherent sheaf $\mathcal{F}$, there is a polynomial Hilbert polynomial $P$ that $\chi(\mathcal{F}(n)) = P(n)$, and $\deg P \leq \dim \text{Supp} \mathcal{F}$.

This Hilbert polynomial is compatible with the definition in (V.8.2.5), because by (V.6.7.32), the higher cohomology group vanishes for $n$ large, so $\chi(\mathcal{F}(n)) = \Gamma(\mathcal{F}(n)) = \Gamma_*(\mathcal{F})_n$.

Prop. (V.6.7.21). Let $f : Y \rightarrow X$ be morphism between schemes proper over field $k$ and $\mathcal{F}$ a coherent sheaf, then we have

\[ \chi(Y, \mathcal{F}) = \sum (-1)^i \chi(X, R^i f_* \mathcal{F}). \]

This comes from the Leray spectral sequence.

Def. (V.6.7.22). The arithmetic genus of a proper scheme of dimension $r$ over a field is defined to be $p_a(X) = (-1)^r(\chi(\mathcal{O}_X) - 1) = (-1)^r(P_X(0) - 1)$ (V.6.7.18). In particular, when $X$ is a curve over a field $k$, then $p_a(X) = \dim_k H^1(X, \mathcal{O}_X)$ (V.8.1.12).
Thus $\chi(X \times_k Y) = \chi(X)\chi(Y)$. In particular, we have

$$p_a(X \times Y) = p_a(X)p_a(Y) + (-1)^{n}p_a(X) + (-1)^{n}p_a(Y).$$

**Prop. (V.6.7.24) (Asymptotic Riemann-Roch).** If $X$ is a proper scheme over a field $k$ of dimension $d$ and $\mathcal{L}$ is an ample invertible sheaf, then $\dim \Gamma(X, \mathcal{L}^n) \sim cn^d$, Cf.[Sta]0BJ8.

**Prop. (V.6.7.25).** Let $X$ be $H$-projective over a Noetherian affine scheme and $\mathcal{F}, \mathcal{G}$ be coherent, then for $n$ large,

$$\text{Ext}^i(\mathcal{F}, \mathcal{G}(n)) \cong \Gamma(X, \mathcal{E}xt^i(\mathcal{F}, \mathcal{G}(n))).$$

**Proof:** This true for $i = 0$, so let $i > 0$. When $\mathcal{F} = mclO_X$, this is true by(V.6.7.32), and hence true for $\mathcal{F}$ locally free(V.6.7.10), and for $\mathcal{F}$ general, choose a locally free surjective $\mathcal{E} \rightarrow \mathcal{F}$ with kernel $\mathcal{G}$, then for $n$ large, there is an exact sequence

$$\text{Hom}(\mathcal{E}, \mathcal{G}(n)) \rightarrow \text{Hom}(\mathcal{R}, \mathcal{R}(n)) \rightarrow \text{Ext}^1(\mathcal{F}, \mathcal{G}(n))$$

and $\text{Ext}^i(\mathcal{R}, \mathcal{G}(n)) \cong \text{Ext}^{i+1}(\mathcal{F}, \mathcal{G}(n))$. And similarly for $\mathcal{E}xt$. When proving $i = 1$, we need to use(V.6.7.33) to choose $n$ even larger to get the corresponding global section exact sequence. $\square$

**Higher Direct Image**

**Prop. (V.6.7.26) (Sheaf Cohomology Commutes with Affine Map).** For $f$ affine and $\mathcal{F}$ $\text{Qco}$, we have $H^n(Y, \mathcal{F}) = H^n(X, f_*\mathcal{F}).$

**Proof:** Because $R^if_*\mathcal{F}(U) = 0$ by(V.6.7.3) and(V.6.1.5), we can then use(V.6.1.7) to conclude. $\square$

**Prop. (V.6.7.27) (Higher Direct Image is $\text{Qco}$ and Local).** If $f$ is qcqs then $R^n f_*\mathcal{F}$ maps $\text{Qco}$ to $\text{Qco}$, and $R^p f_*\mathcal{F}(U) = (H^p(f^{-1}(U), \mathcal{F}))^\sim$.

**Proof:** check local on affine open of $Y$, both side are $\delta$-functors from $\text{Qco}(X)$ to $\text{Mod}_Y$, injectives in $\text{Qco}(X)$ are flabby, thus both are effaceable. We only need to show $f_*\mathcal{F} = \Gamma(X, \mathcal{F})$, and this is(V.3.1.4). Cf.[Hartshorne P251]. $\square$

**Prop. (V.6.7.28).** For a qcqs morphism $f : X \rightarrow S$, if $S$ is qc, there is a $N$ that for every base change $f'$ of $f$, we have $R^n f'_*\mathcal{F} = 0$ for every $\mathcal{F} \text{ Qco}$ and $n \geq N$.

**Proof:** We check local on affine open and use(V.6.7.27), choose an finite affine cover of $X$, their intersection are all f.m. affine opens. Then local on a base change, the number of affine opens are the same. So when $n$ is large enough, using Cech to Sheaf2, we have the cohomology vanish.(This uses the fact that the intersection of affine opens are separated and(V.6.7.2). $\square$

**Cor. (V.6.7.29).** For a qcqs scheme $X$, the cohomology vanish for $n$ large. And when $X$ is separated and can be covered by $r$ affine opens, then $N$ can chosen to be $r$.

**Prop. (V.6.7.30) (Grothendieck’s Coherence Theorem).** If $f : X \rightarrow Y$ is proper and $Y$ locally Noetherian, then $R^n f_*\mathcal{F}$ maps coherent to coherent.
Proof: Cf.[Sta]02O5.

Cor. (V.6.7.31). If $X$ is proper over a Noetherian affine scheme, its global section is a f.g. $A$-module.

Prop. (V.6.7.32) (Serre). If $X \to Y$ is a projective morphism of Noetherian schemes and $\mathcal{F}$ be a coherent sheaf on $X$, then we have $R^i f_* (\mathcal{F}(n)) = 0$ for $i > 0$ and $n$ large enough.

For this it suffice to prove the local case: If $X$ is projective scheme over a Noetherian affine scheme, $H^i (X, \mathcal{F}(n)) = 0$.

Proof: Because $i_* \mathcal{F}$ is coherent on $\mathbb{P}^n_A$, we reduce to the case $X = \mathbb{P}^n_A$. The conclusion is true for $\mathcal{O}_X(n)$ by (V.6.7.16), and for general $\mathcal{F}$, we use descending induction on $i$, choose a $\oplus \mathcal{O}_X(n_i) \to \mathcal{F} \to 0$ with kernel $\mathcal{R}(V.3.3.25)$, then
\[
H^i (X, \oplus \mathcal{O}_X(n_i + n)) \to H^i (X, \mathcal{F}(n)) \to H^{i+1} (X, \mathcal{R}(n)),
\]
and the left term vanish for $n$ large (V.6.7.16) thus the result.

Cor. (V.6.7.33). For any finite exact sequence of coherent sheaves on a $H$-projective scheme over a Noetherian affine scheme, if tensoring it with $\mathcal{O}(n)$ for large $n$, the resulting global section is exact.

Base Change

Prop. (V.6.7.34) (Flat Base Change). For a Cartesian diagram of schemes
\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{f'} & \downarrow{f} & \\
S' & \xrightarrow{g} & S
\end{array}
\]
if $g$ is flat and $f$ is qcqs, then for every complex of $\mathbb{Q}$co sheaves $\mathcal{F}$ on $X$ with base change $\mathcal{F}'$, there is an canonical isomorphism
\[
g^* R^i f_* \mathcal{F} \cong R^i f'_* \mathcal{F}'
\]
when $S, S'$ is affine, this reads:
\[
H^i (X, \mathcal{F}) \otimes_A B \cong H^i (X \otimes_A B, (g')^* \mathcal{F}).
\]

Proof: By (V.6.7.27), we only need to check the results affine opens, so let $S, S'$ be affine open. If $X$ is separated, then the cohomology equals Čech cohomology, and the Čech cohomology of $\mathcal{F}'$ is just the cohomology of the Čech complex tensored with $B$, so it commutes with taking cohomology because $B$ is $A$-flat.

Now if $X$ is only qs, then we choose an open affine cover (finite) $\{U_i\}$, then all the intersections of these $U_i$ are separated. Now we use Čech-to-sheaf2 spectral sequence (V.6.2.11), then by what we proved for separated case, there is an isomorphism of spectral sequences $E_2$, so their limit are the same.

Cor. (V.6.7.35). Let $X \to Y$ be qcqs and $Y$ affine, then for any $y \in Y$, let $X_y$ be the fiber, then $H^i (X_y, \mathcal{F}_y) \cong H^i (X, \mathcal{F} \otimes k(y))$.

Proof: The only problem is to reduce to the case that $Y' = \{y\}$ with the induced reduced structure, because then $\text{Spec } k(y) \to Y$ is flat. All we care is the fiber over $y$, and $X' = X \times_Y Y'$ is a closed scheme of $X$, $\mathcal{F}$ pullbacks to $\mathcal{F} \otimes_A A/p_y$, so $H^i (X', \mathcal{F} \otimes_A A/p_y \otimes k(y)) \cong H^i (X, \mathcal{F} \otimes k(y))$. □
Semicontinuity Theorem

Prop. (V.6.7.36). If $X$ is projective over a Noetherian affine scheme $\text{Spec} A$, and $\mathcal{F}$ is a coherent sheaf on $X$ which is flat over $Y$, then if we define $T^i(M) = H^i(X, \mathcal{F} \otimes_A M)$ as a functor from $A$-modules to $A$-modules, then they form a $\delta$-functor as $\mathcal{F}$ is flat.

And there is a complex $L^\bullet$ of f.g. $A$ modules bounded above that $T^i(M) \cong \gamma(L^\bullet \otimes_A M)$.

Proof: Firstly, the Čech complex satisfies the requirement, but it is not finite free. But, its cohomology equals $H^i(\mathcal{F})(V.6.7.2)$, so (I.11.6.2) can be used.

Prop. (V.6.7.37). $T^i$ is left exact iff $\text{Coker} \ d^{i-1}$ is a projective $A$-module, iff it is representable by a finite $A$-module.

Proof: Denote $W^i = \text{Coker} \ d^{i-1}$, then $\text{Coker} \ d^{i-1} \otimes_A M = W^i \otimes M$, because tensoring is right exact. Thus $T^i(M) = \text{Ker}(W^i \otimes M \to L^{i+1} \otimes M)$. Then for $M' \subset M$, there is a commutative diagram

$$
\begin{array}{c}
0 \to T^i(M) \to W^i \otimes M' \to L^{i+1} \otimes M' \\
\downarrow \alpha \downarrow \beta \downarrow \gamma \\
0 \to T^i(M) \to W^i \otimes M \to L^{i+1} \otimes M
\end{array}
$$

$\gamma$ is injective, so using spectral sequence, its clear $\alpha$ is injective iff $\beta$ is injective, i.e. $W^i$ is flat, which is equivalent to finite projective (I.6.1.7).

To prove $T^i$ is representable, let $Q = \text{Coker}(L^{i+1,*} \to W^{i,*})$, then $Q$ is finite because $W^i$ is finite (I.6.1.17), and $0 \to \text{Hom}(Q, M) \to \text{Hom}(W^{i,*}, M) \to \text{Hom}(L^{i+1,*}, M)$, but by (I.6.1.18), the last two are just $W^i \otimes M$ and $L^{i+1} \otimes M$, $\text{Hom}(Q, M) = T^i(M)$ by what has already be proved.

Prop. (V.6.7.38). $T^i$ is right exact iff the morphism $T^i(A) \otimes_A M \to T^i(M)$ are all isomorphism.

Proof: Because $T^i$ and $\otimes$ commutes with direct limit, it suffices to prove for $M$ finite. In this case, choose a finite presentation $A^r \to A^s \to M \to 0$, then there is a diagram

$$
\begin{array}{ccc}
T^i(A) \otimes A^r & \to & T^i(A) \otimes A^s \\
\downarrow & & \downarrow \\
T^i(A^r) & \to & T^i(A^s) \\
\end{array}
$$

The first two vertical arrows are isomorphisms, so if $T^i$ is right exact, so does the third vertical arrow. Conversely, if $T^i(A) \otimes_A M \to T^i(M)$ are all isomorphism, then by a similar diagram, there $T^i(M) \to T^i(M')$ are surjective for $M \to M'$ surjective.

Cor. (V.6.7.39). $T^i$ is exact iff it is right exact and $T^i(A) = H^i(\mathcal{F})$ is a finite projective $A$-module.

Proof: Because in this case $T^i(M) \cong T_i(A) \otimes_A M$, so it is exact iff $T^i(A)$ is flat, and it is also finite, so it is equivalent to projective (I.6.1.7).

Def. (V.6.7.40). For a prime $p$ of $A$, we define $T^i_p(N) = H^i(L^*_p \otimes N)$, then $T^i$ is (left/right)exact iff $T^i_p$ are all (left/right)exact (exact sequence is stalkwise (I.5.1.55)).

Prop. (V.6.7.41). If $T^i$ is (left/right)exact at a point $y$, then the same is true on a nbhd of $y$. 
Proof: From \((\text{V.6.7.37})\), \((\text{Coker } d^{i-1})_y\) is a finite projective \(A_p\) module, so it is free. Now \(\text{Coker } d^{i-1}\) is a coherent sheaf, so it is free at a nbhd of \(y\), so the same is true on a nbhd of \(y\). Now right exactness of \(T^i\) is equivalent to left exactness of \(T^{i+1}\), and exact is left exact+right exact, so we are done. □

Prop. (\text{V.6.7.42}) (Semicontinuity of Cohomology). Let \(X \to Y\) be a projective morphism of locally Noetherian schemes and \(\mathcal{F}\) is a coherent sheaf on \(X\), flat over \(Y\), then for each \(i\), \(h^i(y, \mathcal{F}) = \dim_{k(y)} H^i(X_y, \mathcal{F}_y)\) is an upper semicontinuous function on \(Y\).

Proof: The question is local on \(Y\), so we may assume \(Y\) is affine Noetherian. By \((\text{V.6.7.34})\), 
\[
H^i(y, \mathcal{F}) = \dim_{k(y)} T^i(k(y)).
\]
And as in the proof of \((\text{V.6.7.37})\), \(T^i(M) = \text{Ker}(W^i \otimes M \to L^{i+1} \otimes M)\), and \(W^i \to L^{i+1} \to W^{i+1} \to 0\) is exact, so 0 → \(T^i(k(y)) \to W^i \otimes k(y) \to L^{i+1} \otimes k(y) \to W^{i+1} \otimes k(y) \to 0\), and counting dimension, 
\[h^i(y, \mathcal{F}) = \dim W^i \otimes k(y) + \dim W^{i+1} \otimes k(y) - \dim L^{i+1} \otimes k(y).\]
Notice the last term is constant as \(L^{i+1}\) is free \(A\)-module and the first two terms are upper semicontinuous by \((\text{V.3.1.21})\), thus \(h^i(y, \mathcal{F})\) is upper-semicontinuous. □

Cor. (\text{V.6.7.43}) (Grauert). If \(Y\) is integral and \(h^i(y, \mathcal{F})\) is constant on \(Y\), then \(R^if_*(\mathcal{F})\) is locally free on \(Y\) and \(R^if_*(\mathcal{F}) \otimes k(y) \cong H^i(X_y, \mathcal{F}_y)\).

Proof: Following the above proof, we get \(\dim W^i \otimes k(y)\) and \(\dim W^{i+1} \otimes k(y)\) are all constant. This implies that \(W^i\) and \(W^{i+1}\) are all locally free, so \(T^i\) and \(T^{i+1}\) are both left exact, so \(T^i\) is exact. So \(T^i(A)\) is finite projective by \((\text{V.6.7.39})\). So \(R^if_*(\mathcal{F})\), as equal to \(\widetilde{T^i}(A)\), is locally free. The last assertion follows from \((\text{V.6.7.35})\) and \((\text{V.6.7.38})\). □

Prop. (\text{V.6.7.44}). To check \(T^i\) is right exact, it suffice to check \(T^i(A) \otimes k(y) \to T^i(k(y))\) is surjective.

Proof: Cf.\,[Hartshorne P289]. □

Theorem of Formal Functions

Basic References are \([\text{Sta}]29.20\).

Prop. (\text{V.6.7.45}).

8 Topological Sheaves

Acyclic sheaves

Def. (\text{V.6.8.1}). An Abelian sheaf on a paracompact Hausdorff topological space \(X\) is called

soft iff is and \(\forall\) closed \(V, \mathcal{F}(X) \to \mathcal{F}(V)\) is surjective. A flabby sheaf is soft.

fine iff the sheaf of rings \(\text{Hom}(\mathcal{F}, \mathcal{F})\) is soft.

Fine and soft are local properties (Use Zorn’s lemma to construct one-by-one).

Prop. (\text{V.6.8.2}). For a sheaf of unital rings over a paracompact Hausdorff space \(X\), the following are equivalent,

1. it is a soft sheaf.
2. for any disjoint closed sets \(V, W\), there is a section of \(X\) that is 0 on \(V\), and 1 on \(W\).
3. it possesses a partition of unity.
4. it is a fine sheaf.
Note any soft sheaf possesses a partition of unity.

**Proof:** 1 $\iff$ 2 is easy and 1 $\rightarrow$ 3 is the to choose a closed locally finite subcover and use Zorn’s lemma to construct one-by-one. For 3 $\rightarrow$ 1, notice a closed section can extend to a slightly larger nbhd.

Because for a sheaf of rings $\mathcal{F}$, a partition of unity is equivalent to a partition of unity $\text{Hom}(\mathcal{F}, \mathcal{F})$, so 34 are equivalent because 13 are equivalent.

**Cor. (V.6.8.3).**
- Note that a fine sheaf possesses a decomposition of section because the previous proposition applies to $\text{Hom}(\mathcal{F}, \mathcal{F})$, and a partition of unity in $\text{Hom}(\mathcal{F}, \mathcal{F})$ yields a decomposition of section in $\mathcal{F}$. Thus a fine sheaf is soft. (extend to a small nbhd and use partition of unity).
- The sheaf of modules over a soft sheaf of rings is soft, by partition of unity.
- The continuous function sheaf on a paracompact Hausdorff space or the smooth function sheaf on a smooth manifold is fine, thus any smooth module is fine (Use bump function).

**Prop. (V.6.8.4).** Soft sheaf, e.g. fine sheaf is adapted to $\Gamma(X, -)$. (Similar as in (V.6.5.2), notice flabby is soft and the others are the same as before).

**Prop. (V.6.8.5).** Let $X$ be a locally compact space of finite compact dimension, when $S$ is a soft sheaf, and one of $S$ and $\mathcal{F}$ is flat, then $S \otimes_k \mathcal{F}$ is soft. Cf.[Cohomology of Sheaves Iversen P319].

**Prop. (V.6.8.6).** Over a locally compact space of finite dimension, any flat sheaf $\mathcal{F}$ on $X$ has a resolution of soft flat sheaves, Cf.[Gelfand P232].

**Comparison Theorems**

**Lemma (V.6.8.7) (Poincare Lemma).** For a smooth manifold $X$ of dimension $n$, there is an exact sequence

$$0 \rightarrow \mathbb{R}^X \xrightarrow{d} \Omega^0 \xrightarrow{d} \Omega^1 \rightarrow \cdots \rightarrow \Omega^n \rightarrow 0$$

**Lemma (V.6.8.8) ($\partial$-Poincare Lemma).** If $X$ is a complex manifold of dimension $n$, there are exact sequences:

$$0 \rightarrow \Omega^p_{\text{hol}} \xrightarrow{\partial} \Omega^{p,0} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^{p,n-p} \rightarrow 0$$

**Proof:** Cf.[Sheaf Cohomology P21].

**Cor. (V.6.8.9).** If $X$ is a complex manifold of dimension $n$, there are exact sequences:

$$0 \rightarrow \underline{\mathbb{C}}^X \xrightarrow{d} \Omega^0_{\text{hol}} \xrightarrow{d} \Omega^1_{\text{hol}} \rightarrow \cdots \rightarrow \Omega^n_{\text{hol}} \rightarrow 0$$

**Proof:** This follows from the Poincare lemma(V.6.8.7) and $\partial$-Poincare lemma(V.6.8.8), by applying the Spectral sequence(I mean in the category of sheaves).

**Cor. (V.6.8.10) (Holomorphic Cohomology).** For a complex manifold $X$,

$$H^p(X, \underline{\mathbb{C}}) = H^p(X, \Omega^*_\text{hol}).$$
Prop. (V.6.8.11) (De Rham). For a smooth manifold and an Abelian group $G$,

$$H^*_d(X, G) \cong H^*(X, G)$$

Where the right is constant sheaf cohomology. (Use the fact that smooth sheaf is fine so adapted to sheaf cohomology (V.6.8.4), and Poincare lemma (V.6.8.7)).

Prop. (V.6.8.12) (Singular). For a locally contractible topological space,

$$H^p_{\text{sing}}(X, G) \cong H^p(X, G)$$

Proof: Shifification of the singular cochain complex is a flabby presheaf resolution of $G$ because it is locally contractible, check on stalks. Then we only have to prove $C^*(X) \rightarrow (C/V)^*(X)$ is quasi-isomorphism, where $V$ is the presheaf of locally vanishing cochain. It suffice to prove $V^*(X)$ is exact.

For any $i$-cocycle $\varphi$, for any $i-1$-complex $\sigma$, use barycentric subdivision, we can construct a $c_\sigma$ whose boundary is $\sigma$ and other simplexes on which $\phi$ vanishes, so we have the coboundary of $\eta: \sigma \rightarrow \varphi(c_\sigma)$ is $\varphi$. \qed

Prop. (V.6.8.13) (Dolbeault). For a complex bundle on a complex manifold,

$$H^{p,q}_\partial(X, E) \cong H^q(M, \Omega^p_{\text{hol}} \otimes \mathcal{O}_X E)$$

where the left is Dolbeault cohomology and the right is sheaf cohomology. (Use the fact that smooth sheaf is fine so adapted to sheaf cohomology (V.6.8.4), and $\partial$-Poincare lemma (V.6.8.8)).

Moreover, there is a spectral sequence of

$$E_1^{p,q} = H^{p,q}_\overline{\partial}(X) \Rightarrow E^n = H^*_d(X, \mathbb{R}) \times \mathbb{C}.$$

Prop. (V.6.8.14) (Cartan). The class of Coh-Acylic subsets of an analytic space is exactly the Stein manifold.

Prop. (V.6.8.15) (Čech and Sheaf Cohomology). For a paracompact Hausdorff space $X$, there are isomorphisms

$$\check{H}^i(X, \mathbb{Z}) \cong H^i(X, \mathbb{Z}).$$

Proof: Cf.[Godement, Prop5.10.1]. \qed

Cohomology with Proper Support

References are [Cohomology of Sheaves Iversen].

Prop. (V.6.8.16). Soft sheaf is adapted to $f_!$ when $X, Y$ are locally compact. Cf.[Gelfand P226]. So we can use soft resolution to define $R^i f_!$, in particular, when $Y = \text{pt}$, we denote it by $H^i_c(X, \mathcal{F})$. Using (V.2.6.2), we get the stalk of $R^i f_!(\mathcal{F})$ at $y$ is just $H^i_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$.

Def. (V.6.8.17). The compact dimension of a locally compact topological space is the smallest $n$ that $H^i_c(X, \mathcal{F}) = 0$ for $i > n$. It is also the maximal length of minimal soft resolution.

$\dim_c \mathbb{R}^n = n$, and when $Y$ is an open or closed subset of $X$, $\dim_c Y \leq \dim_c X$. $\dim_c$ is local in the sense if every point has a nbhd of dimension $\leq n$, then $\dim_c X \leq n$. Cf.[Iversen].
Prop. (V.6.8.18) (Proper Pushforward Commutes with Pullback). For a pullback diagram

\[
\begin{array}{ccc}
X \times_Y Z & \overset{\tau'}{\rightarrow} & X \\
\downarrow{\pi'} & & \downarrow{\pi} \\
Z & \overset{\tau}{\rightarrow} & Y
\end{array}
\]

we have \( \tau^{-1}(\pi_! F) = \pi'_!(\tau')^{-1} F \).

\textit{Proof:} \qed
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Main references are [Sta].

1 Divisors

Weil Divisors

We consider divisors on a locally Noetherian integral scheme. Cartier divisor and Picard Group are more general.

Def. (V.7.1.1) (Weil Divisor). A prime Weil divisor is a closed integral subscheme of codimension 1. A Weil divisor is on a locally Noetherian integral scheme is a formal combination of prime Weil divisors that is locally finite. The collection of Weil Divisors are denoted by Div(X).

Prop. (V.7.1.2) (Principal Weil Divisor). For a rational function \( f \in K \) on a locally Noetherian integral scheme \( X \), for any prime Weil divisor \( Z \) with generic pt \( \eta \), we can define \( \text{ord}_Z(f) = \text{ord}_{O_{X,\eta}}(f)(I.5.1.47) \). It is multiplicative, and the closed integral subschemes \( Z \) that \( \text{ord}_Z(f) \neq 0 \) is locally finite.

So, we can define the principal Weil divisor \( \text{div}(f) = \sum Z \text{ord}_Z(f)[Z] \).

Proof: There is an open subset \( U \) that \( f \in \Gamma(U, O_X^*) \), so all \( Z \) are irreducible components of \( X - U \), which is locally finite because \( X \) is locally Noetherian and (V.4.1.22).

Def. (V.7.1.3). The Weil divisor class group \( \text{Cl}(X) \) of a locally Noetherian integral scheme is defined to be \( \text{Div}(X) \) modulo principal Weil divisors.

Prop. (V.7.1.4). If \( X \) is a Noetherian normal integral scheme regular in codimension 1, let \( Z \) be a proper closed subset of \( X \) and \( U = X \setminus Z \), then

- There is a surjective homomorphism \( \text{Cl}(X) \to \text{Cl}(U) \) defined by \( D = \sum n_i Y_i \to \sum n_i (Y_i \cap U) \).
- If \( \text{codim}(Z, X) \geq 2 \), then \( \text{Cl}(X) \to \text{Cl}(U) \) is an isomorphism.
- If \( Z \) is an irreducible subset of codimension 1, then there is an exact sequence

\[ Z \to \text{Cl}(X) \to \text{Cl}(U) \to 0, \]

where the first map is defined by \( 1 \mapsto [Z] \).

Cor. (V.7.1.5). If \( Y \subset \mathbb{P}^2_k \) is an irreducible curve of dimension \( d \), then \( \text{Cl}(\mathbb{P}^2 \setminus Y) = \mathbb{Z}/d\mathbb{Z} \), by (V.7.1.4) and ??.

Prop. (V.7.1.6). If \( X \) is a Noetherian integral separated scheme regular in codimension 1, then so are \( X \times \text{Spec} \mathbb{Z}[T] \) and \( X \times \mathbb{P}^n_\mathbb{Z} \) (local check), and \( \text{Cl}(X \times \text{Spec} \mathbb{Z}[T]) = \text{Cl}(X) \) and \( \text{Cl}(X \times \mathbb{P}^n_\mathbb{Z}) = Z \times \text{Cl}(X) \).

Proof: It is clearly Noetherian and integral and separated.

We define a map \( \text{Cl}(X) \to \text{Cl}(X \times \mathbb{A}^1) \) by \( D = \sum n_i Y_i \mapsto \pi^*(D) = \sum n_i \pi^{-1}_i(Y_i) \). This is well-defined because \( \pi^*(f) = (f) \). This map is injective: if some \( \pi^*(D) = (f) \), then \( \pi(I(f)) \) doesn’t contain the generic point \( \eta \). Thus \( f \in K \), otherwise \( I(f) \cap \pi^{-1}(\eta) = I(f) \subset \text{Spec} K[t] \), which is non-zero. So \( D = (f) \).
It is also surjective, it suffices to show any prime divisor that lies on the generic point of \(X\) is equivalent to combinations of prime divisors that are not over the generic point. For this, notice \(\mathfrak{p}\) is a prime ideal of \(K[T]\) of codimension 1, thus principle(I.2.3.12), let \(\mathfrak{p} = (f)\). Then \((f)\) contains no other prime divisors over \(\eta\), which means \(\mathfrak{p}\) is equivalent to combinations of prime divisors that are not over the generic point.

For \(X \times \mathbb{P}^n\), notice if \(n \geq 1\), \(X \times \mathbb{P}^n \setminus X \times \mathbb{P}^{n-1} \cong X \times \mathbb{A}^n\), thus we can use(V.7.1.4) to get an exact sequence

\[ \mathbb{Z} \to \text{Cl}(X \times \mathbb{P}^n) \to \text{Cl}(X) \to 0 \]

where \(\mathbb{Z}\) is mapped to \([X \times \mathbb{P}^{n-1}]\), this map is injective: otherwise if \(k[X \times \mathbb{P}^{n-1}] = (f)\), then \(f\) has no pole, which means \(f \in \mathcal{O}(X \times \mathbb{P}^n) = \mathcal{O}(X)\), which means \((f) = D \times \mathbb{P}^n\), which is not equal to \(k[X \times \mathbb{P}^{n-1}]\). Also this exact sequence splits, because there is a pullback function \(\pi^*: \text{Cl}(X) \to \text{Cl}(X \times \mathbb{P}^n)\), which is a section of \(\text{Cl}(X \times \mathbb{P}^n) \to \text{Cl}(X)\).

\[ \square \]

**Prop. (V.7.1.7).** For \(A\) a Noetherian domain, it is a UFD iff \(X = \text{Spec}(A)\) is normal and \(\text{Cl}(X) = 0\).

**Proof:** We only have to show minimal primes of \(A\) is principal iff minimal primes of \(\mathcal{O}(X)\) is a principal divisor. This is done by (I.6.5.9) and (I.2.3.12).

\[ \square \]

**Cor. (V.7.1.8).** The divisors on \(\mathbb{P}^n_k\) is locally defined by a function, this is because the affine opens are UFD.

**Prop. (V.7.1.9).**

**Cartier Divisors**

**Def. (V.7.1.10) (Cartier Divisor).** A **Cartier divisor** on a scheme is an element in \(\Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)\). An **effective Cartier divisor** is a Cartier divisor that is locally defined as \(\{(U_i, f_i)\}\) where \(f_i \in \Gamma(U_i, \mathcal{O}_{U_i})\) are nonzero-divisors, it is equivalent to a closed subscheme whose ideal sheaf is an invertible sheaf.

Notice by definition, \(\mathcal{K}\) is the localization w.r.t. non-zero-divisors, and \(f_i\) is invertible in \(\mathcal{K}^*\) so \(f_i\) must be non-zero-divisors.

The **Cartier group** \(\text{CaCl}\) is the quotient of \(\Gamma(X, \mathcal{K}^*) \to \Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)\).

**Prop. (V.7.1.11) (Weil-Cartier).** For an integral normal locally Noetherian scheme, the (effective)Cartier divisor is the same thing as (effective)Weil divisor.(Immediate from(V.7.1.19) and(V.7.1.16)).

This in particular applies to non-singular varieties, by(I.6.5.16).

**Cor. (V.7.1.12).** If \(X\) is Noetherian and the diagonal map is affine, for a dense affine open \(U\), if all the stalk of \(X - U\) are UFD, then \(U\) is the complement of an effective Cartier divisor.

**Proof:** The irreducible complements of \(X - U\) is finite and has codimension 1 by(V.5.2.5) because \(U \to X\) is affine, and it is an effective Cartier divisor by(V.7.1.11), so their sum will suffice.

\[ \square \]

**Picard Group**

**Remark (V.7.1.13) (Picard Groups).** The Picard group \(\text{Pic}(X)\) of a local ringed space \((X, \mathcal{O}_X)\) is defined in(V.2.5.6), and it is isomorphic to \(H^1(X, \mathcal{O}_X^*)\) by(V.6.1.12).
Prop. (V.7.1.14) (Class Group). If $X = \text{Spec } \mathcal{O}_K$ where $\mathcal{O}_K$ is a Dedekind domain, then by (I.5.10.12), the isomorphism class of invertible sheaves on $X$ is equivalent to the isomorphism class of fractional ideals modulo principle ideals. Thus $\text{Pic}(\mathcal{O}_X)$ equals the class group of $\mathcal{O}_K$ (I.5.10.13).

Def. (V.7.1.15) (Invertible Sheaf Associated to Cartier Divisors). For a Cartier divisor on a scheme $X$, we can define $\mathcal{L}(D)$ the sheaf associated to $D$ as the sub $\mathcal{O}_X$-module of $\mathcal{K}$ locally generated by $(f_i^{-1})$, where $D = (f_i)$ locally.

Prop. (V.7.1.16) (Cartier-Pic). For an integral scheme $X$, the homomorphism $\text{CaCl}(X) \rightarrow \text{Pic}(X) : D \rightarrow \mathcal{L}(D)$ is an isomorphism. (It is alway injective, as it is in fact the $\delta$-functor of the exact sequence of sheaves $0 \rightarrow \mathcal{O}_X^* \rightarrow \mathcal{K}^* \rightarrow \mathcal{O}_X^* \rightarrow 0$.)

Proof: It suffices to show any invertible sheaf can embed into the constant sheaf, tensor with $K$ and restrict to the stalk of the generic point, i.e. there is a compatible choice of homomorphisms into $K(X)$.

Cor. (V.7.1.17). For an integral separated Noetherian scheme $X$ that is locally factorial, by (V.7.1.11), a Weil divisor is equivalent to an effective line bundle, so giving an integral closed subscheme $E$ of $X$, $\mathcal{L}(E)$ can be defined, and there is an exact sequence of sheaves on $X$:

$$0 \rightarrow \mathcal{L}(-E) \rightarrow \mathcal{O}_X \rightarrow i_* \mathcal{O}_E \rightarrow 0$$

Def. (V.7.1.18) (Weil divisor of an invertible module). For $X$ a locally Noetherian integral scheme and $\mathcal{L}$ an invertible module, if $s \in \Gamma(X, \mathcal{K}_X(L))$ is a meromorphic section of $\mathcal{L}$, for any prime Weil divisor $Z$ with generic pt $\eta$, define the $\text{ord}_Z(s) = \text{ord}_{\mathcal{O}_X, \eta}(s/s_\eta)$, for any $s_\eta$ a generator of $\mathcal{L}_\eta$ over $\mathcal{O}_{X, \eta}$. This is independent of $s_\eta$ chosen.

The prime Weil divisors that $\text{ord}_Z(s) \neq 0$ is locally finite, the same as in (V.7.1.2). And any two different sections $s_i$ defines Weil divisors up to a difference of $\text{div}(f)(V.7.1.2)$. So we can define the Weil divisor class associated to $\mathcal{L}$ as $\sum \text{ord}_Z(s)[Z]$ for any meromorphic section $s$ of $\mathcal{L}$.

It is easy to verify that this induces a homomorphism from $\text{Pic}(X)$ to $\text{Cl}(X)$.

Prop. (V.7.1.19) (Cl-Pic). For a normal integral Noetherian scheme, the above map $\text{Pic}(X) \rightarrow \text{Cl}(X)$ (V.7.1.18) is an injection. It is an isomorphism iff all local rings of $X$ are UFD.

In particular, this is true for a smooth prevariety over a field $K$, by (V.5.3.11)/(V.4.2.6).

Proof: If it is not injective, then some meromorphic section on $\mathcal{L}$ has no associated Weil divisors, then it suffices to show $\mathcal{L}$ is trivialized by $s$. Consider on an affine subscheme $\text{Spec } A$, then $\text{ord}_{A_p}(s) = 0$ for each minimal prime $p$ of $A$, but $A_p$ is DVR by (I.6.5.15) and Serre Criterion (I.6.5.24), so $s \in A^*_p$ for each minimal prime $p$, so $s \in A^*$ by (I.6.5.9). This will show $s$ trivialize $\mathcal{L}$.

For the surjectiveness, Cf. [Sta]0BE9. 

Cor. (V.7.1.20) (Picard Group of Projective Spaces). Let $R$ be a UFD, then $\text{Pic}(\mathbb{P}^n_R) = \mathbb{Z}$, and it is generated by $\mathcal{O}_{\mathbb{P}^n_R}(1)$.

Proof: $X = \mathbb{P}^n_R$ is an integral Noetherian scheme whose local rings are all UFD, thus $\text{Pic}(X) \cong \text{Cl}(X)$ by (V.7.1.19). The sheaves $\mathcal{O}_X$ and $\mathcal{O}_X(m)$ are non-isomorphic for $m < 0$ because $H^0(X, \mathcal{O}_X(m)) = 0$ (V.6.7.16). Thus $\mathcal{O}_X(1)$ is non-torsion in $\text{Pic}(X)$.

Now let $\mathcal{L}$ be an arbitrary invertible sheaf on $X$, consider $U = D(T_0) \cong \mathbb{A}^n_R$, then $X \setminus U \cong \mathbb{P}^{n-1}_R$ is a prime divisor of $X$. $H$ is the zero scheme of $T_0$, thus $\mathcal{O}_X(1)$ maps to $[H] \in \text{Cl}(X)$. Because $U = \text{Spec } R[X_1, \ldots, X_n]$ and $R[X_1, \ldots, X_n]$ is a UFD, $\text{Pic}(U) = 0$ by (V.7.1.14) and (I.5.10.14),
so \( \mathcal{L}|_U \cong \mathcal{O}_U \). Choose a trivializing section \( s \) of \( \mathcal{L}|_U \), then \( s \) is a rational section of \( \mathcal{L} \). Now \( \text{div}_C(s) = m[H] \) for some \( m \in \mathbb{Z} \), thus \( \mathcal{L} \) and \( \mathcal{O}_X(m) \) map to the same element in \( \text{Cl}(X) \), which means \( \mathcal{L} \cong \mathcal{O}_X(m) \).

\[ \square \]

**Prop. (V.7.1.21).** For an integral normal projective scheme of dimension \( \geq 2 \) over an alg.closed field, then the support of an effective ample divisor is connected.

**Proof:** We may assume the divisor is very ample, denote \( \mathcal{O}_X(1) = \mathcal{L}(D) \), let \( Y_q \) be the closed subscheme corresponding to the divisor \( qD \), then there is an exact sequence

\[ 0 \to \mathcal{O}_X(-q) \to \mathcal{O}_X \to i_*\mathcal{O}_{Y_q} \to 0 \]

(V.7.1.17), so for \( q \) large, (V.7.4.7) shows that \( \Gamma(X, \mathcal{O}_X) \to \Gamma(Y, \mathcal{O}_{Y_q}) \) is surjective. But the former is \( k \) by (V.8.1.12) and the second contains \( k \), thus the latter is also \( k \), thus it is connected. \( \square \)

2 Blowing Up

**Prop. (V.7.2.1).** On a locally Noetherian scheme, the **blowing up** \( \tilde{X}_I \) along a closed scheme (Corresponding to a coherent sheaf) is defined as \( \text{Proj}(\oplus I^d) \). It has the universal property that any morphism \( Z \to X \) that pulls back \( Y \) to an effective Cartier divisor uniquely factors through \( \tilde{X}_I \).

**Proof:** Notice first an effective divisor is equivalent to an invertible sheaf of ideal. And any morphism \( Z \to X \) pulls back \( I \) to the image of \( f^{-1}I \otimes \mathcal{O}_Z \mathcal{O}_Z \to f^{-1}I \cdot \mathcal{O}_Z \). This is just \( \mathcal{O}(1) \) on \( \tilde{X}_I \) so invertible.

For the construction, the local uniqueness will implie the existence. Notice locally \( \tilde{X}_I \) is projective over \( X \). Now because the \( Z \to X \) pulls back \( I \) to an invertible sheaf and it is generated by \( f^{-1}(a_i) \), we use?? to get another \( Z \to \text{Proj}_X^n \) and it factors through the closed subscheme \( \tilde{X}_I \). If there is another morphism \( g \), then \( f^{-1}I \cdot \mathcal{O}_Z = g^{-1}(\pi^{-1}I \cdot \mathcal{O}_{\tilde{X}_I}) \cdot \mathcal{O}_Z = g^{-1}(\mathcal{O}_{\tilde{X}_I}) \cdot \mathcal{O}_Z \) surjective, and a surjective morphism between two invertible sheave is an isomorphism, and they are both ideal sheaves, thus is the same, so this morphism is unique as it is determined by the map on \( \mathcal{O}_X ?? \). \( \square \)

**Cor. (V.7.2.2).** If the sheaf of ideal is itself invertible, then the blowing up is an isomorphism by the universal property. In particular, on the open set \( U = X - Y \), \( I_U \cong \mathcal{O}_U \), so \( \pi^{-1}(U) \cong U \).

**Cor. (V.7.2.3).** \( \pi : \tilde{X}_I \to X \) is birational, proper thus surjective. If \( X \) is a (complete)variety, then so does \( \tilde{X}_I \).

**Prop. (V.7.2.4) (Strict Transformation).** Same notation as before, for any locally Noetherian scheme \( Z \to X \), we have the pullback sheaf \( J = i^{-1}(I) \cdot \mathcal{O}_Z \) on \( Z \), so \( \tilde{Z}_I \to X \) factors through \( \tilde{X}_I \). This a pullback diagram. (Recall the definition of fiber product, we only need to check for \( Z,X \) affine and glue. For this, check \( B \otimes_A (\oplus I^d) \to \oplus (IB)^d \) defines the fiber map).

**Prop. (V.7.2.5).** If \( X \) is \( H \)-(quasi-)projective, then so does \( \tilde{X}_I \) and \( \pi \) is \( H \)-projective (V.3.2.21). And any birational projective morphism from another variety \( Z \) to \( X \) comes from a blowing-up.

**Proof:** Cf.[Hartshorne P166]. \( \square \)
Blowing up along a regular variety

**Prop. (V.7.2.6).** If $X$ is a regular variety over $k$ and $Y$ is a regular closed subvariety defined by $I$, then blowing up along $I$ is also regular, and the inverse image $Y' = \mathbb{P}(I/I^2)$ of $Y$ is locally principal in it. In fact, $Y' \to Y$ is isomorphic to $\mathbb{P}(I/I^2)$, the projective space associated to the locally free bundle $I/I^2$ on $Y$, and the normal sheaf $N_{Y'//X'} \cong \mathcal{O}_{\mathbb{P}(I/I^2)}(-1)$.

**Proof:** Imagine the blowing up of $k^2$ along $\{(0)\}$). $X' \cong \text{Proj} \oplus I^d$ and $Y' \cong \text{Proj} \oplus I^d/I^{d+1}$. Then since $Y$ is regular, (I.6.4.14) tells us $I$ is locally generated by a regular sequence and (I.6.5.14) tells $Y' = \mathbb{P}(I/I^2)$. $Y'$ is regular by (I.6.5.14), and then (I.6.5.18) shows that $X'$ is regular also. For the normal sheaf, the defining sheaf $I' = \mathcal{O}_{X'}$, and then $I'/I^{d+1} = \mathcal{O}_Y(1)$, thus $N_{Y'/X'} \cong \mathcal{O}_{\mathbb{P}(I/I^2)}(-1)$.

**Def. (V.7.3.1).** A complex of $\mathcal{O}_X$-modules is called **strictly perfect** if it is finite and every term is a direct summand of a finite free sheaf.

**Prop. (V.7.2.7).** In a blowing up along a regular variety of codimension $r \geq 2$, there is an isomorphism $\text{Pic} X' \cong \text{Pic} X \oplus \mathbb{Z}$ induced by the Weil divisor exact sequence of $Y' \subset X'$. This is because $r \geq 2$ and (V.7.2.2).

We also have $\omega_{X'} = f^*\omega_X \otimes \mathcal{O}(r-1)$ by the fact that $\mathcal{O}(r-1)$ is regular. For the normal sheaf, the defining sheaf $I' = \mathcal{O}_{X'}$, and then $I'/I^{d+1} = \mathcal{O}_Y(1)$, thus $N_{Y'/X'} \cong \mathcal{O}_{\mathbb{P}(I/I^2)}(-1)$.

3 Derived Category of Schemes

**Def. (V.7.3.2).** A complex of $\mathcal{O}_X$-modules is called **strictly perfect** if it is finite and every term is a direct summand of a finite free sheaf.

**Prop. (V.7.3.3).** Every mapping from a strictly perfect complex to an acyclic complex has a cover of open sets that on each open set the map is nullhomotopic.

**Proof:** This is true for a single direct summand of a finite free sheaf, and we can use induction to prove, Cf.[Sta]08C7.

**Cor. (V.7.3.4).** The strictly perfect complex is fake ”$K$-projective” object in $K(\mathcal{O}_X)$. Note it is not technically $K$-projective, but it has all the properties of $K$-projective when proven, noticing the fact it is irrelevant when taken shiffification.

**Def. (V.7.3.5).** We say an object $K^\bullet$ in $K(\mathcal{O}_X)$ is **perfect** if there is an open cover that on each open set there is a quasi-iso $K^\bullet_\mathbb{U} \to K^\bullet|_{U_i}$ with $K^\bullet_\mathbb{U}$ strictly perfect.

This is equivalent to $K^\bullet$ is locally represented by perfect objects in $D(\mathcal{O}_X)$ by the fact that perfect object is fake $K$-projective.
Prop. (V.7.3.6). When $X$ is local ringed space, perfectness is equivalent to the fact that it is locally a finite free $\mathcal{O}_{U_i}$-module. This is because direct summand of a finite free module is free, Cf.[[Sta]0BCI].

Resolution Property

Def. (V.7.3.7). A scheme $X$ is said to have resolution property iff every Qco $\mathcal{O}_X$-module of f.t. is a quotient of a locally free sheaf.

Prop. (V.7.3.8). If $X$ is Noetherian scheme and has an ample invertible sheaf, then $X$ has the resolution property(V.3.3.25). In fact, every coherent sheaf is a quotient of a finite direct sum of $\mathcal{O}(-n)$.

Prop. (V.7.3.9). If $X$ is qc regular scheme with an affine diagonal, then $X$ has the resolution property, Cf.[[Sta]0F8A]. Conversely, if $X$ is qcqs with the resolution property, then $X$ has affine diagonal. Cf.[[Sta]0F8C].

Prop. (V.7.3.10) (Kleiman). If $X$ is a qc irreducible and locally factorial scheme with affine diagonal map, then $X$ has the resolution property.

Proof: By(V.7.1.12), we have an basis of the form $X_s$ for $s \in \Gamma(X, \mathcal{L})$ for various invertible sheaves, then for any coherent sheaf, it is generated by f.m. sections in $\Gamma(U_i, \mathcal{F})$ and $U_i = X_s$ for $s \in \Gamma(X, \mathcal{L})$, and for each of them, we can use(V.3.3.2), we can extend these to global sections on $\Gamma(\mathcal{F} \otimes \mathcal{L}_i^{n_i})$ for $n_i$ large. Then tensoring $\mathcal{L}_i^{-n_i}$, we find a $\oplus \mathcal{L}_i^{-n_i} \to \mathcal{F}$ surjective. □

Prop. (V.7.3.11). When $X$ has the resolution property, $\mathcal{E}xt^*(-, \mathcal{G})$ is an universal $\delta$-functor for every Qco $\mathcal{G}$, this is because locally free sheaf is adapted to $\mathcal{E}xt^*(-, \mathcal{G})$ by(V.6.7.7), so we can calculate $\mathcal{E}xt(\mathcal{F}, \mathcal{G})$ using a finite locally free resolution of $\mathcal{F}$.

4 Duality for Schemes

References are [Hartshorne Residues and Duality] and [Sta]Chap46. The following materials are at a low level, should be refreshed with [Sta]Chap46.

Serre Duality Theorem

Def. (V.7.4.1). Let $X$ be a proper scheme of dimension $n$ over a field $k$, then a dualizing sheaf for $X$ is a coherent sheaf $\omega_X$ together with a trace map $H^n(X, \omega_X) \to k$ that for every coherent sheaf $\mathcal{F}$,

$$\text{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X) \to k$$

is a perfect pairing. In other words, $\omega_X$ represents the functor $\mathcal{F} \to (H^n(X, \mathcal{F}))^\vee$.

Prop. (V.7.4.2). If $X$ is proper over a field $k$, then there exists uniquely a dualizing sheaf.

Proof: □

Lemma (V.7.4.3). For $X = \mathbb{P}_k^n$, the canonical sheaf $\omega_X$ is the dualizing sheaf. Moreover, there is a perfect pairing

$$\text{Ext}^i(\mathcal{F}, \omega_X) \times H^{n-i}(X, \mathcal{F}) \to k$$
Proof: In this case, \( \omega_X = \mathcal{O}_X(-n - 1) \). For \( i = 0 \), when \( \mathcal{F} = \mathcal{O}_X(n) \), then this follows from (V.6.7.16)and(V.6.7.10). And \( X \) has the resolution property(V.7.3.8), so we can write \( \mathcal{F} \) as a quotient of two finite direct sum of \( \mathcal{O}(-n) \). then the long exact sequence gives us the result as \( H^{n+1} \) vanish.

For \( i > 0 \), both side are universal \( \delta \)-functors, so we show they are both coeplaceable. write \( \mathcal{F} \) as a quotient of two finite direct sum of \( \mathcal{O}(-n) \) for \( n \) large, then \( \text{Ext}^i(\mathcal{O}(-n), \omega) = H^i(X, \omega(n)) = 0 \) for \( i > 0 \). And \( H^{n-i}(X, \mathcal{O}_{X}(-n)) = 0 \) by(V.6.7.16).

Cor. (V.7.4.4). If \( X \) is a closed subscheme of \( \mathbb{P}^n_k \) of codimension \( r \), then \( X \) has a dualizing sheaf \( \omega_X^0 = \mathcal{E}xt_P^i(i_*\mathcal{O}_X, \omega_P) \).

Proof: It suffices to prove that \( \text{Hom}_X(\mathcal{F}, \omega_X) \cong \mathcal{E}xt_P^i(i_*\mathcal{F}, \omega_P) \), then the above proposition will give the desired result together with the fact pushforward commutes with sheaf cohomology.

For this, we choose an injective resolution \( I^* \) of \( \omega_X \) and let \( J^* = \text{Hom}_P(\mathcal{O}_X, I^*) \). Then \( J^* \) are injective \( \mathcal{O}_X \)-modules because \( \text{Hom}_X(\mathcal{F}, \text{Hom}_P(\mathcal{O}_X, I^*)) = \text{Hom}_P(\mathcal{F}, I^*) \). And by the lemma(V.7.4.5) below, \( J^* \) is exact up to \( r = \text{codim} X \), so it splits and \( \omega_X = \text{Coker} J^n \) hence \( \text{Hom}(\mathcal{F}, \omega_X) = \mathcal{E}xt_P^i(\mathcal{F}, \omega_P) \).

Lemma (V.7.4.5). Let \( X \) be a closed subscheme of \( \mathbb{P}^n_k \) of codimension \( r \), then \( \mathcal{E}xt^i(\mathcal{O}_X, \omega_{\mathbb{P}^n_k}) = 0 \) for \( i < r \).

Proof: Since \( \mathcal{F}^i = \mathcal{E}xt^i(\mathcal{O}_X, \omega_P) = 0 \) is a coherent sheaf, it suffice to show that \( \Gamma(P, \mathcal{F}^i(q)) = 0 \) for \( q \) large enough. But this equals \( \mathcal{E}xt^i_P(\mathcal{O}_X, \omega_P(q)) \), which is the dual of \( H^{n-i}(P, \mathcal{O}_X(-q)) = H^{n-i}(X, \mathcal{O}_X(-q)) \) which vanish by Grothendieck vanishing theorem.

Prop. (V.7.4.6). Let \( X \) be projective of dimension \( n \) over a field \( k \) and \( \omega_X^0 \) be the dualizing sheaf, then for \( \mathcal{F} \) coherent, there is a natural map

\[
\text{Ext}^i(\mathcal{F}, \omega_X^0) \to (H^{n-i}(X, \mathcal{F}))^\vee
\]

And the following are equivalent:

1. For any \( \mathcal{F} \) locally free on \( X \), \( H^i(X, \mathcal{F}(-q)) = 0 \) for \( i < n \) and \( q \) large.
2. \( H^i(X, \mathcal{O}_X(-q)) = 0 \) for \( i < n \) and \( q \) large.
3. This is an isomorphism of \( \delta \)-functors.
4. \( X \) is C.M. and equidimensional.

Proof: Notice the left side is an universal \( \delta \)-functor in \( \mathcal{F} \) by(V.7.3.11), so the map exist, and
\[
2 \to 3: \text{This implies that the right is also universal by(V.7.3.8)}.
3 \to 1: \text{For} \mathcal{F} \text{ locally free, by(V.6.7.10),}
\]

\[
H^i(X, \mathcal{F}(-q)) = (\text{Ext}^{n-i}(\mathcal{F}(-q), \omega_X))^\vee = (H^{n-i}(X, \mathcal{F} \otimes \omega_X(q)))^\vee
\]

which is 0 for \( q \) large.

4 \to 1: Embed \( X \) in \( P = \mathbb{P}^N_k \), for \( \mathcal{F} \) locally free, since \( X \) is catenary, equidimensional is equivalent to \( \dim \mathcal{F}_x = n \) for all closed pt \( x \), and C.M. says depth \( \mathcal{F}_x = n \). Thus by(I.6.5.20), \( pd_{\mathcal{O}_P, x} \mathcal{F}_x = N-n \). Thus \( \mathcal{E}xt^i_P(F, -) \) vanish for \( k > N-n \) checked on stalks.

Now \( H^i(X, \mathcal{F}(-q)) \) is dual to \( \mathcal{E}xt^{N-i}_P(\mathcal{F}, \omega_P(q)) \) by the proof of(V.7.4.4), which is isomorphic to \( \Gamma(P, \mathcal{E}xt^{N-i}_P(\mathcal{F}, \omega_P(q))) \) for \( q \) large by(V.6.7.25), so it vanish when \( i < n \) by what we proved.
1 $\rightarrow$ 4: The same as the proof of 4 $\rightarrow$ 1, then for $i < n$,
\[
\Gamma(P, \mathcal{E}xt_P^{N-i}(\mathcal{F}, \omega_P(q))) = 0 = \Gamma(P, \mathcal{E}xt_P^{N-i}(\mathcal{F}, \omega_P)(q))
\]
for $q$ large, so $\mathcal{E}xt_P^{N-i}(\mathcal{F}, \omega_P) = 0$ as it is coherent. Then the stalk is $\mathcal{E}xt_{O_{X,x}}^{N-i}(\mathcal{O}_{X,x}, \mathcal{O}_{P,x})$, so $pd_{\mathcal{O}_{P,x}}\mathcal{F}_x \leq N - n$ by (I.6.5.21), so depth $\mathcal{O}_{X,x} \geq n$, we must have equality, thus $X$ is C.M. and equidimensional, as it suffice to check at closed pts.

\[\text{Cor. (V.7.4.7) (Enriques-Severi-Zariski).} \text{ Let } X \text{ be a normal projective scheme that every}
\]
irreducible component has dimension $\geq 2$, then for any locally free sheaf $\mathcal{F}$ on $X$, $H^j(X, \mathcal{F}(-q)) = 0$ for $q$ large.

\[\text{Proof:} \text{ Just notice that dim} \mathcal{F}_x \geq 2, \text{ and Serre criterion shows depth} \mathcal{F}_x \geq 2, \text{ the rest is the same as } 4 \rightarrow 1 \text{ in the proof of (V.7.4.6).}\]

\[\text{Prop. (V.7.4.8) (Dualizing Sheaf on Regular Variety).} \text{ When } X \text{ is a closed subscheme of}
\]
$P = \mathbb{P}^n_k$ which is a local complete intersection of dimension $r$, then $\omega_X^n = \omega_P \otimes \mathcal{L}/I^2)^{-1}$, which is an invertible sheaf on $X$. Notice $I/I^2$ is locally free by (V.5.7.3).

In particular, when $X$ is a regular projective over a field $k$, then $\omega_X$ is just the canonical sheaf (V.8.1.18)??

\[\text{Proof:} \text{ Cf.[Harthorne P245]. The basic idea is to use the free resolution of Koszul complex for}
\]
the stalk of $\mathcal{O}_X$ to calculate $\omega_X = \mathcal{E}xt^r(\mathcal{O}_X, \omega_P)$. It depends on the regular sequence, and the transition of $(I/I^2)^{-1}$ neutralize this.

\[\text{Cor. (V.7.4.9) (Serre Duality).} \text{ If } X \text{ is a regular projective variety, then for any locally free}
\]
sheaf $\mathcal{F}$, by (V.4.2.6)(V.7.4.6)(V.7.4.8)(V.6.7.10), there is an isomorphism:
\[
H^i(X, \mathcal{F}) \cong (H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X))^\vee.
\]

\[\text{Cor. (V.7.4.10).} \text{ For a projective regular variety over a field } k, H^n(X, \omega_X) = k, \text{ by (V.8.1.12).}\]

\[\text{Cor. (V.7.4.11).} \text{ Let } X \text{ be a regular projective variety of dimension } n \text{ over an alg.closed field } k,
\]
$\Omega = \Omega_{X/k}$ is locally free by (V.5.3.9), thus by (V.3.1.10), $\Omega^{n-p} \cong (\Omega^p)^\vee \otimes \omega_X$. So by (V.7.4.9):
\[
H^q(X, \Omega^p) \cong (H^{n-q}(X, \Omega^{n-p}))^\vee.
\]

\[\text{Topological Sheaves} \]

\[\text{Prop. (V.7.4.12) (Global Verdier Duality).} \text{ If } f : X \rightarrow Y \text{ is a map between locally compact}
\]
space with finite dimension, then there exists a functor $f^! : D^+(\text{SAb}_Y) \rightarrow D^+(\text{SAb}_X)$ that
\[
R \text{Hom}(Rf_*\mathcal{F}^\bullet, \mathcal{G}^\bullet) \cong R \text{Hom}(\mathcal{F}, f^! \mathcal{G}^\bullet).
\]

In particular, $f^!$ is right adjoint to $Rf_!$. Cf.[Gelfand P228].

There is also a local form of Verdier duality, which implies the global version by taking global section, Cf.[Cohomology of Sheaves Iversen P330].

\[\text{Prop. (V.7.4.13).} \text{ When } X \rightarrow Y \text{ is an inclusion of open subset, } f_! \text{ is just } j_! \text{ defined in (V.2.6.2)}
\]
and $f^!$ is the restriction. When it is an inclusion of closed subset of locally compact spaces, it is the direct image $f_*$ and $f^!$ is the $j_!$ previously defined in (V.2.6.2). They are not barely defined on $D^+(\text{SAb})$ but on $\text{SAb}$. 

Prop. (V.7.4.14). We consider the case where \( f : X \to \text{pt} \), and let \( G = \mathbb{Z} \), denote \( f^!(\mathbb{Z}) \) by \( \mathcal{D}_X^\bullet \), called the dualizing complex, then there is a duality:

\[
R \text{Hom}(R\Gamma_c(X, \mathcal{F}^\bullet), \mathbb{Z}) \cong R \text{Hom}(\mathcal{F}^\bullet, \mathcal{D}_X^\bullet).
\]

for \( \mathcal{F}^\bullet \in D^+(S\text{Ab}_X) \).

Prop. (V.7.4.15). When \( X \) is a \( n \) dimensional topological manifold with boundary, then \( \mathcal{D}_X^\bullet = \omega_X[n] \), where the sheaf \( \omega_X \) is defined by

\[
\Gamma(U, \omega_X) = \text{Hom}_{\text{Ab}}(H^n_c(U, \mathbb{Z}), \mathbb{Z}).
\]

Cf.[Gelfand P234]. If we replace \( \mathbb{Z} \) by a field \( k \), then \( \omega_X \) is the sheaf of \( k \)-orientations of \( \text{int}(X) \), thus the constant sheaf when \( X \) is oriented or \( \text{char } k = 2 \).

In particular, place \( k \) in dimension \( i \) then we get an isomorphism

\[
\text{Hom}_k(H^i_c(X, \mathcal{F}), k) = \text{Ext}^{n-i}(\mathcal{F}, \omega_X)
\]

(because \( k \) is a field thus injective). Gelfand even gives an interpretation of this pair in [Gelfand P236].

And if \( \mathcal{F} = \omega_X \) and \( X \) oriented or char \( k = 2 \), we have \( \text{Ext}^{n-i}(k_X, k_X[n]) = H^{n-i}(X, k_X) \) using the adjointness of constant sheaf, so we get the Poincare duality:

\[
H^i_c(X, k_X)^\vee \cong H^{n-i}(X, k_X).
\]

Prop. (V.7.4.16). Compact cohomology commute with colimits, Cf.[Cohomology of Sheaves Iversen P173].

5 Deformation Theory

Basic references are [[Sta]Chap36].

Def. (V.7.5.1) (Thickening). We call \( X' \) a thickening of a \( X \) iff \( X \) is a closed subscheme of \( X' \) that their underlying topological space are the same. Morphisms of thickenings are defined routinely.

A thickening is said to have order \( n \) iff the ideal sheaf \( \mathcal{I} \) satisfies \( \mathcal{I}^{n+1} = 0 \).

Base change and composition of a (order \( n \))thickening is also a (order \( n \))thickening, because closed immersion and surjective do.

Prop. (V.7.5.2). Any thickening of an affine scheme is also affine.

\[\text{Proof:} \quad \text{This is a special case of(V.4.4.34)}. \]

Def. (V.7.5.3). Let \( X \) be a scheme algebraic over a field \( k \) and \( \mathcal{F} \) is a coherent sheaf on \( X \), then a infinitesimal extension of \( X \) by the sheaf \( \mathcal{F} \) is a scheme \( X' \) over \( k \) that has a sheaf of ideals \( \mathcal{I} \) that \( \mathcal{I}^2 = 0 \) and \( (X', \mathcal{O}_{X'}) / \mathcal{I} \cong (X, \mathcal{O}_X) \), and moreover, \( \mathcal{I} \) with the \( \mathcal{O}_X \)-structure is isomorphic to \( \mathcal{F} \).

There is a trivial extension, that is \( (X', \mathcal{O}_{X'}) \cong (X, \mathcal{O}_X \oplus \mathcal{F}) \), where the multiplication is \( (a, f)(a', f') = (aa', af' + a' f) \).
Def. (V.7.5.4) (Deformation). Let $X$ be a scheme algebraic over a field $k$, an infinitesimal deformation of $X$ is a scheme $X'$ flat over $D = k[t]/(t^2)$ that $X' \otimes_D k = X$. An infinitesimal deformation is a first order thickening, by (V.7.5.1).

If $Y$ is a closed subscheme of $X$, then we define the infinitesimal deformation of $Y$ in $X$ to be a closed subscheme $Y' \subset X \otimes_k D$ which is flat over $D$ and $Y' \otimes_D k = Y$.

A scheme algebraic over a field $k$ is called rigid if it has no infinitesimal deformations.

Prop. (V.7.5.5) (Affine Case). Any thickening of an affine scheme is affine. (Immediate from (V.4.4.34)).

Prop. (V.7.5.6). Let $X$ be a nonsingular variety over an alg.closed field $k$, infinitesimal deformation of $X$ is the same as an infinitesimal extension of $X$ by the sheaf $O_X$. Thus we get the set of infinitesimal deformations of $X$ is parametrized by $H^1(X, T_X)$, by (V.7.5.8) below.

Proof: For an infinitesimal deformation, tensoring $O_{X'}$ with the exact sequence $0 \to k \xto{t} D \to k \to 0$, we get (by flatness)

$$0 \to O_X \xto{t} O_{X'} \to O_X \to 0,$$

and conversely, an extension is locally free (because it is f.g. so flat over $D$ is equivalent to free).

Prop. (V.7.5.7). If $X$ is an affine regular scheme algebraic over an alg.closed field $k$, then any extension by coherent sheaf is trivial.

Proof: For any infinitesimal extension, the morphism $X \to X'$ is a closed immersion and surjection, so $X'$ is also affine by (V.7.5.5), $= \text{Spec } A'$. Now the rest follows from ??.

Cor. (V.7.5.8) (Infinitesimal Extension and Cohomology). Let $X$ be a nonsingular variety over an alg.closed field $k$, then the set of infinitesimal extensions by a coherent sheaf $F$ is parametrized by $H^1(X, F \otimes T_X)$.

If $Y$ is a closed subscheme of $X$, then the set of infinitesimal deformation of $Y$ in $X$ is parametrized by $H^0(Y, N_{Y/X})$.

Proof: By the proposition, we know that an infinitesimal extension is locally isomorphic to $(U, O_X(U) \otimes F(U))$, by a section $F(U) \to O_{X'}(U)$.

But there is a twist, because there can because different sections. But the different sections different at a $\text{Hom}_{O_X(U)}(O_{O_X(U)/k}, F(U)) = (T \otimes F)(U)$. These forms a Čech cocycle for $F \otimes T_X$, and the converse is also true. Finally, use the fact that $X$ is separated so Čech and sheaf cohomology coincide.

For the subscheme, □

6 Quotient of Schemes

Prop. (V.7.6.1). Let $u_0, u_1 : X_1 \to X_0$ be an equivalence relation on the algebraic scheme $X_0$ over $R_0$. Assume that

- $u_0 : X_1 \to X_0$ is locally free of rank $r$.
- For all $x_0 \in X$, $u_0(u_1^{-1}(x))$ is contained in an open affine subscheme of $X_0$.

Then a quotient $u : X_0 \to X$ exists. Moreover, $u$ is locally free of rank $r$.

Proof: Cf[Mil17b]P597.
V.8 Varieties

Basic references are [Sta] and [Hartshorne].

1 Varieties

Classical Variety

Prop. (V.8.1.1). The underlying space of a scheme is sober, Cf. (V.2.7.11).

Prop. (V.8.1.2). For $k$ alg.closed, the soberization functor $t$ induces a fully faithful functor from $\text{Var}(k) \to \text{Sch}(k)$ that maps to quasi-projective integral schemes over $k$. It maps projective varieties to projective integral schemes. And this functor preserves fiber products.

Proof: We assign the irreducible closed subsets space $t(X)$ and show that this embeds $X$ in $t(X)$, and for an affine variety $(V, \mathcal{O}_V)$, the regular function sheaf is isomorphic to the pullback sheaf on $t(V) = \text{Spec}(A)$.

By definition $t(X)$ is quasi-projective, and for a closed subscheme of $\mathbb{P}_k^n$, the closed pt of any closed subscheme are dense so $t(V)$ is homeomorphic to $X$. And because they are both reduced, they are isomorphic. So it is essentially surjective.

It is fully faithful because the closed point are equivalent to $k(x) = k$ and is dense in a f.t scheme over $k$ so it maps closed pt to closed pt. □

Prop. (V.8.1.3). The soberization of a classical variety $X$ is regular at a closed point iff the local defining functions has rank $n - \text{dim} \, X$.

Proof: Consider the space of closed point of $X$, they correspond to classical points because $k$ is alg.closed. Let $a_p = (x_1 - a_1, \ldots, x_n - a_n)$ and $b$ be the locally defining ideal. Then the differential defines an isomorphism of vector space $a_p^2 \cong k^n$, and the local ring at $p$ is $m/m^2 \cong (a_p/b)/(a_p/b)^2 \cong a_p/(b + a_p^2)$. The rank of the defining functions is $b + a_p^2/a_p^2$. Counting dimension gives us the result. (Use (V.5.2.3) also). □

Abstract Variety

Def. (V.8.1.4) (Abstract Variety). An abstract variety is a geo.integral separated scheme algebraic over a field $k$. An prevariety is an integral separated scheme algebraic over a field $k$. It is called complete if it is also proper(i.e. universally closed).

A classical variety is an abstract variety because quasi-projective is f.t. and separated (V.4.5.13).

A variety is called non-singular if it is regular.

A curve over $k$ is an abstract variety of dimension 1.

Remark (V.8.1.5). Notice the prevariety is the same as the variety defined in [Sta].

Cor. (V.8.1.6). An abstract variety is birational to an integral $H$-quasi-projective scheme. A complete variety is birational to an integral projective scheme by Chow’s lemma (V.4.5.16)(V.4.5.3).

Prop. (V.8.1.7). By valuation criterion, for a complete variety, every valuation of the function fields of $K/k$ dominate a unique point of $X$. So the points of $X$ correspond to valuations of $K/k$ (valuation ring is the maximal local ring).
Prop. (V.8.1.8) (Nagata’s Theorem). Any abstract variety can be embedded as an open subset of a complete variety.

Proof: □

Prop. (V.8.1.9) (Product of Varieties). The product of two (complete)varieties over \( k \) is also a (complete)variety.

Proof: It is geometrically integral by (V.4.3.17), it is separated because separatedness is stable under composition and base change (V.4.4.2). So does properness. □

Def. (V.8.1.10) (Geometric Points). A geometric points of a scheme \( X \) over a field \( k \) is an element of \( X(k_{s}) \), where \( k_{s} \) is the separable closure of \( k \). When \( X \) is a variety, the geometric points of \( X \) is dense in \( X \), by (V.4.3.3).

Prop. (V.8.1.11). To verify two morphisms \( f, g \) between two varieties \( X \) and \( Y \) are equal, it suffices to prove that they are equal on the set of geometric points (V.8.1.10). Because the equalizer is a closed subscheme of \( X \) (V.2.7.18), and it contains all closed pts of \( X \), so it must by \( X \), as the geometric-points are dense in \( X \) (V.8.1.10). Thus checking identity of two morphisms between varieties is enough to check on the level of geometric pts.

Prop. (V.8.1.12) (Global Section). If \( X \) is geometrically reduced, connected and proper over a field \( k \), then \( \Gamma(X, \mathcal{O}_X) = k \). In particular, this is true for a complete variety over a field \( k \).

Proof: Cf. [Sta]0BUG]. □

Prop. (V.8.1.13) (Check Properties on Geometric Points). A nice property of varieties is that identity of two morphisms of products of varieties can be checked at the geometric pts (V.8.1.10), by (V.8.1.11) and (V.8.1.9).

Surjectiveness of a map between varieties can be checked on geometric points, by (V.4.1.28).

Also surjective and injective of Qco sheaves need only be checked at geometric pts by (V.3.1.20)(V.4.1.25).

Linear System

Prop. (V.8.1.14). A complete linear system on a prevariety is the set of effective divisors linearly equivalent to \( D_0 \).

When \( X \) is a variety, the equivalent divisors correspond to projective space of \( \Gamma(X, \mathcal{L}(D_0)) \),

Proof: Any divisor equivalent to \( D_0 \) defines a global section on \( \mathcal{L}(D_0) \). And \( \Gamma(X, \mathcal{O}_X^*) = k^* \) by (V.3.2.13). □

Canonical Sheaves

Prop. (V.8.1.15) (Canonical Sheaves). For a geometrically regular (smooth) variety \( X \) over a field \( k \) and \( Y \) a local complete intersection of \( X \) of codimension \( r \), by (V.5.3.2) and (V.5.3.11) and (V.5.7.3), \( K_{X/k} \) and \( K_{Y/k} \) are locally free, and \( \mathcal{I}/\mathcal{I}^2 \) is locally free, so:

1. The canonical sheaf \( K_X = \wedge^n K_{X/k} \) on \( X \).
2. The tangent sheaf \( \mathcal{T}_X = (K_{X/k})^{-1} \) on \( X \).
3. The conormal sheaf \( \mathcal{C}_{Y/X} = \mathcal{I}/\mathcal{I}^2 \) on \( Y \).
The normal sheaf $N_{Y/X} = C_{Y/X}^{-1}$ on $Y$ are all locally free.

**Prop. (V.8.1.16) (Kodaira-Spencer map).** There is another characterization of tangent vector fields. (Note: this should be a special case of Prop 8.5.9 in [FGA]).

Let $X$ be a variety over $k$ and $S = k[\varepsilon]$ the dual numbers. Then $H^0(X, T_X) \cong \text{Aut}^1(X_S/S)$, where $\text{Aut}^1(X_S/S)$ means that the automorphisms of $X_S$ over $S$ that is identity on $X$(inclusion to $X_S$ induced by $\text{Spec} k \subset \text{Spec} S$).

**Proof:** First the case $X = \text{Spec} A$ is affine, then because $H^0(X, T_X) = \text{Hom}_k(K_{A/k}, A) = \text{Der}(A, A)$, so this is equivalent to $\text{Der}(A, A) \cong \text{automorphisms of } A[\varepsilon]$ that is identity under pass to quotients to $A$. For this, a $d \in \text{Der}(A, A)$ is mapped to $a + b\varepsilon \mapsto a + b\varepsilon + d(a)\varepsilon$. This is checked to be a ring morphism, and any desired morphism are like these.

The above construction is natural and functorial in $A$, so it glue together to give the global case. □

**Prop. (V.8.1.17) (Geo.Regular and Conormal Sheaf).** Let $X$ be a regular variety over an alg.closed field $k$, then an irreducible closed subscheme $Y$ of $X$ is regular iff $K_{Y/k}$ is locally free and (V.3.4.13) is exact on the left.

In this case, $I$ is locally generated by $r$ elements and $C_{Y/X}$ is a locally free sheaf of rank $r$ on $Y$ by (V.5.7.3).

**Proof:** Cf.[Hartshorne P178]. Should has something to do with (V.5.3.2), (V.5.3.11) and (V.5.3.9). □

**Prop. (V.8.1.18) (Adjunction Formulas).** For a nonsingular variety $X$ over an alg.closed field $k$ and $Y$ a nonsingular subvariety of codimension $r$. There is an exact sequence

$$0 \to I/I^2 \to K_{X/k} \otimes O_Y \to K_{Y/k} \to 0$$

by (V.3.4.14). Taking the highest exterior power (V.3.1.10), we get:

$$K_Y = K_X \otimes \wedge^r N_{Y/X} = K_X \otimes (\wedge I/I^2)^{-1}$$

In particular, if $r = 1$ then $Y$ is a divisor $D$ in $X$, the canonical sheaf

$$K_Y \cong (K_X \otimes L(D))_Y, \quad K_{P^n/k} = O(-n - 1)(V.3.4.7).$$

because $I_Y \cong \mathcal{L}(-Y)$ in this case so $I_Y/I_Y^2 = L(-Y) \otimes O_Y$.

Taking dual, we get:

$$0 \to T_Y \to T_X \otimes O_Y \to N_{Y/X} \to 0$$

**Prop. (V.8.1.19) (Geometric Genus).** For a regular proper variety over a field $k$, the geometric genus $p_g$ is defined as the rank of the global section of the invertible canonical sheaf $K_X = \wedge^n K_{X/k}$. It is a birational invariance. With the same methods, we can prove the rank of global sections of any other functorially defined bundles of $K_X$ is birational invariance, e.g. Hodge numbers.
Proof: For any rational map $U \to Y$, there is a subset $V \subseteq U$ and a local isomorphism $V$ and $f(V)$, that will define an isomorphism of global sections. Because a nonzero section of an invertible sheaf cannot vanish on a dense open set $f(V)$, the morphism of global sections is injective into $\Gamma(U,\mathcal{O}_U)$. Now we find a $U$ that $\text{codim}(X - U) > 1$, then we can use(I.6.5.9) to get $\Gamma(U) = \Gamma(X)$, then $p_g(X) \geq p_g(X')$, and the converse is also true. For this, we use valuation criterion of properness, then for any codimension 1 point, the stalk is a DVR, thus we find a $\text{Spec} \mathcal{O}_p \to X'$, this extends to a nbhd of $p$ because $X'$ is of f.t..\hfill\Box

Cor. (V.8.1.20). By(V.8.1.18), $K_{\mathbb{P}^n} \cong \mathcal{O}(-n - 1)$, so it has no global section by(V.6.7.16), $p_g(\mathbb{P}^n_k) = 0$. Hence every rational variety over a field $k$, i.e. one that is birational to $\mathbb{P}^n_k$, has geometric genus 0.

Complete Varieties

Lemma (V.8.1.21) (Rigidity Lemma). Let $X,Y$ be varieties over a field $K$. If $X$ is complete and $f : X \times Y \to Z$ that is constant with value $z \in Z(k)$ on some $y \in Y(k)$, then $f$ factors through the projection $\text{pr}_Y : X \times Y \to Y$.

Proof: Notice that checking equality of morphisms can be checked on geometric points(V.8.1.13), so we assume $k = \overline{k}$. Now for any $x_0 \in X(k)$, let $g(y) = f(x_0,y)$, then we want to show $f = g \circ \text{pr}_Y$.

Let $U$ be affine open in $Z$, then because $X$ is universally closed, $\text{pr}_Y$ is closed, so $V = \text{pr}_Y(f^{-1}(Z - U))$ is closed in $Y$. But if $P \notin V$, then $f(X \times P) \subseteq V$, but $X$ is complete and connected, so $f(X \times P)$ is constant(V.4.5.4). Now $f = g \circ \text{pr}_Y$ on an non-empty subset of $X \times Y$, which is irreducible(V.8.1.9), so this is true on all of $X \times Y$.\hfill\Box

Prop. (V.8.1.22). Let $X,Y$ be varieties over $K$ that $X$ is complete. If $L,M$ are two line bundles on $X \times Y$ that $L|_y = M|_y$ for all closed points $y \in Y$, then there exists a line bundle $N$ on $Y$ that $L \cong M \otimes \text{pr}_Y^*(N)$.

Proof: For any $y$ closed in $Y$, $L_y \otimes M^{-1}_y$ is trivial on $X_y$, thus $H^0(X_y,L_y \otimes M^{-1}_y) = k(y)$ by(V.8.1.12). Thus $\text{pr}_{Y*}(L \otimes M^{-1})$ is locally free of rank 1, hence a line bundle.

Now $\text{pr}_Y^*(L \otimes M^{-1}) \cong L \otimes M^{-1}$, because it is isomorphism on the fibers, so by Nakayama, it is surjective as sheaves?, thus isomorphism by comparing rank.\hfill\Box

Cor. (V.8.1.23) (See-Saw Principle). If in addition to the above proposition, $L_x \cong M_x$ for some $x \in X(K)$, then $L \cong M$.

Prop. (V.8.1.24). If $X$ is complete variety over a field $k$ and $Y$ is a $k$-scheme, and $L$ is a line bundle over $X \times Y$, then there is a closed subscheme $Y_0 \to Y$ that is maximal subscheme of $Y$ that $L|_{Y_0}$ is trivial, i.e. $L|_{X \times Y_0}$ is a line bundle $\text{pr}_{Y_0}^*(N)$ for some line bundle $N$ over $Y_0$.

Proof: Cf.[Abelian Variety, van Der Geer, 6.4].\hfill\Box

Prop. (V.8.1.25) (Theorem of the Cube). If $X,Y$ are complete varieties over a field $k$, and $Z$ is a connected, locally Noetherian $k$-scheme, if $x,y,z$ are points of $X,Y,Z$ respectively, and $L$ is a line bundle on $X \times Y \times Z$ that is trivial on $x \times Y \times Z, X \times y \times Z, X \times Y \times z$, then $L$ is trivial.

Proof: A field extension is faithfully flat, thus a line bundle is trivial iff its base change of fields is trivial?. Thus we can assume that $x,y,z$ are both rational points.
Let $Z_0$ be the maximal closed subscheme of $Z$ that $L_2$ is trivial on $Z_0$. We show that $Z_0$ is open, thus it is all of $Z$: If $\zeta \in Z'$, let $I \subset O_{Z,\zeta}$ be the ideal defining $Z'$, we show $I = (0)$. If not, then because $\cap m^n = 0$ by Krull’s theorem (locally Noetherian used), there is an $n$ that $I \subset m^n, I \not\subset m^{n+1}$. Now let $a_1 = (I, m^{n+1})$, and $m^{n+1} \subset a_2 \subset a_1$ that $\dim_{k(\zeta)}(a_1/a_2) = 1$, and let $Z_i \subset \text{Spec} O_{Z,\zeta}$ be the closed subscheme defined by $a_i$. Let $L_i$ be the restriction of $L$ on $X \times Y \times Z_i$. If we show that $L_2$ is trivial, then $Z_2$ is contained in $Z_0$, which is contradiction because $I \not\subset a_2$.

For this, notice that $L_1$ is trivial, and to show that $L_2$ is trivial, it suffices to lift a non-vanishing global section $s$ of $L_1$ to $L_2$, because $Z_1, Z_2$ has the same underlying set.

For this, notice that the obstruction of the lifting is a $\xi \in H^1(X \times Y, \mathcal{O}_{X \times Y})$. But now the conditions show that $\xi$ is zero under the pullback along $x \times Y \hookrightarrow X \times Y$ and $X \times y \hookrightarrow X \times Y$. So by Kunneth formula (V.6.7.13) and (V.8.1.12), $\xi$ vanishes.

## 2 Projective Variety

**Prop. (V.8.2.1) (Bertini).** Any regular projective variety over $k$ alg.closed with f.m singular pt has a hyperplane that intersect it with a regular variety. These hyperplanes form an open dense subset of the complete linear system $|H|$ of $\mathbb{P}_k^n$. Cf.[Hartshorne P179].

**Cor. (V.8.2.2).** When $\dim X \geq 2$, this is even a regular variety by (V.7.1.21) and (V.4.2.4).

**Prop. (V.8.2.3) (Affine Dimension Theorem).** For two affine variety $Y, Z$ of dimension $r, s$ in $\mathbb{A}^n_k$ over fields, there intersection has every component dim $\geq r + s - n$.

**Proof:** The theorem follows from Krull’s theorem (I.5.8.17) when $Y = H$, and for the general case, notice $Y \cap Z \cong (Y \times Z) \cap \Delta$ in $\mathbb{A}^n \times \mathbb{A}^n$.

**Cor. (V.8.2.4) (Projective Dimension Theorem).** For two projective variety $Y, Z$ of codimension $r, s$ in $\mathbb{P}^n_k$ over fields, there intersection has every component of codimension $\leq r + s$.

**Proof:** First prove this for $Y = H$, then we can either induct or directly from the theorem above. For this, we just use Krull’s theorem (I.5.8.17).

### Degree of Projective Varieties

Basic References are [Hartshorne I.7].

**Def. (V.8.2.5) (Hilbert Polynomial).** For a scheme projective over a field $k$ of dimension $r$, we define the **Hilbert polynomial** $P_Y$ as the Hilbert polynomial of its homogenous coordinate ring $\Gamma(Y)$. It has dimension $r$ by (I.5.6.13).

The **degree** of $Y$ is defined as the $r!$ times the leading coefficients of $P_Y$.

**Prop. (V.8.2.6).**

- The degree is a positive integer.
- If $Y = Y_1 \cup Y_2$ and $\dim Y_1 \cap Y_2 < r$, then $\deg Y = \deg Y_1 + \deg Y_2$.
- If $H$ is a hypersurface whose ideal is generated by a homogeneous polynomial of degree $d$, then $\deg H = d$.

**Proof:** Cf.[Hartshorne P52].

**Prop. (V.8.2.7).** For a variety of degree $k$ and a general linear space, the intersection has $k$ points.
3 Birational Geometry

Prop. (V.8.3.1). Any variety over $K$ is birational to a hypersurface in $\mathbb{A}^n_K$ for some $n$.

Proof: Cf. [Diophantine Geometry, P575].

Prop. (V.8.3.2) (Varieties and Function Fields). The following categories are equivalent.
- The category of varieties over $k$ with dominant rational morphisms.
- The dual category of f.g. field extensions over $k$.

Proof: Cf. [Sta]0BXN.

Prop. (V.8.3.3). Let $\varphi : X \to X'$ be a rational map of $K$-varieties with $X$ smooth. If the base change $\varphi_K$ extends to a morphism $X_K \to X'_K$, then $\varphi$ extends to a morphism $X \to X'$.

Proof: Let $U$ be an open dense subset that $\varphi$ is defined. For a point $x$, let $x' = \varphi_K(x)$, then $x'$ is in the closure of $\varphi(U)$. By (V.4.6.2), it suffices to construct a morphism $\mathcal{O}_{X',x'} \to \mathcal{O}_{X,x}$, i.e., to prove for any rational function $f$ regular at $x'$, $\varphi \circ f$ is regular at $x$. The argument is the same as that of (V.4.5.8).

For any such $f$, $f_K \circ \varphi_K$ is regular at $x$, thus no pole of $\text{div}(f_K \circ \varphi_K)$ passes through $x$. For the rest, Cf. [Diophantine Geometry, P576].

4 Surfaces

Prop. (V.8.4.1). Any birational transformation of non-singular surfaces will be factorized into f.m blowing-ups and blowing-downs of points.

Resolution of Surfaces

Cf. [[Sta]Chap51].

5 Others

Prop. (V.8.5.1). Variety is triangulable.

Proof: Cf. [Hironaka Triangulation of Algebraic Sets].

Rationally Connected Varieties

Def. (V.8.5.2) (Rationally Connected Varieties). A variety $X$ over $K$ is called rationally connected if any two points of $X(\overline{K})$ can be connected by a rational curve (IX.12.1.2) over $\overline{K}$. 
V.9 Group Schemes

Main References are [Sta]Chap38, [Mil17] and [A course on finite flat group schemes and p-divisible groups, Jakob Stix].

1 Group Schemes

Def. (V.9.1.1) (Group Scheme). A monoid scheme over $S$ is a monoid object in the category of schemes over $S$. A group scheme over $S$ is a group object in the category of schemes over $S$ (II.1.1.46).

An open/closed subgroup scheme of a group scheme $G◁\ S$ is an open/closed subscheme of $G◁\ S$ that represents a subgroup functor of $G◁\ S$.

A group scheme is called smooth/flat/separated/... if $G/S$ is smooth/flat/separated/....

We have the left(right) translation for an elements in $G(R)$, equivalently, a natural transformation on $G$, and base change $(G ⊗_S S')(T'/S') = G(T'/S)$

Remark (V.9.1.2) (Yoneda Interpretation). We do not need to verify all the relations, whenever we have a functorial commutative group structure on all the set $\text{Hom}(T, G)$, we immediately recover the map $m : G × G \to G$ as $pr_1 · pr_2$ in $G(G × G)$, $inv : G \to G$ as $inv$ in $G(G)$, $u : S \to G$ as $1$ in $G(S)$, by Yoneda lemma.

Cor. (V.9.1.3) (Affine Group Schemes). From the Yoneda Interpretation, an affine group scheme over $\text{Spec}\ R$ is equivalent to a commutative Hopf algebra over $R$ (I.15.2.1). Thus the category of affine group schemes over $R$ embeds fully faithfully into the category of group functors $\text{Aff}_R \rightarrow \text{Grp}$.

Cor. (V.9.1.4) (Multiplication by $n$). For any $n \geq 1$, the natural transformation of commutative groups functors $x \mapsto x^n$ induces a morphism of commutative group schemes $[n] : G \mapsto G$ for any group scheme $G$, and it is functorial.

Def. (V.9.1.5) (Translation Map). For a group scheme $G$ over $R$, for any $a \in G(R)$, there is a (left)translation map

$$l_a : G \cong k × G \xrightarrow{(a, \text{id})} G × G \xrightarrow{m} G.$$  

and it satisfies $l_a, l_b$ by associativity.

Def. (V.9.1.6) (Semi-Direct Product Group Scheme). Let $G$ be a group scheme acting on a group scheme $H$, then we can form a semi-direct product group scheme $G \rtimes H$ representing the functor $T \mapsto G(T) \rtimes H(T)$. and $G \rtimes H$ is isomorphic to $G × H$ as schemes.

Prop. (V.9.1.7) (Common Group Schemes).

- $D(\Gamma) = \text{Spec}\ Z[\Gamma]$ for a group $\Gamma$ (I.15.2.3).
- $\mu_n = Z[T]/(T^n - 1)$ (I.15.2.5)
- $G_a = Z[T]$ (I.15.2.6)
- $\Gamma = \text{Spec}\ \prod_{\gamma \in \Gamma} Z$ (I.15.2.9).
- $\alpha_p = \text{Spec}\ F_p[X]/X^p$ (I.15.2.15).
Prop. (V.9.1.8) (Semi-Direct Product). If $X, Y$ are group schemes over $S$, and there is a natural map of group schemes $X \to \text{Aut}(Y)$, then we can define their semi-direct product $Y \rtimes X$, because of (V.9.1.2).

Prop. (V.9.1.9). For a group scheme $G \triangleleft S$, there is a Cartesian diagram:

$$
\begin{array}{ccc}
G & \xrightarrow{\Delta_{X/S}} & G \times_S G \\
\downarrow & & \downarrow \begin{pmatrix}(g,g') \mapsto m(i(g),g') \end{pmatrix} \\
S & \xrightarrow{e} & G
\end{array}
$$

This can be seen by a testing scheme $T$.

Cor. (V.9.1.10) (Separatedness). $G \triangleleft S$ is (quasi-)separated iff $e$ is qc(closed immersion). In particular, if $S$ is a field, then $G$ is separated.

Def. (V.9.1.11) (Kernel Group). For a homomorphisms of group schemes $\varphi : G \to H$ over $R$, we define $\text{Ker} \varphi$ the kernel of $\varphi$ to be the fibered product

$$
\begin{array}{ccc}
\text{Ker} \varphi & \twoheadrightarrow & \text{Spec } R \\
\downarrow & & \downarrow \varepsilon_H \\
G & \xrightarrow{\varphi} & H
\end{array}
$$

then it is a group scheme over $R$. And we denote by the exact sequence

$$0 \to \text{Ker} \varphi \to G \xrightarrow{\varphi} H$$

When $G, H$ are affine group schemes, it corresponds to the cokernel Hopf algebra defined in (I.15.2.17).

Prop. (V.9.1.12) (Short Exact Sequences). A sequence of algebraic groups

$$e \to N \xrightarrow{i} G \xrightarrow{q} Q$$

is called exact iff $i$ is an isomorphism of $N$ onto the kernel of $q$. A sequence of algebraic groups

$$e \to N \xrightarrow{i} G \xrightarrow{q} Q \to e$$

is called exact if moreover $q$ is faithfully flat. And in this case, $G$ is called an extension of $Q$ by $N$.

Def. (V.9.1.13) (Character Group). A character of a group scheme $G$ is a homomorphism $G \to \mathbb{G}_m$. It is easy to see that a character of $G$ is equivalent to a group-like element (I.15.2.19) in $\Gamma(G)$. The characters of $G$ form a group, denoted by $X(G)$.

Moreover, let $k^s/k$ be a separable closure, then the character group of $G_{k^s}$ is denoted by $X^*(G)$.

In particular, if $G$ is an algebraic group scheme over a field, then the set of characters of $G$ are linearly independent, by (I.15.2.20).

Prop. (V.9.1.14) (Sheaf of Differentials Parallel). If $f : G \to S$ is a group scheme over $S$, then there exists a canonical isomorphism

$$\Omega_{G/S} \cong f^* \mathcal{C}_{G/S} \cong f^* e^* \Omega.$$

In particular, if $S$ is a field, then $\Omega_{G/S}$ is a free $\mathcal{O}_G$-module.
Proof: By base change, \( \Omega_{G \otimes S G/S} = \pi_0^* \Omega_{G/S} \), and the transition map
\[
\tau : G \otimes_S G \to G \otimes_S G : (g, h) \mapsto (m(g, h), h)
\]
is an automorphism of \( G \otimes_S G \) over \( G \), so there is an isomorphism
\[
\tau^* \pi_0^* \Omega_{G/S} \cong \pi_0^* \Omega_{G/S}
\]
but \( \pi_0 \circ \tau = m \), showing this isomorphism is \( m^* \Omega_{G/S} \cong \pi_0^* \Omega_{G/S} \). Now pulling this isomorphism along the isomorphism by \( (e \circ f, \text{id}) : G \to G \otimes_S G \) we obtain the isomorphism
\[
\Omega_{G/S} \cong f^* e^* \Omega_{G/S}.
\]
Finally \( e^* \Omega_{G/S} \cong \mathcal{C}_{S/G} \) by (V.3.4.14). \( \square \)

**Prop. (V.9.1.15).** Any group scheme over a field of char 0 is reduced.

*Proof:* Cf. [Sta] 047O. \( \square \)

**Prop. (V.9.1.16).** Let \( G \) be a group scheme over a field \( k \) and \( K/k \) is a Galois extension with Galois group \( \Gamma \). If \( H' \) is a subgroup of \( G \otimes_k K \), then \( H' \) is stable under the action of \( \Gamma \) iff there exists a subgroup \( H \) of \( G \) that \( H \otimes_k K = H' \). In this case, \( H \) is unique.

*Proof:* Use (V.1.5.18) and (V.1.5.21). \( \square \)

**Classification of groups schemes of height 1 over a field**

**Quotients of Group Schemes**

**Prop. (V.9.1.17) (Grothendieck).** Let \( G \) be a group scheme of f.t. over \( S \) and \( H \) is a closed subgroup scheme of \( G \). If \( H \) is proper and flat over \( S \) and if \( G \) is quasi-projective over \( S \), then the quotient sheaf \( G/H \) in \( \text{Sh}(\text{Sh}_{fppf} \langle S \rangle) \) is representable.

*Proof:* Cf. [Grothendieck, A. Technique de descente et theoremes d’existence en geometrie algebrique, III]. \( \square \)

**2 Finite Locally Free Group Schemes**

**Def. (V.9.2.1) (Finite Locally Free Group Scheme).** A finite Locally Free group scheme is a group scheme that is finite locally free (V.5.8.8). A group scheme \( G \) over \( S \) is said to have order \( d \) if \( G \) is finite locally free of rank \( d \) over \( S \).

**Prop. (V.9.2.2) (Cartier).** Let \( k \) be a field of characteristic 0, then
- Every finite locally free affine commutative group scheme over \( k \) is finite étale.
- If \( k \) is alg.closed, then there is an equivalence of categories between finite locally free affine commutative group schemes over \( k \) and \( \text{Ab} \), by \( G \mapsto G(k) \) and \( \Gamma \mapsto \Gamma_k \).

*Proof:* 1: by (I.15.2.22) and (I.7.7.20).

2: \( \Gamma_k \) is clearly affine commutative group schemes over \( k \). If \( A \) is finite locally free Hopf algebra over \( k \), then it is finite étale by item 1 and isomorphic to a fintie product of copies of \( k \). Now the equivalence is clear. \( \square \)
Prop. (V.9.2.3) (Finite Group Schemes of Order Invertible in $S$ is Finite Étale). A finite group scheme $G$ over $S$ of order invertible in $S$ is a finite étale group scheme.

Proof: Cf.[Jakob, P45]. □

Prop. (V.9.2.4) (Finite Étale Group Schemes). Let $X$ be a connected smooth scheme with a geometric point $x$, then there is an equivalence of categories:

\[ \{\text{Finite étale group schemes over } X\} \leftrightarrow \{\text{Finite groups with a continuous action of } \pi_1(X, x)\} \]

by (VI.2.2.6). In particular, constant group schemes correspond to finite groups with trivial $\pi_1(X, x)$ actions.

Cor. (V.9.2.5). The category of finite étale commutative group schemes over $X$ is Abelian.

The category of commutative group schemes over $X$ of order invertible in $X$ is Abelian, by (V.9.2.3).

Commutative Finite Locally Free Group Schemes

Prop. (V.9.2.6) (Cartier Duality). Let $G$ be a finite commutative locally free group scheme over $S$, then $\mathcal{O}_G$ is a finite locally free $\mathcal{O}_S$-Hopf algebra, then $\text{Hom}_{\mathcal{O}_S}(\mathcal{O}_G, \mathcal{O}_S) = \mathcal{O}_G^\vee$ is again a finite locally free $\mathcal{O}_S$-Hopf algebra (V.2.5.2), and thus $\text{Spec } \mathcal{O}_G$ is a finite locally free group scheme over $S$, called the Cartier dual $G^D$ of $G$.

Moreover, this Cartier dual of $G$ represents the Hom sheaf $\mathcal{H}om_{\text{Sch}/S}(G, \mathbb{G}_m)$.

If $G$ is dual to $G'$, then their base changes are dual, too.

Finally, $(G^D)^D = G$ by (V.2.5.2), so Cartier duality is a contravariant autoequivalence of the category of commutative finite locally free group schemes over $S$.

Proof: For the Hom sheaf, we need to show

\[ G^D(T) \cong \text{Hom}_T(G \otimes_S T, \mathbb{G}_m, T) \]

for any $T/S$. Notice a $g \in G^D(T)$ corresponds to an $R$-algebra morphism $g \in \text{Hom}_{\mathcal{O}_S}(\mathcal{O}_G^\vee \to \mathcal{O}_T) = \mathcal{O}_G \otimes_{\mathcal{O}_S} \mathcal{O}_T$ (V.2.5.2) that satisfies

\[ (\Delta \otimes \text{id}_T)(g) = g \otimes_T g \in (\mathcal{O}_G \otimes_{\mathcal{O}_S} \mathcal{O}_G) \otimes_{\mathcal{O}_S} \mathcal{O}_T, \quad (\varepsilon \otimes \text{id}_T)(g) = 1 \]

Also, $g$ is a unit, as

\[ g \cdot (\iota \times \text{id}_T)(g) = \mu \circ ((\text{id}_G \otimes \iota) \otimes \text{id}_T)(g \otimes g) = \mu \circ (\text{id}_G \otimes (\iota \times \text{id}_T)) \circ (\Delta \otimes \text{id}_T)(g) = (\eta \circ \varepsilon \otimes \text{id}_T)(\varepsilon \otimes \text{id}_T)(g) = 1 \]

so $g$ corresponds to a $\mathcal{O}_T$-Hopf algebra map

\[ \mathcal{O}_T[X, X^{-1}] \to \mathcal{O}_G \otimes_{\mathcal{O}_S} \mathcal{O}_T \]

which maps $X$ to $G$, □

Prop. (V.9.2.7) ($\Gamma$ is Cartier Dual to $D(\Gamma)$). Let $\Gamma$ be a finite commutative group and $S$ be a scheme, then $\Gamma_S$ is Cartier dual to $D(\Gamma)_S$. 

Proof: By (V.9.2.6), it suffices to show for \( S = \text{Spec} \mathbb{Z} \). Now \( \Gamma = \prod_{\gamma \in \Gamma} \mathbb{Z} \), and \( \Delta(e_\gamma) = \sum_{g' = \gamma} e_g \otimes e_{g'} \). Let \( f_\gamma \in \Gamma^\vee \) be dual to \( e_\gamma \), then

\[
\Delta(f_\gamma) = \sum_{g, g' \in \Gamma} \mu(\gamma)(e_g \otimes e_{g'}) f_g \otimes f_{g'} = f_\gamma \otimes f_\gamma.
\]

so \( \Gamma^\vee \cong \mathbb{Z}[\Gamma] \), and

\[
\Delta(f_\gamma) = \sum_{g, g' \in \Gamma} \mu(\gamma)(e_g \otimes e_{g'}) f_g \otimes f_{g'} = f_\gamma \otimes f_\gamma.
\]

\( \square \)

Cor. (V.9.2.8). \( \mathbb{Z}/n\mathbb{Z} \) is Cartier Dual to \( \mu_n \).

Prop. (V.9.2.9) \( (\alpha_p \text{ is Cartier Dual to } \alpha_p) \). Over a \( \mathbb{F}_p \)-scheme \( S \), the group scheme \( \alpha_{p, S} \) is Cartier dual to itself.

Proof: By (V.9.2.6), it suffices to show for \( S = \text{Spec} \mathbb{F}_p \). Then \( \alpha_p = \mathbb{F}_p[X]/X^p \) with

\[
\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \varepsilon(X) = 0, S(X) = -X.
\]

Let \( Y_i \in \alpha_p^\vee \) dual to \( X^i \) then

\[
Y_i Y_j = \sum_{k=0}^{p-1} \Delta(Y_i \otimes Y_j)(X^k)Y_k = \sum Y_i Y_j \Delta(X^k)Y_k = \sum Y_i Y_j \left( \sum_{a + b = k} \binom{k}{a} X^a \otimes X^b \right) Y_k = \binom{i+j}{i}
\]

Now \( \binom{i+j}{i} \) is unit, so \( \alpha_p^\vee = \mathbb{F}_p[Y]/Y^p \) where \( Y = Y_1 \), and

\[
\Delta(Y) = \sum_{a, b} \mu(\gamma)(X^a \otimes X^b) Y_a \otimes Y_b = \sum Y(X^{a+b}) = Y \otimes 1 + 1 \otimes Y,
\]

so \( \alpha_p^\vee = \alpha_p \). \( \square \)

Prop. (V.9.2.10). Over a \( \mathbb{F}_p \)-scheme \( S \), the three group schemes \( \mathbb{Z}/p\mathbb{Z}_S, \mu_{p, S}, \alpha_{p, S} \) are mutually non-isomorphic.

Proof: We may take a fiber and assume \( S = \text{Spec} K \), then \( \mathbb{Z}/p\mathbb{Z}_S \) is reduced, \( \mu_{p, S} \) is non-reduced and \( \alpha_{p, S} \) is non-reduced. Then we can look at the reducedness of the group scheme and its Cartier dual. \( \square \)

Prop. (V.9.2.11) (The Order Kills the Group). If \( G \) is a finite locally free commutative group scheme over \( S \) of constant order \( n \), then \( [n]_G : G \to G \) is 0.

Proof: Cf. [Jakob P12].? \( \square \)

Finite Locally Free Group Schemes over Henselian Local Rings

Remark (V.9.2.12). Throughout this subsubsection, let \( R \) be a Henselian local ring \((R, m)\).

Prop. (V.9.2.13).

Proof:
Cor. (V.9.2.14). Let \( R \) be a equicharacteristic Henselian local ring of characteristic \( p > 0 \), then every finite locally free group scheme over \( R \) of prime order is automatically commutative.

Proof: Cf.[Shatz, P50]. □

Cor. (V.9.2.15). Let \( R \) be a strict Henselian local ring with residue field of characteristic \( p > 0 \), then any connected finite locally free group scheme over \( R \) has order \( p^t \) for some \( t > 0 \).

Proof: Cf.[Shatz, P50]. □

Prop. (V.9.2.16) (Connected-Étale Exact Sequence). Cf.[Fintie Flat Group Schemes, Tate]Section3.7 and [Mil17b]P117.

Commutative \( p \)-Group Schemes

Cf.[Finite Flat Group Schemes, Tate]Section4.

3 Groupoid Schemes

Cf.[Sta]Cha38, 39.

4 Algebraic Groups

Affine Algebraic Groups

Prop. (V.9.4.1). Every binational homomorphism of a connected affine group varieties is an isomorphism.

Proof: Such an isomorphism induces a homomorphism \( A \to B \) of integral Hopf algebras that is an isomorphism on the fraction field, then it is an isomorphism by (I.15.2.24). □

Algebraic Groups over Fields

Prop. (V.9.4.2) (Smoothness and Geo.Reducedness). For a locally algebraic group scheme \( G \) over a field, smoothness is equivalent to geo.reducedness.

Proof: If \( G \) is smooth, then it is geo.regular thus geo.reduced. Conversely, if it is geo.reduced, then by (V.5.3.15), it has an open dense smooth locus. Now smoothness can be checked after base change to alg.closed field (I.7.2.1), but then because \( G(\overline{k}) \) acts transitively on itself, thus all the geometric points are smooth. But geometric points are dense in \( G(V.4.1.25) \), thus \( G \) is smooth. □

Cor. (V.9.4.3) (Cartier’s Theorem). A locally algebraic group scheme over a field of char 0 is smooth.

Proof: This is a consequence of (V.9.1.15) and (V.9.4.2). Alternative proof: \( \Omega_{G/k} \) is free, by (V.9.1.14), so it is smooth by (V.5.3.9). □

Prop. (V.9.4.4) (Smoothness in Characteristic \( p \)). Let \( G \) be an affine algebraic groups over a perfect field \( k \) of characteristic \( p \neq 0 \), and \( r \geq 0 \), then image of the relative Frobenius \( F^r : G \to G(p^r) \) is geo.reduced group scheme when \( r \) is sufficiently large.
Proof: To show it is a group scheme, notice \( F^r \) is a homomorphism and use homomorphism theorem (V.9.6.13). And it corresponds to

\[
\Gamma(G) \otimes_{k,F^r} k \to \Gamma(G) : a \otimes c \mapsto ca^{\text{pr}}.
\]

The image of which is just \( \Gamma(G)^{\text{pr}} \) as \( k \) is perfect. To show it is geo.reduced, we can assume \( k \) is alg.closed, then the nilradical \( N \) of \( \Gamma(G) \) is nilpotent, so some \( N^m = 0 \), and then the image is reduced for any \( p^r > m \).

\[ \square \]

Prop. (V.9.4.5) (Smoothness and Tangent Space). An algebraic group scheme \( G \) is smooth iff \( \dim_k T_e = \dim O_{G,e} \), where \( e \) is the identity element.

Proof: By homogeneity and the fact smooth locus is open, \( G \) is smooth iff it is smooth at \( e \). Now as \( e \) is a rational point, by (I.7.5.27), \( G \) is smooth at \( e \) iff it is regular at \( e \).

\[ \square \]

Prop. (V.9.4.6) (Zariski Closure). Let \( G \) be a locally algebraic subgroup over \( k \) and \( S \subset G(k) \) a closed subgroup, then there is a unique reduce closed subgroup \( H \) of \( X \) that \( H(k) = S \). Moreover, \( H \) is geo.reduced. The algebraic subgroups of \( G \) arising in this way are exactly those that \( H(k) \) is schematically dense in \( H \).

In particular, when \( k \) is sep.closed, then \( H \mapsto H(k) \) is a bijection between closed subgroups of \( G \) and closed subgroups of \( G(k) \), by (V.4.3.10).

Proof: Let \( H \) be a reduced closed subscheme of \( G \) that \( H(k) = S \), then \( S \) is dense in \( |H| \) and \( H \) is reduced, so by (V.4.3.5) and (V.4.3.8) shows \( S \) is schematically dense in \( H \) and \( H \) is geo.reduced. Therefore \( H \times H \) is reduced and thus multiplication map \( H \times H \to G \) factors through \( H \), also does inversion and unit, so \( H \) is a subgroup of \( G \).

The converse is also true.

\[ \square \]

Prop. (V.9.4.7) (Algebraic Group Scheme is Quasi-Projective). An algebraic group scheme over a field \( k \) is quasi-projective.

Proof: Cf. [Sta]0BF7.

\[ \square \]

Prop. (V.9.4.8). For a locally algebraic group scheme \( G \) over a field \( k \), its center is a closed subgp scheme of \( G \).

Proof: Cf. [Sta]0BF8.

\[ \square \]

Prop. (V.9.4.9) (Identity Component). For an algebraic group \( G \) over a field \( k \), its identity component \( G^0 \) is a closed subgroup of \( G \), and the formation of identity component commutes with field base change. In particular, \( G^0 \) is geo.connected.

Proof:

\[ \square \]

Prop. (V.9.4.10). The identity component of an algebraic group \( G \) over a field \( k \) is irreducible.

Proof: By (V.9.4.9), \( G_k \) is connected, thus it is irreducible, otherwise there are some closed point contained in two irreducible components of it. Then \( G \), as a quotient space of \( G_k \), is also irreducible.

\[ \square \]

Cor. (V.9.4.11) (Connected Algebraic Group is Irreducible). An algebraic group \( G \) over a field \( k \) is irreducible iff it is connected iff it is geo.connected, by (V.9.4.9) and (V.9.4.10).

Prop. (V.9.4.12). The identity component \( G^0 \) of an algebraic group \( G \) is a characteristic subgroup of \( G \).

Construction of Algebraic Groups

Prop. (V.9.4.13). Let \( \varphi_i : X_i \to G \) be a family of maps from geo.reduced algebraic schemes \( X_i \) to an algebraic group over a field \( k \). Then there exists a smallest algebraic subgroup \( H \) of \( G \) that all \( \varphi_i \) factors through \( H \). Moreover, \( H \) is smooth.

5 Classical Algebraic Groups

Def. (V.9.5.1) (Examples of (Affine) Classical Groups).
- the general linear group \( \text{GL}_n = \mathbb{Z}[T_{ij}][1/\det] \) representing the group functor \( \text{CAlg}_\mathbb{Z} \to \text{grp} : R \mapsto \text{GL}(n, R) \).
- the multiplicative group \( \text{G}_m = \text{Spec} \mathbb{Z}[T, T^{-1}] \), which is just \( \text{GL}_1 \).
- the special linear group \( \text{SL}_n \) is the closed subgroup scheme of \( \text{GL}_n \) defined by the ideal \( (\det - 1) \), representing the group functor \( \text{CAlg}_\mathbb{Z} \to \text{grp} : R \mapsto \text{SL}(n, R) \).
- the symplectic group \( \text{Sp}_{2n} \) is the closed subgroup scheme of \( \text{GL}_n \) defined by the ideal generated by the \( n \times n \) entries of the equation \( (T_{ij})^T J_n (T_{ij}) = J_n \), where \( J_n = \begin{bmatrix} 0 & 1_n \\ 1_n & 0 \end{bmatrix} \), representing the functor \( \text{CAlg}_\mathbb{Z} \to \text{grp} : R \mapsto \{ g \in \text{GL}(2n, R) | g^T J_n g = J_n \} \).
- the general symplectic group \( \text{GSp}_{2n} \) is the closed subgroup scheme of \( \text{GL}_{2n} \times \text{G}_m \subset \text{GL}_{2n+1} \) generated by the entries of the equation \( (T_{ij})^T J_n (T_{ij}) = T J_n \), representing the functor \( \text{CAlg}_\mathbb{Z} \to \text{grp} : R \mapsto \{(g, r) \in \text{GL}(2n, R) \times \mathbb{R}^\times | g^T J_n g = r J_n \} \).
- orthogonal groups.
- special orthogonal groups.
- unitary groups.
- special unitary groups.

Proof: By (V.9.1.3), it suffices to show \( \text{Hom}(-, G) \) is a group functor when restricted to affine schemes.

Prop. (V.9.5.2) (Amplitude Character). There is an amplitude character \( \text{GSp}_{2n} \to \text{G}_m : T \mapsto T \), which represents the natural transformation \( R \mapsto ((g, r) \mapsto r) \).

6 Group Theory Aspects

Prop. (V.9.6.1) (Extensions of Affine Algebraic Groups are Affine). Let
\[
e \to N \to G \to Q \to e
\]
be an exact sequence of algebraic groups over \( k \), if \( N, Q \) are affine, then \( G \) is also affine.


Isomorphism Theorems

Def. (V.9.6.2) (Properties of Algebraic Subgroups). For a group scheme \( G \), an subgroup \( H \) is called normal if \( H(R) \) is normal in \( G(R) \) for any \( k \)-algebra \( R \).
It is called characteristic if $H_R = \alpha(H_R)$ for all $k$-algebra $R$ and automorphism $\alpha$ of $G_R$.

Let $H, N$ be subgroups of $G$, then we say $H$ normalizes $N$ if $H(R)$ normalizes $N(R)$ for any $k$-algebra $R$.

**Prop. (V.9.6.3) (Algebraic Subgroups are Closed).** Algebraic subgroups $H$ of an algebraic group $G$ are closed subgroups. In particular, an algebraic subgroup of an affine algebraic group is affine.

**WARNING:** $H$ must first be an algebraic group, so we can use Chevalley theorem to show it is locally closed.

**Proof:** As $H_{k'} \to H$ is a quotient map (V.4.6.3), we can assume $k$ is alg.closed, and also assume $H, G$ are reduced. Now Chevalley shows that the image is a constructible set of $G$. Then we can consider all on the level of $k$ points, because $G$ is Jacobson. Then $H$ contains an open subset of $\overline{H}$ by (IX.1.14.16), which implies $H(\overline{k})$ is open in $\overline{H} \cap G(\overline{k})$. Now $\overline{H} \cap G(\overline{k})$ is the closure of $H(\overline{k})$ in $G(\overline{k})$, thus it is also a subgroup of $G(\overline{k})$, and we can consider the coset of $H(\overline{k})$ in $\overline{H} \cap G(\overline{k})$. $H$ is open in $\overline{H}$, and $\overline{H}$ is compact, and also $\overline{H} \cap G(\overline{k})$ is compact because $G$ is Jacobson, so there are only f.m. coset, thus $H(\overline{k})$ is also closed in $\overline{H} \cap G(\overline{k})$, thus $H$ is also closed in $\overline{H}$, so $H = \overline{H}$. □

**Def. (V.9.6.4) (Quotient Map).** A quotient map of algebraic groups is a group homomorphism that is faithfully flat.

**Prop. (V.9.6.5).** Let $\varphi : X \to Y$ be a quotient map of algebraic groups over $k$ (V.9.6.4), then $\varphi(\overline{X})$ is a fat subfunctor of $\overline{Y}$ (V.3.5.14).

**Proof:** For any map $\text{Spec} R \to Y$, base change of $\varphi$ along it provides a covering of elements Spec $R_i \to Y$ that factor through $X$. □

**Prop. (V.9.6.6) (Equivalent Characterizations of Quotient Maps).** The following conditions on a homomorphism $\varphi : G \to Q$ of algebraic groups are equivalent:

- $\varphi$ is a quotient map.
- $\overline{G}$ is a fat subfunctor of $\overline{Q}$.
- The homomorphism $O_Q \to \varphi_* O_G$ is injective.

**Proof:** 1 $\to$ 2 is (V.9.6.5).


**Prop. (V.9.6.7) (Quotient Map is a Cokernel).** Let $q : G \to Q$ be a quotient map of algebraic groups over $k$ and let $N$ be the kernel, then every homomorphism $G \to H$ whose kernel contains $N$ factors uniquely through $q$.

**Proof:** This is because the shifification of the functor $R \mapsto G(R)/N(R)$ is just $Q$, so we can use shifification to get a unique functor through $q$ which is also a homomorphism. □

**Def. (V.9.6.8) (Monomorphisms).** A homomorphism $\varphi : G \to H$ of algebraic groups over $k$ is called a monomorphism if it satisfies the following equivalent conditions:

- $G(R) \to H(R)$ is injective for any $k$-algebra $R$.
- $\text{Ker}(\varphi) = e$ (V.9.1.11).
- $\varphi$ is a monomorphism in the category of algebraic groups over $k$. 
• $\varphi$ is a monomorphism in the category of algebraic schemes over $k$.

**Proof:**  $1 \rightarrow 4 \rightarrow 3$ is obvious.

$3 \rightarrow 2$: the composition of $\text{Ker}(\varphi) \rightarrow G$ with $\varphi$ is trivial, thus $\text{Ker}(\varphi)$ is trivial.

$2 \iff 1$: This follows from the definition of $\text{Ker}(\varphi)$ (V.9.1.11). □

**Def. (V.9.6.9) (Embedding).** An embedding of algebraic groups is a closed immersion of algebraic groups.

**Lemma (V.9.6.10) (Monomorphism is an Embedding).** A monomorphism of algebraic groups over $k$ is just a closed embedding.

**Proof:** Let $\varphi : H \rightarrow G$ be a monomorphism, and let $X$ be the quotient space (V.9.8.13), then $H$ is the fiber of $G \rightarrow X$ at $o$, thus it is a closed subscheme of $X$. □

**Cor. (V.9.6.11) (Quotient by Normal Subgroups).** Every normal algebraic group $N$ of an algebraic group $G$ arises as the kernel of a quotient map of algebraic groups $G \rightarrow Q$.

**Proof:** Cf. [Mil17]P105. This is a corollary of (V.9.8.14). □

**Prop. (V.9.6.12).** The quotient $G/H$ is affine if $G$ is affine and $H$ is normal.

**Proof:** C,[Mil17]P105. □

**Thm. (V.9.6.13) (Homomorphism Theorem).** Every homomorphism of affine algebraic groups $G \rightarrow H$ factors as

$$G \xrightarrow{q} I \xrightarrow{i} H$$

where $q$ is a f.f. and $i$ is a closed immersion.

**Proof:** Clear, the point is $q$ is f.f., by (I.15.2.23). □

**Cor. (V.9.6.14).** Commutative algebraic groups over $k$ form an Abelian category. And the set of affine commutative algebraic groups over $k$ form a Serre subcategory of this category, by (V.9.6.12)(V.9.6.3) and (V.9.6.1),

**Cor. (V.9.6.15) (Kernel and Closed Immersion).** A homomorphism $\varphi : G \rightarrow H$ of affine algebraic groups with $\text{Ker}(\varphi) = e$ is a closed immersion.

**Proof:** Factor $\varphi$ as by (V.9.6.13), then we can assume $\varphi$ is f.f., then $\varphi(R) : G(R) \rightarrow H(R)$ is injective for any $R$, and it suffices to show it is also surjective.

Let $a \in H(R)$, then it base change to an element $b \in G(R')$ where $R' = R \otimes R(\hat{H}) \Gamma(G)$ is f.f. over $R$. As $R \rightarrow R'$ is f.f., there are diagrams

$$
\begin{array}{ccc}
G(R) & \longrightarrow & G(R') \\
\downarrow & & \downarrow \\
H(R) & \longrightarrow & H(R')
\end{array}
\quad
\begin{array}{ccc}
G(R') & \longrightarrow & G(R' \otimes R R') \\
\downarrow & & \downarrow \\
H(R') & \longrightarrow & H(R' \otimes R R')
\end{array}
$$

where the rows are exact because the fpqc site are canonical (V.1.3.21), so by diagram chasing $a$ is in the image of $G(R) \rightarrow H(R)$, so we are done. □
Prop. (V.9.6.16) (Isomorphism Theorem). Let $H, N$ be algebraic subgroups of an algebraic group $G$ such that $H$ normalizes $N$, then $H \cap N$ is a normal algebraic subgroup of $H$, and the natural map

$$H/H \cap N \rightarrow HN/N$$

is an isomorphism.

Proof: Cf.[Mil17]P112. □

Prop. (V.9.6.17) (Correspondence Theorem). Let $N$ be a normal algebraic subgroup of an algebraic group $G$, then the map $H \mapsto H \triangleleft N$ is a bijection from the set of algebraic subgroups of $G$ containing $N$ to the set of algebraic subgroups of $G/N$. An algebraic group of $G$ containing $N$ is normal in $G$ iff $H \triangleleft N$ is normal in $G/N$, in which case the natural map $G/H \rightarrow (G/N)/(H/N)$ is an isomorphism.


Def. (V.9.6.18) (Simply Connected Groups). A simply connected Algebraic group is a connected algebraic group of $G$ of characteristic 0 that every isogeny (surjective homomorphism with finite kernel) is an isomorphism.

Prop. (V.9.6.19). Every semisimple algebraic group admits an essentially unique isogeny $\tilde{G} \rightarrow G$ that $\tilde{G}$ is simply connected.

Proof: □

Subnormal Series

Def. (V.9.6.20). An algebraic group is called perfect if it equals its derived group.

Def. (V.9.6.21) (Isogeny). A homomorphism of algebraic groups $G \rightarrow H$ is called an isogeny if its kernel is a finite group scheme and its image is of finite index in $H$.

7 Representations of Algebraic Groups

Throughout this subsection, $G$ is an affine algebraic group over a field $k$.

Def. (V.9.7.1) (Linear Representations of Algebraic Groups). A linear representation of an algebraic group $G$ over $k$ is a homomorphism $r : G \rightarrow GL(V)$ for some f.d. $k$-vector space $V$. It is called faithful if $r(R) : G(R) \rightarrow GL(V_R)$ is injective for all $k$-algebras $R$. By(V.9.6.15), this is equivalent to $G \rightarrow GL(V)$ is a closed immersion.

Prop. (V.9.7.2) (Representations and Co-modules). A representation of $G$ on $V$ is equivalent to a right $\Gamma(G)$-comodule structure on $V$(I.15.1.10):

Let $A = \Gamma(G)$. For any representation $r : G \rightarrow GL(V)$, it induces a map $G(A) \rightarrow GL_A(V \otimes A)$, which maps $\text{id}_A$ to a map $p : V \rightarrow V \otimes A$. Now if $\rho(e_j) = \sum e_i \otimes a_{ij}$, then by functoriality, the map $G(R) \rightarrow GL_R(V \otimes R)$ is given by $g \mapsto (e_j \mapsto e_i \otimes a_{ij}(g))$.

And it can be verified that this is a group homomorphism iff the comodule condition is satisfied.

Def. (V.9.7.3) (Stabilizer). Let $r$ be an action of $G$ on $V$, $W$ a subspace of $V$, then the functor

$$R \mapsto G_W(R) = \{g \in G(R)| g(W_R) = W_R\}$$

is representable by a subgroup of $G$, called the stabilizer $\text{Stab}_G(W)$ of $W$ in $V$. 

Proof: Let $\rho : \Gamma(G) \to V \otimes \Gamma(G)$ be the comodule action corresponding to $r$, let $\{e_i\}_{i \in I}$ be a basis for $W$, and extends it to a basis $\{e_i\}_{i \in I}$ of $V$, and

$$\rho(e_j) = \sum e_i \otimes a_{ij}, \quad a_{ij} \in \Gamma(G).$$

Let $g \in \text{Hom}(\Gamma(G), R)$, then $ge_j = \sum e_i \otimes g(a_{ij})$, and then clearly $G_W$ is represented by the quotient of $\Gamma(G)$ by the ideal generated by $\{a_{ij} | i \in I, j \in J\}$. □

Cor. (V.9.7.4). Let $(\rho, V)$ be a f.d. representation of $G$ and $S \subset G(k)$ be a subset schematically dense in $G$, then a subspace $W \subset V$ is stable under $G$ iff it is stable under $S$.

Prop. (V.9.7.5) (Constructing All F.D. Representations). Let $V$ be a faithful f.d. representation of $G$, then every f.d. representation of $G$ is a subquotient of $V^m \otimes (V^\vee)^n$.


Def. (V.9.7.6) (Diagonalizable Representation). A representation of an algebraic group is called diagonalizable if it is a direct sum of 1-dimensional representations.

Let $G$ be an algebraic group over a field $k$ and $r : G \to GL(V)$ be a representation. If $V$ is a sum of 1-dimensional representations, then $r$ is diagonalizable(V.9.7.6).

Proof: Let $V = \sum_{\chi \in \chi(G)} V_\chi$. If the sum is not direct, then there is some relation $v_1 + v_2 + \ldots + v_m = 0$ for $v_i \in V_\chi$. Applying $\rho$ shows

$$0 = v_1 \otimes a(\chi_1) + \ldots + v_m a(\chi_m)$$

so any coordinate of $v_i$ is 0 by(V.9.1.13). □

Prop. (V.9.7.7) (Chevalley). Let $G$ be an algebraic group, then every algebraic subgroup $H \subset G$ arises as the stabilizer of a 1-dimensional subspace in a f.d. representation of $G$.

Proof: Cf.[Mil17] P94. □

Linear Algebraic Group

Def. (V.9.7.8) (Linear Algebraic Groups). Let $k$ be a field, then a linear algebraic group over $k$ is a closed subgroup scheme of $GL_n$ for some $n$.

Notice a linear algebraic group over a field of characteristic 0 is automatically smooth, by Cartier theorem(V.9.4.3).

Prop. (V.9.7.9) (Affine Algebraic Group is Linear). If $G$ is an affine group scheme, then the regular representation(V.9.8.2) contains a faithful f.d. subrepresentation. In particular, the regular representation is itself faithful.

Proof: Let $e_i$ be a generator of $\Gamma(G)$ as a $k$-algebra, let $V$ be a f.d. subrepresentation of the regular representation containing $e_i$, let $v_i$ be a basis for $V$, and suppose $\Delta(e_j) = \sum e_i \otimes a_{ij}$, then the image of $\Gamma(GL(V)) \to \Gamma(G)$ contains $a_{ij}$. Now because $\varepsilon : A \to k$ is the counit,

$$e_j = (\varepsilon \otimes \text{id})\Delta(e_j) = \sum \varepsilon(e_i)a_{ij},$$

so the image contains $V$, so it contains $\Gamma(G)$, so this is a closed immersion, thus a faithful representation, by(V.9.7.1). □

Cor. (V.9.7.10) (Affine Group Schemes are Linear). Every affine algebraic group is linear.
8 Group Actions

In this subsection, all schemes are algebraic over a field $k$, and all functors are from the small category of $k$-algebras to Grps.

**Def. (V.9.8.1).** Group action of an algebraic group on an algebraic scheme is defined in (II.1.1.47).

**Prop. (V.9.8.2) (Group Action).** An action of an algebraic group $G$ on an algebraic group $X$ is equivalent to a right $\Gamma(G)$-comodule structure on $\Gamma(X)$ as $\Gamma(G)$-modules. This action will induce a right comodule structure on $\Gamma(X)$.

The action of $G$ on itself is called the regular representation of $G$.

**Prop. (V.9.8.3).** Let $\mu : G \times X \to X$ be an action of an algebraic group $G$ on a scheme $X$, then it is faithfully flat, and it is smooth/finite/... if $G$ is smooth/finite/....

**Proof:** We can see this from the commutative diagram (II.1.1.48).

**Prop. (V.9.8.4) (Image of Equivariant Map).** Let $G$ be a group functor and $X, Y$ be algebraic schemes on which $G$ acts, and $f : X \to Y$ is an equivariant map.

- If $Y$ is reduced and $G(\bar{k})$ acts transitively on $X(\bar{k})$, then $f$ is faithfully flat.
- If $G(\bar{k})$ acts transitively on $X(\bar{k})$, then the set $f(|X|)$ is locally closed in $|Y|$, so we can let $f(X)_{\text{red}}$ denote its reduced subscheme structure (V.4.1.13).
- If $X$ is reduced and $G(\bar{k})$ acts transitively on $X(\bar{k})$, then $f$ factors into

$$X \xrightarrow{\text{faithfully flat}} f(X)_{\text{red}} \xrightarrow{\text{immersion}} Y.$$  

Moreover, $f(X)_{\text{red}}$ is stable under the action of $G$.

**Proof:** 1: 

2: 

3: $f$ factors through $f(X)_{\text{red}}$ because $X$ is reduced (V.4.1.13). Then the first assertion follows from 1 and 2. The last assertion follows from universal property again.

**Def. (V.9.8.5) (Orbit Map).** Let $\mu : G \times X \to X$ be an action of an algebraic group $G$ on an algebraic scheme $X$. For any $x \in X(k)$, the orbit map

$$\mu_x : G \to X : g \mapsto gx$$

is defined to be the restriction to $\mu$ to $G \times \{x\} \cong G$. The image of the orbit map is locally closed in $X$ by (V.9.8.4), and then its reduced structure subscheme is called the orbit $O_x$ of $x$.

**Prop. (V.9.8.6) (Fixed Subscheme).** Let $\mu : G \times X \to X$ be an action of a group functor $G$ on a separated algebraic scheme over $k$, then the functor

$$\bar{X}^G : R \mapsto \{x \in X(R) | \mu(g, x_{R'}) = x_{R'}, \forall R - \text{algebra} \ R', g \in G(R')\}$$

is representable by a closed subscheme $X^G$ of $X$, called the fixed subscheme of this action. Then it can be seen directly that the formation of fixed subscheme commutes with extension of base fields.
Proof: An element \( x \in X(R) \) defines two functors
\[
G(R') \to X(R') : g \mapsto gx_{R'}
\]
which are both natural in \( R' \). Thus we get a map \( X(R) \to \text{Hom}(G_R, X_R \times X_R) \) which is also natural in \( R \), thus induce a map \( X \mapsto \text{Hom}(G, X \times X) \).

Then there is a Cartesian diagram
\[
\begin{array}{ccc}
\tilde{X}^G & \longrightarrow & \text{Hom}(G, \Delta_X) \\
\downarrow & & \downarrow \text{closed} \\
\tilde{X} & \longrightarrow & \text{Hom}(G, X \times X)
\end{array}
\]

The right vertical map is a closed subfunctor by (V.3.5.13), as \( \Delta_X \) is closed in \( X \times X \) because \( X \) is separated. Hence \( \tilde{X}^G \) is a closed subfunctor of \( \tilde{X} \), thus represented by a closed subscheme of \( X \), by (V.3.5.10).

\[\square\]

Def. (V.9.8.7) (Isotropy Group). Let \( G \) be an algebraic group acting on an algebraic scheme \( X \), and \( x \in X(k) \), then the \textit{isotropy group} \( G_x \) is defined to be the fiber of the orbit map \( \mu_x : G \to X \) over \( x \).

Prop. (V.9.8.8). Let \( \mu : G \times X \to X \) be an action of an algebraic group \( G \) on an algebraic scheme \( X \) and \( x \in X(k) \).

- If \( G \) is reduced and \( G(k) \) acts transitively on \( X(k) \), then the orbit map \( \mu_x : G \to X \) is faithfully flat.

- If \( G \) is smooth, then \( O_x \) is stable under \( G \), and the map \( \mu_x : G \to O_x \) is faithfully flat. If \( G \) is smooth, then \( O_x \) is smooth.

Proof: 1: this follows from (V.9.8.4).

2: The first statement follows from (V.9.8.4)2, and then \( O_{O_x} \to \mu_x_* (O_G) \) is universally injective. Therefore if \( G \) is smooth, then \( O_{O_x} \) is geometrically reduced.

\[\square\]

Prop. (V.9.8.9). Let \( \mu : G \times X \to X \) be an action of a smooth algebraic group on an algebraic scheme.

- A reduced closed subscheme \( Y \) of \( X \) is stable under \( G \) iff \( Y(k) \) is stable under \( G(k) \).

- Let \( Y \) be a locally closed subscheme of \( X \), then if \( Y \) is stable under \( G \), then \( (Y)_{\text{red}} \) and \( (Y \setminus Y)_{\text{red}} \) is also stable under \( G \).

Proof: 1: Because \( G \) is geo.reduced and \( Y \) is reduced, \( G \times Y \) is reduced (V.4.3.2), thus \( \mu : G \times Y \to X \) factors through \( Y \) iff \( \mu(k) \) factors through \( Y(k) \).

2: \( Y_{\text{red}}(k) \) is the closure of \( Y(k) \) in \( X(k) \). So as \( G(k) \) acts continuously on \( X(k) \), if it fixes \( Y(k) \), then it also fixes \( (Y)(k) \) and \( (Y \setminus Y)(k) \), thus we finish by 1.

\[\square\]

Cor. (V.9.8.10). Let \( G \) be a smooth algebraic group acting on an algebraic scheme \( X \) and let \( Y \) be a non-empty locally closed subscheme of \( X \) stable under the action of \( G \) of the smallest dimension, then it is closed.
Proof: This is because \( \dim Y > \dim(\overline{\mathbb{Y}} \setminus Y)_{\text{red}} \) (Because irreducible components of \( \overline{\mathbb{Y}} \) is not contained in \( \overline{\mathbb{Y}} \setminus Y \)).

Cor. (V.9.8.11). Let \( G \) be a group variety acting on a variety over an alg.closed field \( k \), then every orbit of minimal dimension is closed.

Proof: This is because in this case every locally closed subscheme of \( X \) stable under the action of \( G \) of minimal dimension is an orbit.

Def. (V.9.8.12) (Homogenous Space). A non-empty algebraic scheme \( X \) with an action of \( G \) is called a homogenous space for \( G \) if for any \( x \in G(\bar{k}) \), \( \mu_x \) is faithfully flat (thus surjective).

Quotients by Algebraic Groups

Def. (V.9.8.13) (Quotient Space). Let \( i : H \to G \) be a monomorphism of algebraic groups over a field \( k \) (V.9.6.8), then a quotient space of \( G \) by \( H \) is an algebraic scheme together with an action \( G \times X \to X \) of \( G \) and a rational point \( o \in X(k) \) that the orbit map \( \mu_o : G \to X \) realizes the functor \( \bar{\mathcal{X}} \) as a fat subfunctor of \( \bar{\mathcal{X}} \) (V.3.5.14).

Prop. (V.9.8.14) (Existence of Quotient Space). Let \( H \to G \) be a monomorphism of algebraic groups over a field \( k \), then the quotient space of \( G \) by \( H \) exists. And the orbit map \( \mu_o \) is faithfully flat.

Proof: Cf. [Mil17b] P605.

Prop. (V.9.8.15). A representation \((V, \rho)\) of an algebraic group \( G \) induces an action of \( G \) on the affine algebraic scheme \( V_\alpha \), and also an action of \( G \) on the projective algebraic scheme \( \mathbb{P}(V) \).

Prop. (V.9.8.16). Let \( G \times X \to X \) be an action of an affine algebraic group \( G \) on an affine algebraic scheme \( X \) over \( k \), then there exists a f.d. representation \((V, \rho)\) of \( G \) and an equivariant closed embedding \( X \hookrightarrow V_\alpha \).


Def. (V.9.8.17) (Linear Action). An action of an affine algebraic group \( G \) on an algebraic scheme \( X \) is called linear if there exists a f.d. representation \((V, \rho)\) of \( G \) and an equivariant immersion \( X \hookrightarrow \mathbb{P}(V) \).

Prop. (V.9.8.18). If \( G \times X \to X \) is a transitive action of an affine algebraic group \( G \) on an algebraic variety \( X \) that \( X(k) \) is non-empty, then this action is linear.


Def. (V.9.8.19) (Grassmannian Variety). The Grassmannian variety \( G(n, k) \) is defined to be the quotient of \( GL_n \) by the algebraic subgroup \( B \) fixing a subspace of dimension \( k \) (V.9.8.13). It represents the functor that maps a scheme \( S \) to the set of direct sums of \( \mathcal{O}_S^g \) of dimension \( k \), because this is already a fpfp sheaf.

Prop. (V.9.8.20). The Grassmannian variety is projective.

Proof: ?
The flag variety is defined to be the quotient of \( GL_n \) by the algebraic subgroup \( B_F \) fixing a flag \( F \). It represents the functor that maps a scheme \( S \) to the set of flags in \( O_S^n \), because this is already a fppf sheaf.

The flag variety is projective, because the functor it represents can be embedded to a product of Grassmannian varieties.

Let \( F \) be a local field and \( X \) is an algebraic variety over \( F \), then the \( F \)-topology makes \( X(F) \) into a locally profinite space (because varieties are closed, and use (IX.1.4.6)). Let \( G \) be a linear algebraic group over \( F \) and \( G \times X \to X \) is a \( F \)-rational map, then \( G(F) \times X(F) \to X(F) \) is a continuous action, and this action is constructible (IX.1.12.21).

In this subsection, let \( G \) be an affine group scheme, and \( k \) is a field.

Let \( G \) be an algebraic group over \( k \) and \( R \) is a \( k \)-algebra. Suppose that for any f.d. representation \( (V, r_V) \) of \( G \), we are given an \( R \)-linear map \( \lambda_R : V_R \to V_R \) that satisfies:

- \( \lambda_V \circ \omega = \lambda_V \otimes \lambda_W \).
- \( \lambda_1 = \text{id} \).

For \( G \)-invariant maps \( u : V \to W \), \( \lambda_W \circ u_R = u_R \circ \lambda_V \).

then there exists a unique \( g \in G(R) \) that \( \lambda_V = r_V(g) \) for any \( V \).

Cor. (V.9.9.3) (Reconstruction Theorem). Let \( \omega : \text{Rep}_k(G) \to \text{Vect}_k \) be the forgetful functor, and for any \( k \)-algebra \( R \), let \( \omega_R = R \otimes \omega \), then this proposition says the canonical morphism \( G(R) \to \text{End}^\otimes(\omega_R) \) is an isomorphism. Now if \( \text{Aut}^\otimes(\omega) \) is the functor \( R \mapsto \text{End}^\otimes(\omega_R) \), then \( G \cong \text{Aut}^\otimes(\omega) \).

Cor. (V.9.9.4). Let \( G, G' \) be affine algebraic groups over \( k \) and let \( F : \text{Rep}_k(G') \to \text{Rep}_k(G) \) be a tensor functor that \( \omega^G \circ F = \omega^{G'} \), then there exists a unique homomorphism \( f : G \to G' \) that \( F = \omega^f \).
Proof: Such a tensor functor defines a homomorphism
\[ F^* : \text{Aut}^{\otimes}(\omega^G)(R) \to \text{Aut}^{\otimes}(\omega^{G'})(R) \]
functorial in \( R \), thus defines a homomorphism \( f : G \to G' \) by Yoneda lemma and (V.9.9.4).

Lemma (V.9.9.5). Let \( \mu \) be a cocharacter \( \mathbb{G}_m \to G \) over \( \overline{\mathbb{Q}_p} \), then the conjugacy class \( \{ \mu \} \) of \( \mu \) is defined over a finite extension \( E/\mathbb{Q}_p \).

Proof: By Tannakian duality, two cocharacters are conjugate over a field \( K \) iff the filtrations they defined on \( V \otimes \overline{\mathbb{Q}_p} \) for some faithful \( V \in \text{Rep}_{\mathbb{Q}_p}(G) \) are isomorphic by an action of \( G(K) \). Now this action of \( G(\overline{\mathbb{Q}_p}) \) on \( V_{\overline{\mathbb{Q}_p}} \) is defined over a f.d. field extension \( E/\mathbb{Q}_p \), so the filtrations are isomorphic by an action of \( G(E) \).

Prop. (V.9.9.6) (Jordan Decomposition). Let \( G \) be an algebraic group over a perfect field \( k \) and \( g \in G(k) \), then there exists unique elements \( g_s, g_u \in G(k) \) that for any representation \( (V, r_V) \) of \( G \), \( r_V(g_s) = r_V(g)_s \) and \( r_V(g_u) = r_V(g)_u \). Furthermore,
\[ g = g_sg_u = g_ug_s. \]
The elements \( g_s, g_u \) are called the semisimple and unipotent part of \( g \), and this decomposition is called the Jordan decomposition of \( g \). \( g \in G(k) \) is called semisimple or unipotent if \( g = g_s \) or \( g = g_u \).

Proof: This follows from the functoriality of Jordan decompositions (I.1.6.7) and the reconstruction theorem (V.9.9.2).

Cor. (V.9.9.7). To check a decomposition is Jordan decomposition, it suffices to check for a single faithful representation of \( G \).

Remark (V.9.9.8). Let \( G \) be a group variety over an alg.closed field \( k \). In general, the set \( G(k)_s \) of semisimple elements in \( G(k) \) is not closed for the Zariski topology, but the set \( G(k)_u \) of unipotent elements are closed for the Zariski topology. To see this, embed \( G \) into \( GL_n \) for some \( n \), then the set of unipotent elements are the matrices with characteristic polynomial \((T - 1)^{n-1}\), and this is a polynomial condition.

Tannakian Reconstruction

Prop. (V.9.9.9) (Tannakian Reconstruction). Let \( (\mathcal{C}, \otimes) \) be a rigid Abelian tensor category that \( k = \text{End}(1) \) and \( \omega : \mathcal{C} \to \text{Vect}_k \) an exact faithful \( k \)-linear tensor functor, then the functor \( \text{Aut}^{\otimes}(\omega) \) of \( k \)-algebras is representable by an affine groups scheme \( G \), and \( \mathcal{C} \cong \text{Rep}_k(G) \).

Proof: Cf.[Milne, Tannakian category, P21].

Cor. (V.9.9.10) (Tannakian Reconstruction). Let \( \mathcal{C} \) be a \( k \)-linear Abelian category where \( k \) is a field, and \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) a \( k \)-bilinear functor. Suppose there are given a faithful exact \( k \)-linear functor \( \mathcal{C} \to \text{Vect}_k \) and functorial isomorphisms \( \varphi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z \) and \( \psi_{X,Y} : X \otimes Y \to Y \otimes X \) that
- \( F \) commutes with \( \otimes \), and maps \( \varphi \) and \( \psi \) to the natural associativity and commutativity isomorphism in \( \text{Vect}_k \).
- There exists an identity object \( 1 \in \mathcal{C} \) that \( k \to \text{End}(1) \) is an isomorphism and \( F(1) \) has dimension 1.
• Any object $L \in \mathcal{C}$ that $F(L)$ has dimension 1 is an invertible object.

Then $\mathcal{C}$ is equivalent to $\text{Rep}_k(G)$ for some affine group scheme $G$ over $k$. In fact, $G \cong \text{Aut}^\otimes(\omega)$ as in (V.9.9.3)

**Proof:** The proof of (V.9.9.9) shows $F$ defines an equivalence of categories $\mathcal{C} \to \text{Rep}_k(G)$ where $G$ is an affine monoid scheme representing $\text{End}_k^\otimes(\omega)$. Thus we may assume $\mathcal{C} = \text{Rep}_k(G)$. For the rest, Cf. [Tannakian Categories, Milne, P24].

**Cor. (V.9.9.11) (Real Algebraic Envelope).** Let $K$ be a topological group, then the category $\text{Rep}_\mathbb{R}(K)$ of f.d. continuous real representations, together with the forgetful functor satisfies the hypothesis of (V.9.9.10), thus there is an algebraic group $\tilde{K}$ over $\mathbb{R}$, called the real algebraic envelope of $K$, and an equivalence of categories

$$\text{Rep}_\mathbb{R}(\tilde{K}) \to \text{Rep}_\mathbb{R}(K)$$

induced by a homomorphism $K \to \tilde{K}(R)$, which is an isomorphism when $K$ is compact.

**Cor. (V.9.9.12) (Hochschild-Mostow Group).** Similar as (V.9.9.11), if $G$ is a complex Lie group or a f.g. abstract group, and $\mathcal{C}$ the category of f.d. complex representations, then it satisfies the hypothesis of (V.9.9.10), thus it is the category of representations of an affine group scheme $A(G)$ over $\mathbb{C}$, together with a homomorphism $P : G \to A(G)$, called the Hochschild-Mostow group of $G$.

**Prop. (V.9.9.13).** Let $\mathcal{C}$ be a small $k$-linear Abelian category, and let $\omega : \mathcal{C} \to \text{Vect}_k$ be an exact faithful $k$-linear functor, then there exists a coalgebra $C$ s.t. $\mathcal{C}$ is equivalent to the category of $C$-comodules of f.d.

**Proof:** Cf. [Mil17] P175.

**Properties of $G$ and $\text{Rep}_k(G)$**

**Prop. (V.9.9.14).** Let $G$ be an affine group scheme over $k$, then

- $G$ is finite iff there exists an object $X \in \text{Rep}_k(G)$ that every object of $\text{Rep}_k(G)$ is isomorphic to a subquotient of $X^n$ for some $n > 0$.

- $G$ is algebraic iff there exists an object $X \in \text{Rep}_k(G)$ that every object of $\text{Rep}_k(G)$ is isomorphic to a subquotient of $X^n \otimes (X')^m$ for some $m, n \geq 0$.

**Proof:** Cf. [Milne, Tannakian categories, P25].

**Prop. (V.9.9.15).** Let $f : G \to G'$ be a homomorphism of affine group schemes over $k$, and let $\omega^f$ be the corresponding functor $\text{Rep}_k(G') \to \text{Rep}_k(G)$. Then

- $f$ is faithfully flat iff $\omega^f$ is fully faithful and each $\omega^f$ induces an equivalence of subobjects of $X'$ and $\omega^f(X')$.

- $f$ is a closed immersion iff every object of $\text{Rep}_k(G)$ is isomorphic to a subquotient of an object $\omega^f(X')$.

**Proof:** Cf. [Milne, Tannakian categories, P25].

**Cor. (V.9.9.16).** Let $k$ has characteristic 0, then $G$ is connected iff for any non-trivial representation $X$ of $G$, $\langle X \rangle$ is not stable under $\otimes$.

**Proof:** Cf. [Milne, Tannakian categories, P25].
10 Lie Algebras of Algebraic Groups

In this subsection, all algebraic groups $G$ is affine over a field $k$.

Def. (V.9.10.1) (Lie Algebra of an Algebraic Group). Let $k$ be a field and $G$ an algebraic group, then the tangent space at the unit element $e \subset G$, as it is defined, is

$$L(G) = \ker(G(k[\varepsilon]) \to G(k)), \varepsilon^2 = 0.$$  

In particular, it is the set of homomorphisms $\Gamma(G) \to k[\varepsilon]$ that the composition with $k[\varepsilon] \to k$ is the counit map $\varepsilon : \mathcal{O}(G) \to k(V.9.1.2)$. In particular, $\varphi$ maps the augmentation ideal $I_G = \ker(\varepsilon)$ into $(\varepsilon)$, and thus is trivial on $I_G^2$. So $\varphi$ factors through $\Gamma(G)/I_G^2 \cong k \oplus I_G/I_G^2$ by (I.15.2.11), so

$$L(G) \cong \text{Hom}_k(I_G/I_G^2, k).$$

And we define $\text{Lie}(G)$ to be $L(G)$.

for any homomorphism $f : G \to H$, there is a Lie algebra map

$$\text{Lie}(f) : \text{Lie}(G) \to \text{Lie}(H)$$

induced by $f$.

In general, if $R$ is any $k$-algebra, then we define $g(R) = \ker(G(R[\varepsilon]) \to G(R)$, then similarly

$$g(R) = \text{Hom}_R(I_R/I_R^2, R) = \text{Hom}_k(I_G/I_G^2, k) \otimes R = g \otimes R.$$  

Now $G(R[\varepsilon])$ acts on $g(R)$ by inner automorphism, so also does $G(R)$. So we get a homomorphism of algebraic groups

$$\text{Ad} : G \to GL_g.$$  

This homomorphism commutes with Lie algebra homomorphism: If $f : G \to H$ is a homomorphism of algebraic groups, then there is a commutative diagram

$$\begin{array}{ccc}
G \times g & \xrightarrow{(x, X) \mapsto \text{Ad}(x)X} & g \\
\downarrow f & & \downarrow \text{Lie}(f) \\
H \times h & \xrightarrow{(y, Y) \mapsto \text{Ad}(y)Y} & h
\end{array}$$

Then we define a Lie bracket on $g$ as follows: $[X, Y] = \text{ad}(X)(Y) = \text{Lie}(\text{Ad})(X)(Y)$. Then this is a Lie algebra structure on $g$, and it commutes with arbitrary base change.

Proof: To verify this is truly a Lie algebra, we take a faithful embedding of $G$ into some $GL_V(V.9.7.10)$. Thus it suffices to prove the Lie algebra of $GL_V$ is a Lie algebra. Now for $A, B \in M_n(R)$, pondering the definition shows

$$(1 + \delta A)(1 + \varepsilon B)(1 - \delta A) = 1 + \varepsilon B + \varepsilon \delta[X, Y] \in k[\varepsilon, \delta]/(\varepsilon^2, \delta^2).$$

So in fact $[X, Y] = XY - YX$, so it is truly a Lie algebra.  

$\square$
Cor. (V.9.10.2) (Exponential Map). As there are natural isomorphisms \( \mathfrak{g}(R) \cong \mathfrak{g} \otimes R \), we can write \( e^{\varepsilon X} \) the element of \( \mathfrak{g}(R) \subset G(R[\varepsilon]) \) corresponding to \( X \in \mathfrak{g} \times R \). Then \( e^{\varepsilon X + \varepsilon Y} = e^{\varepsilon X}e^{\varepsilon Y} \), and by functoriality, for any homomorphism \( f : G \to H \),

\[
f(e^{\varepsilon X}) = e^{\varepsilon \text{Lie}(f)(X)}.
\]

Also

\[
x \cdot e^{\varepsilon Y} x^{-1} = e^{\varepsilon \text{Ad}(x)Y}
\]

and also the commutative diagram in (V.9.10.1) means

\[
f(e^{\varepsilon X}) = e^{\varepsilon \text{Lie}(f)(X)}.
\]

Cor. (V.9.10.3) (Lie algebra commutes with Limits). It can be seen from the definition that the Lie algebra construction commutes with limits of algebraic groups. In particular it commutes with kernel map.

Prop. (V.9.10.4). Let \( H \subset G \) be algebraic groups s.t. \( \text{Lie}(H) = \text{Lie}(G) \). If \( H \) is smooth and \( G \) is connected, then \( H = G \).

Proof: Recall that \( \dim \mathfrak{g} \geq \dim G \), with equality iff \( G \) is smooth (V.9.4.5), so the condition forces \( G \) to be smooth. Now \( G \) is smooth and connected thus irreducible (V.9.4.11) and \( \dim G = \dim H \), so \( H = G \).

Cor. (V.9.10.5). Let \( H_1, H_2 \) be connected algebraic subgroups of \( G \) and \( H_1 \cap H_2 \) is smooth. If \( \text{Lie}(H_1) = \text{Lie}(H_2) \), then \( H_1 = H_2 \).

Cor. (V.9.10.6). If \( G \) is an algebraic group over a field of characteristic 0, then the connected subgroups of \( G \) corresponds 1 to 1 to Lie subalgebras of \( \text{Lie}(G) \), because every subgroup is smooth, by (V.9.4.3).

Cor. (V.9.10.7). Let \( H_i \) be a family of smooth algebraic subgroups of an algebraic subgroup \( G \) over a field \( k \). If \( \text{Lie}(H_i) \) generate \( \text{Lie}(G) \) as a Lie algebra, then \( H_i \) generates \( G \) (V.9.4.13).

Proof: Let \( H \) be the subgroup they generate, then \( H \) is smooth (V.9.4.13) and \( \text{Lie}(H) = \text{Lie}(G) \), thus \( H = G \) by (V.9.10.4).

Stabilizer, Center and Centralizer

Prop. (V.9.10.8) (Lie Algebra of Stabilizer). Let \( G \) be an algebraic group and \((V, r)\) be a representation of \( G \), then it induces an action of \( \mathfrak{g} \) on \( W \) (V.9.10.1). Let \( W \subset V \) be a subspace, then the stabilizer \( \text{Stab}_G(W) \) is a subgroup of \( G \) (V.9.7.3)

\[
\text{Lie}(\text{Stab}_G(W)) = \text{Stab}_\mathfrak{g}(W).
\]

In particular, \( \dim(\text{Stab}_G(W)) \leq \dim \text{Stab}_\mathfrak{g}(W) \), with equation iff \( \text{Stab}_G(W) \) is smooth.

Proof: By (V.9.10.2),

\[
X \in \text{Lie}(\text{Stab}_G(W)) \iff r(e^{\varepsilon X})W[\varepsilon] \subset W[\varepsilon]
\]

\[
\iff e^{\varepsilon \text{Lie}(r)(X)}W[\varepsilon] \subset W_R[\varepsilon]
\]
\(\Leftrightarrow (1 + \varepsilon \text{Lie}(r)(X))(W_R + \varepsilon W_R) \subset (W_R + \varepsilon W_R)\)
\(\Leftrightarrow \text{Lie}(r)(X)(W_R) \subset W_R\)
\(\Leftrightarrow X \subset Stab_g(W)\)

\(\square\)

**Prop. (V.9.10.9) (Lie Algebra of Center).** Let \(G\) be a smooth connected algebraic group, then
\[\text{dim } z(\mathfrak{g}) \geq \text{dim } Z(G),\]
and if equality holds, then \(Z(G)\) is smooth and \(\text{Lie}(Z(G)) = z(\mathfrak{g})\).

**Proof:** There are maps
\[\text{Ad} : G \mapsto GL_{\mathfrak{g}}, Z(G) \subset \text{Ker(Ad)},\]
\[\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}_g, \text{Ker(ad)} = z(\mathfrak{g}).\]
Because Lie algebra commutes with kernel(V.9.10.3), \(\text{Lie}(Z(G)) \subset \text{Lie(Ker(Ad))) = Ker(ad)}\).

\[\text{dim } z(\mathfrak{g}) = \text{dim Ker(ad)} = \text{dim Lie(Ker(Ad)))} \geq \text{dim Ker(Ad)} \geq \text{dim } Z(G)\]
with equality iff \(\text{Ker(Ad)}\) is smooth and \(\text{dim Ker(Ad)} = \text{dim } Z(G)\), so \(Z(G)\) is also smooth. Finally, \(\text{Lie}(Z(G)) \subset z(\mathfrak{g})\), so they are equal if they have the same dimensions.
\(\square\)

**Prop. (V.9.10.10) (Lie Algebra of Centralizer).** Let \(G\) be an algebraic group and \(H\) a subgroup, then \(H\) acts on \(\mathfrak{g}\) by \(\text{Ad}\). Then
\[\text{Lie}(C_G(H)) = \mathfrak{g}^H, \quad \text{Lie}(N_G(H))/\text{Lie}(H) = (\text{Lie}(G)/\text{Lie}(H))^H.\]

**Proof:**
\[X \in \text{Lie}(C_G(H)) \iff x(e^{\varepsilon X})_S x^{-1} = e^{\varepsilon X}_S, \quad \forall k[\varepsilon] \rightarrow S, x \in H(S)\]
\[X \in \mathfrak{g}^H \iff ye^{\varepsilon X_R} y^{-1} = e^{\varepsilon X_R}, \quad \forall k \rightarrow R, x \in H(R)\]
And it can be shown these two are equal. Similarly, there is a natural map \(\text{Lie}(N_G(H)) \rightarrow \text{Lie}(G)/\text{Lie}(H)\), and the image lies in the fixed subgroup of \(H\), because
\[X \in \text{Lie}(N_G(H)) \iff (e^{\varepsilon X})_S x(e^{\varepsilon X})_S^{-1} \subset H(S) \iff e^{\varepsilon \text{ad}(x)X_S} \subset H(S)e^{\varepsilon X_S}, \quad \forall k[\varepsilon] \rightarrow S, x \in H(S)\]
\[X \in (\text{Lie}(G)/\text{Lie}(H))^H \iff e^{\varepsilon \text{ad}(x)X_R} \subset e^{\varepsilon (X_R + \text{Lie}(H)(R)), \quad \forall k \rightarrow R, x \in H(R)\]
Then it can be shown that \(X\) satisfies condition in 1 iff \(\overline{X}\) satisfies condition in 2, thus we are done.
\(\square\)

11 Groups of Multiplicative Type

Throughout this subsection, \(G\) is an affine(linear) algebraic group over a field \(k\).
Diagonalizability

**Def. (V.9.11.1) (Diagonalizable Groups).** An algebraic group \( G \) is called **diagonalizable** if the group-like elements in \( \Gamma(G) \) generate \( \Gamma(G) \) as a \( k \)-vector space.

**Prop. (V.9.11.2).** An algebraic group \( G \) is diagonalizable iff it is isomorphic to the the algebraic group corresponding to a group algebra \( D(M) \) for some commutative group \( M \).

**Proof:** For the group algebra \( D(M) \), its group-like elements are just \( \{m|m \in M\} \) by (I.15.2.21), and they clearly span \( D(M) \). Conversely, if the set \( M \) of group-like elements in \( \Gamma(G) \) spans \( \Gamma(G) \), then by (I.15.2.20) they form a basis of \( \Gamma(G) \), so there is an isomorphism of vector spaces \( D(M) \to \Gamma(G) \). But this is also a homomorphism, because they are on a basis. \( \square \)

**Cor. (V.9.11.3).**
- The functor \( M \mapsto D(M) \) is a contravariant equivalence from the category of f.g. commutative groups to the category of diagonalizable algebraic groups, with inverse \( G \mapsto X(G) \).
- This functor preserves exact sequences.
- Algebraic subgroups and quotient groups of diagonalizable groups are diagonalizable.

**Proof:**
1. By (V.9.11.2), it suffices to show that \( \text{Hom}(M, M') \to \text{Hom}(D(M'), D(M)) \) is an isomorphism. Because \( D \) sends direct sums to direct products, it suffices to check the case that \( M, M' \) is cyclic. This is easy to check, just notice that a group homomorphism maps group-like elements to group-like elements. Thus \( \Gamma(G) \) is also spanned by group-like elements, and \( H \) is diagonalizable.

2. If \( M' \to M \) is injective, then \( k[M'] \to k[M] \) is injective, thus faithfully flat by (I.15.2.23), and \( D(M) \to D(M') \) is a quotient map. Conversely, \( D(M) \to D(M') \) is a quotient map iff \( k[M'] \to k[M] \) is faithfully flat thus injective thus \( M' \to M \) is injective. Now the kernel of \( D(M) \to D(M') \) is represented by \( k[M]/I_{k[M']} \), where \( I_{k[M']} \) is the augmentation ideal. Then it is isomorphic to \( k[M/M'] \).

3. Let \( H \) be an algebraic subgroup of \( G \), then the map \( \Gamma(G) \to \Gamma(H) \) is surjective, and sends group-like elements to group-like elements, thus \( \Gamma(H) \) is also spanned by group-like elements, and \( H \) is diagonalizable.

**Prop. (V.9.11.4) (Representation of Diagonalizable Groups).** The following conditions are equivalent for an algebraic group \( G \) over a field \( k \):

1. \( G \) is diagonalizable.
2. Every representation of \( G \) is diagonalizable.
3. Every f.d. representation of \( G \) is diagonalizable.

**Proof:**
1. \( G \) is diagonalizable.
2. Every representation of \( G \) is diagonalizable.
3. Every f.d. representation of \( G \) is diagonalizable.

**Proof:** 1 \( \to \) 2: We need to show for any comodule \( \rho : V \to V \otimes \Gamma(G) \), \( V \) is a sum of 1-dimensional representations, or equivalently, it is spanned by vectors \( u \) that \( \rho(u) \in k u \otimes \Gamma(G) \). Let \( v \in V \), we can write \( \rho(v) = \sum u_i \otimes e_i \) where \( e_i \) are group-like in \( G \).

Applying comodule relations, we get

\[
\sum u_i \otimes e_i \otimes e_i = \sum \rho(u_i) \otimes e_i, \quad v = \sum u_i (I.15.2.19).
\]
so \( \rho(u_i) = u_i \otimes e_i \) and they span \( V \).

2 \( \rightarrow \) 1: The regular representation of \( G \) is diagonalizable, so \( \Gamma(G) \) is spanned by its eigenvectors, for any eigenvector \( f \in \Gamma(G) \), so \( \mu(f) = f \otimes e \) where \( e \) is group-like. Applying \( \varepsilon \otimes \text{id} \) shows \( f = \varepsilon(f)e \), so \( G \) is diagonalizable.

2 \( \rightarrow \) 3: trivial.

3 \( \rightarrow \) 2: Every representation of \( G \) is a sum of f.d. representation, so it is a sum of 1-dimensional representations, so it is diagonalizable by(V.9.7.6).

\[ \square \]

**Linear Tori**

**Def.** (V.9.11.5) (Linear Tori). Let \( k \) be a field, then a split torus over \( k \) is a linear group scheme of the form \( G = \mathbb{G}_{n,k} \), and a linear torus over \( k \) is defined to be a linear algebraic group \( T \) over \( k \) that is a split torus over \( k \).

**Prop.** (V.9.11.6). Any torus over a separably closed field is split.

**Proof:**

**Def.** (V.9.11.7) (Quasi-Split Tori). Let \( A \) be a f.d. separable \( k \)-algebra, then there is a linear algebraic group \( G \) defined by \( G(B) = G_{m}(A \otimes_{k} B) = (A \otimes_{k} B)^{\ast} \), denoted by \( \text{res}_{A/k} \mathbb{G}_{m} \).

This is a linear algebraic group because if we choose a basis \( \{v_1, \ldots, v_r\} \) of \( A \) over \( k \), which induces a ring homomorphism \( \phi : A \rightarrow M_n(k) \). Then \( f = \det(\phi(x_1v_1 + \ldots + x_rv_r)) \) is a polynomial in \( x_1, \ldots, x_r \). Thus \( G(B) \) is the set of points in \( \mathbb{A}^r(B) \) that \( f(x_1, \ldots, x_r) \) is invertible in \( B \). So \( G \) is a linear algebraic variety. Moreover, as \( A \) is separable, \( A \otimes_{k} \overline{k} \cong \oplus_{i=1}^{r} M_n_{n}(\overline{k}) \), so \( G_{\overline{k}} \cong \mathbb{G}_{m,\overline{k}}^r \), so \( G \) is a torus, called a quasi-split torus over \( k \).

**Def.** (V.9.11.8) (Monoidal Transformation). For a matrix \( A \in SL(n, \mathbb{Z}) \) with \( \det A = \pm 1 \), we define an isomorphism of \( \mathbb{G}_{n,k} \):

\[ \varphi_A(x) = (x_1^{a_{11}} x_2^{a_{12}} \ldots x_n^{a_{1n}}, \ldots, x_1^{a_{n1}} \ldots x_n^{a_{nn}}). \]

these isomorphisms \( \varphi_A \) is called the monoidal transformations.

**Groups of Multiplicative Type**

**Def.** (V.9.11.9) (Groups of Multiplicative Type). An algebraic group over of multiplicative type over a field \( k \) is an algebraic group \( G \) that \( G_{K} \) is diagonalizable over \( K \) for some field \( K \) containing \( k \).

Subgroups and quotient groups of groups of multiplicative type are also of multiplicative type, because this is true for diagonalizable groups(V.9.11.3).

**Prop.** (V.9.11.10) (Characterization of Groups of Multiplicative Type). The follows are equivalent for an algebraic group \( G \) over \( k \):

- \( G \) is of multiplicative type.
- \( G \) is commutative and \( \text{Hom}(G, \mathbb{G}_{a}) = 0 \).
- \( G \) is commutative and \( \Gamma(G) \) is coétale.
- \( G \) becomes diagonalizable over \( k^s \).
Proof: Cf.[?].

Cor. (V.9.11.11). An algebraic group over \( k \) becomes diagonalizable over some field extension of \( k \) iff it becomes diagonalizable over some finite separable extension of \( k \).

Cor. (V.9.11.12). If a group of multiplicative type splits over a purely inseparable extension of \( k \), then it splits over \( k \).

Proof: Cf.[Mil17]P238.

Cor. (V.9.11.13). A smooth commutative algebraic group \( G \) over \( k \) is of multiplicative type iff \( G(\overline{k}) \) consists of semisimple elements.

Proof: We can assume that \( k = \overline{k} \), and embed \( G \) into \( GL_n \) for some \( n(V.9.7.10) \). If \( G \) is of multiplicative type, then by (V.9.11.4), there is a basis that \( G \subset \mathbb{D}_n \), so all the elements in \( G(k) \) is diagonalizable hence semisimple. Conversely, if \( G(k) \) are all semisimple, then they form a commutative family of semisimple elements, so \( G(k) \subset \mathbb{D}_n(k) \) in some basis. Because \( G \) is smooth thus reduced, \( G \subset \mathbb{D}_n \).

Cor. (V.9.11.14). An extension of algebraic groups of multiplicative type is of multiplicative type iff it is commutative.

Proof: A exact sequence \( e \to G' \to G \to G'' \to e \) of commutative group schemes gives rise to an exact sequence

\[ 0 \to \text{Hom}(G'',G_a) \to \text{Hom}(G,G_a) \to \text{Hom}(G',G_a) \]

of Abelian groups, by (V.9.6.7), thus we can use characterization (V.9.11.10).

Prop. (V.9.11.15) (Representation of Groups of Multiplicative Type). Let \( G \) be an algebraic group over \( k \), then \( \text{Rep}(G) \) is a semisimple Abelian category, and the isomorphism classes of simple objects in \( \text{Rep}(G) \) are classified by the orbits of \( G(k^s/k) \) acting on \( X^*(G) \).

Let \((V, r)\) be a representation corresponding to an orbit \( \Sigma \), and let \( \chi \in \Sigma \), then \( \text{End}(V, r) \cong k_\chi \), where \( k_\chi \) is the subfield of \( k^s \) fixed by the subgroup of \( G(k^s/k) \) fixing \( \chi \).

Proof: The group \( G \) is split by a finite Galois extension \( K/k \) by (V.9.11.11). Let \( \overline{\Gamma} = G(K/k) \), then \( \Gamma \) acts on \( \Gamma(G_K) \) through its action on \( K \). Let \( (V, r) \) be a representation of \( G_K \) and let \( \rho \) be the corresponding co-action, then by (V.1.5.23), the functor \( V \mapsto V \otimes_k K \) induces an equivalence between \( \text{Rep}(G) \) and \( \text{Rep}(G_K) \) with a semi-linear action of \( \overline{\Gamma} \) fixing \( \rho \).

Let \( V \) be a representation of \( G \) over \( k \), then \( K \otimes V \) decomposes as a representation of \( G_K \) into

\[ K \otimes V = \oplus_{\chi \in X(G_K)} V_\chi. \]

and an element \( \gamma \in \Gamma \) maps \( V_\chi \) isomorphically onto \( V_{\sigma \chi} \). Thus the set of \( \chi \) occurring in \( K \otimes V \) is stable under the action of \( \Gamma \).

Conversely, if \( \Sigma \) is an orbit of \( \Gamma \) in \( X(G_K) \) and \( V \) is a 1-dimensional \( K \) vector space, then \( \oplus_{\chi \in \Sigma} V_\chi \) has a natural semi-linear action of \( \Gamma \), so it arises from a simple representation of \( G \) over \( k \).

Prop. (V.9.11.16) (Density Theorem for Groups of Multiplicative Type). Let \( G \) be a smooth algebraic group of multiplicative type, thus \( G \) is commutative. Let \( G_n \) be the kernel of multiplication by \( n \) on \( G \).

- The only closed subscheme containing every \( G_n \) is \( G \) itself.
• If $G$ is smooth, then the only closed subscheme containing $G_n$ for $n$ prime to characteristic of $k$, is $G$ itself.


Cor. (V.9.11.17). Let $G$ be an algebraic group of multiplicative type. If two homomorphisms from $G$ to another algebraic group $H$ coincides on $G_n$ for all $n \geq 1$, then they are equal.

Proof: This is because the equalizer is a closed subscheme of $G$, as $H$ is separated (V.9.1.10). □

Prop. (V.9.11.18) (Rigidity Theorem for Groups of Multiplicative Type). Let $G, H$ be diagonalizable groups over $k$, and let $X$ be a connected group scheme over $k$. Let $\varphi : X \times G \to H$ be a morphism that for all $k$-algebra $R$ and $x \in X(R)$, the map $g \mapsto \varphi(x, g) : G(R) \to H(R)$ is a homomorphism. Then for any $x_0 \in X(k)$, we have $\varphi(x, g) = \varphi(x_0, g)$ for any $k$-algebra $R$ and $(x, g) \in X(R) \times G(R)$.


Cor. (V.9.11.19). Every action of a connected algebraic group $G$ on an algebraic group $H$ by group homomorphisms is a trivial action.

Cor. (V.9.11.20). Every normal algebraic subgroup of multiplicative type of a connected algebraic group $G$ is contained in the center of $G$.

Proof: The action of $G$ on $N$ by inner automorphism is trivial. □

Cor. (V.9.11.21). Let $H$ be a subgroup of multiplicative type of an algebraic group $G$, then $N_G(H) = C_G(H)$, i.e. $C_G(H)$ is an open subgroup of $N_G(H)$.

Proof: The inner action of $N_G(H)$ on $H$ by inner automorphism is trivial. □

Cor. (V.9.11.22). If $N$ is a normal subgroup of an algebraic group $H$ that $N$ and $H/N$ are of multiplicative type, then every action of a connected algebraic group $G$ on $H$ by group homomorphisms preserving $N$ is trivial.

Proof: The action of $G$ on $N$ is trivial, thus the action factors through $G \times H/N \to H$, thus also factors through $G \times H/N \to N$. Now the action is trivial, by (V.9.11.18). □

Cor. (V.9.11.23). An extension of algebraic groups of multiplicative type is of multiplicative type if it is connected.

Proof: The adjoint action of $G$ is trivial, by (V.9.11.22), thus $G$ is commutative. Thus it is of multiplicative type by (V.9.11.14). □

12 Actions of Tori

13 Unipotent Algebraic Groups

Def. (V.9.13.1) (Unipotent Linear Group Scheme). A unipotent linear algebraic group is a closed subgroup of the upper triangular subgroup of some $U_n \subset GL_n(K)$ for some $n$.

For a linear algebraic group $G$, it has a maximal connected smooth normal unipotent subgroup $R_u(G)$, which is called the unipotent radical of $G$.

Proof:

Prop. (V.9.13.2). The unipotent radical of $G$ commutes with finite base change of fields.

Proof:
14 Solvable Algebraic Groups
V.10  Reductive Groups

1  Borel Subgroups

2  Reductive Groups

Def. (V.10.2.1) (Linearly Reductive Groups). An algebraic group over a field is called **linearly reductive** if every f.d. representation of $G$ is semisimple.

Prop. (V.10.2.2). Let $G$ be an algebraic group over $k$, and $k'$ a field containing $k$. If $G_{k'}$ is linearly reductive, then so is $G$. Conversely, if $G$ is linearly reductive and $k'$ is separable over $k$, then $G_{k'}$ is linearly reductive.

Proof: Cf.[Mil17]P248.

Prop. (V.10.2.3). A commutative algebraic group is linearly reductive iff it is of multiplicative type.

Proof: Cf.[Mil17]P248. □

Prop. (V.10.2.4) (Hilbert). Let $G$ be a linearly reductive group of $GL_n$ and let $A = k[T_1, \ldots, T_n]$, then $A^G$ is f.g. as a $k$-algebra.

Proof: Cf.[Mil17]P249. □

Def. (V.10.2.5) (Radical). The **radical** $R(G)$ of a linear algebraic group $G$ is the maximal smooth connected solvable normal subgroup of $G$. $G$ is called **semisimple** iff $R(G_K) = \{1\}$. Semisimple linear algebraic group is reductive, because $U_n$ is solvable.

Prop. (V.10.2.6). The radical commutes with finite base change of fields.

Prop. (V.10.2.7). If $G$ is reductive, then $[G,G]$ is semisimple. And $G/Z(G)$ is semisimple.

Prop. (V.10.2.8) (Maximal Central Torus). If $G$ is a connected, reductive linear algebraic group over a perfect field $K$, then $R(G) = Z(G)^\text{red}$, which is the maximal central torus by (V.10.2.11). In particular, $G$ is semisimple if and only if $Z(G)$ is finite.

Proof: □

Prop. (V.10.2.9) (Maximal Quotient). If $G$ is reductive, then $G/[G,G]$ is finite iff $Z(G)$ is finite iff $G$ is semisimple.

Proof: □

Def. (V.10.2.10) (Reductive Algebraic Group). A linear algebraic group over a field $k$ of characteristic 0 is called **reductive** if $\text{Rep}(G)$ is semisimple.

This condition is equivalent to $R_u(G_K) = \{1\}$.

Proof: □

Prop. (V.10.2.11). A connected smooth commutative linear algebraic group $G$ is of the form $U \times T$ where $U$ is unipotent and $T$ is a torus.

Proof: □

3  Representations of Reductive Groups

Prop. (V.10.3.1). Let $G$ be a representation of a
V.11 Formal Groups and $p$-Divisible Groups

Main References are [Zin84].

1 Formal Power Series

Prop. (V.11.1.1) (Automorphisms). If $F_i$ are power series without constant terms that the matrix degree 1 terms of $(F_i)$ (the Jacobi matrix) is invertible, then there are unique power series $G_i$ without constant terms that $G \circ F = id$ and $F \circ G = id$.

Proof: It $F_i$ induces a map $F : K[[X_1, \ldots, X_n]] \to K[[X_1, \ldots, X_n]]$ which in turn induces a graded map $K[X_1, \ldots, X_n] \to K[X_1, \ldots, X_n]$. It is clear that $(\frac{\partial F_i}{\partial X_j})_{ij}$ is invertible iff the induced graded ring map is an isomorphism, and because $K[[X_1, \ldots, X_n]]$, a map is an isomorphism iff its induced graded map is an isomorphism. □

Def. (V.11.1.2) (Formal Logarithm). The formal exponential and formal logarithm is defined to be elements in $\mathbb{Q}[[x]]$:

$$\exp(x) = \sum_{n \geq 0} \frac{x^n}{n!}, \quad \log(1 + x) = -\sum_{n > 0} \frac{(-x)^n}{n}.$$  

They satisfies $\exp(\log(1 + x)) = 1 + x$, $\log(\exp(x)) = x$.

Proof: It suffices to prove $\text{Exp}(x) = \exp(x) - 1$ and $\text{Log}(x) = \log(1 + x)$ are inverse to each other. It suffices to show $\log(\exp(x)) = x$, because then by (V.11.1.1) the inverse of Log must be just Exp by (V.11.1.1).

We notice Exp are the unique formal power series without constant term that satisfied $d(\text{Exp}) = \text{Exp} + 1$, and Log is the unique formal power series that satisfies $d(\text{Log}) = \frac{1}{1+x}$. Thus

$$d(\log(\exp(x))) = \frac{\exp(x)}{\exp(x)} = 1,$$

so $\log(\exp(x)) = x$, because it has no constant term. □

Prop. (V.11.1.3). In $\mathbb{Q}[[x]]$,

$$\exp(x) = \prod_{d > 0} (\frac{1}{1-x^d})^{\mu(d)}.$$

Proof: Taking log, we prove its convergence and equality at once:

$$\sum_{d > 0} \log((\frac{1}{1-x^d})^{\mu(d)}) = \sum d \frac{\mu(d)}{d} \log(\frac{1}{1-x^d}) = \sum \frac{\mu(d)}{d} \sum_{d' > 0} \frac{x^{dd'}}{d'} = \sum_{n > 0} \frac{x^n}{n} \sum_{n|d} \mu(d) = x.$$  

□
2 Formal Group Law

Def. (V.11.2.1) (Formal Group Law). A formal group law of dimension $n$ over a commutative ring $R$ is a set of $n$ power series $G = (G_1, \ldots, G_n)$ in $R[[X_1, \ldots, X_n, Y_1, \ldots, Y_n]]$ that

\[ G(X, 0) = G(0, X) = X, \quad G(G(X, Y), Z) = G(X, G(Y, Z)). \]

A morphism of formal groups is a vector of power series $\varphi(X)$ that $\varphi(G(X, Y)) = H(\varphi(X), \varphi(Y))$.

A formal $R$-module is a formal group $G$ over $R$ together with a ring homomorphism $R \to \text{End}_R(G)$ that $[a](X) = aX + \ldots$.

Cor. (V.11.2.2). Note this immediately induce an inverse $\text{inv}(X)$ that $G(X, \text{inv}(X)) = G(\text{inv}(X), X) = 0$. This can be constructed noticing $G(X, Y) = X + Y + o(X, Y)$.

Proof: □

Cor. (V.11.2.3) (Formal Group Laws and Group Schemes). A (commutative) formal group law is equivalent to a (commutative)$R$-group scheme structure on $\text{Spec } R[[X_1, \ldots, X_n]]$ (V.9.1.1), and morphisms of formal group laws are equivalent to morphisms of group schemes.

Prop. (V.11.2.4). $G_\alpha$ is the one-dimensional formal group with $G_\alpha(X, Y) = X + Y$, $G_m$ is the one-dimensional formal group with $G_m(X, Y) = X + Y + XY$. Over a $\mathbb{Q}$-algebra $K$, there is an isomorphism between $G_\alpha$ and $G_m$, giving by $X \to \exp(X) - 1$.

Def. (V.11.2.5) (Differential Operators). A continuous $R$-linear mapping $D : R[[X]] \to R[[X]]$ is called a differential operator of order $N$ iff

\[ L_D : R[[X, Z]] \to R[[X]] : \sum p_\alpha(X)Z^\alpha \to \sum p_\alpha(X)D(X^\alpha) \]

vanish on $J^{N+1}$, where $J = (X_i - Z_i)$.

Then $D$ is an operator of order $N$ if $fD - Df$ is of order $N - 1$ for any $f \in R[[X]]$.

Proof: Let $D$ be a differential operator of order $N$, since $f(X) - f(Z) \in J$, for all $g(X, Z) \in J^N$, we have $L_D((f(X) - f(Z))g(X, Z)) = 0$, which is equivalent to $L_{D\circ f - f \circ D}(g) = 0$, so $D \circ f - f \circ D$ is an operator of order $N - 1$. Conversely, if $D \circ f - f \circ D$ is an operator of degree $N - 1$, then $L_D((f(X) - f(Z))g) = 0$ for all $g \in J^N$, then $D$ is an operator of order $N$. □

Prop. (V.11.2.6) (Graded Module of Differential Operators). Let $D_1, D_2$ be differential forms of order $N_1, N_2$, then $D_1 \circ D_2$ is a differential form of order $N_1 + N_2$, and $[D_1, D_2]$ is a differential form of order $N_1 + N_2 - 1$.

In particular, the graded module of differential operators

\[ \text{grDO} = \oplus DO_N / DO_{N-1} \]

is a commutative graded ring.

Cor. (V.11.2.7). There is a representation $g(X + Y) = \sum D_\alpha g(X)Y^\alpha$ for any $g \in R[[X, Y]]$, where $D_\alpha$ is a differential operator of degree $|\alpha|$. And $D_\alpha$ forms a basis for the module of differential operators (V.11.2.6).

**Def. (V.11.2.8) (Invariant Differential Forms).** An **invariant differential form** on a formal group law \( G = R[[X]] \) is an element \( \omega = \sum a_i(X) dX_i \in \Omega^1_{R[[X]]} \) that \( \mu_i \omega = (i_2)_i \omega \).

**Def. (V.11.2.9) (Invariant Derivations).** An **invariant derivation** on a formal group \( R[[X]] \) is a derivation that

\[
\mu \circ D = (1 \otimes D) \circ \mu.
\]

Equivalently it is an element of \( \text{Hom}_{R[[X]]}(\Omega^1_{R[[X]]}, R[[X]]) \).

**Prop. (V.11.2.10).** The module of invariant differential forms and invariant derivations on a formal group of dimension \( n \) are all isomorphic to \( R^n \).

Proof: Cf.[Zin84]P16.

**Prop. (V.11.2.11) (Q-Theorem).** Any commutative connected formal group over a \( \mathbb{Q} \)-algebra \( R \) is a direct sum of \( \mathbb{G}_a \).

Proof: Cf.[Cartier Theory of Commutative Formal Groups Zink P19].

**1-dimensional Formal Groups**

**Def. (V.11.2.12) (Normalized Invariant Differential in Dimension 1).** For a 1-dimensional formal group \( F = \text{Spec } R[[X]] \) over \( R \), the module of invariant differentials is isomorphic to \( R(V.11.2.10) \).

An invariant differential form is called **normalized** if \( P(0) = 1 \).

The unique invariant differential on \( G \) is given by \( F_X(0,T)^{-1}dT \).

Proof: We need to check \( F_X(0,F(T,S))^{-1}F_X(T,S) = F_X(0,T)^{-1} \), and this is just \( F(U,F(T,S)) = F(F(U,T),S) \) differentiated at \( U \) and let \( U = 0 \).

**Prop. (V.11.2.13).** For a morphism \( f : \mathcal{F} \to \mathcal{G} \) of 1-dimensional formal groups over \( R \), \( \omega_{\mathcal{G}} \circ f = f'(0) \omega_{\mathcal{F}} \).

Proof: We only need to show that \( \omega_{\mathcal{G}} \circ f \) is an invariant differential for \( \mathcal{F} \) and then compare their constant coefficients. For this, notice

\[
\omega_{\mathcal{G}} \circ f(F(T,S)) = \omega_{\mathcal{G}}(G(f(T),f(S))) = \omega_{\mathcal{G}}(f(T)) = \omega_{\mathcal{G}} \circ f(T).
\]

**Def. (V.11.2.14).** When \( R \) has characteristic 0, the **formal logarithm** \( \log_{\mathcal{F}} \) for a 1-dimensional formal group is the integration of invariant differential \( \int_0^T \omega_{\mathcal{F}} = T + c_1/2T^2 + \cdots \).

Then the **formal power exponential** is the the unique power series \( \exp_{\mathcal{F}} \) that is the inverse of \( \log_{\mathcal{F}} \). It exists uniquely by(V.11.1.1).

**Prop. (V.11.2.15).** For \( R \) char \( = 0 \) and an 1 dimensional formal group \( \mathcal{F} \) over \( R \), \( \log_{\mathcal{F}} : \mathcal{F} \to \mathbb{G}_a \) is an isomorphism of formal groups over \( R \otimes \mathbb{Q} \).

And if \( \mathcal{F} \) is a formal \( R \)-module, then it is an isomorphism of \( R \)-modules, because from(V.11.2.13) that \( \omega_{\mathcal{F}} \circ [a] = a \omega_{\mathcal{F}} \), thus \( \log_{\mathcal{F}} \circ [a] = a \cdot \log_{\mathcal{F}} \).

Proof: From \( \omega_{\mathcal{F}}(F(T,S)) = \omega_{\mathcal{F}}(T) \), we get that \( \log_{\mathcal{F}}(F(T,S)) = \log_{\mathcal{F}}(T) + \log_{\mathcal{F}}(S) \). So it is a homomorphism. Now the inverse \( \exp_{\mathcal{F}} \) is already given, so it is an isomorphism.

**Cor. (V.11.2.16).** A 1-dimensional formal group over a ring \( R \) that has no torsion nilpotents is commutative.

Proof: We only prove for \( R \) torsion free. \( F(T,S) = \exp_{\mathcal{F}}(\log_{\mathcal{F}}(T) + \log_{\mathcal{F}}(S)) \).
Lubin-Tate Formal Group

Def. (V.11.2.17). For a $p$-adic number field $K$ with a uniformizer $\pi_K$ with residue field $\mathbb{F}_q$, a Lubin-Tate power series for $\pi_K$ is a $\varphi(X) \in \mathcal{O}_K[[X]]$ that $\varphi(X) \equiv \pi_K X \mod X^2$ and $\varphi(X) \equiv X^q \mod \pi_K$.

A Lubin-Tate module $G$ over $\mathcal{O}_K$ is a formal $\mathcal{O}_K$-module that $[\pi_K](X)$ is a Lubin-Tate power series.

Prop. (V.11.2.18). Given a $p$-adic number field $K$ with residue field $\mathbb{F}_q$, we consider the set $\xi_\pi$ of all Lubin-Tate power series for $\pi$.

If $f, g \in \xi_\pi$ and $L(X) = \sum a_i X_i$ be a linear form, then there exists a unique power series $F(X)$ that $F(X) \equiv L(X) \mod \deg 2$ and $f(F(X)) = F(g(X_1), \ldots, g(X_n))$.

Proof: Choose $F$ consecutively, if $F_{r+1} = F_r + \Delta_r$, then must

$$\Delta \equiv \frac{f(F_r(X)) - F_r(g(X))}{\pi^{r+1} - \pi} \mod \deg (r+2).$$

This has coefficient in $\mathcal{O}$ because $f \equiv g \equiv Z^q \mod \pi$. \qed

Cor. (V.11.2.19). If we let $f = g, L = X + Y$ to get $F_f$ and $f, g, L = aX$ to get $a_{f,g}$, then

- $F_f(X, Y) = F_f(Y, X)$.
- $F_f(F_f(X, Y), Z) = F_f(X, F_f(Y, Z))$.
- $a_{f,g}(F_g(X, Y)) = F_f(a_{f,g}(X), a_{f,g}(Y))$.
- $a_f b_f(Z) = (ab_f)(Z)$.
- $(a + b)_f(Z) = F_f(a_f(Z), b_f(Z))$.
- $\pi_f(Z) = f(Z)$.

all follow from the unicity of the last proposition.

Cor. (V.11.2.20) (Existence of Lubin-Tate Module). We get a commutative formal $\mathcal{O}$-module $F_f$ for every $f$. And this group can act on $\mathfrak{p}_L$ for an alg.ext $L/K$. The set of zeros $\Lambda_{f,n}$ of $f^n$ in $L$, as the elements annihilated by $\pi^n$, is a submodule of $\mathfrak{p}_L^{(f)}$.

And $u_{f,g}$ for any unit $u \in \mathcal{O}$ defines an isomorphism between $F_f$ and $F_g$, thus this formal group only depends on $\pi$, called $F_\pi$. Hence $L_{f,n} = K(\Lambda_{f,n})$ only depends on $\pi$, with Galois group $G_{\pi,n}$.

Prop. (V.11.2.21) (Different Uniformizers). Now consider different $\pi$, it is proven that $F_{\pi}$ and $F_{\pi'}$ are isomorphic, but the coefficient in $\mathcal{O}_T$ where $T$ is the maximal unramified extension.

Thus $L_{\pi,n}$ and $L_{\pi',n}$ may not be isomorphic, but $T \cdot L_{\pi,n} = T \cdot L_{\pi',n}$ since $T \cdot L_{\pi,n} = T \cdot L_{\pi',n}$ and both of them is the algebraic closure of $K$ in it.

Proof: Cf.[Neukirch CFT P105]. \qed

Lemma (V.11.2.22). The Newton polygon of $[\pi_K^n]/\pi_K^m$ has vertices

$$\{(1,0), (q, -1/e_K), (q^2, -2/e_K), \ldots\}.$$ 

Proof: Notice $[\pi_K^n]$ has no infinite edge of negative slope because all its coefficient are in $\mathcal{O}_K$. Now look at its roots, it has a root 0, and $q - 1$ roots of valuation $v_p(\pi_K)/(q - 1)$, $q(q - 1)$ roots of valuation $v_p(\pi_K)/q(q - 1)$, and so on. So by factor out these roots, $[\pi_K^n]/\pi_K^m$ is left with a power series whose Newton polygon is a single line, which shows the desired result. \qed
Prop. (V.11.2.23). The formal logarithm of the Lubin-Tate formal group $F_\pi$ satisfies:

$$\log_{F_\pi}(T) = \lim_{\to} \left( \pi^n F_\pi \right) \circ \pi^n F_\pi.$$

Proof: By (V.11.2.15) we have

$$\log_{F_\pi}(T) = \log_{F_\pi} \left( \pi^n F_\pi \right) \circ \pi^n F_\pi = \left( \pi^n F_\pi + a_2/2[\pi^n F_\pi]^2 + \ldots \right) \circ \pi^n F_\pi$$

and for any degree $n$, the coefficient of $[\pi^{2n} F_\pi] / \pi^{2n} F_\pi$ is bounded below by a $c(n)$, so $[\pi^{2n} F_\pi] / \pi^{2n} F_\pi$ converges to 0, thus the result. $\square$

Cor. (V.11.2.24). The Newton polygon of $\log_{F_\pi}(T)$ has vertices $(1, 0), (q, -1/e_K), (q^2, -2/e_K), \ldots$.

The discussion is continued at 1.

3 Cartier Theory for Formal Groups

Main References are [Zin84].

4 $p$-divisible Groups

Def. (V.11.4.1) ($\Lambda$-Formal Schemes). Let $\Lambda$ be a local complete Noetherian ring and $A^f_\Lambda$ be the category of finite length(Artinian) $\Lambda$-algebra, then a $\Lambda$-formal functor is a functor $A^f_\Lambda \to \text{Sets}$.

The formal completion of a functor $A_\Lambda \to \text{Sets}$ is its restriction on $A^f_\Lambda$. We denote the formal completion of Spec $A$ by Spf $A$.

Then a $\Lambda$-formal scheme is a filtered colimits of functors $\lim_{\to} \text{Spf} A_i$ or equivalently a profinite $\Lambda$-algebra $A = \lim_{\leftarrow} A_i$ with profinite topology.

Def. (V.11.4.2) ($\Lambda$-Formal Group Schemes). A $\Lambda$-formal group is a $\Lambda$-formal scheme with values in groups.

Def. (V.11.4.3) ($p$-Divisible Formal Lie Group Schemes). A formal Lie group $G$ over $\Lambda$ is a connected formally smooth $\Lambda$-formal group. It is necessarily isomorphic to $G = \text{Spf} \Lambda[[X_1, \ldots, X_n]]$ where $n = \dim G$. The number $n$ is called the dimension of $G$.

A $p$-divisible formal Lie group is a commutative formal Lie group $G = \text{Spf} \Lambda[[X_1, \ldots, X_n]]$ that multiplication by $p : [p]^n$ is a finite flat morphism on $\Lambda[[X_1, \ldots, X_n]]$.

Def. (V.11.4.4) ($p$-Divisible Groups). Let $p$ be a prime and $S$ a scheme, a $p$-divisible group is a commutative group functor on $\text{Sch}_{fppf}/S$ that

- $G$ is $p$-divisible: $[p]^n G$ is an epimorphism.
- $G$ is $p$-torsion: $G = \lim_{\to} G(n)$, where $G(n) = \text{Ker}([p]^n G \to G)$.
- $G(n)$ are representable as sheaves on $\text{Sch}_{fppf}/S$.

The category of $p$-divisible groups over $S$ is denoted by $p\text{div}(S)$.

Prop. (V.11.4.5) (Equivalent Definitions of $p$-Divisible Groups). Let $p$ be a prime and $S$ a scheme, then a $p$-divisible group over $S$ is an ind system $(G_v, i_v)$ of finite commutative groups schemes over $S$ s.t.

- $0 \to G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{\delta} G_{v+1}$ is an exact sequence of group schemes over $S$. 

the rank of fiber of \(G(n)\) at \(s \in S\) is \(p^{nh(s)}\) where \(h\) is a locally constant function on \(S\). and \((G_v, i_v)\) is called a \textbf{\(p\)-divisible group of height \(h\)} over \(S\).

\textit{Proof:} \ Cf.[Shatz, P61], [P-Divisible Groups, Haoran Wang]. \hfill \Box

\textbf{Prop. (V.11.4.6) (Connected \(p\)-Divisible Groups and Formal Lie Groups).} \ Cf.[Shatz, P62].

\textbf{Def. (V.11.4.7) (Tate Module).} Let \(G\) be a \(p\)-divisible group over an integral domain \(O\) with fraction field \(K\) of characteristic 0, then the \textbf{Tate module} of \(G\) is defined to be

\[ T_p(G) = \lim_{\longrightarrow} G_n(K), \]

and the \textbf{Tate comodule} of \(G\) is defined to be

\[ \Phi_p(G) = \lim_{\longleftarrow} G_n(K). \]

\textbf{Hodge-Tate Decomposition}

\textbf{Prop. (V.11.4.8) (Hodge-Tate Decomposition).} If \(O\) is a CDVR of mixed characteristic with perfect residue field \(k\) and fraction field \(K\), then there is an isomorphism of f.d. \(\mathbb{Q}_p\)-representation of \(G_K\):

\[ T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \text{tangent} \oplus \text{cotangent spaces of } G. \]

\textit{Proof:} \ Cf.[p-divisible Groups, Morrow]. \hfill \Box
V.12 Geometric Invariant Theory
V.13 Algebraic Stacks

Basic references are [Algebraic Spaces and Stacks Olsson], [Fibered Categories and Descent Theory Vistoli] and [Fibered Category to Algebraic Stacks Lamb].

1 Algebraic Stacks

Def. (V.13.1.1) (Schematic Morphism). A morphism of fibered categories $\mathcal{F} \to \mathcal{G}$ over a scheme $S$ is called schematic if it is representable as morphisms of fibered categories over $\text{Sch}/S$(II.1.7.32).

Prop. (V.13.1.2). Let $\mathcal{F}$ be a fibered category equipped with a morphism to a scheme $S$, then the diagonal $\mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is schematic iff every morphism $S \to \mathcal{F}$ for $S$ a scheme is schematic.

Proof: The key to this proposition is the fibered product diagram (II.1.7.13)(II.1.1.36)

$$
\begin{array}{ccc}
X \times_{\mathcal{F}} Y & \longrightarrow & \mathcal{F} \\
\downarrow & & \downarrow \Delta \\
X \times_S Y & \overset{f \times g}{\longrightarrow} & \mathcal{F} \times_S \mathcal{F}
\end{array}
$$

which still holds in the 2-commutative sense.

So if $\Delta_{\mathcal{F}}$ is schematic, then $X \times_{\mathcal{F}} Y$ is a scheme, so $X \to \mathcal{F}$ is schematic, for any scheme $X$. Conversely, consider the fibered products

$$
\begin{array}{ccc}
\mathcal{F} \times_{\mathcal{F} \times_S \mathcal{F}} X & \longrightarrow & X \times_{\mathcal{F}} X \\
\downarrow & & \downarrow \Delta \\
X & \overset{\Delta}{\longrightarrow} & X \times_S X \overset{f \times g}{\longrightarrow} \mathcal{F} \times_S \mathcal{F}
\end{array}
$$

induced by $h = f \times g : X \to \mathcal{F} \times_S \mathcal{F}$. So in order to prove $\Delta_{\mathcal{F}}$ is schematic, it suffices to prove $\mathcal{F} \times_{\mathcal{F} \times_S \mathcal{F}} X$ is a scheme, and for this, it suffices to prove $X \times_{\mathcal{F}} X$ is a scheme. But $X \times_{\mathcal{F}} X \to X$ is a pullback of $X \to \mathcal{F}$, so it is a scheme. □

Def. (V.13.1.3). For a property $P$ of maps of schemes which is stable under base change, we say that a schematic map $X \to Y$ has that property iff for every scheme $S$, the map $S \times_Y X \to S$ has that property.

Def. (V.13.1.4) (Algebraic Stack). An algebraic stack is a stack $X$ with schematic, quasi-compact and separated diagonals, such that there exists some scheme $U$ with an étale surjective map $\text{Sch}/U \to X$ of fibered categories, called an atlas for the stack $X$.

Prop. (V.13.1.5) (Algebraic Stacks and Étale Groupoid Object). There is a bijection of étale groupoid objects on a scheme with the category of algebraic stacks.

Proof: Cf.[Lamb, P39]. □

Def. (V.13.1.6) (Algebraic Space). An algebraic space is a sheaf on $\text{Aff}$ which is also an algebraic stack, as in (II.1.7.25).

Prop. (V.13.1.7) (Algebraic Spaces and Étale Equivalence Relations). There is a bijection of étale equivalence relations on a scheme with the category of algebraic spaces.

Proof: This is a corollary of(V.13.1.5). □
Examples of Algebraic Stacks

Def. (V.13.2.1) (Smooth Curves of Genus $g$). Given a scheme $S$, there is a category $\mathcal{M}_g$ fibered in sets over $\text{Sch}_S$ where $\mathcal{M}_g(T)$ is the set of smooth and proper morphisms of schemes $C \to T$ that the fibered are all geometrically connected curves of genus $g$.

Similarly there is a category $Z_g$ fibered in sets over $\text{Sch}_S$ of smooth pointed curves of genus $g$.

3 Sheaves on Algebraic Stacks

4 Representability

5 Artin’s Axioms

6 Quot and Hilbert Stacks

7 Properties of Algebraic Stacks

8 Morphisms of Algebraic Stacks

9 Limits of Algebraic Stacks

10 Cohomology of Algebraic Stacks

11 Derived Categories of Stacks
Chapter VI
Weil Cohomologies and Motives

VI.1 Intersection Theory

Main references are [Sta] and [Intersection Theories, Fulton]. [Hartshorne Ex2.6.2] might be useful.

Setup

Def. (VI.1.0.1). The setup is a universally catenary locally Noetherian scheme $S$ endowed with a dimension function $\delta$.

Def. (VI.1.0.2). For $f : X \to S$ locally of f.t., the function

$$\delta(x) = \delta(f(x)) + \text{tr.deg}_k(f(x))k(x)$$

is a dimension function on $X$. In particular, this equation is satisfied for any morphisms between schemes of f.t. over $S$.

For a closed subscheme $Z$ of $X$, let $\text{dim}_\delta(Z) = \sup \text{dim}_\delta(\eta)$ where $\eta$ are generic pts of irreducible components of $Z$.

Proof: Cf.[[Sta]02JW].

Def. (VI.1.0.3) (Cycle). A cycle on a scheme $X$ locally of f.t. over $S$ is an formal sum of integral closed subschemes of $X$ with integer coefficients that is locally finite. A k-cycle is a cycle that is a sum of integral closed subschemes of dimension $k$.

Prop. (VI.1.0.4). Let $X$ be a scheme locally of f.t. over $S$, and $X = X_1 \cup X_2$ is a decomposition as closed subschemes, then there are exact sequences

$$Z_k(X_1 \cap X_2) \to Z_k(X_1) \oplus Z_k(X_2) \to Z_k(X) \to 0$$

Prop. (VI.1.0.5) (Cycle associated to a Closed Subscheme). For a closed subscheme $Z$ of a scheme $X$ locally of f.t. over $S$, if $\text{dim}_\delta(Z) = k$ and $\eta \in Z$ has dimension $k$, then $\eta$ is a generic pt of an irreducible component $Z'$ of $Z$, and $m_{Z,Z'} = \text{length}_{\mathcal{O}_{X,\eta}}\mathcal{O}_{Z,\eta}$ is finite.

So we may define the $k$-cycle associated to $Z$ as: $[Z]_k = \sum_{Z' \subset Z} m_{Z,Z'}[Z']$.

Proof: $m_{Z,Z'}$ is finite because $\text{length}_{\mathcal{O}_{X,\eta}}\mathcal{O}_{Z,\eta}$ is finite because it is Noetherian and have 0 dimension(I.5.1.51), so also $\text{length}_{\mathcal{O}_{X,\eta}}\mathcal{O}_{Z,\eta}$ is finite. The sum is locally finite by(V.4.1.22).

Prop. (VI.1.0.6) (Cycles associated to a Coherent Sheaf).
Pushforward and Pullback

Lemma (VI.1.0.7) (Degree of Maps). Let \( f : X \to Y \) be a map between schemes integral and locally of f.t. over \( S \), if \( \dim_\delta X = \dim_\delta Y \), then either \( f(X) \) not dominant or the function field extension is finite. If \( f \) is dominant, the extension degree \( d \) is called the degree of \( f \).

Proof: Because \( X \) is irreducible, so does \( f(X) \) and \( f(X) \). If \( f(X) \) is dominant, then \( f \) maps the generic point of \( X \) to that of \( Y \). Now \( \deg_K(Y) = 0 \) and \( K(X) \prec K(Y) \) is f.g., thus it is a finite extension. \( \square \)

Lemma (VI.1.0.8). Let \( f : X \to Y \) be a qc map between schemes integral and locally of f.t. over \( S \), and \( \{ Z_i \} \) is a locally finite collection of closed subschemes of \( X \), then \( \{ f(Z_i) \} \) is also a locally finite collection of closed subschemes of \( X \).

Proof: This is a simple topological proof and omitted. \( \square \)

Def. (VI.1.0.9) (Proper Pushforward). Let \( f : X \to Y \) be a proper morphism of schemes locally of f.t. over \( S \), \( Z \subset X \) be an integral closed subscheme of \( X \) that \( \dim_\delta(Z) = k \), then we define

\[
\begin{align*}
    f_*[Z] &= \begin{cases} 
        0 & \text{dim}_\delta(f(Z)) \leq k \\
        \deg(Z/f(Z))[f(Z)](VI.1.0.7) & \text{dim}_\delta(f(Z)) = k
    \end{cases}
\end{align*}
\]

where we regard \( f(Z) \) as an integral subscheme of \( Y \) using its scheme-theoretical image. In general

\[
    f_* \left( \sum n_Z [Z] \right) = \sum n_Z f_*([Z]).
\]

The sum is locally finite by (VI.1.0.8).

It can be easily verified that \( f_* \circ g_* = (f \circ g)_* \).

Prop. (VI.1.0.10) (Pushforward of Coherent Sheaves). Let \( f : X \to Y \) be a proper morphism of schemes locally of f.t. over \( S \), then

- If \( Z \subset X \) is an integral closed subscheme of \( X \) that \( \dim_\delta(Z) \leq k \), then
  \[
  f_*[Z]_k = [f_*O_Z]_k.
  \]

- If \( \mathcal{F} \) is a coherent sheaf on \( X \) that \( \dim_\delta(\text{Supp}(\mathcal{F})) \leq k \), then
  \[
  f_*[\mathcal{F}]_k = [f_*\mathcal{F}]_k.
  \]

Proof: Cf.[Sta]02R6. \( \square \)

Rational Equivalence

Def. (VI.1.0.11) (Principle Divisors). Let \( X \) be an integral scheme locally of f.t. over \( S \) of \( \delta \)-dimension \( n \), and \( f \) a rational function on \( X \), then the principle divisor associated to \( f \) is defined to be the \((n - 1)\)-cycle

\[
(f) = \sum \text{ord}_Z(f)[Z]
\]

as defined in (V.7.1.2). This is truly a \( k \)-cycle.
Cor. (VI.1.0.12) (Rational Equivalence). If $X$ is a scheme locally of f.t. over $S$. Given any locally finite collection of integrally closed subschemes $W_i \subset X$ of $\delta$-dimension $k + 1$ and rational functions $f_i$ on $W_i$, we can consider the $k$-cycle $\sum (i_j)_* (\text{div}(f_i))$ on $X$. This is a cycle because $\coprod W_i \to X$ is proper. Two $k$-cycles are called rational equivalent if they differ by a $k$-cycle of this form. And we can define $CH_k(X)$ the Chow group of $k$-cycles to be $\mathbb{Z}_k(X)$ modulo the rational equivalence relation.

Prop. (VI.1.0.13). Let $X$ be locally of f.t. over $S$, $U$ an open subscheme of $X$, and $Y = X \setminus U$ be given the reduced closed subscheme structure, then there is an exact sequence

$$CH_k(Y) \xrightarrow{i_*} CH_k(X) \xrightarrow{j^*} CH_k(U) \to 0.$$ 

Proof: Cf. [Sta]02RX. □

Chern Classs

Prop. (VI.1.0.14) (Grothendieck-Riemann-Roch).

Proper Intersection

Chow Ring

Def. (VI.1.0.15) (Setup). Let $X$ be locally of f.t. over $S$, then the following are equivalent:

- There exists a decomposition $X = \coprod_n X_n$ where $X_n$ is pure of dimension $n$.

These conditions are satisfied if $X$ is normal or Cohen-Macaulay.

Proof: □

Def. (VI.1.0.16) (Cohomological Chow Cycles). Let $X$ be locally of f.t. over $S$ satisfying the hypothesis of (VI.1.0.15), we define

$$Z^p(X) = \prod_n Z_{n-p}(X_n),$$

and the Chow group of codimension $p$ cycle

$$CH^p(X) = \prod_n CH_{n-p}(X_n).$$

Def. (VI.1.0.17) (Fundamental Class). Let $X$ be locally of f.t. over $S$ satisfying the hypothesis of (VI.1.0.15), we define

$$[X] = \prod [X_n] \in CH^0(X)$$

to be the fundamental class of $X$

Prop. (VI.1.0.18) (Bezout). The Chow ring of $\mathbb{P}^n_k$ is isomorphic to $\mathbb{Z}[x]/(x^{n+1})$. The degree of an irreducible closed variety corresponds to the coefficient of it.
Def. (VI.1.0.19). Let $X$ be a prevariety over $K$ and $L_1, L_2$ be line bundles on $X$. $L_1, L_2$ are called algebraically equivalent if there is a smooth variety $T$ called the parameter space and a line bundle $L$ on $X \times T$ that

$$L_1 \cong L|_{X \times t_1}, \quad L_2 \cong L|_{X \times t_2}.$$  

This is an equivalence, because the product of smooth varieties is also a smooth variety by (V.8.1.9) and (V.5.3.1).

Cor. (VI.1.0.20) (Pic\(^0\)(X)). The element in Pic(X) algebraically equivalent to 0 is a subgroup of Pic(X), denoted by Pic\(^0\)(X).

Prop. (VI.1.0.21). Let $L_1, L_2$ be algebraically equivalent line bundles on a projective prevariety $X$ over $K$, then $\deg_{L_1}(X) = \deg_{L_2}(X)$. More generally, this holds for a complete prevariety over $K$.

Proof: Cf.[Diophantine Geometry, P562]. □

Cor. (VI.1.0.22). Let $X$ be a complete prevariety over $K$ and $L_i$ be line bundles and $Z \in \mathbb{Z}_r(X)$, then $\deg(c_1(L_1).c_1(L_2).\ldots.c_r(L_r).Z)$ only depends on algebraic equivalence classes of $L_i$.

Proof: Cf.[Diophantine Geometry, P562]. □

Prop. (VI.1.0.23) (Algebraically Equivalent Divisors). Let $X$ be a smooth $K$-prevariety, then $\text{Pic}(X) \cong CH^1(V.7.1.19)$. We call two divisors $D_1, D_2$ on $X$ algebraically equivalent if $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$ are algebraically equivalent.

Then(VI.1.0.21) shows algebraic equivalent divisors have the same degree. And for a smooth projective curve over an alg.closed field $K$, the converse is also true.

Proof: Cf.[Diophantine Geometry, P563]. □

Def. (VI.1.0.24) (Numerically Equivalence). Tow line bundles $L_1, L_2$ on a smooth complete prevariety over a field $K$ is called numerically equivalent if $c_1(L_1).\alpha = c_1(L_2).\alpha$ for any $\alpha \in CH^1(X)$. Then(VI.1.0.22) shows algebraically equivalent line bundles are numerically equivalent. Two divisors on $X$ is called numerically equivalent if their corresponding line bundle do.
VI.2 ÉTALE COHOMOLOGY

Basic references are [Fu11], [Sta] and [Étale Cohomology Tamme].

1 Basics (Tamme Level Stuff)

Prop. (VI.2.1.1) (Zariski-Étale Comparison). Considering the inclusion \( \varepsilon : X_{zar} \to X_{ét} \) of topologies, for any Abelian sheaf \( F \) on \( X_{ét} \), there is a Leray spectral sequence (V.6.1.7)

\[ E_2^{pq} = H^p_{zar}(X, R^q\varepsilon^*(F)) \Rightarrow H^{p+q}_{ét}(X, F). \]

Def. (VI.2.1.2) (Pushforward & Pullback). Denote \( \widetilde{X}_{ét} \) as the category of sheaves on \( X_{ét} \).

For a morphism of schemes \( X \to Y \), there is a morphism of topologies \( f_{ét} : Y_{ét} \to X_{ét} \), and we define

\[ f_* = (f_{ét})^* : \widetilde{X}_{ét} \to \widetilde{Y}_{ét}, \quad f^* = (f_{ét})_* : \widetilde{Y}_{ét} \to \widetilde{X}_{ét} \]

\( f^* \) is called the inverse image, it is exact because \( f_{ét} \) preserves fiber product and final object (V.1.2.13). So it is a morphisms of sites \( X_{ét} \to Y_{ét} \). \( f^*G(X') = f^*F(Z') = F(Z') \), and \( H^q(X_{ét}; Z', F) \cong H^q(X'_{ét}; Z', F/X') \), by (V.6.1.6) and (V.1.2.22).

Cor. (VI.2.1.3). For a \( f : X' \to X \) étale, \( f^* \) induce a morphism of topoi, that \( F/X'(Z') = f^*F(Z') = F(Z') \), and \( H^q(X_{ét}; Z', F) \cong H^q(X'_{ét}; Z', F/X') \), by (V.6.1.6) and (V.1.2.22).

Prop. (VI.2.1.4). If \( Z \) is étalé over \( X \), then the canonical morphism

\[ f^* \text{Hom}_X(-, Z) \to \text{Hom}_Y(-, Z \times_X Y) \]

is an isomorphism.

Proof: By definition, \( f^* \text{Hom}_X(-, Z) \) is the sheaf associated to the presheaf \( f_* \text{Hom}_X(-, Z) \) (V.1.2.10), which is identical to the presheaf \( \text{Hom}_Y(-, Z) \) on \( Y_{ét} \), but it is already a sheaf (V.1.3.21).

Prop. (VI.2.1.5) (Leray Spectral Sequence). For any Abelian sheaf on \( X_{ét} \) and any étale scheme \( Y'/Y \), there is a Leray spectral sequence??:

\[ E_2^{pq} = H^p(Y', R^qf_*(F)) \Rightarrow H^{p+q}(Y' \times_Y X, F) \]

Prop. (VI.2.1.6) (Leray Spectral Sequence). If \( f : X \to Y, Y \to Z \) is a morphism of schemes, then for any sheaf on \( X_{ét} \), there is a Leray spectral sequence (V.6.1.7)

\[ E_2^{pq} = R^pg_*\text{(R}^qf_*(F)) \Rightarrow R^{p+q}(gf)_*(F) \]

Prop. (VI.2.1.7) (Commutes with Colimits). If \( X \) is qcqs, then by (V.1.3.8) (V.1.3.9) and (V.6.1.10), \( H^q_{ét}(X, -) \) commutes with filtered colimits.
Field Case

Prop. (VI.2.1.8) (Étale Site on Fields). The functor \( f : X' \to X'(k_s) \) is an equivalence of topologies from the small étale site \((\text{Spec}(k))_{\text{ét}}\) to the canonical topology \( T_G \) on the category of \( G \)-sets, where \( G = G(k_s/k) \).

In particular, any Abelian sheaf on \((\text{Spec}(k))_{\text{ét}}\) is representable by \(?\).

Proof: First \( f \) maps a family of morphisms of schemes to a covering if this family is a covering itself. This is because both are defined by set-theoretical surjectivity, and this is by(V.5.5.11).

Next we need to show this is an equivalence of categories. \( f \) has a left adjoint \( g \) because \( X' \to \text{Hom}_G(U, X'(k_s)) \) is representable for any \( G \)-set \( U \), because any \( G \)-set is equivalent to disjoint sums of \( G/H \), and both category has arbitrary sums, so it suffice to prove for \( G/H \), but this is represented by \( \text{Spec} k' \), where \( k' \) is the fixed field of \( H \).

To prove \( fg \cong \text{id} \) and \( gf \cong \text{id} \), they commutes with direct sums, so the first one is true because \( G/H \to fg(G/H) = \text{Spec}(k_s)(k) \) is an isomorphism, and the second follows from(V.5.5.6) as all étale schemes over field \( k \) is a disjoint union of spectra of finite separable field extensions of \( k \). □

Cor. (VI.2.1.9). By(IV.3.2.1), \( F \to \lim_{k \subset k'} F(\text{Spec} k') \) is an equivalence between the category of Abelian sheaves on \((\text{Spec} k)_{\text{ét}}\) to the category of continuous \( G \)-modules. So

\[
H^q_{\text{ét}}(\text{Spec} k, F) \cong H^q(G, \lim_{k \subset k'} F(\text{Spec} k_s)).
\]

And if \( k \) is separably closed, then \((\text{Spec} k)_{\text{ét}}\) is equivalent to \( \text{Ab} \), and \( H^p(\text{Spec} k, F) = 0 \) for \( p > 0 \).

Stalks

Def. (VI.2.1.10) (Stalk). By(VI.2.1.8), for any scheme \( X \) and a geometric point \( P \), the section functor \( F \to F(P) \) is an equivalence of categories from \((\text{Spec} k)_{\text{ét}}\) to \( \text{Ab} \), thus for any \( F \in X_{\text{ét}} \), we can define the stalk \( F_P = u^* F(P) \).

Prop. (VI.2.1.11). For any geometric point \( P \) of \( X \),

- the stalk map is exact and commutes with colimits.
- For any morphism \( u : P' \to P \) of geometric points over \( X \), \( F_P \cong F_{P'} \).
- If \( X \to Y \) is a morphism, then \( (f^* F)_P \cong F_P \).

Proof: 1: taking stalk is a composition of \( f^* \) and taking section over \( P \)(which is an equivalence), so it is exact and commutes with colimits(VI.2.1.2).

2,3: Trivial. □

Prop. (VI.2.1.12) (Stalk is Defined Naturally). By the definition of \( f^*(\text{V.1.2.10}) \), if \( X' \) be an étale nbhd of \( P \) in \( X \), i.e. \( P \to X' \to X \), then

\[
(f_{\text{ét}})_P(F(P)) = \lim_{X' \to X} F(X')
\]

and \( F_P = f^* F(P) \), thus there is a natural map \( \lim_{X' \to X} F(X') \to F_P \).

Then we have:

\[
\lim_{X' \to X} G(X') \to (G^2)_P
\]

for any presheaf \( G \) on \( X_{\text{ét}} \).
VI.2. ÉTALE COHOMOLOGY

Proof: Firstly \((f^\sharp)(G) \cong (f^*G)^\sharp\) by (VI.1.2.11). Then it suffices to prove that \(G(P) \to G^\sharp(P)\) is an isomorphism for any presheaf \(G\) on \(P_{\text{et}}\). But this is because \(P_{\text{et}}\) is just \(\text{Ab}(\text{VI.2.1.8})\), and \(P \overset{\text{id}}{\to} P\) is cofinal in the category of coverings of \(P\).

\[\text{Cor. (VI.2.1.13)}}\] For a morphism of schemes \(X \to Y\) and \(P\) is a geometric point of \(Y\), then

\[R^pf_*F_p \cong \lim_{\overrightarrow{P \in \mathcal{Y}}} H^p(X \times_Y Y', F)\]

\[\text{Cor. (VI.2.1.14)}}\] For \(X = \text{Spec } k\), the equivalence (VI.2.1.8) of \(X_{\text{et}}\) with continuous \(G\)-modules are just induced by taking the stalk at \(\text{Spec } k_s\).

\[\text{Prop. (VI.2.1.15)}} (\text{Exactness and Stalks)}\] The exactness, injectivity and surjectivity of maps of sheaves \(F' \to F\) on \(X_{\text{et}}\) can be checked on stalks.

\[\text{Proof: \ It suffices to prove the isomorphism case, because taking stalks are exact(VI.2.1.11) and other maps can be characterized by isomorphisms.}\]

Monomorphism: suppose not, if \(s \in F'(X')\) is mapped to 0, by taking base change, we can assume \(X' = X\), and then \(0 = v(s)_\mathfrak{p} = v(s_\mathfrak{p})\), thus \(s_\mathfrak{p} = 0\) by assumption. Now by (VI.2.1.12), for any \(x\) there is an étale nbhd of \(x\) that \(s\) vanishes on it. So we find an étale covering of \(X\) that \(s\) vanishes, thus \(s = 0\) because \(F'\) is a sheaf.

Epimorphism: Similarly, for any \(v \in F(X')\), we can pass to the base change and assume \(X' = X\), then find for each \(x\) a nbhd that comes from some \(v(s_x)\), and they glue together to be a global section of \(F'(X)\).

\[\text{Prop. (VI.2.1.16)}} (\text{Finite Morphism is Exact)}\] For a finite morphism \(f, f_*\) are exact on étale topos.

\[\text{Proof: \ Check on stalks,}\]

Artin-Schreier Theory and Kummer Theory

\[\text{Prop. (VI.2.1.17)}} (\text{Étale Sheaves of } \mathcal{O}_X\text{-Modules)}\] Recall by (VI.1.3.23) if \(M\) is a Qco \(\mathcal{O}_X\)-sheaf, then \(\mathcal{M}\) is a fpqc sheaf on \(X\), in particular an étale sheaf on \(X\). Now the edge map of the Zariski-étale comparison for \(\mathcal{M}\) is an isomorphism:

\[H^p_{\text{Zar}}(X, M) \cong H^p_{\text{et}}(X, \mathcal{M})\]

In particular, \(H^p_{\text{et}}(X, (\mathbb{G}_a)_X) \cong H^p(X, \mathcal{O}_X)\), and the étale cohomology for Qco sheaves vanishes on affine schemes.

\[\text{Proof: \ We show that } R^p\varepsilon^*(\mathcal{M}) = 0 \text{ for } p > 0, \text{ Cf.}[\text{Tamme P109}]. \text{ Not hard.}\]

\[\text{Prop. (VI.2.1.18)}} (\text{Artin-Schreier Sequence)}\] Let \(X\) be a scheme that has char \(p\), let \(F : (\mathbb{G}_a)_X \to (\mathbb{G}_a)_X\) be the Frobenius map, and let \(P = \text{id} - F\), then there is an Artin-Schreier exact sequence

\[0 \to (\mathbb{Z}/p\mathbb{Z})_X \to (\mathbb{G}_a)_X \overset{P}{\to} (\mathbb{G}_a)_X \to 0\]

\[\text{Proof: \ If } s \in \mathcal{O}_X\text{'s kernel, then } s = s^p\text{, so it is locally constant and comes from the map } \mathbb{Z}/p\mathbb{Z} \to \mathcal{O}_X\text{'. Conversely, for any } s \in \mathcal{O}_X\text{'s kernel, it suffices to find an étale cover that } s\text{ is a } p\text{-th power in } \mathcal{O}_X\text{'. For this, it suffices to notice that for any } p\text{-ring } A, \ A[t]/(t^p - t - s)\text{ is free of rank } p\text{ and étale over } A.\]
Cor. (VI.2.1.19). If $X$ has char $p$, then by the long exact sequence and (VI.2.1.17), there is an exact sequence
\[ 0 \to H^0(X, \mathcal{O}_X)/P(H^0(X, \mathcal{O}_X)) \to H^1(X, \mathbb{Z}/p\mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to 0 \]
where the last one is the fixed elements.

Cor. (VI.2.1.20). If $X = \text{Spec } A$ and $pA = 0$, then $H^q(X, (\mathbb{Z}/p\mathbb{Z})_X) = A/P(A)$ for $p = 0$ and vanish for $p > 0$.

Cor. (VI.2.1.21). If $k$ is separably closed field of char $p$ and $X$ is a reduced proper $k$-scheme, then $H^1(X, (\mathbb{Z}/p\mathbb{Z})_X) = (H^1(X, \mathcal{O}_X))^F$.

Prop. (VI.2.1.22) (Hilbert’s Theorem 90). $H^1(X, (\mathbb{G}_m)_X) \cong \text{Pic}(X)$. Equivalently, $H^1(X, \mathcal{O}_X^*) \to H^1_{\text{ét}}(X, (\mathbb{G}_m)_X)$ is an isomorphism.

Proof: Using the lower five term of the Leray spectral sequence (VI.2.1.1), it will suffice to prove that $R^1\mathcal{I}^*(\mathbb{G}_m)_X = 0$. For this, Cf. [Tamme P107].

Prop. (VI.2.1.23) (Kummer Sequence). If $n$ is invertible on $X$, then there is an exact sequence
\[ 0 \to (\mu_n)_X \to (\mathbb{G}_m)_X \xrightarrow{n} (\mathbb{G}_m)_X \to 0 \]

Proof: The proof is similar to that of Artin-Schreier sequence (VI.2.1.18), noticing that $A \to A[t]/(t^n - s)$ is an étale map.

Cor. (VI.2.1.24). If $n$ is invertible on $X$, then there is an exact sequence
\[ 0 \to H^0(X, \mathcal{O}_X^n) \to H^1(X, (\mu_n)_X) \to n\text{Pic}(X) \to 0 \]

Cor. (VI.2.1.25). If $X = \text{Spec } A$ and $n$ is invertible in $A$, then $H^1(X, (\mu_n)_X) \cong A^*/(A^*)^n$.

Strict Henselization

Def. (VI.2.1.26).

Torsion Sheaves

Def. (VI.2.1.27) (Torsion Sheaf). An Abelian sheaf $\mathcal{F}$ on a topology is called a torsion sheaf if it is associated to a presheaf of torsion Abelian groups. Equivalently, the canonical morphism $\lim_n \mathcal{F} \to \mathcal{F}$ is an isomorphism.

Proof: If $\mathcal{F} = P^{\sharp}$, then consider $0 \to nP \to P \xrightarrow{n} P \to 0$. Because $\sharp$ is exact, $\mathcal{F} = (nP)^{\sharp}$. Because $\sharp$ commutes with inductive limits and $P = \lim_n nP$, $\mathcal{F} = \lim_n \mathcal{F}$.

Conversely, if $\mathcal{F} = \lim_n nP$, then $\mathcal{F} = \lim_n (nP)^{\sharp} = (\lim_n nP)^{\sharp}$ which is presheaf of torsion Abelian groups.

Remark (VI.2.1.28). For a torsion sheaf, $\mathcal{F}(U)$ need not be torsion Abelian, but this is the case if $U$ is quasi-compact, Cf. [Tamme P146].

Prop. (VI.2.1.29) (Being Torsion is Local). An Abelian sheaf $F$ on $X_{\text{ét}}$ is a torsion sheaf iff all stalks $F_x$ are torsion groups.
Proof: Use the definition of torsion sheaf $F = \varprojlim_n nF$ and the fact isomorphisms are checked on stalks (VI.2.1.15) and stalk maps are exact (VI.2.1.11).

Prop. (VI.2.1.30).
- If $X \to Y$ is a morphism of schemes and $F$ is a torsion sheaf on $Y$, then $f^*F$ is torsion sheaf on $X$.
- If $X \to Y$ is a qcqs morphism of schemes and $F$ is a torsion sheaf on $X$, then $R^q f_*F$ are torsion sheaves on $Y$.
- In particular, if $X$ is qcqs and $F$ is a torsion sheaf on $X$, then $H^q_{\text{ét}}(X, F)$ are torsion for all $q$.

Proof: 1: This follows immediately from (VI.2.1.29) and (VI.2.1.11).
2: For any $x \in Y$, $(R^q f_*F)_{\overline{x}} \cong H^q(\overline{X}, \overline{F})$, where $\overline{X} = X \otimes_Y \overline{Y}$ and $\overline{Y}$ is the strict localization of $Y$ in $\overline{y}$ by ?. Now $\overline{F}$ is torsion sheaf by item1, and $\overline{X} \to \overline{Y}$ is also qcqs with $\overline{Y}$ being affine, so $H^q(\overline{X}, \overline{F})$ is torsion by item3, so $R^q f_*F$ is torsion by (VI.2.1.29).
3: By (VI.2.1.7), in this case, $H^q_{\text{ét}}(X, -)$ commutes with filtered colimits, so we can replace $F$ by $nF$. Then multiplying by $n$ is zero on $F$, so also it is zero on $H^q_{\text{ét}}(X, F)$, so $H^q_{\text{ét}}(X, F)$ is torsion.

Prop. (VI.2.1.31). If $X$ is Noetherian scheme and $x$ is a point of $X$, let $i : \text{Spec}(k(x)) \to X$ be the structure map, then

- for any Abelian sheaf $F$ on $\text{Spec}(k(x))_{\text{ét}}$, the sheaves $R^p i_*F$ are torsion sheaf for $p > 0$.
- $H^p_{\text{ét}}(X, i_*F)$ are torsion for all $p > 0$.

2: Consider the Leray spectral sequence $H^p_{\text{ét}}(X, R^q i_*F) \Rightarrow H^{p+q}_{\text{ét}}(\text{Spec}(k(x)), F)$, the left term vanishes for $p \geq 0, q > 0$ by item1 and (VI.2.1.30), and the right hand side vanish for $p+q > 0$ by (VI.2.1.30), then it can be checked that $H^p_{\text{ét}}(X, i_*F)$ are torsion for $p > 0$.

Prop. (VI.2.1.32). For a closed subscheme $i : Y \subset X$, $R^pi^!$ preserves torsion sheaves.

Proof: Cf. [Tamme P148].

Prop. (VI.2.1.33). For a regular Noetherian scheme $X$, $H^q_{\text{ét}}(X, (\mathbb{G}_m)_X)$ are torsion for $q \geq 2$.

Proof: Cf. [Tamme P149].

Prop. (VI.2.1.34). For a torsion sheaf $F$, define $F(l) = \varinjlim_n nF$, so it is a $l$-torsion sheaf, and in fact

$$\oplus F(l) = F$$

this is because this is true at the stalks, because stalks is exact and commutes with colimits (VI.2.1.11).

So if $X$ is qcqs, then $H^p(X, F) \cong H^p(X, F(l))$, which is the primary decomposition of $H^p(X, F)$.

Def. (VI.2.1.35) (Cohomological Dimension). If $X$ is qcqs, then we define the cohomological $l$-dimension of $X$ as the smallest number $cd_l(X) = n$ that $H^p(X, F)(l) = 0$ for all $p > n$ and $F$ torsion sheaf on $X$, and define the cohomological dimension of $X$ as the smallest number $cd(X) = n$ that $H^p(X, F) = 0$ for all $p > n$ and $F$ torsion sheaf on $X$. Equivalently, $cd(X) = \sup_l \{cd_l(X)\}$. 
Prop. (VI.2.1.36). If $X$ is an algebraic scheme over a field $k$ of char $p$, then

$$cd_l(X) \leq \begin{cases} 2 \dim X + cd_l(k) & l \neq p \\ \dim X + 1 & l = p \end{cases}.$$  

Proof:

Cor. (VI.2.1.37). If $k$ is separably closed, then $cd(X) \leq 2 \dim X$.

Proof:

Prop. (VI.2.1.38) (Artin Vanishing theorem). If $X$ is an affine algebraic scheme over a separably closed field $k$, then $cd(X) \leq \dim X$.

Proof: Cf.[Milne Étale Cohomology P153].

Prop. (VI.2.1.39) (Arc Descent for Étale Cohomology). Let $R$ be a ring and $G$ a torsion sheaf on $(\text{Spec} R)_{\text{ét}}$, and $\mathcal{F} : \text{Sch}_{\text{qcqs},R}^{\text{op}} \to \mathcal{D}(\Lambda)^{\geq 0}$ be the functor $(f : X \to \text{Spec} R) \mapsto R\Gamma(X_{\text{ét}}, f^*G)$, then $\mathcal{F}$ satisfies arc-descent.

Proof: Cf.[Arc Topology, Bhatt, 5.4.]

Locally Constructible Sheaves

Prop. (VI.2.1.40). $\mu_{n,X}$ is étale over $X$ iff $n$ is prime to the characteristic of all local residue fields of $X$. (Only unramifiedness is concerned, and it is fiberwise(I.7.6.6). And we can compute the Kahler differential of $k[T]/(T^n - 1)$ vanish iff $n \neq 0$ in $k$.

In this case, $\mu_n$ is locally isomorphic to $(\mathbb{Z}/n\mathbb{Z})_X$, because for any affine open $U = \text{Spec} A$, $U' = \text{Spec} A[t]/(t^n - 1) \to U$ is étale and surjective(I.7.1.14) and $U'$ has all $n$-th roots of unity, so $(\mu_n)_{\text{Spec} U'} \cong (\mathbb{Z}/n\mathbb{Z})_{\text{Spec} U'}$.

Prop. (VI.2.1.41). If $G$ is a commutative, finite and étale group scheme on $X$, the sheaf $G_X$ represented by $G$ is locally finite on $X_{\text{ét}}$.

Conversely, any locally constant sheaf on $X_{\text{ét}}$ is represented by a unique commutative étale group scheme over $X$, and it is finite if $F$ has finite stalks.

Proof: Cf.[Tamme P152].

Def. (VI.2.1.42). An Abelian sheaf is called finite iff all its stalks are finite.

Def. (VI.2.1.43) (Constructible Sheaf). An Abelian sheaf $F$ on $X_{\text{ét}}$ is called constructible if each affine open subset $U$ has a decomposition into f.m. constructible reduced subschemes $U_i$ of $U$ that $F/U_i$ are locally constant and finite(i.e. stalks are finite) for each $i$.

Prop. (VI.2.1.44) (Properties of Constructible Sheaves).

- If $F$ is an Abelian sheaf on $X_{\text{ét}}$ that $X$ has a finite decomposition into constructible reduced subschemes $X_i$ that $F/X_i$ are locally constant, then $F$ is constructible. The converse is also true if $X$ is qcqs.
- Constructible is a local property.
- Constructibility is stable under pullback, pushout and finite direct limits.
• Constructibility is stable \( j_i \) for a qc étale map.
• Subsheaves of a constructible sheaf is constructible.

**Proof:** Cf.[Tamme P155], [Conrad L3 P2], [Étale Cohomology and Weil Conjecture P42]. □

**Prop. (VI.2.1.45) (lcc Sheaves and Finite Étale Schemes).** The functor \( X \mapsto \text{Hom}_S(-, X) \) defines an equivalence of categories between the category of finite étale \( S \)-schemes to the category of locally constant finite sheaves.

**Proof:** The Yoneda functor is fully faithful, thus we need to show the essentially surjectivity. Notice first \( \text{Hom}_S(-, X) \) is locally constant finite: we can restrict to an open subset of \( S \) that the fiber are of fixed order \( n \), and \( X \to X \times_S X \) is étale and a closed immersion, thus \( X \times_S X = X \amalg Y \), and \( Y \) is finite étale over \( X \) through \( \pi_1 \). Now by induction on the order of the fiber, \( Y = X \otimes \Sigma' \) locally. So \( X = S \times \Sigma \) locally, which means \( X \) represents the constant sheaf \( \Sigma \) locally.

To show that every locally constant finite étale sheaf is represented by a finite étale scheme, Cf.[Conrad, P19]. □

**Prop. (VI.2.1.46).** If \( G \) is a commutative étale group scheme over \( X \), then the sheaf \( G_X \) represented by \( G \) is constructible iff \( G \) is f.p. over \( X \).

**Proof:** □

**Prop. (VI.2.1.47) (Locally Constancy and Stalks).** If \( F \) is a finite sheaf over a Noetherian scheme \( S \), then \( F \) is locally constant iff all the specialization maps for geometric points \( F_\pi \to F_\eta \) are bijective.

**Proof:** If \( F \) is locally constant, because the conclusion is local, we may assume \( F \) is constant, then \( F_\pi \to F_\eta \) are all identities.

Conversely, for any geometric point \( s, \Sigma = F_s \) is finite by definition, thus there is an étale nbhd \( U \) of \( s \) that the map \( \Sigma \to F \) induces an isomorphism on \( s \)-stalks, so this is an isomorphism for any geometric point linked to \( s \) by specialization, in particular the generic point of the irreducible component containing \( s \) and all the points in this irreducible component, thus \( F \) is constant on an open nbhd of \( s \) (because \( X \) is Noetherian thus has f.m. irreducible components), so \( F \) is locally constant because \( X \) is Noetherian. □

**Prop. (VI.2.1.48) (Constructible Sheaves are Noetherian).** The constructible sheaves of Abelian groups are exactly the Noetherian objects in the category of torsion sheaves.

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### 2 Étale Fundamental Group

Basic references are [Sta]Chap53 and [Fu11]Chap3.

**Def. (VI.2.2.1) (Geometric Point).** A geometric point of a scheme over a field \( k \) is a map \( \overline{x} : \text{Spec}(k_s) \to X \) where \( k_s \) is a separable closure of \( k \).

**Lemma (VI.2.2.2) (Rigidity Lemma).** If \( f, g : S' \to S'' \) are two \( S \)-morphisms where \( S'' \) is a separated étale \( S \)-scheme and \( (S', \overline{x}') \) is a pointed scheme that \( f(\overline{x}') = g(\overline{x}') \), then \( f = g \).

**Proof:** The diagonal \( S'' \to S'' \otimes_S S'' \) is a closed immersion and also étale hence open(V.5.5.3), so the diagonal is an clopen subset. And now \( f \times g : S' \to S'' \otimes_S S'' \) intersects the diagonal, and \( S' \) is connected, so \( f, g \) are identical on the diagonal. □
Def. (VI.2.2.3) (Galois Cover). If \((S, \pi)\) is a pointed connected scheme, \(S' \to S\) is a finite étale cover of degree \(n\), then there are at most \(n\) point over \(\pi\), so by (VI.2.2.2), \(|\text{Aut}(S'/S)| \leq n\). If the equality holds, then we call \(S'/S\) a Galois cover and define \(\text{Gal}(S'/S) = \text{Aut}(S'/S)^{op}\).

Prop. (VI.2.2.4). If \(S' \to S\) is a connected finite étale cover, then there is a finite étale cover \(S'' \to S'\) that \(S'' \to S\) is Galois.

Proof: Cf. [SGA1, Exp.V, §2 – §4]. □

Def. (VI.2.2.5) (Étale Fundamental Group). For any two finite Galois étale cover \(S'/S, S''/S\), if there is a \(S\)-morphism \(S'' \to S'\), then it induces a morphism of Galois groups because the Galois group of \(S'\) acts transitively on the fiber over a closed point. And it is surjective by the same reason for \(S''\).

Then we define the étale cohomology group
\[
\pi_1(S, x) = \lim_{\rightarrow} \text{Gal}(S'/S)
\]

Prop. (VI.2.2.6) (Fundamental Group and Covers). For \(X\) connected smooth scheme and \(\pi \to X\) a geometric point, there is a profinite group \(\pi_1(X, \pi)\) that there is a correspondence:
\[
\{\text{finite étale covers } Y \to X\} \leftrightarrow \{\text{Finite sets with a continuous action of } \pi_1(X, x)\}
\]
Such a group \(\pi_1(X, \pi)\) is called the étale fundamental group of \(X\) w.r.t \(\pi\).

Proof: □

Prop. (VI.2.2.7). Let \((S, s)\) be a connected scheme, then the functor \(S' \to S_\pi\) induces an equivalence of categories between the finite étale covers \(S' \to S\) with the category of finite discrete \(\pi_1(X, x)\)-sets.

Proof: We may assume \(S'\) is connected, then use (VI.2.2.4) to find a Galois cover \(S'' \to S'\) that \(S''/S\) is Galois, then clearly there is a bijection
\[
\text{Gal}(S''/S)/\text{Gal}(S''/S') \cong S'(s')
\]
And any transitive discrete \(\pi_1(X, x)\)-sets arise this way.

To prove the essentially surjectivity and fully faithfulness, \_

Cor. (VI.2.2.8). The étale fundamental group is independent of the base point \(\pi\) chosen.

Proof: This is because for two profinite groups, if the categories of their finite sets are equivalent, then they are isomorphic. □

Cor. (VI.2.2.9) (Locally Constant Sheaves and Fundamental Group). By (VI.2.1.44), if \(X\) is a connected scheme and \(\pi\) be a geometric point of \(X\), then there is an equivalence of categories between finite locally constant Abelian sheaves on \(X\) and finite \(\pi_1(X, \pi)\)-modules.

Prop. (VI.2.2.10). For \(k\) alg.closed, \(\pi_1(\mathbb{P}^1_k) = 0\).

Proof: □

Prop. (VI.2.2.11) (Arithmetic Geometric Exact Sequence). If \(X_0\) is a variety over \(\mathbb{F}_q\), then there is an exact sequence
\[
1 \to \pi_1(X, \pi) \to \pi_1(X_0, \pi) \to G(\overline{\mathbb{F}}/k) \to 1.
\]
3 Curve Case

Prop. (VI.2.3.1). If $X$ is separated f.t. scheme of dimension $\leq 1$ over a separably closed field $k$, and $\mathcal{F}$ is a torsion sheaf on $X_\text{ét}$, then $H^i_\text{ét}(X, \mathcal{F}) = 0$ for $i > 2$, and if $\mathcal{F}$ is constructible, then $H^i_\text{ét}(X, \mathcal{F})$ are finite.

Moreover if $X$ is affine and $\mathcal{F}$ is locally killed by $n$ not divisible by $\text{char } k$ or $X$ is proper and sections of $\mathcal{F}$ are locally $p$-torsions with $p = \text{char } k > 0$, then $H^2_\text{ét}(X, \mathcal{F}) = 0$.

Proof: Cf.[Conrad L4 P4] and [Tamme]. □

Lemma (VI.2.3.2). If $X$ is a connected smooth projective curve over an alg.closed field $k$ of $\text{char } p > 0$ and $n$ is not divisible by $p$, $M$ is a finite Abelian group, then

$$H^1_\text{ét}(X, \mu_n) \cong \text{Pic}(X)[n], \quad H^2_\text{ét}(X, \mu_n) = \mathbb{Z}/n\mathbb{Z}, \quad H^0(X, M) = M, \quad H^{>2}_\text{ét}(X, \mu_n) = 0;$$

$$H^1_\text{ét}(X, \mathbb{Z}/p\mathbb{Z}) \text{ is finite, } \quad H^2_\text{ét}(X, \mathbb{Z}/p\mathbb{Z}) = 0, \quad H^{>2}_\text{ét}(X, \mathbb{Z}/p\mathbb{Z}) = 0.$$  

Proof: Cf.[Conrad L4 P6] □

Duality

$l$-adic Cohomology

4 Base Changes

Prop. (VI.2.4.1) (Proper Base Change). If there is a Cartesian diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{g'} & Y' \\
\downarrow{f'} & & \downarrow{f} \\
X & \xrightarrow{g} & Y
\end{array}
$$

that $f$ is proper, then for any torsion Abelian sheaf $\mathcal{F}$, the base change map (V.6.1.15)

$$g^* R^q f_* \mathcal{F} \to R^q f'_* (g^* \mathcal{F})$$

is an isomorphism.

Proof: Cf.[Conrad L6]. □

5 Cohomology with Compact Support

Cf.[Weil1, P18].

Lemma (VI.2.5.1). Extension by 0 commutes with pullback Cf.[KF Lemma4.9].

Def. (VI.2.5.2). For a scheme $X$ any étale sheaf $\mathcal{F}$ on $X$, and any morphism $f : X \to Y$, if it can be extended to a proper morphism $f^c : X^c \to Y$ where $j : X \to X^c$ is an open dense subscheme, then we define the higher direct image with compact support as

$$Rf_! = Rf^c_* j_! : D^+(X, \text{tor}) \to D^+(Y, \text{tor}).$$

And this is the case of $Y = \text{Spec } k$ and we define $H^i_c(X, \mathcal{F}) = H^i(\text{Spec } k, Rf_! \mathcal{F})$. 

Prop. (VI.2.5.3). Any separated morphism of f.t. $X \to Y$ that $Y$ is qcqs has higher direct image with compact support. And it is independent of the compactification chosen.

Thus $f_i$ can be defined even in case $Y$ is not qcqs, because we can define it on each affine opens $U_i$ of $S$ and glue them together by uniqueness.

Proof: The compactification exists because of Nagata compactification. For the uniqueness, notice for any two compactification, we can find a common compactification that dominates them both ?, so using lemma(VI.2.5.4), we easily show they are isomorphic. \[\] □

Lemma (VI.2.5.4) ($i_{!}$ and Higher Pushforward). If there is a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i} & X \\
\downarrow{f} & & \downarrow{f} \\
Y & \xrightarrow{j} & Y
\end{array}
$$

where $i, j$ are open immersion and $f, f$ are proper, then there is a natural transformation $j_{!}f_{*} \to f_{*}i_{!}$

that induces a natural transformation

$$j_{!}Rf_{*} \to Rf_{*}i_{!},$$

which is an isomorphism iff $f$ is proper.

Proof: If this is a Cartesian diagram, then the natural transformation is give by

$$j_{!}f_{*} \to f_{*}f_{!} \cong f_{*}i_{!}f_{*} \to f_{*}i_{!}.$$ (the second isomorphism is by(VI.2.5.1)). The rest is by proper base change Cf.[KF P88].

The general case is also easily reduced to the Cartesian case. \[\]

Prop. (VI.2.5.5) (Proper Map Induces Map on Proper Pushforward). If $g : Y \to X$ is a proper morphism between schemes separated of f.t. over a Noetherian scheme $S$, then for any étale Abelian sheaf $\mathcal{F}$ on $X$, there is a canonical mop

$$g: R(f_{1})!(\mathcal{F}) \to R(f_{2})!$$

Proof: Choose a compactification $X_{2} \xrightarrow{i_{2}} \overline{X}_{2}$, then choose a compactification $X_{1} \xrightarrow{i_{1}} \overline{X}_{1}$ of $i \circ g$, now


Prop. (VI.2.5.6) (Properties of Compact Pushforward).

• (Base Change) If there is a Cartesian diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{h} & X \\
\downarrow{f'} & & \downarrow{f} \\
S' & \xrightarrow{g} & S
\end{array}
$$

that $f$ is separated of f.t., then there is a natural isomorphism

$$g^{*}Rf_{!} \cong R(f'_{!})h^{*}.$$
• (Composition) For two separated morphisms of f.t, \( R(f_1f_2)! = Rf_1!Rf_2! \), which induces Leray spectral sequence.

• (Excision) Let \( f : X \to S \) be a separated morphism of f.t, and \( \mathcal{F}^* \in D^+(X,\text{tor}) \). Let \( Z \subset X \) be a closed subscheme and \( U = X - Z \), then there is a long exact sequence

\[
\cdots \to R^p(f_U)!((\mathcal{F}^*)_U) \to R^p f_! \mathcal{F}^* \to R^p(f_Z)!((\mathcal{F}^*)_Z) \to R^{p+1}(f_U)!((\mathcal{F}^*)_U) \to \cdots
\]

Proof: Cf.[Conrad L10 P3].

1: Choose a compactification of \( f \), then it suffices to show \( Rf_c! \) and \( j! \) both commutes with base change, which is by proper base change(VI.2.4.1) and (VI.2.5.1).

2: Two compactification can be splinted, and use (VI.2.5.4).

3: Use the long exact sequence applied to the exact sequence(checkered on stalks(VI.2.1.15))

\[
0 \to j!j^*F \to F \to i_*i^*F \to 0
\]

□

Prop. (VI.2.5.7) (Proper Pushforward to Direct Image). There is a natural map from \( R^i f_! \to R^i f_* \), which is induced by

\[
R^i f_! = R^i f_c^! j! \to R^i f_c^! j_* \to R^i f_*
\]

where the second one is edge map of Leray spectral sequence.

In particular, there is a map \( H^i_{c,\text{et}}(X,F) \to H^i_{\text{et}}(X,F) \).

Prop. (VI.2.5.8) (Vanishing Result). If \( f : X \to S \) is separated of f.t. and let \( d = \sup_{s \in S} \dim X_s \), then if \( \mathcal{F} \in D(X,\text{tor}) \) satisfies \( pH^p(\mathcal{F}) = 0 \) for \( p \geq r \), then

\[
R^p f_! \mathcal{F} = 0, \quad p \geq r + 2d.
\]

Proof: Cf.[Conrad L10 P4].

□

Prop. (VI.2.5.9) (Projection Formula). If \( X \to S \) is a quasi-projective morphism, \( \mathcal{F} \in D^-(S) \) and \( \mathcal{G} \in D(X) \), then we have a natural isomorphism

\[
\mathcal{F} \otimes^L Rf_! \mathcal{G} \cong Rf_!(f^* \mathcal{F} \otimes^L \mathcal{G})
\]

Proof: We may pass to the compactification, as \( j_! \) commutes with \( f^* \).

□

Finiteness Theorems

Prop. (VI.2.5.10). If \( f : X \to S \) is a separated morphism of f.t., \( S \) is Noetherian, and \( \mathcal{F} \) is a constructible Abelian sheaf on \( X \) whose torsion is order is invertible in \( S \), then \( R^p f_! F \) are all constructible on \( Y \).

Proof: Cf.[Conrad L10 P5].

□

Cor. (VI.2.5.11). If \( X \) is a proper scheme over a separably closed field \( k \) and \( F \) is a constructible Abelian sheaf on \( X_{\text{et}} \), then \( H^q(X,F) \) are finite for all \( p \geq 0 \).
Lemma (VI.2.5.12). If $X \to S$ is smooth and proper, and $F$ is a locally constant finite Abelian sheaf with torsion order invertible on $S$, suppose $S$ is Noetherian, then all specialization maps for $R^p f_* F$ are isomorphisms.

Proof: Cf.[Conrad L10 P5].

Prop. (VI.2.5.13). If $X \to S$ is smooth and proper, and $F$ is a locally constant finite Abelian sheaf (VI.2.1.42) with torsion order invertible on $S$, then $R^p f_* F$ are locally constant finite sheaves for any $p \geq 0$.

Proof: By (VI.2.1.45), we may assume $F = X'$ for some finite étale scheme $X' \to X$. By Noetherian descent, together with proper base change, we may reduce to the case $S$ is Noetherian. Thus by (VI.2.5.10), $R^p f_* F = R^p f_* F$ are constructible, and (VI.2.5.12) shows that the stalk maps are isomorphisms. So (VI.2.1.47) shows that $R^p f_* F$ are locally constant finite.

Comparison Theorems

6 Poincare Duality

Prop. (VI.2.6.1) (Poincare Duality). If $F$ is lisse, $X$ is smooth separated of dimension $d$, then we have a perfect pairing

$$H^*_\text{ét}(X_k, F) \times H^{2d-n}_{\text{ét}}(X_k, F^\vee(d)) \to H^{2d}_{\text{ét}}(X, \bar{\mathbb{Q}}_l(d)) \xrightarrow{\text{tr}_X} \bar{\mathbb{Q}}_l$$

which is Galois Invariant.

Proof: Cf.[Conrad L12-13]. Use Duality, Cf.[Weil 2Bhatt P5].

Cor. (VI.2.6.2) (Weak Lefschetz). Is a consequence of poincare duality and localization sequence.

Proof: Cf.[Weil 1 Proof P19].
VI.3 Pro-Étale Cohomology

Main references are [B-S14] and [Sta]Chap56.

1 Introduction

In his second paper on the Weil conjectures ([Del80]), Deligne introduced a derived category of \( l \)-adic sheaves as a certain 2-limit of categories of complexes of sheaves of \( \mathbb{Z}/l^n \mathbb{Z} \)-modules on the étale site of a scheme \( X \). This approach is used in the paper by Beilinson, Bernstein, and Deligne ([BBD82]) as the basis for their beautiful theory of perverse sheaves. In a paper entitled "Continuous Étale Cohomology" ([Jan88]) Uwe Jannsen discusses an important variant of the cohomology of a \( l \)-adic sheaf on a variety over a field. His paper is followed up by a paper of Torsten Ekedahl ([Eke90]) who discusses the adic formalism needed to work comfortably with derived categories defined as limits.

2 Ring-Theoretical Stuff

Def. (VI.3.2.1) (étale Local Isomorphism). A ring map \( A \to B \) is called a local isomorphism if for every prime \( q \in \text{Spec } B \), there is a nbhd \( \text{Spec } B_g \) that \( \text{Spec } B_g \to \text{Spec } A \) is an open immersion.

Prop. (VI.3.2.2). The class of local isomorphisms is stable under base change and compositions. (This follows from (V.4.4.44)). Moreover, if \( A \to B \to C \) are ring maps that \( A \to B, A \to C \) are both local isomorphisms, then \( B \to C \) is also a local isomorphism.

Def. (VI.3.2.3) (w-Local Rings). A ring \( A \) is called w-local if \( \text{Spec } A \) is w-local (IX.1.15.16). It is called strictly w-local if it is w-local and every f.f. étale map \( A \to B \) has a section. A map of rings is called w-local if it induces a w-local map (IX.1.15.16) on the Spec.

Prop. (VI.3.2.4). A w-local ring \( A \) is strictly w-local iff all local rings of \( A \) at closed pts are strictly Henselian.

Proof: Cf.[Pro-Etale Cohomology, Scholze, P10].

Ind-Zariski Algebra

Def. (VI.3.2.5) (Ind-Zariski Algebra). A ring map \( A \to B \) is called ind-Zariski/smooth/étale if \( B \) is a filtered colimit of local isomorphisms/smooth ring maps/étale ring maps \( A \to B_i \).

Prop. (VI.3.2.6) (Properties of Ind-Zariski Algebras). Ind-Zariski ring maps are stable under base change and composition, by (VI.3.2.2). If \( A \to B \to C \) are ring maps that \( A \to B, A \to C \) are both ind-Zariski, then \( B \to C \) is also ind-Zariski.

A \to B \text{ be ind-Zariski, then it identifies local rings.}

Proof:

Def. (VI.3.2.7) (Ind-(Zariski Localization)). A ring map \( A \to B \) is called a Zariski localization if \( B = \prod_{i} A_{f_i} \). An ind-(Zariski localization) of \( A \) is a colimit of Zariski localizations of \( A \).
Ind-Smooth Algebra

Ind-Étale Algebra
VI.4 Crystalline Cohomology

Main references are [Sta].

1 Algebraic deRham Cohomology

Prop. (VI.4.1.1) (Algebraic de Rham Cohomology). Let $X \rightarrow S$ be a morphism of rings, then we define the algebraic de Rham cohomology of $X$ over $S$ as the image of the de Rham complex $\Omega^\bullet_{X/S}$ in $D(\text{Mod}(\mathcal{O}_S))$.

Prop. (VI.4.1.2). There is a similar construction of connections on a f.g. projective $R$-module $M$ and Weil-Chern theory parallel to that of 9 and 3.

But in this case, the trace map is defined only when $M$ is f.g. projective, which is called the Hattoris-Stallings trace: If $A$ is f.g. projective, the natural map $\text{Hom}_R(A, R) \otimes_R A \rightarrow \text{End}_A(P)$ is an isomorphism (Because locally it is an isomorphism??), and the inverse composed with $\text{Hom}_R(A, R) \otimes_R A \rightarrow A$, we get the desired map.

Also, when $M$ is f.g. projective, there is a Levi-Cevita connection induced by the $A \rightarrow \Omega^1_A \otimes_R A$ because $M$ is a direct summand of some $A^n$. This is verified to be independent of $n$, or one can more algeoly use the fact that projective module is locally free.

The Chern character is important, it defines a ring map from $K_0(R)$ to $H_{dR}^\text{ev}(A)$. In fact, this can be lifted to a morphism $K_0(A) \rightarrow HC_0^\text{per} \rightarrow H_{dR}^\text{ev}(A)$, Cf.[阴阳林 循环同调 Dennis trace].

de Rham Complexes

Def. (VI.4.1.3) (Absolute de Rham Complex). Let $B$ be a ring, let $\Omega_B = \Omega_{B/\mathbb{Z}}$, and $\Omega_B^n = \wedge^n \Omega_B$, then there is a total de Rham complex of $B$:

$$B \rightarrow \Omega^1_B \rightarrow \Omega^2_B \rightarrow \cdots$$

as $B$-modules, which is a complex, where $d(b_0 db_1 \wedge \cdots \wedge db_p) = db_0 \wedge db_1 \ldots db_p$.

Proof: $d$ is well-defined on $\Omega^1_B$ because it vanishes on the element $d(a + b) - da - db$ and $d(ab) - ad(b) - bd(a)$ by Leibniz rule, and the we get a map

$$\bigotimes_{i=1}^p \Omega_B \rightarrow \Omega_B^{p+1} : \omega_1 \otimes \ldots \otimes \omega_p \mapsto \sum (-1)^{i+1} w_1 \wedge \ldots \wedge d(\omega_i) \wedge \ldots \wedge \omega_p.$$

We want to descend this to a map on $\Omega_B^n$ using(I.5.1.16): it is clearly alternating, and it suffices to show it is $f$-linear, and this is clear by direct calculation.

Finally $d^2 = 0$. □

Prop. (VI.4.1.4) (Quotient of de Rham Complexes). Let $B$ be a ring and $\pi: \Omega_B \rightarrow \Omega$ be a surjective map of $B$-modules. Denote $d: B \rightarrow \Omega_B \rightarrow \Omega$, and $\Omega^i = \wedge^i \Omega$. Assume that the kernel of $\pi$ is generated as a $B$-module by elements $\omega$ that $\wedge^2(\pi)(d_B(\omega)) = 0$ in $\Omega^2$, then there is a de Rham complex

$$B \rightarrow \Omega^1 \rightarrow \Omega^2 \rightarrow \cdots$$

whose differential is defined by the rules similar to that of(VI.4.1.3).
Proof: Because \( \pi \) is surjective, so do \( \land^i \pi \), and it suffices to show \( \land^i \pi \) gives a connecting morphism between \( \Omega^*_B \) and \( \Omega^* \), then the well-definedness of \( d \) is automatic. Cf.\([Sta]07HY\).

Cor. (VI.4.1.5) (Relative de Rham Complex). If \( B \) is an \( A \)-algebra, the surjection \( \Omega_B \to \Omega_{B/A} \) satisfies the condition of(VI.4.1.4) thus we can define the relative de Rham complex \( \Omega^*_B/A \).

Proof: The verification of the condition is routine.

Prop. (VI.4.1.6) (Universality of de Rham Complexes). Let \( C \) be a \( B \)-algebra and \( (E^*, d) \) a non-negatively graded commutative \( B \)-dga and we are given a \( B \)-algebra map \( \eta : C \to E^0 \) that for every \( x \in C \), the element \( d(\eta(x)) \in E^1 \) satisfies \( d^2 = 0 \), then the map \( C \to E^0 \) extends uniquely to a map \( \Omega^*_C/B \to E^* \) of \( B \)-dga.

Proof: One direction is trivial as \( \Omega^*_C/B \) is strict by construction. Conversely, the composite map \( C \to E^0 \to E^1 \) is a \( B \)-derivation thus extends to a map \( \eta^1 : \Omega^1_{C/B} \to E^1 \), then the universal property of the exterior product(I.5.1.16) gives maps \( \eta^i : \Omega^i_{C/B} \to E^i \), and this gives the desired extension.

Def. (VI.4.1.7) (Connection). Let \( B \) be a ring and \( \Omega_B \to \Omega \) be a quotient satisfying the condition of(VI.4.1.4), then a connection on \( M \) is an additive map
\[
\nabla : M \to M \otimes_B \Omega : \quad \nabla(b \otimes m) = b \nabla(m) + m \otimes db
\]
Given a connection on \( M \), we can define maps
\[
\nabla : M \otimes_B \Omega^i \to M \otimes_B \Omega^{i+1} : \quad \nabla(b \otimes \omega) = \nabla(b) \wedge \omega + m \wedge d\omega
\]
This is well defined because it commutes it commutes with \( B \)-action. The connection is called integrable if \( \nabla^2 = 0 \).

2 PD-Schemes

Def. (VI.4.2.1) (PD-Schemes). A pd-scheme is a triple \((S, I, \gamma)\) where \( S \) is a scheme, \( I \) is a \( Qco \) sheaf of ideals, and \( \gamma \) is a pd-structure on \( I(I.13.0.1) \). A morphism of pd-schemes is a morphism that all the structure morphisms are morphisms of pd-structures.

Def. (VI.4.2.2) (PD-Thickening). A pd-thickening is a \((U, T, \delta)\) where \( T \) is a thickening of \( U(V.7.5.1) \) with sheaf of ideals \( I(U = \text{Spec}(I)) \) that \((T, I, \delta)\) is a pd-structure.

Prop. (VI.4.2.3). The fibered product of two morphisms in the category of pd-schemes exists if one of them is a pd-thickening.

Proof: Cf.\([Sta]07ME\).

3 Crystalline Site

Def. (VI.4.3.1) (Coverings). A family of morphisms \( \{(U_i, T_i, \delta_i) \to (U, T, \delta)\} \) of pd-thickenings is called a Zariski/smooth/étale/syntomic/fppf . . . iff
- \( U_i = U \otimes_T T_i \),
- \( \{T_i \to T\} \) is a Zariski/smooth/étale/syntomic/fppf . . . covering of \( T \).
Def. (VI.4.3.2) (Big Crystalline Site). Let $p$ be a prime and $(S, I, \gamma)$ be a pd-scheme over $\mathbb{Z}_{(p)}$, let $S_0 = V(I) \subset S$, and $X \to S_0$ a morphism of schemes that $p$ is nilpotent on $X$, then the big-crystalline site $(X/S)_{\text{crys}}$ consists of pd-thickenings $(U, \delta)$ over $(S, I, \delta)$ and a morphism of schemes $U \to X$, and the topology is the Zariski topology (VI.4.3.1). In fact for any $(U, T, \delta) \in (X/S)_{\text{crys}}$, $p$ is locally nilpotent in $T$, by (I.13.0.5).

The crystalline site $(X/S)_{\text{crys}}$ is the strictly full subcategory consisting of objects that $U \to X$ is an open immersion.

Notice the structure sheaf that maps $(U, T, \delta)$ to $\Gamma(O_U, U)$ is a sheaf of rings on $(X/S)_{\text{crys}}$, called the structure sheaf $O_{X/S}$.

Prop. (VI.4.3.3) (Comparing with Zariski Site). The functor

$$u_{X/S} : (X/S)_{\text{crys}} \to X_{\text{Zar}} : (U, T, \delta) \to U$$

is cocontinuous (easy to verify), thus defines a morphism of topoi $\text{Sh}((X/S)_{\text{crys}}) \to \text{Sh}(X_{\text{Zar}})$ by (V.1.2.19), which is functorial in $X$ and $S$.

Prop. (VI.4.3.4) (Finite Limits). The category $(S/X)_{\text{crys}}$ has all finite limits, and the forgetful functor $(U, T, \delta) \to U$ preserves finite limits.

Proof: Cf. ([Sta]07I9). \qed

Def. (VI.4.3.5) (Affine Crystalline Site). Let $(A, I, \gamma)$ be a pd-structure that $A$ is a $\mathbb{Z}_{(p)}$-algebra, and $C$ is an $A/I$-algebra that $p$ is nilpotent in $C$, then the crystalline site (VI.4.3.2) $(C/A)_{\text{crys}}$ is the site whose object are pd-structures $(B, J, \delta)$ over $(A, I, \gamma)$ that $p$ is nilpotent in $B$ (I.13.0.5), together with a map of rings $C \to B/J$ over $A/I$, and $(C/A)_{\text{crys}}$ the full subcategory of objects $B$ that $C \to B/J$ is an isomorphism.

Notice for any object $(B, J, \delta)$ in $(C/A)_{\text{crys}}$, $J$ is nilpotent, by (I.13.0.5).

Sheaf of Differentials

Def. (VI.4.3.6) (S-Derivations). Let $\mathcal{F}$ be a sheaf of $O_{X/S}$-modules on $(X/S)_{\text{crys}}$, then an $S$-derivation $D : O_{X/S} \to \mathcal{F}$ is a map of sheaves that for any object $(U, T, \delta)$ of $(X/S)_{\text{crys}}$, the map $D : \Gamma(T, O_U) \to \Gamma(T, \mathcal{F})$ is a pd-derivation over $\Gamma(V, O_V)$ for any open subset $V \subset S$ that $T \to S$ factors through $V$.

Prop. (VI.4.3.7) (Sheaf of PD-Differentials). Similar to the construction of sheaf of differentials (V.2.4.2), we can construct of sheaf of pd-differentials $\Omega_{X/S, \delta}$ on $(X/S)_{\text{crys}}$, which is a quotient of the sheaf of differentials $\Omega_{X/S}$. And similar to (V.2.4.3), for any $(U, T, \delta) \in (X/S)_{\text{crys}}$, $\Omega_{X/S, \delta}(T/S)_{\text{crys}} = \Omega_{T/S, \delta}$.

Prop. (VI.4.3.8) (A First Order PD-Thickening). Let $(U, T, \delta) \in (X/S)_{\text{crys}}$, $\mathcal{J}$ the ideal sheaf of $U$, we define a first order thickening $T'$ of $T$: let $O_{T'} = O_T \oplus \Omega_{T/S, \delta}$ with the algebraic structure that $\Omega^2_{T/S, \delta} = 0$, and let $\mathcal{J}' = \mathcal{J} \otimes \Omega_{T/S, \delta}$, and define the pd-structure as

$$\delta'_n(f, \omega) = (\delta_n(f), \delta_{n-1}(f)\omega).$$

Then $(U, T, \delta')$ is a pd-thickening and $(U, T, \delta) \to (U, T', \delta')$ is a morphism in $(X/S)_{\text{crys}}$. Moreover, there are two ring maps

$$p_0, p_1 : O_T \to O_{T'} : p_0(f) = (f, 0), \quad p_1(f) = (f, d_{T/S, \delta}(f)).$$
Then we get two contraction of the morphism $T \to T'$, and $p_0^\ast - p_1^\ast$ is the universal derivation $d_{T/S,\delta}$ included in $\mathcal{O}_{T'}$.

This construction is functorial in $T/S$ by (VI.4.3.7) and hence gives a functor of sites $(X/S)_{crys} \to (X/S)_{crys}$.

Proof: The verification of the pd-axioms in in [(Sta)[07HH]]. □

Prop. (VI.4.3.9) (Second Order PD-Thickening). There is a further thickening $T''$ of $T'$, which is a second order thickening of $T$:

$$\Omega_{T''} = \mathcal{O}_T \oplus \Omega_{T/S,\delta} \oplus \Omega_{T/S,\delta} \oplus \Omega_{T/S,\delta}$$

with the algebra structure given by

$$(f, \omega_1, \omega_2, \eta)(f', \omega_1', \omega_2', \eta') = (ff', f\omega_1', f\omega_2' + f'\omega_2, f\eta' + f'\eta + \omega_1 \wedge \omega_2')$$

Let $J'' = J \oplus \Omega_{T/S,\delta} \oplus \Omega_{T/S,\delta} \oplus \Omega_{T/S,\delta}^2$, then there is a PD-structure on $J''$ given by

$$\delta''(f, \omega_1, \omega_2, \eta) = (\delta(f), \delta_{n-1}(f)\omega_1, \delta_{n-1}(f)\omega_2, \delta_{n-1}(f)\eta + \delta_{n-1}(f)\omega_1 \wedge \omega_2)$$

This construction is functorial in $T/S$ by (VI.4.3.7) and hence gives a functor of sites $(X/S)_{crys} \to (X/S)_{crys}$.

Proof: For the details, Cf.[Sta]07J3. □

4 Crystals

Def. (VI.4.4.1) (Crystals). In situation (VI.4.3.2), if $\mathcal{F}$ is a sheaf of $\mathcal{O}_{X/S}$-modules on $(X/S)_{crys}$, then it restricts to a sheaf $f_T$ on $T$ for every object $(U, T, \delta) \in (X/S)_{crys}$. And it is functorial. Then $\mathcal{F}$ is called

- a crystal in $\mathcal{O}_{X/S}$-modules if for any morphism $u : (U', T', \delta') \to (U, T, \delta)$ in $(X/S)_{crys}$, the morphism of $\mathcal{O}_{T'}$-modules $u^* \mathcal{F}_T \to \mathcal{F}_{T'}$ is an isomorphism.
- locally Qco if $\mathcal{F}_T$ is a Qco $\mathcal{O}_T$-module for any $(U, T, \delta) \in (X/S)_{crys}$.
- Qco as defined in (V.2.2.22).

In particular, $\mathcal{O}_{X/S}$ is a crystal in $\mathcal{O}_{X/S}$-modules.

Connections

Def. (VI.4.4.2) (de Rham Complex for $(X/S)_{crys}$). On a crystalline site $(X/S)_{crys}$, if we define $\Omega^i_{X/S,\delta} = \wedge^i \Omega_{X/S,\delta}$ (VI.4.3.7), then by (VI.4.1.4), the universal $S$-derivative $d_{X/S}$ give rise to the de Rham complex

$$\mathcal{O}_{X/S} \to \Omega^1_{X/S,\delta} \to \Omega^2_{X/S,\delta} \to \ldots$$

as $\mathcal{O}_{X/S}$-modules on $(X/S)_{crys}$.

Proof: The verification of the condition for the quotient $\Omega_X \to \Omega_{X/S,\delta}$ is routine. □

Def. (VI.4.4.3) (Connections on $(X/S)_{crys}$). We define the notion of connection on $(X/S)_{crys}$ of an $\mathcal{O}_{X/S}$-module $\mathcal{F}$ on $(X/S)_{crys}$ w.r.t. the differential $\Omega_{X/S,\delta}$, as in (VI.4.1.7).
VI.4. CRYSTALLINE COHOMOLOGY

Prop. (VI.4.4.4) (Connection of a Crystal). Any crystal in $\mathcal{O}_{X/S}$-modules $\mathcal{F}$ are equipped with a canonical integrable connection.

Proof: For any $(U,T,\delta) \in (X/S)_{\text{crys}}$, consider the first order thickening $(U,T',\delta')$ given in (VI.4.3.8), then there are two projections $p_0,p_1 : T' \to T$ and a inclusion $i : T \to T'$, then by the property of crystals we get isomorphisms

$$p_0^*\mathcal{F}_T \xrightarrow{c_0} \mathcal{F}_{T'}, \xleftarrow{c_1} p_1^*\mathcal{F}_T$$

then $\nabla(s) = p_1^*(s) - c_1^{-1}c_0(p_0^*(s))$ vanishes after pulling back to $T$ via $i^*$, so it is in the kernel of $i^*$, which is

$$\mathcal{F}_T \otimes_{\mathcal{O}_T} \Omega_{T/S}$$

by the construction of $T'(VI.4.3.8)$. This $\nabla$ is functorial in $T$ as everything is functorial, hence gives a connection on $\mathcal{F}$.

For the integrability, Cf.[[Sta07J6], ?]

□

Cor. (VI.4.4.5). If $\mathcal{F}$ is a crystal in in $\text{Qco}$ modules, then we can define a de Rham complex

$$\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega_{X/S,\delta} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S,\delta}} \Omega^2_{X/S,\delta} \to \ldots$$

Crystals in $\text{Qco}$ modules

Def. (VI.4.4.6) (Crystals in $\text{Qco}$ Modules). An $\mathcal{O}_X$-module $\mathcal{F}$ on $(X/S)_{\text{crys}}$ is called a crystal in quasi-coherent modules if it satisfies the following equivalent conditions:

- $\mathcal{F}$ is $\text{Qco}$.
- $\mathcal{F}$ is locally $\text{Qco}(VI.4.4.1)$ and it is a crystal in $\mathcal{O}_{X/S}$-modules.

Moreover, $\mathcal{F}$ is called a crystal in finite locally free modules if $\mathcal{F}$ is finite locally free.

Proof: Cf.[[Sta07IT]].

□

Def. (VI.4.4.7) (Notations in Polynomial case). If in situation (VI.4.3.5), $S = \text{Spec} A, X = \text{Spec} C$, we let $P \to C$ be a surjection of $A$-algebras with $P = A[X_i]$, and the kernel is $J$. Set $D = D_{P,\gamma}(J)^\wedge$ be the $p$-adically completed pd-envelope, then $(D,\tilde{J},\tilde{\gamma})$ admits a natural pd-structure, by $(I.13.0.12)$. Let $D_e = D/p^e$ and $J_e$ the image of $\tilde{J}$ in $D_e$. Denote

$$\Omega_D = (\Omega_{D/A,\gamma})^\wedge = (\Omega_{D_{P,\gamma}(J)/A,\gamma})^\wedge = \lim_e \Omega_{D(n)_e/A,\gamma}(I.13.1.4).$$

By $(I.13.1.4)$, $\Omega_{D_{P,\gamma}(J)/A,\gamma} = \Omega_{P/A,\gamma} \otimes_P D_P(J)$ which is free over $D_{P,\gamma}(J)$ on $dx_i$, so $\Omega_D$ is topologically free over $D$ on $dx_i$, and there is a universal derivation $d : D \to \Omega_D$.

Now let $J(n) = \text{Ker}(P \otimes_A \ldots \otimes_A P \to C)$ where the tensor has $n + 1$ factors, and

$$D(n) = (DP \otimes_A \ldots \otimes_A P/A,\gamma(J(n)))^\wedge$$

with divided ideals $\tilde{J}(n)$, and also $D(n)_e = D(n)/p^e$, $T(n)_e = \text{Spec } D(n)_e$, then $(X,T(n)_e,\overline{\gamma}(n))$ is a pd-thickening by $(I.13.0.5)$ for $e$ large$(I.13.0.12)$, by $(I.13.0.5)$ as $p$ is nilpotent in $X$. And

$$\Omega_{D(n)} = (\Omega_{D(n)/A,\overline{\gamma}(n)})^\wedge.$$

Then $D(0) = D, D(1), \ldots$ form a cosimplicial pd-structures.

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{X/S}$-modules, denote

$$M(n) = \lim_e \Gamma((X,T(n)_e,\overline{\gamma}(n)), \mathcal{F}).$$
Prop. (VI.4.4.8). Notation as in (VI.4.4.7), there is an isomorphism
\[ D(n) \cong (D(\xi_i(j)))^\wedge \]
where \( \xi_i(j) = X_i \otimes 1 \otimes \ldots 1 - 1 \otimes 1 \ldots X_i \otimes 1 \otimes \ldots 1 \).

Proof: There is an isomorphism \( P \otimes_A \ldots \otimes_A P \cong P[\xi_i(j)] \), and \( J(n) \) is just generated by \( J_P \otimes_A \ldots \otimes_A P + (\xi_i(j)) \), so this theorem follows from (I.13.0.17). \( \square \)

Crystals and Connections

Lemma (VI.4.4.9) (Crystals and Connections). Notation as in (VI.4.4.7), there is a functor from the category of crystals in \( \mathcal{O}_{X/S} \)-modules to the category of pairs \( (M, \nabla) \) that
- \( M \) is a \( p \)-adically complete \( D \)-module.
- \( \nabla : M \to M \hat{\otimes} D \Omega_D \) is an integrable connection.
- \( \nabla \) is topological quasi-nilpotent: for any \( m \in M \), there are only f.m. pairs \( (i, k) \) that \( \nabla_k^i \hat{\otimes} \partial / \partial x_i (m) \in pM \).

Proof: For a crystal, we associate to it \( M = M(0) \) defined in (VI.4.4.7), let \( \Gamma((X,T(0)e,7(0))) \cdot F = M_e \), then because \( F \) is a crystal in qco sheaves, \( F_{T(0)e} = \check{M}_e \), and \( M_e = M_{e+1}/p^e M_{e+1} \), thus \( M_e = M/p^e M \) and \( M \) is \( p \)-adically complete by (I.5.7.6). By evaluating the natural connection defined in (VI.4.4.5) on \( T_e \) and take limit, then we get an integral connection \( \nabla : M \to M \hat{\otimes} D \Omega_D \).

To show this integral is topologically quasi-nilpotent, we can do the same thing for \( M = M(n) \) for any \( n \), and using the crystal property of \( F \) and take limits, we get isomorphisms
\[ M \hat{\otimes}_{D,p_0} D(1) \to M(1) \to M \hat{\otimes}_{D,p_1} D(1) \]
For the rest, Cf. [Sta]07JG.

Prop. (VI.4.4.10). The functor defined in (VI.4.4.9) is an equivalence of categories.

Proof: Cf. [Sta]07JH.

Def. (VI.4.4.11) (Notations in Smooth Case). Situation as in (VI.4.4.7), but this time we choose a smooth \( A \)-algebra \( P' \) and \( A \to P' \to C \) with \( \text{Ker}(P' \to C) = J' \) and do the same as (VI.4.4.7) again, to get
\[ D' = D_{P',\gamma}(J')^\wedge \]
and
\[ \Omega_{D'} = (\Omega_{D'/A,\gamma})^\wedge = (\Omega_{D_{P',\gamma}(J)/A,\gamma})^\wedge = \lim e \Omega_{D'(n)_{e}/A,\gamma} \]

Prop. (VI.4.4.12) (Crystals and Connections in Smooth case). Situation as in (VI.4.4.7) and (VI.4.4.7), then we can find a \( P = A[X_i] \) that there are maps \( a : D \to D' \), \( b : D' \to D \) between the completed pd-envelope of \( P, P' \) that \( a \circ b = \text{id} \) and compatible with the maps \( D \to C \) and \( D' \to C \), such that the base change along \( a,b \) induces an equivalence of categories between the categories of modules with an integrable connection over \( D \) as in (VI.4.4.9) and the category of modules over \( D' \) with an integrable connection.
we find a lift $P$ is surjective), we can use the smooth of $\text{pd-envelope}$ and the fact $
abla$ to lift inductively the map $P' \to D_e$ to a map $P' \to D$, thus by universal property of completed pd-envelope extends to a map $b : D' \to D$. It is clear that $a \circ b = \text{id}$.

For the equivalence of categories, Cf.

**Remark (VI.4.4.13).** In fact this proposition holds with $P'$ being any ring that $A \to P'$ satisfies the strong lifting property (I.7.5.18). In particular, this holds for ind-smooth $A$-algebras (VI.3.2.5).

5 Computing Cohomology

**Prop. (VI.4.5.1) (Affine Thickening is Acyclic).** If $T$ is a locally Qco sheaf of $\mathcal{O}_X$-modules on $(X/S)$, then $H^p ((U, T, \delta), \mathcal{F}) = 0$ for any $p > 0$ and $U$ or $T$ affine.

**Proof:** Firstly notice $U$ is affine iff $T$ is affine, by (V.7.5.2), then we use (V.6.2.14) with $\mathcal{G}$ being the affine thickenings and $\text{Cov}$ the affine coverings of affine thickenings, then $\text{Cov}$ is cofinal, and it suffices to check that $H^q (T, \mathcal{F}) = 0$ for an affine thickening $T$ and $q > 0$, and this is just the usual cohomology for Qco sheaves as the affine covering is cofinal, and it follows from (V.6.7.1).

**Lemma (VI.4.5.2).** Situation as in (VI.4.4.7), then the morphism

$$\text{colim}_e h_{(X, T, \delta)}^\mathcal{G} \to *$$

of sheaves on $(X/S)_{\text{crys}}$ is surjective.

**Proof:** We need to show that for any $(U, B, \delta) \in (X/A)_{\text{crys}}$, there is a Zariski covering $(U_i, B_i, \delta)$ of it that there are maps $(U_i, B_i, \delta) \to (X, T_{e_i}, \delta)$ that are compatible, But this is in fact equivalent to the existence of a morphism $(U, B, \delta) \to (X, \text{Spec } D, \delta)$ of pd-structures. For this, notice the morphology $U \to X$ can be extended to a morphism $X \to \text{Spec } P$ by strong lifting property (I.7.5.18) of smooth morphism (I.7.5.22), and this extends to the desired morphism by the universal property of pd-envelope and the fact $p$ is locally nilpotent on $B$ (VI.4.3.2) thus $B$ is locally $p$-complete.

**Lemma (VI.4.5.3).** Let $K' = (\text{colim}_e h_{(X, T, \delta)}^\mathcal{G})^\mathcal{G}$, then the product sheaf $(K')^\mathcal{G}$ is in fact isomorphic to $(\text{colim}_e h_{(X, T, \delta)}^\mathcal{G})^\mathcal{G}$ on $(X/S)_{\text{crys}}$.

**Proof:** This follows from the definition and the universal properties of completion, pd-envelope and $P \otimes_A \otimes_A \ldots \otimes_A P$ is a coproduct. Compare with the proof of (VI.4.5.2).

**Prop. (VI.4.5.4).** Situation as in (VI.4.4.7), if $\mathcal{F}$ is locally Qco and satisfies: for any morphism $f : (U, T, \delta) \to (U', T', \delta') \in (X/S)_{\text{crys}}$ that $f : T \to T'$ is a closed immersion the map $f^* \mathcal{F}_{T'} \to \mathcal{F}_T$ is surjective, then the complex $M(0) \to M(1) \to \ldots$

computes $R\Gamma((X/S)_{\text{crys}}, \mathcal{F})$. Moreover,

$$R\Gamma((X/S)_{\text{crys}}, \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^1_{X/S, \delta}) = 0$$

for $i > 0$. 


Proof: We use (V.6.1.16) for the presheaf $K' = \text{colim}_e h_{(X,T,e)}$ and $K = \ast$, which satisfies the condition by (VI.4.5.2). Then we get a spectral sequence

$$E_1^{p,q} = H^q(K'_p, \mathcal{F}) \Rightarrow H^{p+q}(K, \mathcal{F})$$

Notice $K'_n = (\text{colim}_e h_{(X,T(n),e)})^\sharp$ by (VI.4.5.3), so the cohomology

$$R\Gamma(K'_n, \mathcal{F}) = R \lim_e(\Gamma((X,T(n),e), \mathcal{F}))$$

Now the surjectivity $f^*\mathcal{F}_T' \rightarrow \mathcal{F}_T$ is equivalent to the surjectivity $\mathcal{F}_T' \rightarrow f_*\mathcal{F}_T$, so there is an exact sequence of $\mathbb{Q}_{\text{co}}$ $\mathcal{T}'$-sheaves

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{F}_T' \rightarrow f_*\mathcal{F}_T \rightarrow 0$$

which implies that $\mathcal{F}((U', T', \delta')) \rightarrow \mathcal{F}((U, T, \delta))$ is surjective, by (V.6.7.26) and (V.6.7.3). Then by (I.10.3.3),

$$R\Gamma(K'_n, \mathcal{F}) = R \lim_e(\Gamma((X,T(n),e), \mathcal{F})) = M(n)$$

thus we are done. □

Lemma (VI.4.5.5). Situation as in (VI.4.4.7), the complex $\Omega_{D(*)}$ is homotopic to 0 as a $D(*)$-cosimplicial module.

Proof: This complex is the $p$-adic completion of the base change of the cosimplicial module $M_* = (\Omega_{P^\otimes \mathbb{A}/\mathbb{A}})$ under the cosimplicial ring map $P^\otimes \mathbb{A} \rightarrow D(*)$. Then it suffices to show $M_*$ is homotopic to 0. For this, the whole thing can be written down clearly, Cf. [Sta]07LA. □

Lemma (VI.4.5.6). In situation (VI.4.4.7), for any cosimplicial module $M_*$ over the cosimplicial ring $D(*)$ and $i > 0$, the cosimplicial module

$$M_0 \hat{\otimes}_{D(0)} \Omega_{D(0)}^i \rightarrow M_1 \hat{\otimes}_{D(1)} \Omega_{D(1)}^i \rightarrow \ldots$$

is homotopic to 0.

Proof: □

Crystal Case

Prop. (VI.4.5.7). Situation as in (VI.4.4.7), and let $\mathcal{F}$ be a crystal in $\mathbb{Q}_{\text{co}}$ modules, and let $(M, \nabla)$ be the corresponding module with connection over $D$ by (VI.4.4.9), then the complex

$$M \hat{\otimes}_D \Omega_D^*$$

computes $R\Gamma((X/S)_{\text{crys}}, \mathcal{F})$.

Proof: Use the spectral sequence associated to the double complex

$$K^{a,b} = M \hat{\otimes}_D \Omega_{D(b)}^a$$
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Then the rows $K^{a \bullet}$ is acyclic for $a > 0$ by (VI.4.5.6) and (XI.1.2.21), and $K^{0 \bullet}$ is quasi-isomorphic to $R\Gamma((X/S)_{\text{crys}}, F)$ by (VI.4.5.4). Now we look at the other direction, (I.13.1.7) and (VI.4.4.8) show that each of the $b$ maps $D \to D(b)$ determines the same quasi-isomorphism

$$M \hat{\otimes}_D \Omega^*_D \cong M \hat{\otimes}_{D(b)} \Omega^*_D(b)$$

as their inverse is given by the same $D(b) \to D$. Then it is clear that the $E_2$ page in this direction is $H^a(M \hat{\otimes}_D \Omega^*_D)$ in the zero-th row and vanish otherwise, so we get the desired isomorphism by edge morphisms.

Prop. (VI.4.5.8) (de Rham Comparison for Crystalline Cohomology). In situation (VI.4.4.11), let $F$ be a crystal in $\text{Qco}$ modules, and let $(M', \nabla')$ be the corresponding module with connection over $D'$ by (VI.4.4.12), then the complex

$$M \hat{\otimes}_{D'} \Omega^*_D$$

computes $R\Gamma((X/S)_{\text{crys}}, F)$.

Proof: Let $b : D' \to D$, $a : D \to D'$ be the maps defined in (VI.4.4.12), then by (VI.4.5.7), it suffices to prove the base change along $a, b$ induces quasi-isomorphisms

$$M \hat{\otimes}_D \Omega^*_D \cong M \hat{\otimes}_{D'} \Omega^*_D.$$  

$a \circ b$ is trivial, thus it suffices to prove that $b \circ a$ induces an automorphism of $M \hat{\otimes}_D \Omega^*_D$. In fact, this is true for any morphism $\rho : D \to D$ of pd-algebras over $A$ compatible with the map $D \to C$;

Write $\rho(x_i) = x_i + z_i$, where $z_i \in J$ because $\rho$ is compatible with $D \to C$. Then we can factor $\rho$ as

$$D \xrightarrow{\sigma} D\langle \xi \rangle \xrightarrow{\tau} C$$

where $\sigma(x_i) = x_i + \xi_i$ and $\tau(\xi_i) = z_i$.

Notice that there exists an automorphism $\alpha$ of $D\langle x_i \rangle$ that maps $x_i$ to $x_i - \xi_i$ and $\xi_i$ to $\xi_i$. (Such a map exists because by universal property, it suffices to give a map of pairs

$$(P, J) \to D_{P, \gamma}(J)(\xi_i) = D_{P(\xi_i), \gamma}(JP(\xi_i) + (\xi_i))$$

by (I.13.0.17), and we surely have.

Now $\alpha$ is an automorphism, we have a quasi-isomorphism

$$M \hat{\otimes}_D \Omega^*_D \cong M \hat{\otimes}_{D, \sigma} \Omega^*_D(\xi)$$

by (I.13.1.7). Also $\tau$ induces an isomorphism because it has a right inverse, which is an isomorphism by (I.13.1.7) again, so $\rho$ induces an isomorphism.

Cor. (VI.4.5.9) (Crystalline-de Rham Comparison modulo p). In situation (VI.4.4.11), if $R$ is a smooth $A/p$-algebra, then there is a natural quasi-isomorphism

$$R\Gamma_{\text{crys}}(R/A) \otimes^L_A A/p \cong \Omega^*_R/(A/p)$$

of commutative algebra objects.

Proof: we choose $P$ to be a smooth lift of $R^?_p$ to $A$, then $J = IP$, and by (I.13.0.11), the pd-structure extends to $P$, thus $D_{P/A, \gamma} = P$, and notice $\Omega^i_{P/A}$ is finite projective hence flat, so $\Omega^*_P/A$ is $K$-flat (V.6.5.14), and the left side of is just $\hat{\Omega}^*_P/A \otimes_A A/p = \hat{\Omega}^*_R/(A/p)$ by (VI.4.5.8)(I.13.1.2) and (I.7.3.6).
6 Properties of Crystalline Cohomology


Def. (VI.4.6.1) (Higher Direct Images). Let \( p \) be a prime number, \((S, I, \gamma) \to (S', I', \gamma')\) be a morphism of PD-schemes over \( \mathbb{Z}(p) \) and \( f : X/S_0 \to X'/S'_0 \) be a morphism of schemes that \( p \) is locally nilpotent on \( X \) and \( X' \). For the rest, Cf. [Sta]07MJ.

Def. (VI.4.6.2) (F-Crystals). In situation (VI.4.3.2), let \( S = \text{Spec} \ A \) where \((A, I, \gamma)\) is a divided power algebra with \( p \in I \), and there is a Frobenius \( \sigma \) on \( A \) extending that of \( A/I \). Since the absolute Frobenius on \( X \) and \( S_0 \) are compatible, thus there is a morphism of crystalline site \((F_X)_{\text{cris}} : (X/S)_{\text{cris}} \to (X/S)_{\text{cris}}\).

Then an \( F \)-crystal on \( X/S \) relative to \( \sigma \) is a pair \((\mathcal{E}, F_{\mathcal{E}})\) given by a crystal in finite locally free \( \mathcal{O}_{X/S} \)-modules (VI.4.4.6) together with a map

\[
F_{\mathcal{E}} : (F_X)^{\ast}_{\text{cris}} \mathcal{E} \to \mathcal{E}.
\]

A non-degenerate \( F \)-crystal is an \( F \)-crystal that there exists a map \( V : \mathcal{E} \to (F_X)^{\ast}_{\text{cris}} \mathcal{E} \) that \( V \circ F_{\mathcal{E}} = p^i \text{id} \) for \( i \geq 0 \).
VI.5 Prismatic Cohomology

Main references are [[Sta]], [Prisms and Prismatic Cohomology, Scholze], [Notes on Prismatic Cohomology, Bhatt], [Prismatic Cohomology notes, Kedlaya].

1 Prisms

Def. (VI.5.1.1) (Prisms). A prism is a pair $(A,I)$ where $A$ is a δ-ring and $I$ is an ideal of $A$, that $V(I)$ is a Cartier divisor on $\text{Spec} \ A$, $A$ is derived $(p,I)$-complete, and $p \in I + \varphi(I)A$.

A prism $(A,I)$ is called perfect prism if $A$ is a perfect δ-ring (I.8.4.23). It is called bounded prism if $A/I$ has bounded $p^\infty$-torsion. It is called crystalline prism if $I = (p)$. It is called orientable if $I = (d)$ is of the free δ-de Rham cohomology $A$ and $d, \phi(d)$ is a unit.

Remark (VI.5.1.3) (Examples of Prisms). There is a f.f. morphism of δ-rings $\delta$-ring $A\to A'$ where $A'$ is a finite product of localizations of $A$ $\varphi$-stable multiplicative subsets that $IA'$ is generated by a distinguished element $d$ and $d,p \in \text{rad}(A')$.

Lemma (VI.5.1.2). Let $A$ be a δ-ring and $I$ be a locally principal ideal contained in $\text{rad}(A)$ that $(p,I) \subset \text{rad}(A)$, then the following are equivalent:

- $p \in I^p + \varphi(I)A$.
- $p \in I + \varphi(I)A$.
- There is a f.f. morphism of δ-rings $A\to A'$ where $A'$ is a finite product of localizations of $A$ $\varphi$-stable multiplicative subsets that $IA'$ is generated by a distinguished element $d$ and $d,p \in \text{rad}(A')$.

Proof: 1 → 2 is trivial, for 2 → 3: Choose $(g_1,\ldots,g_r) = A$ that $IA_{g_i}$ is principal. Let $B = \prod_{i=1}^r A_{g_i}$, so $A \to B$ is f.f. and $IB = (f)$ is principal. Let $A'$ be the localization of $B$ along the ideal $(p,f)$(I.5.1.26), then $p,f \in \text{rad}(A')$. Then $A \to A'$ is still f.f., because it is flat hence the image is stable under generalization by(I.7.1.19), so it must be all of $\text{Spec} \ A$ because it contains $(p,I)$ by construction, and $(p,I) \in \text{rad}(A)$ by hypothesis.

Now because $p \in \text{rad}(A)$, each localization of $A$ has a compatible δ-structure, and $\tilde A$ is a finite product of localizations of $A$, thus it has a δ-structure, and $\tilde A$ is distinguished by(I.8.4.21).

3 → 1: We need to check that $p = 0$ in $A/(I^p + \varphi(I)A)$, but this can be checked after base change to $A'$, which is $p = 0 \in A'/d^p, \varphi(d))$. This is true, because $d^p = d^p + p\delta(d)$ and $\delta(d)$ is a unit. □

Remark (VI.5.1.3) (Examples of Prisms).

- If $A$ is a $p$-torsionfree and $p$-adically complete δ-ring $A$, the pair $(A,(p))$ is a bounded crystalline prism.
- $q$-de Rham cohomology(I.8.4.22) determines a bounded prism. (completeness and boundedness is clear, and $p \in (d,\varphi(d))$ because?)
- Breuil-Kisin cohomology(I.8.4.22) determines a bounded prism. (boundedness is clear, and?)
- $A_{m,f}$-cohomology(I.8.4.22) determines a bounded prism. (The same reason as item2).

Prop. (VI.5.1.4) (Universal Oriented Prism). Let $A_0 = \mathbb{Z}_p\{d,\delta(d)^{-1}\}$ be the localization of the variable $d$, and let $A$ be the derived $(p,d)$-completion of $A_0$(it is discrete by(I.10.7.7)), and let $I = (d)$, then $(A,I)$ is a bounded oriented prism, and it is the initial object in the category of bounded oriented prisms.

Moreover, the sequence $(p,d)$ is regular and the Frobenius $\varphi : A/p \to A/p$ is $d$-completely flat.
Proof: It is clearly a prism and universal. For the assertions, firstly we show \( A/(p, d) = A_0/(p, d) \): notice \( A \otimes_{A_0} A_0/(p, d) = A_0/(p, d) \) by (I.10.7.4), so we can replace \( \otimes \) with \( \oplus \). Similarly for \( A/(p, d^p) \) because \( A \) is \( (p, d^p) \)-complete. Now this map is.

\((p, d)\) is regular by (I.10.7.7) applied to \((\mathbb{Z}/p)[d], A\)?, for the last assertion, it suffices to show \( A/(p, d) \overset{\text{Frob}}{\to} A/(p, d^p) \) is f.f.. □

Prop. (VI.5.1.5). Let \( (A, (d)) \) be the universal prism (VI.5.1.4) and let \( B = A\{\frac{\varphi(d)}{p}\} \) (derived \( (p, d) \)-completion), then \( B \) is \( (p, d) \)-complete, \( p \) torsionfree and it equals the derived \( (p, d) \)-completion of the \( \text{pd}-\text{envelope} \ D_{A, \delta((d))} \) of \( (A, (d)) \). In particular, \( (B, (p)) \) is a crystalline prism, by (I.8.4.16).

Proof: \( B \) is \( p \)-complete by (I.10.6.16), as \( A \) is \( p \)-torsionfree. Also \( d^p = p(\varphi(d) - \delta(d)) \), so \( B \) is also \((p, d)\)-complete. The last assertion follows from (I.13.0.21) and (VI.5.1.4).

□

Cor. (VI.5.1.6) (Connection with Crystalline Prism). In the above situation, \( B \) is also a \( \delta \)-ring by (I.8.4.15), and both \( \varphi(d), p \) are distinguished in \( B \) and \( \varphi(d) \) divides \( p \) in \( B \), so by (I.8.4.20) \( (\varphi(d)) = (p) \).

Now the composition of maps of \( \delta \)-rings (I.8.4.5) \( \alpha: A \overset{\varphi}{\to} B \) promotes to a morphism of prisms (VI.5.1.5) \( (A, (d)) \to (B, (p)) \) which decompose as

\[(A, (d)) \overset{\varphi}{\to} (A, (\varphi(d)) \to (B, (\varphi(d))) = (B, (p))\]

Prop. (VI.5.1.7) (Rigidity of Maps). If \( (A, I) \to (B, J) \) is a morphism of prisms, then \( I \otimes_A B = J \), in particular, \( IB = J \). Conversely, if \( B \) is a derived \( (p, I) \)-complete \( \delta \)-A-algebra, then \( (B, IB) \) is a prism iff \( B[I] = 0 \).

Proof: For the first assertion, it suffices to show that \( I \otimes_A B \to J \) is surjective, because they are both invertible sheaves on \( B \). Choose f.f. ring morphisms \( A \to A', B \to B' \) as in (VI.5.1.2) and there is a morphism \( A' \to B' \) extending \( A \to B \), (by taking \( B \) as the localization of \( A' \times_A B \) along \( (p, J) \) and do the construction again). Then \( IB' \subset JB' \) are an inclusion of principal ideals generated by distinguished elements, thus they are equal, by (I.8.4.20), finally we use faithfully flatness.

For the second assertion, notice \( B[I] = 0 \) iff \( I \otimes_A B \to IB \) is an isomorphism. If \( (B, IB) \) is a prism, then clearly \( I \otimes_A B \to IB \) is an isomorphism, because they are both invertible sheaves. The converse is also trivial. □

Prop. (VI.5.1.8) (Prism is Nearly Principal). Let \( (A, I) \) be a prism, then the ideal \( \varphi(I)A \) is a principal ideal, and any generator is a distinguished element. In particular, if it is a perfect prism, then \( I = (f) \) where \( f \) is a distinguished element.

Proof: It suffices to prove \( \varphi(I)A \) is generated by a single distinguished element, and then use (I.8.4.19). By (VI.5.1.2), we can assume \( p = a + b \) where \( a \in I^p, b \in \varphi(I)A \). Now we show \( b \) generate \( \varphi(I)A \): choose a f.f. map as in (VI.5.1.2), and then it suffices to check that \( b : A' \to \varphi(I)A' \) is surjective. Now \( \varphi(I)A' = (d), a = xd^p, b = y\varphi(d), \) so it suffices to show \( y \) is a unit in \( A' \). Now \( (p, d) \in \text{rad} A' \), it suffices to show \( A'/(p, f, y) = 0 \). If not, we localize along \( (p, f, y) \), then we may assume \( (p, f, y) \in \text{rad} A' \).

The equation \( p = a + b \) implies \( p(1 - y\varphi(d)) = d^p(x + y) \), and the left side is distinguished because \( p \) does and \( 1 - y\varphi(d) \) is a unit, by (I.8.4.19). Then (I.8.4.20) shows \( d^{p-1}(x + y) \) is a unit, then so does \( d \), contradicting \( d \in \text{rad} A' \). □

Remark (VI.5.1.9). Notice the proof goes through even \( I \) is only a locally principal ideal of \( A \).
Cor. (VI.5.1.10). If \((A, I)\) is a prism, the invertible \(A\)-modules \(\varphi^*(I) = I \otimes_{A,\varphi} A\) and \(I^p\) are trivial.

Proof: Cf.[Prisms, Scholze, P25]. \qed

Prop. (VI.5.1.11) (Properties of Bounded Prisms). Let \((A, I)\) be a bounded prism, then

1. Any derived \((p, I)\)-complete and \((p, I)\)-completely flat \(A\)-complex \(M \in D(A)\) is discrete and \((p, I)\)-complete. For any \(n \geq 0\), we have \(M[I^n] = 0\) and \(M/I^nM\) has bounded \(p^\infty\)-torsion.

2. \(A\) is \((p, I)\)-complete, \(A/I\) is \(p\)-adically complete, \(A/I^n = 0\) and \(A/I^n\) has bounded \(p^\infty\)-torsions.

3. The category of (faithfully)flat prisms over \((A, I)\) identifies with the category of \((p, I)\)-completely (faithfully)flat \(\delta\)-\(A\)-algebras \(B\) by the bijection \(B \leftrightarrow (B, IB)\).

4. (Bounded Prisms are fpqc-Locally Orientable) There is a \((p, I)\)-completely faithfully flat \(\delta\)-\(A\)-algebra \(B\) that \(IB = (d)\), where \(d\) is distinguished and determines a nonzero divisor of \(B\). Also \((B, IB)\) is bounded.

Proof: 1: By(I.10.7.7).

2 follows from1. For \(A/I\), it is derived \(p\)-complete by(I.10.6.8), then it is \(p\)-adically complete by(I.10.6.16).

3: By definition, a (faithfully)flat \((A, I)\)-prism is \((p, I)\)-completely (faithfully)flat(I.10.7.1). Conversely, by(VI.5.1.7), it suffices to show that \(B[I] = 0\), and this follows from item1.

4: We may choose \(B\) to be the derived \((p, I)\)-completion of the f.f. \(\delta\)-ring defined in(VI.5.1.2), then it is also \((p, I)\)-completely faithfully flat(I.10.7.4), and by item2 and3 it determines a bounded prism \((B, IB)\). \qed

Prop. (VI.5.1.12) (The Site of Bounded Prisms). The prismatic site is the opposite category of the category of bounded prisms where the covers are determined by f.f. map of prisms.

Then the functors that maps \((A, I)\) to \(A\) or \(A/I\) are sheaves on this cohomology with vanishing higher cohomologies.

Proof: To show this is a site, we need to check the base change of covers. If \((C, IC) \leftarrow (A, IA) \rightarrow (B, IB)\) is a diagram that \(b\) is f.f., then we let \(D\) be the derived \((p, I)\)-completion of \(B \otimes_A^L C\), then \(C \rightarrow D\) is also \((p, I)\)-completely f.f. by(I.10.7.5), so by(VI.5.1.11) and(VI.5.1.7) \(D\) is discrete and \((D, ID)\) is a bounded prism over \((A, IA)\). It is clear this is a base change in the category of bounded prisms.

The assertion about cohomology follows from [Scholze, Prism, 3.12]. \qed

Prismatic Envelopes

Prop. (VI.5.1.13) (Prismatic Envelopes). Let \((A, I)\) be a prism, then the forgetful functor from the prisms over \((A, I)\) to \(\delta\)-pairs over \((A, I)\) admits a left adjoint, called the prismatic envelope which maps \((B, J)\) to \(B_1\).

Proof: If we can construct this locally, then we can construct it globally by gluing and the universal property, so we can localize and assume \(I = (d)\) where \(d\) is distinguished. Let \(B'\) be the free \(\delta\)-ring over \(A\) generated by \(\{x/d| x \in J\}\) and \(B_1\) the derived \((p, d)\)-completion module of \(B\) which is a \(\delta\)-algebra by(I.8.4.16).

If \(d\) is torsion-free in \(B_1\), then \((B_1, (d))\) is a prism that satisfies the universal property. Otherwise we choose the maximal \(d\)-torsion-free quotient(I.5.1.13)(I.8.4.12) and taking the derived
$(p,d)$-completion module, and we can do this to $\mathbb{N}_0$, where we take the filleted colimit, then it is $d$-torsion-free and $(p,d)$-complete, by(I.10.6.5) and any prism over $A$ map factors through this chain.

\textbf{Cor.} (VI.5.1.14). In the above situation, if $(B,J)$ is $(p,I)$-completely flat over $A$, and $J = (I, x_1, \ldots, x_n)$ where $(x_1, \ldots, x_n)$ is a $(p,I)$-completely regular sequence w.r.t. $A$, then the prismatic envelope of $(B\{\frac{x}{d}\}^\wedge, IB\{\frac{x}{d}\}^\wedge)$ is flat over $(A, I)(VI.5.1.1)$.

Moreover, It is compatible with completed derived base change on $(A, I)$, by universal properties and the fact the completed derived base change of it is discrete(I.10.7.5). Also, it is compatible with completed derived base change along a $(p, I)$-completely flat map $(B, J) \to (B', J')$.

\textbf{Proof:} It suffices to check locally for $I = (d)$ that $B_1 = (B\{\frac{x}{d}, \ldots, \frac{x_n}{d}\})^\wedge$ as a simplicial $\delta$-ring is $(p, I)$-completely flat over $A$, then it is discrete and is torsionfree by(VI.5.1.11), thus it is a prism over $(A, (d))$ by(VI.5.1.7). And for the flat localization, notice the image of $x_1, \ldots, x_n$ is also $(p, I)$-completely regular w.r.t. $A$.

Consider the following diagram of derived $(p, d)$-complete simplicial $\delta$-rings:

\[
\begin{array}{cccc}
\mathbb{Z}_p\{z\}^\wedge & \xrightarrow{z \mapsto d} & A & \xrightarrow{B} B\{\frac{x_1}{d}, \ldots, \frac{x_n}{d}\}^\wedge = C \\
\downarrow \binom{z \mapsto \varphi(y)}{z \mapsto \varphi(y)} & & \downarrow & \downarrow \\
\mathbb{Z}_p\{y\}^\wedge & \xrightarrow{A'} & B' & \xrightarrow{B'} B'\{\frac{x_1}{\varphi(d)}, \ldots, \frac{x_n}{\varphi(d)}\}^\wedge = C' \\
\downarrow & & \downarrow & \downarrow \\
D = A'\{\frac{\varphi(y)}{p}\}^\wedge & \xrightarrow{B''} & B'' & \xrightarrow{B''} B''\{\frac{x_1}{p}, \ldots, \frac{x_n}{p}\} = C''
\end{array}
\]

where each square is completed derived tensor product. Notice the lat term has denominator $p$ because $\varphi(y)$ and $p$ are both distinguished in $\pi_0(D)$, so by(I.8.4.20) $\frac{\varphi(y)}{p}$ is a unit in $D$.

The leftmost arrow is $(p, z)$-completely f.f. by(I.8.4.9)(I.10.7.4), so all the vertical arrow in the upper row is $(p, z)$-completely flat by(I.10.7.5). Now the map $D \to C''$ is $(p, z)$-completely flat by(I.13.0.24), noticing that the conditions holds, by(I.10.7.5).

Now by definition the $(p, d)$-completely flatness is defined by flatness after base change to $Kos(A, p, d)(XI.1.3.5)$, it suffices to show that there is a map $D \to Kos(A', p, d)$. To show this, it suffices to assume $A' = \mathbb{Z}_p\{y\} = \mathbb{Z}_p[y, y_1, \ldots, y_n, \ldots]$ and base change. In this case, $p, y$ is a regular sequence, so by(I.13.0.21) $D$ is the derived completion of $D_{\mathbb{Z}_p\{y\}}((y))$, and $Kos(A', p, y) = F_p[y_1, \ldots, y_n]$, so $A' \to Kos(A', p, d)$ factors through $D$ by universal property.

\[\square\]

\section{Perfect Prisms}

\textbf{Prop.} (VI.5.2.1) (Properties of Perfect Prisms). Let $(A, I)$ be a perfect prism(VI.5.1.1), then:

- $I = (d)$ where $d$ is distinguished and is a nonzero-divisor.
- $A$ is $p$-torsionfree and $p$-adically complete, hence there is a natural isomorphism $A \cong W(A/p)$ of $\delta$-rings.
- $A$ is $(p, I)$-complete.
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Proof: 1: $I$ is principal by (VI.5.1.8). $d$ is distinguished by (I.8.4.21), it is nonzero-divisor by definition of prisms.

2: This is because $A$ is $p$-torsionfree by (I.8.4.25) and thus $p$-adically complete by (I.10.6.16), then $A \cong W(A/p)$ by the equivalence in (I.8.4.26).


4: This follows from item 3 and (VI.5.1.11).

Prop. (VI.5.2.2) (Perfection of Prisms). There is a perfection of prisms functor that maps a prism $(A, I)$ to a perfect prism $(A_\infty, IA_\infty)$ left adjoint to the inclusion functor.

Proof: Let $A'_\infty = A_{\text{perf}}$ be the perfection of $A$ as a $\delta$-ring (I.8.4.24), and $A_\infty$ be the derived $(p, I)$-complete of $A'_\infty$ as a $\delta$-ring (I.8.4.16), then the universal property follows form that of derived completion and perfection once we proved that $A_\infty$ is perfect and $IA_\infty = (d)$ where $d$ is a nonzero-divisor.

$A_\infty$ is perfect because the Frobenius is isomorphism on $A_{\text{perf}}$ and derived $(p, I)$-completion and $(p, \varphi(I))$-completion coincide (they have the same radical (I.10.6.15)).

$A_\infty$ is $p$-adically complete because it is $p$-torsionfree (I.8.4.25), and then use (I.10.6.16).

Now (VI.5.1.9) and the fact $A_\infty$ is perfect shows $IA_\infty = (d)$ where $d$ is distinguished, and (I.8.4.28) shows $d$ is a nonzero-divisor, so we are done.

Prop. (VI.5.2.3) (Perfect Prisms are Final). Let $(A, I)$ be a perfect prism, then for any prism $(B, J)$, a map $A/I \to B/J$ will induce a map of prisms $(A, I) \to (B, J)$.


This map will induce a map $A \cong W((A/I)^\delta) \to W((B/J)^\delta)$. And there is a Fontaine’s functor $W((B/J)^\delta) \to B$ (I.9.9.7) which is a $\delta$-map. Thus we obtain a map $A \to B$. And this map can be seen to be lifting $A/I \to B/J$.

3 Integral Perfectoid Rings

Def. (VI.5.3.1) (Integral Perfectoid Rings). A commutative ring $R$ is called an integral perfectoid ring if it has the form $A/I$ for a perfect prism $(A, I)$. An equivalent definition of an integral perfectoid ring is given in (VI.5.3.6).

Def. (VI.5.3.2) (Special Fiber). For an integral perfectoid ring $R$, then special fiber of $R$ is defined to be $\overline{R} = R/\sqrt{pR}$. It is perfect, by (VI.5.3.5).

Prop. (VI.5.3.3) (Perfect Prisms and Integral Perfectoid Rings). The mapping $(A, I) \to A/I$ defines an equivalence of categories between perfect prisms and integral perfectoid rings, where the converse is given by $R \mapsto (A_{\text{inf}}(R), \text{Ker}(A_{\text{inf}}(R) \to R))$ (I.8.1.15).

Proof: To show $A \cong A_{\text{inf}}(A/I)$, by (I.8.4.26), it suffices to show there is a natural isomorphism $A/p \cong (A/I)^\delta$. By (I.8.1.18), $(A/I)^\delta$ identifies with $d$-adic completion of $A/p$. Then it suffices to show $A/p$ is $I$-adically complete, which is by (I.10.6.16), as $A/p$ is derived $d$-complete because $A$ does (I.10.6.8).

Now we have the Fontaine’s map $A = A_{\text{inf}}(A/I) \to A/I$, this map is surjective because $\varphi : A/(p, I) \to A/(p, I)$ is surjective as $A/p$ is perfect. Also this is just the quotient map $A \to A/I$ because they are equal when modulo $p$, and then use (I.8.4.26).
Cor. (VI.5.3.4). Any perfect $\mathbb{F}_p$-algebra is an integral perfectoid ring corresponding to a crystalline prism, by (I.8.4.26).

Any integral perfectoid ring is $p$-adically complete, by (VI.5.1.11).

Prop. (VI.5.3.5). If $R$ is a perfectoid ring, then

- $R$ is semiperfect.
- There exists an element $\varpi \in R$ that admits a compatible system $\varpi^{1/p^n}$ of $p$-power roots s.t. $\varpi = pu$ for a unit $u$ and the kernel of the Frobenius $\varphi: R/p \to R/p$ is generated by $\varpi^{1/p}$.
- $\sqrt[p]{R} = \cup_n(\varpi^{1/p^n})$, and it is flat.

Proof: 1: Let $R = A/d$ where $A = A_{inf}(R)$, then $R/p = A/(p, d)$, so $\varphi_{R/p}$ is surjective, as $A/p = R^\#$ is perfect.

2: Notice $d = [a_0] - pu$ for a unit $u \in A$ by (I.8.4.28), then we can take $\varpi$ to be the image of $[a_0]$ in $R$, and then $\varpi^{1/p^n} = [a_0^{1/p^n}]$.

3: firstly the LHS contains the RHS, and $R/\cup_n(\varpi^{1/p^n})$ is perfect hence reduced, so the two sides are equal. To check it is flat, we need to check that $M \otimes R^\# \sqrt[p]{R}$ is discrete, or equivalently $M \otimes R^\# \sqrt[p]{R} \in D^{\geq 1}$ where $R = R/\sqrt[p]{R}$ is perfect. But there is a distinguished triangle $M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p[-1] \to M \to M[\zeta_{p^2}]$ flat (I.7.1.5)) and the fact $\mathbb{R}[\mathbb{Z}_p] = 0$, it suffices to prove $M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \otimes R^\# \sqrt[p]{R} \in D^{\geq 2}(R)$. Now $M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ has cohomology groups $p^\infty$-torsion, so using canonical truncation, it suffices to show $M \otimes R^\# \sqrt[p]{R} \in D^{\geq -1}$ for any $p^\infty$-torsion $M$. Because tensoring commutes with filtered colimits, it suffices to show for $M$ an $R/p^n$-module. Now using exact sequences like

$$0 \to M[p]/M[p^2] \to M/M[p^2] \to M[p]/M[p^2] \to 0,$$

we can reduce to the case $M$ is an $R/p$-module.

Now there is a commutative diagram

$$
\begin{array}{ccc}
A_{inf}(R) & \longrightarrow & R \\
\downarrow & & \downarrow \\
A_{inf}(R) & \longrightarrow & R
\end{array}
$$

and $d$ is $p$-torsionfree in both $A_{inf}(R)$ and $A_{inf}(R)(I.8.4.28)$, and $d = [a_0] - pu = pu \in W(R)(as a_0 = 0 \in \overline{R})$, so $\overline{R} = W(R)/d$ and this is a Tor-independent pushout square. Thus $M \otimes R^\# \overline{R} \cong M \otimes_{W(R^\#)} W(\overline{R})$. As $p$ is nonzero divisor in both $A_{inf}(R)$ and $A_{inf}(\overline{R})$, and $pM = 0$, we have a similar diagram quotient by $p$, and by the same reason $M \otimes_{W(R^\#)} W(\overline{R}) \cong M \otimes_{\overline{R}} \overline{R}$. Now the kernel of $R^\# \to \overline{R}$ is of the form $(f^{1/p^\infty})$, where $f$ corresponds to $(\omega^{1/p^n})$, so the claim follows from (I.8.1.8).

4: notation as in the proof of item 2, it suffices to show the $A$-module $R[p]$ is annihilated by $[a_0^{1/p^n}]$ for $n \geq 0$. But $R[p] = A/d[p] = A/p[d] = R^d[d]$, and $d = [a_0]$ on $R^d$, which is perfect, so we are done.

Prop. (VI.5.3.6) (Equivalent Definition of Integral Perfectoid Rings). A commutative ring $R$ is an integral perfectoid ring iff the following are satisfied:

- $R$ is $p$-adically complete and $R/p$ is semiperfect.
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The kernel of $\theta_R : A_{inf}(R) \to R$ (I.8.1.15), notice $R$ is $p$-adically complete) is principal.

There exists some $\varpi \in R$ that $(\varpi^p) = (p)$.

And if $R$ is $p$-torsionfree, the condition2 can be replaced by: $R$ is $p$-normal?.

**Proof:** If $R$ is an integral perfectoid ring, then these are true by (VI.5.3.5). Now if these are satisfied, then firstly $\theta$ is surjective by (I.8.1.16). Next, let $d \in A_{inf}(R)$ be the generator of $\theta$, we show $A_{inf}(R)$ is derived $(p,d)$-complete. It is derived $p$-complete by (I.10.6.16). Now $R^p$ is derived $d$-complete by (I.8.1.18).

By induction and (I.10.6.8), $A_{inf}(R)/p^n$ are all derived $d$-complete, and also by induction $A_{inf}(R)/p^n$ has bounded $d^\infty$-torsion, as $R^p$ is perfect. Then $A_{inf}(R)/p^n$ are all $d$-adically complete. Thus

$$A_{inf}(R) = \lim_{\inf} A_{inf}(R)/p^n = \lim_{\inf} \lim_{\inf} A_{inf}(R)/(p^n,d^n),$$

which means $A_{inf}(R)$ is $(p,d)$-adically complete thus derived $(p,d)$ adically complete.

Then it suffices to show $d$ is distinguished, by (I.8.4.28). Let $\varpi^p = u$, and lift $\varpi, u$ to $x, v \in A_{inf}(R)$. Since $A_{inf}(R)$ is $d$-adically complete, $v$ is unit in $A_{inf}(R)$. Then $d|x^p - pv$ and $x^p - pv$ is distinguished (I.8.4.28). Now $d,p \in \text{rad}(A_{inf}(R))$ as $A_{inf}(R)$ is $(d,p)$-adically complete, so by (I.8.4.20), $d$ is distinguished.

Now if $R$ is a $p$-torsionfree integral perfectoid ring, if $x \in R[1/p]$ satisfies $x^p \in R$, let $n \geq 0$ minimal that $y = \varpi^n x \in R$, then we show $n = 0$: if $n > 0$, then $(\varpi^n x)^p = \varpi^{np} x^p \in \varpi^n R$. Then we get $\varpi^n x \in \varpi^n R$, and then $x \in R$ as $R$ is $p$-torsionfree.

Conversely, we use condition1,2,4 to prove 3: We first show the kernel of $\varphi : R/p \to R/p$ is generated by $\varpi$ as in condition4: if $x^p \in pR = \varpi^p y$, then $(x/\varpi)^p = y$, Thus $x \in \varpi R$ by hypothesis. Since $R/p$ is semiperfect, $\varpi$ admits a compatible $p^n$-th roots $\{\varpi^1/p^n\}$. It can be shown by induction that $\text{Ker}(\varphi^n) = (\varpi^1/p^n)$. This implies that the kernel of $\vartheta_R : R^p \to R/p$ is generated by the element $\varpi^b$ determined by the system $\{\varpi^1/p^n\}$. As $W(R^p)$ and $R$ are both $p$-torsionfree and $p$-adically complete, they kernel of $\theta_R$ is generated by any element in the kernel that lifts $\varpi^b$. In particular, the kernel is principal.

**Prop. (VI.5.3.7) (Pushout of Integral Perfectoid Rings).** Integral perfectoid rings are closed under pushouts in the category of derived $p$-complete rings: i.e. if $C \leftarrow A \to B$ are maps of integral perfectoid rings, then $B \hat{\otimes}_A^L C$ is also an integral perfectoid ring.

**Proof:** Let $R = C^b \hat{\otimes}_A^L B^b = C^b \otimes_A^L B^b$ which is a perfect ring, by (I.8.1.3) and (I.8.1.8). Then we have

$$W(R) = W(A^b) \hat{\otimes}_{W(A^b)}^L W(B^b),$$

because this can be checked via derived Nakayama (I.10.6.10). Now use the fact $A = W(A^b)/d$ for some distinguished element $d$, and then $B = W(B^b)/d, C = W(C^b)/d$ by rigidity (VI.5.1.7), and $d$ is nonzero-divisor in $W(B^b), W(C^b), W(R)$ by (I.8.4.26), so taking derived base change along $W(A^b) \to A$, we get

$$D = W(R)/d = B \hat{\otimes}_A^L C,$$

and $W(R)$ is a perfect prism, by (VI.5.1.7), so $D = W(R)/d$ is an integral perfectoid ring and equals $B \hat{\otimes}_A^L C$. 

**Cor. (VI.5.3.8).** The category of integral perfectoid rings is closed under arbitrary colimits and products in the category of derived $p$-complete rings.

But it is not closed under equalizers: Notice by Ax-Sen-Tate, $(\mathcal{O}_{C_p})^{Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)} = \mathbb{Z}_p$, but $\mathbb{Z}_p$ is not an integral perfectoid by (VI.5.3.6).
Proof: It suffices to show it is closed under products and sums.

Prop. (VI.5.3.9) (Gluing). Let $R$ be an integral perfectoid ring, $\overline{R} = R/\sqrt{pR}, S = R/R[\sqrt{pR}], \overline{S} = S/\sqrt{pS}$, then $\overline{R}, S, \overline{S}$ are perfectoids, and the square

\[
\begin{array}{ccc}
R & \to & S \\
\downarrow & & \downarrow \\
\overline{R} & \to & \overline{S}
\end{array}
\]

is both a homotopy fiber square (I.10.0.2) and pullback square. Moreover,

- $S$ is $p$-torsionfree.
- $\sqrt{pR}$ maps isomorphically onto $\sqrt{pS}$.
- $R[\sqrt{pR}]$ maps isomorphically to $\text{Ker}(\overline{R} \to \overline{S})$, thus $x \mapsto x^p$ is bijective on $R[\sqrt{pR}]$.

In particular, any integral perfectoid ring is a fiber product of integral rings that is either perfect or $p$-torsionfree.

Proof: By (VI.5.3.5), $R[\sqrt{pR}] = R[p^\infty]$. In particular, $S$ is $p$-torsionfree. Now if we know this is a homotopy fiber square, then we get 2, 3 by comparing the kernel. And if we know the kernel, then this is a pushout square by (I.5.1.19). So it suffices to show this is a homotopy fiber square.

Let $d = [a_0] - pu$ for a distinguished element of $A = A_{inf}(R)$ that $R = A/(d)$, and the ideal $I = (a_0^{1/p^\infty}) \subset R^\wedge$ and $J = R^\wedge[I]$. Then the square

\[
\begin{array}{ccc}
W(R^\wedge) & \longrightarrow & W(R^\wedge/J) \\
\downarrow & & \downarrow \\
W(R^\wedge/I) & \longrightarrow & W(R^\wedge/I + J)
\end{array}
\]

is a homotopy fiber square: all vertices are $p$-torsionfree and $p$-adically complete, and the square gives a fiber square when modulo $p^n$(use induction on $n$ and (I.8.1.9)), and then take derived $p$-completions. Next we apply.

Cor. (VI.5.3.10). Integral perfectoid rings are reduced.

Proof: By (VI.5.3.9), we may assume $R$ is $p$-torsionfree or perfect. Thus it suffices to assume $R$ is $p$-torsionfree. Let $\varpi \in R$ that $\varpi^p = pu$ as in (VI.5.3.6). If $x^p = 0$, we show inductively that $x \in \varpi^nR$. If $x = \varpi^ny$, then $y^p = 0$ as $R$ is $p$-torsionfree. Now the kernel of Frobenius $\varphi : R/p \to R/p$ is generated by $\varpi$, thus we have $y \in \varpi R$, so we can use induction.

4 Prismatic Site

Remark (VI.5.4.1). In this subsection, fix a bounded prism $(A, I)$, all formal schemes over $A$ are assumed to have the $(p, I)$-adic topology, and formal schemes over $A/I$ are assumed to have the $p$-adic topology.

Def. (VI.5.4.2) (Prismatic Site). Let $(A, I)$ be a bounded prism and $X$ be a smooth $p$-adic formal scheme over $A/I$, let $(X/A)_\Delta$ be the site whose objects are bounded prisms $(B, IB)$ over $(A, I)$ together with a map $\text{Spf}(B/IB) \to X$ over $A/I$. The morphisms are the natural one, and
the coverings in \((X/A)_{\Delta}\) are f.f. maps of prisms \((B, IB) \to (C, IC)\). There are structure sheaves \(\mathcal{O}_\Delta((B, IB)) = B\) and \(\overline{\mathcal{O}}((B, IB)) = B/IB\). They are sheaves by (VI.5.1.12).

Thus \(\mathcal{O}_\Delta\) is valued in \((p, I)\)-complete \(\delta\)-\(A\)-algebras and \(\overline{\mathcal{O}}_\Delta\) is valued over \(p\)-complete-algebras (I.10.6.16).

**Def. (VI.5.4.3) (Perfect Prismatic Site).** The perfect prismatic site \((X\triangleleft A)_{\Delta\text{perf}}\) is the full subcategory of \((X/A)_{\Delta}\) consisting of perfect prisms. By (VI.5.3.3), objects in this site are equivalent to the category of perfectoid rings \(R\) over \(A\triangleleft I\) with a map \(\text{Spf} R \to X\).

**Remark (VI.5.4.4).** If we further restrict to the site of perfect prisms \((S, I)\) that \(S\triangleleft I\) is integrally closed in \(S[i[\frac{1}{p}]\], then we will get the notion of diamond of \((X[i[\frac{1}{p}]], X)\), in sense of [?].

**Def. (VI.5.4.5) (Absolute Prismatic Site).** For a \(p\)-adic formal scheme \(X\) the absolute prismatic site consisting of bounded prisms \((B, J)\) with a map \(\text{Spf} B\triangleleft J \to X\).

**Prop. (VI.5.4.6) (Prismatic Site and Étale Site).** Let \(f\text{Sch}/X\) be the category of \(p\)-adic formal schemes over \(X\) with the étale topology, then there is a natural functor \(\mu : (X/A)_{\Delta} \to f\text{Sch}/X\) sending \((B, IB)\) over \(X\) to \(\text{Spf} B\triangleleft IB \to X\).

This functor is cocontinuous: for any \(p\)-completely étale map \(B\triangleleft IB \to C\), it is a derived \(p\)-completion of some étale map \(B\triangleleft IB \to C\) by (I.10.7.9), and this can be lifted to a map \(B \to S'\) by (VI.5.1.11) (I.6.10.6) and (I.6.10.9), and we choose the \((p, I)\)-completion of \(S'\), then it is a prism that lifts \(C\). Thus by (V.1.2.19) defines a morphism of topoi:

\[
\mu : \text{Sh}((X/A)_{\Delta}) \to \text{Sh}(f\text{Sch}/X).
\]

Also there is a natural map of topoi \(\text{Sh}(f\text{Sch}/X) \to \text{Sh}(X_{\text{ét}})\) by restriction, by (I.10.7.1). So we get a morphism of topoi

\[

\nu : \text{Sh}((X/A)_{\Delta}) \to \text{Sh}(X_{\text{ét}}).
\]

In particular, for any étale formal scheme \(U\) over \(X\), by definition of \(s\mu(V.1.2.10)\), for any sheaf \(\mathcal{F}\),

\[
(\nu_* \mathcal{F})(U/X) = H^0((U/A)_{\Delta}, \mathcal{F}|_{(U/A)_{\Delta}}).
\]

**Cor. (VI.5.4.7) (Prismatic Complex and Hodge-Tate Complex).** In the above situation, we define prismatic complexes

\[
\Delta_{X/A} = R\nu_* \mathcal{O}_\Delta \in \text{D}(X_{\text{ét}}, A)
\]

and the Hodge-Tate complex

\[
\overline{\Delta}_{X/A} = R\nu_* \overline{\mathcal{O}}_\Delta \in \text{D}(X_{\text{ét}}, \mathcal{O}_X).
\]

The Frobenius action on \(\mathcal{O}_\Delta\) induces a \(\varphi\)-semi-linear map \(\Delta_{X/A} \to \Delta_{X/A}\). And there is a relation

\[
\overline{\Delta}_{X/A} \cong \Delta_{X/A} \otimes^L_A A/I \in \text{D}(X_{\text{ét}}, A/I)
\]

by the Grothendieck spectral sequences associated to the diagram of functors:

\[
\begin{array}{ccc}
\text{Mod}((X/A)_{\Delta}, \mathcal{O}_\Delta) & \xrightarrow{\otimes^L_{\mathcal{O}_\Delta} \overline{\mathcal{O}}_\Delta} & \text{Mod}((X/A)_{\Delta}, \overline{\mathcal{O}}_\Delta) \\
\Gamma((X/A)_{\Delta}, -) & \downarrow & \Gamma((X/A)_{\Delta}, -) \\
\text{Mod}(X_{\text{ét}}, A) & \xrightarrow{\varphi} & \text{Mod}(X_{\text{ét}}, A/I)
\end{array}
\]
Affine Case

Def. (VI.5.4.8) (Situation). In this subsubsection, fix a bounded prism \((A, I)\) and a \(p\)-completely smooth (I.10.7.1) \(A/I\)-algebra \(R\) (or equivalently the \(p\)-adic completion of an étale \(A\)-algebra, by (I.10.7.9) and (I.10.6.16), and they define the same site by the universal property of completion).

Prop. (VI.5.4.9) (Prismatic Site over Affine Formal Scheme). The prismatic site of \(R\) relative to \(A\), denoted by \((R/A)_\Delta\) is the site whose objects are bounded prisms over \((A, I)\) together with an \(A/I\)-algebra map \(R \to B/IB\). And it is endowed with the indiscrete topology (V.1.1.2), so sheaves on this site is just presheaves.

There are two natural sheaves on this site, \(\mathcal{O}_\Delta\) maps a prism \((B, IB)\) to \(B\) which is valued in \((p, I)\)-complete \(\delta\)-\(A\)-algebras, and \(\overline{\mathcal{O}}_\Delta\) which maps a prism \((B, IB)\) to \(B/IB\) which is valued in \(p\)-complete \(R\)-algebras (I.10.6.16)

Prop. (VI.5.4.10) (Compare with Indiscrete Topology). If \((X/A)_\Delta \subset (X/A)\) is a continuous map of sites that the former is endowed with the indiscrete topology, then it is a morphism of sites, by (V.1.2.13), so this induces a morphism of topoi

\[
\text{Sh}((X/A)_\Delta) \to \text{Sh}((X/A)'_\Delta)
\]

by (V.1.2.18), then we have the Leray spectral sequence (V.6.1.8)

\[
E^{p,q}_2 = H^p(C, R^q f_*(\mathcal{F}^\bullet)) \Rightarrow H^{p+q}(C, \mathcal{F}^\bullet).
\]

and by (V.6.1.5)(V.6.1.4) and (VI.5.1.12), we have a natural isomorphism

\[
R\Gamma((X/A)'_\Delta, \mathcal{F}) \to R\Gamma((X/A)_\Delta, \mathcal{F})
\]

for \(\mathcal{F} = \mathcal{O}_\Delta\) or \(\overline{\mathcal{O}}_\Delta\).

How to Compute the Prismatic Complex in the Affine Case

Lemma (VI.5.4.11) (Weakly Final Object). Let \((A, I)\) be a prism and let \(R\) be a \(p\)-completely smooth \(A/I\)-algebra, then the category \((R/A)_\Delta\) admits a weakly final object. Moreover, we can choose it to be flat over \((A, I)\).

Proof: Let \(F_0\) be the derived \((p, I)\)-completion of a free \(\delta\)-ring over \(A\) on the set \(R\), then there is a surjection of \(A\)-algebras \(F_0 \to R\), with kernel \(J\) derived \((p, I)\)-complete?. Then (VI.5.1.13) applied to the \(\delta\)-ring \((F_0, J)\) gives a prism \((F, IF)\) over \((A, I)\), and by construction it is an object of \((R/A)_\Delta\).

And it is weakly final because of the universal properties of \(F_0\) (I.8.4.9) and \(F\) (VI.5.1.13).

For the flatness, we temporarily call a \(\delta\)-pair \((B, J)\) good if

- \(B\) is \((p, I)\)-completely flat over \(A\) and \(J\) is \((p, I)\)-complete.

- the prismatic envelope is flat over \((A, I)\) and its formation commutes with completed derived base change on a \((p, I)\)-completely flat map \(B \to B'\).

Then we need to show that \((F_0, J)\) is good. We have the following observations:

- Good pairs are stable under filtered colimit in the category of \(\delta\)-pairs \((B, J)\) that \(J\) is derived \((p, I)\)-complete. (Because filtered colimits of flat modules are flat (I.7.1.5)).

- If \((B, J)\) is a \(\delta\)-pair over \(A\) with \(B\) completely \((p, I)\)-flat over \(A\), and \(B \to B'\) is a \((p, I)\)-completely f.f. map that \((B', JB')\) is good, then \((B, J)\) is good. (This follows from (VI.5.1.13) and the f.f. descent (VI.5.1.13)).
Then we can write $B$ as a filtered colimit of $(p,I)$-complete algebras $B_n \to R$, and the kernel of each of them is locally generate by a $(p,I)$-completely regular sequence, so we can use the observations to pass to f.f. localization and filtered colimit to show that $(B,J)$ is good. Cf. [Prism, Scholze, 3.14].

\[
\]

Lemma (VI.5.4.12) (Products). The category $(R/A)_\Delta$ admits products.

**Proof:** For $\delta$-rings $B, C \in (R/A)_\Delta$, we can take the $\delta$-ring colimit $D_0 = B \otimes_A C$ (I.8.4.8), but it may not be compatible with $R$-actions. Instead, let $J$ be the kernel of the natural map

\[
D_0 \to D_0/ID_0 \to B/IB \otimes_{A/IA} C/IC \to B/IB \otimes_R C/IC,
\]

then $(D_0,J)$ is a $\delta$-ring over $(A,I)$, and then we can use (VI.5.13) to get a prism $(D,ID)$ over $(A,I)$, then the maps $R \to B/IB \to D_0/ID_0 \to D/ID$ and $R \to C/IC \to D_0/ID_0 \to D/ID$ are equal (because they all factor through $D/J$), thus giving a product object in $(R/A)_\Delta$. □

Prop. (VI.5.4.13) (Čech-Alexander Construction for Prismatic Cohomology). By (V.6.1.17)(VI.5.4.10) and the lemmas (VI.5.4.11)(VI.5.4.12) above, the prismatic complex $\Delta_{R/A}$ is represented by the complex

\[
F^0 \to F^1 \to F^2 \to \ldots
\]

In particular, $F^0 = F$ as constructed in (VI.5.4.11) and each $F^n$ are $(p,I)$-completely $A$-flat, $I$-torsion-free and $(p,I)$-complete $\delta$-rings by (VI.5.1.11) and (VI.5.1.14).

Moreover, this complex is the prismatic envelope functor applied to the Čech nerve of $A \to F_0$, w.r.t. the Čech nerve $F^\bullet$ of ideals $J^\bullet = \text{Ker}(F^\bullet \to F_0 \to R)$, because prismatic envelope is a left adjoint hence commutes with tensor product of pairs (VI.5.1.13).

Cor. (VI.5.4.14). $\Delta_{R/A}$ is derived $(p,I)$-complete and $\overline{\Delta}_{R/A}$ is derived $p$-complete, because each term of the complex $F^\bullet$ is derived $(p,I)$-complete, so does its cohomology groups (I.10.6.8), thus so does $\Delta_{R/A}$ itself. Similarly for $\overline{\Delta}_{R/A}$, because each term of $F^\bullet/I$ is $p$-complete (VI.5.4.13) hence derived $p$-complete, by (I.10.6.16).

5 Hodge-Tate Comparison

Def. (VI.5.5.1) (Breuil-Kisin Twist). Let $I$ be an invertible ideal of $A$, then for any $A/I$-module $M$, we define the Breuil-Kisin twist of $M$ as $M\{n\} = M \otimes (I/I^2)^n$. Notice this definable for $n \in \mathbb{Z}$ because $I/I^2$ is an invertible $\mathcal{O}_{A/I}$-module, by definition (VI.5.1.1). Also it is definable in the level of $D(A/I)$, as $(I/I^2)^n$ is locally free thus flat.

Def. (VI.5.5.2) (Completed de Rham Complex). The completed de Rham complex is the derived $p$-completion of the de Rham complex of $\Omega_{X/A/I}$. We will use the derived $p$-completed de Rham complex in the sequel. It has a property that it coincides with the $p$-completion of its separate terms by (I.10.7.8) and (I.10.6.16) and the fact $\Omega_{X/A/I}$ is finite projective hence flat (I.7.5.14).

In fact, as $\Omega_{X/S}$ is finite locally free, the derived $p$-completion is just by tensoring $- \otimes_{R_0} R$, where $R_0$ is a smooth $A$-algebra that $R_0^n = R$ by (I.10.7.9). In particular, the completed de Rham complex is compatible with base change and $p$-completely étale extension, because the ordinary de Rham complex does (I.7.3.6)(I.7.7.6).
Prop. (VI.5.5.3) (Bockstein Differential). Let $I$ be an invertible ideal of $A$, for any $M^\bullet \in D(A)$, we use the Breuil-Kisin Twist notation (VI.5.5.1) and consider the exact triangle

$$M^\bullet \otimes^L_A A/I\{i+1\} \to M^\bullet \otimes^L_A I^i/I^{i+2} \to M^\bullet \otimes^L_A A/I\{i\}$$

obtained from the exact triangle

$$I^{n+1}/I^{n+2} \to I^n/I^{n+2} \to I^n/I^{n+1}$$
tensoring $O_\Delta$. Then we get a Bockstein differential

$$\beta^n : H^n(M^\bullet \otimes^L_A A/I\{n\}) \to H^{n+1}(M^\bullet \otimes^L_A A/I\{n+1\})$$

Then these maps satisfy $\beta^{n+1} \circ \beta^n = 0$.

Proof: Consider the morphism of distinguished triangles:

$$
\begin{array}{ccc}
I^{n+1}/I^{n+3} & \to & I^n/I^{n+3} \\
\downarrow & & \downarrow \\
I^{n+1}/I^{n+2} & \to & I^n/I^{n+2}
\end{array}
$$

then we see for any $M^\bullet \in D(A)$, $\beta^n$ factors as

$$H^n(M^\bullet \otimes^L_A I^n/I^{n+1}) \to H^{n+1}(M^\bullet \otimes^L_A I^{n+1}/I^{n+3}) \to H^{n+1}(M^\bullet \otimes^L_A I^{n+1}/I^{n+2})$$

and also we consider the distinguished triangle

$$I^{n+2}/I^{n+3} \to I^{n+1}/I^{n+3} \to I^{n+1}/I^{n+2}$$

to see that the composition

$$H^{n+1}(M^\bullet \otimes^L_A I^{n+1}/I^{n+3}) \to H^{n+1}(M^\bullet \otimes^L_A I^{n+1}/I^{n+2}) \xrightarrow{\beta^{n+1}} H^{n+2}(M^\bullet \otimes^L_A I^{n+2}/I^{n+3})$$

is 0, and this two observation gives the result. □

Prop. (VI.5.5.4) (Hodge-Tate Comparison Theorem). We have a structure map $\eta^0 : O_X \to H^0(\Delta_X/A)$, and $H^\bullet(\Delta_X/A)$ is a dga by (VI.5.5.3) applied to $M^\bullet = \Delta_X/A$ and (I.10.1.1) noticing $\Delta_A$ is a sheaf of algebras. then the universal property of de Rham complex (VI.4.1.6) and lemma (VI.5.5.5) shows $\eta^0$ extends to a map

$$\eta^0_H : \Omega^\bullet_X/(A/I) \to H^\bullet(\Delta_X/A)\{\bullet\}$$

of sheaves of $A/I$-dgas on $X_{\acute{e}t}$.

Then this is an isomorphism of differential graded $A/I$-algebra. In particular $\Delta_X/A \in D(X_{\acute{e}t}, A/I)$ is a perfect complex with $H^i(\Delta_X/A) \cong \Omega^i_X/(A/I)\{-i\}$.

Proof: The proof of the isomorphism is given in (VI.5.6.10). □

Lemma (VI.5.5.5). For any local section $f \in O_X(U)$, the differential $\beta_I(f) \in H^1(\Delta_X/A)\{1\}(U)$ squares to 0.
VI.5. PRISMATIC COHOMOLOGY

Proof: This follows from (VI.5.6.9) in the affine line case because \( H^2(\Delta/X_A)\{1\}(U) = 0 \), and for the general case, use the étale localization (VI.5.6.2) and base change theorem (VI.5.6.1), noticing that cup product survives through derived tensor product.

Cor. (VI.5.5.6) (Base Change). The formation of \( \Delta X_A \in D(X_{\text{ét}}, A) \) commutes with base change along a bounded prism \((A, IA) \to (B, IB)\): Let \( g : X_B = X \otimes_{\text{Spf} A/IA} \text{Spf} B/IB \to X \) be the projection, then

\[
(g^* \Delta X_A) \wedge \sim = \Delta X_B / B, \quad (g^* \Delta X_A) \wedge \sim = \Delta X_B / B
\]

where \((\wedge \sim)\) is the derived \((p, I)\)-completion or \(p\)-completion.

Proof: Because both side are derived \((p, I)\)-complete, we take their cylinder object and by derived Nakayama w.r.t. \( B/(p, I)\) (I.10.6.10) it suffices to show the second, which is true because this is true for completed de Rham complexes (VI.5.5.2).

\[\square\]

Proof of Hodge-Tate Comparison

The strategy is as follows: we study the affine case to construct and prove the Hodge-Tate comparison isomorphism in the affine case, and then this gives the construction of Hodge-Tate map in the general case, then we can also prove the general Hodge-Tate comparison by localizing at affine subschemes, so the affine case is important.

To prove the Hodge-Tate comparison isomorphism in the affine case, we use étale localization to reduce to the polynomial case, and then use flat base change to reduce to the oriented case. Then we use a slick strategy to reduce to the crystalline case, and finally reduce to crystalline comparison.

Lemma (VI.5.6.1) (Base Change). Let \( R \) be a \( p \)-completely smooth \( A/I\)-algebra and \((A, I) \to (A', I')\) be a map of bounded prisms that \( A \to A' \) has finite \((p, I)\)-complete Tor amplitude (I.10.7.1). If \( R' = R \otimes_A^{L} A' \) for the the base change, then the natural map induces an isomorphism

\[
\Delta R/A \otimes^{L} A' \sim \Delta R'/A', \quad \Delta R/A \otimes^{L} A' \sim \Delta R'/A'
\]

Proof: We use the Čech nerve of a weakly final object (VI.5.4.13) to compute the cohomology, then we notice the \((p, I)\)-completed base change \(- \otimes^{L} A'\) applied termwise to the Čech nerve of \( A \to F^0 \) is the Čech nerve of \( A \to F^0 \otimes A A' \), which is weakly final in \((R'/A')_{\Delta}\), by the universal property and the fact \((p, I)\)-completed base change is a left adjoint.

Finally we use (I.10.7.8) to see that this termwise completed derived base change just represents \( \Delta R/A \otimes^{L} A' \) because each term of the prismatic envelope \( F \) is \((p, I)\)-completely flat over \( A \) and the completed derived base change of \( F^n \) is discrete by (VI.5.4.13).

\[\square\]

Lemma (VI.5.6.2) (Étale Localization). Let \( R \to S \) be a \( p \)-completely étale map of \( p \)-completely smooth algebras, then the natural map

\[
\Delta R/A \otimes^{L} R S \to \Delta S/A
\]

is an isomorphism.

Proof: Firstly the forgetful functor \((R/A)_{\Delta} \to (S/A)_{\Delta}\) has a right adjoint, described as follows: a primis \((B, IB) \in (B/S)_{\Delta}\) induces a \( p \)-completely étale morphism of (discrete) rings \( B/IB \to B/IB \otimes^{L} R S\) by (I.10.7.9), and by Elkik’s algebrization (I.10.7.9), this is a derived \( p \)-completion of...
some étale map \( B/IB \to T_0 \), and we can lift it to some étale map \( B \to S_0 \) by Henselian pair property(I.10.6.12)(I.6.10.9), then we can also take the derived \((p, I)\)-completion(discrete by(VI.5.1.11)) \( S_B \) of \( S_0 \), then \( S_B/IS_B \cong B/IB \otimes^L_R S \) [. So we have the following base change diagram:

\[
\begin{array}{ccc}
B & \to & S_0 \\
\downarrow & \downarrow & \downarrow \\
B/IB & \to & T_0 \end{array}
\]

\[
B/IB \otimes^L_R S
\]

For the adjointness, it suffices for every prism \((T, IT)\) with morphisms \( B \to T, B/IB \otimes^L_R S \to T/IT \), we can lift to a map \( S_B \to T \). But if we consider the \( p \)-completely étale map \( T \to T \otimes_B B_S \) and its base change, it suffices to find a section of this map, and this is by Henselian pair \((T, IT)(I.6.10.6)(VI.5.1.11)(I.6.10.9)\). Moreover, \( S_B \) has a \( \delta \)-structure by(I.8.4.17) so it is clearly a prism, and the right adjoint \( F \) just takes \((T, IT) \to (S_B, IS_B)\).

This right adjoint preserves weakly final objects and products, and it is just the completed derived tensor \( -\otimes^L_R S \) when modulo \( I \) by construction, so when combined with(I.10.7.8) we get the conclusion.

\[\square\]

Remark (VI.5.6.3). Notice these two lemmas(VI.5.6.1)(VI.5.6.2) are consequences of Hodge-Tate comparison isomorphism, once we proved it!

Crystalline Comparison in Characteristic \( p \)

Prop. (VI.5.6.4) (Crystalline Comparison in Characteristic \( p \)). Let \((A, (p))\) be a crystalline prism and let \( I \subset A \) be a pd-ideal with \( p \in I \), in particular the Frobenius \( A/p \to A/p \) factors through \( A/I \) by(I.13.0.3), inducing a map \( \psi = \psi_I : A/I \to A/p \). Let \( R \) be a smooth \( A/I \)-algebra and let \( R^{(1)} = R \otimes_{A/I, \psi} A/p \), then there is a canonical

\[
\Delta_{R^{(1)}/A} \cong R_T \Gamma_{\text{crys}}(R/A)
\]

of \( E_\infty \)-A-algebras compatible with Frobenius action.

Proof: \[\square\]

Cor. (VI.5.6.5). If we have a smooth \( R \) over \( A/p \) and let \( \tilde{R} = R \otimes_{A/p} A/I \), then \( \tilde{R}^{(1)} = \varphi_* R \), and we can apply this theorem to \( \tilde{R} \) to get a canonical isomorphism

\[
\varphi^* \Delta_{R/A} \cong R \Gamma_{\text{crys}}(\tilde{R}/A)
\]

of \( E_\infty \)-A-algebras compatible with Frobenius action.

Lemma (VI.5.6.6) (Hodge-Tate Comparison for the Affine Line over \((Z_p, (p))\)). If \((A, (p))\) is a \( p \)-torsionfree crystalline prism and \( R = \mathbb{F}_p(X) \), then the Hodge-Tate map constructed in(VI.5.5.4) is an isomorphism.

Proof: WARNING: This proof will not use the construction of the Hodge-Tate map in degree> 1 and this lemma will be used in the proof of Hodge-Tate map in the general case, so there is no cycle in the reasoning.
The map
\[ \Omega_{R^{(1)}/(A/p)}^* \to H^*(R_{\text{crys}}(R/A) \otimes^L_A A/p)^* \cong H^*(\Delta_{R^{(1)}/A} \otimes^L_A A/I)^* = H^*(\Delta_{R^{(1)}/A})^* \]
is an isomorphism by Cartier isomorphism, where the middle is by prismatic-crystalline comparison (VI.5.6.4).

It suffices to check this is the Hodge-Tate map for \( R^{(1)}/(A/p) \). But the Hodge-Tate map is induced by the inclusion \( \mathcal{O}_X \to H^0(\Delta_{X/S}) \). It is fairly easy to check the proof of (VI.5.6.4) and (VI.4.5.9) the composition is also the canonical one. Finally if we choose \( \mathbb{F}_p(X) = R = R^{(1)} \), then we get the desired Hodge-Tate isomorphism.

Lemma (VI.5.6.7) (Hodge-Tate Comparison for \((\mathbb{Z}_p, (p))\)). If \((A, (p))\) is a \(p\)-torsionfree crystalline prism and \( R = \mathbb{F}_p[X_1, \ldots, X_n] \), then the Hodge-Tate map constructed in (VI.5.4.4) is an isomorphism.

**Proof:** The proof is the same as that of (VI.5.6.6), notice now we already have the Hodge-Tate Comparison map.

Direct proof of the Hodge-Tate comparison for \((\mathbb{Z}_p, (p))\)

The proof of (VI.5.6.6) and (VI.5.6.7) is sloppy because we somehow lose track of whether the composition morphism is the Hodge-Tate comparison map. As the situation is so explicit, we decided to give a direct proof.

Mix Characteristic Case

Prop. (VI.5.6.8) (Comparing with the Characteristic \(p\)). Let \( A \) be a universal oriented prism in any characteristic and \( A \to B \) as in (VI.5.1.5), let \( \alpha : (A, (d)) \xrightarrow{\varphi} (A, (d)) \to (B, (p)) \), then \( \alpha \) is a map of prisms by (VI.5.1.6), and:

1. \( \alpha/p \) factors as \( A/p \to A/(p, d) \xrightarrow{\varphi} A/(p, d^p) \to B/p = D_{A/p}((d)) \) where the first map has finite Tor amplitude and the last two maps are f.f., thus \( \alpha/p \) has finite Tor amplitude.
2. The functor \( \hat{\alpha}^* : D_{\text{comp}}(A, (p, d)) \to D_{\text{comp}}(B, (p)) \) reflects isomorphisms.
3. For any \( p \)-completely smooth \( A/I \)-algebra \( R \), let \( R_B = R \otimes_A B \), then the map
\[
\hat{\alpha}^* \Delta_{R/A} \to \Delta_{R_B/B}
\]
is an isomorphism.

**Proof:**

1. It factors because \( d^p = p! \gamma_p(d) \in pB, \) and \( B/p = D_{A/p}((d)) \) by (I.13.0.16), noticing \( D_A((d)) = D_A((p, d)) \). The first map is of finite Tor amplitude (I.10.7.1) because \( d \) is a nonzerodivisor in \( A/p \) by (VI.5.1.4). \( \varphi \) is f.f. because it is a base change of \( \varphi_{A/p} \) and the latter is f.f. (VI.5.1.4). The last one is f.f. because it is a free summand as \( D_{A/p}((d)) = A/p[X_1, X_2, \ldots]/(a^p, X_1^p, X_2^p, \ldots) \) by (I.13.0.20).
2. Because \( D_{\text{comp}}(A, I) \) is a weak Serre subcategory of \( D(A) \) (I.10.6.8), to show it reflects isomorphisms, by item1 and derived Nakayama applied to \( \xrightarrow{- \otimes^L_B B/p} \), it suffices to show if \( X \in D_{\text{comp}}(A, (p, d)) \), if \( X \otimes^L_A A/p \otimes^L_{A/p} A/(p, d) = 0 \), then \( X = 0 \), but \( X \otimes^L_A A/p \otimes^L_{A/p} A/(p, d) = X \otimes^L_A A/(p, d) \), and this follows from derived Nakayama again.
3. \( \alpha \) has finite \((p, d)\) Tor amplitude by (I.10.7.3) and (VI.5.1.4), so 3 follows from (VI.5.6.1).
Final Proof

Prop. (VI.5.6.9) (Hodge-Tate Comparison in the Affine Line Case). In situation (VI.5.4.8), if \( R = A/I(X) \), \( \eta^R_0 : R \to H^0(\Delta_{R/A}) \) and the twisted morphisms \( \eta^R_i \{ -1 \} : \Omega^R_{(A/I)} \{ -1 \} \to H^i(\Delta_{R/A}) \) defined by the universal property of \( \Omega^R_{A/I} \) are isomorphisms, and \( H^i(\Delta_{R/A}) = 0 \) for \( i > 1 \).

In particular, by étale localization of prismatic cohomology (VI.5.6.2), lemma (VI.5.5.5) holds for any \( p \)-completely smooth algebra \( R \) over \( A/I \). But this doesn’t say that higher cohomologies vanish for any \( R \), this is because cup product can survive derived tensor but cohomology groups cannot.

Proof: In this case, \( \Omega_{R/S} \) is topologically free over \( R \), thus we can choose a map

\[
\eta : R \oplus \Omega^1_{R/A} \{ -1 \}[{-1}] \to \Delta_{R/A}
\]

lifting \( \eta^R_0 \oplus \eta^1_{R/I} \{ -1 \} \).

Firstly if \( (A, I) = (\mathbb{Z}_p, (p)) \), this case is done by (VI.5.6.6).

Next, if \( (A, (d)) \) is oriented, then there is a map of prisms from the universal oriented prism (VI.5.1.4) \( A_0 \to A \), then we have a pushout diagram of simplicial commutative rings

\[
\begin{array}{ccc}
A_0 & \longrightarrow & A \\
\downarrow^{\alpha} & & \downarrow^{\beta} \\
\mathbb{Z}_p & \longrightarrow & D_0 \\
& & \longrightarrow E = A\hat{\otimes}_{A_0}^L D_0
\end{array}
\]

and \( (E, (p)) \) is a simplicial prism.

Now the warning is what we have done so far can all be extended to the derived algebraic geometry setting, or at least to the "animated commutative algebra" setting!

We denote the composite of the lower row as \( \gamma \), because \( \hat{\alpha}^* \) reflects isomorphisms, so does \( \hat{\beta}^* \), and we can show \( \beta^* \Delta_{R/A} \cong \gamma^* \Delta_{\mathbb{Z}_p(X)/\mathbb{Z}_p} \), and identifies the Hodge-Tate map: This is because we are in the polynomial case, so we can make the construction of \( \Delta_{R/A} \) clear: we just take the \( F^0 \) to be the derived \( (p, I) \)-completion of the free \( \delta \)-rings \( A\{X\} \) in the construction of weakly final object (VI.5.4.11), then the resulting Čech-Alexander complex is free and compatible with base change in the complex level.

Also \( \beta \) has finite \( (p, d) \)-Tor amplitude because \( \alpha \) has because \( (p, d) \) is regular in \( A_0 \)(VI.5.1.4), also \( \gamma \) has finite \( p \)-Tor amplitude because \( E \) is \( p \)-torsionfree by (VI.5.1.11), and use (I.10.7.3). So we are reduced to the \( (\mathbb{Z}_p, (p)) \) case, which we have done.

Finally, for a general bounded prism \( (A, I) \), we can reduce to the oriented case by base change along the f.f. extension defined in (VI.5.1.11), then we reduce to the orientable case by (VI.5.6.1). □

Prop. (VI.5.6.10) (Hodge-Tate Comparison Theorem in General). The map

\[
\eta^R_* : \Omega^\bullet_{X/(A/I)} \to H^\bullet(\Delta_{X/A})\bullet
\]

constructed in (VI.5.5.4) is an isomorphism of sheaves of \( A/I \)-dgas on \( X_{\text{ét}} \).

Proof: Because we already have the Hodge-Tate comparison map, it suffices to prove the theorem for affine subscheme \( \text{Spf} \ R \), and because both prismatic cohomology and de Rham complex are étale
it suffices to prove for the polynomial case. In this case, \( \hat{\Omega}^i_{R/A} \) is topological free over \( R \), then we can lift the Hodge-Tate map (VI.5.5.4) to the level of chain complexes:

\[
\eta : \bigotimes_{i=0}^{n} \Omega_R^{\ast} \{ -i \} \rightarrow \Delta_{R/A}.
\]

And then the rest is the same as the proof of (VI.5.6.9), where in the \((\mathbb{Z}_p, (p))\) case, we use (VI.5.6.7) instead of (VI.5.6.6).

\[\square\]

**de Rham Comparisons**

**Prop. (VI.5.6.11) (de Rham Comparison).** In situation (VI.5.4.2), if \( W(A/I) \) is \( p \)-torsionfree, then there is a natural isomorphism

\[
\Delta_{X/A} \hat{\otimes}_{A, \varphi} A/I \cong \Omega_{X/(A/I)}^\ast
\]

of commutative algebra objects in \( D(A/I) \).

**Proof:** Cf. [Sholze, Prism, 6.4].

It suffices to construct locally a functorial isomorphism \( \Delta_{R/A} \hat{\otimes}_{A, \varphi} A/I \cong \Omega_{R/(A/I)}^\ast \) and then glue.

Let \( A \rightarrow W(A/I) \) be the canonical map, and \( \psi : A \xrightarrow{\varphi} A \rightarrow W(A/I) \), then it takes \( I \) into \((p)^?\). But the map \( \psi/p : A/I \rightarrow W(A/I)/p = A/I \) factors through \( A/(p, I) \), thus \( R' \) also equals \( R/p \otimes_{A/(p, I)} W(A/I)/p \), so there is a base change diagram:

\[
\begin{array}{ccc}
R & \longrightarrow & R/p \\
\uparrow & & \uparrow \\
A/I & \longrightarrow & A/(p, I) \xrightarrow{\psi/p} W(A/I)/p = A/I
\end{array}
\]

Base change for prismatic cohomology (VI.5.5.6) gives an isomorphism

\[
\Delta_{R/A} \hat{\otimes}_{A, \varphi} W(A/I) \cong \Delta_{R'/W(A/I)}.
\]

Note that \( W(A/I) \rightarrow A/(p, I) \) is a pd-thickening with ideal \((p, [p])^?\), so we can use crystalline comparison w.r.t. the crystalline prism \((W(A/I), (p))^?\) to show that

\[
\Delta_{R'(1)/W(A/I)} \cong R\Gamma_{\text{crys}}((R/p)/W(A/I)).
\]

Then finally we use the crystalline de Rham comparison to get the desired result. \[\square\]

**Remark (VI.5.6.12).** The technical condition \( W(A/I) \) is \( p \)-torsionfree can be removed, by [Scholze, Prism, 15.4].

### 7 Derived de Rham Cohomology

**Def. (VI.5.7.1) (Derived de Rham Cohomology).** For an \( \mathbb{F}_p \)-algebra \( k \), the **derived de Rham cohomology** functor \( dR_{-}/k : \text{CAAlg}/k \rightarrow D(k) \) is the left derived functor of the functor \( \text{Poly}_k \rightarrow D(k) \) given by \( R \rightarrow \Omega_{R/k}^\ast \) via (XI.1.4.1).

**Prop. (VI.5.7.2) (Derived Cartier Isomorphism).**
Regular Semiperfect Rings

Def. (VI.5.7.3) (Regular Semiperfect Rings). Let $k$ be a perfect ring, a regular semiperfect ring over $k$ is an $k$-algebra of the form $R/I$ where $R$ is a perfect $k$-algebra and $I$ is an ideal generated by a regular sequence.

Prop. (VI.5.7.4). Let $k$ be a perfect field and $S$ be a regular semiperfect ring, then

8 Derived Prismatic Cohomology

Prop. (VI.5.8.1) (Derived Hodge-Tate Comparison).

9 $q$-de Rham Cohomology

Prop. (VI.5.9.1) (Hodge-Tate isomorphism via $q$-de Rham Complex). Cf.[Bhatt, Prism, 5.3.9].

10 Étale Comparison

Prop. (VI.5.10.1) (Frobenius Fixed Points). Fix an $\mathbb{F}_p$-algebra $B$ with an element $t$, let $D(B[F])$ be the derived category of Frobenius $B$-modules: this is a category whose objects are $(M, \varphi)$ where $M \in D(B)$ and $\varphi$ is a morphism $M \to M \otimes_{B, \varphi} B$ in $D(B)$. And let $D_{comp}(B[F])$ be the full subcategory spanned by pairs $(M, \varphi)$ where $M \in D_{comp}(B,(t))(I.10.6.4)$.

Given $(M, \varphi) \in D_{comp}(B[F])$, let $M^{\varphi=1} = R\text{Hom}_{D(B[F])}((B, \varphi), (M, \varphi)) \in D(F_p)$?, called the Frobenius fixed pts of $M$.

Prop. (VI.5.10.2). Fix an $\mathbb{F}_p$-algebra $B$ with an element $t$, then

- The functor $D_{comp}(B[F]) \to D(F_p)$ given by $M \mapsto M^{\varphi=1}$ and $M \mapsto (M[t^{-1}])^{\varphi=1}$ commute with colimits.
- For any $(M, \varphi) \in D_{comp}(B[F])$ and

11 Almost Purity
VI.6 Motives

1 Weil Cohomology Theories

Def. (VI.6.1.1) (Correspondences). Let $X, Y$ be smooth projective schemes over $k$, and $X = \bigsqcup_n X_n$ be decomposition of $X$ into open and closed subschemes of $X$ that $X_n$ is equidimensional of dimension $n$(this is possible by (V.4.2.6)(V.5.3.11) and (VI.1.0.15)).

Prop. (VI.6.1.2) (Category of Correspondences over $k$). The category of correspondences over $k$ is a symmetric monoidal category with unit.
Chapter VII

Arithmetic Geometry

VII.1 Abelian Varieties

Main References are [Sta], [Abelian Variety Mumford], [Abelian Varieties notes Conrad], [Abelian Varieties Milne], [Heights in Diophantine Geometry, Bombieri] and [Abelian Variety van der Geer].

1 Basics

Abelian Variety

Def. (VII.1.1.1). A group variety over a field $k$ is a $k$-variety that is also a group scheme. Denote $e \subset X$ the identity of $X(k)$, which is a rational point.

Def. (VII.1.1.2) (Abelian Variety). An Abelian variety $A$ over a field $k$ is a group variety over $k$ that is a complete variety over $k$(i.e. proper and geometrically integral).

For each field extension $K/k$, $A_K$ is an Abelian variety over $K$, because proper and geometrically integral is stable under base change.

Prop. (VII.1.1.3) (Group Variety Smooth). A group variety over $k$ is smooth. In particular, it is also geo.regular.

Proof: As the smooth locus of a variety is open dense(V.5.3.15), and it has a smooth closed pts because closed pts are dense(V.4.1.25). The closed pts are transitive by translation, so all pts of $X$ is smooth.

□

Prop. (VII.1.1.4) (Tangent Bundle Trivial). For a group variety over a field $k$, $T_{X,e}$ is the tangent space at $e$, then there is a natural isomorphism $\Omega_{X/k} \cong T^*_{X,e} \otimes O_X$. Also true for $T_X$ (because $\Omega_{X/k}$ is locally free as $X$ is smooth(VII.1.1.3)(V.8.1.15)).

Proof: There should be another proof using relation in(V.3.4.14).?

Use a dual number characterization of tangent spaces and tangent vector fields(V.3.5.15)(V.8.1.16), then notice a tangent vector $\tau \in T_{X,e}$ is a $S = k[\varepsilon]$-point of $X$, then right translation gives a translation $X_S \to X_S$ that is invariant on $X$, which gives a tangent vector on $X$. 
So there is a map $T_{X,e} \otimes \mathcal{O}_X \to \Omega_{X/k}$. To check isomorphism, both are locally free of the same rank, so it suffices to show it is surjective. But on closed pts, pass to Nakayama, this is clearly true, so it is surjective by (V.8.1.13).

**Cor. (VII.1.1.5).** If $X$ is an Abelian variety, any global tangent vector on a group variety is left invariant.

**Proof:** Because $\Gamma(X, \mathcal{O}_X) = k$ (V.8.1.12), so $\Gamma(X, \mathcal{O}_X \otimes T_{X,e}) = T_{X,e}$ are all generated by left invariant vectors (left and right translation commutes).

**Prop. (VII.1.1.6) (Dimension Theorem).** Let $\varphi : G \to H$ be a surjective homomorphism of group varieties, then

$$\dim(G) = \dim(H) + \dim(\text{Ker}(\varphi)).$$

**Proof:** This is a consequence of (V.5.2.18), as the fiber over any closed pt is isomorphic to a field base change of Ker($\varphi$).

**Prop. (VII.1.1.7) (Abelian Variety is Projective).** Abelian variety is projective, by (V.9.4.7) and (V.4.5.3).

**Prop. (VII.1.1.8).** The analytification of a complex Abelian variety is a complex tori with a Riemann form. And the reverse is also true.

**Proof:** The analytification is compact because an Abelian variety is proper (IX.10.6.3). It is a smooth manifold by (IX.10.6.3)(VII.1.1.3). It is connected by (IX.10.6.3) and the fact it is projective (VII.1.1.7). It is then a compact complex Lie group, which is then a complex tori by (IX.8.5.1). Then notice it is projective, so by (IX.11.8.10) it has a Riemann bilinear form.

**Prop. (VII.1.1.9) (Rigidity Theorem).** Let $f : X \to Y$ be a morphism of an Abelian varieties to a group variety, then it is a group homomorphism followed by a translation $t_{f(e_X)}$.

**Proof:** Set $y = i_Y(f(e_X))$ and consider $t_y \circ f$, then $h(e_x) = e_Y$, and consider the morphism:

$$g : X \times X \to Y \times Y \to Y,$$

then $g(e_X, X) = g(Y, e_Y) = e_Y$, so the rigidity lemma (V.8.1.21) shows $g$ is constant with value $e_Y$. Thus $h \circ m_X = m_Y \circ (h \times h)$, thus a group homomorphism.

**Cor. (VII.1.1.10) (Abelian Variety is Commutative).**

- Let $X$ be a variety over $k$, then there is at most one one structure of Abelian variety on $X$.
- The group law of an Abelian variety is commutative, justifying the name.

**Remark (VII.1.1.11).** The completeness of $X$ is essential for the proof. In fact, there are many non-commutative algebraic groups, like $GL_n$.

From now, use the additive notation for Abelian varieties.

**Proof:**

1: If there are two structure $(m, i), (n, j)$, then consider $X \times X \to X \to X : (x, y) \mapsto m(x, y)(n(x, y))^{-1}$, then it is constant on $e_X \times X$ and $X \times e_X$, thus it is constant, so $m = n$. And $i = j$ is also clear by the associativity.

2: The inverse $i$ is a group homomorphism by (VII.1.1.9), thus it is Commutative.
VII.1. ABELIAN VARIETIES

Prop. (VII.1.1.12). Let $X, Y$ be varieties over a field $K$ that both have at least one $K$-point, assume $X$ is complete. Then any morphism $X \times Y \to G$ to a group variety $G$ factorizes as $f(x, y) = g(x)h(y)$, where $f : X \to G$ and $h : Y \to G$.

Proof: Fix a $y_0 \in Y(K)$ and define a morphism $g : X \to G : x \mapsto f(x, y_0)$, then the morphism $F : X \times Y \to G : (x, y) \mapsto g(x)^{-1}f(x, y)$ is constant on $X \times \{y_0\}$. Then the rigidity lemma (V.8.1.21) shows $F((x, y)) = h(y)$ where $y$ is a morphism. Thus we are done.

Cor. (VII.1.1.13). Any morphism from a $\mathbb{P}_K^1$ to a group variety $G$ is constant.

Proof: Let $(x_0, x_1)$ be a homogenous coordinate of $\mathbb{P}^1$, consider the morphism $s : \mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{P}^1 : (x_0, x_1) \times y \mapsto (x_0, x_0 + x_1y)$. Let $f : \mathbb{P}^1 \to G$ be a morphism, consider the composition

$$\mathbb{P}^1 \times \mathbb{A}^1 \xrightarrow{s} \mathbb{P}^1 \xrightarrow{f} G$$

then by (VII.1.1.12), $f \circ s$ factors as $f(s(x, y)) = g(x)h(y)$.

We take $y = 0$, then $s(x, 0) = x$, and $g(x) = f(x)h(0)^{-1}$. Thus $f(s(x, y)) = f(x)h(0)^{-1}h(y)$.

Next we take $x = (0, 1)$, then $s((0, 1), y) = (0, 1)$, and $f((0, 1)) = f((0, 1))h(0)^{-1}h(y)$. This shows $h(y) = h(0)$ is constant, thus $f(s(x, y)) = f(x)$. Finally, let $x = (0, 1)$, then $s((0, 1), y) = y$, and $f(y) = h(0)$ is constant.

Cor. (VII.1.1.14). Let $U$ be an open subset of $\mathbb{P}^1$, then any morphism from $U$ to $G$ is constant. In particular, $G$ contains no rational curve, and any morphism from a rationally connected variety to $G$ is constant, in particular $\mathbb{P}_K^n$.

Proof: Any rational map from $\mathbb{P}^1$ to $G$ can be extended to a morphism, by (IX.12.1.12), thus it is constant, by the proposition above.

Prop. (VII.1.1.15) (Singularity in Codimension1). Let $\varphi : X \to G$ be a rational map from a smooth variety $X$ to a group scheme and let $U$ be a maximal definition set, then every irreducible component of $X \setminus U$ is of codimension 1.

Proof: Cf. [Diophantine Geometry, P237] ?. The proof need to consider the group scheme case. □

Cor. (VII.1.1.16) (Rigidity of Morphism from Smooth Varieties). If $X$ is an Abelian variety over a field $K$, then any rational map from another smooth $K$-variety $V$ extends to a morphism $V \to X$.

Proof: By (V.5.3.11), $V$ is regular, thus by (V.4.5.9), the rational map is defined on a set that the complement has codimension $\geq 2$, but then the proposition above shows it is a morphism. □

2 Line Bundles

Remark (VII.1.2.1). As Abelian varieties are regular, $Cl(X) \cong Pic(X)$, by (V.7.1.19).

Prop. (VII.1.2.2) (Theorem of the Cube). If $X$ is an Abelian variety, and $L$ is a line bundle over $X$, then

$$\Theta(L) = p_{123}(L) \otimes p_{12}^*(L^{-1}) \otimes p_{13}^*(L^{-1}) \otimes p_{23}^*(L^{-1}) \otimes p_1^*(L) \otimes p_2^*(L) \otimes p_3^*(L)$$

is trivial.
**Prop. (VII.1.2.3)**. There is a form of morphisms from a scheme to $X$, just by considering $(f, g, h) : Y \times Y \times Y \to X \times X \times X$, i.e.,

$$(f + g + h)^* L \otimes (f + g)^* L^{-1} \otimes (f + h)^* L^{-1} \otimes (g + h)^* L^{-1} \otimes f^* L \otimes g^* L \otimes h^* L$$

is trivial.

**Cor. (VII.1.2.4) (Theorem of the Square).** Let $X$ be an Abelian variety that $L$ is a line bundle, then for any $x, y \in X$,

$$t^*_x y \Leftrightarrow L \cong t^*_y L \otimes t^*_x L.$$

Notice this isomorphism is defined under the field generated by the residue fields of $x$ and $y$.

**Proof:** Apply the theorem of the cube(VII.1.2.3) for $f : \text{id}_X$ and $g, h$ the function with constant value $x, y$. □

**Cor. (VII.1.2.5).** For a line bundle $L$ on an Abelian variety $X$, then the map $\varphi_L : X \to \text{Pic}(X) : x \mapsto [t^*_x L \otimes L^{-1}]$ is a homomorphism.

**Cor. (VII.1.2.6).** For any line bundle $L$, $[n]^* L \cong L^{n(n+1)/2} \otimes (-1)^* L^{n(n-1)/2}$.

**Proof:** Use(VII.1.2.3) in case $f = [n], g = [1], h = [-1]$, then we have:

$$n^* L^2 \otimes (n + 1)^* L^{-1} \otimes (n - 1)^* L^{-1} \cong (L \otimes (-1)^* L)^{-1}.$$

So we can use induction. □

**Def. (VII.1.2.7) (Even Line Bundle).** Consider the involution $[-1]$ of $X$ and its action on the line bundles, then a **even/odd line bundle** is defined to be a line bundle $L$ that $[-1]^* L \cong L(\text{or} L^{-1})$.

**Prop. (VII.1.2.8).** On an Abelian Variety, there is an even very ample line bundle.

**Proof:** Abelian variety is projective by(VII.1.1.7), thus there is a very ample line bundle $L$, and $[-1]^* L$ is also very ample, so $L \otimes [-1]^* L$ is even and very ample, by(V.3.3.20). □

**Prop. (VII.1.2.9).** For an Abelian variety over a number field $K$ and any line bundle $c \in \text{Pic}(X)$, there are an odd line bundle $c^-$ and an even line bundle $c^+$ that $c = c_- + c_+$.

**Proof:** Consider $2c = (c + [-1]^* c) + (c - [-1]^* c)$. $c - [-1]^* c$ is odd thus is in $\text{Pic}^0(X)$ by(VII.1.5.12), thus by(VII.1.3.4), there is a $c^d \in \text{Pic}^0(X)$ that $2c^d = c - [-1]^* c$. Then $c = (c - c^d) + c^d$ satisfies the requirement. □

### 3 Isogenies

**Def. (VII.1.3.1) (Isogenies).** A homomorphism $f : X \to Y$ between Abelian varieties is called an **isogeny** iff it satisfies the following equivalent conditions:

- $f$ is surjective and $\dim X = \dim Y$.
- $\text{Ker} f$ is a finite group scheme and $\dim X = \dim Y$.
- $f$ is finite flat and surjective.

If $f$ is an isogeny, then we define $\deg f = [K(X) : K(Y)]$. 

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Proof: Cf.[van der Geer P72]. □

Cor. (VII.1.3.2). Isogenies are stable under composition, and degree is multiplicative.

Def. (VII.1.3.3) (Separable Isogenies). An isogeny \( f: X \to Y \) is called separable isogeny iff it satisfies the following conditions:
- \( K(X)/K(Y) \) is separable.
- \( f \) is étale.
- \( \ker f \) is an étale group scheme.

Proof: Cf.[van der Geer P73]. □

Prop. (VII.1.3.4) (Multiplication Isogeny). Let \( A \) be an Abelian variety over \( K \) and \( n \neq 0 \in \mathbb{Z} \), then \( [n] \) is a finite flat surjective morphism of degree \( n^{2\dim A} \). In particular it is an isogeny. The separable degree of \( [n] \) equals the number of points in any fiber, and if \( \text{char}(K) \nmid n \), then \( [n] \) is étale and \( A[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2\dim A} \). if \( \text{char}(K) \mid n \), then \( [n] \) is not separable.

Proof: Cf.[Diophantine Geometry, P264].?

Cor. (VII.1.3.5). For an Abelian variety \( X \) and any \( n > 0 \), the group of geometric points of order \( n \) in \( X \) is finite.

Proof: Because \( \ker[n] \) is a finite flat scheme?. □

4 Algebraically Closed Field Case

All Abelian variety \( X \) in this subsection is over an alg.closed field \( k \).

Prop. (VII.1.4.1). There is a closed pt \( 0 \) in \( X \) that corresponds to \( 0 \) in the group \( X(k) \), if we denote \( \Omega_0 \) the cotangent space at \( 0 \), it is the stalk of the differential \( \Omega_{X/k} \) at \( 0 \)(V.3.4.6).

\( p \)-divisible Groups

Prop. (VII.1.4.2). For a field \( K \) of characteristic \( p \), then \( A(K^{sep}) \) is an Abelian group and its \( l^n \) torsion is isomorphic to \( (\mathbb{Z}/l^n\mathbb{Z})^{2g} \) and its \( p^n \) torsion is isomorphic to \( (\mathbb{Z}/p^n\mathbb{Z})^r \).

Prop. (VII.1.4.3). There is an isomorphism

\[
H^m_l(A_{K^{sep}}, \mathbb{Q}_l) \cong \bigwedge^m (V_i(A))^\ast.
\]

Cf.[Grothendieck Monodromy theorem].

5 Picard Schemes

Main References are [FGA explained, Chap9].
**Pic⁰(X)**

**Def.** (VII.1.5.1). Let $X$ be a smooth complete variety over $K$ and $X(K)$ is not empty. Let $P_0 \in X(K)$. Let $T$ be a variety, then $c \in \text{Pic}(X \times T)$ is called a **subfamily of Pic⁰(X)** parametrized by $T$ if

- $c_t \in \text{Pic}^0(X_{K(t)})$ for any $t \in T$.
- $c_{P_0} = 0 \in \text{Pic}(T)$.

Notice $c$ is uniquely determined by these two conditions, by see-saw principle (V.8.1.23).

**Prop.** (VII.1.5.2) (**Picard Variety**). There is a subfamily $p$ of Pic⁰($X$) parametrized by a smooth complete variety $B$ that satisfies the following universal property: any other subfamily parametrized by $T$ is the pullback along a morphism $T \to B$. This $B$ is called the **Picard variety** of $X$, denoted by Pic⁰($X$), and $p$ the **Poincaré class**.

Equivalently, $(X, p)$ represents the functor

$$T \mapsto \{ \mathcal{L} \in \text{Pic}(X \times T) \mid L_t \in \text{Pic}^0(X) \} / \pi_2^*\text{Pic}(T).$$

**Proof:** Cf. [Diophantine Geometry, P249].

**Cor.** (VII.1.5.3). Let $F$ be a field extension of $K$, $p$ be the Poincaré class in Pic($X \times \text{Pic}^0(X)$), then the following holds:

- By base change, we have Pic($X$) $\subset$ Pic($X_F$).
- Pic⁰($X_F$) = Pic⁰($X$)$_F$, and the Poincaré class is also the base change.
- Pic⁰($X$)($F$) = Pic⁰($X_F$), by identifying $b$ with $p_b$.

**Proof:** These are all abstract nonsense.

**Cor.** (VII.1.5.4) (**Subfamily and Morphisms Relations**). The Poincaré class $p$ in Pic($X \times \text{Pic}^0(X)$) satisfies $p_b = b$ for a point $b \in \text{Pic}(X)$, and $p_{P_0}$ is trivial.

For a subfamily $c$ of Pic⁰($X$) parametrized by an irreducible prevariety $T$ over $K$, the map

$$T \to \text{Pic}^0(X) : t \mapsto c_t$$

is a morphism over $K$.

**Proof:** A point $b \in \text{Pic}^0(X)$ is a morphism $K(b) \rightarrow \text{Pic}^0(X)$, and the restriction of $p$ to $b$ is just $b$, by definition.

The second assertion follows from the first, because this subfamily corresponds to a morphism $T \rightarrow \text{Pic}^0(X)$, and the restriction of $p$ at the image of $t$ in Pic⁰($X$) is just the subfamily restricted at $t$, which is $c_t$.

**Prop.** (VII.1.5.5) (**Pic⁰(X) is an Abelian Variety**). Together with the canonical group structure on Pic⁰($X$) induced by tensor product makes it an Abelian variety.

**Proof:** Pic⁰($X$) is clearly a group scheme because it represents a group functor. It is also a smooth complete variety (VII.1.5.2), thus it is an Abelian variety.

**Prop.** (VII.1.5.6) (**Dual Map**). By functoriality, if $X, X'$ are smooth complete varieties over $K$ and $P_0 \in X(K), P'_0 \in X'(K)$, and there is a morphism $\varphi X \rightarrow X'$ that maps $P_0$ to $P'_0$, then the pullback along $\varphi$ induces a map from subfamilies of Pic⁰($X$) parametrized $T$ to subfamilies of Pic⁰($X'$) parametrized $T$ which is functorial and preserves group structure, thus induces a dual homomorphism of Abelian varieties:

$$\hat{\varphi} : \text{Pic}^0(X') \rightarrow \text{Pic}^0(X).$$
**Def. (VII.1.5.7) (Dual Abelian Variety).** For an Abelian variety $A$, the dual Abelian variety $\hat{A}$ is defined to be its Picard variety.

**Prop. (VII.1.5.8) (Line Bundle Induces a Map $\varphi_c$).** If $A$ is an Abelian variety, for any line bundle $c$, there is a subfamily of $\text{Pic}^0$ parametrized by $A$: $m^*(c) - \pi_1^*c$, where $m : A \times A \to A$ is the product. It is in $\text{Pic}^0(A)$ because of the obvious parametrization. Then by definition, this subfamily corresponds to a morphism $\varphi_c : A \to \hat{A}$, and by (VII.1.5.4) $\varphi_c(a) = \tau_a^*c - c \in \text{Pic}^0(A)(K(a))$. This is also a homomorphism of Abelian variety, by (VII.1.1.9).

**Prop. (VII.1.5.9) (Kernel of $\varphi_c$).** A class $c \in \text{Pic}(A)$ is ample iff $\text{Ker} \varphi_c$ is finite and $H^0(A, nc) \neq 0$ for some $n > 0$.

*Proof:* Cf.[Diophantine Geometry, P253].

**Lemma (VII.1.5.10).** Let $A$ be an Abelian variety over $K$ and $p_i : A \times A \to A$ be the projections and $m$ be the multiplication, then the following are equivalent:

- $m^*(c) \cong p_1^*(c) + p_2^*(c)$.
- $\tau_a^*(c) \cong c$ for $a \in A$.

And if these are satisfied, $[-1]^*(c) \cong c$.

*Proof:* The equivalence is a consequence of the equation

$$ (m^*(c) - p_1^*(c) - p_2^*(c))|_{A \times \{a\}} = \tau_a^*(c) - c $$

and the see-saw principle(V.8.1.23). the last assertion is a consequence of the first equation pulled back via the morphism

$$ A \to A \times A : a \mapsto (a, -a). $$

**Lemma (VII.1.5.11).** If $b \in \text{Pic}(A)$ that $\varphi_b = 0$, then for any ample $c \in \text{Pic}(A)$, there is some $a \in A(K)$ that $b = \tau_a^*(c) - c$.

*Proof:* [Mumford, P77].

**Prop. (VII.1.5.12) (Equivalent Definitions of $\text{Pic}^0(A)$).** For $c \in \text{Pic}(A)$, $[-1]^*(c) - c \in \text{Pic}^0(A)$, the following are equivalent:

1. $b \in \text{Pic}^0(A)$.
2. $\text{Ker}(\varphi_b) = A$.
3. For every ample line bundle $c$, there is an $a \in A$ that $b \cong \tau_a^*(c) - c$.
4. There is an ample line bundle $c$ and an $a \in A$ that $b \cong \tau_a^*(c) - c$.
5. $c$ is odd.
6. For any scheme $X$, the map $\text{Mor}(X, A) \to \text{Pic}(X) : \varphi \mapsto \varphi^*(c)$ is linear.
2 \rightarrow 3 is by the lemma (VII.1.5.11). 4 \rightarrow 1 is by VII.1.5.8). Now we prove 1 \rightarrow 2:

By (VII.1.5.3), It suffices to prove for K alg. closed. Firstly we shows there is a morphism underlying the map

$$\varphi : A \times P\text{ic}^0(A) \rightarrow P\text{ic}^0(A) : (a,b) \mapsto \tau_a^*(b).$$

For $$T = A \times P\text{ic}^0(A)$$, consider the line bundle $$c = (m \times \text{id}_{P\text{ic}^0(A)})^*(p)$$ on $$A \times T$$. Notice the restriction of $$m \times \text{id}_{P\text{ic}^0(A)}$$ on $$A \times \{a\} \times \{b\}$$ is given by $$\tau_a$$,

$$c|_{A \times \{a\} \times \{b\}} = \tau_a^*(b),$$

and also $$c|_{\{0\} \times T} = p$$, so the family $$c - \tau_a^*(p)$$ of $$\text{Pic}^0(A)$$ parametrized by $$T$$ gives a morphism $$T \rightarrow \text{Pic}^0(A)$$ extending $$\varphi$$.

Next, because $$\varphi(A \times \{0\}) = 0$$, by rigidity lemma (V.8.1.21) we have $$\tau_a^*(b) \equiv b$$ for any line bundle $$b$$.

6 \rightarrow 5 is trivial, and 6 is clearly equivalent to the assertion when $$X = A$$, and it is equivalent to 2 by (VII.1.5.10).

We next prove $$[-1]^*(c) - c \in \text{Pic}^0(A)$$: because $$[-1] \tau_a = \tau_a[-1]$$, we have

$$\tau_a^*([-1]^*(c)) - [-1]^*(c) = [-1](\tau_a^*(c) - c).$$

since $$\tau_a^*(c) - c \in \text{Pic}^0(X)$$ by (VII.1.5.8), the equation is equal to $$c - \tau_a^*(c)$$ by implication 1 \rightarrow 2. and this further equals $$\tau_a^*(c) - c$$ by the theorem of square (VII.1.2.4). Then

$$\tau_a^*([-1]^*(c) - c) - ([1]^*(c) - c) = 0.$$ 

Hence $$[-1]^*(c) - c \in \text{Pic}^0(A)$$ by the implication 2 \rightarrow 3 \rightarrow 4 \rightarrow 1.

Finally we prove 5 \rightarrow 1: Let $$c$$ be an odd element, then $$-2c = [-1]^*(c) - c \in \text{Pic}^0(A)$$ by what just proved, we have $$c \in \text{Pic}^0(A)$$, then $$\varphi_c$$ has image in the kernel of $$[2]$$ on $$\hat{A}$$, by the implication 1 \rightarrow 2, thus it has trivial image, by (VII.1.3.4).

\[ \square \]

**Cor. (VII.1.5.13).** For an Abelian variety $$A$$, $$\dim A = \dim \hat{A}$$.

**Proof:** Choose an ample line bundle $$c$$ on $$A$$, then $$\varphi_c : A \rightarrow \hat{A}$$ is surjective (surjective on closed points by (VII.1.5.12) applied to base changes of $$K$$ and use (VII.1.5.3)), and the kernel is finite by (VII.1.5.9), so we are done by dimension theorem (VII.1.1.6).

\[ \square \]

**Cor. (VII.1.5.14).** For an ample line bundle $$c$$ on $$A$$, the morphism $$\varphi_c : A \rightarrow \hat{A}$$ is an isogeny (it is surjective because of (VII.1.5.12)).

**Prop. (VII.1.5.15).** Let $$p \in \text{Pic}(A \times \hat{A})$$ be the Poincaré class of $$A$$, then $$p$$ is even in $$\text{Pic}(A \times \hat{A})$$.

**Proof:** Let $$b \in \hat{A}$$, then

$$([-1]^*(p))|_{A \times \{b\}} = [-1]^*(p|_{A \times \{b\}}) = [-1]^*(-b) = b.$$ 

and

$$([-1]^*(p))|_{\{0\} \times \hat{A}} = [-1]^*(p|_{\{0\} \times \hat{A}}) = 0.$$ 

Thus $$[-1]^*p \equiv p$$ by (VII.1.5.4) and the see-saw principle (V.8.1.23).

\[ \square \]
VII.1. ABELIAN VARIETIES

Jacobians of Curves

Def. (VII.1.5.16) (Jacobians of Curves). For a nonsingular complete (hence projective by (IX.12.1.7)) curve over a field \( K \), its Jacobian variety is just its Picard variety.

Prop. (VII.1.5.17).

6 Hom\((X, X)\) and \( l \)-adic Representation

Def. (VII.1.6.1). A simple Abelian variety is an Abelian variety that has no non-trivial Abelian subvarieties.

Prop. (VII.1.6.2). Let \( B \) be an Abelian subvariety of \( A \), then there exists an Abelian subvariety \( C \) of \( A \) that the addition gives an isogeny

\[
B \times C \rightarrow A.
\]

Proof: Choose an ample line bundle \( c \) on \( A \), let \( \iota : B \rightarrow A \) be the inclusion and \( \hat{\iota} : \hat{A} \rightarrow \hat{B} \) the dual map, then

\[
(\hat{\iota} \circ \varphi_c)|_B = \varphi_{\iota^*(c)}.
\]

Since \( \iota^*(c) \) is also ample, \( \varphi_{\iota^*(c)} \) is an isogeny, thus has finite kernel. Let \( C = \text{Ker}(\iota \circ \varphi_c) \), then we have \( C \cap B \) is finite, whence \( B \times C \rightarrow A \) has finite kernel. The dimension theorem (VII.1.1.6) applied to \( \iota \circ \varphi_c \) shows

\[
\dim C + \dim \hat{B} = \dim \hat{A}.
\]

and this together with (VII.1.5.13) shows \( B \times C \rightarrow A \) is a surjection, because \( A \) is irreducible. Thus it is an isogeny. \( \square \)

Cor. (VII.1.6.3) (Poincaré’s Complete Reducibility Theorem). For an Abelian variety \( A \), there are simple Abelian subvarieties \( B_1, \ldots, B_n \) of \( A \) that additions give an isogeny

\[
B_1 \times \ldots \times B_n \rightarrow A.
\]

Prop. (VII.1.6.4). For two Abelian varieties \( A_1, A_2 \) over \( K \), then \( \text{Hom}(A_1, A_2) \) w.r.t. the addition is a torsion-free Abelian group.

Proof: Assume \( [m] \circ \varphi = 0 \), take a prime \( l \) that is coprime to \( \text{char} K \) and \( m \), then \( \varphi \) induces a map \( A_1[l^r] \rightarrow A_2[l^r] \). Because

\[
A_i[l^r] \cong (\mathbb{Z}/l^r \mathbb{Z})^{2 \dim A_i},
\]

\( \varphi \) vanishes on \( A_1[l^r] \). Apply this argument to any simple Abelian variety \( B \) of \( A_1 \), we see \( \varphi \) vanishes on infinitely many points of \( B \), thus \( \text{Ker}(\varphi) \) is not of dimension 0, so it equals \( B \). Now Poincaré’s reducibility theorem (VII.1.6.3) shows \( \varphi \) vanishes on \( A_1 \). \( \square \)

7 Mordell-Weil Theorem

Chevalley-Weil Theorem

Prop. (VII.1.7.1) (Local Chevalley-Weil Theorem). Let \( K \) be a number field, \( | \cdot | \) be a non-Archimedean valuation on \( \overline{K} \) and \( R \) be the valuation ring of \( K \). Let \( \varphi : Y \rightarrow X \) be a finite unramified morphism of \( K \)-varieties and \( E \) a bounded set in \( X(\overline{K}) \). Then there is an \( \alpha \neq 0 \in R \) that \( \alpha \in \delta_{\overline{K}_P/\overline{K}_Q} \) whenever \( P \in Y(\overline{K}) \) and \( Q = \varphi(P) \in E \).
Proof: An unramified map is locally of the form a closed embedding of a standard étale morphism, thus there are f.m. $U_i, V_i$ covering $X, Y$ respectively that $V_i \rightarrow U_i$ is closed embedding $V_i \rightarrow W_i$ of a standard étale morphism $W_i \rightarrow U_i$. Because $\varphi$ is finite hence proper, $\varphi^{-1}(E)$ is bounded by (VII.3.2.12), thus there is a decomposition of $\varphi^{-1}(E)$ into bounded sets $E_1 \subset V_i$. Then it suffices to prove for standard étale morphisms, because we can then multiply them.

The image of $E_1$ in $W_i$ is also bounded. Let $W_i \rightarrow U_i : \text{Spec}(A[t]/f) \rightarrow \text{Spec} A$, where

$$f = t^d + a_1 t^{d-1} + \ldots + a_d, \quad a_i \in A.$$ 

By boundedness, there is an $a \neq 0 \in R$ that

$$\max_{i=1, \ldots, d} \sup_{P \in E'} |a_i(\varphi(P))| \leq |a|^{-1}.$$

Then $\xi = a t_P$ is a root of the polynomial

$$g_Q(t) = t^d + a a_1(Q) t^{d-1} + \ldots + a_d(Q),$$

and it is easily verified that $|\xi| \leq 1$ thus $\xi \in R_P$. Now let $g_\xi$ be the minimal polynomial of $\xi$ over $\overline{K(Q)}$, then $g_\xi = g_\xi h, h \in \overline{K(Q)}[t]$]. By Gauss lemma in fact $g_\xi, h \in \overline{R_Q}[t]$.

Now the elements $1, \xi, \ldots, \xi^{d-1}$ form a basis of $\overline{K(P)}/\overline{K(Q)}$, so the discriminant $D_{g_\xi} \in \delta_{\overline{K(P)}/\overline{K(Q)}}$. By ?? and (IV.2.1.37), $|D_{g_\xi}| = |N_{\overline{K(P)}/\overline{K(Q)}}(g_\xi(\xi))| = |g_\xi(\xi)|^d$. But

$$g_\xi(\xi) h(\xi) = g_\xi'(\xi) = a^{d-1} f_P'(\xi) = a^{d-1} f_P'(t_P) = a^{d-1} f'(P).$$

Because $f'$ is unit, we have $|D_{g_\xi}|$ is bounded on $E'$. So there is an $\alpha \neq 0 \in R$ that $|D_{g_\xi}| \geq |\alpha|$, independent of $P$. Then we are done, as $D_{g_\xi} \in \delta_{\overline{K(P)}/\overline{K(Q)}}$.

Lemma (VII.1.7.2) (Global Chevalley-Weil Theorem). Let $M_K$ be a set of discrete valuations of a field $K$ that any element $\alpha$ has only f.m. nonzero valuations, $\varphi : Y \rightarrow X$ be an unramified finite $K$-morphism of complete $K$-varieties, and $(E^u)_{u \in M}$ is an $M$-bounded family in $X$ (VII.3.2.13), then for every $v \in M_K$ there is a nonzero $\alpha_v \in R_v$ that $\alpha_v \in \delta_{P/Q}^\circ$ whenever $u|v$ and $P \in Y(\overline{K})$ with $\varphi(P) = Q \in E^u$. Moreover, $\alpha_v = 1$ for a.e. $v \in M_K$.

Proof: The proof is exactly the same as that of (VII.1.7.1), noticing that $a, \alpha$ depend only on $v$ but not $u|v$, and also $a = \alpha = 1$ for a.e. $v$ (VII.3.2.13).

Lemma (VII.1.7.3) (Global Chevalley-Weil Theorem for Discrete Valuations). Let $M_K$ be a set of discrete valuations of a field $K$ that any element $\alpha$ has only f.m. nonzero valuations, $\varphi : Y \rightarrow X$ be an unramified finite $K$-morphism of complete $K$-varieties, then there are a finite set $S \subset M_K$ and for any $v \in S$ a nonzero element $\alpha_v \in m_v$ that for any $P \in Y(\overline{K}), Q = \varphi(P)$ and any place $w_0|v$ of $K(Q), K(P)/K(Q)$ is unramified outside $S$ and if $v \in S, \alpha_v \in \delta_{P/Q}^\circ$.

Proof: We may assume $\varphi$ is surjective because it is closed(finite is proper). Notice $X(\overline{K})$ is $M$-bounded in (VII.3.2.12), where $M$ is the set of valuations of $\overline{K}$ extending that of $M_K$. Then we can use global Chevalley-Weil theorem (VII.1.7.2) to find elements $\alpha_v$ for $v \in M_K$. Now notice

$$\delta_{P/Q}^\circ \overline{R_{w_0}} = \prod_{u|v_0} \delta_{P/Q}^u$$

by (IV.2.1.40), and the number of $w \in M_{K(P)}, w|w_0$ is bounded by $|K(P) : K(Q)|$, which is further bounded by $\deg(f)$ as in (VII.1.7.1). Hence we can take $\alpha_v = \alpha_v^{\deg(f)}$ to finish the proof.
Prop. (VII.1.7.4) (Chevalley-Weil Theorem for Number Fields). Let \( K \) be a number field and let \( \varphi : Y \to X \) be an unramified finite morphism of \( K \)-varieties. If \( X \) is complete, then there is an \( \alpha \neq 0 \in \mathcal{O}_K \) that for any \( P \in X(K) \) that \( Q = \varphi(P) \), the discriminant \( \delta_{P/Q} \) of \( \mathcal{O}_{K(P)} \) over \( \mathcal{O}_{K(Q)} \) contains \( \alpha \).

Proof: Use the above lemma (VII.1.7.3) to the set of all non-Archimedean valuations of \( K \). Notice because \( \delta_{P/Q} = \prod_{v \in S} (\delta_{P/Q}^v \cap \mathcal{O}_{K(Q)}) \), we can assume \( \alpha_v \in \mathcal{O}_{K(Q)} \), then take

\[
\alpha = \prod_{v \in S} \alpha_v.
\]

\[ \square \]

Prop. (VII.1.7.5) (Chevalley-Weil Theorem). Let \( K \) be a number field, \( \varphi : Y \to X \) be a finite unramified morphisms of \( K \)-varieties. If \( X \) is complete, then there is a finite extension \( L/K \) that \( P \in Y(L) \) for any \( P \in Y(K) \) that \( \varphi(P) \in X(K) \).

Proof: By Chevalley-Weil for number fields (VII.1.7.4), there is an \( \alpha \in \mathcal{O}_K \) that \( \alpha \in \delta_{K(P)/K(Q)} \) for any \( P \in Y(K) \) that \( \varphi(P) \in X(K) \), thus \( K(P)/K(Q) \) is unramified outside the primes dividing \( \alpha \). But then (IV.2.1.41) shows there are only f.m. possibilities of \( K(P) \). Thus we are done. \[ \square \]

Prop. (VII.1.7.6) (Local Chevalley-Weil Theorem for Abelian Varieties). Let \( A \) be an Abelian variety over \( K \) and \( v \) a discrete valuation on \( K \), \( m \) a nonzero integer. Suppose \( A \) has good reduction in \( v \) and the characteristic of the residue field doesn’t divide \( m \), then for any \( P \in A(K) \), the extension \( K(P)/K([m]P) \) is unramified at all places over \( v \).

Proof: By base change, we may assume \( Q = [m]P \) is \( K \)-rational. Let \( w \) be a place of \( K(P) \) that \( w/v \) and valuation ring \( R_w \), then the valuation criterion of properness shows \( P \) extends to a \( R_w \)-valued point of \( \overline{A} \). Now the theorem follows from (VII.1.3.4) and (V.5.4.13). \[ \square \]

Weak Modell-Weil

Lemma (VII.1.7.7) (Fundamental Lemma). Let \( L = K([n]^{-1}X(K)) \), i.e. the composite of all fields of \( K \) obtained by adjoining \( n^{-1}x, x \in X(K) \), then \( L \) is a finite field extension of \( K \).

Proof: This follows from Chevalley-Weil theorem (VII.1.7.5), as \( [n] \) is finite étale (VII.1.3.4) (\( K \) is a number field). \[ \square \]

Lemma (VII.1.7.8) ((Fake)Weak Mordell-Weil Theorem). Let \( K \) be a number field and \( X \) be an Abelian variety over \( K \), then (passing to a finite field extension) \( X(K)/nX(K) \) is finite for any \( n \geq 1 \).

Proof: By (VII.1.3.5), we can take a field extension \( K \) that \( X(K) \) contains the group of geometrical points of order \( n \) in \( X \) and \( K \) contains the \( n \)-th roots of unity.

Let \( L \) as in (VII.1.7.7), and let \( G = G(L/K) \). We consider a map

\[
f : X(K) \to \text{Hom}(G, X_n) : f(x)(s) = s(n^{-1}x) - n^{-1}x.
\]

(\( n^{-1}x \) exists by (VII.1.3.4)). Notice this is independent of the choice of \( n^{-1}x \) because \( X_n \in X(K) \). In particular, \( f(x_1 + x_2) = f(x_1) + f(x_2) \), and

\[
f(x) = 0 \iff n^{-1}x \in X(K) \iff x \in nX(K)
\]
thus $f$ induces an embedding

$$X(L)/nX(K) \hookrightarrow \text{Hom}(G, X_n).$$

The latter one is finite by (VII.1.7.7) and this gives the required. \hfill $\square$

**Prop. (VII.1.7.9) (Mordell-Weil Theorem).** The group $X(K)$ of rational points of an Abelian variety $X$ is f.g.

**Proof:** First notice it suffices to prove this theorem for a finite field extension, so we will chose a field extension $K$ that (VII.1.7.8) and (VII.3.3.4) hold.

Denote $\Gamma = X(K)$. Consider the bilinear product in $\Gamma$ defined in (VII.3.3.4). Let $n > 1$, choose a generator $x_i$ for $X(K)/nX(K)$ (VII.1.7.8). Now there is a constant $C$ that whenever $(x, x) \geq C$, $$(x - x_i, x - x_i) < 2(x, x), \forall i.$$ This is because, by Cauchy inequality, $(x - x_i, x - x_i)$ is similar to $(x, x)$ when $(x, x)$ is large.

Now let $M = \{x_1, \ldots, x_s\} \cup \{x \in \Gamma | (x, x) < C\}$. Then $M$ is finite by (VII.3.3.4). We prove $M$ generates $\Gamma$: Consider the infimum $C_0$ of $(x, x)$ that $x$ is not generated by $M$, then there is a $x$ that $C_0 \leq (x, x) < 2C_0$. Obviously $C_0 \geq C$. Let $x = x_i + ny$ for some $x_i \in M, y \in \Gamma$. Consider

$$(y, y) = \frac{1}{n^2}(x - x_i, x - x_i) < \frac{2}{n^2}(x, x) \leq \frac{1}{2}(x, x) < C_0$$

Thus $y \in M$ thus $x \in M$, contradiction. \hfill $\square$

8 Néron Model

**Def. (VII.1.8.1) (Good Reduction).** Let $K$ be a field and $R_v$ be a DVR of $K$. Let $A$ be an Abelian variety over $K$, $A$ is said to have **good reduction** in $v$ if there is a proper smooth scheme $\overline{A}$ over $R_v$ that the generic fiber $\overline{A}_K \cong A$.

**Def. (VII.1.8.2) (Néron Model).** Let $A$ be an Abelian variety that has good reduction in $v$, then for any smooth scheme $Y$ over $R_v$, let $Y_K \to A$ be a morphism over $K$, then it extends to a rational morphism $Y \to \overline{A}$, and by (V.4.5.9), this is defined on a set of codimension $\geq 2$. But then (VII.1.1.15) shows it extends to a morphism over $R_v$. Hence the reduction of $A$ is unique, called the **Néron model** of $A$.

In the same token, the group structure of $A$ can be extended to $\overline{A}$.

**Lemma (VII.1.8.3) (Existence).** Let $X$ be an Abelian variety over $K$ and $A$ be any ring of integers in the field $K$, then there exists an open subset $Y = \text{Spec} A_S \subset \text{Spec} A$, and a scheme $\overline{X}$ projective over $Y$, and morphisms $\overline{m} : \overline{X} \times \overline{X} \to \overline{X}$ over $Y$ and $\overline{e} : Y \to \overline{X}$ that:

- The fiber of $\overline{X}$ over a generic point of $Y$ with the morphisms $\overline{m}, \overline{e}$ are Abelian varieties isomorphic to $X$.
- $\overline{X}$ is a group scheme over $Y$, and its fiber over any closed point of $Y$ with the morphisms $\overline{m}, \overline{e}$ are Abelian varieties.
- The mapping in item 1 induces an isomorphism of groups $\overline{X}(Y) \cong X(K)$.

**Proof:** 1: Consider for any qc scheme over a field $K$, it is glued together from f.m. affine schemes, and these glueing involves f.m. polynomials and rational transition functions, and the coefficients of them are contained in a subring $A_S \subset K$ of f.t. over $A$. Thus this variety can be seen as a variety
over $A_S$ with the same equations, satisfying item1. The situation is similar for morphisms between qc schemes, in particular $\tilde{m}$ and $\tilde{e}$, thus constructing $\tilde{X}$.

2: Cf.[Mumford, P265].

3: Cf.[Mumford, P265].

\[ \square \]

\textbf{Cor. (VII.1.8.4).} Let $x \in \tilde{X}(Y)$, consider $x$ as a closed subscheme of $\tilde{X}$ and denote $n^{-1}(x)$ the closed subscheme in $\tilde{X}$ the inverse image of $x$ under the morphism $[n]_Y$. Then the natural projection $n^{-1}(x) \to Y$ is étale over all points $y \in Y$ that $\text{char}(y) \nmid n$.

\textit{Proof:} [Mumford, P265].

\[ \square \]

\section{9 Abelian Schemes}

\textbf{Def. (VII.1.9.1) (Abelian Schemes).} Let $S$ be a scheme, then a scheme $A$ over $S$ is called an \textbf{Abelian scheme} over $S$ if $A$ is proper and smooth group scheme over $S$ and that all the fibers $A_s$ are Abelian varieties over the resp. residue field $k(s)$.
VII.2 Elliptic Curves

Basic References are [Sil16] and [Sil99].

Materials that need to be added in the Algebraic Geometry Part

Prop. (VII.2.0.1). Prop 4.2, 4.3 in [Silverman1] needs clarification.

Def. (VII.2.0.2) (Elliptic Curves). An elliptic curve is a complete regular curve of genus 1, together with a rational pt.

Prop. (VII.2.0.3). If X is an Abelian variety of dimension 1, then X is an elliptic curve. The converse is also true, by (VII.2.0.4).

Proof: By (VII.1.1.4) (VII.1.1.3), the tangent space $T_{X/k}$ is trivial of rank 1, thus it is a curve of genus 1 by (IX.12.1.21). It is also regular, by (V.5.3.11). □

Prop. (VII.2.0.4) (Explicit Embedding of Elliptic Curves). If $E$ is an elliptic curve, consider a rational point $P \in E(k)$. Now Riemann-Roch tells use $l(nP) = \deg(nP) = n$ for $n \geq 1$. Now $L(kP) = k$ by Riemann Roch (IX.12.1.30). So we choose a basis $1, x$ for $L(2P)$, and extend it to a basis $1, x, y$ of $L(3P)$. Since $L(6P) = 6$, there is a linear relation between the seven elements $1, x, x^2, xy, y^2, x^3$. And $y^2, x^3$ must occur by observing the pole order at $P$. Thus by rescaling, we can write the relation as

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.$$

So $x, y$ defines a rational map of $E$ to $\mathbb{P}^2 : a \mapsto (x(a), y(a), 1)$. This map extends to an embedding of $E$ into $\mathbb{P}^2$, by (IX.12.1.12).

To define an Abelian structure on $E$, first notice that

$$E(k) \rightarrow \text{Cl}^0(E) : Q \mapsto [Q] - [P]$$

is an isomorphism. Using Riemann-Roch, it is injective because $l(Q) = 1$ for $Q \in E(k)$, and for any divisor $A$ of degree 0, $L(A + P) > 0$, so there exists an effective divisor that is equivalent to $A + P$, but this must be a rational point $Q \in E(k)$. Thus we can endow $E$ with a group structure inherited from $\text{Cl}^0(E)$. This makes $E$ an Abelian variety, by (VII.2.0.5).

Prop. (VII.2.0.5). The group actions defined in (VII.2.0.4) are all morphisms.

Proof: Cf. [Silverman1 Chap 3.2]. □

Prop. (VII.2.0.6). By (X.2.7.8), the Weierstrass functions (X.2.7.7) $\wp, \wp'$ are just the rational functions in (VII.2.0.4), so the map

$$z \mapsto (\wp'(z), \wp(z))$$

is biholomorphic from $\mathbb{C}/\Lambda$ to the elliptic curve defined by $y^2 = 4x^3 - g_2 x - g_3$. And it also preserves the group structure.

Proof: Finally, to show that the homomorphism preserves group structure, notice that if $z_1, z_2, z_3$ maps to three points that is colinear, then they satisfy a equation

$$f(z) = a\wp(z) + b\wp'(z) + c.$$

If $b \neq 0$, then this is a meromorphic function with three poles, thus three zeros, which is exactly $z_1, z_2, z_3$, so by (X.2.7.4), $z_1 + z_2 + z_3 \equiv 0 \mod \Lambda$. If $b = 0$, then $z_3 = 0$, which corresponds to the point $(1, 0, 0) \in \mathbb{P}^2$, then the same argument shows $z_1 + z_2 + 0 \equiv 0 \mod \Lambda$. □
Prop. (VII.2.4.1). For an Elliptic curve $E$ over $\overline{\mathbb{Q}}$, there is a bijection of $\mathbb{Q}$-isomorphism classes of elliptic curves which become isomorphic to $E$ over $\overline{\mathbb{Q}}$ with $H^1(Gal(\overline{\mathbb{Q}}/\mathbb{Q}), \text{Aut } E)$, where $E$ is seen as a presheaf so $\text{Aut } E$ is also a presheaf.

Proof: This is an immediate corollary of (V.1.5.2). □
VII.3 Diophantine Geometry

Main references are [Serre Galois Cohomology Chap2.4.5], [Heights in Diophantine Geometry, Bombieri].

1 Artin’s Conjecture

Def. (VII.3.1.1). A field $K$ is called $C_1$ or for any homogenous polynomial $F(X_1,\ldots, X_n)$ of degree $d$ with coefficient in $K$ that $d^k < n$ has a non-zero solution in $K^n$.

$C_0$ fields are just alg.closed fields, $C_1$ fields are also called quasi-algebraically closed.

Prop. (VII.3.1.2). Any field $L$ algebraic over a quasi-alg.closed field is quasi-alg.closed.

Proof: For a homogenous polynomial $F(X_1,\ldots, X_n)$, its coefficient lies in a finite extension of $K$ contained in $L$, so we may assume $L/K$ is finite. Then choose a basis $\{e_1,\ldots, e_m\}$ of $L$ over $K$, then consider the function $f(x_{11},\ldots, x_{1m},\ldots, x_{n1},\ldots, x_{nm}) = N_{L/K}(F(x_{11}e_1 + \ldots + x_{1m}e_m,\ldots, x_{n1}e_1 + \ldots + x_{nm}e_m)$, which is a homogenous polynomial of degree $mn$ with coefficient in $K$, because it has values all in $K$. So it has a nonzero solution in $K^{nm}$ by (V.8.2.4), Krull’s height theorem and $k$ is alg.closed.

Prop. (VII.3.1.3) (Chevalley-Warning). Any finite field $\mathbb{F}_q$ is quasi-algebraically closed. In fact, for any system of polynomials $f_i$, if $\sum_{i=1}^r \deg f_i < d$, then the number of solutions to this equation on $\mathbb{F}_q$ is divisible by $p$.

Proof: The number of solutions to this system is equivalent to

$$\sum_{x \in \mathbb{F}_q^n} \prod (1 - f_i^{q-1}(x))$$

modulo $p$.

But notice that if $i < q - 1$, then $\sum_{x \in \mathbb{F}_q^n} x^i = 0$ in $\mathbb{F}_q$ by (XIV.1.2.7), but as the degree of the highest term of $\sum_{x \in \mathbb{F}_q^n} \prod (1 - f_i^{q-1}(x))$ modulo $p$ is smaller than $n(q-1)$, some $x_i$ has power smaller than $q-1$, thus when summed over $\mathbb{F}_q$, it vanishes.

Prop. (VII.3.1.4) (Tsen). Algebraic function fields of dimension 1 over an alg.closed field $K$ is quasi-alg.closed.

Proof: By (VII.3.1.2), it suffice to consider the case $K = k(t)$ purely transcendental. For a polynomial $F$ with coefficient in $k(t)$, we can assume it has coefficient in $k[t]$, then let $\delta$ be their maximal degree. If substituted with $X_i = \sum_{j=0}^N a_{ij}jt^j$, the function becomes a system of $\delta + dN + 1$ homogeneous equation with $n(N+1)$ unknowns $a_{ij}$, since $d < n$, $\delta + dN + 1 < n(N+1)$ for $N$ large. In this case,

Prop. (VII.3.1.5). If $K$ is quasi-alg.closed, then $H^2(G(K_s/K), K_s^*) = 0$.

Proof: Cf.[Etale Cohomology Fulei 5.7.15].

Cor. (VII.3.1.6). By this and (VII.3.1.2), the condition of (IV.3.2.13) are satisfied. So if $K$ is quasi-alg.closed, then $cd(G(K_s/K)) \leq 1$ and $H^i(G(K_s/K), K_s^*) = 0$ for $i \geq 1$.

Prop. (VII.3.1.7) (Ax-Kochen). For any $d$, there is a $N_d$ that if $p > N_d$, any homogenous polynomial $f(X_1,\ldots, X_n)$ of degree $d$ with coefficient in $\mathbb{Q}_p$ that $d^k < n$ has a non-zero solution in $\mathbb{Q}_p^n$.

Proof: The proof uses Model theory.
2 Heights

Def. (VII.3.2.1) (Equivalent Height function). We call two height function equivalent iff they differ by a bounded function.

Prop. (VII.3.2.2). There is a way of constructing the Weil heights as a special case of the global heights, which are sum of local heights, Cf.[Diophantine Geometry, Chap2].

Heights of Projective Spaces

Def. (VII.3.2.3) (Height of Projective Spaces). For a number field \( K \), let \( (x_0, \ldots, x_n) \in K^{n+1} \) be a point on \( \mathbb{P}^n(K) \), then the number

\[
h(x_0, \ldots, x_n) = \frac{1}{[K : \mathbb{Q}]} \sum_v \max \{ \log |x_i|_v \}
\]

has the following properties:
- \( h(\lambda x_0, \ldots, \lambda x_n) = h(x_0, \ldots, x_n) \).
- \( h(x) \geq 0 \).
- \( h(x) \) is independent of the base change of fields \( K \subset K' \).

Thus \( h(x) \) is well-defined and called the **height** on \( \mathbb{P}^n \). We also use the **absolute heights** \( H(x) = e^{h(x)} \).

Proof: 1 is a consequence of the product formula (IV.2.4.15). 2 follows from 1 because we can divide a constant to make a coordinate unit in \( K^* \), thus clearly \( h(x) \geq 0 \). The third assertion is also clear.

Def. (VII.3.2.4) (Height of Algebraic Numbers). For an algebraic number \( \alpha \), the height is in particular:

\[
h(\alpha) = \sum_v \max \{ 0, \log |\alpha|_v \}.
\]

Lemma (VII.3.2.5). For \( \alpha \in \overline{\mathbb{Q}} \neq 0 \) and \( \lambda \in \overline{\mathbb{Q}} \), we have \( h(\alpha^\lambda) = |\lambda|h(\alpha) \).

Proof: For \( \lambda > 0 \), this is easy. So it suffices to consider \( \lambda = -1 \). Notice

\[
\log |\alpha|_v = \max \{ 0, \log |\alpha|_v \} - \max \{ 0, \log |1/\alpha|_v \},
\]

summing over all places \( v \) and use the product formula, we get the desired results.

Prop. (VII.3.2.6) (Special Case of Northcott’s Theorem). Let \( C > 0, d > 0 \), then \( \{ x \in \mathbb{P}^n(\overline{K}) | h(x) \leq C, [K(x) : \mathbb{Q}] \leq d \} \) is finite.

Proof: We first reduce to the case \( K(x) = \mathbb{Q} \): for a point \( x \) with \( [K(x) : \mathbb{Q}] \leq d \), consider the point \( (X_0, \ldots, X_m) \) in the projective space of forms of degree \( [K(x) : K] \) in \( n + 1 \) variables corresponding to \( \text{Nm}(\sum x_i T_i) \). Notice this mapping for points with the same degree is fibered in finite sets, because the norm form splits into linear factors. Next we prove the height of \( (X_0, \ldots, X_m) \) is bounded by a function of the height of \( x \), which is nearly obvious.

Now we have f.m. \( [K(x) : \mathbb{Q}] \), thus it suffices to prove for \( \mathbb{P}^m(\mathbb{Q}) \). In this case, we normalize a point to a unique point with integral coordinates with gcd1, then \( h(x) = \max_i \log |x_i| \), thus clearly only f.m. points has bounded heights.
Prop. (VII.3.2.7) (Change of Coordinates). Let \( h_1, h_2 \) be heights of \( \mathbb{P}(K) \) defined w.r.t. two coordinate systems, then \( h_1 \sim h_2 \). Thus we can consider heights in any particular coordinates that is convenient.

**Proof:** The proof is nearly obvious.

Heights associated with an invertible sheaf

Prop. (VII.3.2.8). Let \( X \) be a complete variety over \( K \) and \( \varphi : X \to \mathbb{P}^k, \psi : X \to \mathbb{P}^l \) are two \( K \)-morphisms. If \( \varphi^* (\mathcal{O}_{\mathbb{P}^k}(1)) \cong \psi^* (\mathcal{O}_{\mathbb{P}^l}(1)) \), then the induced height function \( h_\varphi - h_\psi \) is bounded on \( X(K) \).

**Proof:** Let \( \varphi^* (\mathcal{O}_{\mathbb{P}^k}(1)) \cong \psi^* (\mathcal{O}_{\mathbb{P}^l}(1)) \cong L \), then \( L \) is very ample. Consider a basis \( \{s_0, \ldots, s_n\} \) of \( \Gamma(X, L) \), then it induces a closed embedding \( \chi : X \to \mathbb{P}^n : x \mapsto [s_0(x), \ldots, s_n(x)] \) and \( \chi^* (\mathcal{O}_{\mathbb{P}^n}(1)) \cong L \). Then it suffices by symmetry to prove \( h_\varphi \sim h_\chi \).

We may assume \( \varphi(X) \) is not contained in any proper linear subspace of \( X \), so we can choose a basis \( T_0, \ldots, T_k \) of \( \varphi^* (\Gamma(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))) \), and let \( s_0, \ldots, s_k = T_0, \ldots, T_k \). Let \( x \in X(K) \), let \( (x_0, \ldots, x_n) \) be the coordinate of \( \chi(x) \) and \( (x_0, \ldots, x_k) \) the coordinates of \( \varphi(x) \). Then by the formula (VII.3.2.3) clearly \( h_\varphi \leq h_\chi \).

For the converse, let \( I \) be the homogenous ideal corresponding to \( X \subset \mathbb{P}^n \), then \( X \) has coordinate ring \( R = K[T_0, \ldots, T_n]/I \). The \( T_0, \ldots, T_k \) generates a radical ideal equal to \( (T_0, \ldots, T_n) \), because they have no common zero on \( X \). Now there is an integer \( q \) and homogenous ideals \( F_{ij} \) that

\[
T_{k+i}^q - \sum_{j=0}^{k} F_{ij}(T_0, \ldots, T_n)T_j \in I
\]

so

\[
q \log |x_{k+i}|_v \leq (q - 1) \max_{j \leq n} \log |x_j|_v + \max_{j \leq k} \log |x_j|_v + C_v
\]

where \( C_v = 0 \) unless \( v \) is Archimedean. Hence

\[
\max_{j \leq n} \log |x_j|_v \leq \max_{j \leq k} \log |x_j|_v + C_v \Rightarrow h_\chi \leq h_\varphi + C.
\]

Prop. (VII.3.2.9) (Weil Height). Let \( X \) be a projective variety over a number field \( K \), then for each element \( L \in \text{Pic}(X) \), we can assign a unique Weil height \( h_L \), determined up to equivalence, that

- \( h_{L_1 \otimes L_2} \cong h_{L_1} + h_{L_2} \).
- If \( X = \mathbb{P}^n \), then \( h_{\mathcal{O}(1)} \) is defined as in (VII.3.2.3).
- For any \( K \)-morphism \( \varphi : X \to Y \) and \( L \in \text{Pic}(Y) \), \( h_{\varphi^*(L)} \sim h_L \circ \varphi \).

**Proof:** The uniqueness follows from the fact that \( \text{Pic}(X) \) is generated by very ample line bundles because \( X \) is projective. For the existence, we may take item3 as definition, extend it to all \( \text{Pic}(X) \), and it is essential to verify item1.

Let \( L_1 = \varphi^*(\mathcal{O}(1)), L_2 = \psi^*(\mathcal{O}(1)) \), where \( \varphi : X \to \mathbb{P}^k, \psi : X \to \mathbb{P}^l \), Denote \( \sigma : \mathbb{P}^k \times \mathbb{P}^l \to \mathbb{P}^{kl+k+l} \) the Segre embedding, then \( L_1 \otimes L_2 \cong \chi^*(\mathcal{O}_{\mathbb{P}^{kl+k+l}}(1)) \) where \( \chi : X \xrightarrow{(\varphi, \psi)} \mathbb{P}^k \times \mathbb{P}^l \xrightarrow{\sigma} \mathbb{P}^{kl+k+l} \). And we check \( h_\chi \sim h_\varphi + h_\psi \).
Prop. (VII.3.2.10) (Northcott’s Theorem). Let $X$ be a projective variety over a number field $K$ and let $h_c$ be a height function associated to an ample class $c \in \text{Pic}(X)$, then the set
\[ \{ P \in X(\overline{K}) | h_c(P) \leq C, |K(P) : K| \leq d \} \]
is finite for any constant $C, d$.

Proof: There is a $m > 0$ that $mc$ is very ample. Because $mh_c$ is the height function associated to $mc$, we can assume that $c$ is very ample, thus proving for $X = \mathbb{P}^n$ and $c = O(1)$. This follows from (VII.3.2.6).

Bounded Sets

Def. (VII.3.2.11) (Bounded Sets). Let $K$ be a field and $| \cdot |$ a valuation on its alg.closure $\overline{K}$, $X$ be a variety over $K$. Then:
- If $X$ is an affine variety, then a subset $E \subset X(\overline{K})$ is called bounded if for any $f \in K[X]$, $|f|$ is bounded on $E$.
- If $X$ is arbitrary, then a subset $E \subset X(\overline{K})$ is called bounded if there is a finite open affine covering $U_i$ of $X$ and sets $E_i \subset U_i(\overline{K})$ that $E_i$ is bounded in $U_i$ and $E = \cup E_i$.

Prop. (VII.3.2.12) (Properties of Bounded Sets).
- For a bounded set $E$ in $X$ and any finite open affine covering $U_i$ of $X$, there is a division of $E$:
  \[ E = \cup E_i, \quad E_i \subset U_i(\overline{K}) \]
  and $E_i$ being bounded in $U_i$.
- If $E$ is bounded in $X$ and $Y$ is a closed subscheme of $X$, then $E \cap Y(\overline{K})$ is bounded in $Y$.
- The image of a bounded set under a morphism is also bounded.
- $\mathbb{P}^n(\overline{K})$ is bounded in $\mathbb{P}^n$.
- The inverse image of bounded set under a proper morphism is bounded. In particular, if $X$ is a complete variety over a field $K$, then $X(\overline{K})$ is bounded in $X$.

Proof: 1: Because $X$ is separated, it suffices to prove for $X$ affine. And we can also take a refinement of the covering, thus assuming $U_i = X_{h_i}$. Suppose $\sum g_i h_i = 1$, and let $E_i = \{ P \in E | |h_i(P)| = \max_k |h_k(P)| \}$, then $E_i \subset U_i(\overline{K})$. To show $E_i$ is bounded in $U_i$, it suffices to show $|1/h_i|$ is bounded in $E_i$. But this is clear, because $|g_i|$ is bounded on $E$.
  2: Use local coordinates.
  4: Use the affine open covering $X_i = \{ x_i \neq 0 \}$, and let $E_i = \{|x_i| = \max_{j=0,...,n} |x_j| \}$, then clearly $E_i$ is bounded in $X_i$.
  5: Cf.[Diophantine Geometry, P55], may use Chow’s lemma?.

Prop. (VII.3.2.13) (M-Bounded). We can define the notion of $M$-boundedness similar to that of (VII.3.2.11), where $M$ is a set of places on $K$ that for any $\alpha \neq 0 \in K$, only f.m. of $M$ have nontrivial valuation. Then $(E^u)$ is said to be $M$-bounded in an affine variety $X$ if for any $f \in K[X],$
\[ C_v(f) = \sup_{u \in M, u | v} \sup_{P \in E^u} |f(P)|_u \]
is finite for any $v \in M_K$ and $C_v(f) > 1$ for only f.m. $v$.

Then similar properties as in (VII.3.2.12) hold for $M$-bounded sets.
Def. (VII.3.2.14). A real function \( f \) on \( X(\overline{K}) \) is called **locally bounded** if \( f(E) \) is bounded for every bounded set \( E \) in \( X \).

Metricized Line Bundles

3 Néron-Tate Heights

Prop. (VII.3.3.1). Let \( h : \Gamma \to \mathbb{R} \) be a function on an Abelian group that satisfies the condition

\[
h(\sum_{i=1}^{3} x_i) - \sum_{1 \leq i < j \leq 3} h(x_i + x_j) + \sum_{i=1}^{3} h(x_i) \sim 0,
\]

Then there exists a unique symmetric bilinear pairing \( b \) on \( \Gamma \) and \( l \) a homomorphism \( \Gamma \to \mathbb{R} \), that

\[
h(x) \sim \hat{h}(x) = \frac{1}{2} h(x, x) + l(x).
\]

**Proof:** Set \( \beta(x_1, x_2) = h(x_1 + x_2) - h(x_1) - h(x_2) \), then

\[
\beta(x_1 + x_2, x_3) - \beta(x_1, x_3) - \beta(x_2, x_3) \sim 0,
\]

and similarly for the second term.

Set now \( b(x_1, x_2) = \lim_{n \to \infty} 4^{-n} \beta(2^n x_1, 2^n x_2) \). This limit exists, as

\[
4^{-(n+1)} \beta(2^{n+1} x_1, 2^{n+1} x_2) = 4^{-n} \beta(2^n x_1, 2^n x_2) + 4^{-(n+1)} \theta_n.
\]

where \( \theta_n \) is bounded. And the linearity also follows easily from the property of \( \beta(x_1, x_2) \). Also \( b(x_1, x_2) \sim \beta(x_1, x_2) \).

Let \( \lambda(x) = h(x) - \frac{1}{2} b(x, x) \), then

\[
\lambda(x_1 + x_2) - \lambda(x_1) - \lambda(x_2) = \beta(x_1, x_2) - b(x_1, x_2) \sim 0
\]

Thus similarly we can define \( l(x) = \lim_{n \to \infty} 2^{-n} \lambda(2^n x) \), then we are finished. The uniqueness of \( b, l \) is implicit in the proof. \( \square \)

Prop. (VII.3.3.2) (Néron-Tate Heights). Let \( X \) be an Abelian variety over a number field \( K \), \( L \in \text{Pic}X \). Then the Weil height (VII.3.2.9) \( h_L \) on \( X(\overline{K}) \) satisfies the condition of (VII.3.3.1), thus there are uniquely defined bilinear form \( b_L \), homomorphism \( l_L \), and \( \hat{h}_L(x) = \frac{1}{2} b_L(x, x) + l_L(x) \) is called the **Néron-Tate height** of \( L \).

**Proof:** We want to use (VII.3.3.1). Apply the theorem of the cube to the projections \( p_i : X \times X \times X \to X \), then take the height, we will get the desired relation

\[
h_L(\sum_{i=1}^{3} x_i) - \sum_{1 \leq i < j \leq 3} h_L(x_i + x_j) + \sum_{i=1}^{3} h_L(x_i) \sim 0,
\]

\( \square \)

Cor. (VII.3.3). The Néron-Tate height of an Abelian variety satisfies:

- The map \( \hat{h} : \text{Pic}(X) \to \mathbb{R}^{X(\overline{K})} : c \mapsto \hat{h}_c \) is a homomorphism.
- If \( \varphi : A \to B \) is a homomorphisms of Abelian varieties, then \( \hat{h}_{\varphi^*(c)} = \hat{h}_c \circ \varphi \).
Let \( c \in \text{Pic}(X) \) be even. If \( c \) is base-free or ample, then \( \hat{h}_c \geq 0 \).

**Proof:** The first two follow from the uniqueness of \( \text{Néron Tate height} \), for the third, notice that the fact \( c \) is even implies \( \hat{h}_c = \frac{1}{b}(x, x) \). The ample case reduces to the base-free case, because there is a multiple \( m \) that is very ample, and then \( \hat{h}_{mc} = m\hat{h}_c \). Now for the base free case, \( c \) is the pullback of \( O(1) \) for some \( \mathbb{P}^1 \), thus it is non-negative (VII.3.2.3), and it is equivalent to \( \hat{h}_c \), thus \( \hat{h}_c \) must be non-negative, because it is homogenous of degree 2.

**Lemma (VII.3.3.4).** For an Abelian \( X \), there is a symmetric bilinear scalar product \( X(K) \times X(K) \to R \) that \( \langle x, x \rangle \geq 0 \), and \( \{ x | \langle x, x \rangle < C \} \) is finite for all \( C > 0 \).

**Proof:** Take a symmetric very ample line bundle \( L \), because \( X \) is projective (VII.1.1.7), and set \( (x, y) = b_L(x, y) \) as in (VII.3.2), then \( \langle x, x \rangle \geq 0 \), because otherwise \( \hat{h}_L(nx) = \frac{n^2}{2}(x, x) \to -\infty \), contradicting the fact \( h_L \sim \hat{h}_L \) and \( h_L \) can be non-negative.

The last assertion follows from Northcott’s theorem (VII.3.2.10).

**Prop. (VII.3.3.5) (Tate’s Limiting argument).**

**Proof:** Cf. [Diophantine Geometry, P285].

**Cor. (VII.3.3.6).** Tate’s limiting argument gives another way of constructing \( \text{Néron-Tate heights} \). More generally, for any projective variety over a number field \( K \) and \( \varphi : X \to X \) a morphism with a line bundle \( c \) and \( k, l \in \mathbb{Z}, |k| > |l| \) that

\[
    l\varphi^*(c) = kc,
\]

then there is a unique function \( \hat{h}_{\varphi} \) in the equivalent class of \( h_c \) that

\[
    \hat{h}_{\varphi}(\varphi(x)) = kh_{\varphi}(x).
\]

In particular, the \( \text{Néron-Tate heights} \) on an Abelian variety for an even or odd line bundle is obtained by taking \( \varphi = [m] \) for some \( m \geq 2 \).

**Proof:** Assume \( l \neq 0 \), consider the subgroup \( N = \{ \lambda^r | r \in \mathbb{N} \} \), where \( \lambda = k/l \), and \( N \) acts on \( X(K) \) by \( \lambda^r \cdot x = \varphi^r(x) \), then the Wei height function for \( c \) is quasi-homogenous of degree 1, so Tate’s limiting argument shows

\[
    \hat{h}_{\varphi}(c) = \lim_{r \to \infty} \lambda^{-r}h_c(\varphi^r(x))
\]

satisfies the requirement. And similarly, it is non-negative if \( c \) is ample or base-free.

**Prop. (VII.3.3.7).** If \( K \) is a number field and \( c \) is ample, then \( \hat{h}_{\varphi}(x) = 0 \) iff \( x \) is preperiodic, i.e. the sequence \( \{ x, \varphi(x), \varphi^2(x), \ldots \} \) is finite.

**Proof:** Assume \( l \neq 0 \). If \( x \) is preperiodic, then clearly \( \hat{h}_{\varphi}(c) = \lim_{r \to \infty} \lambda^{-r}h_c(\varphi^r(x)) = 0 \). Conversely, if \( \hat{h}_{\varphi}(c) = 0 \), then \( \hat{h}_{\varphi}(\varphi^r(x)) = 0 \) for any \( r \), then \( |h_c(\varphi^r(x))| = |\hat{h}_{\varphi}(\varphi(x))| + C(\varphi) = C(\varphi) \) is bounded and also \( \varphi^r(x) \in X(K(x)) \), thus of bounded degree, so by Northcott’s theorem (VII.3.2.10), there are f.m. points.

**Cor. (VII.3.3.8) (Kronecker).** The height (VII.3.2.4) of a \( \zeta \in \overline{Q} \) equals 0 iff it is a root of unity.

**Proof:** This is a special case of (VII.3.3.7), where \( \varphi : \mathbb{P}^1 \to \mathbb{P}^1 : x \mapsto x^n \), thus the preperiodic points are just 0, \( \infty \) and all the roots of unity.
Hilbert’s Irreducibility Theorem

Prop. (VII.3.3.9) (Runge’s Theorem).

Prop. (VII.3.3.10) (Hilbert’s Irreducibility Theorem). Let $C$ be a smooth irreducible projective curve over a number field $K$ and let $f : C \to \mathbb{P}^1$ be a surjective rational function on $C$ over $K$, then for all $n \in \mathbb{N}$ except for a set of natural density $0$, the divisor $f^*[n]$ is a prime divisor over $K$.

Proof: Cf.[Diophantine Geometry, P319].

4 Roth’s Theorem

Thm. (VII.3.4.1) (Roth). Let $K$ be a number field with a set $S$ of f.m. places and for each $v \in S$ a $K$-algebraic number $\alpha_v \in K_v$. Then for $k > 2$, there are only f.m. $\beta \in K$ that

$$\prod_{v \in S} \min(1, |\beta - \alpha_v|_v) \leq H(\beta)^{-k}.$$

Warning’s Problem

Cf.[Heights in Diophantine Geometry, P153].

5 abc-Conjecture

Def. (VII.3.5.1) (Radical). The radical of an integer $N$ is the product of distinct primes dividing $N$.

Prop. (VII.3.5.2) (Strong abc-Conjecture). Let $\varepsilon > 0$, then there is a constant $C(\varepsilon)$ that for any coprime positive integers $a, b, c > 0$ that $a + b = c$, there is an inequality:

$$c \leq C(\varepsilon) \text{rad}(abc)^{1+\varepsilon}.$$ 

6 Mordell-Lang Conjecture

Conjecture (VII.3.6.1) (Mordell-Lang). Let $X$ be a closed geometrically integral subvariety of a semi-Abelian variety $A$ defined over a field $K$ of characteristic 0. Let $\Gamma$ be a f.g. subgroup of $A(K)$ and $\Gamma'$ be a subgroup of the divisible hull of $\Gamma$. If $X$ is not a translate of a semi-Abelian subvariety of $A$, then $X(K) \cap \Gamma'$ is not Zariski dense in $X$.

Cor. (VII.3.6.2) (Mordell). Let $K$ be a number field and $C$ be a curve of genus $g \geq 2$ defined over $K$, then $C(K)$ is finite.

Proof: 

Cor. (VII.3.6.3) (Manin-Mumford). Let $K$ be a number field and $C$ be a curve of genus $g \geq 2$ defined over $K$, and $J$ the Jacobian of $C$. Fix an embedding $C \hookrightarrow J$ defined over $K$, then the set $C(K) \cap J(K)_{tor}$ is finite.

Proof:
VII.4 Algebraic Groups and Number Theory

Main references are [Algebraic Groups and Number Theory].

Prop. (VII.4.0.1) (Real Approximation). If $G$ is an algebraic group over $\mathbb{Q}$ that each connected components of $G$ contains a rational point, then $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$.

Proof: Cf.[Mil17b]P54. □
VII.5 Weil 2 Proof

Basic References are [Conrad Seminar note in Princeton], [Seminars on Gross-Zagier over Function Fields, Lei Fu], [Seminars on Weil 2 Bhatt], [Weil conjectures Perverse Sheaves and $l$-adic Fourier Transform Kiehl/Weissauer].

1 $l$-adic Étale Cohomology

Def. (VII.5.1.1) (Notations). $p$ is a prime, $q = p^r$ is a $p$-power, $k = \mathbb{F}_q$ is a finite field, $X$ is a separated algebraic scheme over $k$, $X = X_0 \otimes_k \bar{k}$ is its base change.

Fix a mixed characteristic complete DVR $(\Lambda, \mathfrak{m})$ with residue field finite of char=$l \neq p$, and $K$ its quotient field. Let $A$ be a Noetherian ring that is $I$-adically complete Hausdorff, let $A_n = A/I^n$.

Artin-Rees Formalism

Def. (VII.5.1.2). The pre Artin-Rees category of $A$-modules has objects $M^\bullet = (M_n) \in \mathbb{Z}$ which are projective systems of $A$-modules with $M_n = 0$ for $n << 0$, and the morphisms in this category are the elements of the set

$$\text{Hom}_{A-R}(M^\bullet, N^\bullet) = \lim \text{Hom}(M^\bullet[d], N^\bullet)$$

An object $M^\bullet$ in the Artin-Rees category is called a null system if for some $\geq 0$ the map $M_{n+} \to M_n$ vanishes for all $n$.

Prop. (VII.5.1.3). The pre Artin-Rees category is an Abelian category, and the null systems form a Weak Serre subcategory. Then we define the Arin-Rees category as the quotient category.

Proof: □

Prop. (VII.5.1.4). If the kernel and cokernel of two systems are all null systems, then they induce isomorphism on inverse limit.

Proof: Cf.[Conrad L15, P5]. □

Def. (VII.5.1.5). An object $M^\bullet$ in the A-R category is called Artin-Rees $I$-adic if it is represented by a system $M_n$ that $M_n = 0$ for $n < 0$ and $M_n$ is finite over $A_n$, $M_{n+1} \otimes_{A_{n+1}} A_n \to M_n$ is an isomorphism for $n \geq 0$.

Prop. (VII.5.1.6). The full subcategory of Artin-Rees $I$-adic modules is an Abelian category, and it is equivalent to the category of finite $A$-modules by the stalk functor.

$l$-adic Sheaves

Def. (VII.5.1.7). The Artin-Rees category of $\Lambda$-sheaves on $X_0$, strict $m$-adic sheaves are defined as before. It is called constructible $m$-adic sheaf iff it is isomorphic to a system that $F_n$ are all constructible.

It is called lisse $m$-adic if it is isomorphic to a strict $m$-adic system that $F_n$ are all locally constant finite $\Lambda_n$-modules.

Prop. (VII.5.1.8). Constructibility of Artin-Rees $\Lambda$-Sheaves are étale local, and stratification local.
Proof: Cf.[Conrad L15, P10].

Prop. (VII.5.1.9) (Constructible and Lisse). Let $F$ be a constructible $m$-adic sheaf, then there is a stratification of $X$ that $F$ is locally constant finite on each stratum.

Proof: By the stalk criterion of locally constant finite (VI.2.1.47), a constructible extension of locally constant finite sheaves is also locally constant finite. So by the exact sequence $1 \rightarrow l^{n-1}F_n \rightarrow F_n \rightarrow F_{n-1} \rightarrow 1$, if we show there is a stratification that all $l^{n-1}F_n$ are locally constant finite, then by induction all $F_n$ are locally constant finite. But then $l^{n-1}F_n$ is a descending chain of quotients of $F_1$, thus the kernel is ascending thus stabilizes because $F_1$ is constructible (VI.2.1.48), so there are only f.m. such $l^nF_n$, so there is a common stratification.

Cor. (VII.5.1.10). Exactness of complexes of constructible sheaves can be checked at stalks.

Proof: Cf.[Conrad L16 P3].

Prop. (VII.5.1.11). Constructible $m$-adic sheaves are Noetherian: Ascending chain of subsheaves stabilizes.

Proof: Cf.[Conrad L16 P3].

Prop. (VII.5.1.12) (Direct Pushforward of $m$-adic Sheaves). For a constructible $m$-adic sheaf $F$ and a compatifiable morphism $X_0 \rightarrow S_0$, we can define $R^if_*$ and $R^if_!$ termwisely, and we have $R^if_*F$ is a constructible sheaf, hence also does $R^if_!$.

Proof: Cf.[Weil Conjecture and Étale sheaves, P128].

QuoSheaves

Def. (VII.5.1.13). For a finite extension $E/\mathbb{Q}_l$, the category of $E$-sheaves are the category of constructible $O_E$-sheaves with the homomorphism given by

$$\text{Hom}_E(F,G) = \text{Hom}_{O_E}(F,G) \otimes_{O_E} E.$$ 

We may write $F \otimes E$ for this object, but the tensor is fake.

The category of $\mathbb{Q}_l$-sheaves are the direct limit of categories of $E$-sheaves for $E/\mathbb{Q}_l$ finite.

Def. (VII.5.1.14) (Tate Twist Sheaf). The Tate twist sheaf $\mathbb{Q}_l(1)$ is defined to be the lisse $\mathbb{Q}_l$ sheaf of the limit system of locally constant finite sheaves $\mu_n$ of rank 1. It is invertible, thus we denote its dual by $\mathbb{Q}_l(-1)$. For any $\mathbb{Q}_l$-sheaf $F$, denote $F(1)$ to be the sheaf $F \otimes \mathbb{Q}_l(1)$.

Prop. (VII.5.1.15) (Lisse $\mathbb{Q}_l$-Sheaves and $\pi_1(X_0,\overline{x})$). Assume $X_0$ is connected, then for a geometric point $\overline{x}$ of $X_0$, the functor $\mathcal{F} \rightarrow \mathcal{F}_{\overline{x}}$ induces an equivalence between the category of lisse $\mathbb{Q}_l$-sheaves to the category of continuous f.d. representations of $\pi_1(X_0,\overline{x})$ over $\overline{\mathbb{Q}}_l$.

Proof: By (III.1.7.3) any representation of $\pi_1(X_0,\overline{x})$ is in fact an $O_E$-action for some $E/\mathbb{Q}_l$ finite. So by the equivalence (VI.2.2.6)(VI.2.1.45) and taking limit using (VII.5.1.6), we get the result about $O_E$-sheaves and representations of $\pi_1(X_0,\overline{x})$ over $\overline{\mathbb{Q}}_l$.

Cor. (VII.5.1.16). We can call a lisse $\mathbb{Q}_l$-sheaf irreducible/semisimple if its corresponding representation is. It is called geometrically irreducible/semisimple if $F = (F_0)_{\overline{x}}$ is irreducible, or equivalently, its corresponding representation is irreducible as a $\pi_1(X,\overline{x})$-representation.
2 Frobenius Morphisms

Def. (VII.5.2.1) (Frobenius). Let $q = p^r$, for a $k = \mathbb{F}_q$ scheme $X_0$ with base change $X$,
- The absolute Frobenius for $X_0$ or $X$ is the automorphism $\varphi_{r,X} = \varphi^r : X \to X$ that is $q$-th power on $\mathcal{O}_X$.
- $F_X = \text{id}_{X_0} \times_k \varphi_{r,k}^{-1}$ is called the geometric Frobenius.
- Let $U$ be a $X_0$-scheme, then the relative Frobenius $F_U \triangleright X_0 : U \to \varphi_{X_0}^{-1}(U)$ is defined by the universal property of the base change of $U$ by $F_X$.

Prop. (VII.5.2.2). $F_{rX} = \varphi_{r,X} \circ F_X : X \to X$.

Proof: Easy. □

Prop. (VII.5.2.3). $F_{U/X_0}$ is a universal homeomorphism. In particular, if $U \to X_0$ is étale, then it is an isomorphism.

Proof: Because $U \to X, X \times_{\varphi_{X}} U \to X$ are both étale, $F_{U/X_0}$ is étale. And from the fact both both $\varphi_{X_0}$ and $\varphi_{U_0}$ are universally bijective, we see $F_{U_0/X_0}$ is universally bijective. So it must be an isomorphism. □

Cor. (VII.5.2.4) (Frobenius action on Sheaves). For any étale sheaf $\mathcal{F}$ on $X$, we have an isomorphism $\mathcal{F} \cong (F_{rX})_* \mathcal{F}$ which is the inverse of the isomorphism

\[
(F_{rX})_*(\mathcal{F})(U) = \mathcal{F}(F_{rX}^{-1}(U)) \xrightarrow{\mathcal{F}_{U/X}} \mathcal{F}(U),
\]

and its adjoint $F_{rX}^* \mathcal{F} \to \mathcal{F}$ is denoted by $\text{Frob}_\mathcal{F}$.

Then $\text{Frob}_\mathcal{F}$ commutes with tensor product and it is an isomorphism.

Proof: The adjoints are isomorphism because $(F_{rX})_*, (F_{rX})^*$ induces equivalence of categories of étale site (V.1.3.11). □

Remark (VII.5.2.5). Notice this reverse in the definition of $\text{Frob}_\mathcal{F}$.

Prop. (VII.5.2.6) (Compatibility of $\text{Frob}_\mathcal{F}$ with Higher Direct Image). If $X \to S$ is a separated morphism between $\mathbb{F}_p$-schemes of f.t., so we have a Cartesian diagram about $F_{rX}$ and $F_{rS}$. Then the composition

\[
F_{rS}^i f_* \mathcal{F} \to R^i f_* F_{rX}^* \mathcal{F} \xrightarrow{R^i f_*(\text{Frob}_\mathcal{F})} R^i f_* \mathcal{F}
\]

is just $\text{Frob}_{R^i f_* \mathcal{F}}$.

Proof: Cf. [Conrad L18 P4]? □

Cor. (VII.5.2.7) (Compatibility of $\text{Frob}_\mathcal{F}$ with Proper Pushforward). If $X \to S$ is a separated morphism of f.t. between $k$-schemes and $\mathcal{F}$ is a torsion Abelian sheaf on $X_{\text{ét}}$, then the morphism

\[
F_{rS}^i f_1^* \mathcal{F} \to R^i f_1^* F_{rX}^* \mathcal{F} \xrightarrow{R^i f_1^*(\text{Frob}_\mathcal{F})} R^i f_1^* \mathcal{F}
\]

is just $\text{Frob}_{R^i f_1^* \mathcal{F}}$. □
Prop. (VII.5.2.8) (Frobenius Action on Compact Cohomology). As \( Fr_X \) is proper because it is finite, by (VI.2.5.5) it induces a map \( H^i_c(X, F) \to H^i_c(X, Fr_X^* F) \), which by composing with \( \text{Frob}_X \), gives us an endomorphism \( Fr^*_X : H^i_c(X, F) \to H^i_c(X, F) \), which is called the Frobenius action on \( H^i_c(X, F) \).

Similarly, there are isomorphisms \( F_X^* \pi^* G_0 \cong (\pi \circ F_X)^* G_0 = \pi^* G_0 \), \( \text{Fro}_G : \varphi^*_{r,X} G \cong G \), so we can define the action of \( F_X \) or \( \varphi_{r,X} \) on \( H^i_c(X, G) \):

Then the action of \( \varphi_{r,X} \) on \( H^i_c(X, F) \) is in fact identity because it induces an isomorphism on étale site (V.1.3.11). In particular, (VII.5.2.2) show that \( F^*_X \) agrees with the Frobenius action for \( H^i_c(X, F) \), so we can calculate with either one of them, and denoted by \( F^*_X \).

3 Weil Sheaf

Fundamental Groups

Def. (VII.5.3.1) (Weil Group). For a

Weil Sheaves

Def. (VII.5.3.2) (Weil Sheaf). By Galois descent, the pullback induces an equivalence of categories between the category of constructible \( \mathbb{Q}_l \)-sheaves on \( X_0 \) to the category of constructible \( \mathbb{Q}_l \)-sheaves on \( X \) with a specified \( G(X/X_0) = G(\overline{K}/k) \cong \mathbb{Z} \)-actions. In practice, sometimes it is hard to verify the action of \( \mathbb{Z} \in \mathbb{Z} \) is continuous, which leads to the following definition:

A Weil sheaf \( G_0 \) on an algebraic scheme \( X_0 \) over \( k \) consists of a constructible \( \mathbb{Q}_l \)-sheaf \( G \) on \( X \) and an isomorphism \( F^*_G : F^*_X G \to G \). A lisse Weil sheaf is a Weil sheaf that \( G \) is lisse.

For any constructible \( \mathbb{Q}_l \)-sheaf \( F_0 \) on \( X_0 \), the canonical \( F^*_X \pi^* G_0 \cong (\pi \circ F_X)^* G_0 = \pi^* G_0 \) makes \( F \) into a Weil sheaf.

Prop. (VII.5.3.3) (Weil Sheaf and Representation). When \( X_0 \) is geometrically connected, the functor \( G_0 \to (G_0)_{\overline{\mu}} \) defines an equivalent between the category of Weil sheaves on \( X_0 \) and the category of continuous \( \mathbb{Q}_l \)-representations of \( W(X_0, \overline{\mu}) \). And the correspondence defined in (VII.5.1.15) is a subcorrespondence of this.

Thus the notion of geometric irreducible/semisimple is definable for Weil sheaves.

Proof: Because by the correspondence (VII.5.1.15), \( G \) corresponds to a representation of \( \pi_1(X, \overline{\mu}) \), and \( \pi_1(X, \overline{\mu}) \) acts trivially on the Galois cover \( X/X_0 \). Now a representation of \( W(X_0, \overline{\mu}) \) is equivalent to an automorphism \( \rho(\sigma)( \text{ where } \sigma \in W(X_0, \overline{\mu}) \text{ satisfies } \deg(\sigma) \text{ corresponds to the geometric Frobenius}) \) that \( \rho(\sigma)\rho(\pi_1(X, \overline{\mu}))\rho(\sigma^{-1}) = \rho(\sigma\pi_1(X, \overline{\mu})\sigma^{-1}) \), which is equivalent to an isomorphism \( F^*_X G \to G \).}

Prop. (VII.5.3.4). The constructions like \( R^i f_* , R^i f_! , f^* \) is functorial thus is definable in the category of Weil sheaves by (VI.2.1.44). And the specified isomorphism \( F^*_G : F^*_X G \to G \) gives us an action \( F^*_X \) of \( F_X \) on \( H^i_c(X, G) \), just like in (VII.5.2.8).

Similarly, there is an action of \( F_X \) on \( G_{\overline{\mu}} \) for each \( x \in |X_0| \).

Prop. (VII.5.3.5) (Weil Sheaf and Eigenvalues). If \( X_0 \) is geometrically connected, a lisse Weil sheaf \( G_0 \) on \( X_0 \) is an ordinary \( \mathbb{Q}_l \)-sheaf iff some deg1 element \( \sigma \) in \( W(X, \overline{\mu}) \) acts on \( G_{\overline{\mu}} \) with eigenvalues which are \( l \)-adic units.
Proof: This is purely a Galois representation problem, concerning whether the representation of $W(X_0, \mathfrak{X})$ can be extended to a representation of $\pi_1(X_0, \mathfrak{X})$, and it is a continuity problem.

Firstly the representation of $\pi_1(X, \mathfrak{X})$ stabilizes a lattice $O_E^\sigma$ for some $E/\mathbb{Q}_l$ finite, and then extends $E$ to contain coefficients of $\rho(\sigma)$ and even its rational form. Then notice $\pi_1(X_0, \mathfrak{X})$ is the profinite completion of $W(X_0, \mathfrak{X})$, thus it suffices to see if the image $\rho(W(X_0, \mathfrak{X}))$ is compact, and this is equivalent to eigenvalues of $\rho(\sigma)$ are units.

Prop. (VII.5.3.6) (Determinential Criterion). If $X_0$ is normal and geometrically connected, then an irreducible lisse Weil sheaf on $X_0$ is an actual $\overline{\mathbb{Q}}_l$-sheaf iff its determinant bundle is.

Proof: Use geometric monodromy group. Cf.[Conrad L19 P7].

First assume that $G_0$ is geometrically irreducible, then(VII.5.5.4) shows that there is a nonzero power $\sigma^m = gz$ where $g \in G_{geo}(\overline{\mathbb{Q}}_l)$ and $z \in Z(G(\overline{\mathbb{Q}}_l))$. Now $G_{geo}$ is a semisimple algebraic group(VII.5.5.4), so the determinential character maps $G_{geo}(G(\overline{\mathbb{Q}}_l))$ to a finite group, because a connected semisimple algebraic group has no nontrivial character as $[G, G] = G$. So the determinant of $g$ is an $l$-unit, and det$(\sigma^m) = \det(z)$ is a unit. But $z$ is a scalar by Schur’s lemma, thus $z$ is an $l$-adic unit. Now it suffices to show the eigenvalue of $g$ are all $l$-units.

Now consider $\rho(\pi_1(X, \mathfrak{X}))$ is a compact group in $\text{End}(V)$, thus it generates a finite $O_E$-submodule $A$, which is full-rank lattice in $\text{End}(V)$ by Jacobson density theorem? and the fact $\rho$ is absolutely irreducible. $g$ normalized $A$, because $\sigma$ and $z$ both normalizes $\rho(\pi_1(X, \mathfrak{X}))$, so the eigenvalue of the conjugate action of $g$ are all $l$-units, but its eigenvalue are of the form $\lambda_i \lambda_j^{-1}$ where $\lambda_k$ are eigenvalues of $g$, so this together with the fact det$(g)$ is $l$-units shows that all $\lambda_i$ are $l$-units.

For the general case, Cf.[Conrad L19 P7].

Cor. (VII.5.3.7) (Filtration of Weil Sheaf). If $X_0$ is normal and geometrically connected, then for any lisse Weil sheaf $\mathcal{G}_0$, there is some $b \in \overline{\mathbb{Q}}_l^*$ and a lisse Weil sheaf $\mathcal{F}_0$ that $\mathcal{G}_0 \cong \mathcal{F}_0 \otimes \mathcal{L}_b$, where $\mathcal{L}_b$ is the Weil sheaf corresponding to the character $W(X_0, \mathfrak{X}) \to \overline{\mathbb{Q}}_l^*: x \mapsto b^{\deg(x)}$, which is a pull back from $\text{Spec} \mathbb{F}_q$.

More generally, for any lisse Weil sheaf, there is a filtration that each quotient is of the form $\mathcal{F}_0^{(i)} \otimes \mathcal{L}_{b_i}$ for some $b_i \in \overline{\mathbb{Q}}_l^*$ and $\mathcal{F}_0^{(i)}$ lisse $\overline{\mathbb{Q}}_l$-sheaves.

Proof: Just choose $b = \chi_{\text{det}(\sigma)}^{1/n}$, where $\deg(\sigma) = 1$, then

$$\wedge(\mathcal{G}_0 \otimes \mathcal{L}_{b^{-1}}) \cong \wedge(\mathcal{G}_0) \otimes \mathcal{L}_1^{-1}$$

which has unit eigenvalues thus is a lisse $\overline{\mathbb{Q}}_l$-sheaf.

Grothendieck-Lefschetz Trace Formula

Def. (VII.5.3.8) ($L$-Function). Given a constructible $K$-sheaf $\mathcal{F}$ on $X_{et}$, its $L$-function is defined to be

$$L(X, \mathcal{F}, t) = \prod_{x \in |X|} \det(1 - F_x t^{d_x} |\mathcal{F}_x|^{-1} \in 1 + tA[[t]]).$$

Prop. (VII.5.3.9) (Grothendieck-Lefschetz Trace Formula). For a separated morphism of f.t. $k$-schemes $X \to S$, if $\mathcal{F}$ is any constructible $K$-sheaf $\mathcal{F}$ on $X_{et}$, then we have

$$L(X, \mathcal{F}, t) = \prod_{n=0}^{2 \dim X} L(S, R^n f_! \mathcal{F}, t)(-1)^n.$$
In particular, for $S = \text{Spec } k$, we have
\[ L(X_0, \mathcal{F}, t) = \prod_{n=0}^{2 \dim X_0} \det(1 - F_X^n t | H_{c,\text{ét}}^n (X, \mathcal{F}))^{(-1)^{n+1}} \]

Notice by (VI.2.5.8), the higher proper pushforward just vanish.

**Proof:** First take an open subscheme $U \to S$ and $Z = S - U$, consider $f_U : X_U \to U, f_Z : X_Z \to Z$, then we can use the excision long exact sequence for compact pushforward (VI.2.5.6), we can use Noetherian induction to reduce to the case that $X, S$ are both separated.

Now we reduce to the absolute case $S = \text{Spec } k$: If the absolute case is true, then it suffices to prove that
\[ \prod_{s \geq 0} \det(1 - F_X^s t | H_{c,\text{ét}}^s (X_k, \mathcal{F}))^{(-1)^s} = \prod_{n,m \geq 0} \det(1 - F_X^n t | H_{c,\text{ét}}^m (S_k, R^n f_! \mathcal{F}))^{(-1)^{m+n}} \]

And for this, use Leray spectral sequence, which is Frobenius equivariant by (VII.5.2.7) and a determinantal Euler characteristic of spectral sequences, Cf.[Conrad L18, P9].

The $S = \text{Spec } k$ case is done in (VII.5.3.13). \[ \square \]

**Cor. (VII.5.3.10).** The $L$-function is a rational.

**Remark (VII.5.3.11) (Name of the Trace Formula).** Using the formula
\[ \det(1 - Ft | V)^{-1} = \exp(\sum_{i \geq 1} tr(F^i) \frac{t^i}{i}) \]
for each endomorphism $F \in \text{End}(V)$, we can unwinding the equation that it is equivalent to
\[ \chi(F_X^* | H_{c,\text{ét}}^*(X_k, \mathcal{F})) = \sum_{x \in X(k)} tr(F_x| \mathcal{F}_x), \]
for any $k = \mathbb{F}_{q^n}$, where $\chi(F_X^* | H_{c,\text{ét}}^*(X_k, \mathcal{F})) = \sum_{n \geq 0} (-1)^n tr(F_X^n | H_{c,\text{ét}}^n (X_k, \mathcal{F}))$.

**Proof:** Cf.[Conrad L18 P8]. \[ \square \]

**Lemma (VII.5.3.12) (Weil Trace Formula).** If $C$ is a smooth projective curve over $k = \mathbb{F}_q$ and $\psi : C \to C$ is an endomorphism, then
\[ \Delta \cdot \Gamma_{\psi} = \sum_{i=0}^{2} (-1)^i \text{tr}(\psi^* | H^i (C, \mathcal{O}_C)). \]

**Prop. (VII.5.3.13) (General Trace Formula for Frobenius).** Let $X_0$ be a variety over $k = \mathbb{F}_q$ and $K_0 \in D_{\text{perf}}(X_0)$, then
\[ \sum_{x \in X(k)} \text{tr}(F_x | K_x) = \sum_{i} (-1)^i \text{tr}(F_X^i, H_c^i (X, K)). \]
Prop. (VII.5.3.14) (Grothendieck-Lefschetz Trace Formula for Weil Sheaves). For any Weil sheaf $F_0$ over $X_0$, define

$$L(X_0, F_0, t) = \prod_{x \in |X_0|} \det(1 - t^d F_x|\mathcal{F}_x)^{-1}$$

Then we have

$$L(X_0, F_0, t) = \prod_{i=0}^{2 \dim X_0} \det(1 - t F^*_X|H^i_c(X, G))(-1)^{i+1}$$

Proof: Use the filtration in (VII.5.3.7), notice that the trace is additive for a filtration, so we can reduce to the case $G_0 = F_0 \otimes L_b$ and $G_x = F_x \otimes L_{b,x}$, then the Euler factor is

$$\det(1 - t^d F_x|\mathcal{F}_x)$$

and the cohomology factor is

$$\det(1 - t F^*_X|H^i_c(X, F \otimes L_b)) = \det(1 - t^d F^*_X|H^i_c(X, F))$$

where the projection formula (VI.2.5.9) is used, noticing the $L_b$ is pulled back from $\text{Spec } \mathbb{F}_q$. □

4 Weights and Purity

Determinantal Weights

Prop. (VII.5.4.1) (Structure of Weil Group of Curve). If $X_0$ is a geometrically connected smooth curve over $\mathbb{F}_q$, then the image of $\pi_1(X, \overline{x})$ in $W(X_0, \overline{x})^{ab}$ is a product of a finite group and a pro-$p$ group.

Proof: Let $K$ be the function field of $X_0$, $\overline{X_0}$ be the regular completion of $X_0$, with $S_0 = \overline{X_0} - X_0$, then we have an isomorphism $\pi_1(\overline{X_0}, \overline{x}) \cong G_K$ Cf.[Étale Cohomology Lei Fu P136]?. So we can use global class field theory:

$$\begin{array}{cccccc}
\pi_1(\overline{X}, \overline{x})^{ab} & \to & \pi_1(\overline{X}_0, \overline{x})^{ab} & \to & G_K & \cong \hat{\mathbb{Z}} & \to & 0 \\
\downarrow & & \uparrow & & \uparrow & & \downarrow & & \downarrow \\
0 & \to & I_K & \to & W(\overline{X}_0, \overline{x})^{ab} & \cong W(K, k) & \to & W(k) & \cong \mathbb{Z} & \to & 0 \\
& \cong & & \cong & & \cong & & \cong & & \cong \\
0 & \to & K^*\backslash A_K^1/\prod_v \mathcal{O}_v^* & \to & K^*\backslash A_K^*/\prod_v \mathcal{O}_v^* & \to & q_\mathbb{Z} & \to & 0
\end{array}$$

So the image of $\pi_1(\overline{X}, \overline{x})$ factors through $\pi_1(\overline{X}, \overline{x}) \to \pi_1(\overline{X}_0, \overline{x})^{ab} \to K^*\backslash A_K^*/\prod_v \mathcal{O}_v^*$ which is the class number of $K$, is finite.

In this diagram, $W(X_0, \overline{x})$ corresponds to $K^*\backslash A_K^*/\prod_{v \notin S_0} \mathcal{O}_v^*$, so

$$0 \to \text{Ker}(W(X_0, \overline{x}) \to W(\overline{X}_0, \overline{x})) \to \text{Im}(\pi_1(X_0, \overline{x})) \to \text{Im}(\pi_1(\overline{X}_0, \overline{x})) \to 0$$

But the kernel is a quotient of $\prod_{v \in S_0} \mathcal{O}_v^*$, which is a pro-$p$ group times a finite group, so finally $\text{Im}(\pi_1(X_0, \overline{x}))$ is a product of a pro-$p$-group times a finite group. □
Lemma (VII.5.4.2) (Curve Rank 1 case). If $X_0$ is a geometrically connected smooth curve over $\mathbb{F}_q$ and $\chi : W(X_0, \pi) \to \mathbb{Q}^*_l$ be a continuous character, then there exists a $c \in \mathbb{Q}^*_l$ such that $\chi$ is a product of a character of finite order and the character $\sigma \mapsto e^{c\deg(\sigma)}$.

In particular, the Weil sheaf corresponding to $\chi$ is punctually $\nu$-pure of weight $2 \log_q |\nu(c)|$.

Proof:  By (III.1.7.3), the image of $\chi$ is in $\mathcal{O}^*_E$ for some $E/\mathbb{Q}_l$ finite, so use (VII.5.4.1), it has an open subgroup which is pro-$p$ and pro-$l$ trivial, thus $\pi_1(X_0, \bar{x})$ is mapped to a finite group.

In particular there is an $n$ that $\chi^n = id$ on $\pi_1(X_0, \bar{x})$, so there is some $b$ that $\chi^n = b^{\deg(\sigma)}$, hence if $c$ is an $n$-th roots of $b$ and we let $\chi^l = \chi/e^{\deg(\sigma)}$, then $\chi^m = 1$. □

Cor. (VII.5.4.3) (Rank 1 Lisse Sheaf is Pure). If $X_0$ is a geometrically connected smooth curve, then any lisse Weil sheaf of rank 1 is pure.

Def. (VII.5.4.4) (Determinential Weight). Let $\mathcal{F}_0$ be a lisse Weil sheaf on a geometrically connected smooth scheme $X_0$, and $0 = \mathcal{F}_n \subset \mathcal{F}_{n-1} \subset \ldots \subset \mathcal{F}_0$ be a filtration of lisse sheaves that the quotients are irreducible, we define the determinential $\nu$-weights of $\mathcal{F}_0$ to be that of the $\nu$-weights of the top wedge products of the successive quotients divided by their ranks, which exists by (VII.5.4.3).

Notice that the determinential $\nu$-weights are unchanged when $\mathcal{F}_0$ is replaced by its semisimplification $\mathcal{F}^s_0 = \oplus_{i \geq 0}(\mathcal{F}_i/\mathcal{F}_{i-1})$.

Purity

Def. (VII.5.4.5) (Purity). For an embedding $\bar{Q}_l \to \mathbb{C}$, a constructible sheaf $\mathcal{F}$ on a $k$-scheme $X$ is called $\nu$-pure of weight $w$ if for any closed point $x \in X$, the $\bar{Q}_l$-eigenvalues of $F_x$ on $\mathcal{F}_x$ satisfies $|\nu(\alpha_i)| = (q^{dz})^{w/2}$. It is called pure of weight $w$ iff for any closed point $x \in X$, the $\bar{Q}_l$-eigenvalues are $q^{dz}$-Weil numbers of weight $w$, i.e. $\nu$-pure for any embedding $\nu : \bar{Q}_l \to \mathbb{C}$.

It is said to be $(\nu)$-mixed with weights $w_1, \ldots, w_n$ if it has a successive quotients of constructible $\bar{Q}_l$-sheaves that are pure of weight $w_i$ respectively.

Prop. (VII.5.4.6). $\mathcal{Q}_l(1)$ is pure of weight $-2$, thus $\mathcal{Q}_l(r)$ is pure of weight $-2r$. This is because the geometric Frobenius $F_x$ acts by $1/q^{dz}$-th power, which is additively multiplying by $(q^{dz})^{-2/2}$.

Prop. (VII.5.4.7) (Permanence Properties).

• $f_0 : X_0 \to Y_0$ is a morphism, and $\mathcal{G}_0$ is a Weil sheaf on $Y_0$, then if $\mathcal{G}_0$ is $\nu$-pure, then $f^*_0 \mathcal{G}_0$ is also $\nu$-pure, and the converse is also true if $f$ is surjective.

• If $f_0 : X_0 \to Y_0$ is finite, and $\mathcal{G}_0$ is a Weil sheaf on $X_0$, then

Proof:  1 is because the stalk corresponds.

2: This is because the stalks can be calculated, by □

Semicontinuity of Weights

Def. (VII.5.4.8) (Purity). A Weil sheaf $\mathcal{G}$ on $X_0$ is called pure of weight $w$ if for any closed point $x \in X$, the $\bar{Q}_l$-eigenvalues of $F_{\mathcal{G}}$ on the stalks $\mathcal{F}_x$ are all $q^{dz}$-Weil numbers of weight $w$.

Def. (VII.5.4.9) (Maximal Weight). For a general Weil sheaf $\mathcal{G}_0$ on $X_0$, we can also define the maximal $\nu$-weight of $\mathcal{G}_0$ as

$$w(\mathcal{G}_0) = \sup_{x \in |X_0|} \sup_{\alpha_i} 2 \log_N(x)(|\nu(\alpha_i)|).$$
Lemma (VII.5.4.10). \(|X_0(k_n)| = O(q^{n \dim X})\)

Proof: We can pass to the reduced structure of \(X_0\), then we can use excision to pass to the integral case. Then choose an open affine dense subset \(U_0\) of \(X_0\), then by Noetherian normalization, it factors through a finite map \(f: U_0 \to \mathbb{A}_{k_n}^{\dim X_0}\), so

\[
|U(k_n)| \leq (\deg f)q^{n \dim X_0}
\]

Then we can use induction on dimension, because \(\dim(X_0 - U_0) < \dim X_0\).

Lemma (VII.5.4.11). Let \(\mathcal{G}_0\) be a Weil sheaf on \(X_0\) and \(\beta\) be a real number that \(\beta \geq w(\mathcal{G}_0)\), then the \(L\)-function

\[
\iota(L(X_0, \mathcal{G}_0, t)) = \prod_{x \in |X_0|} \iota((1 - t^{d_x}F_x, \mathcal{G}_0, \pi)^{-1})
\]

converges for \(|t| < q^{-\beta/2 - \dim X_0}\) and has no zero or pole there.

Proof: We can show that it has no zero or pole using the fact that the logarithmic derivative has no poles (when it is convergent). We suppress the isomorphism \(\iota: \overline{\mathbb{Q}}_l \to \mathbb{C}\) and calculate:

\[
\frac{d\log}{dt} L(X_0, \mathcal{G}_0, t) = \sum_{x \in |X_0|} \sum_{n \geq 1} d_x(\text{tr}(F_x^n))t^{d_x n - 1}(\text{VII.5.3.11}) = \sum_{n \geq 1} \sum_{x \in |X_0|, d_x | n} d_x(\text{tr}(F_x^{n/d_x}))t^{n - 1}
\]

Notice by assumption on \(\beta\), \(|\text{tr}(F_x^{n/d_x})| \leq rq^{n/2}\), where \(r = \max_{x \in |X_0|} \dim_{\overline{\mathbb{Q}}_l} \mathcal{G}_0, x\) is finite because it has a stratification by (VII.5.1.9), so

\[
\left| \frac{d\log}{dt} L(X_0, \mathcal{G}_0, t) \right| \leq \sum_{n \geq 1} \sum_{x \in |X_0|, d_x | n} d_x r q^{n/2} t^{n - 1} = \sum_{n \geq 1} \left| X_0(k_n) \right| q^{n\beta/2} t^{n - 1}
\]

converges for \(|t| < q^{-\beta/2 - \dim X_0}\) by (VII.5.4.10).

Lemma (VII.5.4.12) (Semicontinuity of Weights for Curves). If \(X_0\) is a smooth geometrically irreducible curve over \(k\) and \(U_0 \to X_0\) be a nonempty open with \(S_0 = X_0 - U_0\). Let \(\mathcal{G}_0\) be a Weil sheaf on \(X_0\) s.t. the restriction \(j_0^* \mathcal{G}_0\) is lisse and \(H^0_S(X, \mathcal{G}) = 0\), then \(w(j_0^* \mathcal{G}_0) \leq \beta\) implies \(w(\mathcal{G}_0) \leq \beta\).

Proof: Consider an affine open subset of \(X_0\), then reduce to the affine case, and because \(H^0_S(X, \mathcal{G}) = 0\) and the excision sequence (VI.2.5.6), we have \(\mathcal{G} \hookrightarrow j_\ast j^\ast \mathcal{G}\), so the weights of \(\mathcal{G}_0\) are no more than that of \(j_\ast j^\ast \mathcal{G}\), and replacing \(\mathcal{G}_0\) with \(j_0 j_0^* \mathcal{G}_0\), we can assume \(\mathcal{G}_0 = j_0 j_0^* \mathcal{G}_0\). Then

\[
H^0_c(X, \mathcal{G}) = H^0_c(X, j_\ast j^\ast \mathcal{G}) = H^0_c(U, j^\ast \mathcal{G}) = 0
\]

by Poincare duality and the fact \(j_\ast\) is exact because it is finite.

Now by Grothendieck-Lefschetz trace formula,

\[
L(X_0, \mathcal{G}_0, t) = L(U_0, j_\ast^\dagger (\mathcal{G}_0), t) \cdot \prod_{s \in |S_0|} \det(1 - t^{d_s}F_s, \mathcal{G}_s)^{-1} = \frac{\det(1 - F_X t | H^1_c(X, \mathcal{G}))}{\det(1 - F_X t | H^1_c(X, \mathcal{G}))}
\]

Denote \(\mathcal{F}_0 = j_0^* \mathcal{G}_0\), then

\[
H^2_c(X, \mathcal{G}) = H^2_c(U, \mathcal{F}) = (\mathcal{F}_\pi|_{U, \pi})(-1)
\]
So the weights of eigenvalues of $F_X$ on $H^2_c(X, \mathcal{G}) \leq \text{weights of } \mathcal{F} + 2$, hence the $L$-function converges for $|t| < q^{-\beta/2-1}$. Now the LHFS has $L(U_0, j_0^*(\mathcal{G}_0), t)$ converges for $|t| < q^{-\beta/2-1}$ because $w(F_0) \leq \beta$, and so for the points in $S_0$, we also have $\det(1 - t^{d_s} F_s, \mathcal{G}_s)$ has no zero there, which means they have weights $\leq \beta + 1$. Now consider replacing $\mathcal{G}_0$ with $\mathcal{G}_0 \otimes \kappa$ and let $k \to \infty$, then their weights $\leq \beta$. □

**Prop. (VII.5.4.13) (Semicontinuity of Weights).** Let $X_0$ be normal geometrically $\mathcal{G}_0$ be a lisse sheaf on $X_0$ and $j_0 : U_0 \to X_0$ be an open dense subscheme, then

- $w(\mathcal{G}_0) = w(j_0^* \mathcal{G}_0)$.
- If $j_0^* \mathcal{G}_0$ is $\nu$-pure of weights $\beta$, then $\mathcal{G}_0$ is also $\nu$-pure of weights $\beta$.
- Let $X_0$ be irreducible and normal, and $\mathcal{G}_0$ is irreducible, then if $j_0^* \mathcal{G}_0$ is $\nu$-mixed, then $\mathcal{G}_0$ is $\nu$-pure.

**Proof:** 1: The weights is local so we may assume $X_0$ is irreducible, and then for any closed point $x$, we can connect it with $U_0$ with a curve (choose an affine open and use Noetherian Normalization to choose an irreducible component of an arbitrary curve in $\mathbb{A}^n$). Notice $H^0_0(X, \mathcal{G}) = 0$ because it is lisse thus $H^0(X, \mathcal{G})$ is determined by stalk thus $H^0(X, \mathcal{G}) \to H^0(U, \mathcal{G}|_U)$ is injective. So we finish by the curve case (VII.5.4.12).

2: Apply item 1 to $\mathcal{G}_0$ and $\mathcal{G}_0^\vee$.

3: It is $\nu$-mixed so it has $\nu$-pure Weil sheaf constituents. Now by (VII.5.1.9) we can find an open dense $U_0$ that restriction to $U_0$ has constituents $\nu$-pure lisse sheaves. But it is also irreducible because $\pi_1(U_0, \overline{\alpha}) \to \pi_1(X_0, \overline{\alpha})$ is surjective?, so it is $\nu$-pure and item 2 shows $\mathcal{G}_0$ is $\nu$-pure. □

**Def. (VII.5.4.14).** As in (VII.5.9.1), for any Weil sheaf $\mathcal{G}_0$, we have a a function

$$f^{K_0} : X_0(k^n) \to \overline{\mathbb{Q}}_l : x \mapsto \sum_i (-1)^i \text{tr}(F^n_{x/d_x})(|H^i(K_0)|_\pi),$$

Fix an arbitrary isomorphism $\overline{\mathbb{Q}}_l \cong \mathbb{C}$, we can consider the usual $L^2$-norm for functions on $X_0(k_n)$, denoted by $(f, g)_n$.

**Def. (VII.5.4.15).** Notice the equation form (VII.5.4.11) can be rewritten as

$$\frac{d \log}{dt} L(X_0, \mathcal{G}_0, t) = \sum_{n \geq 1} \sum_{x \in |X_0|, d_x|n} d_x(\text{tr}(F^n_{x/d_x})) t^{n-1} = \sum_n (f^{\mathcal{G}_0}, 1)_n t^{n-1}.$$ 

Now we define another closed related function

$$\varphi^{\mathcal{G}_0}(t) = \sum_n ||f^{\mathcal{G}_0}||_n^2 t^{n-1},$$

which works better with Fourier transform we are about to define later.

**Lemma (VII.5.4.16).** There is a constant $C$ that $||f^{\mathcal{G}_0}(x)||^2 \leq C q^{n(w(\mathcal{G}_0) + \dim X_0)}$, so $\varphi^{\mathcal{G}_0}(t)$ converges for $|t| \leq q^{-w(\mathcal{G}_0) - \dim X_0}$.

**Proof:** The proof is similar to that of (VII.5.4.11) thus omitted. □
Def. (VII.5.4.17) (Norm of a Weil Sheaf). Define the Norm of a Weil sheaf as

$$||G_0|| = \sup \{\rho | \limsup_n \frac{||f^{G_0}||^2_n}{q^{n(\rho + \dim X_0)}} > 0\}$$

Then $q^{-||G_0||-\dim X_0}$ is just the radius of convergence of the function $\varphi^{G_0}$, and $||G_0|| \leq w(G_0)$ by Lemma (VII.5.4.16) above.

Prop. (VII.5.4.18) (Radius of Convergence). Let $G_0$ be a $\nu$-mixed sheaf on an algebraic scheme $X_0$ of dimension 1, then if $X_0$ is a smooth curve, and $H^0(X, G) = 0$, then $||G_0|| = w(G_0) = \beta$.

Proof: It suffices to show $w(G_0) \leq ||G_0||$. First notice we can assume $X_0$ is reduced because the nilpotents corresponds to zero Frobenius eigenvalues, and also it is connected, because the function $f^{G_0}$ is additive in $X$. Now we study by cases:

1: If $G_0$ is a lisse $\nu$-pure sheaf on a smooth affine curve $X_0$, we may assume $G_0 \neq 0$, then $G_0 \otimes G_0 (VII.5.6.2)$ is $\nu$-real of weight $2\beta$ and $(f^{G_0} \otimes f^{G_0})_n = ||f^{G_0}||^2_n$, so $\varphi^{G_0}(t)$ is just the logarithmic derivative of the L-function $L(X_0, G_0 \otimes G_0, t)$, thus (VII.5.4.11) shows its convergence radius $\geq q^{-\beta-1}$. And notice the $H^0_c$ terms vanish so the poles can only appear as the zeros of $H^1_c$ term, so (VII.5.6.3) shows the poles of the $L(X_0, G_0 \otimes G_0, t)$ has weight $2\beta + 2$, thus the poles can only appear on $|t| = q^{-\beta-1}$.

Now consider each local Euler factor $\det(1 - F_x t^d)(G_0 \otimes G_0)_x^{-1}$ has non-negative coefficients, they have poles because $G_0 \neq 0$, and their poles have weight $\beta$ because of purity, thus their product also has (real)poles, before this argument, the pole has weight $\beta + 1$, this is its convergence radius at most $q^{-\beta-1}$, so we are done.

2: If $G_0$ is a $\nu$-mixed, consider its semisimplification $G_0^{ss} = F_0 \oplus H_0$, where $F_0$ is $\nu$-pure of weight $w(G_0)$, and $w(H_0) \leq w(F_0)$.

Then $f^{G_0} = f^{F_0} + f^{H_0}$, and

$$\varphi^{G_0}(t) = \varphi^{F_0}(t) + \sum_{n \geq 1} 2 \text{Re}(f^{F_0}, f^{H_0})_n t^n + \varphi^{H_0}(t)$$

then by item1 $\varphi^{F_0}(t)$ has convergence radius $q^{-w(G_0)-1}$, and by (VII.5.4.16) $\varphi^{H_0}(t)$ has radius at least $q^{-w(H_0)-1} > q^{-w(G_0)-1}$, and by Cauchy inequality the middle term satisfies

$$|2 \text{Re}(f^{F_0}, f^{H_0})_n| \leq 2 ||f^{F_0}||_n ||f^{H_0}||_n \leq C q^{n(w(F_0) + w(H_0)/2+1)}$$

So the middle term has convergence radius $> q^{-w(G_0)-1}$, so their sum has convergence radius $q^{-w(G_0)-1}$. \qed

5 Geometric Monodromy

Def. (VII.5.1) (Notations). Let $X_0$ be a geometrically connected normal scheme over $k = \mathbb{F}_q$ in this subsection.

Def. (VII.5.2) (Geometric Monodromy Group). Let $G_0$ be a Weil-sheaf associated to a representation $(V, \rho) = GL(G \otimes \pi_1(X, \bar{X}))$, the geometric monodromy group $G_{geo}$ associated to $G_0$ is the Zariski Closure of $\rho(\pi_1(X, \bar{X})) \subset GL(V)$.

Every element in $\rho(W(X_0, \bar{X}))$ normalizes $G_{geo}$ by continuity, so choosing an arbitrary generator $\sigma \in W(\bar{k}/k)$, we have an action of $W(\bar{k}/k)$ on $G_{geo}$. Define $G = W(\bar{k}/k) \rtimes G_{geo}$ the arithmetic monodromy group of $G_0$. 

**Lemma (VII.5.5.3).** If $G_{\text{geo}}$ is connected, then there is a positive integer $N$ that the semidirect sequence

$$1 \to G_{\text{geo}} \to \text{deg}^{-1}(NZ) \to \text{deg} N \to 1$$

is direct, i.e. $\text{deg}^{-1}(NZ) \cong G_{\text{geo}} \times Z$.

**Proof:** Choose a $\text{deg}(g) = 1$. The representation $G_{\text{geo}}$ splits as a characters of $Z(G)$, and then some $g^n$ stabilizes these characters, hence stabilizes $Z(G)$, which then it descends to an action on $G_{\text{adj}}$, whose automorphism is the automorphism of the Dynkin diagram, so finite, so some $g^m$ fixes $G_{\text{adj}}$ after changing a semidirect product, thus induces a map $\text{Hom}(G_{\text{adj}}, Z(G))$, but $G_{\text{adj}}$ is semisimple(V.10.2.7), so the connected component is mapped to 1 in $Z(G)$ (V.10.2.9), so there are only f.m. such homomorphism, showing $g^k$ is 1, so the product is exact for $N = k$. \hfill $\Box$

**Prop. (VII.5.5.4) (Geometric Monodromy Group is Semisimple).** Let $\mathcal{G}_0$ be a geometrically semisimple lisse Weil sheaf (VII.5.3.3), then

- $G_{\text{geo}}$ and $G^0_{\text{geo}}$ are semisimple algebraic group.
- Let $Z = Z(G(\overline{Q}_t))$, then the map $\psi : Z \to W(\overline{k}/k)$ has finite kernel and cokernel. In particular, $Z$ contains an element of finite degree, and it is surjective after a finite base change of fields.

And notice in fact if $\mathcal{G}_0$ is semisimple, then it is automatically geometrically semisimple by (III.1.1.5).

**Proof:** 1: Let $G_{\text{geo}}$ is semisimple if $G^0_{\text{geo}}$ is semisimple. Pass to a finite étale covering, we may assume $G_{\text{geo}} = G^0_{\text{geo}}$. Let $R(C^0_{\text{geo}})$ be the radical and $R_u(C^0_{\text{geo}})$ be the unipotent radical, then $R$ is normal in $G^0$ and $G^0$ is normal in $G$, so by (III.1.1.5) $V = GL(G_\tau)$ is irreducible $R(C^0_{\text{geo}})$ representation, but it is solvable, so $V$ is a direct sum of 1-dimensional representations, and $R_u(C^0_{\text{geo}})$ is trivial, in particular $G^0_{\text{geo}}$ is reductive. So it is semisimple if the maximal Abelian quotient $G^0_{\text{geo}}$ is finite (V.10.2.9).

Let $T_1$ be the maximal central torus of $G^0_{\text{geo}}$, then lemma (VII.5.5.3) shows after a finite base change of fields, we may assume $G = G_{\text{geo}} \times Z$, consider the composite $W(X_0, \overline{\pi}) \to G_{\text{geo}} \times Z \to G_{\text{geo}} \to G_{\text{geo}}^0$, then $\pi_1(X, \overline{\pi})$ is Zariski dense in $G_{\text{geo}}^0$, and (VII.5.4.3) shows clearly $G_{\text{geo}}^0$ has no maximal torus thus finite.

2: Ker $\psi \subset Z(G_{\text{geo}}(\overline{Q}_t))$ is finite since $G_{\text{geo}}$ is semisimple. To find an element in $Z(G)$ of positive degree, we may use the same method as before to find an element $\zeta$ that commutes with $G_{\text{geo}}$, and pass to a power, we may assume it acts trivially on $G_{\text{geo}}$. For any $g \in G_{\text{geo}}$, consider $vp_g(n) = g\zeta^n g^{-1} \zeta^{-n} \in G_{\text{geo}}$, so $\varphi_g(m + n) = \varphi_g(n) \zeta^n \varphi_g(m) \zeta^{-n} = \varphi_g(n) \varphi_g(m)$, thus it is a homomorphism, and if $g' \in G^0_{\text{geo}}$, then

$$\varphi_g = \varphi_{g^{-1}g'} = \varphi_{g'g} = g' \varphi_g(g')^{-1}$$

so $\varphi_g$ has image in $Z(G^0_{\text{geo}})$, which is finite, so $\varphi_g(n) = 1$ for some $n$, then $\zeta^n$ commutes with $G_{\text{geo}}$ so $\zeta^n \in Z(G)$. \hfill $\Box$

**Cor. (VII.5.5.5) (Weights and Center Element Actions).** Let $\mathcal{G}_0$ be a semisimple lisse Weil sheaf on $X_0$, if $z \in Z(G(Q))$ satisfies $\text{deg}(z) = n \neq 0$, which exists by (VII.5.5.4), then if $z$ acts on $V$ with eigenvalues $\alpha_i$, then $\frac{1}{n} \log_\alpha(|\alpha_i|)$ is just the determinental $\iota$-weights of $\mathcal{G}_0$.

**Proof:** $z$ is in the center, thus by Shur’s lemma, it acts on each irreducible part of $\mathcal{G}_0$ by a constant. Thus the determinental weights are clear, by definition. \hfill $\Box$
Cor. (VII.5.6.6) (Properties of Determinental Weights). Let $X_0$ be a smooth and geometrically connected curve, $\mathcal{F}_0, \mathcal{G}_0$ be lisse Weil sheaves on $X_0$, then

- If $\alpha_i$ are the determinental $\iota$-weights of $\mathcal{F}_0$ and $\beta_j$ be that of $\mathcal{G}_0$, then $\alpha_i + \beta_j$ are those of $\mathcal{F}_0 \otimes \mathcal{G}_0$ with multiplicity.

- For $\gamma \in \mathbb{R}$, let $r(\gamma)$ be the sum of ranks of all irreducible constituents of $\mathcal{F}_0$ which have determinental weight $\gamma$ w.r.t $\iota$, then the determinental weights of $\wedge^r \mathcal{F}_0$ are the numbers $\sum \gamma n(\gamma) \gamma$ with $\sum n(\gamma) = r$ and $0 \leq n(\gamma) \leq r(\gamma)$, $n(\gamma) \in \mathbb{Z}$ with multiplicity.

**Proof:** Firstly notice the determinental weight is unchanged when we change $\mathcal{F}_0, \mathcal{G}_0$ to their semisimplification $\mathcal{F}_0^{ss}, \mathcal{G}_0^{ss}$ (VII.5.4.4). And notice $(\mathcal{F}_0 \otimes \mathcal{G}_0)^{ss} = ((\mathcal{F}_0)^{ss} \otimes (\mathcal{G}_0)^{ss})^{ss}$, thus the determinental weights of $\mathcal{F}_0 \otimes \mathcal{G}_0$ are also unchanged. Similarly for the wedge product.

2: We may assume $\mathcal{F}_0, \mathcal{G}_0$ are irreducible, and let $\mathcal{H}_0 = (\mathcal{F}_0 \otimes \mathcal{G}_0)^{ss}$, $\mathcal{G}_0^{\text{geo}}, \mathcal{G}_0^{\text{ss}}$ be the geometric monodromy group of $\pi_1(X, \mathcal{X})$ in $\text{GL}(\mathcal{F}_\mathcal{X} \oplus \mathcal{G}_\mathcal{X})$ and $\text{GL}(\mathcal{H}_\mathcal{X})$ correspondingly, then $\mathcal{G}_0^{\text{geo}} \to \mathcal{G}_0^{\text{ss}}$ is surjective because they are both the geometric monodromy group of $\mathcal{H}_\mathcal{X}$. So also $\mathcal{G}_0^{\oplus} \to \mathcal{G}^{ss}$ is surjective. So if $g$ be an element in the center of $G^{\oplus}$ that has nonzero degree, then it maps to the center of $G^{ss}$ of nonzero degree. And the action of $g$ on each factor $\mathcal{F}_\mathcal{X}, \mathcal{G}_\mathcal{X}$ is a constant, so action on $\mathcal{H}_\mathcal{X}$ is also a constant, so we are done.

3: Easy from 2.

\[\square\]

6 Real Sheaves

Def. (VII.5.6.1) (\iota-Real Sheaf). Let $\mathcal{F}_0$ be a Weil sheaf on $X_0$, then $\mathcal{F}_0$ is called \iota-real if for any $x \in |X_0|$, the characteristic polynomial $\iota(\det(1 - \mathcal{F}_x t, \mathcal{F}_\mathcal{X}))$ of $\mathcal{F}_x$ real coefficients.

Prop. (VII.5.6.2). Any $\iota$-pure Weil sheaf of weight $w$ is a direct sum of a $\iota$-real $\iota$-pure Weil sheaf.

In fact, $\mathcal{F}_0 \oplus \mathcal{F}_0'(-w) = \mathcal{F}_0 \oplus \mathcal{F}_0'$ is $\iota$-real.

Lemma (VII.5.6.3) (Eigenvalue of Cohomology and Stalk in Curve case). Let $X_0$ be a smooth geometrically connected curve over $\mathbb{F}_q$, $\mathcal{F}_0$ be a lisse Weil sheaf on $X_0$, then the eigenvalues of $F_X$ on $H^0(X, \mathcal{F})$ or $H^2_c(X, \mathcal{F})$ is related to the determinental weights of $\mathcal{F}_0$ and the eigenvalue of $F_x$ on $\mathcal{F}_\mathcal{X}$.

**Proof:** Let $V = \mathcal{F}_\mathcal{X}$, then

\[H^0(X, \mathcal{F}) = V^{\pi_1(X, \mathcal{X})}, \quad H^2_c(X, \mathcal{F}) = V^{\pi_1(X, \mathcal{X})(-1)}.\]

Then the base change sheaf of the sheaf $V^{\pi_1(X, \mathcal{X})}$ or $V^{\pi_1(X, \mathcal{X})(-1)}$ on $\text{Spec} k$ is the maximal subsheaf/quotient lisse sheaf of $\mathcal{F}_0$ that is constant on $X$. Then it has determinental weights just the action of $F_X$ on the stalk by (VII.5.5.5), which are also determinental weights of $\mathcal{F}_0$ by (VII.5.5.6).

\[\square\]

Lemma (VII.5.6.4) (Rankin-Selberg Method). Let $X_0$ be a smooth geometrically connected curve over $\mathbb{F}_q$, $\mathcal{F}_0$ be a lisse Weil sheaf on $X_0$, and $w$ be the largest determinental weight of $\mathcal{F}_0$, then for any $x \in |X_0|$, $w_{N(x)}(\alpha) \leq w$.

**Proof:** By the arbitrariness of $x$, we can replace $X_0$ by an affine open nbhd of $x$. Then $H^0_c(X, \mathcal{G}) = 0$ by Artin vanishing (VI.2.1.38). By Grothendieck trace formula,

\[\prod_{x \in |X_0|} \iota \det(1 - t^{d_x} F_x \otimes 2k \mathcal{F}_\mathcal{X})^{-1} = \frac{\iota \det(1 - t F_X^s | H^1_c(X, \otimes 2k \mathcal{F}))}{\iota \det(1 - t F_X^s | H^2_c(X, \otimes 2k \mathcal{F}))}\]
Now the weight of root $t_0$ of $\det(1 - tF_x^2|H^2_\ast(X, \otimes^2 \mathcal{F}))$ has weight $\leq$ determinential weight of $G_{\otimes^2 k} + 2(VII.5.6.3) \leq 2kw + 1(VII.5.5.6)$, so $|t_0| \geq q^{-k\beta - 1}$.

Now by the formula $(VII.5.3.11)$ and noticing $\text{tr}(F_x^m, \otimes^2 k \mathcal{F}_\pi) = (\text{tr}(F_x, \mathcal{F}_\pi))^{2k}$, so $1-t^{d_x}F_x| \otimes^{2k} \mathcal{F}_\pi)^{-1}$ has non-negative coefficients, which means their convergence radius are no less than $q^{-k\beta - 1}$, equivalently, $(1-t^{d_x}F_x| \otimes^{2k} \mathcal{F}_\pi)$ has no zeros with eigenvalue $< q^{-k\beta - 1}$.

So for any eigenvalue $\alpha$ of $F_x$ acting on $\mathcal{F}_\pi$, $|\alpha^{-2k/d_x}| \leq q^{-k\beta - 1}$, or equivalently,

$$|\alpha|^2 \leq N(x)^{\beta + 1/k}.$$ 

Now let $k \rightarrow \infty$, we are done. \qed

**Lemma (VII.5.6.5) (Real Sheaf Mixed Curve case).** Let $X_0$ be smooth geometrically irreducible curve over $k$ and $G_0$ be an $\nu$-lisse Weil sheaf on $X_0$, then all irreducible constituents of $G_0$ is $\nu$-pure, and their $\nu$-weights coincides with their determinential weights.

**Proof:** For $\beta \in \mathbb{R}$, let $F_0(\beta)$ be the sum of constituents of $F_0$ of determinential weight $\beta$, and let $n(\beta) = \text{rank}(F_0(\beta))$, then we need to show that $w_{N(x)}(\alpha_i(\beta)) = \beta$ for any eigenvalue of $F_x$ on $\mathcal{F}(\beta)_{\pi}$.

By definition of determinential weight, for each $\gamma$, we have $\sum w_{N(x)}(\alpha_j(\gamma)) = n(\gamma)\gamma$. Now let $N = \sum_{\gamma > \beta} n(\gamma)$, then any determinential weight of $\Lambda^{N+1}F_0$ has weight $\leq \beta + \sum_{\gamma > \beta} n(\gamma)\gamma$: This is clear by $(VII.5.5.6)$ as the determinential weights of $\Lambda^{N+1}F_0$ is of the form $\sum_{\gamma} a(\gamma)\gamma$ that $0 \leq a(\gamma) \leq \gamma$ and $\sum_{\gamma} a(\gamma) = N + 1$.

But now $\alpha_i(\beta) \prod_{\gamma > \beta} \prod_{j=1}^{n(\gamma)} \alpha_j(\gamma)$ is an eigenvalue of $(\Lambda^{N+1}F_0)_{\pi}$, but by lemma $(VII.5.6.4)$, $w_{N(x)}(\alpha_i(\beta)) \leq t$. Thus we must have equality $w_{N(x)}(\alpha_i(\beta)) = \beta$. \qed

**Prop. (VII.5.6.6) (Real Sheaf is Mixed).** Let $X_0$ be an algebraic scheme over $\mathbb{F}_q$, then

- Any $\nu$-real Weil sheaf on $X_0$ is $\nu$-mixed.

- If $X_0$ is irreducible and normal, any irreducible constituent of a lisse of an $\nu$-real sheaf is $\nu$-pure.

**Proof:** Cf.[Bhatt P28], [KW, P36].

We have the following devissages:

- Choose an open subset $j_0 : U_0 \hookrightarrow X_0, S_0 = X_0 - U_0$ and consider the fundamental excision sequence $(VI.2.5.5)$, we can reduce to an open affine subscheme $U_0 \subset X_0$.

- We may base change to a finite field extension.?

- So we may reduce to the case $X_0$ is smooth, irreducible affine, and $G_0$ is lisse, with all the irreducible constituents geometrically irreducible(by base change, because they are geometrically semisimple $(VII.5.5.4)$). And we may assume dim $X_0 > 1$ because the curve case is proven.

- Change $k$ to the alg.closure of $k$ in the function field of $X_0$, we can assume $X_0$ is geometrically irreducible by $(V.4.3.16)$.

Embed $X_0$ in some projective space $\mathbb{P}^N_0$, then by a suitable Bertini theorem, the linear subspaces of codimension dim $X - 1$ that intersects $X$ with a non-empty smooth irreducible curve $C_L$ is dense in the Grassmannian. Now the closed points in any $C_L$ is a pure-point for the any irreducible component $F_0$ of $G_0$ of the same weights. Now let $L$ vary, then there is a dense subset of a finite extension of $X_0$ that $F_0$ is pure. So we are done. \qed
7 Deligne’s Purity Theorem

Prop. (VII.5.7.1) (Deligne’s Purity Theorem). If $f : X_0 \to Y_0$ is a separated morphism of algebraic scheme over $\mathbb{F}_q$, and $\mathcal{F}$ is a constructible $\mathbb{Q}_l$-sheaf on $X$ that is $\nu$-mixed weights $\leq n$, then for any integer $i \geq 0$, the sheaf $R^if_*\mathcal{F}$ is also $\nu$-mixed weights $\leq n + i$. Moreover, each $\nu$-weight of of $R^if_*\mathcal{F}$ is equivalent modulo $\mathbb{Z}$ to an $\nu$-weight of $\mathcal{F}$.

Proof: This follows from (VII.5.7.6).

Cor. (VII.5.7.2). If $X$ is a smooth separated algebraic $k$-scheme, $\mathcal{F}$ is mixed of weight $\geq n$, then $H^i_{\acute{e}t}(X, \mathcal{F})$ is mixed of weights $\geq n + i$.

Proof: Use Poincare duality (VI.2.6.1), we know $H^n_{c,\acute{e}t}(X_{\overline{k}}, \mathcal{F}^\vee(d))$ is the Galois dual representation of $H^n_{c,\acute{e}t}(X_{\overline{k}}, \mathcal{F})$, and $\mathcal{F}^\vee(d)$ is still a lisse sheaf pure of weight $-w - 2d$, thus Deligne’s purity theorem (VII.5.7.1) shows that $H^n_{c,\acute{e}t}(X_{\overline{k}}, \mathcal{F}^\vee(d))$ has weight $\leq (-w - 2d) + (2d - n) = -w - n$, thus we are done.

Cor. (VII.5.7.3) (Weil’s Conjecture). Let $X$ be a smooth separated algebraic $k$-scheme, and $\mathcal{F}$ is a lisse $\mathbb{Q}_l$-sheaf which is pure of weight $w$, then the image of $H^w_{c,\acute{e}t}(X_{\overline{k}}, \mathcal{F})$ in $H^w_{\acute{e}t}(X, \mathcal{F})$ is pure of weight $w + n$.

Proof: The morphism $H^w_{c,\acute{e}t}(X_{\overline{k}}, \mathcal{F}) \to H^w_{\acute{e}t}(X, \mathcal{F})$ defined in (VI.2.5.7) is compatible with Frobenius, so from (VII.5.7.2) we know the image has weights $\geq w + n$, so combined with Deligne’s purity theorem (VII.5.7.1), we know it is pure of weight $w + n$.

Cor. (VII.5.7.4). If $f_0 : X_0 \to Y_0$ is a smooth proper map of schemes of f.t. and $\mathcal{F}_0$ is $\nu$-pure of weight $\beta$, then $R^if_0_*\mathcal{F}_0$ $\nu$-pure of weight $\beta + i$.

Proof: Use proper base change of (VI.2.5.6) to reduce to the case of (VII.5.7.3). Notice in the proper case, $Rf_*=Rf_!$.

Cor. (VII.5.7.5) (Riemann Hypothesis). If $X$ is smooth proper $k$-scheme, then $H^n_{c,\acute{e}t}(X, \mathbb{Q}_l)$ is pure of weight $n$.

Reduction to Curve case

Prop. (VII.5.7.6). Deligne’s purity theorem can be reduced to case that $X_0$ is a smooth geometrically connected affine curve $\subset \mathbb{A}_k^1$ and $\mathcal{F}_0$ a lisse $\mathbb{Q}_l$-sheaf.

Proof: We have the following Devissages for Deligne’s theorem:

- It is trivial in case $f_0$ is quasi-finite. This is because of (VI.2.5.8), as the fiber has dimension 0.
- We can replace $X_0$ by an affine open $U_0 \subset X_0$ by Noetherian induction and excision sequence (VI.2.5.6), which commutes with Frobenius action.
- If the conclusion is true for $g_0, h_0$, then it is true for $f_0 = g_0 \circ h_0$, this follows from the Leray spectral sequence (VI.2.5.6), which is Frobenius equivariant by (VII.5.2.7).
- We can replace $Y_0$ with an affine open $U_0 \subset Y_0$: If the image $f_0$ is not dense, then trivial, if it is dense, then choose any affine open $U_0$, then it suffices to prove for $f_0 : f_0^{-1}(U_0) \to Y_0$, by item2, then by item3 it suffice to prove for $f_0 : f_0^{-1}(U_0) \to U_0$, because $U_0 \to Y_0$ is quasi-finite and use item1.
VII.5. WEIL 2 PROOF

Now we claim we can reduce to the case of \( f_0 : X_0 \to Y_0 \) surjective affine smooth with the fibers being geometrically irreducible curves. By devissage2 and 4, we may assume \( X_0, Y_0 \) is affine, thus \( f_0 \) is affine. Take a generic point \( \eta \) of \( Y_0 \), then \( (X_0)_{\eta} \to \text{Spec} k(\eta) \) is affine hence by Noetherian normalization(I.5.8.19) there is a finite map \( X_0 \to k^n_{k(\eta)} \), and this spread out to a finite morphism \( f_0^{-1}(U_0) \to U_0 \) for some affine open \( U_0 \subset Y_0 \) because \( f_0 \) is of f.t.. Then by Devissage1 and 3 we are reduced to the case \( A^1_{Y_0} \to Y_0 \). Now by(VII.5.1.9), there is an affine open \( U_0 \subset A^1_{Y_0} \) that \( F_0|U_0 \) is lisse, so by Devissage2 we may change \( X_0 \) to \( U_0 \).

That is we reduced to the case that \( F_0 \) is lisse and \( X_0 \) is open in \( A^1_{Y_0} \) so \( f_0 \) is smooth affine, in particular open(V.5.3.4), so we can replace \( Y_0 \) by \( f(X_0) \) and assume \( f_0 \) is surjective. Then the fiber are all geometrically irreducible curves.

Then the assertion about weights are clear from proper base change(VI.2.5.6) and the curve case.

For the \( \nu \)-mixedness, we may use(VII.5.6.2) and(VII.5.6.6) to reduce to showing that \( R^i f_! \) maps \( \nu \)-real sheaf to \( \nu \)-real sheaf.

For a geometric point \( \pi \to x \to X_0 \), let \( C \to C_0 \) be the fiber, which is affine irreducible, so \( H^0_\nu(X,F) = 0 \) by Poincare duality(VI.2.6.1) and Artin vanishing theorem(VI.2.1.38), so

\[
\nu L(C_0,G_0,t) = \frac{t \det(1 - t F^1_\nu X | H^1_\nu(C,G_{|C}))}{\nu \det(1 - t F^1_\nu X | H^2_\nu(C,G_{|C}))}
\]

by Grothendieck-Lefschetz formula(VII.5.3.14). Now we can use Poincare duality and the definition that \( G_0 \) is pure of weight \( \beta \), we know \( H^2_\nu(C,G_{|C}) \) is pure of weight \( \beta + 2 \), by(VII.5.6.3). And \( H^2_\nu(C,G_{|C}) \) has weights smaller than \( \beta + 1 \) by the curve case, so the two polynomial is coprime, and both have constant coefficient 1, which shows they are both real. And then by proper base change(VI.2.5.6), this just says \( R^i f_0 ! G_0 \) is \( \nu \)-real.

Third Reduction

Prop. (VII.5.7.7). The final proof of Weil conjecture by proving(VII.5.7.6).

Proof: We have the following devissages:
- We only need to check for \( H^1_\nu(X,F) \), because \( H^0_\nu \) vanish by Poincare duality(VI.2.6.1) and Artin vanishing theorem(VI.2.1.38) and, \( H^2_\nu(X,F) \) is dealt with in(VII.5.6.3).
- We are free to pass to finite base change.
- We may assume \( F_0 \) is geometrically irreducible: By(VII.5.5.4), all the irreducible constituents of \( F_0 \) are geometrically semisimple, so pass to a finite base change, we may assume that its irreducible filtration is just the geometric irreducible filtration, then because \( H^0_\nu = 0 \), \( H^1_\nu \) is left exact.
- We can assume that \( F_0 \) can be extended to a lisse sheaf on \( \infty \). This is because we can choose a closed point and move it to \( \infty \) by using Möbius transform, after a finite base change.
- We can assume \( F_0 \) is not geometrically constant: if \( F_0 \cong \underline{\mathcal{Q}}_l \), then let \( i : U_0 \to \mathbb{P}^1_{F_0} \) and \( Z_0 = \mathbb{P}^1_{F_0} - U_0 \), then there is a short exact sequence

\[
0 \to j_0 ! \underline{\mathcal{Q}}_l \to j_! \underline{\mathcal{Q}}_l \to Q \to 0
\]

where \( Q \) is supported at \( S \), so its higher compact cohomology vanish, and weights of \( H^0(Q) = \prod_{s \in S} (j_0 _*(\mathcal{F}_0))^s \) is no more than the maximal weight of \( \underline{\mathcal{Q}}_l \) on \( X_0 \), which is 0, by
semicontinuity of weights for curves (VII.5.4.12). And \( j_*(\mathcal{U})_l \) is also geometrically constant, thus its cohomology is \( \text{Pic}(\mathbb{P}^1)[n] = 0 \) by (VI.2.3.2), so \( H^1(\mathbb{P}^1, j_0(X_0)) \) has weights zero.

The actual proof will use the following lemma (VII.5.7.8). After that, notice by (VII.5.9.8)

\[
(T_\psi(G_0))|_{\{0\}} = R\Gamma_c(\mathbb{A}^1, \mathcal{G})[1] = R\Gamma_c(U, \mathcal{F})[1] = H^1_c(U, \mathcal{F})
\]

Then to understand the Frobenius eigenvalues of \( H^1_c(U, \mathcal{F}) \), it suffices to understand the weights of \( T_\psi(G_0) \), i.e.

\[
w(T_\psi(G_0)) \leq w + 1
\]

Then we use (VII.5.4.18), notice the condition is satisfied by lemma (VII.5.7.8), so \( w(T_\psi(G_0)) = ||T_\psi(G_0)|| \), and also \( w(G_0) = ||G_0|| \) for the same reason as \( H^0_c(\mathbb{A}^1, \mathcal{G}) = H^0(\mathbb{U}, \mathcal{F}) = 0 \) by Poincaré duality. Now (VII.5.9.12) gives the result.

\[
\square
\]

**Lemma (VII.5.7.8) (Key Assertions of Weil Proof).** If \( \mathcal{G}_0 = j_0!(\mathcal{F}_0) \) where \( j_0 : U_0 \hookrightarrow \mathbb{A}^1_{\overline{\mathbb{Q}}} \), \( \psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_l \) is a fixed non-trivial additive character, then

- \( T_\psi(G_0) \) is a sheaf placed at degree 0.
- \( H^0_c(\mathbb{A}^1, T_\psi(G_0)) = 0. \)
- \( T_\psi(G_0) \) is \( \iota \)-mixed.

**Proof:**

1. By (VII.5.9.8), we need to show \( H^i(\mathbb{A}^1, \mathcal{G}_0 \otimes \mathcal{L}(\psi_a)) = 0 \) for \( i \neq 1 \), and this is equivalent to

\[
H^i(\mathbb{A}^1, j_!F \otimes \mathcal{L}(\psi_a)) = H^i(U, F \otimes \mathcal{L}(\psi_a)) = 0.
\]

Notice by vanishing result proper-pushforward-to-direct-image sheafomqsk, only need to show \( i = 0, i = 2, i = 0 \) case is done by Poincaré duality (VI.2.6.1) and Artin vanishing (VI.2.1.38) because it is smooth and \( \mathcal{F} \) is lisse.

\[
H^2_c(G_0 \otimes \mathcal{L}(\psi_a)) = V_{\rho \otimes \chi_a \mid \pi_1(U, \mathcal{X})}(-1)
\]

by (VII.5.6.3), and \( \rho \otimes \chi_a \) irreducible as \( \rho \) does, so if \( V_{\rho \otimes \chi_a \mid \pi_1(U, \mathcal{X})} \neq 0 \), then \( \rho \otimes \psi_a \) is trivial representation. Then \( \mathcal{G} \cong \mathcal{L}_{-a} \) on \( \mathbb{P}^1_k \) as an étale sheaf on \( \mathbb{A}^1 \cup \{ \infty \} = \mathbb{P}^1 \) by our reduction, so we have the character \( \psi_{-a} \) factors through \( \pi_1(\mathbb{P}^1, \mathcal{X}) \), i.e.

\[
\pi_1(\mathbb{A}^1, \mathcal{X}) \twoheadrightarrow \pi_1(\mathbb{P}^1, \mathcal{X}) = 0.
\]

But this is in contradiction with the fact the Artin-Schreier cover is geometrically irreducible?.

2. Denote \( T_\psi(\mathcal{G}_0) = \mathcal{K}_0 \), then by (VII.5.9.7) and Fourier inversion (VII.5.9.10):

\[
H^0_c(\mathbb{A}^1, \mathcal{K}) = H^{-1}((T_\psi^{-1}(\mathcal{K}_0))_0) = H^{-1}(T_\psi^{-1} \circ T_\psi(j_0!(\mathcal{F}_0))_0) = H^{-1}(j_0!(\mathcal{F}_0)(-1))_0 = 0
\]

because \( \mathcal{F}_0 \) is placed at degree 0.

3. To show \( \iota \)-mixed, the only thing we can do it show it is embedded in a \( \iota \)-real sheaf: Consider the \( \iota \)-real sheaf

\[
\mathcal{H}_0 = \pi^{-2*}(j_0!(\mathcal{F}_0) \otimes m^*(\mathcal{L}(\psi)) \oplus \pi^{2*}(j_0!(\mathcal{F}_0)^\vee) \otimes m^*(\mathcal{L}(\psi^{-1}))(\mathcal{F}_0)(-w)
\]
Then
\[ (R^i \pi_1^!(H_0))_x = H^i((\{x\} \times \mathbb{A}^1, H_0) = H^i(j_0!F_0 \otimes \mathcal{L}(\psi_x)) \oplus H^i(j_0!F_0^\vee \otimes \mathcal{L}(\psi_{x-1}^{-1}))(-w) \]
which we proved to vanish for \( i \neq 1 \). So using Poincare duality on \( \{x\} \times \mathbb{A}^1 \),
\[
\det(1 - tF_{x|H_0}^d) = \prod_{y \in \mathbb{F}_q^n} \det(1 - tF_{y|x|H_0}^d)^{-1}
\]
which is real, so by \((\text{VII.5.6.5})\), the direct summand \( T_\psi(G_0) \) is \( \iota \)-mixed. \( \square \)

**Remark (VII.5.7.9).** If we use the machinery of perverse sheaf and show that Fourier transform commutes preserves perversity, then item 1, 2 will be a direct consequence, Cf.[Bhatt notes, P39]. In fact, this is just the bigger picture, given in [Weil conjectures Perverse Sheaves and \( l \)-adic Fourier Transform Kiehl/Weissauer].

### 8 Semisimplicity and Hard Lefschetz

**Prop. (VII.5.8.1) (Semisimplicity Theorem).** If \( X_0 \) is smooth and \( F_0 \) is a lisse and \( \iota \)-pure \( \mathbb{Q}_l \)-sheaf, then \( F_0 \) is semisimple, thus geometrically semisimple by (VII.5.5.4).  

**Proof:** Let \( F' \) be the sum of irreducible lisse subsheaves of \( F \), then it is the largest semisimple subsheaf of \( F \). It is stable under \( G(F_n^\vee \otimes F'_n) \), thus can be descended to a lisse subsheaf \( F'_0 \) of \( F_0 \), and let \( F'' = F_0/F'_0 \), we want to show the exact sequence
\[
0 \to F'_0 \to F_0 \to F''_0 \to 0
\]
splits geometrically. Notice this exact sequence defines an element in \( \text{Ext}^1_X(F'', F') \). \( F_0 \) is pure, hence so does \( (F''_0)^\vee \otimes F'_0 \), thus \( H^i(X, (F''_0)^\vee \otimes F') \) is \( \iota \)-mixed of weights\( \geq 1 \). But the exact sequence is compatible with Frobenius action, thus it defines a Frobenius fixed element, which then must vanish. \( \square \)

**Cor. (VII.5.8.2).** If \( f : X \to Y \) is proper between smooth sheaves, then the sheaves \( R^i f_* \mathbb{Q}_l \) are semisimple.

**Prop. (VII.5.8.3) (Hard-Lefschetz).** Cf.[Bhatt P42].

### 9 Fourier Transformation

**Sheaf to Functions Correspondence**

**Def. (VII.5.9.1) (Sheaf to Functions Correspondence).** For a complex \( K_0 \in D^b_{cons}(X_0, \mathbb{Q}_l) \), we can associate a function
\[
f^{K_0} : X_0(\mathbb{F}_q^n) \to \mathbb{Q}_l : x \mapsto \sum_i (-1)^i \text{tr}(F_n^{a/d_x})(\mathcal{H}^i(K_0)\pi)
\]

**Prop. (VII.5.9.2).** We can use Grothendieck formula for a constructible sheaf(VII.5.3.9) to relate the function \( f^{K_0} \) to the compact cohomologies of \( \mathcal{H}_K \), and we can translate many know theorems:

- \( f^{f^*K_0} = f^{K_0} \circ f \).
• 
\[ fK_0 \cdot fT_0 = fK_0 \otimes L T_0 ? \]

• (Base Change)(VI.2.5.6) asserts that given a Cartesian diagram
\[
\begin{align*}
X' & \xrightarrow{g'} X \\
\downarrow f' & \downarrow f \\
Y' & \xrightarrow{g} Y
\end{align*}
\]
then it says in case \( Y' \) is a closed point of \( Y \),
\[
f^{Rf_!}K_0(y) = \sum_{x \in X_y(F_q^n)} f^K_0(x)
\]
where \( y \in Y(F_q^n) \), and more generally
\[
\sum_{x' \in X'_y} f^K_0(g'(x')) = \sum_{x \in X_y} f^K_0(x)
\]

• The projection formula(VI.2.5.9) turns out to say something trivial:
\[
\sum_{x \in X_y} (f^K_0(f(x)) \cdot f^{T_0}(x)) = f^K_0(y) \cdot (\sum_{x \in X_y} f^{T_0}(x))
\]

**Artin-Schreier Sheaf**

**Def.** (VII.5.9.3) (Notations). For an arbitrary character \( \psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_l^* \), it can be extended to \( \mathbb{F}_q^n \) by
\[
\mathbb{F}_q^n \xrightarrow{\text{tr}} \mathbb{F}_q \xrightarrow{\psi} \overline{\mathbb{Q}}_l^*
\]

**Prop.** (VII.5.9.4) (Artin-Schreier Sheaf). Let \( y^{q^n} - y - x \in \mathbb{A}_0^1 \) be the finite Galois cover of \( \mathbb{A}_0^1 \) via \( x \) coordinates, with the Galois group isomorphic to \( \mathbb{F}_q \) with \( 1 \mapsto (x \mapsto x + 1) \). Then we get a surjection \( \pi_1(A_0^1, \overline{\mathbb{F}}) \to \mathbb{F}_q \), when composed with \( \psi \), we get a rank1 étale sheaf \( \mathcal{L}_0(\psi) \) called the Artin-Schreier sheaf on \( \mathbb{A}_0^1 \).

**Prop.** (VII.5.9.5). \( f^{L_0(\psi)}(x) = \psi(-x) \).

**Proof:** If \( k(x) = \mathbb{F}_q^n \), then consider the arithmetic Frobenius \( \sigma : (x, y) \mapsto (x^{q^n}, y^{q^n}) \), then if \( y^{q^n} - y = x \), then we have
\[
y^q = y + x, \quad y^{q^2} = y^q + x^q = y + x^q + x, \ldots, \quad y^{q^n} = y + x + x^q + \ldots + x^{q^{n-1}} = y + \text{tr}_{\mathbb{F}_q^n/\mathbb{F}_q}(x)
\]
So in the correspondence(VII.5.9.4), we know \( F_\sigma \) acts on \( \mathcal{L}_\psi \) by multiplication by \( \psi(\text{tr}_{\mathbb{F}_q^n/\mathbb{F}_q}(x)) = \psi(-x) \), so the geometric Frobenius acts by \( \psi(-x) \).

**Def.** (VII.5.9.6) (Deligne-Fourier Transform). Consider the multiplication map \( \mathbb{A}_0 \times \mathbb{A}_0' \to \mathbb{A}_0 \), let the sheaf \( \mathcal{L}(\psi) \) be placed at \( \mathbb{A}_0 \), and \( K_0 \in D_c^b(A_0', \overline{\mathbb{Q}}_l) \) be placed at \( \mathbb{A}_0' \), then define the Deligne-Fourier transform
\[
T_\psi : D_c^b(A_0', \overline{\mathbb{Q}}_l) \to D_c^b(A_0, \overline{\mathbb{Q}}_l) : K_0 \mapsto R\pi_1^!(\pi^{2*}K_0 \otimes L m^*\mathcal{L}_0(\psi))[1]
\]
Lemma (VII.5.9.7). We have \( f^{T_\psi K_0}(x) = -\sum_{y \in F_q^n} f^{K_0}(y)\psi(-xy) \) for any \( x \in F_q^n \).

**Proof:** Use (VII.5.9.2), we have

\[
\begin{align*}
f^{T_\psi K_0}(x) &= \sum_{y \in F_q^n} f^{(\pi L^* m^* L_0(\psi))([1])(x, y))} f^{K_0}((x, y)) \cdot f^{m^* L_0(\psi)}((x, y)) \\
&= -\sum_{y \in F_q^n} f^{K_0(y)}\psi(-xy).
\end{align*}
\]

\( \square \)

Prop. (VII.5.9.8). Let \( a \) be a geometric point of \( \mathbb{A}^1_0 \), then

\[
(T_\psi(K_0))_a = R\Gamma_c(K \otimes L \mathcal{L}(\psi_a))\big|_a
\]

where \( \psi_a : F_q^n \to \overline{\mathbb{Q}_l} \) maps \( x \mapsto \psi(ax) \). In particular, \( \mathcal{H}^1((T_\psi(K_0))_0) = H^1_c(\mathbb{A}^1, K) \), so we placed the complex into a family of deformations.

**Proof:** By base change (VI.2.5.6),

\[
(T_\psi(K_0))_a = R\Gamma_c((\pi^2 K_0 \otimes m^* L_0(\psi))|_{(a) \times \mathbb{A}^1})[1] = R\Gamma_c(K \otimes L \mathcal{L}(\psi_a))[1].
\]

\( \square \)

Lemma (VII.5.9.9). If \( \delta_0 = i_0_* \overline{\mathbb{Q}_l} \) be the skyscraper sheaf, where \( i_0 : \{0\} \to \mathbb{A}^1 \), then

\[
T_\psi(\overline{\mathbb{Q}_l}[1]) = \delta_0(-1).
\]

**Proof:** For the Artin-Schreier cover \( P : x \mapsto x^q - x \), we have

\[
P_\ast \overline{\mathbb{Q}_l} \cong \oplus_{x \in F_q^n} \mathcal{L}(\psi_x) ?
\]

and \( P \) is finite thus proper and \( P_\ast \) is exact (VI.2.1.16), so using the Leray spectral sequence (VI.2.5.6), we can calculate

\[
H^1_c(\mathbb{A}^1, \mathcal{L}(\psi_x)) = 0 = H^1_c(\mathbb{A}^1, \overline{\mathbb{Q}_l}), \quad H^2_c(\mathbb{A}^1, \mathcal{L}(\psi_x)) = 0 (VI.2.5.8), \quad H^2_c(\mathbb{A}^1, \mathcal{L}(\psi_x)) = \delta_0(x)\overline{\mathbb{Q}_l}(-1) (VII.5.6.3)
\]

So

\[
(R\pi_1^\ast(m^* L_0(\psi))[1])_x = R\Gamma_c(\mathcal{L}(\psi_x))[2] = \delta_0(-1).
\]

\( \square \)

Prop. (VII.5.9.10) (Fourier Inversion). \( T_{\psi^{-1}} T_\psi K_0 = K_0(-1) \).

**Proof:** Consider

\[
\begin{array}{cccc}
\mathbb{A}_0^1 \times \mathbb{A}_0^1 \times \mathbb{A}_0^1 & \xrightarrow{\pi_2^3} & \mathbb{A}_0^1 \times \mathbb{A}_0^1 & \xrightarrow{\pi^2} & \mathbb{A}_0^1 \\
\downarrow \pi^{12} & & \downarrow \pi^1 & & \\
\mathbb{A}_0^1 \times \mathbb{A}_0^1 & \xrightarrow{\pi^1} & \mathbb{A}_0^1 \\
\downarrow \pi^1 & & \\
\mathbb{A}_0^1 \\
\end{array}
\]

\[
\]
And we will use the following Cartesian diagrams:

\[
\begin{array}{ccc}
A_0^3 \xrightarrow{\alpha(x,y,z) \mapsto (y,z-x)} A_0^2 & & A_0^1 \xrightarrow{\ast} \\
\downarrow \beta(x,z) \mapsto z-x & & \downarrow \Delta
\end{array}
\]

Then

\[
T_{\psi^{-1}}T_{\psi}K_0
= R\pi_1^1(\pi^{2*} R\pi_1^1(\pi^{2*} K_0 \otimes m^* L_0(\psi)) \otimes m^* L_0(\psi^{-1}))[2]
\]

By base change (VII.2.5.6):

\[
= R\pi_1^1(R\pi_1^{12} \pi^{23*}(\pi^{2*} K_0 \otimes m^* L_0(\psi)) \otimes m^* L_0(\psi^{-1}))[2]
\]

By projection formula (VII.2.5.6):

\[
= R\pi_1^1 R\pi_1^{12} (\pi^{23*}(\pi^{2*} K_0 \otimes m^* L_0(\psi)) \otimes \pi^{12*} m^* L_0(\psi^{-1}))[2]
\]

Combine the character:

\[
= R\pi_1^1 R\pi_1^{12} (\pi^{23*} \pi^{2*} K_0 \otimes \alpha^* m^* L_0(\psi))[2]
\]

Change order of summation:

\[
= R\pi_1^1 R\pi_1^{13} (\pi^{13*} \pi^{2*} K_0 \otimes \alpha^* m^* L_0(\psi))[2]
\]

By projection formula:

\[
= R\pi_1^1 (\pi^{2*} K_0 \otimes R\pi_1^{13} \alpha^* m^* L_0(\psi))[2]
\]

By base change:

\[
= R\pi_1^1 (\pi^{2*} K_0 \otimes \beta^* R\pi_1^2(m^* L_0(\psi)))[2] = R\pi_1^1 (\pi^{2*} K_0 \otimes \beta^* T_{\psi} \bar{Q}_l[-1])[2]
\]

By (VII.5.9.9):

\[
= R\pi_1^1 (\pi^{2*} K_0 \otimes \beta^* \delta_0[-2])[2] = R\pi_1^1 (\pi^{2*} K_0 \otimes \beta^* \delta_0(-1)) = \sum_z f(z) q^n \delta_0(z - x)
\]

Use base change and noticing \(q_0\) is finite thus proper and exact:

\[
= R\pi_1^1 (\pi^{2*} K_0 \otimes R\Delta_l \bar{Q}_l(-1)) = \sum_z \sum_{x=2} q^n
\]

By projection formula:

\[
= R\pi_1^1 R\Delta_l (\Delta^* \pi^{2*} K_0 \otimes \bar{Q}_l)(-1) = q^n \sum_{\{z|z=x\}} f(z) = q^n f(x)
\]

\[\square\]

Prop. (VII.5.9.11) (Plancherel Formula).

\[
\| f^{T_{\psi}(K_0)} \|_n = q^{n/2} \| f^{K_0} \|_n.
\]
Proof: By definition and using (VII.5.9.7),

\[
\|f_{T_{\psi}(K_0)}\|_n^2 = \sum_{x \in \mathbb{F}_q^n} f_{T_{\psi}(K_0)}(x)\overline{f_{T_{\psi}(K_0)}(x)}
\]

\[
= \sum_{x,y,z} f_{K_0}(y)\overline{f_{K_0}(z)}\psi(-xy)\psi(xz)
\]

\[
= q^n \sum_{z=y} f_{K_0}(y)\overline{f_{K_0}(z)}
\]

\[
= q^n (f,f)_n
\]

\[\square\]

Cor. (VII.5.9.12). Notice by the definition of norm of a Weil sheaf \(G_0\), we have

\[
\|T_{\psi}(K_0)\| \leq \|K_0\| + 1
\]
VII.6 Logarithmic Geometry

Main references are [DLLZ19], [LOG p-DIVISIBLE GROUPS].
Chapter VIII

$p$-Adic Geometry

VIII.1 \( p \)-adic Hodge Theory

Main references are [Berger Galois representations and \((\phi, \Gamma)\)-modules], [Introduction to \( p \)-adic Hodge, Xavier Caruso].

1 \((\phi, \Gamma)\)-modules

Main References are [Fontaine90: Représentations \( p \)-adiques des corps locaux], [Fontaine94a: Le corps des périodes \( p \)-adiques] and [Fontaine94b: Repésentations \( p \)-adiques semi-stables] but I cannot understand French so [Foundations of Theory of \((\phi, \Gamma)\)-modules over the Robba Ring] is used and I'm basically following [Berger Galois representations and \((\phi, \Gamma)\)-modules].

Def. (VIII.1.1.1) \((\phi)\)-module. Let \( M \) be a \( A \)-module and \( \sigma : A \to A \) is a ring map. Then an additive map \( \phi : M \to M \) is called \( \sigma \)-semi-linear iff \( \phi(am) = \sigma(a)\phi(m) \) for \( a \in A \). A \( \phi \)-module over \( (A, \sigma) \) is just an \( A \)-module \( M \) with a \( \sigma \)-semi-linear \( \phi \).

Giving a \( A \)-module \( M \) and a \( \phi : M \to M \), there is a map \( \Phi : A \otimes_{\sigma, A} M = \sigma_*M \to M : \lambda \otimes m \to \lambda\phi(m) \), which is an \( A \)-module map iff \( \phi \) is \( \sigma \)-semi-linear.

If we define a ring \( A_\sigma[\phi] \) as the free group \( A[X] \) modulo the relation \( Xa = \sigma(a)X \) and ring relations in \( A \), then it is a ring. Then a \( \phi \)-module over \( (A, \sigma) \) is equivalent to a left \( A_\sigma[\phi] \)-module.

Thus we know that the category of \( \phi \)-modules is a Grothendieck Abelian category \( \Phi M \) with tensor products, and moreover, the kernel as \( A_\sigma[\phi] \)-module is the same as the kernel as a \( A \)-module.

Def. (VIII.1.1.2). If there is a map \( \alpha : (A_1, \sigma_1) \to (A_2, \sigma_2) \) that commutes with \( \sigma_i \), then we have a pullback from \( \Phi M_1 \) to \( \Phi M_2 \): \( \alpha^*(M) = (A_2)_{\sigma_2}[\phi] \otimes_{(A_1)_{\sigma_1}[\phi]} M \) (VIII.1.1.1).

Def. (VIII.1.1.3) (Étale \( \phi \)-Modules). If \( A \) is Noetherian, then a \( \phi \)-module \( M \) is called étale iff it is f.g and the corresponding \( \Phi : \sigma_*M \to M \) in (VIII.1.1.1) is a bijection. The subcategory of étale \( \phi \)-modules is denoted by \( \Phi M^{\text{ét}} \).

In case when \( \sigma \) is a bijection, \( \Phi \) is a bijection iff \( \phi \) is a bijection.

Proof: Note that in this case \( \sigma_*M \to M \) is a bijection by \( \lambda \otimes m \to \sigma^{-1}(\lambda)m \), so the rest is easy. \( \square \)

Prop. (VIII.1.1.4). If \( A \) is Noetherian and \( A_\sigma \) is flat, then \( \Phi M^{\text{ét}} \) is Abelian category.
Prop. (VIII.1.2.4). \( \lambda \geq 0 \) iff \( D \) is effective. And \( \lambda = s/r \), where \( 1 \leq r \leq h \).

Proof: If \( D \) is effective, then \( a_n \geq 0 \), conversely, if \( a_n \geq 1 \), then \( M' = M + \varphi(M) + \ldots + \varphi^{n-1}(M) \) is stable under \( \varphi \), so \( D \) is effective.

For the second assertion, we first notice, if \( \lambda > 0 \), then \( \varphi \) is nilpotent on \( M/pM \), which is a \( k \)-vector space of dimension \( h \), then \( \varphi^h = 0 \) on \( M/pM \), so \( \lambda \geq 1/h \).

Now we find \( s, r \) that \( r\lambda - s \leq 1/(h+1) \), and \( \bar{\varphi} = p^{-s}\varphi' \) has \( |\bar{\lambda}| \leq 1/(h+1) \), so (VIII.1.2.3) shows that \( \bar{\varphi} \) is effective, hence \( \bar{\varphi} \geq 0 \), and by what we have proved, \( \bar{\varphi} = 0 \), hence it is \( \lambda = s/r \).

Lemma (VIII.1.2.5). For a \( \varphi \)-stable \( W(k) \)-lattice \( M \) of \( D \), one has \( M = M_0 \oplus M_{>0} \), where \( \varphi \) is bijection on \( M_0 \) and topologically nilpotent on \( M_{>0} \).

Proof: We consider \( M/p^nM \), then by (I.2.4.5) under slight modification, we have a decomposition for \( M/p^nM \). This decompositions for different \( n \) are compatible, so taking an inverse limit gives a decomposition of \( M \) it self.
Def. (VIII.1.2.6) (Isotypical $\varphi$-Modules). A $\varphi$-module is called pure(isotypical) of slope $\lambda = s/r \in \mathbb{Q}$ if $D$ admits a lattice $M$ on which $p^{-s}\varphi^r$ is a bijection. This is independent of $M$ because $\lambda$ is independent of $M$.

Prop. (VIII.1.2.7) (Dieudonné-Manin). If $M$ is a $\varphi$-module over $W(k)[\frac{1}{p}]$ where $k$ is a perfect field, then $M$ is a finite sum of modules pure of slopes $\lambda_i$. This is called the isocrystal decomposition of $M$.

Proof: We use the $\bar{\varphi}$ as in the proof of(VIII.1.2.4), we see that $M$ has a decomposition $M_0 \oplus M_{>0}$ by(VIII.1.2.5), and $M_0 \neq 0$ by definition. Then we use induction to get the result. □

Lemma (VIII.1.2.8). If $k$ is a separably closed field and $V$ is a $\varphi$-module with $a \geq 1$ of slope 0, then $V$ has a basis of elements fixed by $\varphi$, and $1 - \varphi$ is a surjection.

If $A = W(k)$ is a ring with $k$ a separably close field and $V$ is a $\varphi$-module over $A$ with $a \geq 1$ and slope 0, then $V$ has a basis of elements fixed by $\varphi$, and $1 - \varphi$ is a surjection.

Proof: We choose a $e_0 \in V$, and set $e_i = \varphi^i(e_0)$, and suppose $e_d = a_0e_0 + \ldots + a_{d-1}e_{d-1}$, then if we consider the equation $\varphi(b_0e_0 + \ldots + b_{d-1}e_{d-1}) = b_0e_0 + \ldots + b_{d-1}e_{d-1}$, then we need to assure $b_{d-1}$ is a zero of

$$x = a_0^d x^d + a_1^d x^{d-1} + \ldots + a_{d-1} x^d$$

which is separable, so it has a non-zero solution in $k$, so $\varphi$ has a fixed point $v$. By induction, we have $V/k \cdot v$ admits a basis fixed by $\varphi$. We know that $1 - \varphi : k \cdot v \to k \cdot v : x \mapsto (x - x^q)$ is surjective, so we can adjust the coefficient of $v$ to get a basis of $V$ fixed by $\varphi$. And meanwhile we proved $1 - \varphi$ is surjective.

The second assertion follows from successive approximation, as $x^p - x - a$ always has a root in $k$. □

Def. (VIII.1.2.9). When $k$ is alg.closed, for $\lambda = s/r$, we define a $\varphi$-module over $K = W(k)[1/p]$ $E_\lambda = \oplus_{i=1}^{r-1} Ke_i$ that $\varphi(e_i) = e_{i+1}$, and $\varphi(e_{i+1}) = p^s e_0$. In this case, $E_\lambda$ is irreducible.

Proof: If $D$ is a $W(k)$-lattice stable under $\varphi$, then we may assume it is pure of slope $d/h$ by(VIII.1.2.7), and then we find an element $y = \sum y_ie_i$ fixed by $p^{-d}\varphi^h$, then $p^h\varphi^h(y_i) = p^dy_i$, which by valuation is only possible when $sh = rd$, so $h \geq r$, so $D$ generate $E_\lambda$. □

Prop. (VIII.1.2.10) (Dieudonné-Manin). If $k$ is alg.closed, then any $\varphi$-module over $K$ has a unique decomposition as sums of $E_\lambda$(VIII.1.2.9).

Proof: By(VIII.1.2.7) we assume $D$ is pure, then by(VIII.1.2.8) we find a basis $y_i$ that $\varphi^r(y_i) = p^s y_i$, then there is a map $E_\lambda \to D$. Since $E_\lambda$ is irreducible, this is injective, and we consider all $y_i$ until $E_\lambda^n \to V$ is surjective, then it is an isomorphism(this is like the case of simple modules). □

Def. (VIII.1.2.11) (Tate Twist). The Tate object $1(n), n \in \mathbb{Z}$ is the 1-dimensional isocrystal over $K_0$ that $\varphi = p^n\sigma$, so it is of slope $n$. And the Tate twist isocrystal is tensoring by $1(n)$. It preserves rank and shifts Hodge-Tate weight by $n$.

Filtered ($\varphi,N$)-Modules

Def. (VIII.1.2.12) ($\varphi$-Modules over $K$). Let $K$ be a finite field extension of $\mathbb{Q}_p$ and $K_0$ be the maximal unramified subextension of $K$, $K_0 = W(k)$ where $k$ is the residue field of $K$, then a $\varphi$-module over $K$ as a $\varphi$-module over $(K_0, \sigma)$ where $\sigma$ is the absolute Frobenius.
Def. (VIII.1.2.13) \((\varphi, N, G_{L/K})\)-Modules. Let \(K\) be a CDVR of mixed characteristic \((0, p)\) and residue field \(k\), let \(L\) be a Galois extension of \(K\), with Galois group \(G_{L/K}\) and residue field \(k_L\). Let \(L_0\) be the maximal unramified extension of \(K_0 = W(k)[\frac{1}{p}]\) in \(L\). Let \(\sigma\) be the Frobenius endomorphism of \(L_0\).

Then I define the category of \((\varphi, N, G_{L/K})\)-modules as the category of f.d. \(L_0\)-spaces \(V_0\) with

- a \(\sigma\)-semi-linear endomorphism,
- a \(L_0\)-linear endomorphism \(N\),
- a semi-linear continuous action of \(G_{L/K}\) (w.r.t the discrete topology).

That satisfies:

- \(N \varphi = p \varphi N\),
- \(N, \varphi\) commutes with \(G_{L/K}\) actions.

The category of all \((\varphi, N)\)-modules is an Abelian category, similar to that of \(\varphi\)-modules(VIII.1.1.1).

Def. (VIII.1.2.14) (Filtered \((\varphi, N, G_{L/K})\)-Modules). A filtered \((\varphi, N, G_{L/K})\)-module \((D, \varphi_D, N, G_{L/K}, \text{Fil})\) is a \((\varphi, N, G_{L/K})\)-module \((D, \varphi_D, N, G_{L/K}) \in (\varphi, N) - \text{Mod}_{L/K}\) together with a finite filtration \(\text{Fil}\) on \(D_L = D \otimes_{L_0} L\) in the category of vector spaces over \(L\). The category of filtered \((\varphi, N, G_{L/K})\)-modules over \(K\) is denoted by \(\varphi - \text{FilMod}_{L/K}\).

The category of filtered \(\varphi\)-modules(isocrystals) \((D, \varphi_D)\) over \(K\) is the sub Abelian category of filtered \((\varphi, N)\)-modules that \(N = 0\).

Prop. (VIII.1.2.15) (HN-Formalism for Filtered \(\varphi\)-Modules). The category \(\varphi - \text{FilMod}_{K/K_0}\) of filtered \(\varphi\)-modules(isocrystals) \((D, \varphi_D)\) over \(K\) is the sub Abelian category of \((\varphi, N)\)-modules that \(N = 0\).

Prop. (VIII.1.2.17) (Faltings). The tensor product of two weakly-admissible filtered \((\varphi, N, G_{L/K})\)-modules is also weakly-admissible.

Def. (VIII.1.3.1) (Admissible Representations). Let \(G\) be a topological group and \(E\) a topological field and \(B\) a topological \(E\)-algebra that \(G\) acts on \(B\) and fixing elements in \(E\). Then a f.d. representation \(V \in \text{Rep}_E(G)\) is called \(B\)-admissible if the \(B\)-semi-linear representation \(B \otimes_E V\) is trivial.

3 Admissible Representations
\( G_K \)-Regularity

**Def. (VIII.1.3.2) (\( G_K \)-Regularity).** We want to establish a numerical criterion for recognizing \( B \)-admissible representations. For this goal, suppose \( B \) satisfies the following axioms:

- (H1): \( B \) is a domain.
- (H2): \( (\text{Frac}(B))^{G} = B^{G} \).
- (H3): if \( b \neq 0 \in B \) and \( Eb \) is stable under \( G \)-action, then \( b \in B^{*} \).

\( B \) is called \( G_K \)-regular if it satisfies these three conditions. Notice a field is clearly \( G_K \)-regular.

**Cor. (VIII.1.3.3).** Notice that (H3) implies \( B^{G} \) is a field, because for \( b \in B^{G} \), \( Eb \) is clearly stable under \( G \)-action, thus \( b \) is invertible.

Also the morphism
\[
\alpha_{W}: B \otimes_{B^{G}} W^{G} \to W
\]
is injective for all \( W \in \text{Rep}_{B}(G) \) free of finite rank over \( B \). In particular, this is true for \( W = B \otimes_{E} V \) for a f.d. \( E \)-linear representation \( V \) of \( G \).

**Proof:** To show \( \alpha_{W} \) is injective, it suffices to show a linear basis \( \{e_i\} \) of \( W^{G} \) over \( B^{G} \) is linearly independent over \( B \): Suppose \( \sum a_i e_i = 0 \), where \( a_i \in B \), with the number of nonzero coefficients minimal, and \( a_1 \neq 0 \), then dividing \( a_1 \in \text{Frac}(B) \), we assume \( a_1 = 1 \), and then acting by \( g - \text{id} \), we get
\[
\sum (g(a_i) - a_i)e_i = 0
\]
and this has smaller non-zero elements, unless \( a_i \) is fixed by \( g \) for any \( g \in G \), so \( a_i \in \text{Frac}(B)^{G} = B^{G} \) by (H2), contradiction.

**Prop. (VIII.1.3.4) (Galois Extension).** Let \( L \) be a finite Galois extension of \( E \) and \( G = G_{L/E} \), then any f.d. representation in \( \text{Rep}_{E}(G) \) is \( L \)-admissible.

**Proof:** We prove that the morphism \( \alpha_{W} : B \otimes_{B^{G}} W^{G} \to W \) is an isomorphism: it is an isomorphism by (VIII.1.3.3), for the surjectivity, Let \( \lambda_i \) be a basis of \( L \) over \( E \), then by Artin-theoremolineraindetpten, \( \lambda_i : G \to K \) are linearly independent, so there are some \( \mu_i \in K \) that \( \sum \mu_i g(\lambda_i) = 1 \) for \( g = \text{id} \) and 0 otherwise. Then \( \sum \mu_i \text{tr}(\lambda_i x) = x \) for all \( x \in W \), showing surjectivity.

**Prop. (VIII.1.3.5).** If \( B \) is \( G_K \)-regular(VIII.1.3.2), \( V \in \text{Rep}_{E}(G) \) be f.d. and \( W = B \otimes_{E} V \), then the following are equivalent:

- \( W \) is trivial.
- \( \alpha_{W}(\text{VIII.1.3.3}) \) is an isomorphism.
- \( \dim_{B^{G}} W^{G} = \dim_{E} V \).

**Proof:** 1, 2 are equivalent by (VIII.1.3.3), as \( B^{G} \) is a field. Also 2 \( \Rightarrow \) 3 is clear.

3 \( \Rightarrow \) 2: \( \alpha_{W} : B \otimes_{B^{G}} W^{G} \to B \otimes_{E} V \) is a \( B \)-linear morphism of two finite free \( B \)-modules, then it suffices to show the determinant map is an isomorphism. Let \( v_1, \ldots, v_d \) be a \( E \)-basis of \( V \) and \( w_1, \ldots, w_d \) a \( B^{G} \)-basis of \( W^{G} \). Let \( b \) be the unique element of \( B \) that
\[
\alpha_{W}(v_1) \wedge \cdots \wedge \alpha_{W}(v_d) = bw_1 \wedge \cdots \wedge w_d
\]
then \( gb = \eta b \) for \( g \in G \) where \( \eta \) is determined by the identity \( \alpha_{W}(gv_1) \wedge \cdots \wedge \alpha_{W}(gv_d) = \eta \alpha_{W}(v_1) \wedge \cdots \wedge \alpha_{W}(v_d) \). Now the \( E \)-space of \( v_1, \ldots, v_d \) is \( V \), which is stable under \( G \) action, thus \( \eta \in E \), and then by (H3) \( b \in B^{*} \), so we are done.
Cor. (VIII.1.3.6) (Category of Admissible Representations). If $B$ is $G_K$-regular, then the category of $B$-admissible representations are stable under subobjects and quotients.

Proof: Given an exact sequence $0 \to V_1 \to V \to V_2 \to 0$ in the category $\text{Rep}_E(G)$, tensoring $B$ and taking $G$-fixed points, we get an exact sequence

$$0 \to (B \otimes_E V_1)^G \to (B \otimes_E V)^G \to (B \otimes_E V_2)^G$$

from which we derive the inequality $\dim_{B_C}(B \otimes_E V)^G \leq \dim_{B_C}(B \otimes_E V_1)^G + \dim_{B_C}(B \otimes_E V_2)^G$. Now we have $\dim_{B_C}(B \otimes_E V_1)^G \leq \dim_{E_V} V_1$ by (VIII.1.3.3), so

$$\dim_{B_C}(B \otimes_E V)^G \leq \dim_{B_C}(B \otimes_E V_1)^G + \dim_{B_C}(B \otimes_E V_2)^G \leq \dim_{E_V} V_1 + \dim_{E_V} V_2 = \dim_{E_V} V.$$

But this is an equality because $V$ is $B$-admissible, thus $V_1, V_2$ are all $B$-admissible. \qed

4 $\mathbb{C}_p$-Admissibility and Hodge-Tate Representations

Fixed Fields in $\mathbb{C}_p$

Lemma (VIII.1.4.1). Let $F$ be a complete $p$-adic field. If $P(X) \in \overline{F}[X]$ is a monic polynomial of degree $n$, all of its roots satisfied $\text{val}_p(\alpha) \geq c$ for some constant $c$. Let $q = p^k$ if $n = p^k d, d \neq 1$ or $n = p^{k+1}$.

Then the derivative $P^{(q)}(X)$ has a root $\beta$ with $v_p(\beta) \geq c$ or in case $n = p^{k+1}$, $v_p(\beta) \geq c - \frac{1}{p^k(p-1)}$.

Proof: Let $P = X^n + a_{n-1}X^{n-1} + \ldots + a_0$, then $\text{val}_p(a_i) \geq (n-i)c$. And

$$1/q! \cdot P^{(q)}(X) = \sum_{i=0}^{n-1} C_n^q a_{n-i} X^{-i-q}.$$ 

So at lest one root satisfies

$$v_p(\beta) \geq \frac{1}{n-1}((n-q)c - \text{val}_p(C_n^q)) = c - \frac{1}{p^k(p-1)}.$$ 

\qed

Lemma (VIII.1.4.2). If $K$ is a complete $p$-adic field and $\alpha \in \overline{K}$, let $\Delta_K(\alpha) = \inf_{q \in G_K} v_p(g(\alpha) - \alpha)$, then there exists a $\delta \in K$ that $v_p(\alpha - \delta) \geq \Delta_K(\alpha) - p/(p-1)^2$.

Proof: We strengthen the assertion and use induction on $n = [K(\alpha) : K]$ to prove that there is a $\delta$ that $v_p(\alpha - \delta) \geq \Delta_K(\alpha) - \sum_{k=0}^{m} \frac{1}{p^k(p-1)}$, where $p^{m+1}$ is the largest power of $p$ that $\leq n$.

$n = 1$ is sure, let the minipoly of $\alpha$ over $K$ be $P(X)$. By lemma(VIII.1.4.1), there is a root $\beta$ of $P^{(q)}$ that $v_p(\beta - \alpha) \geq v_p(\alpha)$ or minus a factor when $n = p^{k+1}$. Then for any $\sigma$, $v_p(\sigma(\beta) - \beta) \geq v_p(\sigma(\alpha) - \alpha)$ or minus a factor. Then $\Delta(\beta) \geq \Delta(\alpha)$ or minus a factor. Now $[K(\beta) : K] < [K(\alpha) : K] = n$, so we can use induction hypothesis to get the result. \qed

Remark (VIII.1.4.3). The constant $p/(p-1)^2$ can be replaced by $1/(p-1)$, and it is optimal: this is a theorem of Le Borgne in[Bor10].

Prop. (VIII.1.4.4) (Ax-Sen-Tate). If $F$ is a complete $p$-adic field and if $K \subset \overline{F}$, then $\hat{F}^{G_K} = \overline{K}$.

Proof: Any $\alpha \in \overline{F}$ can be written as $\sum \alpha_n$ with $\alpha_n \in \overline{F}$. Then $\Delta_K(\alpha_n) \to \infty$, and $\alpha_n$ can be approximated by $\delta_n \in K$ by lemma(VIII.1.4.2), thus $\alpha \in \overline{K}$. \qed
Cohomology of $G_K$ action on $\mathbb{C}_p(\psi)$

**Def. (VIII.1.4.5) (Notations).** Let $K$ be a $p$-adic number field, $K_\infty$ is an Abelian extension of $K$ that the Galois group $\Gamma$ has a subgroup $\Gamma_0$ of finite index that $\Gamma_0 \cong \mathbb{Z}_p$, and $H_K = G_{K/K_\infty}$. The natural examples is $K_\infty = K(p^\infty)$. Let $\Gamma_m = \Gamma_0^m$ and $K_m$ the fixed field of $\Gamma_m$.

Decompose $\Gamma = \Sigma \times \Gamma_0$ and let $\gamma$ be a topological generator of $\Gamma_0$, then every element of $\Gamma_0$ can be written as $\gamma^t$ for some $t \in \mathbb{Z}_p$. Denote $\gamma_s = \gamma^{po}$.

$\psi : G_K \to \mathbb{Z}_p$ be a character factoring through $\Gamma$, then we can form a representation $\mathbb{C}_p(\psi)$ of $G_K$ on $\mathbb{C}_p$ that $\rho(\gamma)(x) = \psi(\gamma)x$. This is an action because $G_K$ acts trivially on $\mathbb{Z}_p$.

**Lemma (VIII.1.4.6).** Given an $\sigma \in G_{K/\mathbb{Q}_p}$, if $x, y \in \mathfrak{m}_\mathbb{C}_p$ that $x \equiv y$ mod $\pi_K^n$, then $[\pi_K]\sigma(x) \equiv [\pi_K]\sigma(y)$ mod $\pi_K^{n+1}$, where $f^\sigma$ is given by action of $\sigma$ on the coefficients.

**Proof:** This is because the coefficients of $[\pi_K]\sigma$ are divisible by $\pi_K$ except for degree $q$, where it is $x^q - y^q = (x - y)(x^{q-1} + x^{q-2}y + \ldots + y^{q-1})$ which is divisible by $\pi_K^{q-1}$ because the residue field of $K$ is of order $q$.

**Prop. (VIII.1.4.7).** If we let the action of $\sigma \in G(K/\mathbb{Q}_p)$ on the residue field giving by $\sigma : k_K \to \mathbb{F}_p : x \mapsto x^{q^{\sigma}}$, where $q^\sigma = p^{n^\sigma}$ is a $p$-power, given an element $\eta = (\eta_0, \eta_1, \ldots) \in TG$, we have $\eta^{q^\sigma} \equiv [\pi_K]\sigma(\eta^{q^\sigma}_{n+1})$ mod $\pi_K$, hence the above lemma(VIII.1.4.6) shows that $[\pi_K]\sigma\eta^{q^\sigma}_{n+1} \equiv [\pi_K]^{n+1}\sigma(\eta^{q^\sigma}_{n+1})$ mod $\pi_K^{n+1}$, so $[\pi_K]^{n}\sigma(\eta^{q^\sigma}_{n+1})$ is a Cauchy sequence, converging to an element $\mu_\sigma$ (don’t care about $\eta$).

If $g \in G_K$, then $g(\eta_\sigma) = [\chi_K(g)](\eta_\sigma)$, hence take $q^\sigma$-th power, $g(\eta^{q^\sigma}_\sigma) \equiv [\chi_K(g)]\sigma(\eta^{q^\sigma}_\sigma)$ mod $\pi_K$, then

$$\left[\chi_K(g)\right][\pi_K]^{n}\sigma(\eta^{q^\sigma}_\sigma) \equiv [\pi_K]^{n}\sigma g(\eta^{q^\sigma}_\sigma) = g([\pi_K]^{n}\sigma\eta^{q^\sigma}_\sigma) \equiv [\pi_K]^{n}\sigma(\eta^{q^\sigma}_\sigma) \equiv \pi_K.$$ 

hence by limiting, $g(\mu_\sigma) = [\chi_K(g)]\sigma(\mu_\sigma)$.

**Lemma (VIII.1.4.8).**

$$v_p(\mu_\sigma) = \begin{cases} \frac{q^\sigma}{e_k(q-1)} + \frac{1}{e_k} & n(\sigma) \neq 0 \\ \frac{1}{e_k(q-1)} + v_p(\sigma(p_K - \pi_K)) & n(\sigma) = 0 \end{cases}$$

**Proof:** By(V.11.2.22), we know the Newton polygon of $[\pi_K^{n}]\sigma$. When $n(\sigma) \neq 0$, $v(\eta^{q^\sigma}_{n}) = \frac{q^\sigma}{e_k(q-1)} > \frac{1}{e_k(q-1)} + \frac{1}{e_k}$. Now we have by(VIII.1.4.7), we have $[\pi_K]\sigma\eta^{q^\sigma} \equiv [\pi_K]^{2}\sigma(\eta^{q^\sigma}_{n+1})$ mod $\pi_K^{2}$, and $\frac{q^\sigma}{e_k(q-1)} + \frac{1}{e_k} < 2/e_k$, so valuation already stable at degree $1$, and $v(\mu_\sigma) = v([\pi_K]\sigma(\eta^{q^\sigma}_{n+1}))$.

If $q^\sigma = 1$, it’s more delicate, because degree $1$ and degree $q$ term has the same minimal valuation, so they may jump to higher valuations. Notice $[\pi_K^{n}]\sigma(\eta_\sigma) = 0$, so $[\pi_K^{n}]\sigma(\eta_\sigma) = ([\pi_K^{n}]\sigma - [\pi_K^{n}])\eta_\sigma$. And we have by(IV.2.1.18), for $x \in \mathcal{O}_K$, $v(\sigma(x) - x) \geq v(x) + v(\frac{q^\sigma x}{\pi_K}) - 1 + \delta_{q^\sigma}(x)v(\pi_K)$, with equality when $v(x) = q/e_k$. So by the Newton polygon, the minimum valuation of the coefficient of $[\pi_K^{n}]\sigma - [\pi_K^{n}]$ appear at degree $p^{n-1}$ and possibly $p^n$. The valuation of $\eta_\sigma$ is too small($\frac{1}{e_k(p^{n-1})}$) that we don’t need to consider other degrees but can assure that degree $p^{n-1}$ is of minimum valuation, which is $v(\eta^{p^{n-1}}_\sigma) + v(\sigma(p_L - \sigma_L)) = \frac{1}{e_k(q-1)} + v_p(\sigma(p_K - \pi_K))$.

**Prop. (VIII.1.4.9).** For any $\sigma \in G(K/\mathbb{Q}_p)\backslash \{id\}$, there is an element $\alpha_\sigma \in \mathbb{C}_p^*$ that $\alpha_\sigma \circ \chi_K(g) = g(\alpha_\sigma)/\alpha_\sigma$ for all $g \in G_K$, where $\chi_K$ is the Lubin-Tate character.
Proof: We let \( \alpha_\sigma = \log^*_F(\mu_\sigma) \), by(VIII.1.4.8), \( 1/\epsilon_K < \mu_\sigma < \infty \), so by the Newton polygon analysis of \( \log_{F_n}(V.11.2.23) \), \( \alpha_\sigma \) has the same valuation of \( \mu_\sigma \), in particular, \( \alpha_\sigma \neq 0 \). Then

\[
g(\alpha_\sigma) = \log^*_{\ell}(g(\mu_\sigma)) = (\log_{\ell} \circ [h(\gamma)]^\sigma(\mu_\sigma)) = (\chi_K(g) \cdot \log_{\ell})^\sigma(\mu_\sigma) = \sigma(\chi_K(g)) \cdot \alpha_\sigma.
\]

\[\square\]

Cor. (VIII.1.4.10). \( \log_p(\sigma(\chi_K(g))) = g(\log(\alpha_\sigma)) - \log_p(\alpha_\sigma) \).

Prop. (VIII.1.4.11). Now we compute \( H^1(G_K, \mathbb{C}_p(\psi)) \). There is a inf-res exact sequence

\[
0 \to H^1(\Gamma_K, \hat{K}_{\infty}(\psi)) \to H^1(G_K, \mathbb{C}_p(\psi)) \to H^1(H_K, \mathbb{C}_p(\psi))
\]

Then \( H^1(H_K, \mathbb{C}_p(\psi)) = 0 \). The first two vanish if \( \psi \) is of infinite order, and is a \( K \)-vector space of dimension 1 if \( \psi \) is of finite order.

Proof: For the first assertion, \( \psi \) is trivial on \( H_K \), so \( \mathbb{C}_p(\psi) \cong \mathbb{C}_p \) as \( H_K \)-representation, so it suffice to show for \( \psi = \text{id} \). Let \( f \) be a cocycle, as \( H_K \) is compact, \( f(H_K) \in p^{-k}\mathcal{O}_{\mathbb{C}_p} \) for some integer \( k \). So the lemma below(VIII.1.4.12) shows that we can move \( f \) cohomologically to higher valuation, i.e. \( f(g) = \sum x_i - g(\sum x_i) \), so \( f \) is a coboundary.

For the second assertion, we assume \( \Gamma_K \neq \mathbb{Z}_p^* \), for this case, see remark(VIII.1.4.13) below.

Let \( \gamma \) be a topological generator of \( \Gamma_K = 1 + p^k\mathbb{Z}_p^* \), \( k \geq 0 \), so \( \mathbb{Z}_p^* \) are all topological cyclic groups except for \( \mathbb{Z}_p^* \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}_2 \), and \( \gamma_n \) be a topological generator of \( \Gamma_{F_n} \) which is also a power of \( \gamma \). By(IV.3.3.13) we know \( H^1(\Gamma_K, \hat{K}_{\infty}(\psi)) = \hat{K}_{\infty}(\psi)/1 - \gamma \).

For \( n \) large, we have a decomposition \( \hat{K}_{\infty}(\psi) = K_n(\psi) \oplus X_n(\psi) \) by(IV.2.2.25), and \( 1 - \gamma_n = (1 - \gamma)(1 + \gamma + \ldots + \gamma^{k-1}) \), so \( 1 - \gamma \) is also invertible in \( X_n(\psi) \). And on \( K_n(\psi) \), if \( \psi \) is of infinite order, then \( 1 - \gamma \) is injective, otherwise \( x = \psi(\gamma)^n x = \psi(\gamma)^n x \).

So it is also surjective because it is a K-linear mapping of \( K_n \). So \( \hat{K}_{\infty}(\psi)/1 - \gamma = 0 \). If \( \psi \) is of finite order then \( K_n(\psi) \cong K_n \) as \( \Gamma_K \)-module when \( n \) is large enough that \( \gamma \) factors through \( \Gamma_{K_n} \), by(IV.3.3.1). So \( K_n/1 - \gamma = K_n/\text{Ker(tr}_{K_n/K} = K \).

\[\square\]

Lemma (VIII.1.4.12). If \( f: H_K \to p^n\mathcal{O}_{\mathbb{C}_p} \) is a continuous cocycle, then there exists a \( x \in p^{n-1}\mathcal{O}_{\mathbb{C}_p} \) that the cohomologous cocycle \( g \mapsto f(g) - (x - g(x)) \) has values in \( p^{n+1}\mathcal{O}_{\mathbb{C}_p} \).

Proof: \( p^{n+2}\mathcal{O}_{\mathbb{C}_p} \) is open in \( p^n\mathcal{O}_{\mathbb{C}_p} \), so there is a finite extension \( L/K \) that \( f(H_L) \in p^{n+2}\mathcal{O}_{\mathbb{C}_p} \). By(IV.2.2.21), there is a \( z \) that \( \text{tr}_{L_{\infty}/K}(z) = p \), so there is a \( y \in p^{-1}\mathcal{O}_{L_{\infty}} \) that \( \text{tr}_{L_{\infty}/K}(y) = 1 \).

Now for a set of representatives \( Q \) of \( H_K/H_L \), denote \( x_Q = \sum_{h \in Q} h(y)f(h) \), then for \( g \in H_K \), \( g(Q) \) is also a set of representative, and \( x_Q = \sum_{h \in Q} gh(y)g(h) = \sum_{h \in Q} gh(y)(f(gh) - f(g)) = x_{g(Q)} - f(g) \), as \( \text{tr}(y) = 1 \). So \( f(y) - (x_Q - g(x_Q)) = x_{g(Q)} - x_Q \). The RHS is in \( p^{n+1}\mathcal{O}_{\mathbb{C}_p} \), because:

if we let \( gh_i = h_{g(i)}a_i \), where \( a_i \in H_L \), then \( x_{g(Q)} = \sum_{h \in Q} h_{g(i)}(y)f(h_{g(i)}a_i) = \sum_{h \in Q} h_{g(i)}(y)f(h_{g(i)}) = \sum_{h \in Q} h_{g(i)}(y)h_{g(i)}(f(a_i)) \), which is in \( p^{n+1} \) because \( h_{g(i)}(y) \in p^{-1}\mathcal{O}_{\mathbb{C}_p} \) and \( f(a_i) \in p^{n+2}\mathcal{O}_{\mathbb{C}_p} \) by the choice of \( L \).

\[\square\]

Remark (VIII.1.4.13). In case \( \Gamma_K = \mathbb{Z}_2^* \),

\[
0 \to H^1(\{ \pm 1 \}, K(\psi)) \to H^1(\mathbb{Z}_2^*, \hat{K}_{\infty}(\psi)) \to H^1(1 + 2\mathbb{Z}_2^*, \mathbb{C}_p(\psi))
\]

\( H^1(\{ \pm 1 \}, K(\psi)) = 0 \) whether \( \psi(-1) = 1 \) or \(-1 \). And by the same proof as above, possibly replace \( X_n \) with \( X_{n+1} \), to remedy the singularity of \( p = 2 \), \( H^1(1 + 2\mathbb{Z}_2^*, \mathbb{C}_p(\psi)) = K \), with generator \( [g \mapsto \chi(x)] \) for some \( a \). This cocycle extends to a cocycle of \( \mathbb{Z}_2^* \), so the map is surjective.
Prop. (VIII.1.4.14). The 1-dimensional $K$-vector space $H^1(G_K, \mathbb{C}_p)$ is generated by the cocycle $[g \mapsto \log_p \chi(g)]$.

Proof: By the proof of(VIII.1.4.11), we know that $H^1(\Gamma_K, K_n) \overset{f}{\to} H^1(G_K, \mathbb{C}_p)$ is an isomorphism. For any $\alpha \in K$, if $\chi(g) = \gamma^k$, then $f(\alpha)(g) = (1 + \gamma + \ldots + \gamma^{k-1})(\alpha) = k\alpha = \alpha \cdot \log_p(\chi(g))/\log_p(\gamma)$. So by continuity, $f$ is a multiple of $[g \mapsto \log_p(\chi(g))]$.

Lemma (VIII.1.4.15). And $f \in \text{Hom}(I_K^{ab}, \mathbb{Q}_p)$ is of the form $f(g) = \text{tr}_{K/\mathbb{Q}_p}(\beta_f \log_p \chi_K(g))$ for some $\beta_f \in K$.

Proof: By (IV.4.1.28), $\chi_K$ is a canonical isomorphism $I_K^{ab} \cong \mathcal{O}_K^*$. Any $f \in \text{Hom}(\mathcal{O}_K^*, \mathbb{Q}_p)$ is of the form $f(y) = \text{tr}_{K/\mathbb{Q}_p}(\beta_f \log_p(y))$ for some $\beta_f \in K$, because by (IV.2.2.7), when $n$ is large, $\log_p$ is a bijection between $\mathbb{Q}_p$ and $\mathcal{O}_K$.

$\pi_K^* \mathcal{O}_K \to \mathbb{Q}_p$ can be extended to a map $K \to \mathbb{Q}_p$ as $\mathbb{Q}_p$ is divisible. Now trace is an invertible bilinear form on $K$, so the assertion is true on $U_K^n$ for some $n$, and because $U_K^n$ is of finite index in $\mathcal{O}_K^*$ and $\mathbb{Q}_p$ is of char 0, this is true for all $\mathcal{O}_K^*$.

Prop. (VIII.1.4.16). The map $H^1(G_K, \mathbb{Q}_p) \to H^1(G_K, \mathbb{C}_p)$ is given as follows: as $f \in H^1(G_K, \mathbb{Q}_p)$ must factor through $G_K^{ab}$, if the restriction of $f$ to $I_K^{ab}$ corresponds to $\beta_f$, then $f$ maps to $\beta_f [g \mapsto \log_p \chi(g)]$.

Proof: $f(g) = \text{tr}_{K/\mathbb{Q}_p}(\beta_f \log_p \chi_K(g))$ on $I_K$, but this map extends to map on $G_K$. So $f(g) = \text{tr}_{K/\mathbb{Q}_p}(\beta_f \log_p \chi_K(g)) + c(g)$ for an unramified map $c$ on $G_K$.

Now by (IV.3.3.12), $H^1(G, \hat{\mathbb{Q}}_p/\mathbb{Q}_p)$ vanish because $H^1(G, \mathbb{F}_p)$ vanish (IV.3.3.1), so there is a $z \in \hat{\mathbb{Q}}_p$ that $c(g) = g(z) - z$. And

\[
\text{tr}_{K/\mathbb{Q}_p}(\beta_f \log_p \chi_K(g)) = \sum_{\sigma} \sigma(\beta_f \log_p \chi_K(g)) = \beta_f \text{tr}_{K/\mathbb{Q}_p}(\log_p \chi_K(g)) + \sum_{\sigma} (\sigma(\beta_f) - \beta_f) \sigma(\log_p \chi_K(g)).
\]

Notice (VIII.1.4.9) gives a $\beta_\sigma$ such that $\sigma(\log_p \chi_K(g)) = g(\beta_\sigma) - \beta_\sigma$, and $\text{tr}_{K/\mathbb{Q}_p}(\log_p \chi_K(g)) = \log_p \chi(g)$ because $(N_{K/\mathbb{Q}}(\chi_K(g)))^{-1} = (\chi(g))^{-1}$, as they both correspond via local CFT to the element in $G_K^{ab}$ which acts by $g$ on $L_\pi$ and id on $K^{ur}$. Thus the result.

$\mathbb{C}_p$-Admissibility

Prop. (VIII.1.4.17) (Variant of Hilbert’s Theorem90). Any f.d. $\hat{\mathbb{Q}}^{ur}$-semi-linear representation of $G_{K^{ur}/K}$ is trivial.

Proof: Denote by $\mathcal{O}$ the ring of integers of $\hat{\mathbb{Q}}^{ur}$ and $m$ the maximal ideal, Let $W$ be a f.d. $\hat{\mathbb{Q}}^{ur}$-semi-linear representation, $(v_{1,0}, \ldots, v_{d,0})$ a basis of $W$ over $\hat{\mathbb{Q}}^{ur}$ and $\mathcal{O}W$ the $\mathcal{O}$-span of $(v_{1,0}, \ldots, v_{d,0})$, then we are going to construct a sequence of tuples $(v_{i,n}, \ldots, v_{d,n})$ that $v_{i,n+1} \equiv v_{i,n} \mod m^n$ and $\text{Frob}_q(v_{i,n}) \equiv v_{i,n} \mod m^n$ for all $i$ and $n$.

Use induction on $n$: the case $n = 1$ follows from the fact $\mathcal{O}W/m\mathcal{O}W$ is trivial as a $\bar{\mathbb{F}}$-semi-linear representation of $G(k/k)$. To prove this, notice there is a finite extension of $k$ and an $l$-semi-linear representation $W_L$ of $G_{l/k}$ that $\bar{\mathbb{F}} \otimes_l W_L \cong \mathcal{O}W/m\mathcal{O}W$, then the assertion follows from Hilbert’s theorem90 (IV.3.3.7).

For general $n$, we are looking for vectors $w_1, \ldots, w_d \in \mathcal{O}W$ that $\text{Frob}_q(v_{i,n} + \pi^n w_i) \equiv v_{i,n} + \pi^n w_i \mod m^{n+1}$, which is equivalent to $\text{Frob}_q w - w = \frac{\text{Frob}_q v_{i,n} - v_{i,n}}{\pi^n}$ in $\mathcal{O}W/m\mathcal{O}W$. To prove this,
notice \( \text{Frob}_q - \text{id} \) is surjective on \( \mathcal{O}_W / \mathfrak{m}\mathcal{O}_W \), which follows from the fact \( \mathcal{O}_W / \mathfrak{m}\mathcal{O}_W \) is trivial as proved above and \( \text{Frob}_q - \text{id} \) is surjective on \( \bar{k} \).

Now \( v_{i,n} \) are Cauchy sequences and they converge to a tuple \( v_i \) that \( G_{K^{ur}/K} \) acts trivially and it is an \( \mathcal{O} \)-basis of \( \mathcal{O}_W \), as its reduction modulo \( \mathfrak{m} \) is a basis of \( \mathcal{O}_W / \mathfrak{m}\mathcal{O}_W \), so it is a \( \bar{K}^{\text{ur}} \)-basis of \( W \).

**Remark (VIII.1.4.18).** Note that this proposition implies that any unramified representation of \( G_K \) is \( \bar{K}^{\text{ur}} \)-admissible thus \( \mathbb{C}_p \)-admissible, which is a special case of (VIII.1.4.19).

**Prop. (VIII.1.4.19) (\( \mathbb{C}_p \)-Admissibility).** Let \( K \) be a finite extension of \( \mathbb{Q}_p \) and \( G_K = G_{\bar{K}/K} \), then a f.d. \( \mathbb{Q}_p \)-representation of \( G_K \) is \( \mathbb{C}_p \)-admissible iff the inertia group \( I_K \) acts on \( V \) through a finite quotient.

**Proof:** Cf. [p-adic Period Rings Intro, P18].

**Cor. (VIII.1.4.20) (\( H^0 \)).** \( H^0(G_K, \mathbb{C}_p(\psi)) = (\mathbb{C}_p(\psi))^{G_K} = K \) if \( \psi \) is of finite order, and vanish if \( \psi \) is of infinite order.

**Proof:** The cyclotomic extension of \( K \) thus also the cyclotomic character of \( G_K \) is infinitely unramified, thus \( \chi^\psi_{\text{cycl}} \) factors through a finite quotient iff \( s = 0 \). And \( H^0(G_K, \mathbb{C}_p(\psi)) = K \) iff \( \mathbb{Q}_p(\psi) \) is admissible, by Ax-Sen-Tate (VIII.1.4.4), so we are done.

**Cor. (VIII.1.4.21) (Potentially Unramified).** If \( \eta : G_K \to \mathbb{Z}_p^* \) is a character and there is \( y \in \mathbb{C}_p^* \) that \( \eta(g) = g(y)/y \), then there exists a finite Abelian extension \( L \) of \( K \) that \( \eta|_L \) is unramified, i.e. \( \eta \) is potentially unramified.

**Sen’s Theory and Hodge-Tate Representations**

**Remark (VIII.1.4.22).** Sen’s theory is to go further than \( \mathbb{C}_p \)-admissible representations and classify all \( \mathbb{C}_p \)-semi-linear representations.

**Prop. (VIII.1.4.23) (Hilbert’s Theorem 90 for \( G_{\bar{K}/K_\infty} \)).** Any \( W \in \text{Rep}_{\mathbb{C}_p}(G_K) \) is trivial as a \( \mathbb{C}_p \)-semi-linear representation of \( G_{\bar{K}/K_\infty} \). In particular, there is an isomorphism

\[
\mathbb{C}_p \otimes_{\bar{K}_\infty} W^{G_{\bar{K}/K_\infty}} \cong W.
\]

**Proof:** The proof is similar to that of (VIII.1.4.17).

Let \( \mathcal{O}_W \) be any \( \mathcal{O}_{\mathbb{C}_p} \)-lattice in \( W \). Firstly we construct a \( \mathbb{C}_p \)-basis \( w_1, \ldots, w_d \) of \( W \) that \( w_i \in \mathcal{O}_W \) and \( gw_i \equiv w_i \mod p\mathcal{O}_W \) for all \( g \in G_{\bar{K}/K_\infty} \) and \( p\mathcal{O}_W \subset \mathcal{O}_{\mathbb{C}_p} w_1 \oplus \cdots \oplus \mathcal{O}_{\mathbb{C}_p} w_d \).

By continuity there is a finite Galois extension \( L/K \) that \( ? \). Cf. [p-adic Hodge Intro, P23].

**Prop. (VIII.1.4.24).** The next step in Sen’s theory is to study the \( \bar{K}_\infty \)-semi-linear representation of \( \Gamma_K \). To this attempt, Sen considers the subspace of \( \Gamma_K \)-finite vectors in \( W \), they form a vector space over \( K_\infty \).

Then for any \( \bar{K}_\infty \)-semi-linear representation \( W \) of \( \Gamma_K \), there exists an integer \( r \) and a basis \( (v_1, \ldots, v_d) \) of \( W \) that the \( K^r \)-span of \( v_i \) are stable under \( \Gamma_K \)-actions. Obviously, these \( v_i \) are \( G_K \)-finite.

**Proof:** Cf. [p-adic Hodge Intro, P25].
VIII.1. \emph{P-adic Hodge Theory}

\textbf{Cor.} (VIII.1.4.25) \emph{(Sen’s Operator).} Combining the previous two propositions, let \( W \in \text{Rep}_{\mathbb{C}_p}(G_K) \), denote \( \hat{W}_\infty = W^{G_{\infty}}/K_{\infty} \) and \( W_\infty \) the set of finite vectors of \( \hat{W}_\infty \), then \( \mathbb{C}_p \oplus K_{\infty} W_\infty \to W \) is an isomorphism. Let \( v_1, \ldots, v_d \) be given by (VIII.1.4.24), \( W = \oplus K_{s} v_i \). For \( g \in G_K \), let \( \rho_W(g) \) be the endomorphism of \( W_\infty \) given by action of \( g \), then \( \rho_W(\gamma_s) \) is linear when \( s \geq r \), and because \( \gamma_s \) converges to \( \gamma \), \( \log \rho_W(\gamma_s) \) is defined for \( s \) large. Then \emph{Sen’s operator} \( \Phi_W \) is defined to be \( \Phi_W = \lim_{s \to 0} \frac{\log(\gamma_s)}{p^s} \), or equivalently \( \Phi_W(v) = \lim_{t \to 0} g_s^{-r}(v) - v \).

Sen’s operator is defined over \( K \), as it commutes with \( \Gamma_K \) seen from the limit form, and its kernel is the \( \mathbb{C}_p \)-subspace of \( W \) generated by elements invariant under \( \Gamma_K \).

\textit{Proof:} It is evident that fixed points of \( \Gamma_K \) are killed by \( \Phi_W \). Conversely, the kernel of \( \Phi_W \) on \( W_\infty \) is stable under action of \( G_K \), thus is a sub-representation of \( \Gamma_K \) in \( W_\infty \), and because \( W_\infty \) consists of finite vectors, the \( G_K \)-action is continuous w.r.t the discrete topology, and Hilbert’s theorem90 (IV.3.3.7) shows this subspace is generated by elements invariant under \( G_K \).

\textbf{Prop.} (VIII.1.4.26) \emph{(Sen’s Category).} Let \( \text{Sen}(K, K_{\infty}) \) be the category of f.d. \( K_{\infty} \)-vector spaces equipped with an endomorphism defined over \( K \), then the construction sending \( W \) to \( (W_\infty, \Phi_W) \) induces a functor \( \mathcal{S} : \text{Rep}_{\mathbb{C}_p}(G_K) \to \text{Sen}(K, K_{\infty}) \). This functor commutes with direct sums, and also under tensor products, with the Sen’s operator given by \( \Phi_W \otimes W' = \Phi_W \otimes \text{id}_{W'} + \text{id}_W \otimes \Phi_{W'} \).

This functor is faithful, but in general not full. However, it reflects isomorphisms.

\textit{Proof:} This functor is faithful because \( W_\infty \) generates \( W \) as a \( \mathbb{C}_p \)-vector space. To show it reflects isomorphism, let \( f : W_\infty \to W'_\infty \) be an isomorphism commuting with Sen’s operator, then it extends by linearity to an isomorphism of \( \mathbb{C}_p \)-vector spaces \( W \to W' \), and \( f \) is \( \Gamma_s \)-equivariant for some \( s \) by the definition of Sen’s operator. Then considering the space of \( \Gamma_s \)-equivariant \( \mathbb{C}_p \)-linear morphisms from \( W \) to \( W' \), Hilbert’s theorem90 shows there is a basis \( f_i \) consisting of \( G_K \)-equivariant morphisms. Then it remains to show there exists a linear \( K \)-combination of \( f_i \) that is invertible. This is possible because it is true for \( K_s \), as \( f \) is invertible, and \( K \) is an infinite field.

\textbf{Cor.} (VIII.1.4.27). A representation \( W \in \text{Rep}_{\mathbb{C}_p}(G_K) \) is trivial iff \( \Phi_W = 0 \).

\textit{Proof:} As \( \mathcal{S} \) reflects isomorphisms, compare with the trivial representation \( \mathbb{C}_p^{d} \).

\textbf{Def.} (VIII.1.4.28) \emph{(Hodge-Tate Representations).} Let \( B_{HT} = \mathbb{C}_p[t, t^{-1}] \), \( B'_{HT} = \mathbb{C}_p((t)) \), and let \( G_K \) acts on it by \( g( at^i) = g(a) \lambda_{cycl}(g) t^i \). In addition, there is a filtration on \( B'_{HT} \) given by \( \text{Fil}^m B'_{HT} = t^m \mathbb{C}_p[[t]] \), then the graded ring of \( B'_{HT} \) is isomorphic to \( B_{HT} \). (VIII.1.4.20) shows that \( B'_{HT} = B'_{HT} \).

\( B_{HT} \) and \( B'_{HT} \) are \( G_K \)-regular (VIII.1.3.2), and a \( \mathbb{Q}_p \)-linear representation of \( G_K \) is \( B_{HT} \)-admissible iff it is \( B'_{HT} \)-admissible iff \( \mathbb{C}_p \otimes_{\mathbb{Q}_p} V \) decomposes as \( \mathbb{C}_p \otimes_{\mathbb{Q}_p} V \cong \mathbb{C}_p(\lambda_{cycl}^{n_1}) \oplus \ldots \oplus \mathbb{C}_p(\lambda_{cycl}^{n_d}) \).

And such a representation is called a \emph{Hodge-Tate representation}.

\textit{Proof:} \( B'_{HT} \) is \( G_K \)-regular because it is a field. For \( B_{HT} \), \( B_{HT} \subset \text{Frac}(B_{HT}) \subset B'_{HT} \), taking \( G_K \)-fixed points shows (H2). For (H3), if \( \mathbb{Q}_p x \) is stable under \( G_K \) and \( x \) is not of the form \( at^i \), then we can get a non-trivial \( G_K \)-fixed point of \( \mathbb{C}_p(\lambda_{cycl}^{j-1}) \), which is impossible by (VIII.1.4.20).
Notice $B_{HT} \cong \oplus_{m \in \mathbb{Z}} \mathbb{C}_p(\chi_{cycl}^m)$ as $\mathbb{C}_p$-semi-linear representations, then

$$(V \otimes_{\mathbb{Q}_p} B_{HT})^{G_K} \cong \oplus_{m \in \mathbb{Z}} (V \otimes \mathbb{C}_p(\chi_{cycl}^m))^{G_K}.$$ 

Now if $V$ is Hodge-Tate, then clearly $(V \otimes_{\mathbb{Q}_p} B_{HT})^{G_K}$ has dimension $\dim_{\mathbb{Q}_p}(V)$ thus $V$ is $B_{HT}$-admissible. Conversely, $(V \otimes_{\mathbb{Q}_p} B_{HT})^{G_K}$ has dimension $\dim_{\mathbb{Q}_p}(V)$ implies there are $d \mathbb{C}_p(\chi_{cycl}^{-m})$ contained in $V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$, and they are non-intersecting by (VIII.1.4.29), thus $V$ is Hodge-Tate. □

Prop. (VIII.1.4.29). For any two integers $n, m$,

$$\text{Hom}_{\text{Rep}_{\mathbb{C}_p}(G_K)}(\mathbb{C}_p(\chi_{cycl}^n), \mathbb{C}_p(\chi_{cycl}^m))$$

is of one-dimensional if $n = m$, and vanishes otherwise.

This implies the decomposition for a Hodge-Tate representation (VIII.1.4.28) is uniquely determined up to a permutation.

Proof: Let $W = \text{Hom}_{\mathbb{C}_p}(\mathbb{C}_p(\chi_{cycl}^n), \mathbb{C}_p(\chi_{cycl}^m)) = \mathbb{C}_p(\chi_{cycl}^{-n})$, then the desired space is $W^{G_K}$, and the assertion follows from (VIII.1.4.20). □

Prop. (VIII.1.4.30) (Sen’s Operator and Hodge-Tate Representations). A representation $V \in \text{Rep}_{\mathbb{Q}_p}(G_K)$ is Hodge-Tate iff the Sen’s operator $\Phi_{\mathbb{C}_p \otimes_{\mathbb{Q}_p} V}$ is semisimple with eigenvalues in $\mathbb{Z}$. For a general $V$, the eigenvalue of $\Phi_{\mathbb{C}_p \otimes_{\mathbb{Q}_p} V}$ is called the generalized Hodge-Tate weights of $V$.

Proof: If $V$ is a Hodge-Tate representation, then clearly $\Phi_W(v) = \lim_{t \to 0} \frac{\gamma(t) - v}{t}$ acts by $k$ on $\mathbb{C}_p(\chi_{cycl}^k)$, thus it is semisimple with eigenvalues in $\mathbb{Z}$. Conversely, on the $i$-eigenspace of $\Phi$, tensoring $\chi_{cycl}^{-i}$, $\Phi$ acts trivially, then because the kernel of $\Phi$ are the fixed points of $\Gamma_K$ (VIII.1.4.25), thus this eigenspace is isomorphic to $\mathbb{C}_p(\chi_{cycl}^i)^d$, so $V$ is Hodge-Tate. □

5 Crystalline and deRham Representations

Lemma (VIII.1.5.1). Let $V$ be a finite $\mathbb{Q}_p$-vector space with an action of $G_K$, then the action is continuous iff the induced action on $V \otimes_{\mathbb{Q}_p} B_{dR}^+$ is continuous.

Proof: This is because the action of $G_K$ on $B_{dR}^+$ is continuous, and $V$ has the induced topology in $V \otimes_{\mathbb{Q}_p} B_{dR}^+$. □

Prop. (VIII.1.5.2).

\begin{itemize}
  \item $K = B_{G_K}^+$
  \item $K_0 = B_{crys}^{G_K}$, and the canonical morphism $K \otimes_{K_0} B_{crys} \to B_{dR}$ is injective.
  \item $\mathbb{Q}_p = B_{e}^{G_K}$.
\end{itemize}

Proof: 1: Use the filtration $t^{-n}B_{dR}^+$ on $B_{dR}$, then the graded are just $\mathbb{C}_p(\chi_{cycl})$, the $p$-adic closure twisted by $\chi_{cycl}$-action. Then by (VIII.1.4.20) $H^0(G_K, \mathbb{C}_p(n)) = K$ iff $n = 0$ because $\chi_{cycl}^n$ has finite order iff $n = 0$. Hence induction this implies $K = B_{G_K}^+$.

2: The injectivity of $K \otimes_{K_0} B_{crys} \to B_{dR}$ [Laurent Fargues and Jean-Marc Fontaine Prop10.2.8].

3: From 2 and notice $B_{e} = B_{crys}^{e=\text{id}}$. □
Def. (VIII.1.5.3) (Admissible Representations). Let $K$ be a finite extension of $\mathbb{Q}_p$ with residue field $k$, $K_0 = W(k)[\frac{1}{p}]$ its maximal unramified extension. Let $B$ be a $\mathbb{Q}_p$-algebra that $G_K$ acts continuously that $F = B^{G_K}$ is a field. Given a finite $G_K$-module $V$, consider $D_B(V) = (V \otimes \mathbb{Q}_p B)^{G_K}$, where $G_K$ acts diagonally, then $V$ is called $B$-admissible iff $\dim_F D_B(V) = \dim_{\mathbb{Q}_p} V$, equivalently,

$$D_B(V) \otimes_F B \to V \otimes_{\mathbb{Q}_p} B$$

is an isomorphism.

In particular, if $B = B_{dR}$ or $B_{crys}$, then the representation is called deRham or Crystalline. The category of deRham or Crystalline representations of $G$ are denoted by $\text{Rep}_{dR}^G(G)$ or $\text{Rep}_{crys}^G(G)$.

Def. (VIII.1.5.4) ($\text{Rep}_{B_c}^G(K)$. Denote by $\text{Rep}_{B_c}^G(K)$ the category of finite locally free $B_c$-modules $M$ with a semilinear $G_K$-action that there exists a $G_K$-invariant $B_{dR}^+$-lattice $\Gamma \in M \otimes_{B_c} B_{dR}$ that $G_K$ acts continuously.

Prop. (VIII.1.5.5) (Crystalline Representation). The functor

$$\mathcal{D} : \text{Rep}_{B_c}^G K \to \varphi - \text{Mod}_{K_0} : W \mapsto (W \otimes_{B_c} B_{crys})^{G_K}$$

are left adjoint to the functor

$$\mathcal{V} : \varphi - \text{Mod}_{K_0} \to \text{Rep}_{B_c}^G K : (D, \varphi_D) \mapsto (D \otimes_{K_0} B_{crys})^{\varphi_D \otimes \varphi = \text{id}}$$

Moreover, $\mathcal{V}$ is fully faithful, $\text{id} \cong \mathcal{D} \circ \mathcal{V}$, $\mathcal{V} \circ \mathcal{D} \to \text{id}$, and $M \in \text{Rep}_{B_c}^G(K)$ is in the image of $\mathcal{V}$ iff $\mathcal{V}(\mathcal{D}(M)) \cong M$.

Proof: Cf.[Laurent Fargues and Jean-Marc Fontaine Prop.10.2.12]. □

Cor. (VIII.1.5.6). In particular, a $B_c$-representation is crystalline if it is in the image of $\mathcal{V}$. Now we define a Vector bundle $\mathcal{E}$ on $X$ to be crystalline if $H^0(X - \{\infty\}, \mathcal{E})$ is crystalline.

Lemma (VIII.1.5.7). Let $W$ be a f.d. $K$-vector space, then the map:

$$\{\text{Filtrations on } W\} \to \{G_K\text{-stable } B_{dR}^+\text{-lattice in } W \otimes_K B_{dR}\} : \text{Fil} \mapsto \text{Fil}^0(W \otimes_K B_{dR})$$

is bijective and the inverse is given by $\Gamma \mapsto \{(t^{\alpha} \Gamma)^{G_K} \subset (B_{dR} \otimes_{B_c} \Gamma)^{G_K} = W\}_{n \in \mathbb{Z}}$.

Proof: Cf.[Laurent Fargues and Jean-Marc Fontaine Prop.10.4.3]. □

Comparison Theorems

Prop. (VIII.1.5.8) ($C_{dR}$-Theorem). Let $X$ be a proper smooth variety over $K$, then for any $r$, there exists a canonical isomorphism

$$\gamma_{dR}(X) : B_{dR} \otimes_K H_{dR}^r(X) \cong B_{dR} \otimes_{\mathbb{Q}_p} H_{\text{ét}}^r(X_{\overline{K}}, \mathbb{Q}_p).$$

which respect filtrations and $G_K$-actions on both sides. Moreover $\gamma_{dR}$ is functorial in $X$.

Cor. (VIII.1.5.9). The $\mathbb{Q}_p$-representation $H_{\text{ét}}^r(X_{\overline{K}}, \mathbb{Q}_p)$ is de Rham, and

$$H_{dR}^r(X) \cong (B_{dR} \otimes_{\mathbb{Q}_p} H_{\text{ét}}^r(X_{\overline{K}}, \mathbb{Q}_p))^{G_K}.$$
Conjecture (VIII.1.5.10) (Fontaine-Mazur). The $C_{dR}$-theorem (VIII.1.5.8) implies any representation of $G_K$ of the form $H^r_{dR}(X_{\mathbb{F}_p}, \mathbb{Q}_p)$ is de Rham, thus it is natural to consider the converse.

Let $F$ be a number field and $V$ a f.d. $\mathbb{Q}_p$-representation of $G_F$ that satisfies:
- For almost all prime ideal $p$ of $\mathcal{O}_F$, the representation is unramified at $p$.
- For all primes above $p$, the representation $V|_{G_{F_p}}$ is de Rham.

Then $V$ appears as a subquotient of some $H^r_{dR}(X_{\mathbb{F}_p}, \mathbb{Q}_p)(\chi^{m}_{cycl})$ where $X$ is a proper smooth variety over $\text{Spec } F$.

Proof: ?

Emerton and Kisin proved the two-dimensional case, Cf.[The Fontaine-Mazur conjecture for GL2, Kisin].

Prop. (VIII.1.5.11). $B_{HT}$ is identified with the graded ring of $B_{dR}$, so any deRham representation of $G$ is Hodge Tate. ?

6 $l$-adic representations

Prop. (VIII.1.6.1). Every continuous representation of $G_K$ on a $\mathbb{Q}_l$ vector space (Continuous group morphism to $GL_n(\mathbb{Q}_l)$) has a $\mathbb{Z}_l$ lattice stable under the action.

So the functor $\rho \rightarrow \rho \otimes \mathbb{Q}_l$ from $\text{Rep}_{\mathbb{Z}_l}(G_K)$ to the Tannakian natural category $\text{Rep}_{\mathbb{Q}_l}(G_K)$ is essentially surjective.

Proof: Notice that the stabilizer of the standard lattice is $GL_n(\mathbb{Z}_l)$ which is open and so the inverse image has a finite coset. And the image of the wild ramification group is finite because it is in $GL_n(\mathbb{F}_l)$. ?

Prop. (VIII.1.6.2) (Grothendieck Monodromy theorem). For a local field $K$, the étale representation and the Tate module are all potentially semisimple. i.e. semisimple for a finite extension.

7 $p$-Adic L-Functions

Main references are [Fontaine’s rings and p-adic L-functions, Colmez].

Prop. (VIII.1.7.1) (Kummer’s Theorem). Let $a \geq b$ be coprime to $p$ and $k \geq 1$. If $n_1, n_2 \geq k$ that $n_1 \equiv n_2 \mod \varphi(p)$, then

$$(1 - a^{n_1}) \frac{B_{n_1}}{n_1} \equiv (1 - a^{n_2}) \frac{B_{n_2}}{n_2} \mod p^k.$$ 

Proof: Cf.[p-adic L-functions, Colmez]P5. ?

Remark (VIII.1.7.2). This has vast generalizations in Iwasawa theory, Cf.[Iwasawa, On p-adic L-functions].
VIII.2 p-Adic Period Domains

1 Isocrystals with Additional Structures

Main references are [Period Spaces for p-Divisible Groups, Rapoport/Zink] and [Isocrystals with Additional Structures, Kottwitz].

Def. (VIII.2.1.1) (Isocrystals with G-structures over $K_0$). An isocrystal with $G$-structures over $K_0$ is an exact faithful $\otimes$-functor $\text{Rep}_{Q_p}(G) \to \varphi - \text{Mod}_{K_0}$.

Prop. (VIII.2.1.2) (Associated Isocrystals). Let $L$ be a perfect field of char $p$ and $K_0 = W(L)[\frac{1}{p}]$ and let $b \in G(K_0)$, then to every $Q_p$-linear representation $V$ of $G$, we can associate an isocrystal

$$\text{Rep}_{Q_p}(G) \to \varphi - \text{Mod}_{K_0} : V \mapsto (V \otimes_{Q_p} K_0, b \circ (\text{id} \otimes \sigma)).$$

this is an isocrystal with $G$-structures over $K_0$ associated to $b$.

If $g \in G(K_0)$ and $b' = gb\sigma(g)^{-1}$, then multiplying by $g$ implies a natural isomorphism between $T_b$ and $T_{b'}$.

Prop. (VIII.2.1.3) (Associated Filtered Isocrystals). Let $K$ be a field extension of $K_0$, $G$ be an algebraic group over $Q_p$, and $\mu : \mathbb{G}_{m,K} \to G_K$ be a cocharacter over $K$, then the associated isocrystal over $K_0$ upgrades to a filtered isocrystal over $K$ (VIII.1.2.14),

$$\text{Rep}_{Q_p}(G) \to \varphi - \text{FilMod}_K \mathcal{I} : V \mapsto (V \otimes_{Q_p} K_0, b \circ (\text{id} \otimes \sigma), \text{Fil}^*_b),$$

where the filtration comes from $\mu$ by weight-filtrations($\mathbb{G}_m$ is diagonalizable(V.9.11.1)).

Def. (VIII.2.1.4) (Admissible Pair). Let $G$ be a reductive group, then a pair $(\mu, b)$ in (VIII.2.1.3) is called a (weakly)admissible pair if for any $Q_p$-representation of $G$, the filtered isocrystal $\mathcal{I}(V)$ is (weakly)admissible??(VIII.1.5.3).

It suffices to check this condition for any faithful representation $V$.

Proof: This is because for a faithful representation $V$, any $Q_p$-representation appears as a direct summand of $V^\otimes m \otimes \hat{V}^\otimes m$ (V.9.7.5). Then the assertion follows from the fact direct summands and tensor products of (weakly)admissible filtered isocrystals is (weakly)admissible.

Prop. (VIII.2.1.5) (Slope Morphism). Let $\mathbb{D} = \text{Spec} Q_p[\{T^{1/k}\}_{k \in \mathbb{Z}}] = \mathbb{D}(Q)_{Q_p}$ be the pro-algebraic torus over $Q_p$ with character group $Q$, and $b \in G(K_0)$, then there is a morphism $\nu : D_{K_0} \to G_{K_0}$, called the slope morphism associated to $b$, which is defined as follows:

For any f.d. representation $\rho : G \to GL(V)$, there is an associated isocrystal defined in (VIII.2.1.2), then there is a morphism $\nu_{\rho} \in \text{Hom}_L(\mathbb{D}, GL(V))$ that $\mathbb{D}$ acts on the isotypical component $V_\lambda$ of $V$ by the character $\lambda \in Q = X^*(\mathbb{D})$. Then for any $x \in G(R)$, the mapping $\rho \mapsto \mu_{\rho}$ gives an automorphism of the standard fiber functor on $\text{Rep}(G)$, so by Tannakian duality corresponds to a unique element $y \in G(R)$ that $\rho(y) = \mu_{\rho}(x)$ for any $\rho$. The homomorphism $x \mapsto y$ is functorial in $R$ and thus defines an element $\nu \in \text{Hom}_L(\mathbb{D}, G)$.

Notice the group $Q^*$ acts on $\mathbb{D}$, and for $s \in Q^*$ and $v \in \text{Hom}(\mathbb{D}, G)$, denote by $sv$ the composite $\mathbb{D} \overset{s}{\to} \mathbb{D} \overset{v}{\to} G$, and $D \to \mathbb{G}_m$ the natural morphism, then for any $v$, there is a suitable $s$ that $sv$ factors through a morphism also denoted by $sv : \mathbb{G}_{m,K_0} \to G_{K_0}$, as $G$ is algebraic.

Prop. (VIII.2.1.6) (Characterizing Slope Morphism). The slope morphism can be characterized intrinsically to be the unique morphism $\nu \in \text{Hom}_L(\mathbb{D}, G)$ that there exists some $s > 0, c \in G(L)$ that
• $s\mu \in \text{Hom}_L(\mathbb{G}_m, G)$,
• $c(s\mu)c^{-1}$ is defined over $\mathbb{Q}_p$.
• $c(b\sigma)^n c^{-1} = c(n\nu)(p)c^{-1}\sigma^n$.

Proof: Cf.[Kottwitz, P13]. □

Cor. (VIII.2.1.7) (Conjugate of Slope Morphism). Now we can define the $\sigma$-conjugate of the slope morphism $\nu$, and we have the identity

$$b\nu^\sigma b^{-1} = \nu$$

To check this, replace $\nu$ by some $s\nu$ to assume $\nu$ factors through $\mathbb{G}_m$, then it suffices to show for any $a \in K_0^*$,

$$b\sigma(s\nu(s^{-1}(a))) = s\nu(a)b\sigma$$

It suffices to check for any $G$-representation $\mathbb{Q}$, and it is true as $(V \otimes_{\mathbb{Q}_p} K_0, b\sigma)$ is a $\varphi$-module for $\sigma$ and $s\nu(a)$ acts on the isotypical part of slope $r$ by $a^r$.

More generally enerally, any $g \in G(L)$ commuting with $b\sigma$ also commutes with $\varphi(a), a \in K_0^*$, as it preserves the isotypical decomposition for any isocrystal, and on the isotypical component $V_\lambda$, the $a$ acts by $a^r$.

Def. (VIII.2.1.8) (Descent Conditions). A $\sigma$-conjugacy class $\tilde{b}$ in $G(\mathbb{K})$ is called a descent if there is some $s \geq 0$ and some $b \in \tilde{b}$ that $s\nu$ factors through $D \to \mathbb{G}_m$ and

$$(b\sigma)^s = s\nu(p)\sigma^s$$

as an identity in $G(K_0) \rtimes \langle \sigma \rangle$.

Prop. (VIII.2.1.9). If $G$ is connected and $L$ is alg.closed, then any $\sigma$-conjugacy class is descent (VIII.2.1.8).

Proof: Cf.[Kottwitz]. □

Prop. (VIII.2.1.10). Let $\tilde{b}$ be a descent and $b$ satisfies the descent condition for $s$ in(VIII.2.1.8), then if $\mathbb{Q}_p^s = W(F_p^s)[\frac{1}{p}], b \in G(\mathbb{Q}_p^s)$ and $\nu$ is defined over $\mathbb{Q}_p^s$.

Proof: Set $b_s = b\sigma(b)\ldots\sigma^{s-1}(b)$, then iterating(VIII.2.1.7), $b_s\nu\sigma^s b_s^{-1} = \nu$. And we have $b_s = s\nu(p)$, so $\nu\sigma^s = \nu$, so $\nu$ is defined over $\mathbb{Q}_p^s$.

To show the first assertion, notice $(b\sigma)(b\sigma)^s = (b\sigma)^s(b\sigma)$ shows

$$s\nu(p)\sigma^s b\sigma = b\sigma s\nu(p)\sigma^s = s\nu(p)b\sigma^{s+1}(VIII.2.1.7).$$

and then $b\sigma^s = \sigma^s b$. □

Cor. (VIII.2.1.11). If $b_1, b_2 \in \tilde{b}$ are descent w.r.t the same $s$, then they are conjugate w.r.t. $G(K_0 \cap \mathbb{Q}_p^s)$.

In particular, for any descent $b \in G(\mathbb{Q}_p^s)$ and any $\mathbb{Q}_p$-representation $V$ of $G$, the induced isocrystal is defined over the field $K_s = W(F_p^s \cap L)[\frac{1}{p}], and it only depends on $\tilde{b}$ up to isomorphism.
Proof: Suppose $b_2 = gb_1\sigma(g)^{-1}$, then $\nu_2 = g\nu_1g^{-1}$, and the descent equations are

$$(b_1\sigma)^2 = s\nu_1(p)\sigma^s, \quad g(b_1\sigma)^s g^{-1} = gs\nu_1(p)g^{-1}\sigma^s.$$ \hfill \square

Comparing these two, $g$ commutes with $\sigma^s$, so $g \in G(K_0 \cap \mathbb{Q}_p^*)$.

**Prop. (VIII.2.1.12).** Let $b \in G(K_0)$, then the following functor on the category of $\mathbb{Q}_p$-algebras is representable by a smooth affine group scheme:

$J(R) = \{g \in G(R \otimes_{\mathbb{Q}_p} K_0)|g(b\sigma) = (b\sigma)g\}.$

Moreover, if $b \in G(W(L)[1/p])$ where $L$ is an alg. closed subfield of $L$, and $J'$ be the corresponding functor defined with $L'$, then the canonical morphism $J' \to J$ is an isomorphism.

**Proof:** Choose an embedding $G \subset GL(V, \mathbb{Q}_p)$, consider the functor:

$$F(R) = \{g \in \text{End}(V_{K_0}) \otimes R|g = b\sigma(g)b^{-1}\},$$

then it is representable by an affine space by the lemma (VIII.2.1.13) applied to the $\sigma$-linear map $g \mapsto b\sigma(g)b^{-1}$.

More precisely, there is a f.d. $\mathbb{Q}_p$-vector space $W \subset \text{End}(V_{K_0})$ that $F(R) = W \otimes_{\mathbb{Q}_p} R$. Choose a basis $A_i$ of $W$, then $J(R)$ is just the subfunctor of $R$ that $f_k(\sum r_iA_i) = 0$, and $\det(\sum r_iA_i) \neq 0$. Taking a basis of $K_0$ over $\mathbb{Q}_p$, then these are polynomials with coefficients in $\mathbb{Q}_p$. It is automatically smooth by Cartier’s theorem (V.9.4.3).

The last assertion follows from the proof of (VIII.2.1.13). \hfill \square

**Lemma (VIII.2.1.13).** Let $N$ be a f.d. isocrystal over $K_0 = W(L)[1/p]$ w.r.t. $\sigma^s$ for some $s \neq 0$, then the following functor on the category of $\mathbb{Q}_p$-algebras

$$F(R) = \{n \in N \otimes_{\mathbb{Q}_p} R|\varphi(n) = n\}$$

is representable by an affine space over $\mathbb{Q}_p$.

**Proof:** $F(R)$ is just $N^\varphi \otimes_{\mathbb{Q}_p} R$, so it suffices to show $\text{dim}_{\mathbb{Q}_p} N^\varphi < \infty$. Firstly assume that $L$ is alg. closed, then this is a consequence of Dieudonné-Manin classification (VIII.1.2.10). This functor $F$ doesn’t depend on $L$ once $L$ reaches its alg. closure: if $L$ is alg. closed and $L'$ is a field extension, then the corresponding functor $F'$ defined by $N \otimes_{W(L)[1/p]} W(L')[1/p]$ coincide with $F$. (This is also by Dieudonné-Manin classification.) \hfill \square

**Cor. (VIII.2.1.14).** Assume $b$ satisfies a descent condition for $s$ (VIII.2.1.8), then $J$ is a $\mathbb{Q}_p^*/\mathbb{Q}_p^*$-inner form of the centralizer $G_{s\nu(p)}$ (VIII.2.1.10).

**Proof:** The descent equation shows $b_2 = s\nu(p)$, so the adjoint $b_{ad} : g \mapsto (b\sigma)g(b\sigma)^{-1} = b\sigma(g)b^{-1}$ defines an element in $H^1(G(\mathbb{Q}_p^*/\mathbb{Q}_p^*), \text{Aut}(G_{s\nu(p)}(\mathbb{Q}_p^*))))$, because

$$\sigma^kb_{ad} : g \mapsto \sigma(b\sigma^{-1}(g)b^{-1}) = \sigma^k(b)\sigma(g)b^k(b)^{-1}.$$ \hfill \smash{\square}

So

$$b_{ad} \circ \sigma(b_{ad}) \circ \ldots \circ \sigma^{-1}(b_{ad}) : g \mapsto b_sg^{-1} \sigma^s(g) = s\nu(p)g(s\nu(p))^{-1} = g.$$ \hfill \smash{\square}

So it defines an inner form, which is just

$$J'(R) = G_{s\nu(p)}(\mathbb{Q}_p^*)^{b_{ad}\sigma} = \{g \in G_{s\nu(p)}(R \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^*)|g(b\sigma) = (b\sigma)g\}$$

Now it suffices to show $J'(R)$ is just $J(R)$ defined in (VIII.2.1.12). For this, notice any $g \in J(R)$ commutes with $b\sigma$ thus commutes with $s\nu(p)$ by (VIII.2.1.7), and the descent condition $(b\sigma)^n = s\nu(p)^n$ shows it commutes with $\sigma^n$, so $g \in J'(R)$. \hfill \square
Prop. (VIII.2.1.15). Let $G$ be a connected reductive group and $L$ be alg.closed, then the following are equivalent for $b \in G(K_0)$:
- The slope morphism $\nu$ factors through the center of $G$.
- $b$ is $\sigma$-conjugate to an element in $T(K_0)$ where $T$ is an elliptic maximal torus of $G$.
- The algebraic group $J$ o(VIII.2.1.12) is an inner form on $G$.

In this case, $b$ and its conjugacy class $b$ are called basic.

Proof: Cf.

Prop. (VIII.2.1.16) (Conjugacy Classes and Base Change). Let $b_1, b_2$ be two elements of $G(W(L)[\frac{1}{p}])$, then the functor

$$J(R) = \{ g \in G(R \otimes_{Q_p} W(L)[\frac{1}{p}]) | g(b_1 \sigma) = (b_2 \sigma)g \}$$

is representable by a smooth affine scheme over $Q_p$.

Assume $b_1, b_2 \in G(W(L')[\frac{1}{p}])$ where $L'$ where $L'$ is an alg.closed field of $L$, and $J'$ the corresponding functor, then $J' \to J$ is an isomorphism. In particular, the map from the set of $\sigma$-conjugacy classes in $G(W(L')[\frac{1}{p}])$ to the set of $\sigma$-conjugacy classes in $G(W(L)[\frac{1}{p}])$ is injective, and it is surjective iff $L$ is also alg.closed and $G$ is connected.

Proof: The surjectivity follows from the fact that every conjugacy class is descent (VIII.2.19), and those descent elements are in $G(Q_s)$ for some $s \geq 0$ (VIII.2.10), so in $G(W(L')[\frac{1}{p}]).$

§ 2 Period Domain

Def. (VIII.2.2.1) (Associated Partial Flag Variety). Let $G$ be an algebraic group over $Q_p$ and $\mu: \mathbb{G}_m \to G$ is a conjugacy class of cocharacters defined over a finite extension field $E \gg Q_p$(V.9.9.5), then there is associated a faithful $\otimes$-functor

$$\text{Rep}_{Q_p} \to \mathbb{Z}\text{-graded } R\text{-vector spaces} \to \text{filtered } E\text{-spaces}$$

Now call two cocharacters equivalent if their associated functor are isomorphic. Consider the functor

$$R \mapsto \{ \text{the equivalence classes in the conjugacy class of } \mu_R \text{ under } G(R) \}$$

in the category of $E$-algebras, and also consider the closed algebraic subgroup $P(\mu) \subset G$ over $E$:

$$P(\mu)(R) = \{ g \in G(R) | g \mu_R g^{-1} \text{ is equivalent to } \mu_R \}$$

then the functor above is representable by the homogenous variety $\mathcal{F} = G_E/P(\mu)$ defined over $E$.

Prop. (VIII.2.2.2). $\mathcal{F}$ is a projective variety.

Proof: If $V$ is a faithful representation in $\text{Rep}_{Q_p}(G)$, we denote $\text{Flag}(V)$ the partial flag variety over $Q_p$ which associates to any $Q_p$-algebra $R$ the filtration $\text{Fil}^\bullet$ of $V \otimes_{Q_p} R$ s.t. $\text{gr}^i(R)$ are direct summands and $\text{rank } \text{Fil}^i = \dim E \text{Fil}^i(\text{V}_E)$. Then $\text{Flag}(V)$ is a projective variety, by classical results, and there is a closed immersion

$$\mathcal{F} \hookrightarrow \text{Flag}(V)_E$$

because the isocrystal on other representations are determined by this faithful representation. □
Def. (VIII.2.2.3) (*-adic Period Space). Let \( \tilde{E} = E K_0(\mathcal{F}_p) \) be the completion of the maximal unramified extension of \( E \), then there is a rigid-analytic structure on \( \mathcal{F} = \mathcal{F}_{\tilde{E}} \). Define the \(*\)-adic period space \((\mathcal{F}_b^{wa})^{rig} \subset \mathcal{F}^{rig}\) associated to \((G, b{\mu})\) the set of points \( \xi \) conjugate to \( \mu \) that \((\xi, b)\) is weakly admissible.

Let \( J_b \) be the algebraic group associated to \( b \) as in (VIII.2.1.12), then \( J_b(\mathbb{Q}_p) \subset G(K_0) \) acts on \( \mathcal{F}^{rig} \), and it preserves the set \((\mathcal{F}_b^{wa})^{rig}\).

\((\mathcal{F}_b^{wa})^{rig}\) has a natural structure of an admissible open subset of \( \mathcal{F}^{rig} \). If \( b' = gb\sigma(g)^{-1} \), then \( \mu \mapsto g^{-1} \mu g \) induces an isomorphism from \((\mathcal{F}_b^{wa})^{rig}\) to \((\mathcal{F}_{b'}^{wa})^{rig}\). Moreover, if \( b \) satisfies descent condition w.r.t. \( s > 0 \), then this admissible open subset is defined over \( E \mathbb{Q}_p^s \).

Proof: Cf. [Rapoport Zink, P26]. \( \square \)

3 Algebraic Groups of EL/PEL Types

Def. (VIII.2.3.1) (Algebraic Groups of EL/PEL Types). Let \( F \) be a finite étale algebra over \( \mathbb{Q}_p \), \( B \) a finite central algebra over \( F \), and \( V \) is a f.g. \( B \)-module.

An algebraic group of EL type over \( \mathbb{Q}_p \) is an algebraic group of the form \( GL_B(V) \). They are related to the classification of \( p \)-divisible groups with an endomorphism and level structures.

Let \((-, -)\) be a non-degenerate alternating \( \mathbb{Q}_p \)-bilinear form on \( V \) together with a formal involution \( \ast \) on \( B \) that

\[
(bv, w) = (v, b^\ast w).
\]

Let \( F_0 \) be the field of elements of \( F \) fixed by \( \ast \).

An algebraic group of PEL type over \( \mathbb{Q}_p \) is an algebraic group over \( \mathbb{Q}_p \) given by

\[
G(R) = \{ g \in GL_B(V \otimes_{\mathbb{Q}_p} R) | \exists c \in X(G), \ (gv, gw) = c(g)(v, w), \ \forall v, w \}
\]

Prop. (VIII.2.3.2) (Setups). If \( G \) is an algebraic group of EL/PEL type, \( K_0 = W(\mathbb{F}_p)[\frac{1}{p}] \), \( b \in G(K_0) \), then we associate to \( b \) and the natural representation of \( G \) on \( V \) the isocrystal

\[
(N(V), \Phi) = (V \otimes_{\mathbb{Q}_p} K_0, b(1 \otimes \sigma)).
\]

This isocrystal is equipped with an action of \( B \), and in the PEL case an alternating bilinear form

\[
\psi : N(V) \otimes N(V) \to 1(n).
\]

where \( n = v_p(c(b)) \). In fact, we can find some unit \( u \) that \( c(b) = p^n u \sigma(u)^{-1} \), then the pairing is defined as

\[
\psi(v, v') = u^{-1}(v, v'),
\]

any other choices of \( u \) multiplies \( \psi \) by an element in \( \mathbb{Z}_p^\ast \).

We will fix in addition a conjugacy class of cocharacters \( \mu : \mathbb{G}_m \to G \) defined over a field \( E \), and the associated homogenous algebraic variety \( \mathcal{F} \) defined over \( E \) of filtrations (VIII.2.2.1). \( \mathcal{F} \) is equipped with a \( B \)-action, as \( G \in GL_B(V) \).

Notice in the PEL case, these filtrations satisfy \( \mathcal{F}^i = (\mathcal{F}^{m-i+1})^\perp \), where \( m = c \circ \mu \in \text{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z} \). This is due to the fact \( (kv, kw) = k^m(v, w) \) and the fact the pairing is non-degenerate.
Prop. (VIII.2.3.3) (Shimura Field). Fix a conjugacy class of cocharacters \( \{ \mu \} \) defined over \( E \) and \( \mu_0 \in \{ \mu \} \), its corresponding filtration \( \mathcal{F}_0^* \). The field \( E \) in (VIII.2.3.2) can be described as the field of definition of the isomorphism class of \( \mathcal{F}_0^* \) as a \( B \)-invariant filtration, or equivalently as the finite extension of \( \mathbb{Q}_p \) generated by the traces

\[
\text{tr}(d; \text{gr}_j^F(V \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p})), \quad d \in B, \quad i \in \mathbb{Z}.
\]

And the filtration \( \mathcal{F} \) is described as the functor that for any \( E \)-algebra \( R \), \( \mathcal{F}(R) \) is the set of filtrations \( \mathcal{F}^* \) of \( V \otimes_{\mathbb{Q}_p} R \) by \( R \)-modules that are direct summands that

\[
\text{tr}(d; \text{gr}_j^F(V \otimes_{\mathbb{Q}_p} R)) = \text{tr}(d; \text{gr}_j^F(V \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p})).
\]

and moreover in the PEL case satisfies \( \mathcal{F}^i = (\mathcal{F}^{m-i+1})^\perp \).

**Proof:**

1. The field of definition \( E \) of the conjugacy class \( \{ \mu \} \) is determined by Tannakian duality, so it suffices to check over which field these two filtrations are isomorphic as \( G \)-filtrations, but \( G \) is just the group fixing the \( B \)-module structure, so it suffices to show they are equivalent as \( B \)-modules, which is then determined by the traces, by (I.4.1.27).

2. It suffices to show \( \mathcal{F} \) is a homogenous space under \( G \). We restrict to the PEL case, the EL case is simpler. After base change from \( \mathbb{Q}_p \) to \( \overline{\mathbb{Q}_p} \), the data decomposes to the following types:

- \( (A) : B = \text{End}(W) \times \text{End}(W^\vee) \) where \( W \) is a f.d. \( \overline{\mathbb{Q}_p} \)-vector space and \( (u, v)^* = (v^t, u^t) \).
  And \( V = W \otimes V' \oplus W^\vee \otimes V'^\vee \) where the pairing is natural and makes the sum orthogonal.

  \[
  G = \{(1 \otimes g, c \cdot (1 \otimes g^{-t})) | g \in GL(V'), c \in X(G)\}
  \]

- \( (C) : B = \text{End}(W) \) where \( W \) is a f.d. \( \overline{\mathbb{Q}_p} \)-vector space equipped with a symmetric bilinear form \( (-, -)_W \) and \( * \) is the transposition w.r.t it.
  And \( V = W \otimes V' \) where \( V' \) is equipped with an alternating form \( (-, -)_V \) that \( (-, -)_V = (-, -)_W \otimes (-, -)_V^\vee \).

  \[
  G = \{cg | g \in \text{Sp}(V'), c \in X(G)\}
  \]

- \( (BD) \): As in \( C \), except that \( (-, -)_W \) is skew-symmetric and \( (-, -)_V \) is symmetric.

  \[
  G = \{cg | g \in \text{SO}(V'), c \in C(G)\}
  \]

Under this decomposition, the functor \( \mathcal{F} \) in the proposition is represented by products of partial flags of \( V \):

- \( (A) : \mathcal{F}^i = W \otimes (F')^i \oplus W^\vee \otimes ((F')^{m-i+1})^\perp \) and the correspondence \( \mathcal{F}^* \mapsto (F')^* \) identifies \( \mathcal{F} \) with the partial flag variety of \( V' \) with fixed dimensions dim((\( F' \)^i)^\perp).

- \( (B, CD) : \mathcal{F}^* = W \otimes (F')^* \) and \( \mathcal{F} \) is identified with the partial flag variety of \( V' \) of fixed dimensions dim((\( F' \)^i)^\perp) and \( (F')^i = ((F')^{m-i+1})^\perp \).

  The \( A \) case \( G \) clearly acts transitively on \( \mathcal{F} \), and the \( B, CD \) case \( (F')^i \) is isotropic for \( i \geq (m+1)/2 \), and it determines all other components, so \( G \) acts transitively, by Witt’s theorem (IV.7.2.3).

  The reason is (I.4.1.32) and the fact representations of \( B \) is semisimple, then contemplating on the pairing condition. \( \square \)
Prop. (VIII.2.3.4) (Examples of PEL Type). Let $B = D$ be the quaternion algebra over $\mathbb{Q}_p$ and $\ast$ be the involution, i.e.

$$D = \mathbb{Q}_p^2[\Pi], \quad \Pi^2 = p, \quad \Pi a = \sigma(a)\Pi$$

and

$$a^\ast = \sigma(a), a \in \mathbb{Q}_p^2, \quad \Pi^* = \Pi.$$

Let $(V, \iota)$ be a free $D$-module of rank $n$ with a non-degenerate bilinear form satisfying the conditions in (VIII.2.3.1). Then $G$ is a non-trivial inner form of the group $GSp_{2n}$ of symmetric similitudes:

Firstly $\mathbb{Q}_p^2 \otimes K_0 \cong K_0 \oplus K_0$, then $\mathbb{Q}_p^2$ acts on $K_0 \oplus K_0$ by $a(x, y) = ax, \sigma(a)y$. As $V$ is a $\mathbb{Q}_p^2$-vector space, there is a decomposition

$$V = V_0 \oplus V_1$$

where $\mathbb{Q}_p^2$ acts on $V_i$ by $a(v) = v.\sigma^i(a)$, then $G_{K_0}$ is just $GSp_{2n,K_0}$, and $G \neq GSp_{2n}$ as the Galois action $\sigma$ on $\mathbb{Q}_p^2 \otimes K_0$ and $K_0 \cong K_0 \oplus K_0$ are different.

Take $b \in G(K_0)$ the element with $c(b) = p$ and the corresponding isocrystal $(N, \Phi)$ is isotypical of slope $1/2$. $N$ decomposes as $N_0 \oplus N_1$. Notice now $\Pi$ and $\Phi = b\sigma$ interchanges $N_i$, and $\Pi\Phi = \Phi\Pi$. Also $N_1$ is isotropic: For $v, w \in N_1, a \in \mathbb{Q}_p^2$,

$$a(v, w) = (av, w) = (i(\sigma^i(a))v, w) = (v, i(\sigma^{i+1}(a))w) = (v, \sigma(a)w) = \sigma(a)(v, w)$$

so $(v, w) = 0$.

We can define a new non-degenerate alternating form

$$\langle - , - \rangle : N_0 \times N_0 \to K_0 : \langle v, v' \rangle = \langle v, \Pi v' \rangle$$

and also a $\sigma$-linear endomorphism of $N_0$: $\Phi_0 = \Pi^{-1} \circ \Phi|_{N_0}$. From the condition, $v_p(\det \Phi_0) = 0$, and $\Phi$ has all the slopes $0$. Also $\langle \Phi_0v, \Phi_0w \rangle = \sigma(\langle v, w \rangle)$, as

$$\langle \Phi_0v, \Phi_0w \rangle = \langle \Pi^{-1}\Phi_0v, \Pi\Phi_0w \rangle = \langle \Pi^{-1}b\sigma v, b\sigma w \rangle = \sigma(\langle v, \Pi w \rangle) = \sigma(\langle v, w \rangle).$$

so this alternating form is defined over $\mathbb{Q}_p$, denoted by $(V_0, \langle - , - \rangle)$, and $\Phi_0$ corresponds to $\sigma$. Then $J_0 = GSp(V_0, \langle - , - \rangle)$.

Next we consider

$$(0) = \mathcal{F}^2_0 \subset \mathcal{F}^1_0 \subset \mathcal{F}^0_0 = V \otimes \overline{\mathbb{Q}_p}$$

be a filtration where $\mathcal{F}^1_0$ be a $D$-invariant Lagrangian subspace. This corresponds to a cocharacter $\mu \to G$, and $\mathcal{F}$ is just the $\mathbb{Q}_p^2$ variety of $D$-invariant Lagrangian subspaces of $V_{\mathbb{Q}_p^2}$. By (VIII.2.3.3), the Shimura field is $\mathbb{Q}_p$.

Let $\mathcal{F} \subset \mathcal{F}(K)$ where $K/K_0$ is a field extension, then

$$\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$$

where $\mathcal{F}_i \in N_0 \otimes K_0, K$, as $\mathcal{F}$ is $\Pi$-invariant. Now $\mathcal{F}_0$ is also a Lagrangian subspace of $(V_0, \langle - , - \rangle)$. $\mathcal{F}(K)$ identifies the $K$-points of the Grassmannian of Lagrangian subspaces of $(V_0, \langle - , - \rangle)$.

Cor. (VIII.2.3.5). Under the above identification, the subset $\mathcal{F}^{wa}(K)$ of the Grassmannian of Lagrangian subspaces $\mathcal{F}$ of $(V_0 \otimes K, \langle - , - \rangle)$ is characterized by $\mathcal{F}$ satisfying the following conditions:

For all totally isotropic subspaces $W_0 \subset V_0$, we have $\dim_K \mathcal{F} \cap (W_0 \otimes K) \leq 1/2 \dim W_0$. 

Proof: It’s clear $\mu(N, \Phi, F) = 0$, so weakly-admissibility is equivalent to semi-stability. The uniqueness of the HN-filtration of $F$ implies its $D$-invariance, thus semi-stability is equivalent to the fact that for any subspace $P \subset N$ stable under $\Phi$ and $D$-action, we have

$$\dim_K (F \cap (P \otimes_{K_0} K)) \leq v_p(\det(\Phi; P)),$$

Now $\Phi$ is isotypical with slope $1/2$, $v_p(\det(\Phi; P)) = \frac{1}{2} \dim P$, and the $D$-invariance of $P$ is equivalent to $P = P_0 \oplus P_1$ and the $\Phi$-invariance of $P$ is equivalent to the $\Phi_0$-invariance of $P_0$, i.e. $P_0$ is a $\mathbb{Q}_p$-rational subspace $W_0 \subset V_0$.

Finally we show it suffices to check for totally isotropic subspaces: Let $W'_0$ be the radical of $W_0$, then there is a non-singular alternating form on $W_0/W'_0$, then the image of $F'_0 \cap (P \otimes_{K_0} K)$ in this quotient is a totally isotropic space, thus has dimension $\leq \frac{1}{2} \dim(W'_0/W_0)$. then it suffices to check the condition for $W'_0$. \qed
VIII.3  $\mathbb{F}_p$-Schemes

1  Perfect Schemes

Def. (VIII.3.1.1) (Perfect Schemes). An $\mathbb{F}_p$-scheme is called perfect if the Frobenius is an isomorphism on it. Equivalently, this means every affine subscheme is the spectrum of a perfect scheme.

Let Perf be the category of perfect qcqs $\mathbb{F}_p$-schemes endowed with the V-topology (V.1.3.30).

Def. (VIII.3.1.2) (Perfection). There is a perfection functor $X \mapsto X_{\text{perf}}$ from the category of schemes to the category of perfect schemes, it is defined as the glueing of the perfection $R \rightarrow R_{\text{perf}}$ (I.8.1.3) as it commutes with colimits.

Prop. (VIII.3.1.3) (Perfection and Properties). Let $f : X \rightarrow Y$ be a morphism of $\mathbb{F}_p$-schemes, then the following properties holds true for $f$ iff it holds true for $f_{\text{perf}}$:

1. Qco.
2. Quasiseparated.
3. Affine.
4. Separated.
5. Integral.
6. Universally closed.
7. Universal homeomorphism.

The following properties holds for $f_{\text{perf}}$ if it holds for $f$:

1. Closed immersion.
2. Open immersion.
3. immersion
4. Étale
5. (Faithfully)Flat.

Proof: Cf. [Projectivity of Witt Vectors Affine Grassmannian, 3.4].

Prop. (VIII.3.1.4). If $X$ is an $\mathbb{F}_p$-scheme and $\mathcal{L}$ is a line bundle on $X$, then $\mathcal{L}$ is ample iff the pullback to $X_{\text{perf}}$ is ample.

Proof: Cf. [Projectivity of Witt Vectors Affine Grassmannian, 3.6].

Prop. (VIII.3.1.5). If $X$ is an $\mathbb{F}_p$-scheme, then $X_{\text{ét}} \rightarrow X_{\text{perf,ét}} : Y \rightarrow Y_{\text{perf}}$ is an equivalence of sites.

Proof:

Prop. (VIII.3.1.6) (Perfectly Finitely Presented Morphisms). Let $f : X \rightarrow Y$ be a morphism in Perf (VIII.3.1.1), then $f$ is called a perfectly finitely presented morphism if it satisfies the following equivalent conditions:

- Any open affine subscheme Spec $B \subset X$ mapping to an open affine subscheme Spec $A \subset Y$, $A \rightarrow B$ is perfectly f.p. (I.8.1.6).
• There is an open affine covering $\text{Spec } A_i \to X$ mapping to an open affine covering $\text{Spec } B_i \to Y$ that $B_i \to A_i$ are all perfectly f.p.

• For any cofiltered system $\{Z_i\} \in \text{Perf}_Y$ with affine transition maps, there is a bijection $\colim \text{Hom}_Y(Z_i, X) \cong \text{Hom}_Y(\lim Z_i, X)$.

In particular, perfectly finitely presented is local on the base and target.

Proof: Cf.[Projectivity of Witt Vectors Affine Grassmannian, 3.11].

Prop. (VIII.3.1.7) (Perfect Base Change). If

\[
\begin{array}{ccc}
X' & \overset{g'}{\longrightarrow} & X \\
\downarrow f' & & \downarrow f \\
Y' & \overset{g}{\longrightarrow} & Y
\end{array}
\]

is a pullback diagram of perfect $\mathbb{F}_p$-schemes, then for any complex $K^\bullet$ of $\text{Qco}$ sheaves on $X$, the base change map (V.6.1.15)

\[
Lg^* Rf_* K \to Rf'_* Lg' K
\]

is an isomorphism.

Proof: It is clearly we need only check for the affine case, then let $X = \text{Spec } B, Y = \text{Spec } A, Y' = \text{Spec } A'$, and $X' = \text{Spec } B \otimes_A A'$, then it suffices to prove that

\[
K \otimes^L_A A' \cong K \otimes^L_B B'.
\]

This follows from the fact $B \otimes_A A' = B \otimes^L_A A'$, by (I.8.1.8).

Prop. (VIII.3.1.8) (Cartier Isomorphism). If

Proof:

Cor. (VIII.3.1.9) (Affine Line case). Let $R$ be an $\mathbb{F}_p$-algebra and
VIII.4 Formal and Rigid Geometry

Main references are [Bos15] and [BGR84], but there are other approaches, such as given by Berkovich, or given by Huber, and used in Scholze’s work, which is most natural because it behaves well w.r.t. the formal model.

1 Affinoid $K$-Spaces

Def. (VIII.4.1.1) (Affinoid $K$-Space). an affinoid algebra $A$ can be viewed as the function ring on the space $\text{Sp} A$ of maximal ideals of $A$ with the usual Zariski topology called the affinoid $K$-space associated to $A$. A morphism of affinoid algebras induce a map on their $\text{Sp} A$. This is because residue fields of maximal ideals are finite over $K$. So we define the category of affinoid $K$-spaces as the opposite category of affinoid $K$-algebras.

Cor. (VIII.4.1.2). The category of affinoid spaces admits fiber products, because of (I.9.4.30).

Prop. (VIII.4.1.3). By the properties of a Jacobson space (IX.1.14.22) (IX.1.14.19), the affinoid $K$-space has good properties w.r.t. closed, open hence irreducible compared to $\text{Spec} A$ in Zariski topology. In particular, it is a Noetherian space.

Def. (VIII.4.1.4) (Canonical Topology). The affinoid $K$-space has another topology, called the canonical topology, generated by $X(f, \varepsilon) = \{x | f(x) \leq \varepsilon\}$ as a subbasis. And this topology is in fact generated by $X(f) = X(f, 1)$ as a subbasis.

Proof: For the last assertion, notice $f(x)$ assume value in $|K|$, which is dense in $\mathbb{R}_+$, so we can assume $\varepsilon \in |K|$ (by approximation from below), hence $\varepsilon^n = |c|$, where $c \in K$, so $X(f, \varepsilon) = X(f^n, c) = X(c^{-1} f^n)$.

□

Prop. (VIII.4.1.5). $\{x | f(x) = \varepsilon\}$ is open in $\text{Sp} A$.

Proof: We let $f(x) = \varepsilon$ and $k = A/\mathfrak{m}_x$, let the minipoly of $f$ in $A/\mathfrak{m}_x$ be $P$ of degree $n$, and let $g = P(f)$, then $g(x) = 0$, and if $|g(y)| < \varepsilon^n$, then $|f(y)| = \varepsilon$, otherwise $|f(y) - \alpha_i| \geq |\alpha_i| = \varepsilon$ for every root $\alpha_i$ of $P$, hence $|P(f(y))| \geq \varepsilon^n$, contradiction.

□

Cor. (VIII.4.1.6). By the proof, we have, $X(f_1, \ldots, f_r)$, $f_i \in \mathfrak{m}_x$ forms a basis of $x$ in $\text{Sp} A$. (Replace every $X(f_i)$ by $\{y | f_i(y) = \varepsilon\}$, then by some $X(g_i)$ for $g_i \in \mathfrak{m}_x$.

Def. (VIII.4.1.7) (Affinoid Subdomain). For an affinoid $K$-space $X$, a subset $U$ is called an affinoid subdomain of $X$ if there is a closest affinoid space map $X' \to X$ with image in $U$, i.e. any other these maps factor through it. The definition is weird but the situation is clarified by the following proposition.

Prop. (VIII.4.1.8). For an affinoid subdomain $i : X' \to X$,

- $i$ is injective and $\text{Im} i = U$.
- $i^*$ induce an isomorphism $A/\mathfrak{m}_{i(x)}^{\mathfrak{m}_x} \cong A'/\mathfrak{m}_x^{\mathfrak{m}_x}$.
- $\mathfrak{m}_x = \mathfrak{m}_{i(x)}A'$.
Weierstrass domain

\[ \text{Proof:} \] Consider a point \( y \in U \), there is a commutative diagram

\[ \begin{array}{ccc}
A & \overset{i^*}{\to} & A' \\
\downarrow & \alpha & \downarrow \\
A/m^n_y & \overset{\sigma}{\to} & A'/m^n_y A'
\end{array} \]

there is a map \( \alpha : A' \to A/m^n_y \) that makes the upper diagram commutative by universal property of subdomain, and the lower triangle is commutative by universal properties again. Then we see \( \sigma \) is surjective and notice the kernel of the projection is \( m_y A' \) is in the kernel of \( \alpha \), thus \( \sigma \) is injective.

Now the case \( n = 1 \) shows \( m_y A' \) is maximal, hence \( i \) is surjective and the inverse image is just one point. \( \square \)

Prop. (VIII.4.1.9) (Special Subdomains). There are three special affinoid subdomain of \( X \): Weierstrass domain \( X(f_1, \ldots, f_r) \), Laurent domain \( X(f_1, \ldots, f_r, g_1^{-1}, \ldots, g_s^{-1}) \), rational domain \( X(\frac{f_1}{f_0}, \ldots, \frac{f_r}{f_0}) = \{ x \mid |f_i(x)| \leq |f_0(x)| \} \) for \( (f_0, \ldots, f_r) = (1) \). They are all open by (VIII.4.1.5).

\[ \text{Proof:} \] The Weierstrass domain corresponds to \( A \to A\langle x_1, \ldots, x_r \rangle/(x_i - f_i) \).

The Laurent domain corresponds to \( A \to A\langle x_1, \ldots, x_{r+s} \rangle/(x_i - f_i, 1 - x_{r+j} g_j) \).

The rational domain corresponds to \( A \to A\langle x_1, \ldots, x_r \rangle/(f_i - f_0 x_i) \).

They are affinoid subdomains is in fact, easily checked. \( \square \)

Lemma (VIII.4.1.10). Weierstrass domain are Laurent, and Laurent domain are rational, this is because intersection of rational domains are rational.

Any rational domain is a Weierstrass domain of a Laurent domain.

\[ \text{Proof:} \] Notice a Laurent subdomain is a finite intersections of \( X(\frac{f}{g}) \) and \( X(\frac{1}{g}) \), so it is rational.

For a rational domain \( U \), \( f_0 \) is a unit in \( O(U) \), hence its inverse has a bounded value, then \( |c f_0| > 1 \) for some \( c \in K^* \). Hence \( U \) is Weierstrass in \( X((c f_0)^{-1}) \). \( \square \)

Cor. (VIII.4.1.11) (Pullback & Composition of Affinoid Subdomain). The pullback(hence intersections) of affinoid subdomains is affinoid subdomain and it is just the set-theoretic inverse image, and specialness are preserved.

The affinoid subdomain of an affinoid subdomain is affinoid subdomain, and Weierstrassness and rationalness are preserved(while Laurentness not).

\[ \text{Proof:} \] Pullback: fiber product exist in the category of affinoid \( K \)-spaces, then the universal property is checked. The set-theoretic property follows from (VIII.4.1.8).

Speciality: Clear.

Transitivity: Clear by universal property.

For the speciality, if \( V = X(f_i) \), \( U = V(g_j) \) is Weierstrass, then because by (I.9.4.26) \( A \) is dense in \( A\langle f_i \rangle \), we can replace \( g_j \) by elements from \( A \), by adding elements of small sup-norm, because valuation is non-Archimedean. Then \( U = X(f_i, g_j) \). For the rational subdomain \( V = X(\frac{f_1}{f_0}, \ldots, \frac{f_r}{f_0}) \), use (VIII.4.1.10), it suffices to prove for \( U = V(g) \) or \( U = V(g^{-1}) \). For this, notice the image of \( A[f_0^{-1}] \) is dense in \( A[\frac{f_1}{f_0}, \ldots, \frac{f_r}{f_0}] \), by (I.9.4.26), so as before, we change \( g \) that it \( g_0 f_0^n g \in A \) for some \( n \). Now

\[ V(g) = V \cap \{ x \in X \mid |g_0(x)| \leq |f_0^n(x)| \}, \quad V(g^{-1}) = V \cap \{ x \in X \mid |g_0(x)| \geq |f_0^n(x)| \}. \]

But now \( f_0^n \) is a unit in \( A[\frac{f_1}{f_0}, \ldots, \frac{f_r}{f_0}] \), so \( |f(x)|_{\sup} \geq |c| \) for some \( c \in K^* \), so

\[ V(g) = V \cap X(\frac{g_0}{f_0^n}, \frac{c}{f_0^n}), \quad V(g^{-1}) = V \cap X(\frac{f_0}{g_0}, \frac{c}{f_0}). \]

is rational in \( X \). \( \square \)
Cor. (VIII.4.1.12). For a special subdomain $U$ of $X$, the canonical topology induces the canonical topology of $U$, by the transitivity property of affinoid subdomains and (VIII.4.1.10). In fact, by (VIII.4.1.14), any affinoid subdomain is open and the topology coincides.

Prop. (VIII.4.1.13). Let $\varphi : Y = \text{Sp} B \to X = \text{Sp} A$ be a morphism, if $x$ is a point of $X$ that $A/m_x \to B/m_x B$ is a surjection, then there is an affinoid nbhd $U$ of $x$ that $\varphi$ restricts to a closed immersion on $\varphi^{-1}(U)$. If $A/m_x^n \cong B/m^n$ for all $n$, then there is an affinoid nbhd $U$ of $x$ that $\varphi$ restricts to an isomorphism $\varphi^{-1}(U) \cong U$.

Proof: Cf.[Rigid and Formal Geometry P57]. □

Cor. (VIII.4.1.14). Every affinoid subdomain of $X$ is open and has the restriction topology of $X$ (canonical topology), because it satisfies the second condition of (VIII.4.1.13), by (VIII.4.1.8).

Lemma (VIII.4.1.15). If $f \subset A(X_1, \ldots, X_n)$ is $X_n$-distinguished of order $\leq s$ for each element of $\text{Sp} A$, then the set of elements that $f$ is $X_n$-distinguished of exact order $s$ is a rational subdomain of $A$.

Proof: Let $f = \sum f_v X_v^n$, let the constant coefficient of $f_v$ be $a_v$, then the set is in fact $U = \{x \in \text{Sp} A | |a_v(x)| \leq |a_s(x)|\}$. This is because, if $f$ is distinguished of order $s_x$ at $x$, then $a_{s_x} \neq 0$ because $f_{s_x}$ is a unit, and $|a_v|_x \leq |f_v| \leq |f_{s_x}| = |a_{s_x}|_x$ for $v \leq s_x$ and strict inequality holds for $v > s_x$. In particular, $a_0, \ldots, a_s$ cannot have a common zero, so it is truly a rational subdomain. □

Prop. (VIII.4.1.16). If $f \subset A(X_1, \ldots, X_n)$ is $X_n$-distinguished of order $s$ for each element of $\text{Sp} A$, then the map $A(X_1, \ldots, X_{n-1}) \to A(X_1, \ldots, X_n)/(f)$ is finite.

Proof: Cf.[Rigid And Formal Geometry P79]. □

Presheaf of Affinoid Functions

Def. (VIII.4.1.17). The weak Grothendieck category (affine topology) on an affinoid space $X$ has coverings defined by the finite cover by affinoid subdomains, called affinoid covering.

The strong Grothendieck category (fpqc topology) on an affinoid space $X$ is defined by: objects are unions of affinoid subdomains $U = \bigcup U_i$ that for any morphism from an affinoid space $\varphi : Z \to U \subset X$, the pullback covering $\bigcup \varphi^{-1}(U_i)$ has a finite subcover by affinoid subdomains. A covering is defined by the same finiteness property.

The strong Grothendieck topology satisfies completeness conditions $G_0, G_1, G_2$ defined in (V.1.1.9), as easily verified.

The weak Grothendieck topology is a temporary notion, it will be obsolete after Tate’s acyclicity theorem is proved. Admissible opens and admissible covers are notions w.r.t. the strong Grothendieck topology.

Proof: The weak Grothendieck category is a Grothendieck category by (VIII.4.1.11). The strong Grothendieck category is a Grothendieck category because: the finiteness condition lifts along base change, and also for base change, because we can first choose a finite subcover, then choose a finite subcover of the base change covering of that finite covering. □
Def. (VIII.4.1.18). For \( n \) functions \( f_1, \ldots, f_n \) without common zeros, the rational subdomains \( U_i = X(\{ \frac{1}{f_i} \}) \) is an affinoid covering, called the rational covering. For \( n \) functions \( f_1, \ldots, f_n \), there is a Laurent covering \( X(\prod f_i^{\varepsilon_i}), \varepsilon_i = \pm 1 \).

Prop. (VIII.4.1.19). Morphisms of affinoid spaces are continuous in weak Grothendieck topology by (VIII.4.1.11). It is also continuous in the strong Grothendieck topology, as one can check the finiteness conditions.

Prop. (VIII.4.1.20). Let \( X \) be an affinoid \( K \)-space, for any \( f \in \mathcal{O}_X(X) \), consider the following sets:
\[
U_1 = \{ x|f(x)| < 1 \}, \quad U_2 = \{ x|f(x)| > 1 \}, \quad U_3 = \{ x|f(x)| > 0 \}.
\]
Then any finite union of sets of the form is admissible, and any finite cover by finite union of sets of the form is an admissible covering.

Proof: We first show that \( U_1 \) is admissible open, the others are similar. Let \( \varepsilon_n \) be an ascending sequence of elements in \( \sqrt{K^*} \) converging to 1, then \( U_1 = \bigcup_n X(\varepsilon_n^{-1} f) \) is a union of open subsets because \( \varepsilon_n \in \sqrt{K^*} \). Now for any affinoid space \( Z \) mapping into \( U_1 \), \( |\varphi^*(f)(x)|_{\sup} < 1 \) for all \( z \in Z \), thus by maximal principle (I.9.4.21), \( |f|_{\sup} < 1 \), thus the cover \( U_1 = \bigcup_n X(\varepsilon_n^{-1} f) \) can be refined by a finite cover, thus it is admissible open.

For the admissibility of covering, the proof is similar, but use the following lemma (VIII.4.1.21). \( \square \)

Lemma (VIII.4.1.21). For any affinoid \( K \)-algebra \( A \), if \( f_i, g_j, h_k \) are system of functions on \( A \) that: for every \( x \in A \), either \( |f_i(x)| < 1, |g_j| > 1 \) or \( h_k(x) > 0 \), then we can replace \( >, < \) by \( \geq, \leq \) and elements in \( \sqrt{K^*} \) that the same condition is true.

Proof: Cf. [Rigid and Formal Geometry P97]. \( \square \)

Cor. (VIII.4.1.22). The strong Grothendieck category is finer than the Zariski category, because any standard affine open set is of the form \( U_3 \) and also Zariski covering is open covering because \( \text{Sp}(A) \) is Noetherian (I.9.4.16).

Def. (VIII.4.1.23) (Presheaf of Affinoid Functions). There is a presheaf of affinoid functions defined on the weak Grothendieck topology because of the universal property of the affinoid subdomains.

Then the stalk \( \mathcal{O}_{X,x} \) are local ring with maximal ideal \( m_x\mathcal{O}_{X,x} \). Hense let \( X = \text{Sp} A \), the stalk map factor thorough \( A \to A_{m_x} \to \mathcal{O}_{X,x} \), and
\[
A/m^n_x \cong A_{m_x}/m^n_x A_{m_x} \cong \mathcal{O}_{X,x}/m^n_x \mathcal{O}_{X,x}
\]
so it induces isomorphisms between their \( m_x \)-adic completions.

Proof: By (VIII.4.1.8), there is an isomorphism \( K' = \mathcal{O}_X(X)/m_x \cong \mathcal{O}_X(U)/m_x \mathcal{O}(U) \). Take the converse and pass to direct colimit (it is exact), \( \mathcal{O}_{X,x}/m_x \mathcal{O}_{X,x} \cong K' \). This map will be regarded as evaluation at \( x \). The kernel \( m_x \mathcal{O}_{X,x} \) is a maximal ideal. There are no other maximal ideals in \( \mathcal{O}_{X,x} \) because if \( f \) in not in the kernel, then \( f(x) \neq 0 \), and multiply by an element in \( K^* \), it can be made \( |f(x)| \geq 1 \), and then \( U(f^{-1}) \) is an affinoid subdomain containing \( x \) that \( f \) is invertible in it.

For the second assertion, for an affinoid subdomain \( \text{Sp} A' \), there are maps
\[
A/m^n \to A'/m^n \to \mathcal{O}_{X,x}/m^n \mathcal{O}_{X,x}.
\]
We first show these are isomorphisms: the first map is an isomorphism by (VIII.4.1.8), then take direct colimit, the composition map is also isomorphism.

\[ A/m^n_x \cong A_{m_x}/m^n A_{m_x} \text{ is classical.} \]

\( A_{m_x} \hookrightarrow O_{x,X} \) is injective because by Krull’s intersection theorem (I.5.6.11), \( A_{m_x} \hookrightarrow O_{x,X} \rightarrow \hat{A}_{m_x} \cong A_m \) is injective.

\[ \text{Cor. (VIII.4.1.24). } f \in A = O_X(X) \text{ vanish iff it vanish at every stalk, this is because } A \rightarrow \prod_m A_m \rightarrow \prod O_{X,x} \text{ is injective.} \]

\[ \text{Cor. (VIII.4.1.25). Giving a covering of affinoid subdomain of an affinoid space } X_i \rightarrow X, \text{ then } O_X(X) \rightarrow \prod O_{X_i}(X_i) \text{ is an injection. (This is because the kernel vanishes at each stalk.)} \]

\[ \text{Cor. (VIII.4.1.26). For a subdomain of an affinoid space } X, \text{ the corresponding ring map is flat.} \]

\[ \text{Proof: Cf. [Formal and Rigid Geometry P68].} \]

\[ \text{Prop. (VIII.4.1.27). The stalk } O_{x,x} \text{ is Noetherian, in particular it is } m\text{-adically separated by Krull’s intersection theorem (I.5.6.11).} \]

\[ \text{Proof: First it is } m\text{-adically separated, because by (VIII.4.1.23), for a } f \in \cap m^n O_{x,x}, \text{ we can choose an affinoid subdomain } \text{Sp } A \text{ that } f \in A \text{ (VIII.4.1.8), then } f \in m^n A, \text{ so by Krull’s intersection theorem (I.5.6.11), we have } f = 0 \text{ in } A_m. \]

In the same way, any f.g. ideal \( a \) of \( O_{x,x} \) is \( m \)-adically closed, this is because it is generated by an ideal in the affinoid algebra of a nbhd, and then \( O_{x,X}/a \) is separated as the stalk of an affinoid algebra \( A'/a' \).

Now pass a chain of f.g. ideals to their completion, then that chain is stationary because \( \hat{O}_{X,x} = \hat{A}_{m_x} \) is Noetherian (I.5.1.36). And now this chain is also stationary because ideals are closed in \( m \)-adic topology. \]

**Locally Closed Immersions**

\[ \text{Def. (VIII.4.1.28) (Immersions). A morphism of affinoid spaces is called a closed immersion iff the corresponding ring map is surjective. It is called a locally closed immersion iff it is injective and the stalk map are all surjective. It is called an open immersion iff it is injective and the corresponding stalk maps are isomorphism. All these notions are stable under compositions.} \]

An affinoid subdomain is an open immersion by (VIII.4.1.23) and (VIII.4.1.27).

\[ \text{Lemma (VIII.4.1.29). Base change by affinoid subdomain of closed/locally closed/open immersions are of the same type.} \]

\[ \text{Proof: This is obvious for locally close and open, because affinoid subdomains are open (VIII.4.1.14), for the closed immersion, use (I.9.4.32).} \]

\[ \text{Prop. (VIII.4.1.30). A closed immersion of affinoid spaces is equivalent to a locally closed immersion that the corresponding ring map is finite.} \]

\[ \text{Proof: Cf. [Rigid and Formal Geometry P70]. A closed immersion } X' \rightarrow X \text{ is a locally closed immersion because the canonical topology of } \text{Sp } A \text{ restricts to the canonical topology on } \text{Sp } A/a \text{ (VIII.4.1.4), then use (I.9.4.32), and the fact direct limit is exact.} \]
Prop. (VIII.4.1.31) (Clopen Immersion). The image of an open and closed immersion is Zariski closed and open. In particular, it is a Weierstrass subdomain.

Proof: Cf. [Rigid and Formal Geometry P71]. □

Def. (VIII.4.1.32). A Runge immersion is a closed immersion followed by an open immersion of Weierstrass subdomain. Runge immersion is stable under base change of affinoid subdomains by (VIII.4.1.29)

Prop. (VIII.4.1.33) (Equivalent Definition of Runge Immersions). For a morphism $\sigma : A \to A'$, $\text{Sp} A' \to \text{Sp} A$ is a Runge immersion iff $\sigma(A)$ is dense in $A'$ iff $\sigma(A)$ contains a set of affinoid generator of $A'$ over $A$.

Proof: For a Runge immersion, $\sigma(A)$ is dense in $A'$, because this is true for Weierstrass subdomain and closed immersion.

If $\sigma(A)$ is dense in $A'$, then by (I.9.4.28), we can modify a set of affinoid generators by a set of affinoid generators in $\sigma(A)$.

If $h_i$ is a set of affinoid generators in $\sigma(A)$, then $A \to A(h_i) \to A'$ is a Runge immersion. □

Cor. (VIII.4.1.34). Runge immersion is stable under composition.

Prop. (VIII.4.1.35). An open and Runge immersion is an immersion of Weierstrass subdomain.

Proof: By localizing on this Weierstrass subdomain, and notice Weierstrass subdomain is stable under composition (VIII.4.1.11), we reduce to clopen immersion case, and result follows by (VIII.4.1.31). □

Lemma (VIII.4.1.36) (Extension of Runge Immersion). For a morphism of affinoid spaces $X' \to X = \text{Sp} A$, if $f_1, \ldots, f_n, g$ generate $A$, for $\varepsilon \in \sqrt{|K^*|}$, denote $X_{\varepsilon} = \{ x | |f_i(x)| \leq \varepsilon|g| \}$, this is a rational subdomain. The inverse image of $X_\varepsilon$ is $X'_\varepsilon$, then if $X'_{\varepsilon_0} \to X_{\varepsilon_0}$ is a Runge immersion for some $\varepsilon_0$, then there is an $\varepsilon > \varepsilon_0$ that $X'_{\varepsilon} \to X_{\varepsilon}$ is also a Runge immersion.

Proof: Cf. [Rigid and Formal Geometry P73]. □

Prop. (VIII.4.1.37) (Gerritzen-Grauert). For a locally closed immersion $\varphi : X' \to X$, there is a finite cover of $X$ of rational subdomains $X_i$ that $\varphi^{-1}(X_i) \to X_i$ are Runge immersions.

Proof: Cf. [Formal and Rigid Geometry P79]. □

Cor. (VIII.4.1.38) (Gerritzen-Grauert). Any affinoid subdomain is equivalent to a finite union of rational subdomains.

Proof: An affinoid subdomain is an open immersion by (VIII.4.1.28), so $\varphi^{-1}(X_i) \to X_i$ is open and Runge, so it is Weierstrass by (VIII.4.1.35). In particular, $X \cap X_i$ is rational in $X$ by transitivity, thus the result. □

Tate's Acyclicity

Lemma (VIII.4.1.39) (Reduction of Weak Grothendieck Topology).

- Every affinoid covering has a refinement of rational covering.
- For every rational covering, there is a Laurent covering $\{ V_i \}$ that restriction on each $V_i$ is rational covering generated by units.
• Every rational covering generated by units has a refinement of Laurent covering.

**Proof:** 1: By (VIII.4.1.38), we can assume the covering consists of rational subdomains \( U_i = X(f_{i1} \ldots f_{ik}) \), then consider the elements \( f_{v_1 \ldots v_n} = \prod_{i=1}^{n} f_{iv_i} \), where at least some \( v_i = 0 \). Denote the set of these elements by \( I \).

Firstly, these elements has no common zero on \( X \), thus generating a rational covering of \( X \): for any \( x \in U_i \), \( f_i \) doesn't vanish at \( x \), thus the product \( \prod_{j \neq i} f_j \) vanishes for all choices of \( v_j \), but this is impossible because for each \( j \), \( \{ f_{jk} \}_k = (1) \).

Secondly, this is a refinement of \( U_i \): We show \( X f_{v_1 \ldots v_n} \subset U_k \) where \( v_k = 0 \). For this, consider \( x \in X f_{v_1 \ldots v_n} \), then \( x \in U_j \) for some \( j \). If \( j = k \), we are done, otherwise,

\[
|f_{v_1 \ldots \mu_k \ldots v_n}(x)| \leq |f_{v_1 \ldots 0 \ldots \mu_k \ldots v_n}(x)| \leq |f_{v_1 \ldots v_n}(x)|.
\]

Where the last inequality is because \((v_1, \ldots, 0, \ldots, \mu_k, \ldots, v_n)\) has a 0, thus \( f_{v_1 \ldots 0 \ldots \mu_k \ldots v_n} \in I \).

2: For a rational covering, \( f_i \) is invertible in the ring of \( U = X(\frac{f_0}{f_1}, \ldots, \frac{f_n}{f_1}) \), thus it has a inverse that attains maximum value on \( U(\text{I.9.4.21}) \). Hence there is a \( c \in K^* \) that \( |c|^{-1} < \inf(\max\{|f_i(x)|\}) \).

I claim the Laurent covering w.r.t. the elements \( c f_0, \ldots, c f_n \) satisfies the requirement. Because for example, on \( V = X((c f_0) \ldots (c f_1)(c f_{s+1})^{-1} \ldots (c f_n)^{-1}) \), \( |f_i(x)| < |f_j(x)| \) for \( i < s < j \), so the covering restricted to \( V \) is just the rational covering generated by \( f_{s+1}, \ldots, f_n \), and they are all invertible in \( O(V) \).

3: In fact the Laurent covering generated by the element \( f_i f_j^{-1} \) for \( i < j \) is a refinement of the rational covering generated by \( f_1, \ldots, f_n \), because in any one of this Laurent subdomains \( V \), for any two \( i, j \), either \( |f_i(x)| < |f_j(x)| \) or \( |f_j(x)| < |f_i(x)| \) for all \( x \in V \), so there is a maximal one \( f_s \), then \( V \subset X(\frac{f_0}{f_s}, \ldots, \frac{f_n}{f_s}) \).

**Prop. (VIII.4.1.40) (Tate’s Acyclicity Theorem).** The presheaf of affinoid functions on an affinoid space \( X = \text{Sp} A \) is a sheaf w.r.t the weak Grothendieck category. In fact, for any \( A \)-module \( M \), the presheaf \( \tilde{M} = M \otimes_A O_X \) is a sheaf w.r.t. the weak Grothendieck topology, called the quasi-coherent sheaf on \( X \).

Moreover, for any finite cover of affinoid subdomains, the Čech cohomology group \( \tilde{H}^q(\text{Sp} A, \tilde{M}) \) vanish for \( q \neq 0 \).

**Proof:** It suffices to prove the last assertion. First reduce to the case of Laurent covering by (VIII.4.1.39) and (V.6.2.8)(V.6.2.9). Noticing the base change invariance of the specialities of affinoid subdomains (VIII.4.1.11). Even more, by (V.6.2.10) and an induction process, it suffices to prove for the simple Laurent covering \( X(f), X(f^{-1}) \).

It suffices to prove for the sheaf of affinoid functions \( O_X \), because for any Qco sheaf \( \tilde{M} \), choose a free resolution of \( M \), then use dimension shifting, notice the covering is finite (the flatness of the algebra map (VIII.4.1.26) is used to deduce the long exact sequence).

For the sheaf \( O_X \), the main tool is the following commutative diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & A \\
\downarrow & & \downarrow \\
(X - f)A(X) \times (1 - fY)A(Y) & \xrightarrow{\delta'} & (X - f)A(X, X^{-1}) \\
\downarrow & & \downarrow \\
0 & \rightarrow & A \\
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{e'} & A(X) \times A(Y) \\
\downarrow & & \downarrow \\
A(f) & \xrightarrow{e} & A(f)A(f^{-1}) \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & A \\
\downarrow & & \downarrow \\
A(X, X^{-1}) & \xrightarrow{\delta'} & A(f, f^{-1}) \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\end{array}
\]
where $\delta'$ is given by $(h_1(X), h_2(Y)) \mapsto h_1(X) - h_2(X^{-1})$, and $\delta''$ is induced by $\delta'$. The columns are all exact, and the first row and the second row are exact. $\varepsilon$ is injective by (VIII.4.1.25). Then the last row is also exact, by spectral sequence.

\[\square\]

Prop. (VIII.4.1.41) (Strong/Weak Topos The same). If $X$ is an affinoid $K$-space, the category of sheaves w.r.t. the strong Grothendieck topology is equivalent to the category of sheaves w.r.t. the weak Grothendieck topology by pushforward and pullback of sheaves by (V.1.2.21) because the strong and weak Grothendieck category satisfies the conditions.

In particular, this applies to the case $O_X$ by (VIII.4.1.40), the resulting sheaf is called the sheaf of rigid analytic functions on $X$, also denoted by $O_X$.

2 Rigid Analytic Spaces

Def. (VIII.4.2.1). A $G$-ringed $K$-space is a pair $(X, O_X)$ where $X$ is a $G$-topological space and $O_X$ is a sheaf of $K$-algebras. It is called local $G$-ringed $K$-space if the stalks are all local rings. Their morphisms are defined routinely.

Prop. (VIII.4.2.2) (Morphisms Between Affinoid Spaces). An affinoid $K$-space with the sheaf of analytic functions $(X, O_X)$ (VIII.4.1.41) is an example of local $G$-ringed $K$-space (VIII.4.1.23).

A continuous homomorphism of rings induces a local $G$-ringed morphism. And all morphisms come from these.

Moreover, an affinoid $K$-space is a complete $G$-ringed $K$-space (i.e. rigid) (V.1.1.9).

Proof: It is a $G$-space by (VIII.4.1.40) (VIII.4.1.41), morphisms by (VIII.4.1.19), notice the $m_x$ generate the maximal ideal of $O_{X,x}$ (VIII.4.1.23), so the morphism is local.

To show all morphisms are like these, we need to show a morphism $\sigma : A \rightarrow B$ gives at most one $\text{Sp} B \rightarrow \text{Sp} A$: the morphism is local, so it maps $m_{\varphi(x)}$ to $m_x$, and from the commutative diagram

\[
\begin{array}{ccc}
A/m_{\varphi(x)} & \rightarrow & B/m_x \\
\downarrow\cong & & \downarrow\cong \\
O_{\varphi(x),\text{Sp} A}/m_{\varphi(x)}O_{\varphi(x),\text{Sp} A} & \rightarrow & O_{x,\text{Sp} B}/m_xO_{x,\text{Sp} B}
\end{array}
\]

(VIII.4.1.23) shows $m_{\varphi(x)}$ is mapped into $m_x$, so $m_{\varphi(x)} = (\sigma^*)^{-1}m_x$, which shows $\varphi$ is unique set-theoretically, and on the level of structure sheaf, the uniqueness of $O_{\text{Sp} A}(V) \rightarrow O_{\text{Sp} B}(\varphi^{-1}(V))$ is unique by the definition of affinoid subdomain (VIII.4.1.7).

Def. (VIII.4.2.3) (Rigid Spaces). The category of rigid (analytic) space is a full subcategory of local $G$-ringed $K$-spaces that it is complete $G_0, G_1, G_2$, and it has an admissible covering $\{X_i \rightarrow X\}$ that $(X_i, O_X|_{X_i})$ are affinoid $K$-spaces.

It follows easily that an admissible open subset of a rigid space is again rigid.

Prop. (VIII.4.2.4) (Glueing Rigid Spaces). Glueing rigid analytic spaces is legitimate, so does glueing morphisms on the source.

Proof: First glue the set, then use (V.1.1.11) to glue $G$-topology, finally the glue of structure sheaf is similar to (V.1.5.3).
Cor. (VIII.4.2.5) (Spectrum Adjointness). If $X$ is rigid and $Y$ is affinoid, then $\text{Hom}(X,Y) \cong \text{Hom}(\mathcal{O}_Y(Y),\mathcal{O}_X(X))$. This follows from (VIII.4.2.2) and glue (VIII.4.2.4).

Prop. (VIII.4.2.6) (Fiber Products). Fiber products exist in the category of rigid analytic space. This is fiber product of affinoid spaces are affinoid so we can glue them by universal property, the same as (V.2.7.15).

Prop. (VIII.4.2.7). An affinoid space is connected in the weak Grothendieck topology iff it is connected in the strong Grothendieck topology iff it is connected in the Zariski topology.

Proof: Firstly the weak and strong are equal because any strong covering of $X$ has a refinement of weak covering, and a weak covering is a strong covering. So it suffices to prove the equivalence of the last two.

One direction is trivial, for the other direction, use Tate’s acyclicity, if $X_1, X_2 \to \text{Sp} A$, $X_1 \cap X_2 = 0$, then $A = \mathcal{O}_X(X_1) \times \mathcal{O}_X(X_2)$, so Spec $A$ is not connected, neither do Sp $A$. □

Prop. (VIII.4.2.8). We can define the connected components of $X$ as the equivalence classes of elements that can be reached using connected admissible open subsets of $X$. Then the connected components are admissible and forms an admissible cover of $X$.

Proof: Notice that there exists a finite covering consisting of connected Zariski subsets, by (VIII.4.2.7) and the fact Sp $A$ has f.m. connected components because Spec $A$ does as $A$ is Noetherian (I.9.4.12) and (VIII.4.1.3).

Thus we are done, because by (VIII.4.1.22), a Zariski covering is admissible, and clearly the connected components of $X$ are just this Zariski covering. □

Rigid GAGA

Lemma (VIII.4.2.9). Let $Z$ be an affine scheme algebraic over $K$, and $Y$ a rigid $K$-space, then the set of morphisms of local $G$-ringed spaces $(Y,\mathcal{O}_Y) \to (Z,\mathcal{O}_Z)$ corresponds to $K$-algebra morphisms from $\mathcal{O}_Z(Z)$ to $\mathcal{O}_Y(Y)$.

Proof: Cf. [Formal and Rigid Geometry P111]. □

Def. (VIII.4.2.10) (Rigid Analytification). There is a partial functor $X^{\text{rig}}$ from the category of schemes $X$ locally algebraic over a valued field $K$ to the category of rigid $K$-spaces that are right adjoint to the forgetful functor from the category of rigid $K$-spaces to local ringed $K$-space, called the GAGA functor.

The existence of this functor is proven in (VIII.4.2.14).

Def. (VIII.4.2.11) (Analytification of Affine Schemes). Let $T_n\xi$ be the elements $\sum a_n\zeta^v$ in $T_n$ that $\lim a_n\tau^{[b]} = 0$. Then choose a $c \in K, |c| > 1$, define $T^{(i)}_n = T_n(|c|^i)$. Then $T^{(i)}_n = K(c^{-i}X_1, \ldots, c^{-i}X_n)$, so clearly Sp$(T^{(i)}_n)$ is an affinoid subdomain of Sp$(T^{(i+1)}_n)$ by (VIII.4.1.9). Thus there is a chain of inclusions of affinoid subdomains:

$$B^n = \text{Sp}(T^{(0)}_n) \hookrightarrow \text{Sp}(T^{(1)}_n) \hookrightarrow \text{Sp}(T^{(2)}_n) \hookrightarrow \ldots$$

Then we can use (VIII.4.2.4) to glue them together as $K^{n,\text{rig}}$.

Prop. (VIII.4.2.12). The maximal spectrum Max$(K[X_i]) = \cup_n\text{Sp}(T^{(i)}_n)$ as sets.
Proof: It suffices to show the following two.

- For any maximal ideal $m \subset T_n$, $m' = m \cap K[X_i]$ is maximal.
- For any maximal ideal $m' \subset K[X_i]$, there is some $N$ that $m'T_n^{(i)}$ is maximal in $T_n^{(i)}$ for all $i > N$.

For 1: Consider the $K \subset K[X_i]/m' \subset T_n/m$, $T_n/m$ is a finite extension of $K$ by (I.9.4.10), then so does $K[X_i]/m'$, by (I.5.5.3). To prove $m' = m \cap K[X_i]$, consider the following diagram:

\[
\begin{array}{ccc}
K[X_i]/m' & \to & T_n/m'T_n \\
\downarrow & & \downarrow \\
K[X_i]/m' & \to & T_n/m
\end{array}
\]

As $K[X_i]/m'$ is finite over $K$, it is complete, but $K[X_i]$ is dense in $T_n$, thus the horizontal maps are surjective. But then the lower horizontal is isomorphism, then the upper horizontal is also isomorphism, and then the vertical map is isomorphism, thus we are done.

For 2, $K[X_i]/m'$ is a finite extension of $K$, thus has a unique valuation, let $N$ be large that $|\overline{x}| \leq |x|^N$, then for $i > N$, the quotient map factors uniquely as $K[X_i] \to T_n^{(i)} \to K/m'$. Then the kernel $m$ of $T_n^{(i)}$ is a maximal ideal (same reason as before) that satisfies $m \cap K[X_i] = m'$. Then we finish by item 1.

\[\square\]

Cor. (VIII.4.2.13) (Analytification for Affine Schemes). Similarly, for an affine scheme $Z = \text{Spec} K[X_i]/\mathfrak{a}$ of f.t over $K$, we construct its analytification $Z^{rig}$ as the glue of the inclusions:

\[
\text{Sp}(T_n^{(0)}/\mathfrak{a}) \hookrightarrow \text{Sp}(T_n^{(1)}/\mathfrak{a}) \hookrightarrow \ldots
\]

Then $Z^{rig}$ is the analytification of $K[X_i]/\mathfrak{a}$.

And we see from the proof of (VIII.4.2.12) the maximal spectrum $\text{Max}(K[X_i]/\mathfrak{a}) = \bigcup_n \text{Spa}(T_n^{(i)}/\mathfrak{a})$ as sets.

Proof: The canonical map $K[X_i]/\mathfrak{a} \to T_n^{(i)}/\mathfrak{a}$ glue together to be a morphism $O_Z(Z) \to O_{Z^{rig}}(Z^{rig})$, which by (VIII.4.2.9) corresponds to a map $Z^{rig} \to Z$ of local ringed spaces.

Now any other morphism $Y \to Z$ from a rigid $K$-space $Y$ to $Z$, choose an affinoid $K$-space covering $Y_i$ of $Y$, then the map $Y_i \to Z$ corresponds by (VIII.4.2.9) to a morphism $\sigma : K[X_i]/\mathfrak{a} \to O_{Y_i}(Y_i)$, thus if we choose $i$ large enough that $|\sigma(x_i)| \leq |x|^i$, then $\sigma$ can be extended uniquely to

\[K[X_i]/\mathfrak{a} \to T_n^{(i)}/\mathfrak{a} \xrightarrow{\sigma} O_{Y_i}(Y_i),\]

By the universality of affinoid subdomains. This $\sigma$ corresponds to a morphism $Y_i \to \text{Sp}(T_n^{(i)}) \to Z^{rig}$, and these clearly glue together to give a morphism $Y \to Z^{rig}$, thus proving the universal property. \[\square\]

Prop. (VIII.4.2.14) (General Analytification). For any locally algebra scheme $X$ over $K$, choose an affine covering $Z_i$, consider the analytification of $Z_i$ by (VIII.4.2.13), then $Z_i \cap Z_j$ obviously has the inverse image as the rigid analytification by universal property, thus unique, so the analytifications of $Z_i$ can be glued to an analytification of $X$.

Moreover, the underlying set of $X^{rig}$ is identified with the closed pts of the scheme $X$, because this is the case of $Z_i$ (VIII.4.2.11).
Prop. (VIII.4.2.15). Rigid analytification preserves fiber products.

Proof: This follows from the construction of fibered product of schemes (V.2.7.15), so we only need to prove the affine case. For this, Cf.[U. Köpf, Über eigentliche Familien algebraischer Varietäten über affinoiden Räumen. Schriftenr, Satz 1.8]. □

Prop. (VIII.4.2.16) (Stalks). For a point \( z \in Z^{\text{rig}} \), the completion of \( \mathcal{O}_{Z^{\text{rig}},z} \) and \( \mathcal{O}_{Z,z} \) are the same.

Proof: Cf.[U. Köpf, Über eigentliche Familien algebraischer Varietäten über affinoiden Räumen. Schriftenr, Satz 2.1]. □

3 Coherent Sheaves on Rigid Spaces

Prop. (VIII.4.3.1). For an affinoid \( K \)-space \( X \), there is a \( \text{Qco} \) module construction \( M \to M \otimes_A \mathcal{O}_X \) as in (VIII.4.1.40) in the weak Grothendieck topology, and it extends uniquely to a sheaf w.r.t. the strong Grothendieck topology by (VIII.4.1.41), also denoted by \( M \otimes_A \mathcal{O}_X \). This is a faithfully exact, fully faithful functor between Abelian categories from \( \text{Ab} \) to \( \mathcal{O}_X \)-modules, and it preserves tensor product and direct sums.

Proof: Because \( \Gamma(X, M \otimes_A \mathcal{O}_X) = M \) and obviously fully faithful, this map is fully faithful, and it is exact because the restriction map of an affinoid subdomain is flat (VIII.4.1.26), and shifification is exact. □

Def. (VIII.4.3.2) (Coherent Sheaves). For an \( \mathcal{O}_X \)-module \( \mathcal{F} \) on a rigid space \( X \), finite type, of finite presentation, coherence are defined w.r.t the strong topology as \( X \) is a ringed site. All these notions are stable under passing to an admissible open subspaces.

Proof: For the passing of coherence to admissible open subspaces, use the fact that restriction maps are flat (VIII.4.4.26). □

Cor. (VIII.4.3.3). Notice \( \mathcal{O}_X^n = A^n \otimes_A \mathcal{O}_X \), by (VIII.4.3.1) and the fact \( A \) is Noetherian, passing to a refinement covering, \( \mathcal{F} \) is coherent iff there is an admissible affinoid covering \( \mathcal{U} : X_1 \to X \) that \( F|_{X_1} \) is associated to a finite \( \mathcal{O}_{X_1} \)-module. In this case, \( \mathcal{F} \) is said to be \( \mathcal{U} \)-coherent. Thus the coherent sheaves form a weak Serre subcategory of \( \mathcal{O}_X \)-modules.

In particular, \( \mathcal{O}_X \) is coherent.

Prop. (VIII.4.3.4). If \( \mathcal{F}, \mathcal{G} \) are all \( \mathcal{U} \)-coherent modules, then:

- \( \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \) and \( \mathcal{F} \oplus \mathcal{G} \) are \( \mathcal{U} \)-coherent,
- if \( \mathcal{F} \to \mathcal{G} \) is a \( \mathcal{O}_X \)-module morphism, then the kernel and image are all \( \mathcal{U} \)-coherent.
- If \( \mathcal{I} \) is a \( \mathcal{U} \)-coherent sheaf of ideal of \( \mathcal{O}_X \), then \( \mathcal{I} \mathcal{F} \) is \( \mathcal{U} \)-coherent.

Proof: The first and the second are consequences of (VIII.4.3.1), noticing \( A_i \) is Noetherian. The third is an image of \( \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F} \). □

Lemma (VIII.4.3.5). If \( \mathcal{F} \) is \( \mathcal{U} \) coherent for a simple Laurent covering \( \mathcal{U} \), then \( H^1(\mathcal{U}, \mathcal{F}) = 0 \).

Proof: The goal is to show any element in \( \mathcal{F}(U_1 \cap U_2) \) can be represented by \( u_1 + u_2 \), where \( u_i \in \mathcal{F}(U_i) \). Let \( U_1 = \text{Sp} A(f), U_2 = \text{Sp} A(f^{-1}), U_1 \cap U_2 = \text{Sp} A(f, f^{-1}) \). Now \( A(f) = A(X)/(X - f) \).
And endow them with the residue norm, which is complete.

Now we want to give norms to \( M_1 = \mathcal{F}(U_1), M_2 = \mathcal{F}(U_2), M_{12} = \mathcal{F}(U_1 \cap U_2). \) \( M_i \) are finite \( \mathcal{O}_X(U_i) \)-modules, so there are elements \( v_i, w_j, i \leq m, j \leq n \) that generate \( M_1, M_2 \) respectively. So there are attached morphisms

\[
(A(f))^m \to M_1, \quad (A(f^{-1}))^n \to M_2, \quad (A(f, f^{-1}))^m \to M_{12}
\]

And endow them with the residue norm, which is complete.

Notice that to prove the assertion, it suffice to show for each \( \varepsilon > 0 \), there is an \( \alpha \) that for each \( u \in 12 \), there are \( u_1 \) and \( u_2 \) in \( M_i \) respectively that \( |u_i| < \alpha|u| \) and \( |u - u_1 - u_2| < \varepsilon|u| \), because then we can use iteration and completeness to get the result.

Giving \( \beta > 1 \), any \( g \in A(f, f^{-1}) \) can be lifted to an element \( \sum c_{ij}X^iY^j \) that \( |c_{ij}| \leq \beta|g| \). Then by regrouping terms that \( i \geq j \) or \( i < j \), there are two term \( g^+ \in A(f) \) and \( g^- \in A(f^{-1}) \) that \( g^+ + g^- \) restricts to \( g \) on \( U_1 \cap U_2 \), and \( |g^+|, |g^-| \leq \beta|g| \).

Now that \( \mathcal{F} \) is coherent, so \( v_i \) and \( w_j \) both generate \( M_{12} \) separately. Then there are equations \( v_i = \sum c_{ij}w_j \) and \( w_i = \sum d_{ij}v_j \), where \( c_{ij}, d_{ij} \in A(f, f^{-1}) \). The image of \( A(f) \) is dense in \( A(f, f^{-1})(\mathbb{I}.9.4.26) \), so there are elements \( c_{ij} \in A(f^{-1}) \) s.t. \( \text{max}_{ij} |c_{ij} - d_{ij}|d_{ij}| < \beta^{-2}\varepsilon \).

Now I claim the above approximation process is true for \( \alpha = \beta^2 \text{max}(|c'_{ij}| + 1) \). For this, notice for any \( u = \sum a_iv_i \) with \( a_i \in A(f, f^{-1}) \), which we may assume \( |a_i| \leq \beta|u| \) by the definition of the norm on \( M_{12} \), then \( a_i = a_i^+ + a_i^- \), that \( |a_i^-| \leq \beta|a_i| \). Consider the following element

\[
u^+ = \sum a_i^+v_i \in M_1, \quad u^- = \sum a_i^- \sum c_{ij}w_j \in M_2
\]

Then it is easily verified that \( |u^*| < \alpha|u| \), and

\[
u - u^- - u^+ = \sum \sum a_i^-(c_{ij} - c'_{ij})w_j = \sum \sum a_i^-(c_{ij} - c'_{ij})d_{ij}v_i.
\]

which has norm smaller than \( \text{max} |a_i^- (c_{ij} - c'_{ij})|d_{ij}| \leq \beta^2|u| \cdot \beta^{-2}\varepsilon = \varepsilon|u| \), finishing the proof. \( \square \)

Prop. (VIII.4.3.6) (Kiehl). An \( \mathcal{O}_X \)-module \( \mathcal{F} \) on an affinoid \( K \)-space \( \text{Sp} A \) is coherent iff it is associated to a finite \( A \)-module.

Proof: The converse is obvious, for the other direction, by (VIII.4.1.39), it suffices to prove for \( \mathfrak{U} \) a Laurent covering, and further, it suffices to prove for the simplest Laurent covering \( X(f), X(f^{-1}) \to X \) because: \( U(f, g) \cup U(f, g^{-1}) \cup U(f^{-1}, g) \cup U(f^{-1}, g^{-1}) = (U(f, g) \cup U(f, g^{-1})) \cup (U(f^{-1}, g) \cup U(f^{-1}, g^{-1})) \).

Thus the above lemma shows that \( H^1(\mathfrak{U}, \mathcal{F}) = 0 \). Now I prove that for any finite affinoid covering \( \mathfrak{U} = \cup \text{Sp} A_i, \) if \( H^1(\mathfrak{U}, \mathcal{F}) = 0 \) for any coherent sheaf \( \mathcal{F} \), then any \( \mathfrak{U} \)-coherent sheaf \( \mathcal{F} \) is associated to a finite \( A \)-module, this will finish the proof.

Consider any maximal ideal \( \mathfrak{m}_x \) of \( A \), \( \mathfrak{m}_x \otimes_A \mathcal{O}_X \) is a coherent sheaf as \( \mathfrak{m}_x \) is finite because \( A \) is Noetherian, so there is a short exact sequence

\[
0 \to \mathfrak{m}_x \mathcal{F} \to \mathcal{F} \to \mathcal{F}/\mathfrak{m}_x \mathcal{F} \to 0
\]

of \( \mathfrak{U} \)-coherent sheaves, because \( A/\mathfrak{m}_x \) is a field, thus flat.

Now for any affinoid space \( U' \) in \( U_i \) for some \( i \), the section of this exact sequence is exact, because the ring morphism associated to an affinoid subdomain is flat (VIII.4.1.26). In particular, this can
be applied to any intersections of $U_i$, in particular the Čech complex of these sheaves. Then the long exact sequence and the fact $H^1(\mathcal{U}, \mathcal{m}_x \mathcal{F}) = 0$ shows

$$0 \to \mathcal{m}_x \mathcal{F}(X) \to \mathcal{F}(X) \to \mathcal{F}/\mathcal{m}_x \mathcal{F}(X) \to 0$$

Next we want to show $\mathcal{F}/\mathcal{m}_x \mathcal{F}(X) \to \mathcal{F}/\mathcal{m}_x \mathcal{F}(U_i)$ is isomorphism for any $x \in U_i$. To prove this, first for any affinoid subspace $U' = \text{Sp} \mathcal{B}$ contained in some $U_j$, let $U' \cap U_i = \text{Sp} \mathcal{B}_i$, $\mathcal{F}|_{U'} = M' \otimes_{\mathcal{O}} \mathcal{O}_{U'}$, we show $\mathcal{F}/\mathcal{m}_x \mathcal{F}(U') \cong \mathcal{F}/\mathcal{m}_x \mathcal{F}(U' \cap U_i)$, this is equivalent to

$$M'/\mathcal{m}_x M' \to M'/\mathcal{m}_x M' \otimes_{\mathcal{B}_i} \mathcal{B}_i = M'/\mathcal{m}_x M' \otimes_{\mathcal{B}/\mathcal{B}_j} \mathcal{B}_j/\mathcal{m}_x \mathcal{B}_j$$

is an isomorphism. But $\mathcal{B}/\mathcal{m}_x \mathcal{B} \cong \mathcal{B}_j/\mathcal{m}_x \mathcal{B}_j$: This is true when $x \in U'$ by(VIII.4.1.8), and they are both trivial ring if $x \notin U'$. Then look at the morphism of Čech complex induced by $\mathcal{F}/\mathcal{m}_x \mathcal{F} \to \mathcal{F}/\mathcal{m}_x \mathcal{F}_{U_i}$, then it is an isomorphism, by what we just proved, so its $H^0$ is also isomorphism, which is $\mathcal{F}/\mathcal{m}_x \mathcal{F}(X) \cong \mathcal{F}/\mathcal{m}_x \mathcal{F}(U_i)$.

Finally, by the commutative diagram

$$\begin{array}{ccc}
\mathcal{F}(X) & \longrightarrow & \mathcal{F}/\mathcal{m}_x \mathcal{F}(X) \\
\downarrow & & \downarrow \\
\mathcal{F}(U_i) & \longrightarrow & \mathcal{F}/\mathcal{m}_x \mathcal{F}(U_i)
\end{array}$$

morphism, so if denote $\mathcal{F}(U_i)$ by $M_i$, then $\mathcal{F}(X)$ generate $M_i/\mathcal{m}_x M_i$ for every $x$, then consider $L = M_i/\mathcal{F}(X)$, then $\mathcal{m}_x L = L$ for every $x$, then by Nakayama, for each maximal ideal $\mathcal{m}$, there is a $m \in \mathcal{m}$ that $(1 + m)L = 0$, so $\text{Ann}(L) = (1)$, so $L = 0$, i.e. $\mathcal{F}(X)$ generate $M_i$ for each $i$.

Now choose $f_i$ in $\mathcal{F}(X)$ that generate $M_i$ simultaneously, then the map $\mathcal{O}_X^m \to \mathcal{F}$ is a surjection of $\mathcal{U}$-coherent sheaves, its kernel $\mathcal{G}$ is also coherent by(VIII.4.3.4), now all the above argument works for $\mathcal{G}$, so there is a surjection $\mathcal{O}_X^m \to \mathcal{G}$, so $\mathcal{O}_X^m \to \mathcal{O}_X^m \to \mathcal{F} \to 0$, so $\mathcal{F}$ is associated to the cokernel of the map $A^m \to A^m$. \hfill \Box

Cor. (VIII.4.3.7). Coherence for a $\mathcal{O}_X$-module on a rigid $K$-space is affinoid local on the target.

Cohomology on Rigid Analytic Spaces

Lemma (VIII.4.3.8). The category of $\mathcal{O}_X$-modules on a rigid $K$-space is a Grothendieck category by(I.11.2.28).

Def. (VIII.4.3.9) (Derived Cohomologies). Consider the right derived functor for $\Gamma$ and more general $f_*$, these are left exact by(V.1.2.8). Then $R^p f_* \mathcal{F} = (f_* \mathcal{H}^p(\mathcal{F}))^\ell$ by Grothendieck spectral sequence.

The Čech-to-Derived spectral sequence(V.6.2.11) is applied: $\mathcal{H}^p(\{U_i \to \text{U} \}, \mathcal{H}^q(\mathcal{F})) \Rightarrow \mathcal{H}^{p+q}(U, \mathcal{F})$, $\mathcal{H}^p(U, \mathcal{H}^q(F)) \Rightarrow \mathcal{H}^{p+q}(U, \mathcal{F})$ and $\mathcal{H}^1(U, \mathcal{F}) \cong \mathcal{H}^1(U, \mathcal{F})$.

In particular, if $\mathcal{H}^q(U_{i_{1}\cdots i_r}, \mathcal{F}) = 0$, $q > 0$, then $\mathcal{H}^p(\{U_i \to \text{U} \}, \mathcal{F}) = \mathcal{H}^p(U, \mathcal{F})(V.6.2.13)$. And it is enough to have $\mathcal{H}^q(U_{i_{1}\cdots i_r}, \mathcal{F}) = 0$, $q > 0$ by(V.6.2.14).

Cor. (VIII.4.3.10). A Qco sheaf on an affinoid space has vanishing higher sheaf cohomology by Tate’s acyclicity(VIII.4.1.40) and(V.6.2.14).

Properties of Rigid $K$-Spaces

This subsubsection is strongly suggested to read after reading the parallel part of schemes.
Def. (VIII.4.3.11). A morphism is called a closed immersion if there is an admissible affinoid covering that it restricts to a closed immersion of affinoid spaces (It is compatible with definition (VIII.4.1.28) before by (VIII.4.3.15)). It is called an open immersion iff it is injective and the corresponding stalk maps are isomorphisms. The (quasi-)separatedness, quasi-compactness, finiteness are defined similarly as for schemes.

Lemma (VIII.4.3.12) (Nike’s Trick). In a rigid analytic $K$-space $X$ and $Sp A, Sp B$ be affinoid subspaces, then there is an admissible affinoid covering of $Sp A \cap Sp B$.

Proof: This is analogous to the scheme case (V.4.1.1), but the proof is different: $X$ has an admissible covering, this restricts to an admissible covering of $Sp A \cap Sp B$, and any admissible covering can be refined by an affinoid admissible covering. □

Prop. (VIII.4.3.13) (Affinoid Communication Theorem). A property $P$ of affinoid open subsets of $X$ is called affinoid local if: $Sp A$ has $P$ $\Rightarrow$ all affinoid subdomains of $Sp A$ has $P$, and any admissible affinoid cover of $Sp A$ has $P$ $\Rightarrow$ $Sp A$ has $P$. Notice a stalk-wise property is obviously affine-local.

Now if we call $X$ has $\tilde{P}$ if there is an admissible affinoid covering $A_i \to X$ that $A_i$ has $P$. Then the following are equivalent:

- all open affinoid subsets of $X$ has $P$.
- all open subspace of $X$ has $\tilde{P}$.
- $X$ has a cover of open subspaces that has $\tilde{P}$.
- $X$ has $\tilde{P}$.

Proof: The proof is the same as the scheme case (V.4.1.2). □

Prop. (VIII.4.3.14). Separated morphism is quasi-separated because closed immersion is affinoid hence quasi-compact (VIII.4.1.3).

Prop. (VIII.4.3.15) (Finite Morphism). For a morphism $\varphi : X \to Y$ of rigid $K$-spaces

- It is finite iff the inverse image of any affinoid space is affinoid, and $\varphi_* \mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-module. In particular, finiteness is local on the target because coherence do.
- It is closed immersion iff it is finite and $\mathcal{O}_Y \to \varphi_* \mathcal{O}_X$ is surjective, this shows the definition of closed immersion is compatible with before.

Proof: Coherence is affinoid local on the target by Kiehl’s theorem, so it suffices to prove the inverse image of any affinoid space is affinoid for a finite morphism: Consider any affinoid subdomain $U \subset X$ with inverse image $\varphi^{-1}(U)$, by Kiehl’s theorem, $B = \mathcal{O}_X(f^{-1}(U))$ is finite over $A = \mathcal{O}_Y(U)$, thus can be given an affinoid $K$-algebra structure (I.9.4.33). Now

$$\varphi^{-1}(U) \xrightarrow{\chi} Sp B \xrightarrow{\rho} Sp A$$

$\chi$ is locally an isomorphism, as $\rho$ is finite, so $\chi$ is an isomorphism.

The second assertion is because locally $\mathcal{O}_Y, \varphi_* \mathcal{O}_X$ are both Qco so surjectivity is equivalent to the global section is surjective (VIII.4.3.1).

□

Prop. (VIII.4.3.16). Closed/Open immersion, quasi-compactness, (quasi-)separatedness are all local on the target, and stable under base change.
Proof: Closed immersion is local on the target because finiteness do and surjectiveness of \( \mathcal{O}_Y \to \varphi_*\mathcal{O}_X \) is checked locally. Open immersion is local on the target because stalk and injectivity are all checked locally.

Then Closed/Open immersion are stable under base change because the affinoid case is true (VIII.4.1.29).

Quasi-compact is easily seen local on the target and stable under base change.

(Quasi-)Separateness is local on the target because closed immersion and quasi-compact do.

(Quasi-)Separateness is stable under base change because closed immersion and quasi-compact do, because diagonal commutes with base change (II.1.1.36).

\( \square \)

Prop. (VIII.4.3.17). Morphisms between affinoid \( K \)-spaces are separated. Moreover, because of localness, any finite morphism is separated.

\( \square \)

Prop. (VIII.4.3.18). By (II.1.1.38), for \( X \to S \) and \( Y \to S \), the map \( X = X \times_Y Y \to X \times_S Y \) is an immersion. It is closed immersion if \( Y \to S \) is separated, and it is qc if \( Y \to S \) is quasi-separated.

Cor. (VIII.4.3.19). If \( s : S \to X \) is a section of \( f : X \to S \), the above proposition applies to this case, because \( S = S \times_X X = S \times_S X = X \).

Prop. (VIII.4.3.20). A morphism is quasi-separated iff there is an admissible affinoid covering \( W_i \) that, for any two affinoid open \( U, V \) that are mapped to an affinoid open, their intersection is quasi-compact. This is because quasi-compact is local on the target.

A morphism is separated iff there is an admissible affinoid covering \( W_i \) that, for any two affinoid open \( U, V \) that are mapped to an affinoid open, their intersection is affinoid open and \( \mathcal{O}(U) \hat{\otimes} \mathcal{O}(W_i) \mathcal{O}(V) \to \mathcal{O}(U \cap V) \) is surjective. This is because closed immersion is local on the target (VIII.4.3.16).

Cor. (VIII.4.3.21). If \( g \circ f \) is (quasi-)separated, then so is \( f \).

Cor. (VIII.4.3.22). If \( X \) is (quasi-)separated, then \( X \to Y \) is (quasi-)separated.

4 Proper Mapping Theorem

Def. (VIII.4.4.1). For a rigid space \( X \) over affinoid space \( Y \), if \( U \subset U' \subset X \) be affinoid subspaces, \( U \) is called relatively compact in \( U' \) iff there is a set of affinoid generators \( f_i \) of \( \mathcal{O}_X(U') \) over \( \mathcal{O}_Y(Y) \) that \( |f_i(x)| < 1 \) on \( U \). This is denoted by \( U \subseteq_Y U' \).

Prop. (VIII.4.4.2). If \( X_1, X_2 \) are affinoid spaces over an affinoid space \( Y \), and \( U_i \) are affinoid space of \( X_i \), then

- if \( U_1 \subseteq_Y X_1 \), then \( U_1 \times_Y X_2 \subseteq_X X_1 \times_Y X_2 \).
- if \( U_1 \subseteq_Y X_i \), then \( U_1 \times_Y U_2 \subseteq_Y X_1 \times_Y X_2 \).
- If \( U_i \subseteq_Y X_i \), and \( X_i \) are affinoid subspaces of a rigid space separable over \( Y \), then \( U_1 \cap U_2 \subseteq_Y X_1 \cap X_2 \).
- If \( U_1 \subseteq_Y X_1 \), and \( i : T \to X_1 \) is a closed immersion, then \( i^{-1}(U_1) \subseteq_Y i^{-1}(X_2) \).

The proof is easy. For the last one, should notice \( |f(x)| = |f(i(x))| \), because it is closed immersion, so the residue field is the same.
Def. (VIII.4.4.3) (Proper Morphism). A proper morphism \( \varphi : X \to Y \) of rigid \( K \)-spaces is a separated morphism that there is an admissible affinoid covering \( Y_i \) of \( Y \) that there are two admissible affinoid coverings \( X_{ij}, X'_{ij} \) of \( \varphi^{-1}(Y_i) \) that \( X_{ij} \subseteq Y_i, X'_{ij} \) for any \( i, j \).

Prop. (VIII.4.4.4). Properness is stable under base change and composition

Proof: The base change follows directly from (VIII.4.4.2).

For the composition, Cf.[Formal and Rigid Geometry P131](difficult).

Prop. (VIII.4.4.5). Properness is local on the target.

Proof: This is because separatedness is local on the target (VIII.4.3.16) and the second condition of properness is itself local.

Prop. (VIII.4.4.6). If \( g \circ f : X \to Y \to Z \) is proper and \( g \) is separated, then \( f \) is proper.

Proof: By (VIII.4.3.18), \( \tau : X \to X \times_Z Y \) is closed immersion, and \( f \) is separated by (VIII.4.3.21). Now proper is local, so we may assume \( Z \) is affinoid, so there are two admissible covering \( X_i, X'_i \) of \( X \) that \( X_i \subseteq Z X'_i \), and choose an admissible affinoid covering \( Y_i \to Y \), then \( X_j \times_Z Y_i, X'_j \times_Z Y_i \) are admissible coverings of \( Y_i \) that is \( X_j \times_Z Y_i \subseteq Y_i, X'_j \times_Z Y_i \). And it can be pulled back to an affinoid admissible coverings of \( f^{-1}(Y_i) \) that \( \tau^{-1}(X_j \times_Z Y_i) \subseteq Y_i, \tau^{-1}(X'_j \times_Z Y_i) \), because \( \tau \) is closed immersion. So \( X \to Y \) is proper.

Prop. (VIII.4.4.7). Finite morphism is proper, in particular, closed immersion is proper.

Proof: Finite morphism is separated by (VIII.4.3.17), and locally, assume both space are affinoid, \( X = \text{Sp} B \to \text{Sp} A = Y \), then \( B \) is a finite \( A \)-module, in priori a \( f, g \), \( A \)-algebra, so there is a set of generators \( f_i \) of \( B \) over \( A \) that (by multiplying a constant in \( K^* \) \( |f_i|_{sup} < 1 \)(I.9.4.21), so \( X \subseteq Y X \), hence it is proper.

Prop. (VIII.4.4.8) (Proper and Analytification). For a morphism between schemes locally of f.t. over \( K \), it is proper iff its rigid analytification is proper.


For the following: Cf.[Formal and Rigid Geometry P132].

Prop. (VIII.4.4.9) (Proper Mapping theorem, Kiehl). The higher direct images of a proper map of rigid analytic spaces takes coherent sheaves to coherent sheaves.

Proof:

Prop. (VIII.4.4.10). For a scheme \( X \) locally of f.t. over \( K \), an \( \mathcal{O}_X \)-module \( \mathcal{F} \) on \( X \) gives rise to an \( \mathcal{O}_{X_{\text{rig}}} \)-module on \( X_{\text{rig}} \), and it is coherent iff \( \mathcal{F} \) is coherent.

Proof:

Prop. (VIII.4.4.11). For a proper scheme over \( K \), \( H^q(X, \mathcal{F}) \cong H^q(X_{\text{rig}}, \mathcal{F}_{\text{rig}}) \) for \( \mathcal{F} \) coherent.

Proof:

Prop. (VIII.4.4.12). When \( X \) is proper, coherent sheaves on \( X_{\text{rig}} \) corresponds to coherent sheaves on \( X \). This gives an analog of Chow’s theorem when applied to \( X = \mathbb{P}^n_K \) and \( \mathcal{F}' \) is a sheaf of ideals in \( \mathcal{O}_{X_{\text{rig}}} \).

Proof:
5 Formal Geometry

Main references for this subsection is [Bos15], [Hartshorne] and [Topics in Algebraic Geometry, Illusie].

**Def. (VIII.4.5.1) (Formal Spectrum \(\text{Spf} A\)).** Let \(A\) be a complete adic ring (I.9.1.8) with an ideal of definition \(a\) that the \(A\) is \(a\)-adically complete and separated. Then we let \(\text{Spf} A\) be the topological space \(\text{Spec}(A/a)\) (open-adically complete), and there is a structure sheaf \(\mathcal{O}\) on \(\text{Spf} A\) that \(\mathcal{O}(D(f)) = A\langle f^{-1}\rangle\) (I.9.1.10).

**Proof:** To construct this sheaf, we first check that \(\mathcal{O}(D(f)) = A\langle f^{-1}\rangle = \lim_{n} (A/a^n[f^{-1}])\) defines a sheaf on the site of subspaces of \(\text{Spf} A\) of the form \(D(f)\): For any open covering \(\{D(f_i)\}\) of \(D(f)\), there are exact sequences:

\[
0 \rightarrow (A/a^n)_f \rightarrow \prod_i (A/a^n)_{f_i} \rightarrow \prod_{i,j} (A/a^n)_{f_if_j},
\]

by (I.7.2.3). Then we take inverse limit, which is exact by Mittag-Leffler (I.10.3.2), to get an exact sequence, which is just the sheaf condition of \(\mathcal{O}\). Then, we can use (V.1.2.21) to extend this sheaf to a sheaf on \(\text{Spf} A\).

**Prop. (VIII.4.5.2) (Stalks of \(\text{Spf} A\)).** Let \(x \in \text{Spf} A\) correspond to a prime \(p_x\) in \(A\). Then the stalk of \(\text{Spf} A\) at \(x\) is just \(\mathcal{O}_x = \lim_{x \in D(f)} A\langle f^{-1}\rangle\), which is a local ring with maximal ideal \(m_x\) containing \(p_x \mathcal{O}_x\). Moreover, \(m_x = p_x \mathcal{O}_x\) iff \(a\) is f.g.. So \(\text{Spf} A\) is a local ringed space, called the affine formal scheme of \(A\).


**Remark (VIII.4.5.3).** In many cases, for example in Scholze’s treatment of the \(p\)-adic geometry, the ideal of definition \(a\) is assumed to be f.g., because we need this to show that the \(a\)-adic completion of \(A\langle f^{-1}\rangle\) is \((a)\)-adically complete (I.5.7.6), so that we can interpret \(D(f) \subset \text{Spf} A\) as an affine formal scheme \(\text{Spf}(A\langle f^{-1}\rangle)\) (I.9.1.11).

Another way to get around this finiteness condition is to consider formal spectrum for a larger class of rings, called the admissible rings, which is a complete and separated topological ring with a basis consisting of open ideals, and with an ideal of definition \(a\) that \(a^n \rightarrow 0\) for \(n \rightarrow 0\).

**Prop. (VIII.4.5.4) (Affine Formal Adjunction).** Morphisms between local topologically ringed spaces \(\text{Spf} B \rightarrow \text{Spf} A\) corresponds to continuous homomorphisms \(A \rightarrow B\).

**Def. (VIII.4.5.5) (Formal Schemes).** The category of formal schemes is the full subcategory of the category of local topologically ringed topological spaces \((X, \mathcal{O}_X)\) consisting of objects that is locally isomorphic to an affine formal scheme \(\text{Spf} A\).

The category of formal schemes contains the category of schemes, by mapping \(\text{Spec} A\) to \(\text{Spf} A\), where \(A\) is endowed with the discrete topology.

**Prop. (VIII.4.5.6) (Glueing and Fiber Products).** Formal Schemes can easily be glued, and also spectrum adjointness holds as in (VIII.4.2.5). Finally there are fibered products, constructed as in (V.2.7.15), where the affine case corresponds to completed tensor product (I.9.1.12).

**Def. (VIII.4.5.7) (Formal Completion of Schemes Along a Closed Subscheme).** Let \(X\) be a scheme and \(Y\) a closed subscheme of \(X\) defined by a Qco ideal \(\mathcal{I} \subset \mathcal{O}_X\), then consider the sheaf \(\mathcal{O}_Y\) defined by restricting the projective limit \(\lim_{n} \mathcal{O}_X/\mathcal{T}^n\) to \(Y\), then \((Y, \mathcal{O}_Y)\) is a locally topologically ringed space, called the formal completion of \(X\) along \(Y\).
6 Admissible Formal Schemes

Remark (VIII.4.6.1). Let $R$ be an adic ring as in setup (I.9.1.13).

Def. (VIII.4.6.2) (Admissible Formal Schemes). Let $R$ be an adic ring, then a formal $R$-scheme $X$ is called locally of topologically finite type/finite presentation/admissible if there is an open affine covering $\text{Spf } A_i$ of $X$ that $A_i$ satisfies those properties.

$X$ is called topologically of finite type if it is locally of topologically finite type and quasi-compact. It is called topologically of finite presentation if it is locally of topologically finite presentation, quasi-compact and quasi-separated.

Prop. (VIII.4.6.3) (Induced Admissible Formal Scheme). Let $X$ be a formal $R$-scheme that is locally of topologically finite type, and let $\mathcal{O}_X$ be its structure sheaf. Then we can look at the ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ consisting of all elements locally killed by a power of $I^n$. This is a qco sheaf, as $\mathcal{I}(U) = \{ f \in \mathcal{O}(U) | I^n f = 0 \text{ for some } n \}$, because the quotient by the RHS is locally topologically of finite type and has no $I$-torsion, thus admissible, by (I.9.1.16), and admissibility is local (I.9.1.17).

In particular we can take the closed subscheme $X_{\text{adm}} \subset X$ corresponding to $\mathcal{I}$, then it is an admissible formal scheme, called the induced admissible formal scheme of $R$.

Prop. (VIII.4.6.4) (Generic Fiber Functor). Let $R$ be a complete valuation ring of height 1 with field of fraction $K$, then the functor $A \mapsto A \otimes_R K$ from the category of $R$-algebras topologically of finite type to the category of affinoid $K$-algebras


7 Formal Models

Def. (VIII.4.7.1) (Formal Models). In view of (VIII.4.6.4), one would like to describe all formal $R$-schemes that the generic fiber $X_{\text{rig}}$ is isomorphic to a given rigid $K$-space $X_K$. But first notice $A \mapsto A \otimes_R K$ kills all $R$-torsion, in particular the generic fiber functor only depends on the induced admissible formal scheme $X_{\text{adm}}$ (VIII.4.6.3). So given any rigid $K$-space $X_K$, any admissible formal $R$-scheme $X$ satisfying $X_{\text{rig}} \cong X_K$ is called a formal $R$-model of $X_K$.

Def. (VIII.4.7.2) (Admissible Formal Blowing-up).
VIII.5 Adic Space and Perfectoid Space(Scholze)

Main References are [Hub93], [Hub96], [Mor19], [Bha17], [Wed14], [S-W20], and [Sch12].

1 (Continuous) Valuation Spectrums

Main references are [Mor19]. Notice this should be prior to the definition of adic spaces.

Def. (VIII.5.1.1) (Riemann-Zariski Space). Let $K$ be a field and $A$ be a subring, the Riemann-Zariski space $RZ(K, A)$ is defined to be the set of all valuation subrings of $K$ containing $A$ that has the topology generated by

$$U(x_1, \ldots, x_n) = \{P \in RZ(K, A) | x_1, \ldots, x_n \in P \}.$$  

$RZ(K, 0)$ is also denoted by $RZ(K)$. $RZ(K, A)$ is just isomorphic to $Spa(K, A^{itc})$, so it is spectral by (VIII.5.2.24).

Cor. (VIII.5.1.2). Clearly the specialization relations of $RZ(K, A)$ is identical to inclusion relations.

Def. (VIII.5.1.3) (Valuation Spectrum). Let $A$ be a ring, the valuation spectrum $Spv(A)$ is the set of equivalent classes of valuations on $A$, topologized by the open subsets

$$Spv(A)(\frac{f}{g}) = \{x \in Spa(A) | |f(x)| \leq |g(x)| \neq 0 \}.$$  

$Spv(A)$ is spectral, with sub-basis generated by $Spv(\frac{f}{g})$.

There is a kernel map $Ker : Spv(A) \to Spec A$ sending a valuation to its kernel(support). Then this map is continuous, and the fiber of this map over $p$ is just isomorphic to the Riemann-Zariski space $RZ(k(p))$.

Moreover, the map $Ker : Spv(A) \to Spec A$ is spectral, as the kernel of $D(f)$ is $U(\frac{f}{g})$.

Specialization Relations in $Spv(A)$

Def. (VIII.5.1.4) (Vertical Specializations). Let $x, y \in Spv(A)$. We say that $x$ is a vertical specialization of $y$ if $x$ is a specialization of $y$ and $p_x = p_y$.

Valuations with Support Conditions

Def. (VIII.5.1.5) ($Spv(A, J)$). We define a space $Spv(A, J) \subset Spv(A)$ by

$$Spv(A, J) = \{x \in Spv(A) | r(x) = x \} = \{x \in Spv(A) | c\Gamma_x(J) = \Gamma \}.$$  

with the subset topology from $Spv(A)$.

Prop. (VIII.5.1.6).

- $Spv(A, J)$ is a spectral space.
- A basis of quasi-compact open subsets for the topology is given by the sets $U(\frac{f_1, \ldots, f_n}{g})$ where $J \subset \sqrt{(f_1, \ldots, f_n)}$.
- The retraction $r : Spv(A) \to Spv(A, J)$ is a spectral map.

Proof: Cf.[Mor19]P52. □
Continuous Valuation Spectrum

Def. (VIII.5.1.7) (Continuous Valuations). Let $A$ be a Huber ring, then we define $\text{Cont}(A)$ as the subspace of $\text{Spv}$ consisting of continuous valuations on $A$, then

$$\text{Cont}(A) = \{ x \in \text{Spv}(A, A^{00}) | x(A^{00}) < 1 \}$$

Proof: Cf. [Mor19]P64.

Cor. (VIII.5.1.8) (Cont$(A)$ is Spectral). For any Huber ring $A$, $\text{Cont}(A)$ is a spectral space, with a basis of quasi-compact open subsets given by $U(f_1, \ldots, f_n)$, where $A^{00} \subset \sqrt{(f_1, \ldots, f_n)}$, or equivalently $(f_1, \ldots, f_n)$ is open.

Proof: This is because

$$\text{Cont}(A) = \text{Spv}(A, A^{00}) - \bigcup_{g \in A^{00}} U(1/g),$$

which is an open subset of $\text{Spv}(A, A^{00})$, thus the assertion follows from (VIII.5.1.6) and (IX.1.15.7).

2 Adic Spectrums

Def. (VIII.5.2.1) (Adic Spectrums). Let $(A, A^+)$ be a Huber ring, the adic spectrum is defined to be

$$\text{Spa}(A, A^+) = \{ x \in \text{Cont}(A) | ||A^+||_x \leq 1 \}$$

For a Huber ring $A$, denote $\text{Spa} A = \text{Spa}(A, A^0)$, where $A^0$ is the ring of power-bounded elements (I.9.5.8).

The shape of these open sets is dictated by the desired properties that both $\{ x | f(x) \neq 0 \}$ and $\{ x | f(x) \leq 1 \}$ be open. These desiderata combine features of classical algebraic geometry and rigid geometry, respectively.

Prop. (VIII.5.2.2). The adic spectrum construction defines a contravariant functor from the category of Huber pairs to the category of topological spaces. And for any ring of integers $A^+$, $\text{Spa} A = \text{Spa}(A, A^0) \hookrightarrow \text{Spa}(A, A^+)$ is an immersion of spaces, by (I.9.5.13).

Def. (VIII.5.2.3) (Kernel map). Taking kernels of valuations gives a map $\text{Ker} : \text{Spa}(A, A^+) \to \text{Spec} A$. This map is continuous, as the inverse image of $D(f)$ is $\text{Spa}(A, A^+)(f)$. We call a subset a Zariski open subset of $\text{Spa}(A, A^+)$ iff it is open in the initial topology along $\text{Ker}$.

Def. (VIII.5.2.4) (Rational Subsets). A rational subset of $\text{Spa}(A, A^+)$ is defined to be

$$\text{Spa}(A, A^+)(f_1, \ldots, f_n) = \{ x \in \text{Spa}(A, A^+) | x(f_i) \leq x(g) \},$$

where $(f_i)$ is an open ideal.

Prop. (VIII.5.2.5) (Adic Spectrums are Spectral). The adic spectrum $\text{Spa}(A, A^+)$ is spectral, and a basis of quasi-compact open subsets are given by rational subsets. And the $\text{Spa}$ functor is naturally a functor from the category of Huber rings to the category of spectral spaces.
VIII.5. ADIC SPACE AND PERFECTOID SPACE (SCHOLZE) 803

Proof: Firstly Spa\((A, A^+)\) is closed in the constructible topology of \(\text{Cont}(A)\): for any \(a \in A\),

\[
\{ x \in \text{Cont}(A) | |a|_x \leq 1 \} = U\left( \frac{1}{1}, \frac{a}{1} \right)
\]

is a quasi-compact open subset of \(\text{Cont}(A)\), so constructible, thus \(\text{Spa}(A, A^+) = \bigcap_{a \in A^+} \{ x \in \text{Cont}(A) | |a|_x \leq 1 \} \) is closed in the constructible topology.

So the assertions follow from (IX.1.15.7) and (VIII.5.1.8).

\[\square\]

Remark (VIII.5.2.6). In spite of the proposition that adic spaces are spectral, it is very different from classical algebraic geometry. For example, the generalizations of a point \(y\) is totally ordered (localization of the valuation ring), but this nearly never happen for an affine variety.

Remark (VIII.5.2.7). The rational subsets forms a basis for the topology of \(\text{Spa}(A, A^+)\). But in general \(\text{Spa}(\frac{1}{x})\) is not quasi-compact, in particular, \(\text{Ker} : \text{Spa}(A, A^+) \rightarrow \text{Spec} A\) is not quasi-compact.

Def. (VIII.5.2.8) (Specialization Map). The specialization map

\[
\text{Sp} : \text{Spa}(A, A^+) \rightarrow \text{Spec}(A^+/A^{00})
\]

that maps a point \(x\) to the inverse image of the maximal ideal of \(R_x\) along the valuation map \(A^+ \rightarrow R_x\) it corresponds. It clearly lies in \(\text{Spec}(A^+/A^{00})\) as any pseudo-uniformizer is mapped to a pseudo-uniformizer in \(R_x\) thus in the maximal ideal.

This map is continuous and spectral, unlike that the kernel map (VIII.5.2.23): the inverse image of a \(D(f)\) for \(f \in A^+\) is the set of points \(x \in \text{Spa}(A, A^+)\) that \(x(f)\) is a unit, i.e. \(|x(f)| = 1\). As \(|x(f)| \leq 1\) for all \(f \in A^+\), this set is just \(\text{Spa}(A, A^+)(\frac{1}{f})\), so specialization map \(\text{sp}\) is both continuous and spectral.

Completed Residue Fields

Def. (VIII.5.2.9) (Completed Residue Fields). Let \(x \in \text{Spa}(A, A^+)\), then we denote by \(k(x)\) the fraction field of \(A/p_x\), with a valuation ring \(k^+(x)\). If \(x\) is not analytic, we set \(\kappa(x) = k(x), \kappa^+(x) = k^+(x)\). If \(x\) is analytic, we set \(\kappa(x) = k(x)^\wedge, \kappa^+(x) = k^+(x)^\wedge\). \((\kappa(x), \kappa(x)^+)\) is called the completed residue field of \(x\).

Prop. (VIII.5.2.10) (Vertical Generalizations). The morphism

\[
\text{Spa}(\kappa(x), \kappa(x)^+) \rightarrow \text{Spa}(A, A^+)
\]

induces a homeomorphism onto the set of vertical generalizations of \(x\).

\(x\) is analytic iff \(\kappa(x)\) is microbial.

Proof: For the first assertion, if \(x\) is not analytic, then \(\text{Spa}(k(x), k(x)^+) \cong \text{RZ}(k(x), k(x)^+)\) is homeomorphic to the vertical generalizations of \(x\).

If \(x\) is analytic, then by (VIII.5.2.17), \(\text{Spa}(\kappa(x), \kappa(x)^+) \rightarrow \text{Spa}(k(x), k(x)^+)\) is a homeomorphism. And now we need to check more that if \(R \in \text{RZ}(k(x), k(x)^+)\) corresponds to a vertical generalization \(y\), then \(|\cdot|_R\) is continuous iff \(|\cdot|_y\) is continuous. For this, see [Mor19] 107.

The second assertion follows from (I.9.6.7).
Prop. (VIII.5.2.11) (Residue of Rational Subsets). Let $f : (A, A^+) \to (A(\frac{T}{x}), A(\frac{T}{x})^+)$ be a rational subset, and $y \in \text{Spa}(A(\frac{T}{x}), A(\frac{T}{x})^+)$ with $x = \text{Spa}(f)(y)$, then the canonical map $(k(x), k(x)^+) \to (k(y), k(y)^+)$ induces an isomorphism of Huber pairs $(\kappa(x), \kappa(x)^+) \cong (\kappa(y), \kappa(y)^+)$ after completion.

Proof: Cf.Morel P108. \hfill \square

Def. (VIII.5.2.12) (Adic Points). An adic point is $\text{Spa}(K, K^+)$ where $(K, K^+)$ is a Huber pair that $K$ is either a complete non-Archimedean field or a discrete field, and $K^+$ is an open and bounded valuation subring(hence integrally closed). An analytic adic point is one that $K$ is complete non-Archimedean.

Prop. (VIII.5.2.13). The adic point is not a point in general. In fact, if $K$ is non-Archimedean, $\text{Spa}(K, K^+)$ is totally ordered by inclusion?, with a unique closed point corresponding to $K^+$ and a unique generic point corresponding to $\mathcal{O}_K$.

Prop. (VIII.5.2.14) (Valuation Ring Characterization of Spa). For a Huber pair $(A, A^+)$, there is a natural bijection between $\text{Spa}(A, A^+)$ and the set of maps $\varphi : (A, A^+) \to (K, K^+)$ that $\text{Spa}(K, K^+)$ is an adic point where $K_x = \kappa(x)$ and $x$ corresponds to the image of the closed point of $\text{Spa}(K, K^+)$ under the map $\text{Spa}(\varphi)$. And $x$ is analytic iff the corresponding adic point $\text{Spa}(K, K^+)$ is analytic.

Proof: Let $\varphi : (A, A^+) \to (K, K^+)$ be a map of Huber pairs, then correspond the maps gives a continuous valuation on $A$ which is in $\text{Spa}(A, A^+)$ and $p_x = \text{Ker}(\varphi)$. Notice $x$ is the image of the closed point of $\text{Spa}(K, K^+)$. And we get a map $k(x) \to K$ such that $k(x)^+ = k(x) \cap K^+$, and that has dense image by assumption, so after completion(when non-Archimedean) induces an isomorphism $(\kappa(x), \kappa(x)^+) \cong (K, K^+)$. Thus this is a bijection of sets. \hfill \square

Prop. (VIII.5.2.15) (Uniformity). If $A$ is uniform, then the map

$$A \to \prod_{x \in \text{Spa}(A, A^+)} \kappa(x)$$

is a homeomorphism of $A$ onto its image, where $\kappa(x)$ is the completed residue field(VIII.5.2.9).

Proof: Berkovich, Étale cohomology for non-Archimedean analytic spaces. \hfill \square

Cor. (VIII.5.2.16). Let $\mathcal{O}_X$ be the sheafification of $\mathcal{O}_X$, then if $A$ is uniform, $A \to H^0(X, \mathcal{O}_X)$ is injective.

Proof: In fact, $A \to H^0(X, \mathcal{O}_X) \to \prod_s \kappa(x)$ is injective. \hfill \square

Properties of Adic Spectrums

Prop. (VIII.5.2.17) (Properties of Adic Spectrums).

- The completion map $(A, A^+) \to (\hat{A}, \hat{A}^+)$ induces an homeomorphism on the adic spectrums that preserves rational subsets.
- $\text{Spa}(A, A^+)$ vanishes iff its completion $\hat{A}$ vanishes.
- (Adic Nullstellensatz)$A^+ = \{ f \in A | x(f) \leq 1, \forall x \in \text{Spa}(A, A^+) \}$. 


• If $A$ is complete, then $f \in A$ is a unit iff $|f|_x \neq 0$ for all $x \in \text{Spa}(A, A^+)$.  

• If $A$ is Tate, then $f$ is topologically nilpotent iff $|f|^n \to 0$ for any $x \in \text{Spa}(A, A^+)$.)

Proof: 1: Use the valuation ring characterization (VIII.5.2.14), a point of $x$ determined a continuous map $(A, \hat{A}^+) \to (K, \hat{K}^+)$. Now this extends to a map under completion, thus determines a point of $\text{Spa}(\hat{A}, \hat{A}^+)$, so the Spa map is surjective. And injectivity follows from the fact $A$ is dense in $\hat{A}$.

For the homeomorphism, just notice that if $f_i - f'_i, g - g' \in t^{N+1} \hat{A}$, then

$$\text{Spa}(f_1, \ldots, f_n) = \text{Spa}(f'_1, \ldots, f'_n),$$

where $f_n = t^N$. So now $A$ is dense in $\hat{A}$, if we choose $f_i, g \in A$, then this rational subset is clearly induced from $A$.

2: Cf. [Mor19] P104.
3: Cf. [Mor19] P91.
4: Cf. [Mor19] P106.
5: Cf. [Mor19] P106. \hfill \square

Cor. (VIII.5.2.18) (Generalizations in Adic Spectrum). The above proposition shows that the generalization relations of $\text{Spa}$ are easily determined, for an element $y$, all generalizations of $y$ are in bijection with $\text{Spec}(R_y/(t))$ as a poset, thus totally ordered, and each $y$ has a unique generic point as generalization, because it is microbial.

Moreover, $\text{Spa}(A, A^0)$ is closed under generalization in $\text{Spa}(A, A^+)$, and they have the same set of generic points.

Proof: The last assertion is because the generalizations of a point $y$ is just valuation rings containing $R_y$, and $R_y$ contains $A^0/p_y$, so does its generalizations. And for any generic point $x \in \text{Spa}(A, A^+)$, $A^0$ is mapped to the valuation ring $R_x$, because it is a rank 1 valuation, so if $t^k f^N \subset R_x$, then $f \in R_x$ because otherwise we have $|t| < |f^n|$ for $n$ large. \hfill \square

Def. (VIII.5.2.19) (Specialization Map). The specialization map

$$\text{Sp} : \text{Spa}(A, A^+) \to \text{Spec}(A^+/A^{00})$$

that maps a point $x$ to the inverse image of the maximal ideal of $R_x$ along the valuation map $A^+ \to R_x$ it corresponds. It clearly lies in $\text{Spec}(A^+/A^{00})$ as any pseudo-uniformizer is mapped to a pseudo-uniformizer in $R_x$ thus in the maximal ideal.

This map is continuous and spectral, unlike that the kernel map (VIII.5.2.23): the inverse image of a $D(f)$ for $f \in A^+$ is the set of points $x \in \text{Spa}(A, A^+)$ that $x(f)$ is a unit, i.e. $|x(f)| = 1$. As $|x(f)| \leq 1$ for all $f \in A^+$, this set is just $\text{Spa}(A, A^+)_{(1)}$, so specialization map $\text{sp}$ is both continuous and spectral.

Prop. (VIII.5.2.20) (Maximal Hausdorff Quotient). Let $X = \text{Spa}(A, A^+)$ be an affinoid Tate space, the if $\overline{X}$ is the quotient of $X$ by the equivalence relation generated by specialization, then $\overline{X}$ is the Hausdorffization of $X$, i.e. $\overline{X}$ is Hausdorff.

Proof: To show $\overline{X}$ is Hausdorff, if $x, y \in X$ is not mapped to the same point in $\overline{X}$, then by (VIII.5.2.18), we may assume $x, y$ is generic in $X$, and $\{x\} \cap \{y\} = \emptyset$. Now we must find two disjoint open subsets of $x, y$ that is stable under specialization. Cf. [Bhatt Perfectoid Spaces P75]. \hfill \square
Spectrality of Adic Spectrums

Def. (VIII.5.2.21) (Rational Subsets). A rational subset of \( \text{Spa}(A, A^+) \) is defined to be

\[
\text{Spa}(A, A^+)(\frac{f_1, \ldots, f_n}{g}) = \{ x \in \text{Spa}(A, A^+) | x(f_i) \leq x(g) \},
\]

where \((f_i) = (1)\).

Prop. (VIII.5.2.22). Rational subsets are stable under intersection (Easy).

Prop. (VIII.5.2.23). The rational subsets form a subbasis for the topology of \( \text{Spa}(A, A^+) \). But in general, \( \text{Spa}(\frac{f}{g}) \) is not quasi-compact, in particular, \( \text{Spv}(A) \to \text{Spa}(A, A^+) \) is not quasi-compact (proper).

Prop. (VIII.5.2.24) (Adic Spectrums are Spectral). The adic spectrum \( \text{Spa}(A, A^+) \) is spectral, and a basis of quasi-compact open subsets are given by rational subsets. And the Spa functor is naturally a functor from the category of Huber rings to the category of spectral spaces.

Proof: Firstly \( \text{Spa}(A, A^+) \) is closed in the constructible topology of \( \text{Cont}(A) \): for any \( a \in A \),

\[
\{ x \in \text{Cont}(A) | |a|_x \leq 1 \} = U\left(\frac{1}{a}\right)
\]

is a quasi-compact open subset of \( \text{Cont}(A) \), so constructible, thus

\[
\text{Spa}(A, A^+) = \cap_{a \in A^+} \{ x \in \text{Cont}(A) | |a|_x \leq 1 \}
\]

is closed in the constructible topology.

So the assertions follow from (IX.1.15.7) and (VIII.5.1.8). \( \square \)

Remark (VIII.5.2.25). In spite of the proposition that adic spaces are spectral, it is very different from classical algebraic geometry. For example, the generalizations of a point \( y \) is totally ordered (localization of the valuation ring), but this nearly never happen for an affine variety.

Cor. (VIII.5.2.26) (Detecting Nilpotence Locally). If \((A, A^+)\) is an affinoid Tate ring and \( f \in A \), then \( f \in A^{00} \) iff \( |f(x)|_n \to 0 \) for all \( x \).

Proof: If \( f \) is topological nilpotent, then \( f^N \in tA^+ \) for some \( n \), so \( |f(x)|_n|_x \leq |t(x)|_n \to 0 \) because \( x \) is continuous. Conversely, if \( |f(x)|_n \to 0 \) for all \( x \), then \( X = \cup_n X(\frac{f}{n}) \). But \( X \) is quasi-compact, so \( |f(x)|_n \leq |t(x)|_n \) for all \( x \), for some \( n \). So by (VIII.5.2.17) \( f^n \in tA^+ \). Now \( A^+ \) is a filtered colimits of rings of definitions (I.9.5.14), so \( f^n \in tA_0 \) for some \( tA_0 \), which shows that \( f \in A^{00} \). \( \square \)

Constructions of Adic Spectrums

Prop. (VIII.5.2.27) (Direct Limits of Uniform Affinoids). The direct limits exists in the category of uniform affinoid Tate rings and \( A^+ = \text{colim} A^+_i \).

Moreover,

\[
|\text{Spa}(A, A^+)| \cong \lim_{\rightarrow i} |\text{Spa}(A_i, A^+_i)|
\]

as a homeomorphism of spectral spaces, and each rational subset of \( \text{Spa}(A, A^+) \) is pulled back from some rational subset of \( \text{Spa}(A_i, A^+_i) \).

The same conclusion also hold in the category of complete uniform affinoid Tate rings (For the homeomorphism, (VIII.5.2.17) is used).
Proof: Suppose the colimit index has a minimal element \( i_0 \), let \( t \) be a pseudo-uniformizer, then each \( A^+_i \) is a ring of definition with pseudo-uniformizer \( t \). Now we set \( A = \text{colim} \ A_i \) with ring of definitions \( A^+ = \text{colim} \ A^+_i \), then \( A^+ \) is integrally closed in \( A \), thus \((A, A^+)\) is truly a uniform affinoid Tate ring. Now we check it is the colimit: For any compatible map \((A_i, A^+_i) \to (B, B^+)\), there is a map \( f : (A, A^+) \to (B, B^+) \) as abstract rings. We check it is continuous: we may assume \( B^+ \) is the ring of definition, then \( t^n A^+ \subseteq f^{-1} (t^n B^+) \), thus it is continuous.

For the adic spectrum, now a point \( x \in \text{Spa}(A, A^+) \) is determined by the map of uniform affinoid Tate rings \( (A, A^+) \to (k(p), R_x) \), and by the universal property, it is defined by a compatible set of maps \((A_i, A^+_i) \to (k(p), R_x)\). Now it is easy to see the desired bijection of topological spaces, as the elements defining rational subsets are pullbacks from some \( A_i \).

\[ \square \]

Prop. (VIII.5.2.28) (Perfection of Adic Spectrum). Let \((A, A^+)\) be an affinoid Tate ring of char\( p \), then

- The Frobenius map induces a homeomorphism on the adic spectrum of \((A, A^+)\).
- If \((A, A^+)\) is uniform, then there is a perfection functor, which is left adjoint to the forgetful functor from the category of perfect uniform affinoid Tate rings to the category of affinoid Tate rings. And it is just \((A_{\text{perf}}, A^+_{\text{perf}})\).
- The natural map \((A, A^+) \to (A_{\text{perf}}, A^+_{\text{perf}})\) induces a homeomorphism on the adic spectrum.

\[ \square \]

3 Structure Presheaf and Adic Spaces

Lemma (VIII.5.3.1) (Functions on Rational Subsets). If \( X = \text{Spa}(A, A^+) \) is a Huber ring, and \( U \) is a rational subset, then there is a unique complete affinoid Tate ring \((\mathcal{O}_X(U), \mathcal{O}^+_X(U))\) over \((A, A^+)\) that the the Spa map

\[ \text{Spa}(\mathcal{O}_X(U), \mathcal{O}^+_X(U)) \to \text{Spa}(A, A^+) \]

is universal in all the complete affinoid Tate algebras that has image in \( U \).

And in this case, this Spa map is a homeomorphism identifying the rational subsets contained in \( U \) to rational subsets of \( \text{Spa}(\mathcal{O}_X(U), \mathcal{O}^+_X(U)) \). In particular, \( U \) is quasi-compact.

\[ \square \]

Proof: See Hub94 P1.3 for the proof in the Huber ring case.

Choose a ring of definition \((A_0, t)\), and \( U = \text{Spa}(A, A^+)(\frac{f_i}{g}, \frac{f_n}{n})\) for \( f_i, g \in A_0 \), and \( f_n = t^n \), and let \( B = A[g^{-1}] \) and \( B_0 = A_0[\frac{f}{g}] \). Then \( B = B_0[t^{-1}] \) (notice that \( A_0[t^{-1}] = A \)). So \( B \) is a Tate \( A \)-algebra with ring of definition \((B_0, t)\). Now if \( B^+ \) is the integral closure of the subring of \( B \) generated by \( A^+[\frac{f}{g}] \), then \((B, B^+)\) is an affinoid Tate ring. Set \((\mathcal{O}_X(U), \mathcal{O}^+_X(U))\) to be its completion.
By construction $\text{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ maps into $U$, because $x(g) \neq 0$ because $g$ is a unit, and $|x(f_i)| \leq |x(g)|$ as $f_i/g \in B^+$.

Now check universal property: if $\text{Spa}(C, C^+)$ maps into $U$, then $g$ is a unit in $C$ by (VIII.5.2.17), and then $f_i/g \in C^+$ by (VIII.5.2.17) again. Now $C^0$ is the filtered colimit of all rings of definition, so there is a ring of definition $C_0$ that contains $A_0$ and all $f_i/g$ (I.9.5.17) is used. Then this gives a map $\mathcal{O}_X(U) \to C$ of Tate algebras. Now also $B^+$ is mapped into $C^+$ because $C^+$ is integrally closed, so we are done.

For the last assertion, by (VIII.5.2.17), we only have to prove $\text{Spa}(B, B^+) \to U$ is a homeomorphism preserving rational subsets, for this, the injectivity is clear as $B$ is a localization of $A$. And the surjectivity follows immediately from the valuation ring characterization and universal property. Continuity is also clear.

For the openness, for any rational subset $X(\frac{f_1,\ldots,f_n}{g})$ of $X = \text{Spa}(B, B^+)$, because $B = A[g^{-1}]$, $g$ is unit in $B$, we can assume that $f_i, g \in A_0$. Now we show $U \cap \text{Spa}(A, A^+)\{\frac{f_1,\ldots,f_n}{g}\}$ is rational, for this, it suffices to add a $t^N$ to $f_i$, and this is possible, as $X(\frac{f_1,\ldots,f_n}{g})$ is quasi-compact by (VIII.5.2.24). (This is in fact similar to the proof that continuous bijection from compact to Hausdorff is homeomorphism).

\[\square\]

Remark (VIII.5.3.2). The proof goes through with complete replaced by Zariski or Henselian, because we only use item 5 of (VIII.5.2.17), which is true for all Zariski pairs.

And by looking at the construction, if a rational subset $U$ has a representation $X(\frac{f_1,\ldots,f_n}{g})$, then $f_i \in \mathcal{O}_X^+(U)$, and $g$ is invertible in $\mathcal{O}_X(U)$.

Stalks

Def. (VIII.5.3.3) (Stalks). The stalks of an affinoid adic space is defined as in the case of schemes, i.e. the colimit of the function ring of rational subsets containing $x$, without topology, and similarly for the integral stalk. Notice that the function rings are defined by universal property w.r.t to complete Huber pairs, so the stalks only depend on the completion of $(A, A^+)$.

Lemma (VIII.5.3.4). Let $U$ be an open subset of $X = \text{Spa}(A, A^+)$ and $f, g \in \mathcal{O}_X(U)$, then $V = \{x \in U||f|_x \leq |g|_x \neq 0\}$ is an open subset of $X$.

Proof: Cf. Morel P116.?

Prop. (VIII.5.3.5) (Valuations on the Stalks). Let $X = \text{Spa}(A, A^+)$ be an affinoid adic ring, and $x \in X$, then:

- There is a valuation $x$ on $\mathcal{O}_{X,x}$ extending that on $A$, and $\mathcal{O}_{X,x}^+ = \{f \in \mathcal{O}_{X,x}||f(x)|| \leq 1\}$.

- $\mathcal{O}_{X,x}$ is local with maximal ideal $\mathfrak{m}_x = \text{Ker} x$, and $\mathcal{O}_{X,x}^+$ is local with maximal ideal $\{f \in \mathcal{O}_{X,x}||f(x)|| < 1\}$.

- If $k(x)$ is the residue field of $\mathcal{O}_{X,x}$ and $k(x)^+$ be the image of $\mathcal{O}_{X,x}^+$ in $k(x)$, then $k^+(x)$ is naturally a valuation ring, and $(k, k^+)$ is an affinoid field over $(A, A^+)$. In particular, there is an isomorphism between the residue fields of $\mathcal{O}_{X,x}$ and $k^+(x)$.

- If $\varphi : (A, A^+) \to (B, B^+)$ is a morphism of Huber pairs and $y \in \text{Spa}(B, B^+)$ is a point that $\text{Spa}(\varphi)(y) = x$, then the morphism of rings $\text{Spa}(\varphi)_x^+ : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ induced by $\text{Spa}(\varphi)$ is such that $|\cdot|_x \circ \text{Spa}(\varphi)_x^+ = |\cdot|_y$. In particular, $\text{Spa}(\varphi)_x^+$ is a morphism of local rings. Also it sends $\mathcal{O}_{X,x}^+$ to $\mathcal{O}_{Y,y}^+$ and this is a morphism of local rings.
If moreover $A$ is Tate, then we have:

- The ring $\mathcal{O}_{X,x}^+$ is $t$-adically Henselian, and $\mathcal{O}_{X,x}^+ \to k^+(x)$ induces an isomorphism after $t$-adic completion.

- The pairs $(\mathcal{O}_{X,x}^+, \mathfrak{m}_x)$ and $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ are Henselian.

**Proof:** ? Morel P115.

1: Consider the $t$-adic completion of the valuation ring $R_x$ corresponding to $x$, then $(\hat{k}(p_x), \hat{R}_x)$ is an affinoid Tate ring over $(A, A^+)$ that is mapped to $x$ (and its generalizations), thus by universal property, there are unique maps from every rational subsets containing $x$ to $(\hat{k}_x, \hat{R}_x)$, thus inducing a map $(\mathcal{O}_{X,x}, \mathcal{O}_{X,x}^+) \to (\hat{k}_x, \hat{R}_x)$, which induces the desired valuation. And also we have $\mathcal{O}_{X,x}^+ \subset \{ f \in \mathcal{O}_{X,x} \mid |f(x)| \leq 1 \}$, for the converse, if $|f(x)| \leq 1$, then $U(\frac{1}{t})$ is rational subsets in $U$ containing $x$, so by (VIII.5.3.2), $f \in \mathcal{O}_X(V)^+$, thus $f \in \mathcal{O}_{X,x}^+$.

2: for $g$ not in $\mathfrak{m}_x$, $|g(x)| > |t(x)|^n$ for some $n$, so $g$ is invertible in $U(\frac{1}{g})$ (by (VIII.5.3.2)), hence invertible in $\mathcal{O}_{X,x}$. Similarly for $\mathcal{O}_{X,x}^+$, as $g$ is invertible in $U(\frac{1}{g})$.

3: This is clear from the construction of the valuation on $\mathcal{O}_{X,x}$ in item1.

4:

5: As filtered colimits of Henselian pair is Henselian (I.6.10.3) and the function ring is complete, the stalk is Henselian. As for the completion, notice $\mathfrak{m}_x \subset \mathcal{O}_{X,x}^+$ and is $t$-divisible, thus $\mathcal{O}_{X,x}^+$ has the same $t$-adic completion as $k^+(x)$.

6: We first prove $(\mathcal{O}_{X,x}^+, t)$ is Henselian, for this, it suffices to prove $(\mathcal{O}_X^+(U), t)$ is Henselian, by (I.6.10.3). And $\mathcal{O}_X^+(U)$ is a filtered colimits of rings of definitions (I.9.5.14) and they are $t$-adically complete hence Henselian, so we are done by (I.6.10.3) again. Then so does $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ because the property of being Henselian only depends on $I$. (I.6.10.10).

**Cor. (VIII.5.3.6).** By the construction of the valuation on the stalk, we have an inclusion of rings $k(p_x) \subset k(x) \subset k(x)$ that has the same completions, where the first is induced by the compatible map $A \to \mathcal{O}_X(X) \to \mathcal{O}_X(U)$.

**Def. (VIII.5.3.7) (Huber’s Presheaf).** Now by the universal property of function ring, we have a map between them induced by inclusion of rational subsets, so we can define the **structure presheaf** to be

$$\mathcal{O}_X(W) = \lim_{U \subset W} \text{rational } \mathcal{O}_X(U),$$

and similarly for the **integral structure sheaf** $\mathcal{O}_X^+$.

Then there is a valuation of a point on $\mathcal{O}_X(W)$ by passing to the stalk, and

$$\mathcal{O}_X^+(W) = \{ f \in \mathcal{O}_X(W) \mid |f(x)| \leq 1, \forall x \in W \}.$$

because this is true for all rational subsets by adic nullstellensatz (VIII.5.2.17).

A Huber ring $(A, A^+)$ is called **sheafy** iff the structure sheaf $\mathcal{O}_X$ on $X = \text{Spa}(A, A^+)$ is a sheaf. In this case, $\mathcal{O}_X^+$ is also a sheaf by the above formula, so sheafyness is a property that only depends on $A$.

**Criterion for Sheafyness**

**Def. (VIII.5.3.8) (Stably Uniform Huber Pair).** Let $(A, A^+)$ be a Huber pair that $A$ is analytic, then it is called **stably uniform** if $\mathcal{O}_X(U)$ is uniform for all rational subsets $U \subset X = \text{Spa}(A, A^+)$. 
Prop. (VIII.5.3.9). Let $A \to B$ be a continuous map of Huber rings which splits in the category of topological $A$-modules, then $A$ is stably uniform.

Proof: The splitting means the map is strict (I.9.1.4), so $A$ is also uniform. Also the splitting is preserved under completed tensor product with rational localization, so $A$ is stably uniform. $\square$

Prop. (VIII.5.3.10) (Examples of Sheafy Huber Rings). Let $(A, A^+)$ be a complete Huber pair,

1. (Schemes) If $A$ is discrete, then $A$ is sheafy.
2. (Formal Schemes) If $A$ has a Noetherian ring of definition, then $A$ is sheafy.
3. (Rigid Spaces, Fargues-Fontaine Curves) If $A$ is Tate and Strongly Noetherian (I.9.4.25), then $A$ is sheafy.
4. (Perfectoid Spaces) $A$ is analytic and $(A, A^+)$ is stably uniform (VIII.5.3.8), then $A$ is sheafy and acyclic.


We use the arguments following (VIII.5.4.30).

If $A$ is analytic and $(A, A^+)$ is stably uniform. Firstly the ideals $(f - gT), (g - fT)$ are closed in their rings, Cf. [Ked19]L1.5.26.

Then we have a diagram

\[
\begin{array}{ccccccc}
0 & \to & 0 & \to & B\langle T \rangle \oplus B\langle T^{-1} \rangle & \xrightarrow{\cdot T^{-1}} & B\langle T, T^{-1} \rangle & \to & 0 \\
\downarrow & & & \downarrow & & \downarrow & \downarrow & \\
0 & \to & B & \to & B\langle T \rangle \oplus B\langle T^{-1} \rangle & \xrightarrow{\cdot T^{-1}} & B\langle T, T^{-1} \rangle & \to & 0 \\
\downarrow & & & \downarrow & & \downarrow & \downarrow & \\
0 & \to & B & \to & B\langle \frac{f}{g} \rangle \oplus B\langle \frac{g}{f} \rangle & \to & B\langle \frac{f}{g}, \frac{g}{f} \rangle & \to & 0 \\
\downarrow & & & \downarrow & & \downarrow & \downarrow & \\
0 & & & 0 & & 0 & & 0 \\
\end{array}
\]

where the columns and the first two rows are exact, thus the third row are exact in the middle and right by spectral sequence. Also it is exact at the left by (VIII.5.2.16). $\square$

Remark (VIII.5.3.11). Notice that these contains nearly everything of interest, so Scholze comments that we can somehow pretend that non-sheafy Huber rings doesn’t appear in nature.

Remark (VIII.5.3.12). Stably uniform is hard to check in practice, so a recent paper of Hansen-Kedlaya [Sheafyness Criterion for Huber Rings] studied another class of sousperfectoid rings which can be splitly embedded into a perfectoid ring, and another class of diamaintine rings, which involves a condition on the cohomology of pro-étale site of $A$, closely related to properties of diamonds.


4 Adic Spaces

Def. (VIII.5.4.1) (Huber category). The category $V_{pre}$ is a category that the objects are triples $(X, \mathcal{O}_X, v_x)$ that $X$ is a topological space, $\mathcal{O}_X$ is a sheaf of complete topological rings, and $v_x$ are continuous valuations on the stalk $\mathcal{O}_{X,x}$ with support $m_x$. 
And a morphism in $V^{pre}$ is a pair $(f, f^\flat)$ where $f : X \to Y$ is a map of topological spaces and $f^\flat : O_X \to f_*O_Y$ is a morphism of presheaves of topological rings, and the induced morphism of $f^\flat$ on the stalks are compatible with the valuations.

The Huber category $V$ is the full subcategory of $V^{pre}$ whose objects are triples $(X, O_X, v_x)$ in $V^{pre}$ that $O_X$ is a sheaf.

**Def. (VIII.5.4.2) (Open Immersions).** An open immersion in $V^{pre}$ is a homeomorphism onto an open subset that induces an isomorphism of presheaves.

**Def. (VIII.5.4.3) (Adic Spaces).** The category of affinoid adic spaces is the full subcategory of the Huber category whose objects are isomorphic to $\text{Spa}(A, A^+)$ for some Huber pair $(A, A^+)$, and the category of adic spaces is the full subcategory of Huber category whose objects are locally isomorphic to an affinoid adic space.

**Prop. (VIII.5.4.4) (Adic Spectrum Adjointness).** For any affinoid adic space $X = \text{Spa}(R, R^+)$ and $Y$ an arbitrary adic space, then there is a natural isomorphism

$$\text{Hom}(Y, X) \cong \text{Hom}((R, R^+), (O_Y(Y), O_Y^+(Y))).$$

**Proof:** It suffices to show for $Y = (S, S^+)$ affine, because an morphism from $Y \to X$ is glued from local morphisms, and $O_Y$ is a sheaf.

For this, Cf.Huber94 Prop2.1(2).

**Def. (VIII.5.4.5) (Uniform Adic Spaces).** An adic space $X$ is called uniform if for all open affinoid $U = \text{Spa}(R, R^+) \subset X$, the Huber ring $R$ is uniform.

**Cartier Divisors and Closed Immersions**

**Def. (VIII.5.4.6) (Cartier Divisors).** Let $X$ be a uniform analytic adic space, then a (effective)Cartier divisor on $X$ is an ideal sheaf $\mathcal{I} \subset O_X$ that is locally free of rank 1. The support of a Cartier divisor is $\text{Supp}(O_X/\mathcal{I})$.

The support $Z$ of a Cartier divisor is a nowhere dense closed subset of $X$, and the map $I \mapsto \mathcal{I} = iO_X$ induces a bijection between invertible ideals $I \subset R$ that $V(I)$ is nowhere dense in $X$ and Cartier divisors on $X$.

**Proof:** By(VIII.5.4.21), any Cartier divisor is of the form $I \otimes_R O_X$ for some invertible ideal $I \subset R$. We need to show that $\varphi : I \otimes_R O_X \to O_X$ is injective iff $V(I) \subset X$ is nowhere dense.

By restriction, we can assume $I = (f)$ is principle, and if $V(f)$ contains an open subset, then on an open rational subset, $f = 0$ by uniformity(VIII.5.2.15), so $\varphi$ is not injective. Conversely, if if $V(f)$ is nowhere dense, we show $f$ is a nonzero-divisor: if $fg = 0$ and $g \neq 0$, then $U = U(\frac{g}{f})$ is contained in $V(f)$ and is open, so $U = 0$, which implies $g = 0$ by uniformity(VIII.5.2.15), contradiction.

**Prop. (VIII.5.4.7).** Let $X$ be a uniform analytic adic space and $\mathcal{I} \subset O_X$ a Cartier divisor with support $Z$ and $j : U = X\setminus Z \to X$. There are injective maps of sheaves

$$O_X \leftarrow \lim_{\mathcal{I}^{-n}} \rightarrow j_*O_U.$$
Then it suffices to show quotient commutes, because they are both defined by universal properties. This is exact.

Remark (VIII.5.4.11). Even if \( A \) is Tate and stably uniform, and \( f \in A \) is a nonzero-divisor that \( fA \subset A \) is closed, it may not be true that \( \mathcal{O} = f\mathcal{O}_X \to \mathcal{O}_X \) is a closed Cartier divisor on \( X = \text{Spa}(A, A^+) \). This is because there may be rational localization \( (A, A^+) \to (B, B^+) \) that \( fB \) is not closed in \( B \). Cf. [Ked19]P16.
Examples of Adic Spaces

Prop. (VIII.5.4.12) (Examples of Adic Spaces).
- (Adic Closed Unit Disk) The space $\text{Spa}(\mathbb{Z}[T]) = \text{Spa}(\mathbb{Z}[T], \mathbb{Z}[T])$ represents the functor $X \mapsto \mathcal{O}_X^+(X)$.
- (Adic Affine Line) The functor $X \mapsto \mathcal{O}_X(X)$ is also representable, by $\text{Spa}(\mathbb{Z}[T], \mathbb{Z})$. Notice for any non-Archimedean field $K$,
  \[ \text{Spa}(\mathbb{Z}[T], \mathbb{Z}) \times \text{Spa} K = \cup_{n \geq 1} \text{Spa} K(\varpi^n T) = \lim_{n, T \to \varpi T} \text{Spa} K(T). \]
  This is because For any Huber pair $(R, R^+)$ over $(K, \mathcal{O}_K)$, $R = \cup_n \varpi^{-n} R^+$ because $\varpi$ is topologically nilpotent.
- (The Open Unit Disk) Let $D = \text{Spa} \mathbb{Z}[[T]]$, then for any non-Archimedean field $K$, $D_K = D \times \text{Spa} K$ represents the functor that maps a $K$-algebra $R$ to all its elements of norm $\leq 1$. Then this the open disk over $K$. And $D_K$ is also represented by \[ \cup_{n \geq 1} \text{Spa} K(T, \frac{T^n}{\varpi^T}). \]
- (The Punctured Open Unit Disk) Let $D^* = \text{Spa} \mathbb{Z}(\langle T \rangle)$, then
  \[ D_K^* = D^* \times \text{Spa} K = D_K(\frac{T_T}{\varpi}) = D_K \setminus \{0\}. \]

Prop. (VIII.5.4.13) (The Open Unit Disk over $\mathbb{Z}_p$). Consider $X = \mathbb{Z}_p[[T]]$ with the $(p, T)$-adic topology, there is exactly one non-analytic point $x = x_{\mathbb{F}_p}$. Let $X = \text{Spa}(\mathbb{Z}_p[[T]])$ and $Y = X \setminus \{x_{\mathbb{F}_p}\}$, then for a point $x \in Y$, $T(x)$ and $p(x)$ cannot both be 0 by (I.9.6.2).
  Then there exists a unique continuous map \[ \kappa : |Y| \to [0, \infty] \]
  characterized by the following property: For any rational number $m/n < r, \|T(x)\|^n > |p(x)|^m$, and for any rational number $m/n > r, \|T(x)\|^n < |p(x)|^m$. This map $\kappa$ is surjective.

Proof: Let $\bar{x}$ be a maximal vertical generalization of $x$, then it has rank 1 and we can assume $\bar{x}$ is real valued by (I.9.3.4). Now we define
  \[ \kappa(x) = \frac{\log \|T(\bar{x})\|}{\log |p(\bar{x})|}. \]
  This is definable as $T(\bar{x})$ and $p(\bar{x})$ cannot both be 0.
  The uniqueness of $\kappa(x)$ follows from the fact $\|T(x)\|^n > |p(x)|^m$ implies to $\|T(\bar{x})\|^n \geq |p(\bar{x})|^m$ because $\bar{x}$ is a generalization of $x$.
  To show $\kappa$ satisfies the condition, if $m/n < r$, then $\|T(\bar{x})\|^n > |p(\bar{x})|^m$, so $\|T(x)\|^n > |p(x)|^m$ because $x, \bar{x}$ define the same topology. And it is continuous because \[ \kappa^{-1}((0, 1]) = \cup_{m/n < r} U(\frac{T^n}{p^m}). \]
  To show the surjectivity, if $\kappa = [x_0 : x_1]$, define the valuation as
  \[ v(\sum a_{ij} p^{i\bar{T}_j}) = \sup_{a_{ij} \neq 0} e^{-x_{0i} - x_{1j}} \]
  where $(a_{ij})^p = 1$. \qed
Construction of Adic Spaces


Def. (VIII.5.4.15) (Adic Spaces attached to Schemes).

Def. (VIII.5.4.16) (Adic Spaces attached to Formal Schemes). Cf. [Wed14].

Def. (VIII.5.4.17) (Adic Spaces attached to Rigid Analytic Spaces). Cf. [Wed14].

Sheaves and Vector Bundles

Def. (VIII.5.4.18). Let \((A, A^+)\) be a Huber pair and \(X = \text{Spa}(A, A^+)\), let \(\widetilde{M} = M \otimes_A O_X\) be the presheaf on \(X\).

Prop. (VIII.5.4.19). If \(A\) is sheafy, then for any finite projective \(A\)-module \(M\), the presheaf \(\widetilde{M}\) is a sheaf on \(X = \text{Spa}(A, A^+)\), and \(H^i(U, F) = 0\) for any rational subset of \(X\) and \(i > 0\).

Proof: Because \(M\) is a direct sum of a finite free \(A\)-module, then we reduce to the case \(O_X\) is sheafy. \(\square\)

Remark (VIII.5.4.20). This is a partial analogy with Tate’s acyclicity theorem in rigid analytic geometry (VIII.4.1.40)(VIII.4.3.10), but it only holds for f.p. modules, not even f.g. modules. One impediment is that the rational localization map are generally not flat. To get around this, Kedlaya defined a category of pseudo-coherent modules, with the property that even when flatness fails, tensoring is also exact in this category.

Prop. (VIII.5.4.21) (Vector Bundles). Let \((A, A^+)\) be a sheafy analytic Huber pair and \(X = \text{Spa}(A, A^+)\), then the functor \(M \to \widetilde{M}\) from the category of finite projective \(A\)-modules to the category \(\text{Vect}_X\) of locally finite free \(O_X\)-modules is an equivalence of categories. In particular, \(\text{Vect}_X\) only depends on \(A\).

Proof: Cf. [Ked19], P40? \(\square\)

Pre-Adic Spaces

Main references are [S-W20] L3 and [Ked19].

Def. (VIII.5.4.22) (Pre-Adic Spaces).

Remark (VIII.5.4.23). Pre-adic spaces is an approach to work around the failure of sheafyness of general Huber pair, with techniques from algebraic stacks.

Analytic Points

Prop. (VIII.5.4.24) (Analytic and Tate Rings). Let \((A, A^+)\) be a complete Huber pair, then

- The Huber ring \(A\) is analytic iff all points of \(\text{Spa}(A, A^+)\) are analytic.

- A point \(x \in \text{Spa}(A, A^+)\) is analytic iff there is a rational nbhd \(U \subset \text{Spa}(A, A^+)\) that \(O_X(U)\) is Tate.
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Proof: 1: $x$ is non-analytic iff $A^{00} \subset p_x$ by [I.9.6.5], so all points are analytic iff $A^{00}A = A$, which means $A$ is analytic.

2: Let $x$ be an analytic point, then there exists $f \in I$ where $I$ is an ideal of definition that $|f|_x \neq 0$. Now $\{g \in A||g(x)| \leq |f(x)|\}$ is open (because $|.|_x$ is continuous), so contains some $I^n$. Now let $I^n = (g_1, \ldots, g_k)$, then $U(\frac{g_1 \cdots g_k}{f})$ is a rational subset. Then in $\mathcal{O}_X(U)$, $f$ is a unit, but also it is topologically nilpotent, because it is contained in $I$.

Conversely, if $x \in X$ has a rational nbhd $U = U(\frac{1}{f})$ such that $\mathcal{O}_X(U)$ is Tate, and $x$ is not analytic, then $p_x$ contains an ideal of definition $I \subset A_0$. Now let $f \in \mathcal{O}_X(U)$ be a topologically nilpotent unit, then there exists $m \geq 1$ that $f^m$ lies in the closure of $IA_0[t/s]|t \in T|$ in $\mathcal{O}_X(U)$. Since $x \in U$, the valuation $|.|_x$ extends to $\mathcal{O}_X(U)$, and since $|.|_x$ is continuous, $|f^m(x)| = 0$, contradiction, as $f$ is a unit in $\mathcal{O}_X(U)$.

Def. (VIII.5.4.25) (Analytic Points). Let $X$ be a pre-adic space, then a point $x \in X$ is called an analytic point if there is some open affinoid nbhd $U = \text{Spa}(A, A^+ \subset X$ of $x$ that $A$ is Tate. And $X$ is called analytic if all of its points are analytic.

Prop. (VIII.5.4.26). Let $f : X \to Y$ be a map of analytic pre-adic spaces, then $|f| : |X| \to |Y|$ is generalizing. If $f$ is quasi-compact and surjective, then $|f|$ is a quotient map.

Proof: Cf. [Hub96]1.1.10. and [Étale Cohomology of Diamonds, L2.5].

Def. (VIII.5.4.27). A map $f : Y \to X$ of pre-adic spaces is called analytic if it carries analytic points to analytic points.

Proof of Acyclicity and Sheafyness by Cech Reduction

Def. (VIII.5.4.28) (Standard Rational Coverings). Let $X = \text{Spa}(A, A^+)$ and $f_1, \ldots, f_n \in A$ which generates the unit ideal, then the sets $X(\frac{1}{f_1 \cdots f_n})$ covers $X$, called the standard rational covering of $X$. And if $n = 2$, it is called a standard binary rational covering of $X$.

There are special types of standard binary rational coverings: if $f_1 = f, f_2 = 1$, then it is called a simple Laurent covering. If $f_1 = f, f_2 = 1 - f$, then it is called a simple balanced covering.

Lemma (VIII.5.4.29) (Reduction of Coverings). Let $A$ be an analytic Huber ring,

- (Huber) Every open covering of $X$ can be refined by some standard rational covering.

- (Gabber-Ramero) Every open covering of a rational subspace of $X$ can be refined by some compositions of simple Laurent coverings and simple balanced coverings.


Prop. (VIII.5.4.30) (Cech Reduction). By a Cech cohomological argument the same as Tate’s acyclicity theorem in rigid geometry (VIII.4.1.40), it suffices to prove any presheaf is a sheaf or any sheaf is acyclic on simple Laurent coverings and simple balanced coverings.

That is, for every rational localization $(B, B^+)$ over $(A, A^+)$, every pair $f, g \in B$ that $g = f$ or $1 - f$, if the sequence

$$0 \to B \to B(\frac{f}{g}) \oplus B(\frac{g}{f}) \to B(\frac{f}{g}, \frac{g}{f}) \to 0,$$

- is exact at exact at left and middle, then $\mathcal{O}_X$ is sheafy.
• is exact, then $\mathcal{O}_X$ is acyclic.

Also remember the following equations:

$$B\left(\frac{f}{g}\right) = B\langle T \rangle/(f - gT), \quad B\left(\frac{g}{f}\right) = B\langle T \rangle/(g - fT), \quad B\left(\frac{f}{g}, \frac{g}{f}\right) = B\langle T, T^{-1} \rangle/(f - gT).$$

**Prop. (VIII.5.4.31) (Properties of (Finite)Étale Maps).**

- (Finite)Étale maps are invariant under compositions and pullbacks.
- If $g$ and $gf$ are (finite)étale, then so does $f$.
- $f$ is (finite)étale iff $f^p$ is (finite)étale.

**Proof:** Cf.[?P65]. □

## 5 Perfectoid Spaces

### Affinoid Perfectoid Spaces and Tilting

**Prop. (VIII.5.5.1) ( Tilting Rational Subsets).** For a perfectoid affinoid $K$-algebra $(R, R^+)$ over a perfectoid field $K$,

- The $\sharp$ map induces an isomorphism $X = \operatorname{Spa}(R, R^+) \cong X^\flat = \operatorname{Spa}(R^\flat, R^{\flat 0})$ that identifies rational open subsets.

- For a rational subset $U$ with tilting $U^\flat$, the complete affinoid Tate algebra $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is perfectoid over $R$ with tilt $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$.

**Proof:** This follows from(VIII.5.5.5). □

**Lemma (VIII.5.5.2) (Huber’s Presheaf in Char $p$).** Assume $\operatorname{char} K = p$ and $U = X(\frac{f_1, \ldots, f_n}{g})$ is a rational subset that $f_i, g \in R^+$ and $f_n = \pi^N$, then:

- Consider the subring $R^+[\frac{f_i}{g}]$ its $\pi$-adic completion $(R^+[\langle \frac{f_i}{g} \rangle^{1/\pi}])^a$ is a perfectoid $K^{0a}$-algebra.

- The map $R^+[X_i^{1/\pi}] \to R^+[\frac{f_i}{g}]^{1/\pi}$ has kernel containing and almost equal to $I = (g^{1/\pi}X_i^{1/\pi} - f_i^{1/\pi})$.

- $\mathcal{O}_X(U)$ is a perfectoid $K$-algebra and $\mathcal{O}_X(U)^{0a} \cong (R^+[\langle \frac{f_i}{g} \rangle^{1/\pi}])^a$.

**Proof:**

1. The ring $R^+[\frac{f_i}{g}]$ is perfect and $\pi$-torsion-free, and $R^+$ is semi-perfect, thus its completion is clearly a perfectoid $K^{0a}$-algebra by(I.9.9.1).

2. Clearly $I \subset \ker$ and notice $R^+[\langle \frac{f_i}{g} \rangle^{1/\pi}][\pi^{-1}] = R[g^{-1}]$ as $f_n = \pi^N$, so $I[\pi^{-1}] = \ker[\pi^{-1}]$. Now consider the mapping

$$P_0 = R^+[X_i^{1/\pi}] / I \to R^+[\langle \frac{f_i}{g} \rangle^{1/\pi}].$$

Now this map is an isomorphism after inverting $\pi$, so the kernel is $\pi^\infty$-torsion. But we have $I = I^{[p]}$ because $R^+$ is semi-perfect, so $P_0$ is perfect, so the kernel must be almost zero.

3. Consider the inclusion $R^+[\frac{f_i}{g}] \to R^+[\langle \frac{f_i}{g} \rangle^{1/\pi}]$, we show the cokernel is killed by $\pi^{nN}$: as

$$\pi^{nN} \prod_{i=1}^n \frac{f_i}{g}^{1/\pi} = \prod_{i=1}^n \frac{f_i^{1/\pi} g^{1/\pi} - f_i}{g} f_n \in R^+[\frac{f_i}{g}].$$
So these two ring has the same $\pi$-adic completion, the first one is just $\mathcal{O}_X(U)$ by the construction (VIII.5.3.1), so $\mathcal{O}_X(U)$ is perfectoid $K$-algebra, and the isomorphism is by tilting equivalence $\text{Perf}_K \cong \text{Perf}_{K^0}$ (I.9.9.5).

**Lemma (VIII.5.5.3) (Huber’s Presheaf in Char 0).** Let $U = X(f_1, \ldots, f_n)$ is a rational subset that $f_i, g$ are perfect elements in $R^+$, $f_i = a_it^i, g = b^r$, and $f_n = \pi^N$, so $f_i, g$ have compatible $p^\alpha$-th roots, then let $U^p = X^p((f_1, \ldots, f_n)$ be the tilting of $U$, $U$ is the inverse image of $U^p$ along the map $X \to X^p$. Then the conclusion of (VIII.5.5.2) is also true, and moreover, $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ tilts to $(\mathcal{O}_{X^p}(U^p), \mathcal{O}_{X^p}^+(U^p))$.

**Proof:** 1, 2 of (VIII.5.5.2): Notation as before, there is a map $P_0 = R^+[X_i^{1/\pi^\alpha}]/I \to R^+[(\frac{f_i}{g})^{1/\pi^\alpha}]$, and an inclusion $R^+[(\frac{f_i}{g})^{1/\pi^\alpha}] \to \mathcal{O}_X^+(U)$. Now write $(S, S^+)$ for the untilt of the perfectoid $(R^0, R^+)$-algebra $(\mathcal{O}_{X^p}(U^p), \mathcal{O}_{X^p}^+(U^p))$, then by the tilting process (I.9.9.13), Spa$(S, S^+)$ maps into $U$, so by the universal property, there is a map

$$\mu : (\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to (S, S^+) .$$

Consider the composition

$$P_0 \xrightarrow{a_0} R^+[(\frac{f_i}{g})^{1/\pi^\alpha}] \xrightarrow{d_0} S^+ ,$$

we prove their completion gives the same $K^{0a}$-algebras (notice $S^+$ is already complete): $a_0$ is surjective, thus so does its completion, the map $d_0 \circ a_0$ is almost isomorphism modulo $\pi$ by (VIII.5.5.2) item2 and tilting equivalence, so does its completion. Now (I.16.3.2) tells us the completion of $d_0 \circ a_0$ is almost isomorphism, so does $a$ and $d$ as $a$ is surjective.

By the way, we know that $R^+\langle (\frac{f_i}{g})^{1/\pi^\alpha}\rangle[\pi^{-1}]$ is the untilt of $\mathcal{O}_{X^p}^+(U^p)$.  3 of (VIII.5.5.2) is proved as before.

For the tilting, by the above, we already know $R^+\langle (\frac{f_i}{g})^{1/\pi^\alpha}\rangle$ tilts to the perfectoid $K^{0a}$-algebra $\mathcal{O}_{X^p}(U^p)^{0a}$, and by item3 $\mathcal{O}_X(U)$ tilts to the perfectoid $K^b$-algebra $\mathcal{O}_{X^p}(U^p)$. Now the question is the tilt of $\mathcal{O}_X^+(U)$, notice as in the proof of item1, there is a natural map

$$(R^+\langle (\frac{f_i}{g})^{1/\pi^\alpha}\rangle[\pi^{-1}], R^+\langle (\frac{f_i}{g})^{1/\pi^\alpha}\rangle) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U)) ,$$

whose tilting gives by university of Huber’s presheaf a map

$$\xi : (\mathcal{O}_{X^p}(U^p), \mathcal{O}_{X^p}^+(U^p)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))^b .$$

These two map $\mu, \xi$ are inverse to each other, showing that the tilting of $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is $(\mathcal{O}_{X^p}(U^p), \mathcal{O}_{X^p}^+(U^p))$. □

**Lemma (VIII.5.5.4) (Approximation Lemma).** Assume $R = K\langle T_0^{1/\pi^\alpha}, \ldots, T_n^{1/\pi^\alpha} \rangle$, $f \in R^0$ is homogenous of degree $d \in \mathbb{N}[\pi^{-1}]$, then for any $c > 0, \varepsilon > 0$, there exists some $g_{c, \varepsilon} \in R^{0a}$ homogeneous of degree $d$ that

$$| (f - g^\varepsilon)(x) | \leq | \pi |^{1-\varepsilon} \max\{|f(x)|, |\pi|^c\} .$$

In particular, if $\varepsilon < 1$, then

$$\max\{|f(x)|, |\pi|^c\} = \max\{|g_{c, \varepsilon}(x)|, |\pi|^c\} .$$
Then for the approximation containing \( U \), we associate it an affinoid adic space \( \text{Spa}\( K \)) of \( \text{Spa}(R, R^+) \), thus by untilting \( \pi \) shows that the adic spectrum of a perfectoid affinoid \( \text{Spa}(R, R^+) \) corresponds to a map \( \hat{K}(x) \rightarrow \hat{L}(x) \). For any \( x \in X \), the non-Archimedean field \( \hat{k}(x) \) is perfectoid.

### Proposition (VIII.5.5.5)

For an arbitrary perfectoid \( K \)-algebra \( R \),

- The same conclusion of (VIII.5.4) holds.
- For \( f, g \in R \), there exist \( a, b \in R^0 \) that \( X(\frac{f}{g}) = \frac{a}{b} \). In particular, any rational subsets \( U \) of \( X \) comes from \( X^0 \), thus (VIII.5.5.3) applies for \( U \).
- For any \( x \in X \), the non-Archimedean field \( \hat{k}(x) \) is perfectoid.
- \( X \rightarrow X^0 \) is a homeomorphism preserving rational subsets.

**Proof:**

1. Using the tilting equivalence, we can write \( f = g_0^0 + \pi g_1^0 + \ldots + \pi^c g_c^0 + f_{c+1} \pi^{c+1} \), let \( f_0 = g_0^0 + \pi g_1^0 + \ldots + \pi^c g_c^0 \). Then notice that the solution \( g_{c,\epsilon} \) for \( f_0 \) is suitable for \( f \) as well. So now if we consider the mapping

\[
\mu : K(\frac{T_0}{\pi}, \ldots, \frac{T_n}{\pi}) \rightarrow R_i \rightarrow g_i^0,
\]

together with its tilting

\[
\mu^b : K^0(\frac{T_0}{\pi}, \ldots, \frac{T_n}{\pi}) \rightarrow R^0.
\]

Then for the approximation \( g_{c,\epsilon}^0 \) for \( f' = \sum \pi^i T_i \), \( \mu^b(g) \) is what we are searching for.

2. Using 1, we find \( a, b \in R^0 \) that \( |(g - b\pi^i)(x)| < \max\{|g(x)|, |\pi^i| \} \) and completion of the filtered limits of perfectoid \( K^{0a} \)-algebras is perfectoid (I.9.9.14), and completion of the filtered limits of perfectoid \( K^{0a} \)-algebras is perfectoid (I.9.9.10), we know that \( k^+(x) \) is perfectoid over \( K^{0a} \), thus inverting \( \pi \) shows \( k(x) \) is also perfectoid over \( K \).

3: By (VIII.5.3.5), \( k^+(x) \) equals the completion of the colimit \( \colim \mathcal{O}_X(U) \) over rational subsets \( U \) containing \( x \). As these are all perfectoid \( K^{0a} \)-algebras (I.9.9.14), and completion of the filtered limits of perfectoid \( K^{0a} \)-algebras is perfectoid (I.9.9.10), we know that \( k^+(x) \) is perfectoid over \( K^{0a} \), thus inverting \( \pi \) shows \( k(x) \) is also perfectoid over \( K \).

4: this is injective because \( X \) is \( T_0 \) and a rational subset is the untilt of a rational subset of \( \text{Spa}(R, R^+) \), called an **affinoid perfectoid space**.

Tate’s acyclicity (VIII.5.6.19) shows that the adic spectrum of a perfectoid affinoid \( K \)-algebra is sheafy, so we can defined the category of **perfectoid spaces** is defined to be the full subcategory of adic spaces that is locally isomorphic to an affinoid perfectoid space.

### Remark (VIII.5.5.7)

Notice that it is not true that if \( (A, A^+) \) is a Huber pair over a perfectoid field \( K \) and \( \text{Spa}(A, A^+) \) is a perfectoid space, then \( A \) is a perfectoid ring. Thus there is ambiguity to the term affinoid perfectoid spaces. But we always use this to mean the affinoid adic space associated to a perfectoid Huber pair.

**Proof:**

Prop. (VIII.5.5.8). The absolute product of two perfectoid spaces of char $p$ is also a perfectoid space.

Proof: Cf.[Sch17]P71.

Prop. (VIII.5.5.9) (Fiber Products of Perfectoid Spaces). The category of perfectoid spaces over $K$ admits fiber products.

Proof: Perfectoid spaces are constructed by glueing, thus it suffices to show that the category of perfectoid $K^\flat$-algebras has fiber pushouts, by tilting equivalence. For this, if $X = (A, A^+), Y = (B, B^+), Z = (C, C^+)$, define $X \otimes_Y Z = (D, D^+)$, where $D$ is the completion of $A \otimes_B C$, and $D^+$ is the completion of the integral closure of $A^+ \otimes_{B^+} C^+$ in $D$. Then $D$ is a perfect $K^\flat$-algebra, and it is truly the filtered colimits, by (I.9.5.23).

Notice that we don’t even need a base field $K$, Cf.[Ked19]P57.

Prop. (VIII.5.5.10) (Tilting Equivalence for Perfectoid Spaces). Fix a perfectoid field $K$, then for any perfectoid space $X \triangleleft K$, there is a unique perfectoid space $X^\flat \triangleleft K^\flat$ that satisfies: $X(R, R^+) \cong X^\flat(R^\flat, R^\flat+)$ functorially, called the tilt of $X$. Moreover, this $X^\flat$ satisfies naturally $|X| \cong |X^\flat|$.

When $X$ is an affinoid perfectoid space, this tilting coincides with that of (VIII.5.5.1).

And this tilting induces an equivalence between the category of perfectoid spaces over $K$ and perfectoid spaces over $K^\flat$.

Proof: Firstly, the universal property truly determines the tilt $X^\flat$ uniquely: if there are two tilts $X_1, X_2$, as they are locally affinoid perfectoid like $Spa(R^\flat, R^\flat+)$ by (VIII.5.5.1), the immersion map $Spa(R^\flat, R^\flat+) \rightarrow X_1$ determines via the functorial isomorphism a morphism $Spa(R^\flat, R^\flat+) \rightarrow X_2$. Now $X_1$ has a sheaf structure, so these morphisms glue to give a morphism $X_1 \rightarrow X_2$. The same argument shows conversely there is a morphism $X_2 \rightarrow X_1$, and they are clearly converse to each other, so $X_1 \cong X_2$.

The construction of $X$ is just the glueing of the tilting of the affinoid perfectoid spaces, as the tilting defined in (VIII.5.5.1) is a functor. The universal property is verified by just checking on the affinoid perfectoid spaces, as we can glue using the sheaf property. For the affinoid case, we should use (VIII.5.4.4). The last assertion is by (VIII.5.5.1).

6 Properties of Perfectoid Spaces

TotallyDisconnectedSpaces

Def. (VIII.5.6.1) (TotallyDisconnectedSpaces). A perfectoid space $X$ is called totally disconnected if it is qcqs and any open covering $\{U_i \rightarrow X\}$ splits, i.e. $\bigsqcup U_i \rightarrow X$ splits, or equivalently, there is a refinement covering $\{V_i \rightarrow X\}$ that $X \cong \bigsqcup V_i$.

A perfectoid space $X$ is called strictly totally disconnected if it is qcqs and every étale cover splits.

Prop. (VIII.5.6.2). Let $X$ be a qcqs perfectoid spaces, then $X$ is totally disconnected iff all its connected components are of the form $Spa(K, K^+) \ where \ (K, K^+)$ are perfectoid affinoid fields. And it is strictly totally disconnected if moreover $K$ are all alg.closed.

Proof: Cf.[Sch17].P29, P35.

Prop. (VIII.5.6.3). if $X$ is a totally disconnected perfectoid space, then $X$ is affinoid.

Proof: Cf.[Sch17].P30.
Injections

Def. (VIII.5.6.4) (Injections). A map \( f : X \to Y \) of perfectoid spaces is called an injection if for any perfectoid space \( Z, f_* : \text{Hom}(Z, X) \to \text{Hom}(Z, Y) \) is an injection.

Prop. (VIII.5.6.5) (Residue Field Map is Injection). Let \( X \) be a perfectoid space and \( x \in X \), giving rise to a map of residue fields \( i_x : \text{Spa}(\kappa(x), \kappa(x)^+) \to X \), then \( i_x \) is an injection of perfectoid spaces.

Proof: To show this, firstly we can replace \( X \) with an affinoid nbhd of \( X \). Then notice that \( \text{Spa}(\kappa(x), \kappa(x)^+) \) is the filtered limit over all rational nbhds \( U \) of \( x \) in \( X \), and for each \( U, U \to X \) is an injection by definition (VIII.5.3.1), so \( i_x \) is also an injection. \( \square \)

Prop. (VIII.5.6.6) (Characterizations of Injections). Let \( f : Y \to X \) be a map of perfectoid spaces, then the following conditions are equivalent:

- \( f \) is an injection.
- For any perfectoid adic field \((K, K^+)\), the map of sets \( f_* : Y(K, K^+) \to X(K, K^+) \) is an injection.
- The map \(|f| : |Y| \to |X|\) is injective, and for all rank 1 point \( y \in Y \) with image \( f(y) = x \in X \), the map of completed residue fields \( \kappa(x) \to \kappa(y) \) is an isomorphism.
- The map \(|f| : |Y| \to |X|\) is injective, and \( f \) is final in the category of maps \( Z \to X \) that \(|Z| \to |X|\) factors through the map \(|Y| \to |X|\).

In particular, by item 4, an injection of perfectoid spaces is determined by its topological map.

Proof: \( 4 \to 1 \to 2 \) is trivial. For the rest, Cf.[Sch17]P21. \( \square \)

Prop. (VIII.5.6.7) (Injection and Base Change). Let \( f : Y \to X \) be an injection of perfectoid spaces, and \( X' \to X \) any map of perfectoid spaces, then the pullback \( f' : Y' = Y \times_X X' \to X' \) is also an injection, and the induced map \(|Y'| \to |Y| \times_{|X|} |X'|\) is a homeomorphism.

A map of perfectoids spaces is an injection iff it is universally injective.

Proof: Cf.[Sch17]P24. \( \square \)

Immersions

Def. (VIII.5.6.8) (Immersions). A map of perfectoid spaces \( f : Y \to X \) is called an immersion if \( f \) is an injection and \(|f| : |Y| \to |X|\) is a locally closed immersion. If \(|f|\) is moreover closed or open, then it is called closed/open immersion.

Def. (VIII.5.6.9) (Zariski Closed Immersion). Let \( f : Z \to X \) be a map of perfectoid spaces where \( X = \text{Spa}(R, R^+) \) is affinoid perfectoid, then

- the map \( f \) is called Zariski closed immersion if \( f \) is a closed immersion and \(|Z| = V(I) \subset |X|\), where \( I \subset R \) is an ideal.
the map \( f \) is called strongly Zariski closed immersion if \( Z = \text{Spa}(S, S^+) \) is affinoid perfectoid, \( R \to S \) is surjective, and \( S^+ \) is the closure of \( R^+ \) in \( S \).

**Prop. (VIII.5.6.10).**

- If \( f \) is strongly Zariski closed, then \( f \) is Zariski closed, in particular a closed immersion.
- If \( f \) is Zariski closed, then \( Z \) is affinoid.
- If \( X \) is of characteristic \( p \), and \( f \) is Zariski closed, then \( f \) is strongly Zariski closed.

**Proof:** Cf. [p] Section2.5. □

**Prop. (VIII.5.6.11).** For any map of perfectoid spaces \( Y \to X \), the diagonal map \( \Delta_f : Y \to Y \times_X Y \) is an immersion.

**Proof:** Clearly \( \Delta_f \) is an injection, thus it suffices to show that \( |\Delta_f| \) identifies \( Y \) with a locally closed subset of \( |Y \times_X Y| \). This can be checked locally on the target, so we can assume \( X = \text{Spa}(R, R^+) \) and \( Y = \text{Spa}(S, S^+) \), then the diagonal map is strongly Zariski closed, as \( S \hat{\otimes}_RS \to S \) is surjective and maps the integral closure of \( S^+ \hat{\otimes}_RS^+ \to S^+ \) onto \( S^+ \). Thus by (VIII.7.1.11), \( \Delta_f \) is a closed immersion in this case. □

**Def. (VIII.5.6.12) (Separated Map).** A map \( f : Y \to X \) of perfectoid spaces is called separated if \( \Delta_f \) is a closed immersion.

**Prop. (VIII.5.6.13) (Valuation Criterion).** Let \( f : Y \to X \) be a map of perfectoid spaces. The following are equivalent:

- \( f \) is separated.
- \( |\Delta_f| : |Y| \to |Y \times_X Y| \) is a closed immersion.
- \( |f| \) is quasi-separated, and for any perfectoid adic field \((K, K^+)\) and any diagram

\[
\begin{array}{ccc}
\text{Spa}(K, \mathcal{O}_K) & \to & Y \\
\downarrow & & \downarrow f \\
\text{Spa}(K, K^+) & \to & X
\end{array}
\]

**Almost Acyclicity**

**Def. (VIII.5.6.14) (\( p \)-Finite Tate Ring).** Denote \( L = \mathbb{F}_p[[t]]_{\text{perf}}[t^{-1}] \), an \( \mathbb{F}_p[t] \) algebra \( A^+ \) is called **algebraically admissible** if it is f.p., reduced, \( t \)-torsion-free, and integrally closed in \( A^+[t^{-1}] \). A perfectoid affinoid \( L \)-algebra \((R, R^+)\) is called **\( p \)-finite** if it is the completion of the perfection (VIII.5.2.28) of a uniform Tate ring of the form \((A^+[t^{-1}], A^+)\), where \( A^+ \) is algebraically admissible.

**Lemma (VIII.5.6.15) (Tate’s Acyclicity for Classical Affinoid Algebra).** If \( A^+ \) is an algebraically admissible \( \mathbb{F}_p[t] \)-algebra, then \((A^+[t^{-1}], A^+)\) is a uniform affinoid Tate algebra (because it is finite), and:

- For any rational subset \( U \subset X \), the structure presheaf \((\mathcal{O}_X(U), \mathcal{O}_X^+(U))\) is also uniform, and it is a perfection of an algebraically admissible \( \mathbb{F}_p[t] \)-algebra, so \( \mathcal{O}_X^+(U) = \mathcal{O}_X(U)^0 \).
- For any covering \( \mathcal{U} : U_i \to X \) of rational subsets, the Čech cohomology groups \( H^i(\mathcal{U}, \mathcal{O}_X^+) \) are all killed by \( t^N \) for \( N \) large.
• $(A, A^+)$ is sheafy, with $H^i(X, \mathcal{O}_X^+)$ being $t^\infty$-torsion for all $i$.

**Proof:**

By Proposition (VIII.5.6.18) (Noetherian Approximation in Char $p$).

Let $(R, R^+)$ be a $p$-finite perfectoid $L$-algebra that comes from the completion of perfection of $(A, A^+)$, then:

1. The map $X = \text{Spa}(R, R^+) \rightarrow Y = \text{Spa}(R, R^+)$ is a homeomorphism.
2. For rational subset $V \subset Y$ with preimage $U \subset X$, $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is the completion of the perfection of $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V))$.
3. For any covering $\mathfrak{U} : U_i \rightarrow X$ of rational subsets, the Čech cohomology groups $H^i(\mathfrak{U}, \mathcal{O}_X^0)$ are all almost zero.
4. $(R, R^+)$ is sheafy, with $H^i(X, \mathcal{O}_X^+)$ almost zero for all $i > 0$.

**Lemma (VIII.5.6.16)** (Tate’s Acyclicity for $p$-Finite Perfectoid Algebras). Let $(R, R^+)$ be a $p$-finite perfectoid $L$-algebra that comes from the completion of perfection of $(A, A^+)$, then:

- The map $X = \text{Spa}(R, R^+) \rightarrow Y = \text{Spa}(R, R^+)$ is a homeomorphism.
- For rational subset $V \subset Y$ with preimage $U \subset X$, $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is the completion of the perfection of $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V))$.
- For any covering $\mathfrak{U} : U_i \rightarrow X$ of rational subsets, the Čech cohomology groups $H^i(\mathfrak{U}, \mathcal{O}_X^0)$ are all almost zero.
- $(R, R^+)$ is sheafy, with $H^i(X, \mathcal{O}_X^+)$ almost zero for all $i > 0$.

**Proof:**

1. This is because the adic spectrum is insensitive for perfection (VIII.5.2.28) and completion (VIII.5.2.17).
2. This is by the universal property, as these two are both the universal elements for the complete and affinoid adic spaces mapping to $X$ that factors through $U$.
3. The complex calculating $H^i(\mathfrak{U}, \mathcal{O}_X^0)$ is just the complex calculating $H^i(\mathfrak{U}, \mathcal{O}_X^+)$ under completion of perfection (VIII.5.2.27) used. So (VIII.5.6.15) and (VIII.5.6.17) (applied to every element) shows that the perfection makes the complex almost acyclic, and this is preserved under completion as $(-)^a$ is exact.
4. The complex calculating $H^i(\mathfrak{U}, \mathcal{O}_X)$ is just the complex calculating $H^i(\mathfrak{U}, \mathcal{O}_X^+)$ inverting $t$, thus they are all 0 as localization is exact. For the second, it is because of item 3 and the fact $\mathcal{O}_X^+$ is almost isomorphic to $\mathcal{O}_X^0$ (I.9.9.14).

**Lemma (VIII.5.6.17).** Let $A$ be a ring with an element $t$ that admits compatible $p^n$-th roots, then for an $A$-module $M$ that $t^NM = 0$, consider the Frobenius pushforward $M \rightarrow F_*M$, then the colimit $\text{colim}_F F_*^nM$ is naturally a module over $A_{\text{perf}}$, and it is annihilated by $t^{\frac{1}{p^n}}$ for all $n$.

**Proof:** The $A_{\text{perf}}$ structure is natural, and notice $F_*^nM$ is annihilated by $t^{\frac{1}{p^n}}$, thus naturally the colimit is annihilated by $t^{\frac{1}{p^n}}$ for all $n$.

**Prop. (VIII.5.6.18)** (Noetherian Approximation in Char $p$). If $K$ is a perfectoid field of char $p$ with pseudo-uniformizer $t$, then $K$ is an extension of $L = \mathbb{F}_p[[t]]_{\text{perf}}[t^{-1}]$, and if $A$ is an $K^0$-perfectoid algebra that is integrally closed in $A[t^{-1}]$, then:

- $A$ is a completion of a filtered colimit $\text{colimi} B_i$ that $B_i$ are $p$-finite, that induces an homeomorphism
  $$\text{Spa}(A[t^{-1}], A) \cong \text{lim}_i \text{Spa}(B_i[t^{-1}], B_i)$$

  that each rational subset of $\text{Spa}(A[t^{-1}], A)$ comes from a rational subset of some $\text{Spa}(B_i[t^{-1}], B_i)$.

- If $U_i \subset \text{Spa}(B_i[t^{-1}], B_i)$ is a compatible system of rational subsets that corresponds to $U \subset \text{Spa}(A, A^+) = X$, then
  $$(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \cong \text{lim}_j (\mathcal{O}_j(U_j), \mathcal{O}_j^+(U_j)).$$
Proof: 1: $A = \operatorname{colim}_i A_i$, where $A_i$ are all the f.p. $\mathbb{F}_p[t]$-algebras in $A$. Then each $A_i$ is reduced (as $A$ is complete and integrally closed in $A[t^{-1}]$) and $t$-torsion-free, and we can assume they are integrally closed in $A_i[t^{-1}]$ because $A$ does, by passing to their integral closure.

Then applying the $(-)_{\text{perf}}$ functor gives $\operatorname{colim}_i (A_i)_{\text{perf}} = A$, as $A$ is perfect, and applying the completion gives

$$(\operatorname{colim}_i \widehat{A}_i) = A,$$

as $A$ is already complete, so we are done.

2: This is immediate from 1 and (VIII.5.2.27).

3: This is because by universal property for Huber presheaves, there are pushouts diagrams

$$
\begin{array}{cccc}
(B_i[t^{-1}], B_i) & \longrightarrow & (B_j[t^{-1}], B_j) & \longrightarrow \ldots \longrightarrow (A[t^{-1}], A) \\
\downarrow & & \downarrow & \\
(\mathcal{O}_{X_i}(U_i), \mathcal{O}_{X_i}^+(U_i)) & \longrightarrow & (\mathcal{O}_{X_j}(U_j), \mathcal{O}_{X_j}^+(U_j)) & \longrightarrow \ldots \longrightarrow (\mathcal{O}_X(U), \mathcal{O}_X^+(U))
\end{array}
$$

So the conclusion follows as colimits commutes with colimits. □

Prop. (VIII.5.6.19) (Almost Acyclicity for Perfectoids). Fix a perfectoid field $K$ and a perfectoid affinoid $K$-algebra $(R, R^+)$ with adic spectrum $X = \operatorname{Spa}(R, R^+)$, then

- $(R, R^+)$ is sheafy, i.e. $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are sheaves.
- $\mathcal{O}_X^+(X) = R^+$, and $H^i(X, \mathcal{O}_X^+)$ is almost zero for $i > 0$.
- $\mathcal{O}_X(X) = R$, and $H^i(X, \mathcal{O}_X) = 0$ all $i > 0$.

Proof: As in the proof of (VIII.5.6.16), it suffices to prove $\mathcal{O}_X^0$ is almost exact w.r.t any covering $\mathcal{U}$. For this, notice each term is $\pi$-adically complete and flat by (VIII.5.5.1), so it suffices to prove it is almost exact modulo $\pi(I.16.3.2)$. Then by the tilting equivalence, it suffices to prove for $X^\flat$. So we may assume at first that $K$ is of charp. Then we may replace $K$ by $L = \mathbb{F}_p[[t]]_{\text{perf}}[t^{-1}]$.

But then Noetherian approximation (VIII.5.6.18) shows that the rational subsrings are completion of filtered colimits of $p$-finite $K$-algebras ((VIII.5.2.27) used), and then we reduced to the $p$-finite case, as in the proof of (VIII.5.6.16). □

7 Étale Site of Perfectoid Spaces

Def. (VIII.5.7.1) (Finite Étale Map of Adic Spaces). A map of Huber pairs $(A, A^+) \to (B, B^+)$ is called finite étale if $A \to B$ is finite étale, and $B^+$ is the integral closure of $A^+$ in $B$.

A map $f : X \to Y$ of adic spaces is called finite étale if there is a cover of $Y$ by affinoids $V \subset Y$ that $U = f^{-1}(V)$ are all affinoids, and the map $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is finite étale. Write $Y_{\text{fét}}$ for the category of all such maps.

Def. (VIII.5.7.2) (Étale Maps). A map $X = \operatorname{Spa}(A, A^+) \to Y = \operatorname{Spa}(B, B^+)$ of adic spaces is called étale iff for any $x \in X$, there exists an open $x \in U$ and open $f(U) \subset V$ together with an adic space $W$ that $f : U \to V$ factors through an open immersion $U \to W$ and a finite étale map $W \to V$.

Cf. [?]P65.
Def. (VIII.5.7.3) (Strongly Finite Étale). For convenience, in case of perfectoid affinoid $K$-algebras, we call a map of Huber pairs $(A, A^+) \to (B, B^+)$ **strongly finite étale** if it is finite étale and $B^+a$ is almost finite étale over $A^+a$.

A map $f : X \to Y$ ofadic spaces is called **strongly finite étale** if there is a cover of $Y$ by affinoids $V \subset Y$ that $U = f^{-1}(V)$ are all affinoids, and the map $(\mathcal{O}_Y(V), \mathcal{O}^+_Y(V)) \to (\mathcal{O}_X(U), \mathcal{O}^+_X(U))$ is strongly finite étale. Write $Y_{sf\text{ét}}$ for the category of all such maps.

Finally we will prove that if $(A, A^+)$ is perfectoid, then any finite étale map $(A, A^+) \to (B, B^+)$ is strongly finite étale.

Prop. (VIII.5.7.4) (Strongly Finite Étale Maps Form a Stack). If $f : X \to Y$ is a strongly finite étale map that $Y = \text{Spa}(A, A^+)$ is an affinoid perfectoid, then $X$ is also affinoid perfectoid, and the structure map $(\mathcal{O}_X(X), \mathcal{O}^+_X(X)) \to (\mathcal{O}_Y(Y), \mathcal{O}^+_Y(Y))$ is strongly finite étale.

Proof: By (VIII.5.7.9), it suffices to prove in char $p$. Then we can replace $K$ by $L = \mathbb{F}_p[[t]]_{perf}[t^{-1}]$. Then by Noetherian approximation (VIII.6.18), we can assume that $Y$ is a limit of $p$-finite affinoids $\text{Spa}(B_i, B_i^+)$. As both rational subsets and finite étale algebras pass through filtered colimit, and adic spectrum is quasi-compact (VIII.5.24), we can assume that a finite étale cover of $Y$ arises through base change of some $\text{Spa}(B_i, B_i^+)$. So it suffices to prove the proposition in case of $Y$ $p$-finite. Then $Y$ is a completion of perfection of some algebraically admissible ring over $\mathbb{F}_p$. Then by the above argument again, we can assume that $Y$ is algebraically admissible.

Now a classical theorem (Cf. [Étale cohomology of rigid analytic varieties and adic spaces, Huber 1.6.6(2)]) shows that the finite étale cover of $Y$ is global finite étale $\text{Spa}(S, S^+) \to \text{Spa} (R, R^+)$ in this case. Notice the strongness is not needed because we are working in char $p$, where almost purity theorem is already proven.

Cor. (VIII.5.7.5). For an affinoid perfectoid space $Y = \text{Spa}(R, R^+)$, the functor $X \mapsto \mathcal{O}_X^+(X)$ defined an equivalence of categories $Y_{sf\text{ét}} \cong R_{af\text{ét}}^+$, and the functor $X \mapsto \mathcal{O}_X(X)$ gives a fully faithful functor $Y_{sf\text{ét}} \to R_{f\text{ét}}$.

Def. (VIII.5.7.6) (Étale Site of Perfectoid Spaces). Let $X$ be a perfectoid space, then the étale site of $X$ is the category $X_{\text{ét}}$ of perfectoid spaces that is étale over $X$, and coverings are given by topological coverings. We also consider the following subcategories:

- $X_{\text{off}}$, the category of affinoid perfectoid spaces étale over $X$.
- $X_{\text{ét, qcqs}}$, the full subcategory of qcqs perfectoid spaces étale over $X$.
- $X_{\text{ét, qc, sep}}$, the full subcategory of qc separated perfectoid spaces étale over $X$.

Prop. (VIII.5.7.7) (Gabber-Ramero). If $A$ is a finite $K^0$-algebra that is $\pi$-adically Henselian, then

$$A[\pi^{-1}]_{f\text{ét}} \cong \widehat{A}[\pi^{-1}]_{f\text{ét}}.$$  

Proof: Cf. [Almost Ring Theory P5.4.53].

Cor. (VIII.5.7.8) (Finite Étale Covers and Direct Limits of Complete Uniform Rings). Let $(A_i, A_i^+)$ be a filtered system of complete uniform affinoid $K$-algebras, and $(A, A^+)$ be their colimit in the category of complete uniform affinoid Tate rings, then

$$2 - \colim_i A_i \cong A$$

as categories.
Proof: By(VIII.5.2.27), $A^+$ is the $\pi$-adic completion of the algebraic colimit $B^+$ of $A_i^+$, and $A = A^+[\pi^{-1}]$. Each $A_i$ is complete and $\pi$-torsion-free, thus the colimit is Henselian and $\pi$-torsion-free(I.6.10.3)(I.6.10.6). Then the proposition(VIII.5.7.7) shows that $B^+[\pi^{-1}]_{\text{ét}} \cong A_{\text{ét}}$. Now it remains to show that

$$2 - \text{colim}_i A_{i, \text{ét}} \cong B^+[\frac{1}{\pi}]_{\text{ét}},$$

which is because étale sites commutes with taking filtered colimits.

**Final Proof of Almost Purity Theorem**

**Prop. (VIII.5.7.9).** We have an equivalence of categories $X_{sf\text{ét}} \cong X^b_{sf\text{ét}}$. For this, use(VIII.5.7.5),(VIII.5.5.10) and the proven part of(I.9.10.1) and notice that

$$A^+_{af\text{ét}} = A_{0a\text{ét}} \cong A_{0\text{ét}} = A^+_{af\text{ét}},$$

and the integral closure clearly corresponds.(It get around the problem that $R^0_{f\text{ét}} \to R_{f\text{ét}}$ hasn’t been proven essentially surjective).

**Prop. (VIII.5.7.10) (Proof of Almost Purity Theorem).** Fix a perfectoid affinoid $K$-algebra $(R, R^+)$, if $S \in R_{f\text{ét}}$, then the integral closure of $S^+$ in $R^+$ lies in $R^+_{af\text{ét}}$, and this gives an inverse to the morphism $d$ in(I.9.10.1), thus finishing the proof of almost purity theorem.

Proof: Continuing the proof of(I.9.10.1), it suffices to show that $d : R^+_{af\text{ét}} \to R_{f\text{ét}}$ is essentially surjective, because $S^+ \to \overline{S} \subset R^+$ is the only possible inverse, by the almost purity theorem in charp and tracing the tilting equivalence(I.9.9.6). Given(VIII.5.7.5), it suffices to prove that for $X = \text{Spa}(R, R^+)$, the prestacks $X_{sf\text{ét}} \cong X_{f\text{ét}}$, where $X_{sf\text{ét}}(U) = \mathcal{O}_{X^+}^+(U)_{sf\text{ét}}$, and $X_{f\text{ét}}(U) = \mathcal{O}_X(U)_{f\text{ét}}$.

We use(V.1.4.20), firstly $X_{sf\text{ét}}$ is a stack, by(VIII.5.7.4), and for each $U$, $X_{sf\text{ét}}(U) \to X_{f\text{ét}}(U)$ is simply faithful by almost purity theorem(I.9.10.1). $X_{f\text{ét}}$ is separated by(VIII.5.4.21), because the structure section of an element $S \in X_{f\text{ét}}$ is determined its value on the stalk.

Its left to prove that their stalks are equal, for this, use the formula

$$\text{colim}_{x \in U}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) = (k(x), k^+(x))$$

in the category of complete uniform affinoid $K$-algebras(they are all perfectoids(VIII.5.5.1) thus uniform), by definition. So we get by(VIII.5.7.8):

$$\text{colim}_{x \in U} \mathcal{O}_X(U)_{f\text{ét}} \cong \widehat{k(x)}_{f\text{ét}},$$

and by(VIII.5.7.8) together with the proven part of almost purity theorem(I.9.10.1):

$$\text{colim}_{x \in U} \mathcal{O}_X^+(U)_{af\text{ét}} \cong \text{colim}_{x \in U^+} \mathcal{O}_X^+(U^+)^{af\text{ét}} \cong \kappa^+(x^b)_{af\text{ét}} \cong \kappa^+(x)_{af\text{ét}}.$$

Now we have already proved the almost purity over fields(I.9.10.1) which says $\kappa(x^b)_{af\text{ét}} \cong \kappa^+(x)_{af\text{ét}}$, so their stalks are the same.

**Cor. (VIII.5.7.11) (Invariance of Étale Site under Tilting).** There is a natural isomorphism of categories $X_{\text{ét}} \cong X^b_{\text{ét}}$, by almost purity theorem(I.9.10.1) and the localness of étale maps.

**Prop. (VIII.5.7.12) (Almost Acyclicity).** For any perfectoid space $X$, the functor $U \mapsto \mathcal{O}_X(U)$ is a sheaf on $X_{\text{ét}}$, and $H^i(X_{\text{ét}}, \mathcal{O}_X)$ is almost zero if $X$ is affinoid perfectoid.
VIII.6  Fargues-Fontaine Curve

Basic references are [FF curves Lurie], [FF Curve Johannes], [The Fargues-Fontaine Curve and Diamonds Mathew Morrow], [Laurent Fargues and Jean-Marc Fontaine. Courbes et fibrés vectoriels en théorie de Hodge p-adique]

1  Fontaine’s Rings

Fontaine’s Ring $A_{\text{inf}}$

Def. (VIII.6.1.1) (Fontaine’s Ring $A_{\text{inf}}$). Let $C^0$ be a perfectoid field of char$p$, Fontaine’s ring $A_{\text{inf}}$ is defined to be the Witt vectors $W(O_C^0) = A_{\text{inf}}(O_C)$ (I.8.1.15).

the set of all the char0 untilts of $C$ is denoted by $Y$. and $Y[a,b]$ denotes those untilts that $a \leq |p| K \leq b$.

Prop. (VIII.6.1.2). By (I.9.9.7), if $K$ is perfectoid field of char0 with tilt $C^0$, there is a diagram

$$
\begin{array}{ccc}
A_{\text{inf}} & \xrightarrow{\theta} & \mathcal{O}_K \\
\downarrow & & \downarrow \\
O_C & \xrightarrow{\vartheta} & \mathcal{O}_K / \pi
\end{array}
$$

And the kernel Ker $\theta$ is generated by $\xi = \pi u - [t]$, where $[t]$ is the Techmuller lift, because it generate after modulo $p$, and use Nakayama.

Lemma (VIII.6.1.3). If $R$ is a commutative ring, $x, y \in R$, if $x$ is not a zero-divisor in $R$ and $R$ is $x$-adically complete Hausdorff, and $y$ is not a zero-divisor in $R/x$ and $R/x$ is $y$-adically complete, then the same is true with $x, y$ interchanged.

Proof: May assume $\xi = [t] - up$ and $t \neq 0$. Consider the mapping $\theta : A_{\text{inf}} \rightarrow A_{\text{inf}}/(\xi) = \mathcal{O}_K$, and denote $\theta([x])$ by $x^\xi$.

Firstly, we can apply lemma(VIII.6.1.3) to $\xi$ and $p$ to conclude that $A_{\text{inf}}$ is $\xi$-complete and $\xi$-torsion-free, and $\mathcal{O}_K$ is $p$-adically complete and $p$-torsion-free.

Now for any $y \in \mathcal{O}_K$ is $p$-adically complete, there is a $x \in O_C^0$, that $(y) = (x^\xi)$: multiplying $p$-power, we can assume $y$ is not divisible by $p$, and there is a $x$ that $y \equiv x^\xi \mod p$, thus $x$ is not divisible by $t$. Now $t = xx'$ for some $x' \in m_C$, thus $y = x^\xi + t^2 w = x^\xi(1 + x'w)$, and $1 + x'w$ is invertible in $\mathcal{O}_K$.

Next we prove $\mathcal{O}_K$ is an integral domain: It suffices to show any $y \neq 0 \in \mathcal{O}_K$ is not a zero-divisor. We can assume $y = x^\xi$, by what just proved, and then $x$ divides $t^n$ for some $n$, so it suffices to consider $y = t^{n\xi} = p^n$, and $p^n$ is not a zero-divisor by what just proved.
Now we can endow $\mathcal{O}_K$ with the valuation $|y| = |x|_{C^0}$ for $y = x^p u$, and extend it to the quotient field $K$. Then this is a Non-Archimedean valuation and the residue field has char $p$ because $|p| < 1$, and $K$ has char 0, because $p \neq 0$ in $K$. And it is $p$-adically complete.

Finally, $\mathcal{O}_K / p\mathcal{O}_K \cong A_{inf}/(\xi,p) = \mathcal{O}_{C^0}/\pi$, so the Frobenius is surjective, thus $K = K(\mathcal{O}_K)$ is a perfectoid field.

**Cor. (VIII.6.1.5).** The correspondence $\xi \mapsto Quot(A_{inf}/(\xi))$ induces a bijection

$$\{\text{Distinguished elements}\}/\text{units} \cong \{\text{Untilts of } C^0\}/\text{isomorphisms}.$$

**Prop. (VIII.6.1.6) (A_{inf} as Holomorphic Function in $p$).** Any element in $A_{inf}$ can be written uniquely as a unique Teichmüller representation $|c_0| + |c_1| p + |c_2| p^2 + \ldots$. Now we can regard these elements as holomorphic functions on $B(0,1)$, and any untilts $K$ of $\mathcal{O}_{C^0}$ can be regarded as points, where $A_{inf}$ take value $c_0^k + c_1^k p + \ldots \in \mathcal{O}_K$ at the point $K$.

This map can in fact be extended to $A_{inf}[\frac{1}{p}, \frac{1}{p}]$ that

$$A_{inf} \leftrightarrow A_{inf}[\frac{1}{p}] \leftrightarrow A_{inf}[\frac{1}{p}, \frac{1}{p}] \to K.$$

called the valuation map.

**Ring $B$**

**Def. (VIII.6.1.7) (Fontaine’s Ring $B$).** If compared to the complex case, the elements of $A_{inf}$ are just elements $\sum a_n z^n$ that $|a_n| \leq 1$, this are not all the holomorphic functions on $B(0,1)$, which is $\sum a_n z^n$ that $\lim \sup |a_n| \leq 1$. This leads to an enlargement of $A_{inf}$:

For $0 < a \leq b < 1$ in the value group of $C^0$, $|\pi_a| = a$, $|\pi_b| = b$, define

$$B_{[a,b]} = A_{inf}[\frac{\pi_a}{p}, \frac{p}{\pi_b}]^{[p^{-1}]}||,$$

this is definable at any untilts $K$ that $a \leq |p|_K \leq b$.

Then $B_{[a,b]}$ is an algebra over $A_{inf}[\frac{1}{p}, \frac{1}{p}]$, and define $B = \varprojlim B_{[a,b]}$.

**Prop. (VIII.6.1.8) (Gauss Norm).** Any element $f$ in $A_{inf}[\frac{1}{p}, \frac{1}{p}]$ is of the form $\sum_{n > -\infty} [c_n |p^n]$, where $\{|c_n|\}$ is bounded. So we can define the valuation $|f|_p = \sup \{|c_n| |p^n\}$, it is realizable by some term $|a_n| |p^n|$. Notice that for an until $y = (K, \iota)$, if $\rho = |p|_K$, then $|f(y)| \leq |f|_p$.

Then this is a non-Archimedean valuation on $A_{inf}[\frac{1}{p}, \frac{1}{p}]$.

**Proof:** Firstly $|f + g|_p \leq \max\{|f|_p, |g|_p\}$ for every $\rho$ that is generic for $f + g$ and in the value group of $C^0$: In this case,

$$|f + g|_p = |(f + g)(y)| \leq \max\{|f(y)|, |g(y)|\} \leq \max\{|f|_p, |g|_p\}$$

for some point $y$ by (VIII.6.1.8), then by continuity (VIII.6.1.9), this is true for any $\rho$.

The same method shows that $|f|_p |g|_p = |fg|_p$. □

**Lemma (VIII.6.1.9) (Generic Norms).** $\rho$ is called **generic** for $f$ iff the valuation is realized exactly once. Notice if $\rho$ is generic for $f$ and in the value group of $C^0$, then $|f|_\rho = |f(y)|$ for some $y$(Choose $K = A_{inf}/([c] - p)$ where $|c|_{C^0} = \rho$).

For any $f$, the numbers $\rho$ that $\rho$ is not generic for $f$ is discrete in $\rho$. 


Thus the ring $B$ induces an isomorphism

This is analogous to the complex case

The topology of $\phi$ is the Fontaine’s ring denoted also by $B$. Thus the Frobenius action on the Witt vector $A_{\text{inf}}$ and it satisfies

Thus induces an isomorphism $B_{[a,b]} \cong B_{[a^{\rho},b^{\rho}]}$. Passing to the limit, we get an automorphism of $B$, denoted also by $\varphi$. 

Lemma (VIII.6.1.10). If $y$ is a point that $|p|_K = \rho$, then $|f(y)| \leq |f|_\rho$, and equality holds if either $\rho$ is generic or $f$ is invertible.

Prop. (VIII.6.1.11) (Valuation Map). For $0 < a \leq b < 1$ in the value group of $C^\flat$, $|\pi_a| = a, |\pi_b| = b$,

Thus the ring $B_{[a,b]}$ is identified with the completion of $A_{\text{inf}}[\frac{\pi_a}{p^{n}}, \frac{\pi_b}{p^{n}}]$ w.r.t the valuation $| \cdot |_a + | \cdot |_b$. In particular, for any point $y$ that $a \leq |p|_K \leq b$, the valuation map (VIII.6.1.6) can be extended to a map

Proof: Notice $V_0$ is a subring by (VIII.6.1.8), so clearly $A_{\text{inf}}[\frac{\pi_a}{p^{n}}, \frac{\pi_b}{p^{n}}] \subset V_0$.

For the reverse containment, notice that $\{|c_n|\}$ is bounded, so there is an $m$ that $\pi_a^m c_n \in C^\flat$ for any $n$. Now

so it suffices to prove the case $f$ has finite presentation. Now $c_n \pi_a^n, c_n \pi_b^n \in \mathcal{O}_C^\flat$, thus $[c_n]^n = [c_n \pi_a^n]^n = [c_n \pi_b^n]^n \in A_{\text{inf}}[\frac{\pi_a}{p^{n}}, \frac{\pi_b}{p^{n}}]$, where $n \geq 0$ or $n \leq 0$. Thus the inverse containment is true.

Prop. (VIII.6.1.12) (Topology of $B$). For $0 < a \leq c \leq b < 1$, $|f|_c \leq \max\{|f|_a, |f|_b\}$ (trivial), thus the Fontaine’s ring $B$ can be realized as the completion of all the norms, and endowed with the topology of $p$-adic Fréchet space.

Prop. (VIII.6.1.13) (Teichmüller Expansion). An infinite sum $f = \sum [a_n]p^n$ converges in $B$ iff it converges in any norm $| \cdot |_\rho$ for $0 < \rho < 1$, which is equivalent to

This is analogous to the complex case (X.2.5.4). However, for now, we don’t know iff every element of $B$ is of this form, and whether the representation is unique?

Prop. (VIII.6.1.14) (Frobenius Action). Notice the Frobenius action of $C^\flat$ extends to a Frobenius action on the Witt vector $A_{\text{inf}}$, and it satisfies

Thus induces an isomorphism $B_{[a,b]} \cong B_{[a^{\rho},b^{\rho}]}$. Passing to the limit, we get an automorphism of $B$, denoted also by $\varphi$. 

Proof: Consider the Newton polygon of $f$, then only the slopes of the Newton polygon are not generic. 

□
Fargues-Fontaine Curve

Def. (VIII.6.1.15) (Fargues-Fontaine Curve). The sum $\oplus_n B^{\phi=p^n}$ is a graded ring. In fact, it is non-negatively graded (VIII.6.2.29), and we define the Fargues-Fontaine curve as the scheme

$$\text{Proj}(\oplus_{n \geq 0} B^{\phi=p^n}).$$

Def. (VIII.6.1.16) (Formal Logarithm). For $x \in 1 + m_{C^0}$, $[x] = 1 = [x - 1] + \sum_{n > 0} [c_n]p^n$, thus $|[x] - 1|_p \geq |x - 1| > 0$, thus the formal logarithm

$$\log([x]) = \sum_{k > 0} \frac{(-1)^{k+1}}{k}([x] - 1)^k$$

converges for every Gauss norm $| \cdot |_p$, thus converges to some element in $B$. And clearly $\varphi(\log([x])) = p \log([x])$, thus $\log([x]) \in B^{\phi=p}$. And $\log([xy]) = \log([x]) \log([y])$.

Prop. (VIII.6.1.17) (Artin-Hasse Exponential). There is another way of constructing elements in $B^{\phi=p}$, which is

$$T : a \in m_{C^0} \mapsto \sum_n \frac{[a^{p^n}]}{p^n}.$$

We want to relate this one to the formal logarithm:

There is a bijection of sets $m_{C^0} \cong 1 + m_{C^0}$ that $\log([E(a)]) = T(a)$, which is defined by the Artin-Hasse exponential

$$E(x) = \prod_{(d,p)=1} \left(1 - \frac{1}{1-x^d}\right)^{\mu(d)/d}.$$

Proof: Firstly, it has coefficients in $Z(p)$, because $\sum_{d|k} \mu(d) = (1 - x^d)^{\mu(d)/d} = \sum_{n \geq 0} \frac{[x^{d^n}]}{p^n}$ has coefficient in $Z(p)$. And $[1-x] = \lim_k (1-[x^{1-k}])^{p^k}$, so

$$\log(\prod_{(d,p)=1} \left(1 - \frac{1}{1-x^d}\right)^{\mu(d)/d}) = \sum_{(d,p)=1} \frac{\mu(d)}{d} \log\left(\frac{1}{1-d}\right) = \sum_{(d,p)=1} \frac{\mu(d)}{d} \sum_{\alpha \in p^{-n}_{\mathbb{Z}}} \frac{[x^{d\alpha}]}{d\alpha}$$

Notice the right hand side stablis for any term $[x^{\beta}]$, and if $\beta \neq \frac{1}{p^x}$, it will vanish, thus for $x \in m_{C^0}$, it converges, and the sum equals $\sum_n \frac{[x^{p^n}]}{p^n}$.

Cor. (VIII.6.1.18). The set of elements of the form $\sum_n \frac{[a^{p^n}]}{p^n}$ is closed under addition.

the Field $B_{df}^+$

Prop. (VIII.6.1.19) (Untilts with Roots of Unity). Let $Q_p^{\infty} = Q_p(\mu_p^{\infty})$, and $\varepsilon = (1, \mu_p, \mu_p^2, \ldots)$ be a compatible $p^n$-th roots of unity that is an element of $(Q_p^{cyc})$. Then $\varepsilon - 1$ is a pseudo-uniformizer of $(Q_p^{cyc})$. For any untilts $K$ of $C^0$ and an embedding of $Q_p^{cyc}$ in $K$, the tilting maps $\varepsilon - 1$ to a pseudo-uniformizer of $C^0$. This induces a bijection:

$$\{\text{Untilts } (K, i) \text{ of } C^0 \text{ with an embedding } Q_p^{cyc} \hookrightarrow K \} \cong \{x \in C^0 | 0 < |x-1| < 1\}.$$
Proof: In fact, the left hand side is equivalent to $K$ has a compatible $p^n$-th roots of unity, and we want to prove that for any $x$ in the right hand side, there is a unique untilts $K$ that $(x^{\frac{1}{p^k}})^n$ is a compatible primitive roots of unity, and this is equivalent to $(x^{\frac{1}{p}})^n$ satisfies $1 + x + \ldots + x^{p-1} = 0$, and further equivalent to $\theta: A_{inf} \to \mathcal{O}_K$ annihilates $1 + [x^{\frac{1}{p}}] + \ldots + [x^{\frac{p-1}{p}}]$.

It suffices to show $\xi = 1 + [x^{\frac{1}{p}}] + \ldots + [x^{\frac{p-1}{p}}]$ is distinguished (VIII.6.1.5). Let $\xi = \sum[c_n]p^n$, consider reducing to the residue field: $W(\mathcal{O}_{C^\circ}) \to W(\mathcal{O}_{C^\circ}/\mathfrak{m}_{C^\circ})$, then $\overline{\alpha} = 1$, and $\overline{\xi} = p$, thus $|c_0| < 1, |c_1 - 1| < 1$, so it is distinguished (I.8.4.18).

Cor. (VIII.6.1.20). Considering different $p^n$-th roots of unities, there is a correspondence:

$$\{\text{Char } 0 \text{ Untilts } (K, \iota) \text{ of } C^\circ \text{ with a compatible } p^n - \text{th roots of unity} \} \cong \{x \in \mathbb{C}^0 | 0 < |x - 1| < 1 \}/\mathbb{Z}_p^*$$

where $\mathbb{Z}_p^*$ acts by exponentiation (I.9.8.8).

Furthermore, there is a correspondence:

$$\{\text{Char } 0 \text{ Untilts } (K, \iota) \text{ of } C^\circ \text{ with a compatible } p^n - \text{th roots of unity} \}/\varphi_{C^\circ}^Z \cong \{x \in \mathbb{C}^0 | 0 < |x - 1| < 1 \}/\mathbb{Q}_p^*$$

where the inverse is given by $x \mapsto$ the vanishing locus of $\log([x]) \in B$.

Proof: The only thing needed to be proven is the inverse is given by $N(\log([x]))$. Notice for any untilts $K$, $|(x^{p^n})^{\frac{1}{p}} - 1| < |p|^{n/(p-1)}$ for $n$ large, then $\log((x^{p^n})^{\frac{1}{p}}) = 0$ iff $(x^{p^n})^{\frac{1}{p}} = 1$ by Newton polygon. Now $x^p \neq 1$ because $x \neq 1$ and $\frac{1}{p}$ is injective. Hence composing $\varphi^n$ for some unique $n$, we can assume $x^p = 1, x^\frac{1}{p} \neq 1$, thus it corresponds an untilt as in (VIII.6.1.19).

Def. (VIII.6.1.21) $(B^{+}_{dR})$. For an untilts $K$ of $C^\circ$ that corresponds to a distinguished element $\xi$ (VIII.6.1.5), $p$ is not a zero-divisor in $A_{inf}/(\xi^n)$, as in the proof pf (VIII.6.1.4), so we can define

$$B^{+}_{dR} = \lim_{\substack{\longrightarrow \\n}} A_{inf}/(\xi^n)[p^{-1}]$$

Prop. (VIII.6.1.22). $B^{+}_{dR}$ is a complete discrete valuation ring with $\xi$ a uniformizer, and the residue field is isomorphic to $K$. We can define $B^{+}_{dR}$ as the quotient field of $B^{+}_{dR}$.

Proof: Firstly $\xi$ is not a zero divisor in $B^{+}_{dR}$, because if $\xi x = 0, x = (x_n)$, then for any $n > 0$, and some $k$ that $p^k x_n \in A_{inf}/(\xi^n)$, so $p^k x_n$ is annihilated by $\xi$ in $A_{inf}/(\xi^n)$, thus $p^k x_n = \xi^{n-1} y_n$ for some $y_n$, because $\xi$ is a non-zero-divisor in $A_{inf}$ (VIII.6.1.4). So $p^{n-1} x^{n-1} = 0 \in A_{inf}/(\xi^n)$, thus $x_{n-1} = 0$, because $p$ is non-zero-divisor in $A_{inf}/(\xi^n)$ (VIII.6.1.4).

Next there is a map $B^{+}_{dR}/(\xi^m) \to A_{inf}/(\xi^n)[p^{-1}]$. I claim this is an isomorphism: it is clearly a surjection, and if $x = (x_n)$ is mapped to 0, then for each $n \geq m$, we choose $p^{k(n)} x_n = 0 \in A_{inf}/(\xi^n)$, then $p^{k(n)} x_n = \xi^{n-m} y_n$ for a unique $y_n \in A_{inf}/(\xi^{n-m})$. So $x = \xi^m \cdot (\frac{y_n}{p^{k(n)}}) \in A_{inf}/(\xi^n)$. The uniqueness of $y_n$ shows $(\frac{y_n}{p^{k(n)}})$ is an element in $B^{+}_{dR}$.

Then it follows $B^{+}_{dR} \cong \lim_{\substack{\longrightarrow \\n}} A_{inf}/(\xi^m)$, which shows that $B^{+}_{dR}$ is $\xi$-adically complete, and $m = 1$ shows the residue field is equal to $K$.

Remark (VIII.6.1.23). Remark if $\xi = [t] - pu$, then $A_{inf}/(\xi^n)[p^{-1}] = A_{inf}/(\xi^n)[[t]^{-1}]$, so if $K$ is of char $p$, then $B^{+}_{dR}$ is just $W(C^\circ)$.

When $K$ is of char 0, then $B^{+}_{dR}$ contains a field isomorphic to $K$ by (I.8.3.1), thus $B^{+}_{dR}$ is (non-canonically) isomorphic to $K[[t]]$, and should be thought as the completed local ring at the point $y = (K, \iota)$.
Prop. (VIII.6.1.24) (The Stalk Map). Notice \( A_{\text{inf}} = \lim_{\rightarrow n} A_{\text{inf}}/(\xi^n) \) by (VIII.6.1.4), thus there is a natural map \( A_{\text{inf}} \to B_{dR}^+ \), whose composition with \( B_{dR}^+ \to B_{dR}^+/\xi \cong K \) maps \( p, [t] \) to \( p, t^* \), which shows they are invertible in \( B_{dR}^+ \), so there is a map
\[
e : A_{\text{inf}}[\frac{1}{p}, \frac{1}{p}] \to B_{dR}^+.
\]
In case \( a \leq |p|_K \leq b \), this can be further extended to a map \( e : B_{[a,b]} \to B_{dR}^+ \) (The stalk map).

Proof: It suffices to prove for \( a = |p|_K = b \) because the topology is stronger. In this case, choose \( t = p^b \in C^0 \), then \( |t|_{C^0} = |p|_K \), thus \( \bar{\tau}_n \) determined a map
\[
A_{\text{inf}}[\frac{[t]}{p}, \frac{p}{p}] \to B_{dR}^+/(\xi^n) \cong (A_{\text{inf}}/(\xi^n))[p^{-1}],
\]
It suffices to prove the image is contained in \( p^{-k}(A_{\text{inf}}/(\xi^n)) \) for some \( k = k(n) \), because then \( \bar{\tau}_n \) is \( p \)-adically continuous, and extends to map of \( B_{[a]} = (A_{\text{inf}}[\frac{[t]}{p}, \frac{p}{p}])_p \to (A_{\text{inf}}/(\xi^n))[p^{-1}] \), which is compatible w.r.t \( n \), thus gives a map \( B_{[a]} \to B_{dR}^+ \).

For this, consider \( f = \tau_n([\frac{t}{p}]), g = \tau_n([\frac{p}{p}]) \), then their reduction under \( B_{dR}^+/(\xi^n) \to B_{dR}^+/(\xi) \cong K \) is in \( \mathcal{O}_K \cong A_{\text{inf}}/(\xi) \), thus
\[
f = f_1 + \frac{\xi}{p^c} f_2, \quad g = g_1 + \frac{\xi}{p^c} g_2
\]
for \( f_1, f_2, g_1, g_2 \in A_{\text{inf}}/(\xi^n) \) for some \( c \). Then any
\[
f^m = (f_1 + \frac{\xi}{p^c} f_2)^m = \sum_{i=0}^{n-1} C_m^i f_1^{m-i} (\frac{\xi}{p^c} f_2)^i \in p^{-nc}(A_{\text{inf}}/(\xi^n)).
\]
Thus \( \bar{\tau}_n(A_{\text{inf}}[\frac{[t]}{p}, \frac{p}{p}]) \in p^{-nc}(A_{\text{inf}}/(\xi^n)). \)

Cor. (VIII.6.1.25). The stalk map \( e : B_{[a,b]} \to B_{dR}^+ \) composed with the map \( B_{dR}^+/(\xi) \cong K \) are in fact equivalent to the valuation map (VIII.6.1.11).

2 Divisors

Valuation Function

Def. (VIII.6.2.1) (Exponential Valuation). For any positive real number \( s \), define a valuation on \( A_{\text{inf}}[\frac{1}{p}, \frac{1}{p}] \) by the formula \( v_s(f) = -\log |f|_{\exp(-s)} \), then it is a valuation by (VIII.6.1.8).

If \( f \) has a Teichmüller expansion \( \sum_{n \gg -\infty} c_n p^n \), then
\[
|f|_p = \sup \{|c_n|_{C^0} p^n\}, \quad v_s(f) = \inf \{v(c_n) + ns\}.
\]

Prop. (VIII.6.2.2). For any \( f \neq 0 \in A_{\text{inf}}[\frac{1}{p}, \frac{1}{p}] \), \( s \mapsto v_s(f) \) is a concave function in \( s \) which is piecewise linear with integral slopes.

Proof: Consider the Newton Polygon. \( \square \)
Lemma (VIII.6.2.3). If \( s > 0 \) and \( f_n \) is a Cauchy sequence in \( A_{\text{inf}}[\frac{1}{p}, \frac{1}{p}] \) for the norm \( | \cdot |_{\exp(-s)} \) and doesn’t converge to 0, then the sequences

\[
v_s(f_n), \quad \partial_v v_s(f_n), \quad \partial_{\pm} v_s(f_n)
\]

stabilize.

Proof: Easy, Cf. [ff Curve Lurie P44]. \( \square \)

Prop. (VIII.6.2.4). If \( 0 < a \leq b < 1 \), and \( f_n \) is a Cauchy sequence in \( A_{\text{inf}}[\frac{1}{p}, \frac{1}{p}] \) and doesn’t converge to 0 for either the norm \( | \cdot |_a \) or \( | \cdot |_b \), then the sequence of functions \( s \mapsto v_s(f) \) stabilizes on \( [-\log(b), -\log(a)] \).

Proof: Assume \( f_n \) doesn’t converge to 0 for the form \( | \cdot |_b \), then by (VIII.6.2.3), the sequences \( v_s(f_n), \partial_v v_s(f_n) \) converges, thus \( v_s(f_n) \) is bounded uniformly, thus \( v_s(f) \) is bounded.

Then choose \( N \) large that \( |f - f_m|_\rho \) very small for any \( m > N \) and \( a \leq \rho \leq b \), then \( v_s(f) = v_s(f_m) \) for any \( a \leq s \leq b \), thus it stabilizes. \( \square \)

Cor. (VIII.6.2.5). Let \( f \) be a non-zero element in \( B \), then the construction \( s \mapsto v_s(f) \) is a concave function in \( s \) with piecewise linear function with integral slopes. This is analogous to the Hadamard three circle theorem (X.2.6.10).

Proof: This is true for \( f \in B_{[a,b]} \), because any \( f \) is a limit of a sequence \( f_n \) in both the norm \( | \cdot |_a \) and \( | \cdot |_b \), so by the proposition, there for \( N \) large, \( v_s(f) = v_s(f_n) \) on \( [-\log(b), -\log(a)] \), thus the conclusion is true by (VIII.6.2.2). And for \( f \in B \), for any interval \([a, b]\) we can do the same, thus the conclusion is true on each interval, thus it is true. \( \square \)

Metric Structures on \( Y \)

Def. (VIII.6.2.6) (Metric on \( Y \)). Let \( Y = Y \cup \{0\} \) be the isomorphism classes of untilts of \( C^0 \), where \( 0 \) corresponds to \( C^0 \) itself.

(VIII.6.1.5) show \( Y \) corresponds to distinguished elements in \( A_{\text{inf}} \) up to units. So for any \( x, y \in Y \), we let \( d(x, y) = |\xi_x(y)|_{K_y} \leq 1 \). Then this is a metric, and it is non-Archimedean.

Proof: Firstly, if \( d(x, y) = 0 \), then \( \xi_x \) divides \( \xi_y \), which is equivalent to \( (\xi_x) = (\xi_y) \), by (I.8.4.19).

Secondly, for any \( x, y \), since \( C^0 \) is alg.closed, we can assume \( \xi_x(y) = c^2 \) for some \( c \in C^0 \). Notice \( \xi(y) = t^2 + pu(y) \) is in \( m_K \), thus \( c \in m_{C^0} \). So \( \xi_x - [c] \) is also a distinguished element and vanishes at \( y \), so we may assume that \( \xi_y = \xi_x - [c] \) by (I.8.4.19) again. Then

\[
d(y, x) = |\xi_y(x)|_{K_x} = |c^2|_{K_x} = |c|_{C^0} = |c^2|_{K_y} = d(x, y).
\]

Finally it is non-Archimedean because any valuation field \( K \) is non-Archimedean. \( \square \)

Prop. (VIII.6.2.7) (\( Y \) is Complete). \( Y \) is complete w.r.t this metric.

Proof: Given a Cauchy sequence of points \( y_n \) in \( Y \), as in the proof of (VIII.6.2.6), we can assume that \( \xi_{y_n} = \xi_{y_{n-1}} + [c_n] \) for some \( c_n \in m_{C^0} \), and \( |c_n|_{C^0} = d(y_{n-1}, y_n) \). Now \( A_{\text{inf}} \) is \([t]\) -adically complete for a uniformizer \( t \in C^0 \), thus \( \sum |c_n| \) is definable in \( A_{\text{inf}} \), and \( \xi = \xi_0 + \sum |c_n| \) is also distinguished, and corresponds to a point \( y \) which \( y_n \) clearly converges to. \( \square \)
VIII.6. FARGUES-FONTAINE CURVE

Lemma (VIII.6.2.8). \( B_{[a,b]} \) is an integral domain.

Proof: By (VIII.6.2.5), the valuation function \( v_s(f) \) and \( v_s(g) \) are bounded, thus it is clear that \( v_s(fg) \) is also finite, so \( fg \neq 0 \).

Prop. (VIII.6.2.9) (Divisors). Assume \( C^b \) is alg.closed, then for any \( f \in B_{[a,b]} \) and \( y = (K, \iota) \in Y_{[a,b]} \), we define the order of vanishing \( \text{ord}_K(f) \in \mathbb{Z} \cup \{ \infty \} \) as the valuation of \( e_K(f) \in B_{dR}^+(K) \), then

- if \( f \neq 0 \in B_{[a,b]} \), then \( \text{ord}_K(f) < \infty \) for each \( K \in Y_{[a,b]} \), and there are only f.m. \( K \) that \( \text{ord}_K(f) \neq 0 \). In particular, \( B_{[a,b]} \) is an integral domain.
- if \( x, y \neq 0 \in B_{[a,b]} \), then \( x \) divides \( y \) iff \( \text{ord}_K(x) \geq \text{ord}_K(y) \) for each \( K \in Y_{[a,b]} \).

Thus for each \( f \in B_{[a,b]} \), we can define the divisor of \( K \) as the formal sum \( \sum_{K \in Y_{[a,b]}} \text{ord}_K(f)K \), and this is also definable for \( f \in B \), but it may be an infinite but locally finite sum.

Proof: Firstly, by (VIII.6.2.13) and (VIII.6.2.14), if \( \text{div}(f) \cap Y_{[a,b]} \neq 0 \), then there is a distinguished element \( \xi \) that \( f = \xi f_1 \). And we can iterate this, and eventually end up with \( f = \xi_1 \ldots \xi_n f_n \) that \( \text{div}(f_1) \cap Y_{[a,b]} = 0 \), by (VIII.6.2.15), so by (VIII.6.2.16), \( f_n \) is invertible in \( f \), so \( \text{div}(f) \) is finite. And if \( \text{div}(g) \geq \text{div}(f) \), then \( g \) also divides \( \xi_1 \ldots \xi_n \), so \( g \) divides \( f \).

Remark (VIII.6.2.10). Notice by (VIII.6.1.25), for a \( f \in B_{[a,b]} \), \( \text{ord}_K(f) > 0 \) iff \( f(y) = 0 \in K \).

Cor. (VIII.6.2.11) (B is Integral Domain). B is an integral domain, and if \( C^b \) is alg.closed, then \( x \) is divisible by \( y \) if and only if \( \text{div}(x) \geq \text{div}(y) \).

Cor. (VIII.6.2.12). B is integrally closed.

Proof: \( B \) is an integral domain by (VIII.6.2.11), it is integrally closed because if \( f/g \) is integral over \( B \), then there image in \( B_{dR}^+(y) \) is integral over \( B_{dR}^+(y) \) for all \( y \in Y \), thus in \( B_{dR}^+(y) \) because it is a valuation ring, and then \( f \) is divisible by \( g \) by (VIII.6.2.11).

Prop. (VIII.6.2.13) (Examples of Divisors).
- For a distinguished element \( \xi \), if \( \xi = up \), then \( \xi \) is invertible in \( B \), thus \( \text{div}(\xi) = 0 \). Otherwise \( \xi \) defines a char0 untilts \( K \) of \( C^p \), and \( \xi \) is a uniformizer of \( B_{dR}^+(K) \), and it doesn’t divides other distinguished elements (I.8.4.19), thus \( \text{div}(\xi) = K \).
- \( \text{div}(\log([x])) = \sum_{n \in \mathbb{Z}} \varphi^n(K) \).

Proof: \( 2^p \) \( \text{div}(\log([x])) \) vanishes at a single \( \varphi \)-orbits of \( Y \), and one of them is given by the distinguished element \( \xi = 1 + [x^{1/p}] + \ldots + [x^{p-1/p}] = \frac{x^{\frac{1}{p}} - 1}{x^{\frac{1}{p}} - 1} \). Notice \( [x^{1/p}] - 1 \) is mapped to an invertible element in \( K \), thus it is invertible in \( B_{dR}^+(K) \), so \( [x] - 1 \) is associated to \( \xi \), and notice

\[
\log([x]) = \sum_{k>0} \frac{(-1)^{k+1}}{k} ([x] - 1)^k \equiv [x] - 1 \ mod \ ([x] - 1)^2,
\]

so \( \text{ord}_K(\log([x])) = 1 \), and because \( \varphi(\log([x])) = p \log([x]) \), \( \text{ord}_{\varphi^n(K)}(\log([x])) = 1 \) for any \( n \), so we are done.

Lemma (VIII.6.2.14). Let \( C^b \) be alg.closed. If \( \xi \) is a distinguished element of \( A_{inf} \) vanishes at a point \( y \in Y_{[a,b]} \) and \( g \in B_{[a,b]} \) also vanishes at \( y \), then \( g \) is divisible by \( \xi \) in \( B_{[a,b]} \).
Proof: If \( g \in A_{\text{inf}}[\frac{1}{p}, \frac{1}{p^2}] \), then this is easy by (VIII.6.2.10) and \( A_{\text{inf}}/(\xi) = \mathcal{O}_K \) (VIII.6.1.4).

Now generally \( g \) is a limit of \( g_n \in A_{\text{inf}}[\frac{1}{p}, \frac{1}{p^2}] \), so \( g(y) \) is the limit of \( g_n(y) \in K \). Now \( g(y) = 0 \), so \( \lim_n g_n(y) = 0 \). Now \( K \) is alg. closed by (I.9.8.15), so we can let \( g_n(y) = c^2_i \), so \( c_i \) converges to 0 in \( C^0 \). So \( \{c_n\} \) converges to 0 in norm \( | \cdot |_a \) and \( | \cdot |_b \), so we can replace \( g_n \) by \( g_n - [c_n] \) and assume \( g_n(y) = 0 \).

Now the first part shows \( g_n = \xi h_i \), and now \( h_i \) is a Cauchy sequence for both \( | \cdot |_a \) and \( | \cdot |_b \), so converges to some \( h \), and then \( g = \xi h \).

Lemma (VIII.6.2.15). Given this lemma (VIII.6.2.14), we have a strategy of proving (VIII.6.2.9), that is, decomposing \( f, g \) into distinguished elements, but we need to show this decomposition is finite. And this is true:

If \( f \neq 0 \in B_{[a,b]} \), denote \( \beta = -\log(b), \alpha = -\log(a) \) and let \( N = \partial_- v_{\beta}(f) - \partial_+ v_{\alpha}(f) \geq 0 \), then \( f \) cannot be divisible by a product of \( \xi_1, \ldots, \xi_{N+1} \) of \( N + 1 \) distinguished elements.

Proof: By (I.8.4.18), if \( \xi \) is distinguished, then \( v_a(\xi) = \max\{s, v(v_0)\} \). Now \( v(v_0) = v(t^2) = v([p]\, K) \) in \( \mathcal{O}_K = A_{\text{inf}}/([t] - up) \), so if \( K \) corresponding to \( \xi \) belongs to \( Y_{[a,b]} \), then \( v(v_0) \in [\beta, \alpha] \), so \( \partial_- v_{\beta}(\xi) = 1, \partial_+ v_{\alpha}(\xi) = 0 \).

So if \( f = \xi_1, \ldots, \xi_{N+1} \), then \( N(f) \geq \sum N(\xi_i) \geq N + 1 \).

Lemma (VIII.6.2.16) (Valuation Function And Invertibility). Let \( C^0 \) be alg. closed and \( f \neq 0 \in B_{[a,b]} \), then the following are equivalent:

- \( f \) is invertible.
- \( \partial_- v_{\beta}(f) = \partial_+ v_{\alpha}(f) \).
- \( \text{div}(f) \cap Y_{[a,b]} = \emptyset \).

Proof: \( 2 \rightarrow 3 \): by (VIII.6.2.15).

1 \( \rightarrow 2 \): Because \( N(f) + N(f^{-1}) = 0 \), and \( N(f) \geq 0, N(f^{-1}) \geq 0 \), so \( N(f) = 0 \).

2 \( \rightarrow 1 \): Assume first that \( f = \sum_{n>\infty}[c_n]p^n \in A_{\text{inf}}[\frac{1}{p}, \frac{1}{p^2}] \), then the hypothesis just says that \( s \rightarrow v_a(f) \) is linear in a small nbhd of \( [\beta, \alpha] \); that is, there is an \( n_0 \) that \( v(c_n) + ns > v(c_{n_0}) + n_0s \) for all \( n \neq n_0 \) and \( s \in [\beta, \alpha] \).

Now we can normalize \( f \) that \( n_0 = 0 \) and \( c_0 = 1 \), so \( |f - 1|_p < 1 \) for all \( p \in [\beta, \alpha] \), so \( f - 1 \) is topologically nilpotent in \( B_{[a,b]} \), and thus \( f \) is invertible.

Generally, \( f \) is a limit of a sequence \( f_n \in A_{\text{inf}}[\frac{1}{p}, \frac{1}{p^2}] \), and by (VIII.6.2.4) we can assume the hypothesis holds for all \( f_n \). Then \( f_n \) is invertible, and it is easily shown that \( f_n^{-1} \) is a Cauchy sequence in \( B_{[a,b]} \), so converges to some \( f^{-1} \).

3 \( \rightarrow 2 \): Firstly, if \( \partial_- v_{\beta}(f) > \partial_+ v_{\alpha}(f) \), then we must have \( \partial_- v_p(f) > \partial_+ v_p(f) \) for some \( s \), so wlog, we can assume \( a = b = s \), and we need to show \( f \) vanishes at some point in \( Y_{\exp(s)} \). Now combining with (VIII.6.2.14) and (VIII.6.2.15), this is equivalent to another statement that any element \( y \in B_{[\rho, \rho]} \) has a decomposition \( y = g\xi_1 \ldots \xi_n \), where \( \xi_k \) corresponds to points in \( Y_{\rho} \) and \( g \) is invertible in \( B_{\rho} \). The proof is finished at (VIII.6.2.26).

Primitive Elements and the Proof of 3 \( \rightarrow 2 \) of The Lemma on Valuation Function and Invertibility

Def. (VIII.6.2.17). Let \( C^0 \) be alg. closed, an element in \( B_{[\rho, \rho]} \) is called \textbf{good} iff it has a decomposition as in the proof of 3 \( \rightarrow 2 \) of (VIII.6.2.16).
VIII. FARGUES-FONTAINE CURVE

Prop. (VIII.6.2.18) (Approximating Zero). If \( f \) is a good element having \( n \)-zeros on \( Y_\rho \), and \( g \in B_\rho \) that \( |f - g|_\rho < |f|_\rho \), then for any zero \( y \) of \( g \) on \( Y_\rho \), there exists a zero \( y' \) of \( f \) on \( Y_\rho \) that \( d(y', y) < \rho \left( \frac{|f - g|_\rho}{|g|_\rho} \right)^{1/n} \).

Proof: \(|f - g|_\rho \geq |(f - g)(y)|_K = |f(y)|_K\). Now \( f = g\xi_1 \ldots \xi_n \), and \( \xi \) corresponds to \( y_i \), then

\[
|f(y)|_K = |g(y)|_K |\xi_1(y)|_K \ldots |\xi_n(y)|_K = \lim_{\xi_i \to y_i} \prod_i d(y_i, y) = |f| \prod_i \frac{d(y_i, y)}{\rho}.
\]

(Notice \(|g|_\rho = |g(y)|_K\) because \( g \) is invertible (VIII.6.1.10)) and \( d(y_i, y) \leq \rho \). So at least one \( \xi \) satisfies the desired inequality. \( \Box \)

Cor. (VIII.6.2.19). If \( f \in B_\rho \) is given by a Cauchy sequence of good elements, and \( \partial_+ v_\rho(f) > \partial_+ v_\rho(f) \), then \( f \) has a root on \( Y_\rho \).

Proof: By (VIII.6.2.3), passing to a subsequence, we may assume

\[
v_s(f) = v_s(f_n), \quad \partial_- v_s(f) = \partial_- v_s(f_n), \quad \partial_+ v_s(f) = \partial_+ v_s(f_n), \quad |f_{n+1} - f_n|_\rho < |f|_\rho.
\]

Let \( n = \partial_- v_s(f) - \partial_+ v_s(f) > 0 \), then each \( f_i \) has exactly \( n \) roots on \( Y_\rho \), and applying (VIII.6.2.18), we can find successively roots \( y_n \) of \( f_n \) that \( d(y_{n+1}, y_n) \leq \rho \left( \frac{|f_{n+1} - f_n|_\rho}{|f|_\rho} \right)^{1/n} \), so the sequence \( \{y_n\} \) is Cauchy and converges to some point \( y \in Y \), so

\[
|f_i(y)|_K \leq |f_i|_\rho \frac{d(y_i, y)}{\rho} = |f|_\rho \frac{d(y_i, y)}{\rho} \to 0.
\]

so \( f(y) = 0 \). \( \Box \)

Def. (VIII.6.2.20) (Primitive Elements). An element \( f = \sum_{n \geq 0} c_n p^n \in A_{\inf} \) is called primitive of degree \( d \) if \( c_0 \neq 0 \), \( |c_d| = 1 \) for some smallest element \( d \).

Clearly an element is distinguished of degree 1 iff it is distinguished and corresponds to an untilts of \( X^0 \) of char0.

Prop. (VIII.6.2.21).

- Any element \( f \in A_{\inf} \) of finite Teichmuller expansion can be written uniquely as \( f = p^m[c]g \), where \( c \in C^b \) and \( g \) is primitive.

- For an element \( f \in A_{\inf} \left[ \frac{1}{p}, \frac{1}{p^2} \right] \), \( f \) can be written as \( p^m[c]g \) iff \( v_s(f) \) consists of f.m. line segments iff \( \sup \{|c_n|\} \) is achieved by some \( n \).

- If \( f = gh \) in \( A_{\inf} \) is primitive, then \( g, h \) are also primitive, and \( \deg(f) = \deg(g) + \deg(h) \).

Prop. (VIII.6.2.22). Let \( f = \sum |c_n| p^n \in A_{\inf} \) be primitive of degree \( d > 0 \), and let \( \lambda \in (0, 1) \) be the number that \( s = -\log(\lambda) \) is the minimal number that \( v_s(f) \) is non-differentiable at, i.e. \( s \) is \(-1\) times the slope of the line segment on the left of \( v(c_d) \). Then \( f \) has a zero on \( Y_\lambda \).

Proof: By (VIII.6.2.23) there is a \( y \in Y_\lambda \) that \(|f(y)| \leq \lambda^{d+1}\), and then (VIII.6.2.24) shows we can find successively \( y_n \) that

\[
d(y_n, y_{n+1}) \leq \lambda^{1+\frac{d}{m}}, \quad |f(\lambda_n)| \leq \lambda^{d+m}.
\]

So \( y_n \) is a Cauchy sequence thus converges to some \( y \), and then \( f(y) = 0 \). \( \Box \)
Lemma (VIII.6.2.23) (Lemma for Approaching a Zero). If $C^b$ is alg.closed and $f \in A_{\text{inf}}$ is primitive of degree $d > 0$, and let $\lambda$ as in(VIII.6.2.22), then there is a point $y \in \mathcal{Y}_{\lambda}$ that $|f(y)|_{K_y} \leq \lambda^{d+1}$.

Proof: Let $f = \sum [c_n]p^n$, we may assume $c_d = 1$, and let $F = x^d + c_{d-1}x^{d-1} + \ldots + c_0$, then the largest valuation of the roots of $F$ in $\mathcal{C}^b$ is $\lambda$, by Newton polygon. Let $r$ be such a root, then $c_i$ is divisible by $r^{d-i}$, and let $\xi = p - [r]$ be a distinguished element of $A_{\text{inf}}$ and corresponds to an until $K$, then $|p|_K = \lambda$, and

$$p^{-d}f(y) = \sum_{n \geq 0} c_n^*p^{-n} \equiv \sum_{i=0}^{d-1} (\frac{c_i}{r^{d-i}})^{\xi} \mod p = (r^{-d}F(r)) \mod p = 0$$

thus $f(y)$ is divisible by $p^{d+1}$, which is equivalent to $|f(y)|_K \leq \lambda^{d+1}$.

Lemma (VIII.6.2.24) (Lemma for Approaching a Zero). Situation as in(VIII.6.2.23), if $y \in \mathcal{Y}_{\lambda}$ and $|f(y)| = \lambda^d \cdot \alpha$, then there is a $y'$ that $d(y, y') \leq \lambda \cdot \alpha^{1/d}$ that $|f(y')| \leq \lambda^{d+1} \alpha$.

Proof: Since $A_{\text{inf}}$ is $\xi$-complete and every element of $A_{\text{inf}}/\xi \cong \mathcal{O}_K$ belongs to the image of $\xi : \mathcal{O}_C \rightarrow \mathcal{O}_K$, thus by induction, we can write $f = \sum_{n \geq 0}[c_n]\xi^n$. Because $f$ is primitive of degree $d$, we may assume $c_d = 1$, and $|c_0|_{\mathcal{C}^b} = |f(y)|_K = \lambda^d \alpha$.

Let $F(x) = c_0 + c_1x + \ldots + c_{d-1}x^{d-1} + x^d$, because $C^b$ is alg.closed, let $r$ be a root of minimal absolute value, then $|r|_{C^b}|c_m|_{C^b} \leq \lambda^n \alpha$, in particular $|r|_{C^b} \leq \lambda^n \alpha$. So let $\xi' = \xi - [r]$, then $\xi$ is also distinguished, and $d(y, y') = |r|_{C^b} \leq \lambda \alpha^{1/n}(VIII.6.2.6)$, and $d(0, y') = \lambda$, and $\xi(y') = r^\xi$.

Now

$$\left(\frac{f(y')}{c_0^\xi}\right)^n = \sum_{n \geq 0} (\frac{c_n^\xi}{c_0^\xi})\xi(y')^n = \sum_{n \geq 0} (\frac{c_n^\xi}{c_0^\xi})\xi(y')^n \equiv \sum_{i=0}^{d-1} (\frac{c_n^\xi}{c_0^\xi})^{\xi(y')} \mod p = (\frac{F(r)}{c_0})^\xi = 0,
$$

So $|f(y')|_{K'} \leq |c_0^\xi|_{K'}|p|_{K'} = |c_0|_{C^b} \lambda = \lambda^{d+1} \alpha$.

Cor. (VIII.6.2.25) (Primitive Elements Decompose as Distinguished Elements). If $f \in A_{\text{inf}}$ is a primitive element of degree $d > 0$, then $f$ admits a factorization as products of distinguished elements $\xi$ corresponding to points in $\mathcal{Y}$.

Proof: Use induction on $d$. If $d = 1$, then $f$ is distinguished by(VIII.6.2.20), and if $d > 1$, then by(VIII.6.2.22), $f = \xi g$, so $g$ is primitive of degree $d - 1$ by(VIII.6.2.21), so induction is finished.

Prop. (VIII.6.2.26) (Finite Teichmuller Expansion is Good). Any element of finite Teichmuller expansion is good.

In particular, because any element of $B_\rho$ can be approximated by elements in $A_{\text{inf}}[\frac{1}{p}, \frac{1}{p}]$, and such element can be approximated by elements of finite Teichmuller expansion, by(VIII.6.2.19), we finishes the proof of $3 \rightarrow 2$ of(VIII.6.2.16).

Proof: If $f$ has finite Teichmuller expansion, then $f = p^m[c]g$, where $g$ is primitive of degree $d$. If $d = 0$, then $g$ is invertible in $A_{\text{inf}}$, thus $f$ is invertible in $B_\rho$. Otherwise, we can use(VIII.6.2.25) to factorize $g$ into distinguished elements, and the elements that corresponds to points outside $Y_\rho$ is invertible in $B_\rho$ because $v_\rho(\xi) = \max\{s, v(v_0)\}$ and $2 \rightarrow 1$ of(VIII.6.2.16), so $f$ is good.
Bounded Meromorphic Functions

Prop. (VIII.6.2.27). If \( f \in B \), then \( f \in A_{\inf} \) iff \( |f|_{\rho} \leq 1 \) for any \( 0 < \rho < 1 \).

Easily we can get characterization of \( f \) being in \( A_{\inf}[\frac{1}{p}, \frac{1}{[\rho]}] \) or \( A_{\inf}[\frac{1}{p}, \frac{1}{[\rho]}] \).

Proof: One direction is trivial, for the other, by (VIII.6.2.28), we can find successively \( f_n \) that \( f = \sum_{i<n} |c_i|p^i + f_n \), and \( |f_n|_{\rho} \leq \rho^n \) for all \( 0 < \rho < 1 \). So \( f_n \) converges to 0 in any norm \( \rho \), thus it converges to 0 in \( B \), and \( f = \sum_{n \geq 0} |c_n|p^n \in A_{\inf} \). □

Lemma (VIII.6.2.28). If \( f \in B \) satisfies \( |f|_{\rho} \leq \rho^n \) for all \( 0 < \rho < 1 \), then there is a \( c \in \mathcal{O}_{C^\circ} \) that \( f = [c]p^m + g \) that \( |g|_{\rho} \leq \rho^{m+1} \) for all \( 0 < \rho < 1 \).

Proof: Replace \( f \) by \( \frac{f}{p^m} \), we may assume \( m = 0 \). Choose a sequence \( f_i \) in \( A_{\inf}[\frac{1}{p}, \frac{1}{[\rho]}] \) converging to \( f \) in \( B \), where \( f_i = \sum_{n \gg -\infty} |c_{n,i}|p^n \).

Firstly we want to truncate \( f_i \) with the positive part \( f^+_i \). Notice for each \( \rho \) and any \( 0 < \varepsilon < 1 \), because \( \lim |f_i - f|_{\varepsilon \rho} = 0 \), thus for \( i \) large, \( |f_i|_{\varepsilon \rho} \leq 1 \), thus \( |c_{n,i}p^n|_{\varepsilon \rho} \leq |c_{n,i}|_{\varepsilon \rho} \leq \tilde{\varepsilon} < \varepsilon \), so \( |f_i - f^+_i|_{\rho} < \varepsilon \) for \( i \) large, so \( f^+_i = f \) also in \( B \).

Secondly, \( |c_{0,i} - c_{0,j}|_{\rho} \leq |f_i - f_j|_{\rho} \) for each \( \rho \), thus \( c_{0,i} \) is Cauchy in \( C^\circ \) thus converges to some \( c \in C^\circ \), and when \( i \) is large, \( |c_{0,i}|_{\rho} \leq |f_i|_{\rho} \leq 1, \) so \( c \in \mathcal{O}_{C^\circ} \). Now let \( g_i = \sum_{n \geq 0} |c_{n,i}|p^n \), then \( g_i \) is also Cauchy in \( B \) for any norm \( |\rho| \) and converges to some \( g \), and \( f = g + [c] \).

It’s left to check \( |g|_{\rho} \leq \rho \): each \( v_s(g_i) \) has positive slopes, then so does \( v_s(g) \) because by (VIII.6.2.3), \( v_s(g_i) \) stabilizes to \( v_s(g) \) uniformly on compact intervals. So if \( v_s(g) < s - \varepsilon \) for some \( s \), then \( v_s(g) = v_s(g) - (s - \varepsilon) < 0 \), but this cannot happen because \( v_s(g) \leq \max\{v_s(f), -\log |c|_{C^\circ}\} \geq 0 \). □

Eigenspaces of Frobenius

Prop. (VIII.6.2.29).

- The vector space \( B^{p^n} \) vanish for \( n < 0 \).
- The canonical map \( \mathbb{Q}_p \to B^{p - \mathrm{id}} \) is an isomorphism.

Proof: 1: Consider \( v_{p^n}(\varphi(f)) = pv_s(f)(\text{VIII.6.14}) \), \( v_s(p^n f) = ns + v_s(f) \), so if \( \varphi(f) = p^n f \), then

\[
pv_s(p^n f) = v_s(\varphi(f)) = v_s((p^n f) = ns + v_s(f).
\]

Let \( h(s) = \partial_+ v_s(f) \), then \( h(s/p) = n + h(s) \), but \( h \) must by non-increasing (VIII.6.2.2), so \( n \geq 0 \).

2: Firstly we prove \( B^{p - \mathrm{id}} \) is a field: by (VIII.6.2.11), it suffices to show that \( \mathrm{div}(f) = 0 \) for \( f \neq 0 \in B^{p - \mathrm{id}} \). If \( \mathrm{div}(f) \neq 0 \), because \( f \) is fixed by \( \varphi \), so \( \mathrm{div}(f) \geq \sum_{n \in \mathbb{Z}} \varphi^n(y) \) for some \( y \), and \( \sum_{n \in \mathbb{Z}} \varphi^n(y) = \mathrm{div}(\log([\varepsilon])) \) for some \( \varepsilon \in 1 + m_{C^\circ} \) because \( K \) alg.closed and by (VIII.6.1.20). So by (VIII.6.2.11) again \( f = g \log([\varepsilon]) \), and \( g \in B^{p - \varphi^{-1}} \) by (VIII.6.1.16), then \( g = 0 \) by item1.

3: From (VIII.6.2.23) and (VIII.6.2.27), \( f \in A_{\inf}[\frac{1}{p}] \), thus \( f = \sum_{n > \infty} |c_n|p^n \), so \( \varphi(f) = f \) shows \( c_n^p = c_n \), which is equivalent to \( c_n \in \mathbb{F}_p \). So \( f \in W(\mathbb{F}_p)[\frac{1}{p}] = \mathbb{Q}_p \). □

Lemma (VIII.6.2.30). If \( f \neq 0 \in B^{p - \mathrm{id}} \), then there is an integer \( n \) that \( |f|_{\rho} = \rho^n \).

Proof: Notice \( |f|^p_{\rho} = |\varphi(f)|_{\rho^p} = |f|_{\rho^p} \), so \( v_{p^n}(f) = pv_s(f) \), differentiation shows that \( \partial_- v_p(f) = \partial_- v_s(f) \). This is for all \( s < 0 \), and \( \partial_- v_s(f) \) is non-decreasing, thus it is constant, and \( v_{p^n}(f) = pv_s(f) \) shows \( v_s(f) = ns \) for some integer \( n \). □
Cor. (VIII.6.2.31). For \( n \geq 0 \), any element \( f \in B^{p^n} \) factors uniquely up to action of \( \mathbb{Q}_p^* \) as \( \lambda \log([\varepsilon_1]) \ldots \log([\varepsilon_n]) \) where \( \lambda \in B^{p=\text{id}} \), \( 0 < |\varepsilon_i - 1| < 1 \).

Proof: The existence is by(VIII.6.2.26)

For the uniqueness: it suffices to prove \( \log([\varepsilon]) \) is a prime element in \( \oplus_{n \geq 0} B^{p^n} \). For this, notice for any \( f \in B^{p^n} \), \( \text{div}(f) \) is fixed by \( \varphi \), and \( \text{div}(\log([\varepsilon])) \) is a single orbit of \( \varphi \), thus by(VIII.6.2.11), if \( \log([\varepsilon]) \) divides \( fg \), then \( \log([\varepsilon]) \) divides \( f \) or \( g \).

Applications

Cor. (VIII.6.2.32). If \( C^b \) is alg.closed, then every untilts \( K \) of \( C^b \) belongs to the vanishing locus of \( \log([x]) \) for some \( x \in C^b \) which \( 0 < |x - 1| < 1 \), and the map

\[
\psi : 1 + m_{C^b} \to K : y \mapsto \log(y^x)
\]

is surjective with kernel generated by \( x \) (as a \( \mathbb{Q}_p \)-subspace of \( 1 + m_{C^b} \)).

Proof: By(I.9.8.15), any untilts of \( C^b \) is alg.closed, thus it has a compatible \( p^n \)-th roots of unity. So it belongs to some locus of \( \log([x]) \) by(VIII.6.1.20). Now if \( |z| < |p|^{1/(p-1)} \), then \( z = \log(\exp(z)) \), and \( \exp \equiv y^x \) for some \( y \) because \( K \) is alg.closed. So \( \psi \) contains sufficiently small elements, but it is a map of \( \mathbb{Q}_p \)-vector spaces, thus it is surjective. For the kernel, if \( \log(y^x) = 0 \), then \( \log([y]) \) vanish on \( K \), thus by(VIII.6.1.20), \( y, x \) is in the same \( \mathbb{Q}_p \)-vector space.

Cor. (VIII.6.2.33). If \( C^b \) is alg.closed, then the map

\[
1 + m_{C^b} \log([x]) \to B^{p=x}
\]

is an isomorphism.

Proof: Firstly any untilts of \( C^b \) is alg.closed by(I.9.8.15). It is injective because of the correspondence(VIII.6.1.20), and for the surjectivity, for each \( f \in B^{p=\text{id}} \), if \( f = 0 \), then \( f = \log([1]) \), and if \( f \neq 0 \), then notice \( \text{div}(f) \neq \emptyset \), because in this case \( f \) is invertible in \( B \) by(VIII.6.2.11), thus \( f^{-1} \in B^{p=\text{id}} \), so \( f^{-1} = 0 \) by(VIII.6.2.29), contradiction.

Now if \( \text{ord}_K(f) \geq 1 \), then \( \text{ord}_K(f^n) \geq 1 \) for any \( n \in \mathbb{Z} \) since \( \varphi(f) = pf \). Consider \( \text{div}(\log([x])) = \sum_{n \in \mathbb{Z}} \varphi^n(K) \) by(VIII.6.2.13), then \( f \) is divisible by \( \text{div}(\log([x])) \) by(VIII.6.2.11), \( f = \log([x])g \), then \( g \in B^{p=\text{id}} \), then \( g \in \mathbb{Q}_p^* \) by(VIII.6.2.29), thus \( f = \log([x^n]) \).

Cor. (VIII.6.2.34) (Filtration on \( B_{dR} \)). By(VIII.6.2.13) and(VIII.6.2.32), we see that for any untilt \( K \) of \( C^b \), there is a unique up to \( \mathbb{Q}_p \)-constant \( \varepsilon \) that \( t = \log([\varepsilon]) \) is the uniformizer of \( B_{dR}(y) \). In fact, this \( \varepsilon \) can be to be \( \varepsilon = (1, \xi_p, \ldots, x_p^n, \ldots) \), where \( \xi_p^n \) is a compatible roots of unity in the alg.closed field \( K \).

Now we prefer to use the filtration \( Fil^n = t^{-n} B^+_{dR} \) on \( B_{dR} \) because it is \( \mathbb{G}_{Q_p} \) invariant, as \( \varepsilon \) does.

Prop. (VIII.6.2.35). Let \( C^b \) be alg.closed, then any point \( x \) of the Fargues-Fontaine curve \( X_{FF} \) that is not the generic point corresponds to the prime \( x_K = (\log([\varepsilon])) \) where \( K \in \text{div}(\log([\varepsilon])) \) which \( K \in \text{div}(\log([\varepsilon])) (VIII.6.2.31) \) where \( K \in \text{div}(\log([\varepsilon])) \) and the residue field of \( x_K \) can be identified to \( K \).

Proof: By(VIII.6.2.31), we can cover \( X_{FF} \) by affine schemes of the form \( \text{Spec}(R_f = B[f^{-1}]^{p=\text{id}}) \) for \( f \in B^{p=\wp} \), now for any prime \( p \subset R_f \), let \( \frac{f}{g} \in p \), then \( g = \lambda \log([\varepsilon_1]) \ldots \log([\varepsilon_n]) \), thus some \( \frac{\log([\varepsilon])}{f} \in p \). Let \( K \) be a point that \( \log([\varepsilon]) \) vanish, then we claim \( (\log([\varepsilon])/f) \) is maximal.
In fact, we may assume $f$ doesn’t vanish on $K$, otherwise $\log([\varepsilon])/f$ is a unit, then there is
a map $\rho : B[f^{-1}]^{\varphi=1} \subset B[f^{-1}] \to K$, and this map is surjective with kernel $(\log([\varepsilon])/f)$: it
is surjective even on $f^{-1}B^{\varphi=p}$ by (VIII.6.2.32), and if $\log([\varepsilon])/f$ is mapped to 0, then $\log([\varepsilon])$
differs from $\log([\varepsilon])$ by some $Q_p$ by (VIII.6.1.20).

\begin{proof}
This is trivial using Cech cohomology, as
of its ideal sheaf
by lemma

\begin{corollary}
(VIII.6.2.36).
If $C^\varphi$ is alg.closed, there is a bijection of sets:

\[ \frac{Y}{\varphi_C^Z} \cong \{ \text{Closed points of } X_{FF} \}. \]

by (VIII.6.1.20).

\end{corollary}

\begin{corollary}
(VIII.6.2.37).
$X_{FF}$ is a Dedekind scheme (V.4.2.8).

\end{corollary}

\begin{proof}
Let $\text{Spec}(R_f = B[f^{-1}]^{\varphi=\text{id}})$, two elements $f = \log([\varepsilon]), g = \log([\mu])$ can cover it. The
proof of (VIII.6.2.35) shows that every prime ideals of $R_f$ is maximal principal, in particular f.g, 
thus by (I.5.1.41), it is Noetherian. And it has Krull dimension 1 and it is regular because all of its
maximal ideals are principal, hence normal (I.6.5.25). So $X_{FF}$ is a Dedekind scheme.
\end{proof}

3 Line Bundles and Filtrations

\begin{definition}
(VIII.6.3.1).
By (VIII.6.2.31), the graded algebra $\oplus_{n\geq 0} B^{\varphi=p^n}$ is generated over $Q_p$ by $B^{\varphi=p}$, 
so we can define the Serre twisting sheaf $O(1)$ on $X_{FF}$, which is a line bundle, and on an open affine
scheme $U = X - \{x\}$, where $x$ corresponds to $\log([\varepsilon])$, $O(1)(U) = (B[1/\log([\varepsilon])])^{\varphi=p}$. Similarly we can
define $O(m)$, and $O(m) = O(1)^m$.

\begin{lemma}
(VIII.6.3.2).
There is an isomorphism $\text{Div}(X) \to \text{Pic}(X)$ that maps each $x$ to the inverse
of its ideal sheaf (V.7.1.19) (I.6.5.15). And there is also a degree map $\text{Div}(X) \to \mathbb{Z}$. Then:

\[ \begin{array}{ccc}
\text{Div}(X) & \xrightarrow{\text{deg}} & \mathbb{Z} \\
& \searrow & \downarrow \rho \\
& & \text{Pic}(X)
\end{array} \]

commutes.

\end{lemma}

\begin{proof}
It suffices to show that any $O(x)$ is isomorphic to $O(1)$. As $\log([\varepsilon])$ is a global section of
$O(1)$ that vanishes of order 1 at $x$, it induces an isomorphism $O(1) \cong O(x)$ by (V.7.1.19).
\end{proof}

\begin{lemma}
(VIII.6.3.3) (Cohomology of Line Bundles).
For any integer $m$, $B^{\varphi=p^n} \to H^0(X, O(m))$ is an isomorphism and $H^i(X, O(m)) = 0$ for $i > 0, m > 0$.

\end{lemma}

\begin{proof}
This is trivial using Cech cohomology, as $\oplus_{n\geq 0} B^{\varphi=p^n}$ is PID, so $X$ is separated.
\end{proof}

\begin{proposition}
(VIII.6.3.4).
The construction induces an isomorphism $\rho : \mathbb{Z} \cong \text{Pic}(X) : m \mapsto O(m)$.

\end{proposition}

\begin{proof}
By lemma (VIII.6.3.2), $\rho$ is surjective because $\text{Div}(X) \to \text{Pic}(X)$ does, and it is injection
because if $O(m) \cong O(n)$, then tensoring $O(-m)$, we can assume $O \cong O(-k)$, but they have different
global sections by lemma (VIII.6.3.3) and (VIII.6.2.29) (VIII.6.2.31).
\end{proof}
Harder-Narasimhan Filtration of Vector Bundles

Prop. (VIII.6.3.5) (Harder Narasimhan Formalism for \( Bun_X \)). For a vector bundle \( L \) on \( X \), we can define \( \deg(L) = n \) iff \( L \cong \mathcal{O}(n)(\text{VIII.6.3.2}) \), and for a vector bundle \( E \), define \( \deg(E) = \deg(\wedge(E)) \). And define the generic rank on the category of coherent sheaves on \( X \). Then this is a Harder-Narasimhan formalism on \( C = Bun_X \) with \( A = Vect_k(X) \).

Proof: Only the last axiom needs proof, but if \( E' \subset E \), notice \( \wedge E' \subset \wedge E \) (The stalks are PID), so by taking their top exterior power product, we reduce to the case of line bundles.

But \( \mathcal{O}(m) \) cannot map into \( \mathcal{O}(n) \) if \( m > n \) and must by isomorphism if \( m = n \), by tensoring \( \mathcal{O}(-m) \) and looking at global sections (VIII.6.3.3), so the assertion is true. \( \Box \)

Cor. (VIII.6.3.6). Every vector bundle \( E \) on \( X \) has a unique functorial Harder-Narasimhan filtration, by (II.2.3.22).

4 Base Change of Fields

Prop. (VIII.6.4.1) (Base Change). Let \( C^0 \) be alg.closed. For any finite extension \( E \) of \( \mathbb{Q}_p \) of degree \( n \), \( \text{Spec} E \to \text{Spec} \mathbb{Q}_p \) is finite étale, and finite locally free of degree \( n \), so does \( X_E = X \otimes_{\mathbb{Q}_p} E \to X (\text{V.5.8.10}) \).

In particular, \( X_E \) is also a Dedekind scheme?. For any closed point \( x \) of \( X \) corresponding to an untilt \( K \) of \( C^0 \), which is alg.closed, the fiber of \( X_E \) over \( x \) is identical to the spectrum of \( E \otimes_{\mathbb{Q}_p} K \cong K^n \) as \( K \) is alg.closed.

In this situation and use (VIII.6.2.36), we see that the closed points of \( X_E \) are in bijection with isomorphism classes of \( (K, \iota, u) \) module \( \varphi \)-actions, where \( (K, \iota) \) is an untilt of \( C^0 \), and \( u : E \to K \) is an embedding of \( E \) into \( K \) over \( \mathbb{Q}_p \), isomorphism classes of these triples are denoted by \( Y_E \).

Prop. (VIII.6.4.2). By (VIII.6.4.1) and flat base change (V.6.7.34), we know \( H^0(X_E, \mathcal{O}_{X_E}) = E \), in particular \( X_E \) is connected.

Lemma (VIII.6.4.3). If \( E \) is unramified of degree \( n \) over \( \mathbb{Q}_p \), then \( E \cong W(\mathbb{F}_p^n)[1/p] \). In particular,

\[
\text{Hom}_{\mathbb{Q}_p}(E, K) \cong \text{Hom}_{\mathbb{Z}_p}(W(\mathbb{F}_p^n), \mathcal{O}_K) \cong \text{Hom}_{\mathbb{F}_p}(W(\mathbb{F}_p^n), \mathcal{O}_K/p) \cong \mathcal{O}_{C^0}/[t]) \cong \text{Hom}_{\mathbb{F}_p}(\mathbb{F}_p^n, C)
\]

where the last isomorphism is by Henselian lemma.

Therefore, \( Y_E \cong Y \otimes \text{Hom}_{\mathbb{F}_p}(\mathbb{F}_p^n, C) \), and

\[
\text{Closed points of } Y_E \cong Y_E/\varphi^Z \cong (Y \otimes \text{Hom}_{\mathbb{F}_p}(\mathbb{F}_p^n, C))/\varphi^Z \cong Y/\varphi^nZ.
\]

Prop. (VIII.6.4.4). If \( E \) is unramified of degree \( n \) over \( \mathbb{Q}_p \) and \( U \neq X \) is an affine open defined by a homogenous element \( t \), then

\[
U_E = \text{Spec}((B[t^{-1}] \otimes_{\mathbb{Q}_p} E)^{\varphi = \text{id}}) = \text{Spec}(B[t^{-1}]_{\varphi^n = \text{id}}).
\]

where \( \varphi \) acts trivially on \( E \).

Proof: Each \( u \in \text{Hom}_{\mathbb{F}_p}(\mathbb{F}_p^n, C^0) \) induces a map \( W(\mathbb{F}_p) \to W(\mathcal{O}_{C^0}) = A_{\text{inf}} \to B \), which extends to a map \( \pi : E \to B[t^{-1}] \), and induces a map \( q_u : B[t^{-1}] \otimes_{\mathbb{Q}_p} E \to B[t^{-1}] \). Now

\[
B[t^{-1}] \otimes_{\mathbb{Q}_p} E \to \prod_{\text{Hom}_{\mathbb{F}_p}(\mathbb{F}_p^n, C)} B[t^{-1}]
\]
is an isomorphism, which is just because $x^{p^n} - x$ splits in $B[t^{-1}]$.

And under this isomorphism, the action of $\varphi$ is

$$\varphi((f_0, \ldots, f_{n-1}) = (\varphi(f_{n-1}), \varphi(f_0), \ldots, \varphi(f_{n-2})),$$

so proposition is clear. \qed

**Cor. (VIII.6.4.5).** Fix now a finite extension $E/Q_p$ with uniformizer $\pi$ that has ramification degree $e$ and inertia degree $d$, and $E_0$ is the maximal unramified subextension, then there are maps $E_0 \to B$ by(VIII.6.4.4), fix forever one of them $p_u$, this induces a map

$$B[\frac{1}{t}] \otimes_{Q_p} E \to B[\frac{1}{t}] \otimes_{E_0} E$$

and this induces an isomorphism

$$(B[\frac{1}{t}] \otimes_{Q_p} E)^{\varphi=id} = (B[\frac{1}{t}] \otimes_{Q_p} E_0)^{\varphi=id} \otimes_{E_0} E = (B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi^d=id}$$

**Def. (VIII.6.4.6) ($Y_E^0$).** Define $Y_E^0 \subset Y_E$ is triples $(K, \iota, u)$, where $(K, \iota)$ is an untilt of $C^0$, and $u : E \to K$ is an embedding that $u|E_0$ is identical to $e_K \circ p_u : E_0 \to B \to K$. Notice $Y_E^0$ is not stable under the Frobenius, but it is stable under $\varphi^d$, and induces an isomorphism

$$Y_E^0/\varphi^d \pi \cong Y_E/\varphi^d \pi.$$  

**Prop. (VIII.6.4.7).** Notice for an element $y$ of $Y_E^0$, the map $u : E \to K_y = B_{dR}^+(y)/\xi$ extends uniquely to a map $E \to B_{dR}^+(y)$ that is compatible with $e_K \circ p_u : E_0 \to B_{dR}^+(y)$, because $E$ is separable over $E_0$. i.e.

$$E \xrightarrow{\pi} B_{dR}^+(y) \xrightarrow{\iota} K_y \xrightarrow{u} K$$

Then this defines a map $B \otimes_{E_0} E \to B_{dR}^+(y)$, also called the stalk map.

**Prop. (VIII.6.4.8).** For any finite extension $E/Q_p$, the degree map $\deg : \text{Pic}(X_E) \cong \mathbb{Z}$ is an isomorphism.

**Proof:** It suffices to show that $\mathcal{O}_{X_E}(x) \cong \mathcal{O}_{X_E}(x')$ for each pair of closed points $x, x'$ of $X_E$.

We attempt to construct a line bundle $\mathcal{O}_{X_E}(1)$ on $X_E$ that $\mathcal{O}_{X_E}(1)(U_E) = (B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi^d=\pi}$, because $\mathcal{O}_{X_E}(U_E) = (B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi^d=1}$.

We show simultaneously that $\mathcal{O}_{X_E}(1)$ is a line bundle and it is isomorphic to $\mathcal{O}_{X_E}(x)$ for any closed point $x \in X_E$; For any $x \in X_E$ corresponding to a $\varphi^d$-orbit of $Y_E^0$, let $f$ be the element constructed by lemma(VIII.6.4.9) below, we show that for any affine open $U = D(t)$, multiplying by $f : \mathcal{O}_{X_E}(x)(U_E) \to \mathcal{O}_{X_E}(1)(U_E)(\ast)$ is an isomorphism:

Notice $B \otimes_{E_0} E$ is free over $B$, let $N(f) \in B$ be its norm, the norm is local, so for each $y \in Y$, $N(f)_y = \prod \tilde{g}_y$, where $\tilde{y} \in Y_E$ are over $y$, so it vanishes with order 1 in a $\varphi^d$-orbit of $Y$ for $1$ in $\varphi^d$-orbit of $Y$ (order 1 because $f$ only vanishes at $\tilde{y}$ in the orbit corresponding to $x$), and then $N(f) \varphi(N(f)) \cdots \varphi^{d-1}(N(f))$ vanishes at a single $\varphi$-orbit of $Y$ with order 1, thus equals $u \log([\varepsilon])$ for some $\varepsilon \in m_{C^0}$, by(VIII.6.2.13)(VIII.6.2.9). In particular, $y$ divides $\log([\varepsilon])$. 

Now if \( x \notin U_E \), then \( \log([\varepsilon]) \) divides \( t \), so \( f \) divides \( t \), thus \( f \) is invertible in \( B[\frac{1}{t}] \otimes_{E_0} E \), thus \((\star)\) is an isomorphism.

Otherwise if \( x \in U_E \), then choose some \( x' \) not in \( U_E \), then the same argument shows that \( f' \) is invertible in \( B[\frac{1}{t}] \otimes_{E_0} E \), so \( f/f' \in (B[\frac{1}{t}] \otimes_{E_0} E)^{\varphi=id} \) that vanishes with a single zero at \( x \), so multiplying by \( f'/f \) defines an isomorphism \( \mathcal{O}_{X_E}(U_E) \cong \mathcal{O}_{X_E}(x)(U_E) \), so it suffices to show the composition

\[
\mathcal{O}_{X_E}(U_E) \xrightarrow{f'/f} \mathcal{O}_{X_E}(x)(U_E) \xrightarrow{f} \mathcal{O}_{X_E}(1)(U_E)
\]

is an isomorphism, but this reduces to the first case. \( \square \)

**Lemma (VIII.6.4.9) (Uniformizer Existence).** If \( x \) be a closed point of \( X_E \) corresponding to an orbit of \( \varphi \) in \( Y_E \) thus an orbit \( S \) of \( \varphi^d \) in \( Y^0_E \), then there is an element \( f \in (B \otimes_{E_0} E)^{\varphi^d=\pi} \) that \( \text{ord}_\varphi(f) = 1 \) if \( \varphi \in S \), and 0 otherwise.

**Proof:** The map defined \((\text{VIII.6.4.18})\) composed with the Teichmuller section\((\text{VIII.6.4.16})\) in fact has image in \((B \otimes_{E_0} E)^{\varphi^d=\pi}\) because \( [\pi] = \pi t + t^{p^n} = \varphi^n \) on \( \mathcal{O}_{C^\prime} \), and it is an isomorphism of \( \mathcal{O}_E \) modules. Now there are commutative diagrams:

\[
G_{LT}(\mathcal{O}_{C^\prime}) \xrightarrow{\sigma} G_{LT}(A_{\inf} \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E) \xrightarrow{\log_G} B \otimes_{E_0} E
\]

The map \( G_{LT}(\mathcal{O}_{C^\prime}) \rightarrow G_{LT}(\mathcal{O}_K) \) has kernel \( \mathcal{O}_Eu \) for some \( u \), thus we can let \( f = \log_G(\sigma(u)) \), then the image of \( f \in K \) is 0, which means \( f \) has a zero at the point \( y \in Y^0_E \). And\((\text{VIII.6.4.13})\) shows the the zeros of \( f \) is just the \( \varphi^d \)-orbit containing \( y \). \( \square \)

**Lubin-Tate Formal Groups and the Proof of the Lemma of Uniformizer**

**Prop. (VIII.6.4.10).** The ring \( B \otimes_{E_0} E \) is an integral domain.

**Proof:** Cf.[Lurie P95]. In fact this is the ramified Witt vector, which is by the same reason as before an integral domain, Cf.[FF Curve Johannes]. \( \square \)

**Cor. (VIII.6.4.11).** If \( f \neq 0 \in B \otimes_{E_0} E \), then \( N_{E/E_0}(f) \neq 0 \in B \), in particular, the vanishing locus of \( f \) is finite.

**Cor. (VIII.6.4.12).** If \( f, g \in B \otimes_{E_0} E \), then \( f \) is divisible by \( g \) iff for each \( \overline{y} \in Y^0_E \), \( \text{ord}_\varphi(f) \geq \text{ord}_\varphi(g) \).

**Proof:** If \( \text{ord}_\varphi(f) \geq \text{ord}_\varphi(g) \), suppose \( N_{E/E_0}(g) = gh \), then multiplying by \( h \), we can assume \( g \in B \). Now \( f \) is written uniquely as \( f_0 + f_1 \pi + \ldots + f_{e-1} \pi^{e-1} \) where \( f_k \in B \), thus it suffices to show \( f_i \) is divisible by \( g \), which is equivalent to \( \text{ord}_\varphi(f) \geq \text{ord}_\varphi(g) \) for each \( y \in Y \), by\((\text{VIII.6.2.11})\).

Now if \( \text{ord}_\varphi(g) = n \), the hypothesis shows \( f \) vanishes in

\[
\prod_{\overline{y} \rightarrow y} B_{dR}^+(y)/\xi^n = (B_{dR}^+(y)/\xi^n) \otimes_{E_0} E = B_{dR}^+(y)/\xi^n + \pi B_{dR}^+(y)/\xi^n + \ldots + \pi^{e-1} B_{dR}^+(y)/\xi^n
\]

thus \( \text{ord}_\varphi(f) \geq n = \text{ord}_\varphi(g) \). \( \square \)
Cor. (VIII.6.4.13). If \( f \in (B \otimes_{E_0} E)^{\varphi^n=\pi} \), then the vanishing locus of \( f \) is a single \( \varphi^d \)-orbit, and all zeros are simple.

**Proof:** Set \( N_{E/E_0}(f) = f' \) and \( N_{E/E_0}(\pi) = \pi' \), then \( f \) belongs to \( B^{\varphi^d=\pi'} \), and its divisor is just the image of divisor of \( f \) in \( Y_E^0 \). So it suffices to show that \( f' \) vanishes on a single \( \varphi^{dZ} \)-orbit.

Now for \( 0 < \rho < 1 \),
\[
\rho^{p^d}|f'|_{\rho^d} = |\pi'f'|_{\rho^d} = |f'^{\varphi^d}|_{\rho^d} = |f'|_{\rho^d},
\]
thus
\[
p^d \cdot s + v_{\rho^d}(f') = p^d v_s(f')
\]
for each \( s > 0 \), differentiating, we get
\[
1 + \partial_- v_s(f') = \partial_- v_s(f')
\]
Now the divisor of \( f' \) is \( \varphi^{dZ} \)-invariant, and it has exactly one zero on any annulus \((\rho^n, \rho)\) (VIII.6.2.15), thus its divisor is a single \( \varphi^d \)-orbit.

Def. (VIII.6.4.14) (Universal Lubin-Tate Formal Group). Recall that if \( E \) is a finite extension of \( \mathcal{O}_L \) with uniformizer \( \pi \), for a \( \mathcal{O}_E \)-algebra \( A \) complete w.r.t \( \pi \), \( G_{LT}(A) \) is the Lubin-Tate formal group, with elements the topological nilpotent elements of \( A \).

Now we define the **universal cover of Lubin-Tate formal group** \( \tilde{G}_{LT} \) as the functor
\[
A \mapsto \lim\{\cdots \xrightarrow{[\pi]} G_{LT}(A) \xrightarrow{[\pi]} G_{LT}(A)\}.
\]

Prop. (VIII.6.4.15).

- Notice for \( K \) an alg.closed extension of \( E \), \( G_{LT}(\mathcal{O}_K) \) is in bijection with \( m_K \), and the kernel of \( [\pi^n] \) on \( G_{LT}(\mathcal{O}_K) \) has order \( \mathcal{O}_E/\pi^n \), thus the kernel of \( \tilde{G}_{LT}(\mathcal{O}_K) \to G_{LT}(\mathcal{O}_K) \) is a 1-dimensional \( \mathcal{O}_E \)-module.

- If \( \pi \) vanishes on \( A \) and \( A \) is perfect, then \( [\pi] = \pi t + t^q = t^q \) on \( A \), so it is just the Frobenius, and \( \tilde{G}_{LT}(A) \to G_{LT}(A) \) is a bijection.

- \( \tilde{G}_{LT}(A) \to \tilde{G}_{LT}(A/I) \) is an isomorphism for \( \pi \in I \) and \( A \) is \( I \)-adic.

- \( \tilde{G}_{LT}(A) \to \tilde{G}_{LT}(A/I) \) is an isomorphism for any ideal \( I \) that \( I + (\pi) \neq (1) \), because both of them is isomorphic to \( \tilde{G}_{LT}(A/(I + (\pi)) \).

**Proof:** For 3, it suffices to prove that \( \tilde{G}_{LT}(A/I^{n+1}) \to \tilde{G}_{LT}(A/I^n) \) for \( n \geq 1 \). Notice \( F(u, v) \equiv u + v \mod I^{2n} \), so we have an exact sequence
\[
0 \to I^n/I^{n+1} \to G_{LT}(A/I^{n+1}) \to G_{LT}(A/I^n) \to 0.
\]

In particular the kernel is annihilated by \( \pi \), so there is a commutative diagram
\[
\cdots \xrightarrow{\pi} G_{LT}(A/I^{n+1}) \xrightarrow{\pi} G_{LT}(A/I^{n+1}) \quad \xrightarrow{\pi} G_{LT}(A/I^{n+1}) \xrightarrow{\pi} G_{LT}(A/I^{n+1})
\]
which show that \( \tilde{G}_{LT}(A/I^{n+1}) \cong G_{LT}(A/I^n) \).
Cor. (VIII.6.4.16) (Teichmuller Section). Consider the \( \mathcal{O}_E \)-algebra \( A_{\infty} \otimes \mathcal{O}_{E_0} \mathcal{O}_E \). Because there are isomorphism \( \mathcal{O}_{E_0}/p \cong \mathcal{O}_E/\pi \), we have an isomorphism

\[ C^0 \cong A_{\infty}/p \cong (A_{\infty} \otimes \mathcal{O}_{E_0} \mathcal{O}_E)/\pi \]

Now (VIII.6.4.15) shows the diagram

\[
\begin{array}{ccc}
\mathcal{G}_{LT}(A_{\infty} \otimes \mathcal{O}_{E_0} \mathcal{O}_E) & \cong & \mathcal{G}_{LT}(\mathcal{O}_{C^0}) \\
\downarrow & & \downarrow \\
\mathcal{G}_{LT}(A_{\infty} \otimes \mathcal{O}_{E_0} \mathcal{O}_E) & \rightarrow & \mathcal{G}_{LT}(\mathcal{O}_{C^0})
\end{array}
\]

So the lower horizontal map is surjective, and it even has a canonical section \( \sigma \), called the Teichmuller section.

Cor. (VIII.6.4.17). Given a point of \( Y_0^E \) which corresponds to an untilt of \( C^0 \) together with a \( E_0 \)-map \( E \rightarrow K \), then this gives a commutative diagram

\[
\begin{array}{ccc}
\mathcal{G}_{LT}(A_{\infty} \otimes \mathcal{O}_{E_0} \mathcal{O}_E) & \cong & \mathcal{G}_{LT}(\mathcal{O}_K) \\
\downarrow & & \downarrow \\
\mathcal{G}_{LT}(A_{\infty} \otimes \mathcal{O}_{E_0} \mathcal{O}_E) & \rightarrow & \mathcal{G}_{LT}(\mathcal{O}_K)
\end{array}
\]

where the right vertical arrow is surjective with kernel free of rank 1 over \( \mathcal{O}_E \). So this together with (VIII.6.4.16) shows there is a surjection \( \mathcal{G}_{LT}(\mathcal{O}_{C^0}) \rightarrow \mathcal{G}_{LT}(\mathcal{O}_K) \) with kernel a rank-1 \( \mathcal{O}_E \)-module.

Prop. (VIII.6.4.18). There is a canonical \( \mathcal{O}_E \)-module map

\[ \mathcal{G}_{LT}(A_{\infty} \otimes \mathcal{O}_{E_0} \mathcal{O}_E) \xrightarrow{\log_{\mathcal{G}}} B \otimes_{E_0} E. \]

and it is equivariant w.r.t \( \varphi \).

Proof: \( \mathcal{G}_{LT}(A_{\infty} \otimes \mathcal{O}_{E_0} \mathcal{O}_E) \) are in bijection with the maximal ideal of \( A_{\infty} \otimes \mathcal{O}_{E_0} \mathcal{O}_E \), and \( \log_{\mathcal{G}}(x) \) is of the form \( x + \frac{c_n}{n} x^n + \frac{c_{n-1}}{n} x^{n-1} + \ldots \), with \( c_n \in \mathcal{O}_E \).

Now for \( x \in \mathcal{G}_{LT}(A_{\infty} \otimes \mathcal{O}_{E_0} \mathcal{O}_E) \), we show that \( \log_{\mathcal{G}}(x) \) converges in \( B \otimes_{E_0} E = B + \pi B + \ldots + \pi^{e-1} B \). Let \( c_n x^n = \sum a_{n,i} \pi^i \), then we need to show \( a_{n,i}/n \) converges to 0 for each of the norm \( | \cdot |_\rho \). And this is because if \( x = x_0 + \pi y_0 \), then for \( n \geq em \), \( |x|_\rho \leq \max\{|x_0|_\rho, \rho^{em}, \rho^n \} \), which decays exponential in \( n \), and \( |\frac{1}{n}|_\rho \) decays linearly in \( n \).

Prop. (VIII.6.4.19). The map \( \log_{\mathcal{G}}(\sigma(\cdot)) : \mathcal{G}_{LT}(\mathcal{O}_{C^0}) \rightarrow (B \otimes_{E_0} E)^{\varphi^n=\pi} \) as in (VIII.6.4.9) is an isomorphism.

Proof: For surjectivity, as any \( f \in (B \otimes_{E_0} E)^{\varphi^n=\pi} \) vanishes at a single \( \varphi^\mathbb{Z} \)-orbit, then by (VIII.6.4.9) we can find a \( \log_{\mathcal{G}}(u) \) that vanishes at the same locus, so \( f = \log(u) \lambda \) where \( \lambda \) is a unit in \( B \otimes_{E_0} E (\text{VIII.6.4.12}) \), so

\[ \lambda \in (B \otimes_{E_0} E)^{\varphi^n=\text{id}} = (B \otimes_{\mathcal{Q}_p} E)^{\varphi=\text{id}} (\text{VIII.6.4.5}) = B^{\varphi=\text{id}} \otimes_{\mathcal{Q}_p} E = E. \]

For injectivity, we proved in (VIII.6.4.9) that each \( \log_{\mathcal{G}}(\sigma(u)) \) only vanishes at a single \( \varphi^\mathbb{Z} \)-orbit in \( Y_0^E \), so it cannot by 0, which vanishes at all points.

Cor. (VIII.6.4.20). There are canonical bijections

\[ \{ \text{Closed Points of } X_E \} \cong \{ \varphi^\mathbb{Z} \text{orbits of } Y_0^E \} \cong ((B \otimes_{E_0} E)^{\varphi^n=\pi} - \{0\})/E^* \cong (\mathcal{G}_{LT}(\mathcal{O}_{C^0}) - \{0\})/E^* \]

by (VIII.6.4.9)(VIII.6.4.19),(VIII.6.4.12).


**Vector Bundles and Base Change**

**Prop. (VIII.6.4.21) (Vector Bundles on the Cover).** Let \( \pi : X_E \to X \) be the covering map, for any vector bundle \( \mathcal{E} \) on \( X_E \), \( \pi_* (\mathcal{E}) \) is a vector bundle on \( X \), and this induces an isomorphism

\[
\{ X_E \text{- Bundles} \} \cong \{ X \text{- Bundles with an } E \text{- action} \}.
\]

Now define \( \deg(\mathcal{E}) = \deg(\pi_* \mathcal{E}) \), and slope(\( \mathcal{E} \)) = \( \frac{\deg(\mathcal{E})}{\text{rank}(\mathcal{E})} = \frac{1}{n} \text{slope}(\pi_* \mathcal{E}). \)

Then \( \mathcal{E} \) is semistable of slope \( \lambda \) iff \( \pi_* \mathcal{E} \) is semistable of slope \( \lambda / n \).

**Proof:** One direction is clear, for the other, if \( \mathcal{F} = \pi_* \mathcal{E} \) is not semistable, choose its HN-filtration, then \( \lambda_1 > \lambda / n \). Now the action of \( E \) on \( \mathcal{F} \) preserves the HN-filtration, thus \( \mathcal{F}_1 \) is an \( E \)-vector bundle, thus by the correspondence above, \( \mathcal{F}_1 = \pi_* \mathcal{E}' \) for some subbundle \( \mathcal{E}' \subseteq \mathcal{E} \), and clearly this contradicts the semistability of \( \mathcal{E} \).

**Cor. (VIII.6.4.22).** For any integral number \( d, n \) with \( n > 0 \), there exists a semistable vector bundle on \( X \) with rank \( n \) and degree \( d \).

**Proof:** Let \( E \) be an extension of \( \mathbb{Q}_p \) of degree \( n \), then \( \pi_* (\mathcal{O}_{X_E}(d)) \) is semistable of rank \( n \) and degree \( d \), by (VIII.6.4.1)(V.5.8.9), because \( \mathcal{O}_{X_E}(d) \) is a line bundle (VIII.6.4.8) so clearly semistable, and it is of degree \( d \) because \(?\).

**Isocrystals and Classification of Semistable Vector Bundle over \( X \)**

**Remark (VIII.6.4.23).** Recall the Dieudonné-Manin Classification (VIII.1.2.7)(VIII.1.2.10): Any isocrystal over \( k \) is a finite sum of modules pure of slopes \( \lambda_i \). And if \( k \) is alg. closed, then any isocrystal over \( k \) has a unique decomposition as sums of \( E_{\lambda_i} \).

**Prop. (VIII.6.4.24).** Let \( k = \mathbb{F}_p \subset \mathbb{C} \), then there is an inclusion \( \mathcal{W}(k) \to A_{inf} \), which extends to a map \( K \to B \). Now given an isocrystal \( V \) over \( k \), denote \( \mathcal{E}_V \) the coherent sheaf on \( X \) defined by the graded module \( \oplus_{n \geq 0} \text{Hom}_K(V, B)^{\mathbb{Q}_p = \mathbb{P}^0} \). In other words, on an affine open subscheme \( U = D(t) \), \( \mathcal{E}_V(U) = \{ \varphi - \text{equivariant } K \text{- linear maps } V \to B[\frac{1}{t}] \} \).

And when \( V = E_{m/n} \) is the simple isocrystal, then \( \mathcal{E}_V \) is denoted by \( \mathcal{O}(\frac{m}{n}) \).

**Prop. (VIII.6.4.25).** In fact we have \( \mathcal{O}(\frac{m}{n})(U) \cong (B[t^{-1}])^{\mathbb{Q}_p = \mathbb{P}^0} = (\rho_* \mathcal{O}(m))(U) \), where \( \rho : X_E \to X \), and \( E \) is an unramified extension of \( \mathbb{Q}_p \).

**Prop. (VIII.6.4.26) (Classification of Semistable Vector Bundles over \( X \)).** For every vector bundle on \( X \), the HN-filtration splits non-canonically, and the construction \( V \to \mathcal{E}_V \) induces an equivalence of categories between

\[
\{ \text{Isoclinic Isocrystals of slope } \mu \}^{op} \to \{ \text{Semistable vector bundles on } X \text{ of slope } \mu \}
\]

**Proof:** Cf.[FF Curve Johannes].

**Cor. (VIII.6.4.27).** Any two semistable vector bundles of slope \( \lambda \) over \( X \) is isomorphic, and a semistable vector bundle of slope \( 0 \) is trivial.

**Prop. (VIII.6.4.28).** If \( \mathcal{E}, \mathcal{E}' \) be semistable vector bundles on \( X \) of slopes \( \mu, \mu' \), then \( \mathcal{E} \otimes \mathcal{E}' \) is semistable of slope \( \mu + \mu' \).

**Proof:** We can assume \( \mathcal{E} = \rho_* \mathcal{O}_{X_E}(d) \) for an unramified extension \( E/\mathbb{Q}_p \) by (VIII.6.4.27), and then \( \mathcal{E} \otimes \mathcal{E}' = \rho_* (\mathcal{O}_X(d) \otimes \rho^* \mathcal{E}') \). Since \( \rho_* \rho^* \) preserves semistability(by (VIII.6.4.21) and ?). So it suffices to prove \( \mathcal{O}(d) \otimes - \) preserves semistablity, but this is clear, as \( \mathcal{O}(d) \) shifts degree.
Diamonds Definitions

Def. (VIII.6.4.29) (Diamond). Let Perf denote the site of perfectoid spaces of characteristic $p$ equipped with the pro-étale topology. A diamond $X$ is a sheaf (of sets) on Perf of the form $X = \text{Hom}_{\text{Perf}}(Z)/R$, where $Z \in \text{Perf}$ and $R \in Z \times Z$ is a reasonable representable equivalence relation.

Prop. (VIII.6.4.30) (Scholze). Let $R = (R, R^+)$ be a Huber pair, then
$$\text{Spd}(R) = Z \mapsto \{\text{untilts of}\text{Spa}(R)\text{ over }Z\}$$
is a diamond.

And this construction can be glued to give diamond $X^\diamond$ of any adic space $X$, which is a sheaf.

Def. (VIII.6.4.31) (Adic Fargues-Fontaine Curve). Let $Y$ be the adic space $\text{Spa}(A_{\text{inf}})$ removing the vanishing locus of $p$ and $[t]$, then by what we proved, the Frobenius act totally discontinuous on $Y$, thus the quotient $X^{\text{FF}}$ is an adic space, the FF-curve.

Prop. (VIII.6.4.32). There is an isomorphism of diamonds:
$$Y^\diamond \cong \text{Spd}(C^\diamond) \times \text{Spd}(\mathbb{Q}_p), \quad X^{\text{FF}, \diamond} \cong \text{Spd}(C^\diamond)/\varphi^Z \times \text{Spd}(\mathbb{Q}_p)$$

More generally, over any Huber pair $R$, there is a relative FF-curve which is defined by
$$\text{Spd}(R) \times \text{Spd}(\mathbb{Q}_p)/\varphi^Z$$

Proof: $C^\diamond$ is a perfectoid of char $p$, so for a perfect Huber pair, $S$ point of $C^\diamond$ is just a morphism $u : (C^\diamond, O_{C^\diamond}) \to (S, S^+)$. And a $\mathbb{Q}_p$ point is just a char0 untilts $T$ of $S$.

So for each pairs $(T, u)$, we need to find a morphism $A_{\text{inf}}[\frac{1}{p}, \frac{1}{[t]}] \to T$. For this, consider
$$A_{\text{inf}} = W(O_{C^\diamond}) \to W(S^+) \cong W(T^0) \xrightarrow{\varphi_T} T$$

This is a bijection, as we proved in the beginning (VIII.6.1.2).

Prop. (VIII.6.4.33). There is a morphism of ringed spaces $X^{\text{FF}} \to X^{\text{FF}}$ that regard $X^{\text{FF}}$ as the rigid analytification of $X$, so they have the same category of vector bundles and cohomology, prove by Kedlaya-Liu.

5 Applications

Prop. (VIII.6.5.1). The FF curve $X$ is geometrically simply connected, i.e. the projection defines an isomorphism of étale groups $\pi_1(X) \to \pi_1(\text{Spec }\mathbb{Q}_p) = \text{Gal}(\mathbb{Q}_p)$.

Equivalently, the pullback defines an equivalence of étale sites.

Proof: Let $\tilde{X} \to X$ be an finite étale morphism, we want to prove that $\tilde{X} = X \otimes_{\text{Spec }\mathbb{Q}_p} \text{Spec}(E)$ for some étale $\mathbb{Q}_p$-algebra. Let $A = p_*O_{\tilde{X}}$, and $E = H^0(X, A) = H^0(\tilde{X}, O_{\tilde{X}})$. Now it suffices to show $A = E \otimes_{\mathbb{Q}_p} O_X$, which shows $\tilde{X} = X \otimes_{\text{Spec }\mathbb{Q}_p} \text{Spec}(E)$, and forces $E$ be an étale $\mathbb{Q}_p$-algebra by fpqc descent. Equivalently, $A$ is trivial, and this is equivalent to $A$ being semistable of slope 0 by (VIII.6.4.27).
Because $\rho$ is finite étale, the trace pairing $A \times A \to A \xrightarrow{\tau} O_X$ is non-degenerate (check on stalks), which induces an isomorphism $A \cong A^\vee$, so $\deg(A) = 0$, and if $A$ is not semistable, let $A'$ be the first term of the $HN$-filtration of $A$, then it is of slope $\lambda > 0$, so $A' \otimes A'$ is of slope $2\lambda$ by (VIII.6.4.28), so the composite $A' \otimes A' \to A \otimes A \to A$ must by 0(II.2.3.32), which is impossible, because if $U$ is an affine open that $A$ has a section, then this says $U \otimes_X \bar{X}$ has a section $s$ that $s^2 = 0$. But $U \otimes_X \bar{X}$ is reduced (check on stalks).

**Cor. (VIII.6.5.2).**
- The projection map induces equivalence of categories between Finite Abelian groups with $Gal(\mathbb{Q}_p)$-action and étale Local system on $X$.
- If $M$ is a finite Abelian group with a $Gal(\mathbb{Q}_p)$-action, then
  
  $$H^*(Gal(\mathbb{Q}_p), M) \to H^*_c(X, u^*M)$$

  is an isomorphism for $* = 0, 1$.

**Proof:** 1 is trivial, and 2 Cf. [Lurie P102].

6 Weakly Admissible $\Rightarrow$ Admissible

**Def. (VIII.6.6.1) (Notations).** Let $K$ be a finite extension of $\mathbb{Q}_p$, and $K_0 = W(k)[\frac{1}{p}]$ be the maximal unramified subextension in $K$, let $C = \overline{K}$ and $F = C^\phi$. Denote by $\infty \in X$ the closed point determined by $C$, which is just the vanishing locus of the Galois stable line $\mathbb{Q}_pt$, where $t = \log([\varepsilon])$ and $\varepsilon = (1, \xi, \xi^2, \ldots) \in C^p(VIII.6.2.32)$.

Notice $G_K$ acts on $\mathbb{Q}_p \log([\varepsilon])$ by the cyclotomic character $\chi_{cycl}$. Recall

$$B^+_{dR} = \hat{\Omega}_X(\infty), \ B^+_{crys} = B^+_{crys}[t^{-1}], \ B_c = H^0(X - \{\infty\}, O_X) = (B_{crys})^{\varphi=\text{id}}$$

**Def. (VIII.6.6.2) (Equivariant Action of $G_K$ on Vector Bundles).** Recall an equivariant action of $G_K$ on a bundle $E$ on $X$ is a data of isomorphisms $\sigma^*(E) \cong E$ that $c_{\sigma t} = c_t \circ \tau^*(c_{\sigma})$. Notice any equivariant action of $G_K$ on $E$ induces a semilinear $G_K$ action on $E^\infty = E \otimes_{O_X} B^+_{dR}$, and here we require this action is continuous. The category of equivariant $G_K$-bundles are denoted by $\text{Bun}^{G_K}_X$.

**Cor. (VIII.6.6.3).** By the slope 0 case of the classification of vector bundles on $X$(VIII.6.4.26) and(VIII.1.5.1), we see that the functor:

$$\text{Rep}_{\mathbb{Q}_p}G_K \to \text{Bun}^{G_K}_X : V \mapsto V \otimes_{\mathbb{Q}_p} O_X$$

is fully faithful with essential image the category $\text{Bun}^{G_K, \text{sst}}_X$ of all $G_K$ vector bundles on $X$ that the underlying bundle is semistable of slope 0, i.e. trivial(VIII.6.4.27).

**Prop. (VIII.6.6.4).** There is a pullback diagram of categories:

$$\varphi: FilM_{\text{ad}} \quad \varphi: \text{Mod}_{K_0}$$

$$\begin{array}{ccc}
\text{Bun}^{G_K}_X & \xrightarrow{\varepsilon(-)} & \text{Rep}_{B_c}G_K \\
\downarrow & & \downarrow \\
\varphi \end{array}$$

Where $\varepsilon(-)$ maps a $\varphi$-filtered module $(D, \varphi_D, Fil)$ to the bundle that is the bundle $(\tilde{D}, \varphi_D)$ modified so that the fiber at $\infty$ is $Fil^0(D_K \otimes_K B_{dR})$. 

Proof: 1: By lemma(VIII.1.5.7), $\varphi - FilMod_{K/K_0}$ is equivalent to a $\varphi$-module $V$ with a $G_K$-stable $B_{dR}^+$-lattice in $(V \otimes_{K_0} K) \otimes_K B_{dR}^+ = V \otimes_{K_0} B_{dR}^+$.

2: By (VIII.1.5.5), $\varphi$-Mod is a full subcategory of $\operatorname{Rep}_{B_e} G_K$, where the $G_K$-stable $B_{dR}^+$-lattice is choose to be $V \otimes_{K_0} B_{dR}^+$.

3: Clearly there is a functor

$$Bun_{X}^{G_K} \rightarrow \operatorname{Rep}_{B_e} G_K : \mathcal{E} \rightarrow H^0(X - \{\infty\}, \mathcal{E}).$$

and (V.4.2.9) says in this case $Bun_{X}^{G_K}$ is equivalent to a $B_e$-module with continuous $G_K$-actions and an a $B_{dR}^+$-module with continuous $G_K$-actions that they corresponds as a $B_{dR}$-module with continuous $G_K$-actions.

4: The compatibility in 3 just says that the $B_{dR}^+$-lattice choosen in the definition of (VIII.1.5.4) just comes from that of 2, so this diagram is clearly a pullback.

□

Lemma (VIII.6.6.5). Let $Fil V \in VectFil_K$ and $W = Fil(V \otimes_K B_{dR})$, if $V \otimes B_{dR} = \{e_1, \ldots, e_n\}$ and $Fil^0(V \otimes_K B_{dR}) = \langle t^{-a_1}e_1, \ldots, t^{-a_n}e_n \rangle$, then the Hodge polygon of $Fil V$ has slopes $(a_1, \ldots, a_n)$.

Proof: Use (VIII.1.5.7), notice $(t^a B_{dR})^{G_K} = 0$ for $a > 0$, as in the proof of (VIII.1.5.2).

□

Lemma (VIII.6.6.6). The functor

$$\mathcal{E}(-) : \varphi - FilMod_{K/K_0} \rightarrow Bun_{X}^{G_K}$$

defined in (VIII.6.6.4) preserves degree and HN-filtration, where the HN-filtration on the RHS is induces by the HN-filtration on $Bun_X$ by canonicity.

Proof: $\deg(\mathcal{E}((D, \varphi_D, Fil)) = \deg(\mathcal{E}(D, \varphi_D)) - \dim_K[D \otimes_{K_0} B_{dR}^+ : Fil^0(D_K \otimes_K B_{dR})]$ $= \deg(D_K, Fil) - \deg(D, \varphi_D)$.

Now the degree correspond, for the invariance of HN-filtration, it sufficient to prove the subobjects are in bijection: Given a subobjects of $\mathcal{E}(V)$, we want to show it is a $\mathcal{E}(V')$, but this is because on the affine open $\operatorname{Spec}(B_e)$, by (VIII.1.5.5) any subbundle is also crysatalline, i.e. comes from $\varphi - Mod_{K_0}$.

□

Prop. (VIII.6.6.7) (Weakly Admissible implies Admissible). The category of crystalline Galois representations of $G_K$ is equivalent to the category $\varphi - FilMod_{K/K_0}^{we}$ of weakly admissible filtered $\varphi$-modules for $K$.

Proof: By definition of weakly admissible and (VIII.6.6.6), there is a pullback diagram

$$\begin{array}{ccc}
\varphi - FilMod_{K/K_0}^{we} & \rightarrow & \varphi - FilMod_{K/K_0} \\
\downarrow & & \downarrow \mathcal{E}(-) \\
\operatorname{Rep}_{Q_p} G_K \cong Bun_{X}^{G_K, sst, 0} & \rightarrow & Bun_{X}^{G_K} \\
\end{array}$$

Adjunction with (VIII.6.6.4), we get another pullback diagram

$$\begin{array}{ccc}
\varphi - FilMod_{K/K_0}^{we} & \rightarrow & \varphi - FilMod_{K/K_0} \\
\downarrow & & \downarrow \mathcal{E}(-) \\
\operatorname{Rep}_{Q_p} G_K \cong Bun_{X}^{G_K, sst, 0} & \rightarrow & Bun_{X}^{G_K} & \rightarrow & \operatorname{Rep}_{B_e} G_K \\
\end{array}$$
But this pullback is just the category of crystalline representations: by (VIII.6.6.3), for $V \in \text{Rep}_{\mathbb{Q}_p} G_K$, the condition $V(D)(M) \cong M$ in (VIII.1.5.5) is just saying that $V$ is in the image of $\mathcal{V}$ iff

$$(V \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{\varphi = \text{id}} \otimes B_{\text{crys}} \cong V \otimes_{\mathbb{Q}_p} B_{\text{crys}}$$

which is equivalent to $V$ being crystalline. \hfill $\square$
VIII.7 Pro-Étale Sites on Perfectoids and Diamonds

Main references are [Sch17].

1 Properties of Perfectoid Spaces

TotallyDisconnectedSpaces

Def. (VIII.7.1.1) (TotallyDisconnectedSpaces). A perfectoid space $X$ is called totally disconnected if it is qcqs and any open covering $\{U_i \to X\}$ splits, i.e. $\bigsqcup U_i \to X$ splits, or equivalently, there is a refinement covering $\{V_i \to X\}$ that $X \cong \bigsqcup V_i$.

A perfectoid space $X$ is called strictly totally disconnected if it is qcqs and every étale cover splits.

Prop. (VIII.7.1.2). Let $X$ be a qcqs perfectoid space, then $X$ is totally disconnected iff all its connected components are of the form $\Spa(K, K^+)$ where $(K, K^+)$ are perfectoid affinoid fields. And it is strictly totally disconnected if moreover $K$ are all alg.closed.

Proof: Cf.[Sch17].P29, P35.

Prop. (VIII.7.1.3). if $X$ is a totally disconnected perfectoid space, then $X$ is affinoid.

Proof: Cf.[Sch17].P30.

Injections

Def. (VIII.7.1.4) (Injections). A map $f : X \to Y$ of perfectoid spaces is called an injection if for any perfectoid space $Z$, $f_* : \Hom(Z, X) \to \Hom(Z, Y)$ is an injection.

Prop. (VIII.7.1.5) (ResidueFieldMapisInjection). Let $X$ be a perfectoid space and $x \in X$, giving rise to a map of residue fields

$$i_x : \Spa(\kappa(x), \kappa(x)^+) \to X,$$

then $i_x$ is an injection of perfectoid spaces.

Proof: To show this, firstly we can replace $X$ with an affinoid nbhd of $X$. Then notice that $\Spa(\kappa(x), \kappa(x)^+)$ is the filtered limit over all rational nbhds $U$ of $x$ in $X$, and for each $U, U \to X$ is an injection by definition(VIII.5.3.1), so $i_x$ is also an injection.

Cor. (VIII.7.1.6). In particular, if $X$ is qcqs and has a unique closed point $x \in X$, then $X = \Spa(\kappa(x), \kappa(x)^+)$, as in this case $X$ is the only quasi-compact open subset containing $x$.

Prop. (VIII.7.1.7) (CharacterizationsofInjections). Let $f : Y \to X$ be a map of perfectoid spaces, then the following conditions are equivalent:

- $f$ is an injection.
- For any perfectoid adic field $(K, K^+)$, the map of sets $f_* : Y(K, K^+) \to X(K, K^+)$ is an injection.
- The map $|f| : |Y| \to |X|$ is injective, and for all rank 1 point $y \in Y$ with image $f(y) = x \in X$, the map of completed residue fields $\kappa(x) \to \kappa(y)$ is an isomorphism.
The map $|f| : |Y| \to |X|$ is injective, and $f$ is final in the category of maps $Z \to X$ that $|Z| \to |X|$ factors through the map $|Y| \to |X|$.

In particular, by item 4, an injection of perfectoid spaces is determined by its topological map.

**Proof:** Cf.[Sch17]P21. $4 \to 1 \to 2$ is trivial. $\square$

**Prop. (VIII.7.1.8) (Injection and Base Change).**
- Let $f : Y \to X$ be an injection of perfectoid spaces, and $X' \to X$ any map of perfectoid spaces, then the pullback $f' : Y' = Y \times_X X' \to X'$ is also an injection, and the induced map $|Y'| \to |Y| \times_{|X|} |X'|$

is a homeomorphism.
- A map of perfectoids spaces is an injection iff it is universally injective.

**Proof:** Cf.[Sch17]P24. $\square$

**Immersions**

**Def. (VIII.7.1.9) (Immersions).** A map of perfectoid spaces $f : Y \to X$ is called an immersion if $f$ is an injection and $|f| : |Y| \to |X|$ is a locally closed immersion. If $|f|$ is moreover closed or open, then it is called closed/open immersion.

**Def. (VIII.7.1.10) (Zariski Closed Immersion).** Let $f : Z \to X$ be a map of perfectoid spaces where $X = \text{Spa}(R, R^+)$ is affinoid perfectoid, then
- the map $f$ is called Zariski closed immersion if $f$ is a closed immersion and $|Z| = V(I) \subset |X|$, where $I \subset R$ is an ideal.
- the map $f$ is called strongly Zariski closed immersion if $Z = \text{Spa}(S, S^+)$ is affinoid perfectoid, $R \to S$ is surjective, and $S^+$ is the closure of $R^+$ in $S$.

**Prop. (VIII.7.1.11).**
- If $f$ is strongly Zariski closed, then $f$ is Zariski closed, in particular a closed immersion.
- If $f$ is Zariski closed, then $Z$ is affinoid.
- If $X$ is of characteristic $p$, and $f$ is Zariski closed, then $f$ is strongly Zariski closed.

**Proof:** Cf.[?] Section2.5. $\square$

**Prop. (VIII.7.1.12).** For any map of perfectoid spaces $Y \to X$, the diagonal map $\Delta_f : Y \to Y \times_X Y$ is an immersion.

**Proof:** Clearly $\Delta_f$ is an injection, thus it suffices to show that $|\Delta_f|$ identifies $Y$ with a locally closed subset of $|Y \times_X Y|$. This can be checked locally on the target, so we can assume $X = \text{Spa}(R, R^+)$ and $Y = \text{Spa}(S, S^+)$, then the diagonal map is strongly Zariski closed, as $S \hat{\otimes}_R S \to S$ is surjective and maps the integral closure of $S^+ \hat{\otimes}_R S^+ \to S^+$ onto $S^+$. Thus by(VIII.7.1.11), $\Delta_f$ is a closed immersion in this case. $\square$

**Def. (VIII.7.1.13) (Separated Map).** A map $f : Y \to X$ of perfectoid spaces is called separated if $\Delta_f$ is a closed immersion.
Prop. (VIII.7.1.14) (Valuation Criterion). Let \( f : Y \to X \) be a map of perfectoid spaces. The following are equivalent:

- \( f \) is separated.
- \( |\Delta_f|: |Y| \to |Y \times_X Y| \) is a closed immersion.
- \( |f| \) is quasi-separated, and for any perfectoid adic field \((K, K^+)\) and any diagram
  \[
  \begin{array}{ccc}
  \text{Spa}(K, \mathcal{O}_K) & \to & Y \\
  \downarrow & & \downarrow f \\
  \text{Spa}(K, K^+) & \to & X
  \end{array}
  \]
  there exists at most one dotted arrow making the diagram commutative.

Proof: The equivalence of 1 and 2 is by (VIII.7.1.12).

2 \( \to \) 3: If \( |\Delta_f| \) is a closed immersion, then it is in particular quasi-compact, thus \( f \) is quasi-separated. Now if there are two dotted-arrow making this diagram commutative, they define a point \( z \in (Y \times_X Y)(K, K^+) \) s.t. \( z|_{\text{Spa}(K, \mathcal{O}_K)} \in \Delta_f(Y)(K, \mathcal{O}_K) \). But \( |\Delta_f|(|Y|) \) is closed in \( |Y \times_X Y| \), so \( z \) maps \( \text{Spa}(K, K^+) \) into \( |\Delta_f|(|Y|) \) if it maps \( \text{Spa}(K, \mathcal{O}_K) \) into \( |\Delta_f|(|Y|) \), as \( \text{Spa}(K, \mathcal{O}_K) \subset \text{Spa}(K, K^+) \) is dense. Now \( \Delta_f \) is an injection, so by (VIII.7.1.7), \( z \) factors through \( \Delta_f \) thus the two maps are equal.

3 \( \to \) 2: The condition implies that \( |\Delta_f|: |Y| \to |Y \times_X Y| \) is a quasi-compact locally closed immersion of locally spectral spaces which is moreover specializing. But because \( |\Delta_f| \) is quasi-compact, the image of \( |\Delta_f| \) is pro-constructible, and it is also closed under specialization, thus it is closed. Cf.[Sch17]P25.

2 Pro-Étale Site and \( v \)-Site

Prop. (VIII.7.2.1). If \((R, R^+)\) is the completed filtered colimit of perfectoid Huber pairs \((R_i, R_i^+)\), \(X_i = \text{Spa}(R_i, R_i^+)\), \(X = \text{Spa}(R, R^+)\), then base change induce equivalences of categories:

- 2 \( \lim_i X_{i, \text{ét}} \cong X_{f, \text{ét}} \).
- 2 \( \lim_i X_{i, \text{aff}} \cong X_{\text{aff}} \).
- 2 \( \lim_i X_{i, \text{qcqs}} \cong X_{\text{qcqs}} \).
- 2 \( \lim_i X_{i, \text{sep}} \cong X_{\text{sep}} \).

Proof: Cf.[Sch17]P27.

1 follows from almost purity theorem.

Def. (VIII.7.2.2) (Pro-étale Morphism). A map of perfectoid Huber pairs \((A, A^+) \to (B, B^+)\) is called pro-étale iff it is the completed filtered colimit of étale ring maps \((A, A^+) \to (A_i, A_i^+)\).

A morphism of perfectoid spaces is pro-étale if there is an affinoid covering \( V_i = \text{Spa}(R_i, R_i^+) \) of \( X \) that \( f^{-1}(V_i) \) have coverings \( U_{ij} = \text{Spa}(R_{ij}, R_{ij}^+) \) that \((R_i, R_i^+) \to (R_{ij}, R_{ij}^+)\) are all pro-étale.

In fact, by (VIII.7.2.7), if this is true for one affinoid covering \( V_i \) of \( X \), then this is true for any affinoid covering of \( X \).

Prop. (VIII.7.2.3). If \( S \) is a profinite set and \( X \) is a perfectoid space, then we can define a new perfectoid space \( X \times_S \) as the inverse limit of \( X \times S_i \), where \( S = \varprojlim S_i \). Then \( X \times_S \) is pro-étale over \( X \).
Prop. (VIII.7.2.4) (Immersion are Pro-Étale). If $f : Z \hookrightarrow X$ be a Zariski closed immersion with image $V(I)$, then $f$ is affinoid pro-étale. Then $f(Z)$ can be written as the intersection of rational subsets

$$U_{f_1, \ldots, f_n} = \{|f_1|, \ldots, |f_n| \leq |\varpi|\}$$

for various $n$ and $f_1, \ldots, f_n \in I$. Then $Z = \lim_{\leftarrow} U_{f_1, \ldots, f_n} \rightarrow X$, as it is a closed immersion thus an injection (VIII.7.1.11), which shows $Z \rightarrow X$ is pro-étale.

In particular, an immersion is also pro-étale because pro-étale can be checked analytically locally.

Cor. (VIII.7.2.5) (Diagonal Map is Pro-Étale). If $f : Y \rightarrow X$ is a map of perfectoid spaces, then $\Delta_f : Y \rightarrow Y \times_X Y$ is pro-étale, by (VIII.7.1.12).

Prop. (VIII.7.2.6). If $X$ is an affinoid perfectoid space, then the functor

$$\text{Pro}(X_{\text{aff}}^{\text{aff}}) \rightarrow X_{\text{pro}}^{\text{aff}} : \lim_{\leftarrow i} (X_i) \mapsto \lim_{\leftarrow i} X_i$$

is an equivalence of categories.

Proof: This map is essentially surjective by definition, To show it is fully faithful, it suffices to show that for $Y = \lim_{\leftarrow i} \text{Spa}(Y_i, Y_i^+), Z = \lim_{\leftarrow j} \text{Spa}(Z_j, Z_j^+)$,

$$\text{Hom}_X(Y, Z) = \lim_{\leftarrow j} \lim_{\rightarrow i} \text{Hom}(Y_i, Z_j).$$

We need to show for any $Z \rightarrow X$,

$$\text{Hom}_X(Y, Z) = \lim_{\rightarrow i} \text{Hom}(Y_i, Z).$$

Because $2 - \lim_{\rightarrow i} X_i,_{\text{et,qcqs}} \cong X_{\text{et,qcqs}}$ (VIII.7.2.1), we have

$$\text{Hom}_X(Y, Z) = \text{Hom}_Y(Y, Y \times_X Z) = \lim_{\rightarrow i} \text{Hom}_{Y_i}(Y_i, Y_i \times_X Z) = \lim_{\rightarrow i} \text{Hom}_X(Y_i, Z).$$

Prop. (VIII.7.2.7) (Properties of Pro-Étale Morphisms).

- (Affinoid)Pro-Étale maps are stable under composition and pullbacks.
- Let $f : Y \rightarrow X, f' : Y' \rightarrow X$ be (affinoid)pro-étale, then any map $g : Y \rightarrow Y'$ over $X$ is also (affinoid)pro-étale.
- For any affinoid perfectoid space $X$, the category $X^{\text{aff}}_{\text{pro-ét}}$ has all finite limits.

Proof: 1: Composition is obvious. pro-étale maps are stable under pullbacks because étale maps do.

2: We can factor $g$ as a section of the map $Y \times_X Y' \rightarrow Y$ and the projection map $Y \times_X Y' \rightarrow Y'$. Thus it suffices to show a section of a pro-étale map is pro-étale. But if $Y = \lim_{\rightarrow i} Y_i \rightarrow X$ is pro-étale, then a section is given by compatible sections $s_i : X \rightarrow Y_i$. Then $X = \lim_{\rightarrow i} (X \times_Y Y_i) \rightarrow X$ is pro-étale.

3: This is because $X^{\text{aff}}_{\text{ét}}$ has finite limits, because it has a final object and fiber products (I.9.9.17).
Def. (VIII.7.2.8) (Big Pro-Étale Site). Consider the following categories:
- Perf, the category of perfectoid spaces.
- $X^\text{pro-aff}_{\text{ét}}$, the category of perfectoid spaces pro-étale over $X$, where $X$ is a perfectoid space.
- $X^\text{aff}_{\text{pro-aff}}$, the category of affinoid perfectoid spaces pro-étale over $X$.

The big pro-étale site is the category Perf endowed with the topology that a family of maps \( \{f_i : Y_i \to X\} \) is a covering if all $f_i$ are pro-étale and for any quasi-compact open subset $U \subset X$, there is a finite set $J \subset I$ and quasi-compact opens $V_j \subset Y_j$ that $U = \cup_{j \in J} f_j(V_j)$.

Cor. (VIII.7.2.9). The presheaf $O : X \mapsto O_X(X), O^+ : X \mapsto O^+_X(X)$ on the big étale site are sheaves. If $X$ is affinoid perfectoid, then $H^i(X^\text{pro-aff}_{\text{ét}}, O) = 0$ for $i > 0$, and $H^i(X^\text{pro-aff}_{\text{ét}}, O^+)$ is almost zero for $i > 0$. Moreover, the big pro-étale site is subcanonical.

Proof: Firstly we can assume $X$ is affinoid because we already know $O, O^+$ are sheaves w.r.t. the analytic topology. Let $Y \to X$ be an affinoid pro-étale covering of $X$, where $Y = \text{Spa}(R_{\infty}, R^+_{\infty})$, $(R_{\infty}, R^+_{\infty}) = (\lim_{\leftarrow i} (R_i, R^+_i))^\wedge$. Then fixing a pseudo-uniformizer $\varpi$ of $R$, the complexes

\[
0 \to R^+/\varpi \to R^+_j/\varpi \to \ldots
\]

is almost exact, because $H^i(X^\text{ét}, O^+_X/\varpi)$ is almost zero for $i > 0$. Now take a direct limit over $i$, then

\[
0 \to R^+/\varpi \to R^+_{\infty}/\varpi \to \ldots
\]

is almost exact. Now by induction on $n$, we can prove

\[
0 \to R^+/\varpi^n \to R^+_{\infty}/\varpi^n \to \ldots
\]

is almost exact. Then by passing to the direct limit,

\[
0 \to R^+ \to R^+_{\infty} \to \ldots
\]

is almost exact. Then by inverting $\varpi$,

\[
0 \to R \to R_{\infty} \to \ldots
\]

is exact. These give us the desired results. Notice $O^+$ is a sheaf because it is the elements of valuations $\leq 1$ everywhere by (VIII.5.3.7).

For the final assertion, if \( \{Y_i \to Y\} \) is a pro-étale covering of $Y$, and $g_i : Y_i \to X$ are maps that agree on $Y_i \times_X Y_j$, then firstly we can glue these maps together topologically to a map $|Y| \to |X|$. So this problem can be considered locally on $X$, so we may assume $X = \text{Spa}(R, R^+)$ is affinoid, and maps $(R, R^+) \to (O(Y_i), O^+(Y_i))$ that agree on the overlap (VIII.5.4.4), then they glue to a map $(R, R^+) \to (O(Y), O^+(Y))$, as $O, O^+$ are all sheaves, and this gives a morphism $Y \to X$.

Prop. (VIII.7.2.10) (Strictly Totally Disconnected Pro-Étale Cover). Let $X$ be an affinoid perfectoid space, then there is an affinoid perfectoid space $\tilde{X}$ with an affinoid pro-étale surjective and universally open map $\tilde{X} \to X$ that $\tilde{X}$ is strictly totally disconnected.

Proof: Cf. [Sch17] P35.
Prop. (VIII.7.2.11). The presheaf $\mathcal{O}, \mathcal{O}^+$ are sheaves w.r.t. the $v$-topology. Moreover, the $v$-site is subcanonical.


Prop. (VIII.7.2.12). Let $X$ be an affinoid perfectoid space, then $H^i_v(X, \mathcal{O}) = 0$ for $i > 0$, and $H^i_v(X, \mathcal{O}^+) = 0$ for $i > 0$.

Proof: Cf. [Sch17]P41.

Prop. (VIII.7.2.13) (Descent May Fail). The question is whether the fibered category over Perfd:

\[ X \mapsto \{\text{Perfectoid Spaces } Y \to X\} \]

is a stack for the pro-étale topology. This fails in general. An evidence is that the fibered category

\[ X \mapsto \{\text{Affinoid Perfectoid Spaces } Y \to X\} \]

is not a stack on the category of affinoid Perfectoid spaces with the analytic topology, let along the pro-étale topology:

Let $X = \text{Spa} K\langle x, y \rangle$, and $V \subset X$ be $\{(x, y) | |x| = 1 \text{ or } |y| = 1\}$, then $V$ is covered by two affinoids, but is not an affinoid itself: $H^1(X, \mathcal{O}_X) = \bigoplus_{m,n>0} Kx^{-m}y^{-n} \neq 0$. But there is a standard rational covering $X = \bigcup_i X(\frac{1}{f_i}, \frac{1}{f_i})$, where $f_1 = x$, $f_2 = y$, and

\[ U_0 \cap V = \emptyset, \quad U_1 \cap V = \{|x| = 1\} \subset U_1, \quad U_2 \cap V = \{|y| = 1\} \subset U_2 \]

are all affinoid. There is a similar example in the perfectoid case, thus

\[ X \mapsto \{\text{Affinoid Perfectoid Spaces } Y \to X\} \]

is not a stack.

Prop. (VIII.7.2.14) (Pro-étale is not Pro-étale Local). There is an example of a non-pro-étale map that is pro-étale locally pro-étale.


Prop. (VIII.7.2.15) (Characterization of Locally pro-étale Maps). Let $f : X \to Y$ be a morphism of affinoid perfectoid spaces, then the following are equivalent:

- There exists an affinoid pro-étale cover $Y' \to Y$ s.t. the base change $X' = X \times_Y Y' \to Y'$ is pro-étale.
- For all geometric points $\text{Spa} C \to Y$, $X \times_Y \text{Spa} C = \text{Spa} C \times S$ for some profinite set $S$.


Prop. (VIII.7.2.16) (Descent). Descent data of the the fibered category

\[ X \mapsto \{\text{Perfectoid Spaces } Y \to X\} \]

of a perfectoid space $Y' \to X'$ along a pro-étale cover $X' \to X$ is effective in the following cases:
• If \( X, X', Y' \) are affinoids and \( X \) is totally disconnected.
• If \( f \) is separated and pro-étale and \( X \) is strictly totally disconnected.
• If \( f \) is separated and étale. In particular, the fibered category
  \[
  X \mapsto \{ \text{separated étale } X \to Y \}
  \]
is a stack over the category of perfectoid spaces with the pro-étale topology.
• If \( f \) is finite étale. In particular, the fibered category
  \[
  X \mapsto \{ \text{finite étale } X \to Y \}
  \]
is a stack over the category of perfectoid spaces with the pro-étale topology.


\( \square \)

3 Morphisms of \( V \)-Stacks

Def. (VIII.7.3.1) (Étale Morphism of Stacks). Let \( f : Y' \to Y \) be a map of pro-étale stacks on the category \( \text{Perfd} \),
• Assume \( f \) is locally separated (i.e. there is an open cover of \( Y' \) over which \( f \) becomes separated), then \( f \) is called quasi-pro-étale if for any strictly totally disconnected perfectoid space \( X \) and a map \( X \to Y \), the pullback \( Y' \times_Y X \) is representable and \( Y' \times_Y X \to X \) is pro-étale.
• Assume \( f \) is locally separated, then \( f \) is called étale if for any perfectoid space \( X \) and a map \( X \to Y \), the pullback \( Y' \times_Y X \) is representable and \( Y' \times_Y X \to X \) is étale.
• \( f \) is called finite étale if for any perfectoid space \( X \) and a map \( X \to Y \), the pullback \( Y' \times_Y X \) is representable and \( Y' \times_Y X \to X \) is finite étale.

4 Diamonds

Remark (VIII.7.4.1) (Motivation of Diamonds). The idea of diamonds is that there should be a functor
\[
\diamond : \{ \text{analytic adic spaces over } \mathbb{Z}_p \} \to \{ \text{diamonds} \}
\]
that forgets the structure morphism to \( \mathbb{Z}_p \). For a perfectoid space \( X \), \( X \mapsto \text{X\textquoterightlat} \) has this property, so this functor should coincide on these objects. Now any analytic adic space \( X \) over \( \mathbb{Z}_p \) is pro-étale locally perfectoid:
\[
X = \text{Coeq}(\tilde{X} \times_X \tilde{X} \Rightarrow X),
\]
where \( \tilde{X} \to X \) is a pro-étale perfectoid cover. The equivalence relations \( R = \tilde{X} \times_X \tilde{X} \) is also perfectoid, so this functor should send \( X \) to \( \text{Coeq}(R^\text{an} \Rightarrow \tilde{X}^\text{an}) \).

For example, if \( X = \text{Spa}(\mathbb{Q}_p) \), then a pro-étale cover of \( X \) is \( \text{Spa}((\mathbb{Q}_p^{\text{cycl}})^\wedge) \), and then \( R = \tilde{X} \times_X \tilde{X} \times \mathbb{Z}_p^* \) by Galois theory, and then \( \mathbb{Q}_p^\circ \) should be defined as the coequalizer of \( \text{Spa}((\mathbb{Q}_p^{\text{cycl}})^\wedge) / \mathbb{Z}_p^* \), whose meaning is explained in (VIII.7.4.10).

Def. (VIII.7.4.2) (Diamonds). A diamond is a pro-étale sheaf \( \mathcal{D} \) on \( \text{Perf} \) that can be written as \( \mathcal{D} = X/R \), where \( X \in \text{Perf} \) and \( R \) is a pro-étale equivalence relation in \( X \times X \) (i.e. an equivalence relation that the maps \( s, t : R \to X \) are pro-étale), and also \( R \) is representable.
Prop. (VIII.7.4.3). Let $X \in \text{Perf}$ and $R \subset X \times X$ a representable pro-étale equivalence relation, then

- The quotient sheaf $Y = X/R$ is a diamond.
- The natural map of sheaves $R \to X \times_Y X$ is an isomorphism.
- Let $\tilde{X} \to X$ be a pro-étale cover by a perfectoid space $\tilde{X}$, and $\tilde{R} = R \times_{X \times X} (\tilde{X} \times \tilde{X})$ the induced equivalence relation, then $\tilde{R}$ is a representable pro-étale equivalence relation of $\tilde{X}$, and the natural map $\tilde{Y} = \tilde{X}/\tilde{R} \to Y = X/R$ is an isomorphism.
- The map $X \to Y$ is quasi-pro-étale(VIII.7.3.1).

Proof: 1 is by definition.

For 2, firstly $R \to X \times_Y X$ is injective as subsheaves of $X \times X$. Next, if $Z \to X \times_Y X$ is any map from a perfectoid space $Z$, then we have two maps $a, b : Z \to X$ that their composition with $X \to Y$ agree. This means after passing to a pro-étale covering $\tilde{Z} \to Z$, the composition map $\tilde{Z} \to Z \to X \times X$ factors through $R$. Now this map $\tilde{Z} \to R$ descends to a map $Z \to R$, because pro-étale site is subcanonical and the two projection maps $\tilde{Z} \times Z \tilde{Z} \to R$ coincides because they do after compositing with $R \to X \times X$.

3: Firstly $\tilde{R}$ is representable as fiber products of representable objects(VIII.5.5.9), and the two projections are pro-étale, as they are compositions of base changes of pro-étale morphisms. Also the map $\tilde{Y} \to Y$ of pro-étale sheaves is surjective, as the composition $\tilde{X} \times \tilde{X} \to X \times Y$ is.

To show $\tilde{Y} \to Y$ is injective, let $Z$ be a perfectoid space with two maps $Z \to \tilde{Y}$ that coincide after compositing with $\tilde{Y} \to Y$, we need to show $a = b$. Now because pro-étale site is subcanonical, it suffices to show this after replacing $Z$ with a pro-étale cover $\tilde{Z}$, such that $a, b$ factors over $\tilde{a}, \tilde{b} : Z \to \tilde{X}$. The associated map $Z \to \tilde{X} \times X \tilde{X} \times X$ factors through $R$ by item 2, so we get a map $Z \to R \times_{X \times X} (\tilde{X} \times \tilde{X}) = \tilde{R}$, which means $\tilde{a}, \tilde{b}$ induces the same map $Z \to \tilde{Y}$. So we are done.

4: By 3, we can replace $X$ by $\tilde{X} = \bigsqcup_i U_i \to X$, where $U_i$ is an affinoid cover of $\tilde{X}$, and $R$ by the the induced equivalence relation in $\tilde{X} \times \tilde{X}$, and to pro-étale is analytically local. In this way, we may assume $R \subset X \times X$ and $X \times X \to X$ are separated.

Let $X'$ be a strictly totally disconnected perfectoid space and $X' \to X$ be a map. Because $X \to Y$ is surjective as sheaves on Perf, there is a pro-étale cover $\tilde{X}' \to X'$ and a map $\tilde{X}' \to X$ lying over $X' \to Y$. We can assume $\tilde{X}'$ is affinoid. Let $W = \tilde{X}' \times_{X'} X \to X'$ be the fiber product, then

$$\tilde{X}' \times_{X'} W \times (\tilde{X}' \times_{X'} X \times R) = \tilde{X}' \times_{X} R$$

is representable and pro-étale over $\tilde{X}'$, and also separated. So by(VIII.7.2.16), $W$ is also representable, pro-étale and separated over $X$. \qed

Cor. (VIII.7.4.4) (Equivalent Characterization of Diamonds). Let $Y$ be a pro-étale sheaf on Perf, then $Y$ is a diamond iff there is a surjective quasi-pro-étale morphism $X \to Y$ from a perfectoid space $X$. If $X$ is a disjoint union of strictly totally disconnected spaces, then $R \subset X \times_Y X \subset X \times X$ is a representable pro-étale equivalence relation with $Y = X/R$.

Proof: If $Y$ is a diamond, then $X \to Y$ is quasi-pro-étale by(VIII.7.4.3). Conversely, if there is a quasi-pro-étale morphism $X \to Y$, by(VIII.7.2.10), we can assume that $X$ is a disjoint union of strictly disconnected spaces. In this case, by the definition of quasi-pro-étale, $R$ is representable and the projections $R = X \times_Y X \to X$ is pro-étale, and $X/R \cong Y$: it suffices to show this map is injective: if $a, b : Z \to X$ are two maps that coincide after composing with $X \to Y$, then after
replacing to a pro-étale covering \( \tilde{Z} \to Z \), we can lift to maps \((\tilde{a}, \tilde{b}) : \tilde{Z} \to R \). And this map descend to a map \( Z \to R \), because the pullback to \( \tilde{Z} \times_Z \tilde{Z} \to R \) coincides as they do after composing with \( R \leftarrow X \times X \). So \( X/R \to Y \) is injective. \( \square \)

**Cor. (VIII.7.4.5).**
- Let \( Y \) be a pro-étale sheaf on Perf and there is a quasi-pro-étale map \( Y' \to Y \), where \( Y' \) is a diamond, then \( Y \) is also a diamond.
- Let \( f : Y' \to Y \) be a quasi-pro-étale map of pro-étale sheaves on Perf and \( Y \) is a diamond, then \( Y' \) is also a diamond.

**Proof:** 1: By (VIII.7.4.4), we can choose a quasi-pro-étale map \( X \to Y' \) where \( X \) is a perfectoid space, then \( X \to Y' \to Y \) is also quasi-pro-étale, so \( Y \) is a diamond by (VIII.7.4.4) again.

2: Choose a surjective quasi-pro-étale map \( X \to Y \) where \( X \) is a perfectoid space, then \( X' \times_X Y \) is representable and \( X' \to Y' \) is quasi-pro-étale and surjective, thus \( Y' \) is a diamond by (VIII.7.4.4). \( \square \)

**Lemma (VIII.7.4.6).** The absolute product of two perfectoid spaces of char \( p \) is also a perfectoid space.

**Proof:** Cf. [Sch17]P71. \( \square \)

**Prop. (VIII.7.4.7) (Absolute Product of Diamonds).** Let \( \mathcal{D}, \mathcal{D}' \) be diamonds, then the product sheaf \( \mathcal{D} \times \mathcal{D}' \) is also a diamond.

**Proof:** Let \( \mathcal{D} = X/R \) and \( \mathcal{D}' = X'/R' \), then \( R \times R' \), \( X \times X' \) are also representable by (VIII.7.4.6), \( R \times R' \to (X \times X') \times (X \times X') \) is also an injection, and the projections are pro-étale. Thus \( \mathcal{D} \times \mathcal{D}' = (X \times X')/(R \times R') \) is a diamond, by (VIII.7.4.3). \( \square \)

**Prop. (VIII.7.4.8) (Fiber Products).** Fiber products exist in the category of diamonds.

**Proof:** Let \( Y_1 \to Y_2 \leftarrow Y_3 \) be a diagram of diamonds. Choose representations \( Y_i = X_i/R_i \), and after replacing \( X_1, X_2 \) with a pro-étale covering using (VIII.7.4.3), we can assume there are maps \( X_i \to X_3 \) lying over \( Y_i \to Y_3 \). Moreover, we can replace \( X_i \) by \( X_i \times_{Y_3} X_3 \) to assume that \( X_i \to Y_1 \times Y_3 \) is surjective in the pro-étale topology.

In this case, the map \( X_1 \times_{X_3} X_2 \to Y_1 \times_{Y_3} Y_2 \) is surjective in the pro-étale topology, and the equivalence relation \( R_4 = X_4 \times_{Y_4} X_4 \) can be calculated to be \( R_4 = R_1 \times_{R_3} R_2 \), which is representable. It remains to see that \( R_4 \to X_4 \) is pro-étale. But \( R_1 \times R_2 \to X_1 \times X_2 \) is pro-étale, so does its base change \( R_1 \times_{X_3} R_2 \to X_1 \times_{X_3} X_2 = X_4 \), and also \( R_4 = R_1 \times_{R_3} R_2 \to R_1 \times_{X_3} R_2 \) is pro-étale because it is the base change of \( R_3 \to R_3 \times_{X_3} R_3 \), which is pro-étale by (VIII.7.2.5). \( \square \)

**Prop. (VIII.7.4.9).** Let \( Y \) be a diamond, then \( Y \) is a sheaf for the \( \nu \)-topology.

**Proof:** Cf. [Sch17]P54. \( \square \)

\[ \text{Spd}(\mathbb{Q}_p) \]

**Prop. (VIII.7.4.10).** Let \( \text{Spd}(\mathbb{Q}_p) \) be defined as

\[ \text{Spa}((\mathbb{Q}_p^{cycl})^\flat)/\mathbb{Z}_p^* = \text{Spa}(\mathbb{F}_p((t^{1/\nu})))/\mathbb{Z}_p^*, \]
where \( \mathbb{Z}_p^* \) acts on \( \mathbb{F}_p((t^{1/p\infty})) \) via \( \gamma(t) = (1 + t)^\gamma - 1 \). To be precise, it is the coequalizer of
\[
\mathbb{Z}_p^* \times \text{Spa}(\mathbb{F}_p((t^{1/p\infty}))) \rightrightarrows \text{Spa}(\mathbb{F}_p((t^{1/p\infty})))
\]
where one map is projection and the other map is the group action. To show this is diamond, we first need to verify this is an injection thus an equivalence relation, which is by (VIII.7.4.11).

**Lemma (VIII.7.4.11).** Consider the map
\[
g : \mathbb{Z}_p^* \times \text{Spa}(\mathbb{F}_p((t^{1/p\infty}))) \rightrightarrows \text{Spa}(\mathbb{F}_p((t^{1/p\infty}))) \times \text{Spa}(\mathbb{F}_p((t^{1/p\infty})))
\]
where the first map is group action and the second map is projection, then this is an injection map.

**Proof:** For any perfectoid affinoid field \((K, K^+), \mathbb{Z}_p^* \) acts freely on the topological nilpotent elements of \( K \), thus the map is an injection, by (VIII.7.1.7). \( \square \)

**Prop. (VIII.7.4.12) (Torsor over Affinoid Perfectoid Space).** Let \( G \) be a profinite group and \( f : \mathcal{F} \to \mathcal{F} \) be a \( G \)-torsor, with \( G \) profinite, then for any affinoid \( X = \text{Spa}(B, B^+) \) and any morphism \( X \to \mathcal{F} \), the pullback \( \mathcal{F} \times_{\mathcal{F}} X \) is representable by a perfectoid affinoid \( X' = \text{Spa}(A, A^+) \), where \((A, A^+)\) is the completed filtered colimit of \((A_H, A_H^+)\), where for each open normal subgroup \( H \) of \( G \), \((A_H, A_H^+)\) is finite étale \( G/H \)-torsor.

**Proof:** If \( H \) is an open normal subgroup of \( G \), then \( \mathcal{F} / H \to \mathcal{F} \) is a \( G/H \)-torsor, and \( \mathcal{F}' = \lim_{\leftarrow H} \mathcal{F}' / H \), thus we reduce to the case \( G \) is finite. But for this case, we use the fact \{perfectoid spaces finite étale over \( X \)\} is a stack (VIII.7.2.16), and the definition of \( G \)-torsor. \( \square \)

**Prop. (VIII.7.4.13) (Description of \( \text{Spd}(\mathbb{Q}_p) \)).** If \( X = \text{Spa}(R, R^+) \) is an affinoid perfectoid space of characteristic \( p \), then \( (\text{Spd}(\mathbb{Q}_p))(X) \) is the set of isomorphism classes of data of the following shape.

- \( \mathbb{Z}_p^* \)-torsor \( R \to \tilde{R} \), i.e. \( \tilde{R} = (\lim_{\to \gamma} R_\gamma) \gamma \), where \( R_\gamma / R \) is finite étale with Galois group \( (\mathbb{Z}/p\mathbb{Z})^\gamma \).

- A topological nilpotent unit \( t \in \tilde{R} \) s.t. for all \( \gamma \in \mathbb{Z}_p^* \), \( \gamma(t) = (1 + t)^\gamma - 1 \).

**Proof:** Notice \( \text{Spa}(\mathbb{Q}_p^{cycl})^h \to \text{Spd}(\mathbb{Q}_p) \) is a \( \mathbb{Z}_p^* \)-torsor by definition, so for any morphism \( X \to \text{Spd}(\mathbb{Q}_p) \), the pullback \( \text{Spa}(\mathbb{Q}_p^{cycl})^h \times_{\text{Spd}(\mathbb{Q}_p)} X \to X \) is also a \( \mathbb{Z}_p^* \)-torsor, so it is isomorphic to a torsor \( \text{Spa}(\tilde{R}, \tilde{R}^+) \to \text{Spa}(R, R^+) \) by (VIII.7.4.12), and the map \( \text{Spa}(\mathbb{Q}_p^{cycl})^h \times_{\text{Spd}(\mathbb{Q}_p)} X \to X \to \text{Spa}(\mathbb{Q}_p^{cycl})^h \) is a \( \mathbb{Z}_p^* \)-equivariant map, which is equivalent to a topologically nilpotent element \( t \in \tilde{R} \) that is equivariant, i.e. \( \gamma(t) = (1 + t)^\gamma - 1 \).

Conversely, for any such a \( \mathbb{Z}_p^* \)-torsor \( R \to \tilde{R} \), an equivariant element \( t \in \tilde{R} \) descends to a map \( X \to \text{Spd}(\mathbb{Q}_p) \). \( \square \)

**Prop. (VIII.7.4.14).** The category of perfectoid spaces over \( \mathbb{Q}_p \) is equivalent to the category of perfectoid spaces \( X \) of characteristic \( p \) equipped with a structure morphism \( X \to \text{Spd}(\mathbb{Q}_p) \).

**Proof:** Consider the category of triples \((X^2, X, \iota)\), where \( X^2 \) is a perfectoid space over \( \mathbb{Q}_p \), \( X \) is a perfectoid space of characteristic \( p \), \( \iota : X^{p^2} \cong X \) is an isomorphism. A map of triples is a tuple \((f^2, f) : (X^2, X, \iota) \to (Y^2, Y, \iota')\) that \( \iota' \circ f^{p^2} = f \circ \iota \).

This category is equivalent to the category of perfectoid spaces over \( \mathbb{Q}_p \) by the forgetful functor, where the quasi-inverse is given by \( X^2 \mapsto (X^2, X^{p^2}, \text{id}_{X^{p^2}}) \). And this category is also fibered in
equivalent relations over Perf. So it is equivalent to a presheaf \( \text{Untilt}_{\mathbb{Q}_p} \) which maps \( X \) to the isomorphism classes of untilts \((X^\sharp, \iota)\) over \( \mathbb{Q}_p \), where \( \iota : X^\sharp \cong X \) is an isomorphism, by (II.1.7.28).

Similarly we can show define a functor Untilt which maps \( X \) to the isomorphism classes of untilts \((X^\sharp, \iota)\)(of whatever characteristic), where \( \iota : X^\sharp \cong X \) is an isomorphism.

Let \( X = \operatorname{Spa}(R, R^+) \) be an affinoid perfectoid space of characteristic \( p \). If \( X^\sharp = \operatorname{Spa}(R^\sharp, R^\sharp^+) \) is an untilt. Let \( \tilde{X}^\sharp = X^\sharp \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\text{cycl}} \), then \( \tilde{X}^\sharp \to X^\sharp \) is a pro-étale \( \mathbb{Z}_p^* \)-torsor, whose tilt \( \tilde{X} \to X \) is a pro-étale \( \mathbb{Z}_p^* \)-torsor equipped with a \( \mathbb{Z}_p^* \)-equivariant map \( \tilde{X} \to \operatorname{Spa}(\mathbb{Q}_p^{\text{cycl}}) \) this is a morphism \( X \to \operatorname{Spd}(\mathbb{Q}_p) \) by (VIII.7.4.13).

Conversely, let \( \tilde{X} \to X \) be a pro-étale \( \mathbb{Z}_p^* \)-torsor and \( \tilde{X} \to \operatorname{Spa}(\mathbb{Q}_p^{\text{cycl}}) \) a \( \mathbb{Z}_p^* \)-equivariant map, then by tilting equivalence there exists a morphism \( \tilde{X}^\sharp \to \operatorname{Spa}(\mathbb{Q}_p^{\text{cycl}}) \) which is also \( \mathbb{Z}_p^* \)-equivariant. The equivariance means that it is a descent datum along \( \tilde{X} \to X \), so it descends to an untilt \( X^\sharp \) of \( X \) over \( \mathbb{Q}_p \).

Finally, for general affinoid perfectoid space \( X \), as \( \text{Untilt}_{\mathbb{Q}_p} \) and \( \operatorname{Spd}(\mathbb{Q}_p) \) are all sheaves on Perf, the above construction can be glued to give an isomorphism between them. \( \square \)

**Prop. (VIII.7.4.15) (Untilts is a Sheaf).** Untilt is a v-sheaf on the site Perf. So does \( \text{Untilt}_{\mathbb{Q}_p} \), because the invertibility of \( p \) can be verified locally as \( \mathcal{O} \) is sheaf.

**Proof:** Firstly Untilt is clearly an analytic sheaf, so it suffices to show that if \( X = \operatorname{Spa}(R, R^+) \) is a perfectoid space of characteristic \( p \) with a v-cover \( Y = \operatorname{Spa}(S, S^+) \to X \) and \( Y^\sharp = \operatorname{Spa}(S^\sharp, S^\sharp^+) \) is an untilt of \( Y \) that the corresponding two untilts of \( Z = Y \times_X Y \) agree, then there is a unique untilt \( X^\sharp = \operatorname{Spa}(R^\sharp, R^\sharp^+) \) whose pullback to \( Y \) is \( Y^\sharp \).

Cf. [Sch17]P86.  \( \square \)
Chapter IX

Geometry

IX.1 Topology

Main references are [Mun00], [Mor19], [H-J99].

1 Basics

Def. (IX.1.1.1). A function from $X$ to $\mathbb{R} \cup \{-\infty, \infty\}$ is called upper semicontinuous iff $f^{-1}((-\infty, a))$ are all open. It is called lower semicontinuous iff $f^{-1}((a, \infty])$ are all open.

Def. (IX.1.1.2). A topological space is called separable if it has a countable dense subset.

Def. (IX.1.1.3) (Product Topology). Arbitrary product exists in the category of topological spaces. It is constructed as follows: for a family of topology spaces $X_i$ indexed over an index set $I$, the product $\prod_I X_i$ is the set-theoretic product endowed with the topology generated by the basis $\pi_i^{-1}(U_i)$ for $U_i$ open in $X_i$.

Prop. (IX.1.1.4) (Limits and Colimits). Arbitrary limits and colimits exist in the category of topological spaces. The limits is given as a subspace of the product topology, and the colimits $X = \text{colim} X_i$ is given a topology that $U \subset X$ is open iff $U \cap X_i$ is open for each $i$.

Prop. (IX.1.1.5) (Pullback Space). Let $E \rightarrow X$ and $f : X' \rightarrow X$ be maps of spaces, then there is a pullback map $f^* E = E \times_X X' \rightarrow X'$, called the pullback space.

Def. (IX.1.1.6) (Quotient Topology). Let $f : X \rightarrow Y$ be a surjective map of spaces, and $X$ has a topology, then we can define a quotient topology on $Y$ that $U \subset Y$ is open iff $f^{-1}(U)$ is open in $X$. It has the universal property that any continuous map $X \rightarrow Z$ that factors through $f$ set-theoretically factors through $f$ as a continuous map.

Such a map is called a quotient map.

Prop. (IX.1.1.7). A surjective open map $\pi$ is a quotient map.

Proof: It is clearly that a subset $U$ is open iff $\pi^{-1}(U)$ is open. \qed

Def. (IX.1.1.8) (Glueing Space). Let $A \subset X$ and $f : A \rightarrow Y$, then we have the glueing space $Y \amalg_f X$. 
Def. (IX.1.1.9) (Mapping Cylinder). Let \( f : X \to Y \) be a map, then we define the mapping cylinder \( M_f = Y \coprod_f X \times I \), where \( X \times \{1\} \subset X \times I \) mapsto \( Y \) by \( f \).

Def. (IX.1.1.10) (Mapping Cone). Let \( f : X \to Y \) be a map, then we define the mapping cone \( C_f = M_f / X \times \{0\} \).

Lemma (IX.1.1.11). If \( f : X \to Y \) is a surjective continuous map that \( f(E) \neq Y \) for any proper closed subspace of \( X \), then for any \( U \subset X \) open, \( f(U) \subset Y \setminus f(X \setminus U) \).

Proof: Take \( y \in f(U) \) and a nbhd \( V \) of \( y \) in \( Y \), we show that \( V \) intersect \( Y \setminus f(X \setminus U) \): \( W = U \cap f^{-1}(V) \) is nonempty, thus \( f(X \setminus W) \neq Y \), take \( y' \in Y \setminus f(X \setminus W) \), then it is clear \( y' \in Y \setminus f(X \setminus U) \), and \( y' \in V \).

Def. (IX.1.1.12) (Locally Closed Subset). A subset \( Z \) of \( X \) is called locally closed if for any \( z \in Z \), there is a nbhd \( U \) of \( z \) in \( X \) that \( U \cap Z \) is closed in \( Z \). Equivalently, a locally closed subset is the intersection of an open subset with a closed subset.

Proof: An intersection of an open subset and a closed subset is clearly locally closed. Conversely, if \( Z \) is locally closed, we choose for each \( z \in Z \) a nbhd \( U_z \) that \( U_z \cap Z \) is closed in \( U_z \), then we can show \( Z = Z \cap \bigcup_z U_z \). This is because if \( x \in Z \cap \bigcup_z U_z \), then \( x \in U_z \) for some \( z \), and also \( x \in Z \), thus \( x \) is in \( Z \).

Filter Langrange

Def. (IX.1.1.13) (Convergence and Filter). For a filter \( \mathcal{F} \) on a topological space, \( \mathcal{F} \) converges to a point \( y \) iff any open set containing \( y \) is in \( \mathcal{F} \).

If \( X \) is a set and \( Y \) is a topological space and \( X \to Y \) is a function, then \( y \in Y \) is a \( \mathcal{F} \)-limit of \( \mathcal{F} \) if \( f_* \mathcal{F} \) converges to \( Y \).

Prop. (IX.1.1.14) (Ultrafilter Convergence Theorem). Let \( X, Y \) be a topological space, then:

1. \( Y \) is compact iff any ultrafilter on \( Y \) has a limit point.
2. \( Y \) is Hausdorff iff any ultrafilter on \( Y \) has at most one limit point.
3. a function \( f : X \to Y \) is continuous iff for any filter on \( X \) converging to \( x \), the filter \( f_* \mathcal{F} \) converges to \( y \).

Proof:

1. If \( Y \) is compact but every point is not a limit point, then for any \( x \), there is an open set \( U_x \) that \( U_x \notin \mathcal{F} \), but then f.m. of them covers \( Y \), which is in \( \mathcal{F} \), so one of them must be in \( \mathcal{F} \) by(XII.1.8.7), contradiction.

Conversely, if \( \cup_i U_i = X \) but no finite union of them cover \( X \), then \( X - U_i \) satisfies the finite intersection property, so there is an ultrafilter containing all \( X - U_i \) by(XII.1.8.4) and(XII.1.8.5). Then clearly any point \( x \) is not a limit point of \( \mathcal{F} \).

2. If \( Y \) is Hausdorff and \( x, y \) are both limit point of a filter \( \mathcal{F} \), then there are two non-intersecting nbhd of them in \( \mathcal{F} \), so its intersection \( \emptyset \in \mathcal{F} \), contradiction.

Conversely, if \( x, y \) are two point that their nbhds both intersect, then their nbhds together satisfies the finite intersection property, so there is an ultrafilter containing all of them, by(XII.1.8.4) and(XII.1.8.5), thus converging to both \( x \) and \( y \).

3. This is an easy consequence considering the filter of all the nbhd containing \( x \).
**IX.1. TOPOLOGY**

**Connected Component**

**Def. (IX.1.1.15) (Connectedness).** A space $X$ is called **connected** if for any open subset $U, V$ of $X$ that $U \cup V = X$ and $U \cap V = \emptyset$, $U = \emptyset$ or $V = \emptyset$.

$X$ is called **locally connected** if there is a basis of $X$ consisting of connected open subsets of $X$.

**Def. (IX.1.1.16) (Path-Connectedness).** A space $X$ is called **path-connected** if any two points of $X$ can be connected by an arc.

$X$ is called **locally path-connected** if there is a basis of $X$ consisting of path-connected open subsets of $X$.

**Def. (IX.1.1.17) (Connected Components).** In a topological space $X$, if $x \in X$, the **connected component** of $x$ is the maximal connected subspace of $X$ containing $x$. The **path-connected component** of $x$ is the maximal path-connected subspace of $X$ containing $x$.

**Prop. (IX.1.1.18).**
- Any connected component of $X$ is closed.
- If $X$ is locally connected, then any connected component of $X$ is clopen.
- If $X$ is locally path-connected, then any path-connected component of $X$ is open, and any connected component is also open.

**Prop. (IX.1.1.19) (Clopen Subsets and Connected Components).** Let $X$ be a normal topological space and $x \in X$, then the connected component of $X$ containing $x$ is the intersection of clopen subsets containing $x$, denoted by $A$.

**Proof:** Assume $A$ splits into two components $B, D$. Since $A$ is closed, $B$ and $D$ are both closed, because $X$ is normal there are disjoint open neighborhoods $U$ and $V$ around $B$ and $D$, respectively. The open sets $U$ and $V$ cover the intersection of all clopen neighborhoods of $A$, so cause $X$ is compact, there must exist a finite number of clopen sets around $A$, say $A_1, \ldots, A_n$ such that $U \cup V$ covers $K = \bigcap_i A_i$.

Note that $K$ is clopen. We can assume that $x \in U$. It is not difficult to see that $K \cap U$ is clopen and does not contain all of $A$, contradicting the definition of $A$. \qed

**Cor. (IX.1.1.20).** For a compact Hausdorff topological space $X$ and a point $x \in X$, the connected component of $X$ containing $x$ is the intersection of all compact open neighborhoods of $x$, because $X$ is normal(IX.1.6.1).

**Def. (IX.1.1.21) (Totally Disconnected Space).** A space is called **totally disconnected** iff any connected subset of $X$ contains only one point.

**Prop. (IX.1.1.22).** A subspace of a totally disconnected space is totally disconnected, because totally disconnected is equivalent to the only connected subsets are pt sets.

**Extremally Disconnected Space**

**Def. (IX.1.1.23) (Extremally Disconnected Space).** A topological space $S$ is called **extremally disconnected** if the closure of any open subset of $X$ is open.
Prop. (IX.1.1.24). Let \( X \) be an extremally disconnected space. If \( U, V \) are disjoint open subsets of \( X \), then \( U, V \)

\[ \text{Proof: } \] Because \( V \cap U = \emptyset \), thus similarly \( V \cap U = \emptyset \). □

Lemma (IX.1.1.25). Let \( f : X \to Y \) be a continuous surjective map of compact Hausdorff spaces that \( Y \) is extremally disconnected and \( f(Z) \neq Y \) for any proper closed subspace of \( X \), then \( f \) is a homeomorphism.

\[ \text{Proof: } \] By (IX.1.2.11) it suffices to show that \( f \) is injective. Suppose \( f(x) = f(x') = y \), then choose disjoint nbhd \( U, U' \) of \( x, x' \), and \( T = f(X \setminus U), T' = f(X \setminus U') \) closed in \( Y \), then \( Y = U \cup T \cup T' \) by (IX.1.1.11), but this contradicts (IX.1.1.24). □

Prop. (IX.1.1.26) (Projective Space). Compact Hausdorff extremally disconnected spaces are exactly the projective objects in the category of compact Hausdorff spaces.

\[ \text{Proof: } \] Assume \( X \) is projective, let \( U \subset X \) be open, and the complement by \( Z \), then consider the surjection \( U \coprod Z \to X \), and let \( \sigma \) be the projection, then \( \sigma(U) \subset U \), thus \( \sigma^{-1}(U) = U \), and it is open.

Conversely, if \( X \) is extremally disconnected, then by (IX.1.2.13), there is a compact subset \( E \subset Y \) that \( f(E) = X \) and \( f(E') \neq X \) for all closed subspace \( E' \subset E \). Then (IX.1.1.25) says \( f|_E \) is a homeomorphism, and the inverse of it gives a desired section. □

Prop. (IX.1.1.27) (Gleason). In an extremally disconnected space \( X \), a convergent sequence is eventually constant. In particular, \( \mathbb{Z}_p \) is a profinite group that is not extremally disconnected.

\[ \text{Proof: } \] Cf. [Projective Topological Spaces] □

2 Compactness

Def. (IX.1.2.1) (Quasi-Compact Space). A topological space is called compact or quasi-compact iff any open covering of it has a finite sub-covering. A subspace of a topological space is called precompact if its closure is compact.

Def. (IX.1.2.2) (Quasi-Compact Morphism). A map of topological spaces is called a quasi-compact morphism if the inverse image of any quasi-compact open subset is quasi-compact open.

Prop. (IX.1.2.3) (Alexander Subbase Theorem). A topological space is compact iff the closed subsets has the finite intersection property (XII.1.8.3). In fact, it suffices to show that the family of complements of a subsbasis of open sets has the finite intersection property.

\[ \text{Proof: } \] Cf. [Sta]08ZP. □

Prop. (IX.1.2.4). Let \( X \) be a totally ordered set with the least upper bound property, then each closed interval of \( X \) is closed in the order topology. In particular, this applies to a complete totally ordered set \( X \) (XII.1.3.19).

\[ \text{Proof: } \] Let \( a < b \) and \( \mathcal{U} \) a covering of \( [a, b] \). We first show that for any \( x \in [a, b] \), there exists some \( x < y \leq b \) that \( [x, y] \) can be covered by at most two elements of \( \mathcal{U} \): If \( x \) has an immediate successor, then take \( y \) to be it. If \( x \) has no immediate successor, choose an element \( U \) of \( \mathcal{U} \) containing \( x \), then \( U \) contains some \( [x, c) \). Choose \( y \in [x, c) \), then \( [x, y] \) is covered by a single element of \( \mathcal{U} \).
Now let $C$ be the set of points $y \in (a, b]$ that $[a, y]$ can be covered by f.m. elements of $\mathcal{U}$. We showed before this set is non-empty. Let $c$ be the least upper bound of $C$, then $a < c \leq b$.

Next, we show $c \in C$: take an element $U$ of $\mathcal{U}$ containing $c$, then $U$ contains some $(d, c]$. There must be some element of $C$ lying in the interval $(d, c]$, otherwise $d$ is an upper bound of $C$. Then $[a, y]$ is covered by f.m. elements of $\mathcal{U}$, so $c \in C$.

Finally, we show $c = b$, but this is because otherwise we can find $c < y \leq b$ that $[c, y]$ is covered by 2 elements, thus $y \in C$, so $y \leq c$, contradiction. \hfill \Box

**Prop. (IX.1.2.5) (Tychonoff).** An arbitrary direct product of compact topological groups is compact.

**Proof:** We prove the finite intersection property. If $A$ is a family of subsets that any finite intersection of closure of them is nonempty, then consider a maximal family $\mathcal{D}$ of subsets containing $A$ that any finite intersection of closures of them is nonempty, it exists by Zorn’s lemma. Consider the projection of $\mathcal{D}$ onto a coordinate, then by Hypothesis, it has an intersection $x_\alpha$. Now we want to show $x = (x_\alpha)$ belongs to each $D \in \mathcal{D}$.

If $U_\beta$ is any subbasis element containing $x$, then $U_\beta$ intersect each $D$ because $x_\beta \in \pi_\beta(D)$, so it is in $\mathcal{D}$, by maximality of $\mathcal{D}$. So the finite intersections are also in $\mathcal{D}$, so all local basis of $x$ are in $\mathcal{D}$. This means that local basis intersect each element of $\mathcal{D}$, that is, all closure of elements in $\mathcal{D}$ contains $x$. \hfill \Box

**Def. (IX.1.2.6) (Sequentially Compact).** A subset $A$ in a space $X$ is called **sequentially compact** iff any sequence of points in $A$ has a convergent subsequence in $X$. It is called **self sequentially compact** if it is sequentially compact in itself.

**Prop. (IX.1.2.7).** $f: X \to Y$, $X$ is compact and $Y$ is Hausdorff, then for a descending chain $Y_i$ of closed subsets of $X$,

$$f\left(\bigcap_n Y_n\right) = \bigcap_n f(Y_n).$$

**Proof:** The left side is compact, so if $x \notin f\left(\bigcap_n Y_n\right)$, there is a closed subsets $x \in T$ that $T \cap f\left(\bigcap_n Y_n\right) = 0$, so $f^{-1}(T) \cap Y_n = 0$, so $f^{-1}(T) \cap Y_n = 0$ for some $n$, hence $x \notin f(Y_n)$. \hfill \Box

**Prop. (IX.1.2.8) (Fixed Point Theorem).** If $X$ is a compact metric space $M$, $T$ is a continuous map $X \to X$ that $d(x, y) < d(Tx, Ty)$, then $T$ has a unique fixed point in $X$.

**Proof:** The uniqueness is obvious, for the existence, first notice $T$ is obviously continuous, so consider $d(x, Tx)$, this is a continuous function on $M$, so it contains a minimum value, if it not 0, then $d(Tx, T^2x) < d(x, Tx)$, which is a contradiction. \hfill \Box

**Def. (IX.1.2.9) (Proper Map).** A map is called **proper** if the inverse image of compact subsets are compact.

**Prop. (IX.1.2.10).** If $X$ is compact and $Y$ is Hausdorff, then a continuous map $f: X \to Y$ is proper.

**Cor. (IX.1.2.11).** A continuous bijective map from a compact space to a Hausdorff space is a homeomorphism.

**Prop. (IX.1.2.12) (Proper Continuous Maps Closed).** Let $f: X \to Y$ be a proper continuous map with $Y$ locally compact Hausdorff, then $f$ is a closed map.
Proof: Let $K \subset X$ closed, and $y \in \overline{f(K)}$. Choose precompact open nbhd $U$ of $y$, then $f^{-1}(U)$ is compact in $X$, so $f(K \cap f^{-1}(U)) = f(K) \cap U$ is compact and thus closed in $Y$. Then $y \in f(K)$, and $f(K)$ is closed.

Lemma (IX.1.2.13). Let $f : X \to Y$ be a continuous map of compact Hausdorff spaces, then there exists a smallest closed subset $E$ of $X$ that $f(E) = Y$.

Proof: Use Zorn’s lemma, noticing that the intersection of a chain of possible $E_i$s also maps to $Y$, by the intersection property (IX.1.2.3).

Stone-Čech Compactification

Def. (IX.1.2.14) (Stone-Čech Compactification). The Stone-Čech Compactification $\beta$ is defined to be a functor from the category of sets to the category of compact Hausdorff space that is left adjoint to the forgetful functor.

The construction of $\beta(X)$ is as follows: $\beta(X)$ = the set of all ultrafilters on $X$, and the topology is generate by $U_A = \{F | A \in F\}$ as a basis of clopen subsets. For a map $f : X \to Y$, the map $\beta X \to \beta Y$ is given by $f_*$.  

Proof: First $\beta X$ is a compact Hausdorff space: it is compact because if there are sets $A_i$ that any ultrafilter contains at least one of them, then f.m. of them must cover $X$, otherwise $X - A_i$ satisfies the finite intersection property thus is contained in some ultrafilter, by (XII.1.8.4) and (XII.1.8.5), contradiction. Then by (XII.1.8.7) shows that any ultrafilter contains one of them. It is Hausdorff because for any two different ultrafilter, there must be an $A$ that $A \in F_1$ and $X - A \in F_2$. $f_*$ is continuous because $f^{-1}(U_A) = U_{f^{-1}(A)}$.

Now for any map $f : X \to Y$ where $Y$ is a topological space, map an ultrafilter $F$ to the unique limit point of $f_*F$ in $Y$ (existence and uniqueness by (IX.1.1.14)). This map is continuous from $\beta X$ to $Y$ because for any open set $V \subset Y$, $U_{f^{-1}(V)}$ is mapped into $V$. And for any $\beta X \to Y$ continuous, consider $X \to Y$ which maps $x$ to the image of the principle ultrafilter $F_x$ in $Y$.

This two map are mutually converses to each other, first for a $f : X \to Y$, $X \to \beta X \to Y$ is $f$ itself, because the pushout of the principle ultrafilter clearly converges to $f(x)$. And for a $\beta X \to Y$, if $F$ doesn’t map to $\lim f_*F$ but mapped to some $t$, then by definition, there is a nbhd $U$ of $t$ that $f^{-1}(U) \notin F$, but by continuity, there is a $F \in U_B$ mapped into $U$. But then $B \in f^{-1}(U)$, otherwise if $x \in B - f^{-1}(U)$, then $F_x$ is mapped to $U$, contradiction, so $f^{-1}(U)$ containing $B$ is also in $F$, contradiction.

Lemma (IX.1.2.15). In fact, the spaces in the image of the Stone-Čech compactification are all profinite spaces.

Proof: As shown before, for any two different ultrafilter, there must be an $A$ that $A \in F_1$ and $X - A \in F_2$, $U_A$ is open and closed.

Prop. (IX.1.2.16) (Stone Representation Theorem). The Stone-Čech compactification $\beta$ gives an equivalent of categories from the category of Boolean algebras to the category of profinite spaces.

Proof: $\beta B$ is a profinite space by lemma (IX.1.2.15), and $B$ can be recovered from $\beta B$ as the Boolean algebra of all clopen subsets of $\beta B$, because $\beta B$ is compact. This is a inverse isomorphism because ?.
Prop. (IX.1.2.17). The Stone-Čech compactification of a set $\beta S_0$ is extremally disconnected.

Proof: We check condition (IX.1.1.26): For any surjection $S' \to \beta S_0$, we may take a lift $S_0 \to S$ arbitrarily, then by definition there is a morphism $\beta S_0 \to S'$ extending this morphism. It is a section by the universal property. □

Locally Compact Space

Def. (IX.1.2.18) (Locally Compact Space). A Hausdorff space is called locally compact if for any point $x$, there is a compact subset containing a nbhd of $x$.

Prop. (IX.1.2.19). If $X$ is Hausdorff, then $X$ is locally compact iff for any $x \in X$ and $x \in U$ open, there exists a precompact nbhd $V$ that $x \in V \subset V' \subset U$.

Proof: One direction is trivial. For the other, for any $x \in X$ and a nbhd $U$ of $x$, let $U_0$ be a precompact nbhd of $x$, then $U_0 \setminus U$ is closed thus compact and disjoint from $x$. Because $X$ is Hausdorff, we can find a nbhd $V'$ of $x$ that $V'$ is disjoint from $U$. Now let $V = V' \cap U_0$, then $V \subset U_0$ is closed thus compact, and $V \subset V' \cap U_0 \subset U$. □

Cor. (IX.1.2.20). Open subsets and closed subsets of a locally compact Hausdorff space $X$ is locally compact Hausdorff.

Proof: If $A \subset X$ is closed and $x \in A$, then there exists a precompact nbhd $U$ of $x \in X$, then $A \cap U$ is closed in $U$ thus compact, and contains the nbhd $U \cap A$ of $x \in A$, so $A$ is locally compact.

If $A \subset X$ is open, $x \in A$, let $x \in U \subset A$, and $U$ open in $X$, then we can apply (IX.1.2.20) to find a precompact nbhd $V$ that $x \in V \subset V' \subset U$, so $A$ is locally compact. □

Prop. (IX.1.2.21) (One-Point Compactification). Let $X$ be a space, then $X$ is locally compact Hausdorff iff there exists a compact Hausdorff space $Y$ containing $X$ s.t. $Y \setminus X$ is a single point. Moreover, in this case, $Y$ is unique up to homeomorphisms, called the one-point compactification of $X$.

Proof: Cf. [Mun00]P183. □

Cor. (IX.1.2.22). A space $X$ is homeomorphic to an open subset of a compact open subspace iff it is locally compact Hausdorff, by (IX.1.2.21) and (IX.1.2.20).

Prop. (IX.1.2.23). A locally compact second countable Hausdorff space $X$ has a countable basis consisting of precompact open subsets. In particular, $X$ is $\sigma$-compact.

Proof: Choose a countable basis $\{U_n\}$ of $X$. For any $x \in X$ and any nbhd $U$ of $x$, there exists a precompact $U_x \subset U$ containing $x$ by (IX.1.2.19). Then for some $n(x)$, $U_n(x) \subset U$, containing $x$, so $U_{n(x)}$ is precompact. Thus the set of $U_n$ that is precompact forms a countable basis of $X$. □

Prop. (IX.1.2.24). If $f : X \to Y$ is a quotient map and $Z$ is locally compact, then $X \times Z \to Y \times Z$ is also a quotient map.

Proof: Consider $W$ as $Y \times Z$ in the quotient map topology, then $X \times Z \to W$ is continuous, which means $X \to \text{Map}(Z,W)$ is continuous. But then $Y \to \text{Map}(Z,W)$ is continuous because $Y$ is a quotient map. Applying (IX.1.3.6), we see $Y \times Z \to W$ is continuous. $W \to Y \times Z$ is continuous by quotient map hypothesis, so $Y \times Z \cong W$. □
Compactly Generated Space

Def. (IX.1.2.25) (Compactly Generated Space). A compactly generated space is a space that is the colimit of its compact subspaces.

Prop. (IX.1.2.26). If $X$ is a compactly generated Hausdorff space and $Y$ is locally compact Hausdorff, then the product topology $X \times Y$ is compactly generated Hausdorff.

Proof: Notice by (IX.1.3.6), a map $X \times Y \to Z$ is continuous iff $C \times Y \to Z$ is continuous for any compact subset $Z \subset X$. The assertion is equivalent to $X \times Y \to (X \times Y)_{c}$ is continuous, which is then equivalent to $C \times Y \to (X \times Y)_{c}$ continuous for any compact subset $C \subset X$. But $Y$ is compactly generated, and $C$ is locally compact, thus it suffices to show $C \times C' \to (X \times Y)_{c}$ is continuous for any compact subset $C' \subset Y$. Then this is true because $C \times C'$ is compact. □

3 Function Spaces

Compact-Open Topology on Function Spaces

Def. (IX.1.3.1) (Compact-Open Topology). The compact-open topology on a function space $\text{Map}(X,Y)$ is a topology generated by the sets $(K,U) = \{ f : f(K) \subset U \}$, where $K$ is compact in $X$ and $U$ is open in $Y$.

Prop. (IX.1.3.2). If $X' \to X$ and $Y \to Y'$ are continuous maps, then the induced map $\text{Map}(X,Y) \to \text{Map}(X',Y')$ is continuous. In particular, if $X \cong X'$ and $Y \cong Y'$, then $\text{Map}(X,Y) \to \text{Map}(X',Y')$.

Prop. (IX.1.3.3). If $Y$ is compact and $X$ a metric space, this the compact open topology on $\text{Map}(X,Y)$ coincides with the uniform topology on functions.

Proof: If $f \in (K,U)$, then $f(K) \subset U$, then there is a $\varepsilon > 0$ that $B(K,\varepsilon) \subset U$, thus $B(f,\varepsilon) \subset (K,U)$.

For any $f \in \text{Map}(Y,X)$, $f(Y)$ is compact thus there are f.m. open balls $B(f(y),\frac{\varepsilon}{2})$ that covers $f(Y)$. Now if $K_{i} = f^{-1}(B(f(y_{i}),\frac{\varepsilon}{2}))$, then $\cup K_{i} = Y$, and $f \in \cap_{i}(K_{i},B(f(y_{i}),\frac{\varepsilon}{2}))$. Also $\cap_{i}(K_{i},B(f(y_{i}),\frac{\varepsilon}{2})) \subset B(f,\varepsilon)$, because for any $g \in \cap_{i}(K_{i},B(f(y_{i}),\frac{\varepsilon}{2}))$ and $y \in K_{i}$, $|f(y) - g(y)| \leq |f(y) - f(y_{i})| + |f(y_{i}) - g(y)| < \varepsilon + \frac{\varepsilon}{2} = \varepsilon$. □

Def. (IX.1.3.4) (Relative Maps). Let $A \subset X$ and $B \subset Y$, we denote by $\text{Map}(X,A;Y,B)$ the subspace of $\text{Map}(X,Y)$ consisting of continuous functions $f : X \to Y$ that maps $A$ into $B$.

Lemma (IX.1.3.5) (Subbasis). Let $X$ be Hausdorff, $W_{a}$ be a subbasis of $Y$, then $(K,W_{a})$ where $K \subset X$ compact is a subbasis of $\text{Map}(X,Y)$.

Proof: If $f \in (K,U) \subset \text{Map}(X,Y)$, let $U = \bigcup_{\beta}U_{\beta}$, where $U_{\beta} = \bigcap_{j=1}^{k(\beta)}W_{\beta,j}$. Now $K \subset \bigcup_{\beta}f^{-1}(U_{\beta})$, because $K$ is compact Hausdorff thus the partition of unity (IX.1.7.10) gives us f.m. compact subsets $K_{1},\ldots,K_{n}$ of $K$ that $K_{i} \subset f^{-1}(U_{\beta,i})$ for some $\beta$. Then

$$f \in \bigcap_{i=1}^{n} \bigcap_{j=1}^{k(\beta)}(K_{i},W_{\beta,i,j}) = \bigcap_{i=1}^{n}(K_{i},U_{\beta,i}) \subset (K,\bigcup_{i=1}^{n}U_{\beta,i}) \subset (K,U).$$

□
Prop. (IX.1.3.6) (Adjointness of Mapping Space). Let $Y$ be a locally compact Hausdorff space, then

$$\text{Map}(X \times Y, Z) \cong \text{Map}(X, \text{Map}(Y, Z))$$

as sets. And if moreover $X$ is Hausdorff, then it is a homeomorphism.

Proof: We may assume $X$ is non-empty. Let $\varphi \in \text{Map}(X \times Y, Z)$, for any $x \in X$, take $\varphi(x) : Y \to Z : \varphi(x)(y) = \varphi(x, y)$, then $\varphi(x)$ is continuous, and $\tilde{\varphi} : X \mapsto \text{Map}(Y, Z) : x \mapsto \varphi(x)$ is continuous: for any compact $K \subset Y$ and open $U \subset Z$, $\tilde{\varphi}^{-1}((K, U)) = \{x \in X | \varphi(x \times K) \subset U\}$, which is compact.

Conversely, for any $\psi : X \mapsto \text{Map}(Y, Z)$, let $\tilde{\psi} : X \times Y \to Z$ be given by $\tilde{\psi}(x, y) = \psi(x, y)$. Then $\tilde{\psi}$ is continuous: For any open $U \subset Z$, if $(x, y) \in \tilde{\psi}^{-1}(U)$, then $\psi(x)y \subset U$. Because $\psi(x)$ is continuous, there is a nbhd $W$ of $y \in Y$ that $\varphi(x)(W) \subset U$. Then because $Y$ is locally compact Hausdorff, there is a precompact nbhd $V$ of $y \in Y$ that $y \in V \subset \overline{V} \subset W$. Then $\psi^{-1}(\overline{V}, U))$ is a nbhd of $x \in X$, and $(x, y) \in (\psi^{-1}(\overline{V}, U), V) \subset \psi^{-1}(U)$, thus $\tilde{\psi}^{-1}(U)$ is open, and $\tilde{\psi}$ is continuous.

Now let $F : \varphi \mapsto \tilde{\varphi}$ and $G : \psi \mapsto \tilde{\psi}$, then we show $F, G$ are both continuous: We use (IX.1.3.5), $F$ is continuous because $F((K \times L, U)) \subset (K, (L, U))$. $G$ is continuous because for any $J$ compact in $X \times Y$ and $U$ open in $Z$, let $J_1 = \pi_1(J)$ and $J_2 = \pi_2(J)$, then $(J_1 \times J_2, U) \subset (J, U)$, so $G((J_1, (J_2, U))) \subset (J, U)$.

Cor. (IX.1.3.7). If $Y$ is locally compact and Hausdorff, $A \subset X, B \subset Y, C \subset Z$, then

$$\text{Map}(X \times Y, X \times B \cup A \times Y; Z, C) \cong \text{Map}(X, A; \text{Map}(Y, B; Z, C))$$

as sets, and if moreover $X$ is Hausdorff, then it is a homeomorphism.

Prop. (IX.1.3.8) (Composition is Continuous). If $Y$ is locally compact Hausdorff, then the composition map

$$\text{Map}(Y, Z) \times \text{Map}(X, Y) \to \text{Map}(X, Z)$$

is continuous.

Proof: For any compact $K \subset X$, open $U \subset Y$ and $g \circ f \in (K, U)$, $f(K) \subset g^{-1}(U) \subset Y$. Because $Y$ is locally compact, there is an precompact open subset $V$ that $f(K) \subset V \subset \overline{V} \subset g^{-1}(U)$, then $(\overline{V}, U) \times (K, V)$ maps into $(K, U)$.

Def. (IX.1.3.9) (Admissible Topology). For continuous spaces $X, Y$, let $M(X, Y)$ be the set of continuous functions from $X$ to $Y$. Then an admissible topology on $M(X, Y)$ is a topology that makes the evaluation map $e : M(X, Y) \times X \to Y$ continuous.

Prop. (IX.1.3.10). The compact-open topology on $M(X, Y)$ is coarser than admissible topology on it (IX.1.3.9). And if $X$ is locally compact Hausdorff, then it is admissible.

Proof: It suffices to show any $(K, U)$ is open in $M(X, Y)$. Let $f \in (K, U)$, then for any $x \in K$, $e(f, x) \in U$, thus there is a nbhd $U_x$ of $f$ and a nbhd $W_x$ of $x$ that $e(U_x \times W_x) \subset U$. Now because $K$ is compact, we can choose a nbhd $V$ of $f$ that $e(V \times K) \subset U$. Thus $f \in V \subset (K, U)$, and $(K, U)$ is open in $M(X, Y)$. If $Y$ is locally compact Hausdorff, then $\text{Map}(Y, Z) \times Y \to Z$ continuous by applying (IX.1.3.8) with $X = \text{pt}$.
Construction of Spaces

**Def. (IX.1.3.11) (Pathspace).** Let \( f : A \to B \) be an arbitrary map, let \( E_f \) be the subspace of \( A \times \text{Map}(I, B) \) consisting of pairs \((a, \gamma)\) that \( \gamma(0) = a \).

The fibers of \( E_f \to B \) are called **homotopy fibers** of \( f \).

**Def. (IX.1.3.12) (**n-Loop Space**).** The \( n \)-loop space of \( X \) is defined to be \( \Omega^n(X, x_0) = M(I^n, \partial I^n; X, x_0) \). Then by (IX.1.3.7) we have

\[
\Omega(\Omega^n(X, x_0), \tilde{x}_0) \cong \Omega^{n+1}(X, x_0).
\]

**Prop. (IX.1.3.13) (Loop space is a Homotopy Fiber).** Let \( f : x_0 \to B \) be a point, then \( E_f \) is the path space \( PB \) of all paths starting from \( x_0 \), and the homotopy fiber over \( x_0 \) is just the loop space \( \Omega(X, x_0) \). \( PB \) is contractible, with the contraction given by \( H : PB \times I \to PB : \gamma t = \gamma(tx) \).

4 Profinite Space

**Def. (IX.1.4.1) (Profinite Space).** A space is called **profinite** if it is a cofiltered limit of discrete topological spaces.

A profinite space is the same thing as a totally disconnected, compact Hausdorff topological space. Thus a closed subspace of a profinite space is profinite.

**Proof:** The profinite spaces are clearly totally disconnected, compact Hausdorff (by Tychonoff).

Conversely, if it is totally disconnected and compact Hausdorff, let \( I \) be the set of clopen decompositions \( X = \bigsqcup I U_i \) of \( X \), then for each \( I \subset I \), there is a map \( X \to I \), and there is a partial order on the decompositions of \( X \). We show that the map \( X \to \lim_{I \subset I} I \) is a homeomorphism. It is injective by (IX.1.1.21)(IX.1.1.19)(IX.1.6.1). It is surjective by compactness of \( X \), and it is clearly open, thus homeomorphism by (IX.1.2.11).

**Cor. (IX.1.4.2).** A cofiltered limit of profinite spaces is profinite.

**Prop. (IX.1.4.3).** Any open covering of a profinite space has a clopen disjoint subcover.

**Proof:** By (IX.1.4.1), we may assume that \( X = \lim_{i \in I} X_i \), where \( X_i \) is finite. Let \( f_i \) be the projection, as the limit is filtered, a fundamental family of nbhds of a point \( (x_i) \ f_i(x_i) \), then for each covering, we may assume it is finite \( X = \bigcup_{i \in I} f_i^{-1}(x_i) \), choose a \( j \) greater than \( i \) for each \( i \), as \( I \) is cofiltered, then \( X = \bigsqcup_{i \in X_j} f_j^{-1}(x) \) satisfies the desired property.

**Prop. (IX.1.4.4).** If \( X \) is quasi-compact and any connected component of \( X \) is the intersection of clopen sets containing it (e.g. \( X \) is normal (IX.1.1.19)), then \( \pi_0(X) \) is a profinite space.

**Proof:** \( \pi_0(X) \) is an image of \( X \), so it is quasi-compact, also it is clearly totally disconnected. To show it is Hausdorff, let \( C, D \) be disjoint connected components of \( X \), then \( C = \cap U_{\alpha} \), where \( U_{\alpha} \) are clopen. Since \( C \cap D = \emptyset \), \( U_{\alpha} \cap D = \emptyset \) for some \( \alpha \). and then the image of \( U_{\alpha} \) separates \( C \) and \( D \) in \( \pi_0(X) \).

**Locally Profinite Space**

**Def. (IX.1.4.5) (Locally Profinite Space).** A space is called **locally profinite** iff it is a totally disconnected, locally compact Hausdorff topological space.
Prop. (IX.1.4.6). A locally closed subsets of a locally profinite space is locally profinite. And compact subsets are profinite.

Proof: Closed subsets are clearly locally profinite, for the open subsets, it is also locally compact. □

Cor. (IX.1.4.7). Any open covering of a compact subsets of a locally profinite space has an clopen disjoint subcover, by (IX.1.4.3).

Prop. (IX.1.4.8). The set of all compact open subsets form a basis of the topology of $G$.

Prop. (IX.1.4.9). If $X$ is locally profinite and $K$ is a compact subspace of $X$. Let $K \subset \bigcup U_\alpha$ be an open covering, then there exist f.m. disjoint open compact subsets $V_i \subset X$ that each $V_i \cap K \subset U_\alpha$ for some $\alpha$, and $K \subset \bigcup V_i$.

Proof: Because $K$ is profinite (IX.1.4.6), (IX.1.4.3) shows there is a finite disjoint compact open subcover $W_i$ of $U_\alpha$, then $W_i = V_i \cap K$ for $V_i$ compact, and using compactness of $W_i$, we can assume $V_i$ is compact. Finally to make $V_i$ disjoint, we can let $V_i = V_i \setminus (\bigcup_{k=1}^{i-1} V_k)$. □

5 Real Numbers

Cf.[H-J99]Chap10 or [Mun00].

Topology

Prop. (IX.1.5.1). $\mathbb{R}^n$ satisfies the Heine-Borel property (X.3.1.7), i.e. bounded closed sets are compact.

Proof: It suffices to consider the square metric. If $A$ is bounded, then $A$ is in $[-N,N]^n$ for some number $N$. $[-N,N]^n$ is compact by (IX.1.2.4) and (IX.1.2.5). □

Borel Set

Def. (IX.1.5.2). Let $U$ be a ultrafilter on a set $I$ and $\{a_I\}$ be a bounded sequence of real numbers. Then a number $a$ is called the $U$-limit of $\{a_I\}$ is for every $\varepsilon > 0, \{i \in I | |a_i - a| < \varepsilon\} \in U$.

There is at most one limit, because $\{i \in I | |a_i - a| < \varepsilon\}, \{i \in I | |a_i - b| < \varepsilon\}$ will be disjoint hence cannot both be in $U$.

Prop. (IX.1.5.3) (Generalized Limit). Let $U$ be an ultrafilter on $\mathbb{N}$, then for any bounded sequence of real numbers $\{a_n\}$, $\lim_U a_n$ exists. i.e. There is a functional from $l^\infty$ to $\mathbb{R}$.

And if $\{a_n\}$ has a limit pt $a$ in the usual sense, then $\lim_U a_n = a$ for any non-principal ultrafilter $U$, because any $\{i \in I | |a_i - a| < \varepsilon\}$ is cofinite hence in $U$(XII.1.8.9).

Proof: Let $A_x = \{n | a_n < x\}$. Then $A_x$ is monotone, then we can choose $c = \sup\{x | A_x \notin U\}$(XII.1.3.19). And it is easily verified that $c = \lim_U a_n$. □

Cor. (IX.1.5.4) (Density Measure). There exists a measure $m$ on $\mathbb{N}$ that $m(A) = d(A)$ for each set $A \subset \mathbb{N}$ that has a density $d(A)$.

Proof: Let $U$ be a non-principal ultrafilter on $\mathbb{N}$(XII.1.8.9), let $m(A) = \lim_U A(n)/n$. It is clearly additive and monotone. And it equals the density by (IX.1.5.3) □
6 Separation Axioms

Hausdorff

Hausdorffization

Cf.[the Hausdorff Quotient].

Regular

Completely Regular

Normal (T4)

Prop. (IX.1.6.1). A paracompact Hausdorff space is normal. In particular, a compact Hausdorff space is normal.

Proof: □

Prop. (IX.1.6.2) (Urysohn lemma). Let $X$ be normal, $A$ and $B$ two closed subset of $X$, then there exists a continuous map from $X$ to $[0,1]$ that maps $A$ to 0 and $B$ to 1.

Proof: Use the countability of rational numbers to construct a family of $U_q$ s.t. $p < q \Rightarrow \bar{U}_p \subset U_q$

Then choose $f(x) = \inf\{p \in \mathbb{Q} | x \in U_p\}$, then this $f$ meets the requirement. □

Cor. (IX.1.6.3) (Tietze extension). If $X$ is normal and $Y$ is a closed subspace, then any continuous function $f$ on $Y$ can be extended to a continuous function on $X$.

Proof: □

7 Paracompactness

Def. (IX.1.7.1) (Paracompactness). A space $X$ is called paracompact if any open covering $\mathcal{U}$ has a locally finite open refinement covering.

Prop. (IX.1.7.2) (Characterization of Paracompactness). If $X$ is regular, then TFAE:

1. Each open cover of $X$ has an open locally finite refinement.
2. Each open cover of $X$ has a locally finite refinement.
3. Each open cover of $X$ has a closed locally finite refinement.
4. Each open cover of $X$ is even. i.e. for any cover, there is an open nbhd $V$ of diagonal of $X \times X$ such that $\forall x, V[x] = \{y | (x, y) \in V\}$ refines the cover.
5. Each open cover of $X$ has an open $\sigma$-discrete refinement.
6. Each open cover of $X$ has an open $\sigma$-locally finite refinement.

If this is satisfied, then $X$ is called paracompact.

Proof: 6 $\rightarrow$ 2:Just minus every open set the part of open sets that appeared in families that ordered before it. 2 $+ 4 \rightarrow$ 1:Use the lemma below, we can transform the cover $\mathcal{A}$ into $V[\mathcal{A}] \cap U_A$ which is an open locally finite cover.

Cf.[General Topology Kelley] and [Mun00]P254. □
Lemma (IX.1.7.3). If $X$ satisfies 4, let $U$ be a nbhd of diagonal of $X \times X$, then there exists a symmetric nbhd of diagonal s.t. $V \circ V \subset U$, where $U \circ V = \{(x, z) \mid (x, y) \in U, (y, z) \in V, \exists y\}$.

Proof: \(\forall x \in X\), there is a nbhd s.t. $W[x] \times W[x] \subset U$, this is an open cover, so there is a nbhd $R$ of diagonal s.t. $R[x]$ refines it. Hence $R[x] \times R[x] \subset U$. Let $V = R \cap R^{-1}$, $V \circ V$ is the union of sets $V[x] \times V[x]$, so $V \circ V \subset U$. \(\square\)

Lemma (IX.1.7.4). In the preceding proposition, if $X$ satisfies 4, Let $\mathcal{A}$ be a locally finite(resp. discrete i.e. intersect only one) family of subsets of $X$, then use the last lemma, there is a nbhd $V$ of diagonal of $X \times X$ such that $V[A] = \{y|(x, y) \in V, \exists x \in A\}$ is locally finite(resp. discrete).

Proof: Choose for every pt a nbhd satisfy the property, then it is an open cover. Choose a diagonal nbhd $U$ for the property 4, then choose coordinate symmetric nbhd $V$ of diagonal s.t. $V \circ V \subset U$. If $V[x]$ intersect $V[A]$, then $V \circ V[x]$ intersect $A$. Done. \(\square\)

Prop. (IX.1.7.5). A locally compact second countable Hausdorff space $X$ is paracompact.

Proof: Let $\mathcal{U}$ be a covering of $X$, because $X$ is second countable and locally compact, by(IX.1.2:23), we may assume $\mathcal{U}$ is a countable covering and consisting of precompact subsets. Moreover, we can change $U_n$ to $U'_n = \cup_{i=1}^{n} U_i$, because if $\{B_\alpha\}$ is a locally finite refinement of $\{\cup_{i=1}^{n} U_i\}$, then $\{B_\alpha \cap U_\alpha\}$ is a locally finite refinement of $\{U_i\}$. So $U_n \subset U_{n+1}$, and because $\overline{U}_n$ is compact, we may assume $\overline{U}_n \subset U_{n+1}$.

Now let $K_n = \overline{U}_{n+1} \setminus U_n$ (where $U_0 = \emptyset$), then $K_n$ are all compact, and $W_n = U_{n+1} \setminus \overline{U}_{n-2}$ is an open subset containing $K_n$. So there are f.m. open subsets of $W_n$ that cover $K_n$. There open subsets is then a refinement covering of $U$, and they are locally finite, because any $x \in X$ is contained in $U_n$ for some $n$. \(\square\)

Prop. (IX.1.7.6) (Paracompact Spaces are Normal). A Hausdorff paracompact space $X$ is normal.

Proof: Firstly $X$ is regular: Let $x \in X$ and $B$ a closed subset disjoint from $x$, then because $X$ is Hausdorff, there is a covering of $B$ by open subsets $V_\alpha$ that $x \notin \overline{V_\alpha}$. Now consider the open covering $\{X \setminus B, V_\alpha\}$ of $X$, then there is a locally finite refinement $\{B_\beta\}$. Those $B_\alpha$ that intersect $B$ is then a locally finite covering of $B$. Let $U$ be the union of these open subsets, then $B \subset U$ and $x \notin \overline{U}$, because of the locally finiteness.

To prove $X$ is normal, we do the same but $x$ replaced by a closed subset disjoint from $B$ and use regularity. \(\square\)

Prop. (IX.1.7.7) (Paracompactness for Manifolds). For a connected Hausdorff locally Euclidean space, the condition of paracompact, second countable and a compact exhaustion is equivalent.

Proof: Cf.[Paracompactness and second countable]. \(\square\)

Prop. (IX.1.7.8). A compact Hausdorff space is paracompact.

Lemma (IX.1.7.9) (Shrinking Lemma). Let $X$ be a paracompact Hausdorff space and $\{U_\alpha\}_{\alpha \in I}$ an open covering of $X$, then there is an open covering $\{V_\alpha\}$ of $X$ that $\overline{V}_\alpha \subset U_\alpha$ for any $\alpha$.

Proof: Let $\mathcal{A}$ be the family of open subsets $A$ of $X$ that $\overline{A} \subset U_\alpha$ for some $\alpha$, then because $X$ is normal(IX.1.7.6), $A$ is a covering of $X$. Then we find a locally finite open covering $\mathcal{B}$ of $\mathcal{A}$, and let the covering map be $B_\beta \subset U_{f(\beta)}$, then we can get a covering $V$ of $X$ indexed by $I$ that $V_\alpha = \cup_{f(\beta) = \alpha} B_\beta$. This is also locally finite, and $\overline{V}_\alpha \subset U_\alpha$ by the locally finiteness of $\mathcal{B}$. \(\square\)
Prop. (IX.1.7.10) (Partition of unity). In a paracompact Hausdorff space, given any open cover \( \{U_\alpha\} \), there exists a partition of unity \( \{\rho_\alpha\} \) that \( \text{supp}\rho_\alpha \subset U_\alpha \), and \( \{\text{Supp}(\rho_\alpha)\} \) is locally finite. Moreover, if \( X \) is locally compact, we can assume

Proof: Using shrinking lemma (IX.1.7.9) twice, we can find locally finite open coverings \( \{W_\alpha\}, \{V_\alpha\} \) that \( W_\alpha \subset V_\alpha \subset U_\alpha \). Because \( X \) is normal, we can find functions \( \psi_\alpha \) on \( X \) that \( \psi(W_\alpha) = 1 \), \( \psi_\alpha(X \setminus V_\alpha) = 0 \). Then \( \text{Supp}(\psi) \subset V_\alpha \subset U_\alpha \), so \( \{\text{Supp}(\psi_\alpha)\} \) is locally finite. So we can define \( \Phi(x) = \sum \alpha \psi_\alpha(x) \). \( \Phi(x) > 0 \) for any \( x \) because \( \{W_\alpha\} \) is a covering of \( X \). Finally, we define \( \rho_\alpha = \psi_\alpha / \Phi \), then this is a partition of unity dominated by \( \{U_\alpha\} \). \( \square \)

8 Metric Space

Prop. (IX.1.8.1) (Metric Space is Paracompact). Any metric space is paracompact.

Proof: Cf. [Mun00]P257. \( \square \)

Complete Metric Space

Def. (IX.1.8.2). A set \( E \) in a metric space is called totally bounded iff for every \( \varepsilon > 0 \), there exists a finite set \( F \) that \( E \subset B(F, \varepsilon) \). This definition is compatible with that in the case of a topological vector space when it is metrizable.

Prop. (IX.1.8.3). The closure of a totally bounded set in a metric space is totally bounded.

Proof: For each \( \varepsilon > 0 \), choose a finite set \( F \) that \( E \subset B(F, \varepsilon/2) \), then \( \overline{E} \subset B(F, \varepsilon) \). \( \square \)

Prop. (IX.1.8.4). A totally bounded metric space \( X \) is separable.

Proof: \( \bigcup N_n \) is dense and countable in \( X \), where \( N_n \) is a finite \( 1/n \)-net of \( X \). \( \square \)

Prop. (IX.1.8.5) (Hausdorff). Let \( X \) be a metric space, then:

1. A sequentially compact subset \( M \) is totally bounded and the converse is true if \( X \) is complete.

2. A subset \( M \) is compact iff it is self-sequentially compact iff it is closed and sequentially compact.

3. \( M \) is precompact iff it is sequentially compact.

Proof:

1. If \( M \) is not totally bounded, then for some \( \varepsilon > 0 \), we can choose consecutively a sequence of points \( x_i \) that \( d(x_i, x_j) \geq \varepsilon \), this cannot has a convergent subsequence in \( X \). Conversely, if \( M \) is totally bounded, choose a \( 1/k \)-net for each \( k \), then for any sequence in \( M \), there is a \( y_i \) that some infinite subsequence \( \{x^{(1)}_n\} \subset B(y_1, 1) \), and consecutively find infinite subsequences \( \{x^{(m)}_n\} \subset B(y_k, 1/k) \), then finally choose the diagonal, then it is a Cauchy sequence.

2. If it is compact, given a sequence, if no point is a convergent point, then each point has a nbhd that contains at most one point of the sequence. Then by compactness, there are at most f.m. points, contradiction. A compact set must has the convergent point in itself because it is closed as \( M \) is Hausdorff.

Conversely, if it is self-sequentially compact, then it is totally bounded by 1. so if \( M \) is not compact, then for each \( n \) it has \( 1/n \)-net \( N_n \), then there is at least one \( x_n \) that \( B(x_n, 1/n) \)
cannot by covered by f.m. of the covering. The sequence \( \{x_n\} \) has a subsequence \( \{x_{n_k}\} \) that is convergent to \( x \). But \( x \in M \) is in some open cover, so \( B(x_{n_k}, 1/n_k) \) is contained in some open cover, contradiction.

That closed and sequentially compact is equivalent to self sequentially compact is obvious.

3. If it is precompact, then it is sequentially compact by 2, conversely, if \( x_i \) is a sequence in \( \overline{M} \), then choose \( |y_n - x_n| \leq 1/n \), so some sequence \( y_{n_k} \) is convergent to \( y_0 \in \overline{M} \), so \( x_{n_k} \) also converges to \( y_0 \). So \( \overline{M} \) is self-sequentially compact, so it is compact by 2.

\[ \square \]

Cor. (IX.1.8.6) (Arzela-Ascoli). For \( M \) compact Hausdorff, \( F \subset C(M) \) is a sequentially compact (precompact, by (IX.1.8.5)) subset iff it is uniformly bounded and equicontinuous.

Proof: As \( C(M) \) is complete metric space, sequentially compact is equivalent to totally bounded. If it is totally bounded, then it is clearly uniformly bounded, and for every \( \varepsilon > 0 \), find a \( \varepsilon/3 \)-net for \( F \), which means f.m. functions in \( F \) that any other function is \( \varepsilon/3 \)-close to one of them. So they are equicontinuous.

Conversely, if it is uniformly bounded and equicontinuous, for every \( \varepsilon > 0 \), find a finite covering of \( M \) that for any two points \( x, y \) in one cover of them, \( |f(x) - f(y)| < \varepsilon/3 \) for all \( f \in F \). Then choose for each covering a point \( x_i \), consider \( f : F \to C^n : \varphi \mapsto (\varphi(x_1), \ldots, \varphi(x_n)) \), then the image is bounded, hence precompact by (IX.1.5.1), so it is totally bounded by (IX.1.8.5). So we can choose a \( \varepsilon/3 \)-net \( \varphi_k \) for \( x_i \) simultaneously, and it is by \( \varepsilon/3 \) argument that these \( \varphi_k \) is a \( \varepsilon \)-net for \( F \). \[ \square \]

Prop. (IX.1.8.7) (Fixed point theorem). If \( X \) is a complete metric space and \( f : X \to X \) satisfies \( d(f(x), f(y)) \leq \lambda d(x, y) \) for some \( 0 \leq \lambda < 1 \), then \( f \) has a unique fixed point in \( X \). If \( X \) is moreover compact, then and \( f \) that \( d(f(x), f(y)) < d(x, y) \) will have a unique fixed point.

Proof: \( x + f(x) + f^2(x) + \cdots \) is the fixed point. And uniqueness is easy. For compact case, notice the image \( \operatorname{Im} f^n \) is a descending chain, it must stable to some \( T \). If \( x, y \in Y \) attains the diameter of \( Y \), and let \( x = f(X), y = f(Y) \), where \( X, Y \in T \), then \( d(x, y) < d(X, Y) \leq d(x, y) \), contradiction. \[ \square \]

Prop. (IX.1.8.8) (Dilation Closed). If \( X, Y \) are metric spaces that \( X \) is complete metric space, then if \( f : X \to Y \) is continuous function that is a dilation, i.e. \( d(f(x_1), f(x_2)) \geq d(x_1, x_2) \), then \( f(X) \) is closed.

So a continuous dilation map on a complete metric space is a closed map.

Proof: If \( y \in f(X) \), then because \( Y \) is metric, there are \( x_n \) that \( y = \lim f(x_n) \). Thus \( \{f(x_n)\} \) is Cauchy in \( Y \), and \( \{x_n\} \) is Cauchy too. So there is a \( x = \lim x_n \), and clearly \( f(x) = y \). \[ \square \]

Compact Metric Space

Lemma (IX.1.8.9) (Lebesgue Number Lemma). For any open covering \( U_i \) of a compact metric space \( X \), there exists a \( \delta > 0 \) that any subset \( X \) of diameter smaller than \( \delta \) is contained in some \( U_i \).

Proof: If \( X \) is in the covering \( U_i \), then there is nothing to prove, otherwise, it suffices to assume the covering is a finite covering, let \( C_i = X - U_i \), and let \( f(x) = \frac{1}{n} \sum d(x, C_i) \). Notice \( f(x) > 0 \), because \( x \notin C_i \) for some \( C_i \). Now it is also continuous, so it has a minimal value \( > \delta > 0 \).

Now if \( B \) has diameter smaller than \( \delta \), then if \( x_0 \in B \), \( \delta < f(x_0) < d(x, C_{i_0}) \), where \( d(x, C_{i_0}) \) is the maximal among \( d(x, C_k) \), and \( B \in B(x, \delta) \), thus \( B \in U_{i_0} \). \[ \square \]
Prop. (IX.1.8.10) (Uniform Continuity Theorem). If \( f : X \to Y \) is a continuous map between two metric spaces that \( X \) is compact, then \( f \) is uniformly continuous.

}\[ \text{Proof: } \text{Take an open covering of } Y \text{ with balls } B(y_i, \varepsilon/2) \text{ of diameter } \varepsilon/2, \text{ and consider their inverse image, then choose the lebesgue number } \delta \text{ for this covering (IX.1.8.9), we see that for any } d(x,y) < \delta, d(f(x),f(y)) < \varepsilon. \]

Prop. (IX.1.8.11). If \( f \) is an isometry of a compact metric space \( X \), then it is a bijection thus an homeomorphism.

}\[ \text{Proof: It is clearly injective. If it is not surjective, then choose a } x \notin \text{ Im}(f), \text{ because Im}(f) \text{ is compact hence closed in } X, d(x, \text{ Im}(f)) = \varepsilon > 0. \text{ Now consider the minimal } N \text{ that we can cover } X \text{ with open subsets of diameter smaller than } \varepsilon, \text{ this } N \text{ exists because } X \text{ is compact. Now if } U_i \text{ covers } X, \text{ then the one that contains } x \text{ cannot intersect with Im}(f). \text{ But } f^{-1}(U_i) \text{ is an open cover of } X \text{ with smaller numbers of open subsets, contradiction.} \]

9 Baire Space

Def. (IX.1.9.1). A subset of a topological space \( X \) is called of first category if it is contained in some countable union of closed subsets of \( X \) having no interior point. It is called of second category if it is not of first category. A Baire space is a topological space that any nonempty open subsets of \( X \) is of second category.

Prop. (IX.1.9.2) (Baire Category Theorem). Every complete metric space & locally compact Hausdorff space is a Baire space.

}\[ \text{Proof: Choose consecutively (precompact)open subsets that doesn’t intersect } E_n \text{ to find a limit point.} \]

10 Uniform Space

Def. (IX.1.10.1) (Uniform Spaces).

Def. (IX.1.10.2) (Cauchy Filter in the Topological Group Case). A Cauchy filter is a topological Abelian group is a filter \( F \) that for any nbhd \( U \) of 0, there exists \( E \in F \) that \( x - y \in U \) if \( x, y \in E \).

Def. (IX.1.10.3) (Complete Uniform Spaces). A topological Abelian group is called complete uniform space iff it is Hausdorff, and any Cauchy filter has a limit.

11 Manifold

Def. (IX.1.11.1). A manifold of dimension \( n \) is a Hausdorff topological space that is locally subsets of \( \mathbb{R}^n \) and it is second countable. By (IX.1.7.7), the last condition is equivalent to say it is paracompact.
12 Topological Groups

Def. (IX.1.12.1). A topological group is a group object in the category of groups (II.1.1.46).

Prop. (IX.1.12.2). If $U$ is a nbhd of 1 in a topological group, then there is a nbhd $V$ of 1 that $VV \subset U$.

Proof: Consider the map $G \times G \rightarrow G$ continuous, and it maps $(1, 1)$ to $1 \in U$, so the preimage of $U$ contains a nbhd of $(1, 1)$, thus some $V_1 \times V_2$, then choose $V = V_1 \cap V_2$. □

Prop. (IX.1.12.3). Let $G$ be a connected topological group, then for any nbhd $U$ of $e$, $G = \bigcup_{n=1}^{\infty} (U \cup U^{-1})^n$. In particular, any open subgroup of $G$ equals $G$.

Proof: This is because the RHS is an open subgroup of $G$, so all its cosets in $G$ are also open, so it equals $G$ as $G$ is connected. □

Prop. (IX.1.12.4) (Separating Axioms). For a topological group $G$, the following are equivalent:

- $e$ is a closed pt.
- $G$ is $T_1$.
- $G$ is Hausdorff($T_2$).
- $G$ is regular.
- $G$ is completely regular

Proof: □

Prop. (IX.1.12.5). Hausdorff topological group is completely regular.

Proof: Use a sequence of neighbourhood of identity to construct a uniform metric on $G$. Then set $\phi(x) = \min\{1, 2\sigma(a, x)\}$. Cf.[Abstract Harmonic Analysis Ross §8.4] □

Prop. (IX.1.12.6). For a compact subset $K$ and a nbhd $U$ of $e$ in a topological group, there exists a nbhd $V$ of $e$ that $xVx^{-1} \subset U$ for any $x \in K$.

Proof: For any $x$, there exists a nbhd $W_x$ of $x$ and a nbhd $V_x$ of $e$ that $txt^{-1} \in U$ for any $t \in W_x$ and $y \in V_x$. Let f.m. $W_x$ cover $K$, then $V = \cap V_x$ satisfies the condition. □

Prop. (IX.1.12.7). A compact open nbhd of $e$ in a Hausdorff topological group contains an open subgroup of $G$.

Proof: Cf.[Etale Cohomology Fulei P147] □

Prop. (IX.1.12.8) (Homogenous Space). Let $G$ be a topological group and $H$ a closed subgroup. $G/H$ is the quotient space in the quotient topology (IX.1.1.6), then it is Hausdorff.

Proof: If $x \neq y$, then consider a preimage $xy^{-1} \in G \setminus H$, then we can find some open subset $V$ that $VV \subset G \setminus H$ by (IX.1.12.2), thus $x + V \cap y + V = \emptyset$. Hence $G/H$ is Hausdorff. □
Group Actions

Prop. (IX.1.12.9) (Quotient by Group Action is Open). Let $G \times X \to X$ be a group action, then the quotient map $\pi : X \to X/G$ is open.

Proof: This is because the $\pi^{-1}(\pi(U)) = \bigcup_{g \in G} gU$. □

Def. (IX.1.12.10) (Regular Action). An regular action is an action $\gamma : G \times X \to X$ that satisfies the following equivalent conditions:
- the graph of $\gamma$ in $X \times X$ is closed.
- The diagonal $\Delta_{X/G} \subset X/G \times X/G$ is closed.
- $X/G$ is Hausdorff.

Proof: $2 \iff 3$ is clear, for $1, 2$, notice $X/G \times X/G \equiv (X \times X)/(G \times G)$, and the inverse image of $\Delta$ in $X \times X$ is just the graph of $\gamma$. □

Def. (IX.1.12.11) (Proper Action). A proper action is an action $\gamma : G \times X \to X$ that the graph map $\Gamma : G \times X \to X \times X$ is a proper map.

Prop. (IX.1.12.12) (Proper Action is Regular). A proper action of a group $G$ on a locally compact Hausdorff space $X$ is a regular action.

Proof: This follows from (IX.1.2.12). □

Def. (IX.1.12.13) (Proper Discontinuous Actions). A group action is called proper discontinuous iff any elements $x, y \in H$ there are nbhds $U_x, U_y$ that $\{g \in G | g(U_x) \cap U_y \neq \emptyset\}$ is finite.

Def. (IX.1.12.14) (Covering Space Action). A covering space action is action of a topological group $G$ on a topological space $Y$ is called if for any $y \in Y$, there is a nbhd $U$ that $g(U) \cap U = \emptyset$ if $g \neq 1$.

Prop. (IX.1.12.15) (Characterization of Proper Actions). Let $\gamma : G \times X \to X$ be a group action that $X$ is Hausdorff, then $\gamma$ is a proper action iff any $K \subset X$ compact, the set $G_K = \{g \in G | g(K) \cap K \neq \emptyset\}$ is compact.

Proof: Let $\Gamma : G \times X \to X \times X$ be the graph.
- $1 \iff 2$: $G_K = \pi_1(\Gamma^{-1}(K \times K))$, thus it is compact.
- $2 \iff 1$: Let $L \subset M \times M$ be compact, then $L \subset \pi_1(L) \times \pi_2(L)$, and $L$ is closed. Let $K = \pi_1(L) \cup \pi_2(L)$, then $\Gamma^{-1}(L) \subset \Gamma^{-1}(\pi_1(L) \times \pi_2(L)) \subset G_K \times K$, which is a closed subset of a compact set, so $\Gamma^{-1}(L)$ is compact, and $\Gamma$ is a proper map. □

Prop. (IX.1.12.16). If $G$ is a compact topological group, then any group action $\gamma : G \times X \to X$ on a Hausdorff space $X$ is proper.

Prop. (IX.1.12.17) (Orbit of Proper Maps). Let $\theta$ be a proper action of $G$ on a Hausdorff space $X$, then each orbit map $\theta^{(p)}$ is proper. In particular, if $X$ is locally compact, then the orbits are all closed (IX.1.2.12).

Proof: For any compact subset $K \subset X$, $(\theta^{(p)})^{-1}(K)$ is closed by continuity and is contained in $G_{K \cup \{p\}}$, thus is compact. □
Prop. (IX.1.12.18) (Properly Discontinuous Map is Proper). If $G$ acts properly discontinuously on a topological space $H$, then for any compact subsets $K_1, K_2 \in H$, $\{ g \in G | K_2 \cap g(K_1) \neq \emptyset \}$ is finite. In particular, if $H$ is Hausdorff, then it is a proper action.

Proof: Notice for any two points we can find nbhds that f.m. $g$ intersects these two nbhds, so we can use the compactness to find f.m. pair of nbhds to cover $K_2$, and then use these nbhds to cover $K_1$ and finite the proof. □

Prop. (IX.1.12.19) (Proper Free Action is a Covering Space Action). Let $G \times X \rightarrow X$ be a proper free action on a locally compact Hausdorff space, then it is a covering space action.

Proof: For any $p \in X$, choose a precompact nbhd $U$ of $x$, then $G_U$ is finite, then shrink $U$. □

Prop. (IX.1.12.20) (Continuity). Let a topological group $G$ acts freely and properly on a space $X$. If $G$ and $X/G$ are both connected, then $X$ is connected.

Proof: If $U, V$ are open subsets of $X$ that $U \cap V = \emptyset, U \cup V = X$, then $\pi(U), \pi(V)$ are open by (IX.1.12.9). Now $\pi(U) \cap \pi(V) = \emptyset$, because otherwise there is a $G$-orbit of $X$ intersecting both $U$ and $V$, contradiction the fact $G$ is connected. Then $\pi_1(U) = \emptyset$ of $\pi(V) = \emptyset$, thus $U = \emptyset$ or $V = \emptyset$, and $X$ is connected. □

Def. (IX.1.12.21) (Constructible Action). An action of a topological group $G$ on a topological space $X$ is called constructible if its graph is constructible in $X \times X$ (IX.1.14.9).

Prop. (IX.1.12.22). Let $\gamma : G \times X \rightarrow X$ be an action

- $\gamma$ is constructible iff the diagonal $\Delta_{X/G} \subset X/G \times X/G$ is constructible.

- If $\gamma$ is constructible and $X$ is non-empty, then there is a $G$-invariant open subset $U \subset X$ that $G$ acts regularly.

- For a constructible action, each $G$-orbit is locally closed.

Proof: Cf.[Bernstein-Zelevinsky, P54]. □

Totally Disconnectedness

Prop. (IX.1.12.23). A compact topological group is totally disconnected iff the intersection of all compact open nbhds of $e$ equals $\{e\}$.

Proof: If it is totally disconnected, then $\{1\}$ is closed, so $G$ is Hausdorff (IX.1.12.4), so by (IX.1.12.20), the assertion is true. Conversely, if the intersection of all compact open nbhds of $e$ equals $\{e\}$, then $\{1\}$ is closed because $G$ is a group. □


Proof: Cf.[Etale Cohomology Fulei P147]. □
13 Hausdorff Geometry

Def. (IX.1.13.1). The Hausdorff distance for two subset $Y_1, Y_2 \in X$ is the

$$d^H_X(Y_1, Y_2) = \inf \{ \varepsilon | Y_2 \subset B(Y_1, \varepsilon), Y_1 \subset B(Y_2, \varepsilon) \}.$$ 

The Gromov-Hausdorff metric for two metric space is

$$d^{GH}(X_1, X_2) = \inf \{ d^H_Z(i_1(X_1), i_2(X_2)) \}$$

where $i_1, i_2$ are isometry of $X_1, X_2$ into a metric space $Z$.

This metric makes the set of all compact metric space into a complete Hausdorff space $\mathcal{M}^E\mathcal{T}$.

Def. (IX.1.13.2). A map from $X$ to $Y$ is called a $\varepsilon$ approximation iff $B(f(X), \varepsilon) = Y$ and $|d(x, y) - d(f(x), f(y))| \leq \varepsilon$.

We have: if there is a $\varepsilon$ approximation, then $d^{GH}(X, Y) \leq 3\varepsilon$, and if $d^{GH}(X, Y) \leq \varepsilon$, there is a $3\varepsilon$ approximation.

Prop. (IX.1.13.3). The set of isometries of

14 Spaces from Algebraic Geometry

Noetherian Space

Def. (IX.1.14.1) (Noetherian Spaces). A Noetherian space is a space that any descending chain of closed subsets stabilizes.

Prop. (IX.1.14.2). A Noetherian space is quasi-compact and all subsets of it in the induced topology is Noetherian hence quasi-compact.

Proof: Let $T \subset X$, for a chain of closed subsets $Z_i \cap T$ of $T$, $Z_1, Z_1 \cap Z_2, \ldots$ stabilize in $X$, hence the chain stabilize in $T$. □

Prop. (IX.1.14.3). If $X$ can be covered by f.m. Noetherian subspaces, then $X$ is Noetherian.

Proof: □

Prop. (IX.1.14.4) (Noetherian Space F.M. Irreducible Components). A Noetherian space has only f.m. irreducible component, hence it has only f.m. connected components.

Proof: Let $C$ be the family of closed subset that has infinitely many component, then there is a minimal object, but it is not irreducible, one of the component has infinitely many components and be smaller. □

Quasi-Separated

Def. (IX.1.14.5) (Quasi-Separated). A space $X$ is called quasi-seperated if the diagonal morphism is quasi-compact (IX.1.2.2). If $X$ has a basis consisting of quasi-compact open subsets, then this is equivalent to any intersection of two quasi-compact open subsets is quasi-separated open.
Specialization & Generalization

Def. (IX.1.14.6) (Going Up and Down). A map $f$ of spaces is said to satisfy the going-up property iff specialization lifts along $f$. It is said to satisfy the going-down property iff generalization lifts along $f$.


Proof: If $y \to y'$, $f(x) = y$, consider $f(\{x\})$, it is closed and contains $y$, so it contains $y'$, thus the result. □

Constructible Set

Def. (IX.1.14.8) (Retrocompact Subset). A subset of $X$ is called retrocompact if the inclusion map is quasi-compact (IX.1.2.2).

Def. (IX.1.14.9) (Constructible Subset). A subset of $X$ is called constructible if it is a finite union of sets of the form $U \cap V^c$ where $U, V$ are open and retrocompact in $X$. In the case when $X$ is Noetherian, by (IX.1.14.2), all subsets are retrocompact hence constructible sets are just union of locally closed subsets of $X$.

A set of $X$ is called locally constructible if locally it is constructible. If $X$ is quasi-compact, then a locally constructible set is just a constructible set.


Proof: Cf.[Sta][051H]. □

Prop. (IX.1.14.11) (Constructible and Subsets).

- If $U$ is open in $X$, then for any $E$ constructible in $X$, $E \cap U$ is constructible in $U$.
- If $U$ is retrocompact open and $E$ is constructible in $U$, then $E$ is constructible in $X$.

Proof: Easy. □


Proof: It suffices to prove $U_i \cap V_i^c \cap W$ is quasi-compact for $W$ quasi-compact, but this is because it is a closed subspace of the quasi-compact subspace $U_i \cap W$. □

Cor. (IX.1.14.13). An open subset of $X$ is constructible iff it is retrocompact, a closed subset of $X$ is constructible iff its complement is retrocompact. (IX.1.14.10) used).

Def. (IX.1.14.14) (Constructible topology). The constructible topology $X_{cons}$ on a quasi-compact space $X$ is generated by the $U, U^c$, where $U$ is a quasi-compact open.

Notice that the space is quasi-compact, so the constructible topology is the coarsest topology that every constructible subset of $X$ is both open and closed.

Prop. (IX.1.14.15). Let $X$ be quasi-compact and quasi-separated, then any constructible subset of $X$ is quasi-compact. In particular, if $Y$ is closed in $X$, then $Y$ is constructible iff it is quasi-compact.

Proof: For $Y = \bigcup_{i=1}^n (U_i - V_i)$, with $U_i, V_j$ quasi-compact open in $X$, then $U_i - V_i$ is closed in $U_i$ thus quasi-compact, and then $Y$ is quasi-compact. □

Prop. (IX.1.14.16). Let $E$ be a constructible subset of a space $X$, then $E$ contains some open dense subset of its closure.

Proof: Cf.[Rigid Structure, Jinpeng An, Lemma2.1]. □
Irreducible

Def. (IX.1.14.17) (Irreducible Space). A space is called irreducible iff there are no two nonempty nonintersecting open subsets. Thus an open subset of an irreducible set is dense and irreducible.

Prop. (IX.1.14.18). If $Y$ is irreducible in $X$, then $\overline{Y}$ is also irreducible.

Proof: Any two nonempty open sets of $\overline{Y}$ must intersect $Y$ thus must intersect. □

Jacobson Space

Def. (IX.1.14.19). Let $X$ be a space and $X_0$ the set of closed pts of $X$, then $X$ is called Jacobson iff $Z \cap X_0 = Z$ for every closed subset $Z$ of $X$. This is equivalent to every non-empty locally closed subset of $X$ contains a closed pt.

Thus there is a correspondence between closed subsets of $X_0$ and closed subsets of $X$, so they have the same Krull dimension.

Prop. (IX.1.14.20). Being Jacobson is local. And for an open covering $U_i$ of $X$, $X_0 = \cup U_{i,0}$.

Proof: Firstly, if $X = \cup U_i$ where $U_i$ are Jacobson, $X_0 \cap U_i = U_{i,0}$. One direction is trivial, for the other, let $x$ be closed in $U_i$, then consider $\{x\} \cap U_j$. If $x \notin U_j$, this is empty, if $x \in U_j$, consider $T = \{x\} \cup (U_j - U_i \cap U_j)$, then $T$ is closed in $U_j$, so by hypothesis, closed pts of $U_j$ are dense in $T$, so $x$ must be closed in $U_j$, so $x$ is closed in $X$. Now clearly $X$ is Jacobson.

Conversely, if $X$ is Jacobson, for a closed subset $Z$ of $U_i$, $X_0 \cap Z$ is dense in $Z$, so $X_0 \cap Z$ is dense in $Z$, then clearly $U_i$ is Jacobson. □

Cor. (IX.1.14.21). If $X$ is Jacobson, then any locally constructible sets of $X$ is Jacobson. And its closed pts are closed in $X$.

Proof: By the proposition, we only have to prove for constructible sets. For $T = \cup T_i$ where $T_i$ is locally closed, then a locally closed set in $T$ has a non-empty intersection $T \cap T_i$ which is also locally closed for some $i$.

Hence is has a closed pt in $X$ hence in $T$, so $T$ is Jacobson. The second assertion is implicit in the proof. □

Prop. (IX.1.14.22). If $X$ is Jacobson, then an open set $U$ of $X$ is compact iff $U \cap X_0$ is compact, hence an open set $U$ is retrocompact iff $U \cap X_0$ is retrocompact.

Hence the constructible sets of $X$ correspond to the constructible sets of $X_0$.

And Irreducible closed subsets correspond to irreducible subsets of $X_0$.

Krull Dimension

Def. (IX.1.14.23). The Krull dimension of a topological space is the length of the longest chain of closed irreducible subsets.

The local dimension $\dim_x(X) = \min\{\dim U| x \in U \subset X \text{ open in } X\}$.

Prop. (IX.1.14.24). If $Y \subset X$, then $\dim Y \leq \dim X$, because the closure of any chain of $Y$ is a chain of $X$ by (IX.1.14.18).

For an open covering $\{U_i\}$ of $X$, $\dim X = \sup \dim U_i$, because for any chain of closed irreducible subsets, if $U_i$ intersects the minimal one, then $\dim U_i =$ length of this chain.
Prop. (IX.1.14.25). $\dim X = \sup \dim_x (X)$.

Proof: The right is smaller than the left by (IX.1.14.24), and for any chain $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ of irreducible closed subset of $X$, if I choose a point $x \in Z_0$, then $\dim_x (X) \geq n$. □

Prop. (IX.1.14.26). In case $X = \text{Spec} A$ for a Noetherian ring $A$, $\dim X = \sup \dim A_p$, because $A$ is of finite?

Catenary space

Def. (IX.1.14.27) (Catenary Space). A space $X$ is called catenary iff for any inclusion of irreducible closed subsets of $X$, their codimension is finite and every maximal chain of irreducible closed subsets has the same dimension. This is equivalent to $\text{codim}(X, Y) + \text{codim}(Y, Z) = \text{codim}(X, Z)$.


Sober Spaces

Def. (IX.1.14.29) (Sober Spaces). A space $X$ is called sober if every irreducible closed subset has a unique generic point.

Prop. (IX.1.14.30). A sober space is $T_1$. Conversely, a finite $T_0$ space is sober.

Proof: The first assertion is because if $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$, then $\overline{\{x\}} = \overline{\{y\}}$, and this irreducible closed subset has two generic point, contradiction.

If the space is finite, then for a closed irreducible subset $T = \overline{\{x_1, \ldots, x_n\}}$, $T = \bigcup \overline{\{x_i\}}$, as it is irreducible, $T = \overline{\{x_i\}}$ for some $x_i$, and $i$ is unique as it is $T_1$, so $X$ is sober. □

Prop. (IX.1.14.31) (Soberization). There is a left adjoint $t$ to the forgetful functor from the Sober spaces. $t(X)$ consists of irreducible closed subsets of $X$, and use $t(Y)$ for $Y$ closed as closed subsets. for a map $f : X \to Z$ to a sober space $Z$, the extension maps the generic point of an irreducible $Y$ to the generic point of the closure of $f(Y)$.

Def. (IX.1.14.32) (Zariski Space). A Noetherian Sober space is called a Zariski space.

Dimension Function

The dimension function is usually considered when the space is sober.

Def. (IX.1.14.33) (Dimension Function). On a topological space, we consider the specialization relation, a dimension function $\delta$ on $X$ is one that if $y$ is a specialization of $x$, then $\delta(y) < \delta(x)$, and if it is a direct specialization, then $\delta(y) = \delta(x) - 1$.

Def. (IX.1.14.34) (pro-Zariski Localization). A map of spectral spaces $f : W \to V$ is called a Zariski localization if $W = \bigsqcup_i U_i$ where $U_i \to V$ is a quasi-compact open immersion. A pro-Zariski localization is a cofiltered limit of Zariski localizations of $V$. 
15 Spectral Spaces

References are [Sta]5.23 and [Adic Spaces].

Def. (IX.1.15.1) (Spectral Space). A space is called spectral iff it is quasi-compact, quasi-separated (IX.1.14.5), sober and the quasi-compact opens form a basis for the topology.

A space is called locally spectral iff it has an open covering by spectral spaces.

A morphism \( f : X \to Y \) between locally spectral spaces is called spectral if for any open spectral spaces \( U \subset f^{-1}(V) \), \( f : U \to V \) is quasi-compact.

Prop. (IX.1.15.2) (Connected Components). Let \( X \) be a spectral space, then any connected subset of \( X \) is an intersection of clopen subsets.

Proof: Let \( x \in X \) and \( S \) be the intersection of all clopen subsets of \( X \) containing \( x \), then it suffices to show \( S \) is connected. Suppose \( S = B \bigcup C \) with \( B, C \) closed, then \( B, C \) are compact, thus there exist quasi-compact opens \( U, V \subset X \) that \( B = U \cap S, C = V \cap S \). Then \( U \cap V \cap S = \emptyset \). Now \( U \cap V \) is quasi-compact also, so there exists some clopen \( Z_\alpha \) containing \( x \) that \( Z_\alpha \cap U \cap V = \emptyset \). Similarly, there exists some clopen \( Z_\beta \) containing \( x \) that \( Z_\beta \subset U \cup V \). Then \( Z_\gamma = Z_\alpha \cap Z_\beta \) is clopen and contained in \( U \Delta V \), Then both \( Z_\gamma \cap U \) and \( Z_\gamma \cap V \) is clopen, so \( U = \emptyset \) or \( V = \emptyset \).

Cor. (IX.1.15.3). Let \( X \) be a spectral space, then for a subset \( T \) of \( X \), \( T \) is an intersection of clopen subsets of \( X \) iff \( T \) is closed in \( X \) and is a union of connected components of \( X \).

Proof: If \( T \) is an intersection of clopen subsets, then \( T \) is clearly a union of connected components of \( X \). Conversely, if \( T \) is a union of connected components of \( X \), if \( x \not\in T \), let \( C \) be a connected component containing \( x \). Then \( C \) is an intersection of clopen subsets, by (IX.1.15.2). These subsets are closed under finite intersections, so by the compactness of \( T \), there is a clopen subset containing \( T \) but not \( x \), so we are done.

Constructible Topology

Lemma (IX.1.15.4). If \( X \) is a finite \( T_0 \) space, then it is spectral and every subset of \( X \) is constructible.

Proof: Cf. [Adic Space Morel, P26].

Prop. (IX.1.15.5). If \( X \) is a spectral space, then the constructible topology (IX.1.14.14) is Hausdorff, totally disconnected and quasi-compact.

Proof: The space is sober hence \( T_0(IX.1.14.30) \), and then the constructible topology is Hausdorff and totally disconnected.

To show quasi-compactness, it suffices to show that the family \( \mathcal{C} \) of quasi-compact open and complement quasi-compact open subsets has the finite intersection property (IX.1.2.3). Notice elements in \( \mathcal{C} \) are all quasi-compact. Now if there is a family that has the finite intersection property by has intersection 0, by Zorn’s lemma, there is a maximal one of them, \( \mathcal{B} \). Now let \( Z \) be the intersection of all the closed subsets in \( \mathcal{B} \), then it is non-empty as \( X \) is quasi-compact. And we claim \( Z \) is irreducible: otherwise \( Z = Z_1 \cup Z_2 \), thus there are quasi-compact open sets \( U_1, U_2 \) that \( U_1 \cap Z_1 \neq \emptyset, U_1 \cap Z_2 = \emptyset, U_2 \cap Z_2 \neq \emptyset, U_2 \cap Z_1 = \emptyset \). Then let \( B_i = X - U_i \), then \( B_1, B_2 \) cannot by added to \( \mathcal{B} \) by maximality, so there is a finite intersection \( T_1, T_2 \) that \( B_i \cap T_i = 0 \). But then \( Z \cap T_1 \cap T_2 = 0 \), but \( Z \) is an intersection of closed subsets, thus some finite intersection of closed subsets in \( \mathcal{B} \) will \( \cap T_1 \cap T_2 = 0 \), contradiction.
So now $Z$ is irreducible, but then for every element $B \in \mathcal{B}$, $Z \cap B$ contains the generic point of $Z$, thus the intersection of $B$ is not empty, contradiction. □

**Cor. (IX.1.15.6).** Let $X$ be a spectral space, then

- The constructible topology is finer than the original topology.
- A subset $X$ is constructible iff it is clopen in the constructible topology of $X$.
- If $U$ is open in $X$, then the constructible topology induces the constructible topology on $U$.

**Proof:**

1: Every open subset of $X$ is a union of its quasi-compact open subsets, so it is open in the constructible topology.

2: Clearly constructible subset is clopen in the constructible topology. Conversely, if $Y$ is clopen, then $Y$ is a union of constructible subsets, but also it is quasi-compact, so it is a finite union of constructible subsets, thus constructible.


**Prop. (IX.1.15.7).** If $E \subset X$ is closed in the constructible topology, then it is a spectral space with the induced topology, and the inclusion map is spectral.

**Proof:** Cf. [Mor19]P34. □

**Prop. (IX.1.15.8).** For a set $E$ closed in the constructible topology in a spectral space,

- every point of $E$ is a specialization of elements in $E$. Thus if $E$ is stable under specialization, then it is closed.

- If $E$ is also open in the constructible topology and stable under generalization, then it is open.

**Proof:** Cf. [Sta]0903. □

**Prop. (IX.1.15.9).** For a map between spectral spaces $f : X \to Y$, the following are equivalent:

- $f$ is spectral.
- $f$ is quasi-compact.
- $f : X_{cons} \to Y_{cons}$ is continuous.

And if this is true, then $f : X_{cons} \to Y_{cons}$ is proper.

**Proof:** $1 \to 2 \to 3$ is trivial, an open subset of $X$ is quasi-compact iff it is clopen in the constructible topology (IX.1.15.6), so $3 \to 2$. For $2 \to 1$, notice that if $U \subset f^{-1}(V)$ are open spectrals, and $W \subset U$ is quasi-compact open, then $f^{-1}W \cap U$ is quasi-compact open, because $X$ is quasi-separated.

Finally, $f$ is proper because $X_{cons}, Y_{cons}$ is compact Hausdorff (IX.1.15.5), then use (IX.1.2.10). □

**Criterion of Spectralness**

**Lemma (IX.1.15.10).** Let $X$ be a quasi-compact $T_0$ space that is has a subbasis consisting of quasi-compact open subsets that is stable under finite intersections. Let $X'$ be the topology generated by the quasi-compact open subsets and their complements, then the following are equivalent:

- $X$ is spectral.
- $X'$ is compact Hausdorff, and its topology has a basis consisting of open and closed subsets.
• $X'$ is quasi-compact.

Proof: Cf.[Adic Space Morel P30]. □

Prop. (IX.1.15.11) (Hochster’s Criterion of Spectrality). Let $X' = (X_0, \mathcal{T}')$ be a quasi-compact topological space, and let $\mathcal{U}$ be the family of clopen subsets of $\mathcal{T}'$, let $\mathcal{T}$ be the topology generated by $\mathcal{U}$, let $X = (X_0, \mathcal{T})$.

Then if $X$ is $T_0$, then it is spectral, and every element of $\mathcal{U}$ is quasi-compact open in $X$, and $X' = X_{cons}$.

Proof: Cf.[Adic Space Morel, P29]. □

Prop. (IX.1.15.12) (Spectral and Inverse Limit). A space is spectral iff it is a direct limit of finite sober(finite $T_1$) spaces.

Proof: Cf.[Sta]09XX. □

Prop. (IX.1.15.13) (Spec and Spectral Space). A spectral space is exactly the underlying space of spectrum of some ring.

Proof: The spectrum of a ring is qc: if $\cup D(f_i) = \text{Spec } A$, then $(f_i) = (1)$, so f.m. of them generate (1). And similarly it is quasi-separable and $D(f) = \text{Spec } A_f$ is quasi-compact. For the other direction, Cf.[M. Hochster. Prime ideal structure in commutative rings, Thm6]?

Cor. (IX.1.15.14). Every quasi-compact irreducible scheme is homeomorphic to an affine scheme.

Cor. (IX.1.15.15) (Characterization of Spectral Spaces). The following are equivalent for a topological space $T$:

1. $T \cong \text{Spec } R$ for some ring $R$.
2. $T \cong \lim_{\leftarrow} \{T_i\}$ where $\{T_i\}$ is an inverse system of finite $T_0$ spaces.
3. $T$ is spectral.

Proof: This follows from(IX.1.15.13) and(IX.1.15.12). □

w-localness

Def. (IX.1.15.16) (w-Local). A spectral space is called w-local if the set of closed pts of $X$ is closed and any point of $X$ specializes to a unique closed pt. A morphism of w-local spaces are called w-local if it is spectral and maps closed pts to closed pts.

Prop. (IX.1.15.17). If $X$ is w-local and $Y \subset X$ is a closed subset, then $Y$ is also w-local.

Proof: $Y$ is spectral by(IX.1.15.7). $Y_0$ is closed because $Y_0 = Y \cap X_0$. And the second assertion is also trivial. □

Prop. (IX.1.15.18). Let $X$ be a spectral space and $T$ profinite, then $Y = X \times_{\pi_0(X)} T$ is also spectral and $T = \pi_0(Y)$. If moreover $X$ is $w$-local, then $Y$ is also $w$-local and $Y \to X$ is $w$-local.

Proof: Cf.[Sta]096C. □
Def. (IX.1.15.19) (Localization Along a Closed Set). Given a closed set $Z$ of a spectral set $X$, the pro-open subset of $X$ consisting of all points that specializes to a point of $Z$ is called the localization of $X$ along $Z$. And $X$ is called local along $Z$ if $X_0 \subset Z$.

Prop. (IX.1.15.20). A spectral space that is local along a closed w-local subset $Z \subset X$ with $\pi_0(Z) \cong \pi_0(X)$, is also w-local.

Proof: $X_0 = Z_0$ is clearly closed, and if a pt $x$ of $X$ specializes to two closed pts of $Z$, then the $\pi_0$ map is not injective, contradiction. $\square$
IX.2 Geometric Analysis

Basic references are [Lee13], [Differential Topology Pollack](Good) and [Geometric Analysis Jost].

All manifolds in this section is assumed to be smooth over $\mathbb{R}$ or $\mathbb{C}$. Where the complex case is just the complex analytic manifold.

1 Smooth Manifolds

Prop. (IX.2.1.1) (Rank Theorem). Let $F : M \to N$ be a smooth map between manifolds of dimensions $m$ and $n$ with constant rank $r$. Then for any $p \in M$, there exists smooth charts centered at $p, F(p)$ that the coordinate representation of $F$ is

$$F(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) \mapsto (x^1, \ldots, x^r, 0, \ldots, 0).$$


Def. (IX.2.1.2) (Immersion). A smooth immersion of manifolds $f : M \to N$ is a smooth map that the differential is injective at every point.

A smooth submersion of manifolds $f : M \to N$ is a smooth map that the differential is surjective at every point.

Def. (IX.2.1.3) (Local Diffeomorphism). A local diffeomorphism $f : M \to N$ is a smooth map that for any $p \in M$, there exists an open subset $U$ that $U \to f(U)$ is a diffeomorphism.

Prop. (IX.2.1.4) (Local Section Theorem). Let $F : M \to N$ be a smooth map between smooth manifolds, then $\pi$ is a smooth submersion iff each point of $M$ is in the image of a local section of $F$.

Proof: If each point of $M$ is in the image of a local section of $F$, the differential is surjective at every point. Conversely, use the rank theorem(IX.2.1.1).

Prop. (IX.2.1.5). A smooth submersion $F : M \to N$ is an open map, and a surjective smooth submersion is a quotient map.

Proof: Let $W \subset M$, and $q = \pi(p)$ where $p \in W$, then there is a local section $\sigma : U \to M$ that $\sigma(q) = p$(IX.2.1.4), thus $\sigma^{-1}(W)$ is open in $N$. But for any $y \in \sigma^{-1}(W)$, $y = \pi(\sigma(y)) \subset \pi(W)$. So $p \in \sigma^{-1}(W) \subset \pi(U)$, which means $\pi(W)$ is open, and $\pi$ is an open map. t The last assertion follows as any open surjective map is a quotient map(IX.1.1.7).

Prop. (IX.2.1.6) (Characteristic Property of Surjective Smooth Submersions). Let $\pi : M \to N$ be a smooth submersion of manifolds, $P$ another smooth manifold, then

- Any map $F : N \to P$ is smooth iff $F \circ \pi$ is smooth.
- Any smooth map $\tilde{F} : M \to P$ that is constant on the fibers of $\pi$ induces a smooth map $F : N \to P$ that $\tilde{F} = F \circ \pi$.

Proof: 1: Use local section theorem(IX.2.1.4).

2: There is a constant map $F : N \to P$ that $\tilde{F} = F \circ \pi$ by(IX.2.1.5) and universal property of quotient maps, and it is smooth by item1.
Def. (IX.2.1.7) (Smooth Covering Space). A smooth covering space of a smooth manifold $X$ is a space $\tilde{X}$ together with a smooth map $\pi : \tilde{X} \rightarrow X$ that there is a covering $U_\alpha$ of $X$ that for each $\alpha$, $\pi^{-1}(U_\alpha)$ is a disjoint union of open subsets of $\tilde{X}$, each of which is mapped diffeomorphically onto $U_\alpha$.

Prop. (IX.2.1.8) (Proper Free Action). Let $\pi : E \rightarrow M$ be a smooth covering map, then with the discrete topology, $\text{Aut}_\pi(E)$ is a discrete Lie group acting smoothly, freely and properly on $E$.

Conversely, suppose $M$ is a smooth manifold and $\Gamma$ is discrete group acting smoothly, freely and properly on a manifold, then the quotient space $M/\Gamma$ is a topological manifold, and it has a unique smooth structure that the quotient map $\pi : M \rightarrow M/\Gamma$ is a smooth normal covering map.

Proof: If $\pi : E \rightarrow M$ is a smooth covering map, then the action is continuously, freely and properly by (IX.6.1.18). Smoothness can be seen by applying (IX.2.1.6). $\text{Aut}_\pi(E)$ is a Lie group because it is countable: Let $q \in M$, and $U$ an evenly covered nbhd of $q$, then $\pi^{-1}(U)$ is a union of open subsets each containing one element of $\pi^{-1}(q)$. so $|\pi^{-1}(q)|$ is countable, and because $\text{Aut}_\pi(E)$ acts freely, it is also countable.

Because this action is a covering map action by (IX.1.12.19), (IX.6.1.21) shows this is a normal covering map. The quotient space is locally Euclidean, and also Hausdorff by (IX.1.12.12)(IX.1.12.10), so it is a topological manifold. The smooth structure is clear. □

Def. (IX.2.1.9) (Smooth Embedding). A smooth embedding of manifolds $f : M \rightarrow N$ is a smooth immersion that is also a homeomorphism onto its image.

Prop. (IX.2.1.10) (Global Rank Theorem). Let $F : M \rightarrow N$ be a smooth map of manifolds of constant rank, then:

- if it is an injection, then it is a submersion.
- if it is an surjection, then it is a submersion.
- if it is a bijection, then it is an diffeomorphism.

Proof: Cf.[Lee Smooth Manifold P83]. □

Prop. (IX.2.1.11) (Local Embedding Theorem). If $F : M \rightarrow N$ is a smooth morphism of manifolds, then it is a smooth immersion if it is locally a smooth embedding on the source.

Proof: Let $p \in M$, then there exists a nbhd $U_1$ of $p \in M$ that $F$ is injective. Now choose another precompact nbhd $U$ of $p$ that $\overline{U} \subset V$, then $F|_U$ is an injective map with compact domain, so it is a topological embedding by (IX.1.2.11). Thus $F|_U$ is a smooth embedding. □

Submanifolds

Def. (IX.2.1.12) (Submanifolds). For a smooth manifold $M$, an embedded submanifold $S \subset M$ is a subset $S$ that is a manifold in the induced topology together with a smooth structure that the inclusion is a smooth embedding of manifolds (IX.2.1.9).

An immersed submanifold $S \subset M$ is a subset endowed with a topology and a smooth manifold structure that the inclusion is a smooth immersion (IX.2.1.2).

An weakly embedded submanifold $S \subset M$ is an immersed submanifold that any smooth map $F : N \rightarrow M$ from some space $N$ that has image in $S$ is smooth as a map from $N$ to $S$. 
Remark (IX.2.1.13). Examples of immersed submanifolds that is not an embedded submanifolds are the 8-figure and the dense curve in a torus. However, an immerse submanifold is locally embedded on the source, by (IX.2.1.11).

Def. (IX.2.1.14) (Slice Charts). Let $U \subset \mathbb{R}^n$, then a $k$-slice of $U$ is the set $S = \{(x^1, \ldots, x^n) \in U | x^{k+1} = \ldots = x^n = 0\}$.

Let $M$ be a manifold, a slice chart of a subset $S \subset M$ is a smooth chart $(U, \varphi)$ that $\varphi(U \cap S)$ is a $k$-slice of $\varphi(U)$ for some $k$.

Prop. (IX.2.1.15) (Local Slice Criterion for Embedded Submanifolds). Let $M$ be a smooth $n$-manifold and $S$ an embedded $k$-submanifold, then each point of $S$ is contained in the domain of a slice chart. Conversely, if $S \subset M$ is a subset that each point of $S$ is contained in the domain of a slice chart, then the induced topology makes $S$ a topological manifold, and there is a smooth structure on $S$ that makes it an embedded submanifold of $M$.

Proof: Suppose $S$ is an embedded submanifold, then the rank theorem (IX.2.1.1) shows there exists coordinates that the image of $i(S)$ is contained in a $k$-slice of $M$. Shrinking the open subset a little bit, we get a slice chart of $S$.

Conversely, if each point of $S$ is contained in the domain of a slice chart, then we can use these smooth charts to get an atlas for $S$, which makes $S$ an embedded topological submanifold of $M$. The transition maps are also smooth because they are restrictions of the corresponding transition map of $M$, so $S$ is an embedded submanifold of $M$. □

Prop. (IX.2.1.16). If $M$ is a compact manifold, then any injective immersion $f : M \hookrightarrow M$ is an embedding of submanifolds.

Proof: The topology of $M$ is equivalent to the induced topology by (IX.1.2.10). □

Prop. (IX.2.1.17) (Immersed Submanifolds are Locally Embedded). If $S \subset M$ is an immersed submanifold, then for any $p \in S$, there is a nbhd $U$ of $p \in S$ that $U \subset M$ is an embedded submanifold.

Proof: This follows immediately from (IX.2.1.11). Notice that the topology on $U$ must be the induced topology, because it is a smooth embedding thus a homeomorphism onto its image (IX.2.1.9). □

Lemma (IX.2.1.18). If $i : S \hookrightarrow M$ is an immersed submanifold, if $F : N \to M$ is a smooth morphism of manifolds that has image in $S$, if $F : N \to S$ is continuous, then $N \to S$ is smooth.

Proof: Let $p \in N$ mapping to $q = F(p)$. By (IX.2.1.17), there is a nbhd $V$ of $q \in S$ that $i|_V$ is an smooth embedding. Thus there is a slice chart (IX.2.1.15) $(W, \psi)$ of $M$ that $(V_0, \tilde{\psi})$ is a smooth chart for $V$, where $V_0 = W \cap V$ and $\tilde{\psi} = \pi \circ \psi$, where $\pi$ is the projection $\mathbb{R}^n \to \mathbb{R}^k$ onto the first $k$ coordinates, and also a smooth chart for $S$.

Let $U = F^{-1}(V_0)$ be open in $F$, then there is a smooth chart of $N$ contained in $U$. Now the coordinate representation of $F$ in the slice chart of $S$ is just the representation of $F : N \to M$ composed with the projection $\pi$, so it is smooth. □

Remark (IX.2.1.19). $N \to S$ being continuous is a necessary condition, otherwise consider the figure 8.
Prop. (IX.2.1.20) (Restricting Codomain of Smooth Morphism). If $S$ is an embedded submanifold in $M$, if if $N \to M$ is a smooth that has image in $S$, then $N \to S$ is smooth.

Proof: Because in this case, $S$ has the induced topology, so it is easily seen that $N \to S$ is continuous.

Sard’s Theorem

Lemma (IX.2.1.21) (Invariance of Measure Zero Sets). If $S \in \mathbb{R}^n$ has measure zero, then for any smooth map $g : \mathbb{R}^n \to \mathbb{R}^m$, $g(S)$ has measure zero.

Thus the notion of measure zero is definable for arbitrary smooth manifolds.

Lemma (IX.2.1.22). Cf.[Pollack Appendix A].

Def. (IX.2.1.23). For a map of schemes $f : X \to Y$, a point $y \in Y$ is called critical iff $df_x$ is not surjective, for some $x \in f^{-1}(y)$, otherwise it is called a regular value.

Prop. (IX.2.1.24) (Regular Value Theorem). If $y$ is a regular value for a map $f : X \to Y$, then $f^{-1}(Y)$ has a natural submanifold structure.

Proof: □

Prop. (IX.2.1.25) (Stack of Records Theorem). If $y$ is regular value of a map $f : X \to Y$, where $X$ is compact and $\dim X = \dim Y$, then $f$ is a covering map locally on $f^{-1}(U)$ for some nbhd $U$ of $y$.

Proof: □

Prop. (IX.2.1.26) (Sard Theorem). For a map $X \to Y$ of smooth manifolds, the set of critical values is of measure zero $Y$.

Proof: Cf.[Pollack Appendix A]. □

Prop. (IX.2.1.27) (Whitney Embedding Theorem). Any $k$-dimensional manifold $M$ can be embedded into $\mathbb{R}^{2k+1}$.

Proof: Cf.[Pollack P51]. □

Cor. (IX.2.1.28) (Whitney Immersion Theorem). Any smooth manifold $M$ of dimension $k$ can be immersed into $\mathbb{R}^{2k}$.

Proof: □

1-dimensional Smooth Manifold with Boundaries

Prop. (IX.2.1.29). Any smooth manifold of dimension 1 with boundary is isomorphic to $[0, 1]$ or $S^1$.

Proof: Cf.[Pollack Appendix]. □

Cor. (IX.2.1.30). The boundary of any smooth manifold of dimension 1 consists of points of even number.
Simplifications

Prop. (IX.2.1.31). For every vector field $X$ and every point $X(p) \neq 0$, there exists a coordinate nbhd $(x_1, \ldots, x_{n-1}, t)$ such that $X = \partial \over \partial t$.

2 Smooth Vector Bundles

Def. (IX.2.2.1) (Smooth Vector Bundle). A smooth vector bundle over a smooth manifold is a vector bundle over $X$ that the trivialization maps are all smooth.

Def. (IX.2.2.2) (Smooth Fiber Bundle).

Tangent and Cotangent Bundles

Lemma (IX.2.2.3) (Differential in Coordinates). Let $F : U \to V$ be a smooth map where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$, with corresponding coordinates $(x^i)$ and $(y^j)$, then

$$dF_p(\partial \over \partial x^i|_p) = \sum_j \frac{\partial F^j}{\partial y^j}(p) \frac{\partial}{\partial y^j}|_{F(p)}.$$  

Proof: for any smooth function $f$,

$$dF_p(\partial \over \partial x^i|_p)(f) = \partial \over \partial x^i|_p(f \circ F) = \sum_j \frac{\partial f}{\partial y^j}(F(p)) \frac{\partial F^j}{\partial x^i}(p) = \left(\sum_j \frac{\partial F^j}{\partial x^i}(p) \frac{\partial}{\partial y^j}|_{F(p)}\right)(f).$$  

\[ \square \]

Lemma (IX.2.2.4) (Change of Coordinates). Suppose $(U, \varphi), (V, \psi)$ be two smooth charts of a smooth manifold, and the transition function on $U \cap V$ is denoted by

$$\psi \circ \varphi^{-1}(x) = (\tilde{x}^1(x), \ldots, \tilde{x}^n(x)),$$

then(IX.2.2.3) shows

$$\frac{\partial}{\partial x^i}|_p = d(\varphi^{-1})(\varphi(p)) \frac{\partial}{\partial \tilde{x}^i}|_{\varphi(p)} = d(\varphi^{-1})(\varphi(p)) \cdot d(\psi \cdot \varphi^{-1})(\varphi(p)) \frac{\partial}{\partial \tilde{x}^i}|_{\varphi(p)}$$

$$= d(\psi^{-1})(\varphi(p)) \left(\sum_j \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial \tilde{x}^j}|_{\varphi(p)}\right) = \sum_j \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial \tilde{x}^j}|_{\varphi(p)}.$$

Def. (IX.2.2.5) (Tangent Vectors). Let $M$ be a smooth manifolds, then a tangent vector at a point $p$ is a linear maps $v : C^\infty(M) \to \mathbb{R}$ satisfying:

$$v(fg) = f(p)v(g) + g(p)v(f).$$

The set of tangent vectors at $p$ is a vector space, denoted by $T_pM$.

Def. (IX.2.2.6) (Tangent Bundle). Let $M$ be a $n$-dimensional smooth manifold, the tangent bundle of $M$ is defined to be the set $TM = \bigsqcup T_pM$. It has a smooth manifold structure that makes it into a $2n$-dimensional manifold, and the projection $\pi : TM \to M$ is smooth. And it is a $n$-dimensional vector bundle over $M$.
Proof: Let \((U, \varphi)\) be a smooth chart of \(M\), with coordinate functions \(\varphi^1, \ldots, \varphi^n\), then we define a map \(\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}\) by
\[
\tilde{\varphi}(\sum v^i \frac{\partial}{\partial x^i}|_p) = (x^1(p), \ldots, x^n(p), v^1, \ldots, v^n).
\]
Then for two different open subset \(U, V\), the transition map is
\[
\tilde{\psi} \cdot \tilde{\varphi}^{-1}(x^1, \ldots, x^n, v^1, \ldots, v^n) = (\tilde{x}^1(x), \ldots, \tilde{x}^n(x), \sum_j \frac{\partial \tilde{x}^1}{\partial x^j}(x)v^j, \ldots, \sum_j \frac{\partial \tilde{x}^n}{\partial x^j}(x)v^j)
\]
which is clearly smooth. So this defines a smooth vector bundle over \(M\), called the **tangent bundle** of \(M\). □

Def. (IX.2.2.7) (Cotangent Bundle). The **cotangent bundle** \(T^*M\) of a smooth manifold \(M\) is the dual of the tangent bundle \(TM\).

Def. (IX.2.2.8) (Parallelizable manifold). A manifold is called **parallelizable** iff the tangent bundle is trivial.

**Vector Fields**

Def. (IX.2.2.9) (Smooth Vector Field). A **smooth vector field** on a smooth manifold is a smooth global section of the tangent bundle \(TM \to M\).

Prop. (IX.2.2.10) (Check Smoothness). Let \(M\) be a smooth manifold and \(X\) be a section of the vector bundle \(TM \to M\), then \(X\) is a smooth vector field iff for any \(f \in C^\infty(M)\), \(Xf \in C^\infty(M)\).

Proof: If for any \(f \in C^\infty(M)\), \(Xf \in C^\infty(M)\), let \(U\) be a trivializing nbhd of \(M\) with coordinate functions \(x^i\), then near any point \(p \in U\), we can use bump function to extend \(x^i\) to a smooth function on \(M\). Then \(X(x^i) = X^i\) near \(p\), thus the coordinates of \(X\) in this trivialization are all smooth, so \(X\) is smooth near \(p\), thus smooth everywhere.

Conversely, for any \(f \in C^\infty(M)\), in a trivializing nbhd \(U\) of \(M\), \(Xf(x) = (\sum X^i(x) \frac{\partial}{\partial x^i}|_x)(f) = \sum X^i(x) \frac{\partial f}{\partial x^i}(x)\) is smooth. □

Def. (IX.2.2.11) (Pushforward of Vector Fields). Let \(F : M \to N\) be a diffeomorphism, then for any \(X \in \mathfrak{X}(M)\), there exists a \(Y \in \mathfrak{X}(N)\) that \(dF_p(X_p) = Y_{F(p)}\).

Proof: We just define \(Y_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)})\), it suffices to show this a smooth vector field. But \(Y : N \to TN\) is the composition
\[
N \xrightarrow{F^{-1}} M \xrightarrow{X} TM \xrightarrow{TF} TN,
\]
so it is smooth. □


Prop. (IX.2.2.13) (Pushforward of Lie Bracket). Let \(F : M \to N\) be a diffeomorphism and \(X_1, X_2 \in \mathfrak{X}(M)\), then \(F_*[X_1, X_2] = [F_*X_1, F_*X_2]\)(IX.2.2.11).

Proof: For any \(f \in C^\infty(N)\), \(F_*X_i = Y_i\), then
\[
[X_1, X_2](f \circ F) = X_1X_2(f \circ F) - X_2X_1(f \circ F) = X_1((Y_2(f) \circ F) - X_2((Y_1(f) \circ F) = (Y_1Y_2(f) - Y_2Y_1(f)) \circ F,
\]
which means exactly \(F_*[X_1, X_2] = [Y_1, Y_2]\). □
Tensor Fields


3 Differential Forms

Prop. (IX.2.3.1) (Frobenius Theorem). If $X$ is an involutive distribution on a manifold $M$, then there is a unique maximal integration manifold passing through it. Where a distribution is involutive if it is closed under Lie bracket.

Proof: The key to the proof is to prove that involutive is equivalent to integrable, i.e. flat locally as $\{\frac{\partial}{\partial x_i}\}$ for some local coordinate. Cf. [李群讲义 项武义]P226 □

Cor. (IX.2.3.2). $X, Y$ in a Lie algebra commute iff their corresponding vector fields commute.

Interior and Exterior Derivatives

Lie Derivatives

Def. (IX.2.3.3). The Lie bracket of two vector fields $X, Y$ is defined to be $[X, Y](f) = (XY - YX)f$, then if $X = \sum a_i \partial/\partial x_i$, $Y = \sum b_i \partial/\partial x_i$, then $[X, Y] = \sum (X(b_i) - Y(a_i)) \partial/\partial x_i$.

Lemma (IX.2.3.4). $[X, Y] = \frac{\partial}{\partial t} (d(\phi_t) Y)|_{t=0}$.

Proof: For any function $f$, set $g(t, q) = \int_0^1 X f(\phi_{ts}(p)) ds$, and:

$$\lim_{t \to 0} d(\phi_t) Y f(p) = \lim_{t \to 0} \frac{Y f(p) - Y f(\phi_t(p))}{t}$$

$$= \lim_{t \to 0} \frac{Y f(p) - Y f(\phi_t(p))}{t}$$

$$= (XY - YX)f(p)$$

$$= [X, Y] f(p)$$ □

Prop. (IX.2.3.5). $[fu, v] = f[u, v] - df(u)v$.

Proof: Direct from the definition (IX.2.3.3). □

Prop. (IX.2.3.6) (Derivative formula).

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

Proof: □

Prop. (IX.2.3.7) (Cartan’s magic formula).

$$L_X \omega = \iota_X (d\omega) + d(\iota_X \omega)$$

$$\iota([X, Y]) = [L_X, \iota_Y]$$
Proof: Notice that four of them are derivatives (check because $\iota_X (w \wedge v) = \iota_X w \wedge v + (-1)^{|w|} w \wedge \iota_X v$). So by induction, we only has to verify them on dimension 0 and 1. □

Prop. (IX.2.3.8) (Stoke’s theorem).

$$\oint_{\Omega} d\omega = \oint_{\partial \Omega} i^* \omega.$$  

In a 3-dimensional Riemannian manifold, if we set:

$$df = \omega^1_{\text{grad} f}, \quad d\omega^1_A = \omega^2_{\text{curl} A}, \quad d\omega^2_A = (\nabla A) \omega^3,$$

Then:

$$f(y) - f(x) = \int_l \text{grad} f \cdot dl.$$

$$\int_l A \cdot dl = \int_S \text{curl} A \cdot dn.$$

$$\int_U \nabla \cdot F dV = \int_{\partial U} F \cdot ndS.$$

Proof: □

**Hodge Star**

Def. (IX.2.3.9) (Hodge Star Operator). Given a volume-form $\omega$ on a vector space, the Hodge star operator $\star$ is an operator from $\Lambda^k V \rightarrow \Lambda^{n-k} V$ such that:

$$\alpha \wedge (\star \beta) = \langle \alpha, \beta \rangle \omega.$$

On a closed oriented Riemannian manifold, given a volume form $\omega$, the star operator satisfies:

$$\langle \alpha, \beta \rangle = \int_M \langle \alpha, \beta \rangle \omega = \int_M \alpha \wedge \star \beta.$$  

And $\star \star = (-1)^{p(n-p)}$ on $\Omega^p M$.

Def. (IX.2.3.10). For an operator $d$ on $\Omega^* M$, we define the adjoint $d^* = (-1)^{n(p+1)+1} \star d \star$ on $\Omega^p$, which satisfies the adjoint property by calculation:

$$\langle d^* \alpha, \beta \rangle = \langle \alpha, d^* \beta \rangle.$$  

The laplacian $\Delta = d^* d + dd^*$. It can be verified that $\Delta$ commutes with $\star$ and $d$.

4 Differential Topology

**Transversality**

Def. (IX.2.4.1) (Transversality).

Prop. (IX.2.4.2) (Transveral Stable under Pertabations). The property of transversal for a map $f : X \rightarrow Y$ for a compact manifold $X$ to a fixed submanifold $Z$ of $Y$ is stable under smooth deformation.
Proof: We can assume the submanifold is defined by a slice, so the transversality is in fact equivalent to locally submersion in the vertical direction. Thus it is clearly stable under deformation. □

Prop. (IX.2.4.3). If a smooth map \( f : X \to Y \) is transversal to a submanifold \( Z \subset Y \) of codimension \( r \), then the preimage \( f^{-1}(Z) \) is a submanifold of \( X \) of codimension \( r \).

Proof: Cf.[Pollack P28]. □

Cor. (IX.2.4.4). If two submanifolds are transversal at every point is again a submanifold, and the codimension is the sum of them.

Prop. (IX.2.4.5) (Parametric Transversality Theorem). Suppose \( N \) and \( M \) are smooth manifolds, \( X \subset M \) is an embedded submanifold, and \( F_s \) is a smooth family of maps from \( N \) to \( M \). If the map \( F : N \times S \to M \) is transverse to \( X \), then for almost every \( s \), the map \( F_s : N \to M \) is transverse to \( X \).

Proof: Cf.[Smooth Manifold Lee T6.35]. □

Prop. (IX.2.4.6) (Transversality Homotopy Theorem). Suppose \( N \) and \( M \) are smooth manifolds and \( X \subset M \) is an embedded submanifold. Every smooth map \( f : N \to M \) is homotopic to a smooth map \( g : N \to M \) that is transverse to \( X \).

Proof: Embed \( M \) into an \( R^k \) and take a tubular neighbourhood, then we can construct a \( N \times D^k \) transversal to \( M \). Cf.[Smooth Manifold Lee T6.36]. □

Prop. (IX.2.4.7) (Transversality Extension Theorem). Let \( X \) be a manifold with boundary and \( C \subset X \) is a closed subscheme, \( Z \) is a closed submanifold of \( Y \). If \( f : X \to Y \) is a smooth map that is transversal to \( Z \) on \( C \) and transversal to \( Z \) on \( C \cap \partial X \), then there is a map \( g : X \to Y \) that is homotopic to \( f \), and \( g = f \) on a nbhd of \( C \).

Proof: Cf.[Pollack P72]. □

Intersection Numbers Modulo 2

Prop. (IX.2.4.8) (Intersection Number Modulo 2). Let \( X \) be a compact manifold, and \( Z \) is a closed submanifold of \( Y \), where \( \dim X + \dim Z = \dim Y \), then for any smooth map \( f : X \to Y \) transversal to \( Z \), define \( I_2(f, Z) \) as the number of points of \( f^{-1}(Z) \) modulo 2.

Prop. (IX.2.4.9) (Boundary Theorem). If \( X \) is the boundary of a smooth manifold \( W \), \( Z \) is a closed subscheme of \( Y \) that \( \dim X + \dim Z = \dim Y \). If \( g : X \to Y \) is a map of smooth manifolds that can be extended to \( W \to Y \), then \( I_2(g, Z) = 0 \).

Proof: Use extension theorem(IX.2.4.7), (IX.2.4.3) and(IX.2.1.30). □

Cor. (IX.2.4.10). Let \( X \) be a compact manifold, and \( Z \) is a closed submanifold of \( Y \), where \( \dim X + \dim Z = \dim Y \), then for any smooth maps \( f, g : X \to Y \) transversal to \( Z \). If \( f \) is homotopic to \( g \), then \( I_2(f, Z) = I_2(g, Z) \).

Proof: Immediate from boundary theorem(IX.2.4.9). □

Prop. (IX.2.4.11) (Mod 2 Degree of Maps). If \( X, Y \) are manifolds of the same dimension and \( X \) is compact, then \( I_2(f, \{y\}) \) is the same for each \( y \in Y \), called the mod 2 degree of \( f \). This number is 0 for the boundary of a map, by(IX.2.4.9).

Proof: Cf.[Pollack P80]. □
Orientable Intersection Numbers

Prop. (IX.2.4.12) (Preimage Orientation). Let $X, Y$ be orientable and $Z$ is an orientable closed subscheme in $Y$. If $f : X \to Y$ is transversal to $Z$, then the orientation of $Z, Y, Z$ defines canonically an orientation on $f^{-1}(Z)$, called the *preimage orientation* of $f^{-1}(Z)$.

Def. (IX.2.4.13) (Intersection Number). If $X$ is an orientable smooth manifold, $Z$ is an orientable closed subscheme of an orientable manifold $Y$ that $\dim X + \dim Z = \dim Y$. If $g : X \to Y$ is a map of smooth manifolds that is transversal to $Z$, then we defined the $I(g, Z)$ to be the sum of the orientations of $f^{-1}(Z)$.

Lemma (IX.2.4.14) (Boundary Theorem). If $X$ is the boundary of an orientable compact smooth manifold $W$, $Z$ is an orientable closed subscheme of an orientable manifold $Y$ that $\dim X + \dim Z = \dim Y$. If $g : X \to Y$ is a map of smooth manifolds that is transversal to $Z$ and can be extended to $W \to Y$, then $I(g, Z) = 0$.

Proof: The same as the proof of (IX.2.4.9).

Prop. (IX.2.4.15). Homotopic transversal maps always have the same intersection number w.r.t $Z$.

Prop. (IX.2.4.16) (Degree of Maps). If $X, Y$ are orientable manifolds of the same dimension and $X$ is compact, then $I_2(f, \{y\})$ is the same for each $y \in Y$, called the *degree of $f$*. This number is 0 for a boundary map, by (IX.2.4.14).

Proof: The same as that of (IX.2.4.11).

Cor. (IX.2.4.17). The only finite group $G$ that can act freely on $S^{2n}$ is $\mathbb{Z}/2\mathbb{Z}$ or 1.

Proof: Consider the degree map, then it is a homomorphism from $G$ to $\mathbb{Z}$, thus the image is just $\pm 1$. Now it is by Lefschetz fixed point theorem that $\deg(g) = -1$ for $g \neq 1$, thus it is injective to $\pm$.

Prop. (IX.2.4.18) (General Intersection Number). The intersection number can be generalized to the case that $g : Z \to Y$ is an arbitrary map of the complementary dimension, and we can define $I(f, g)$. Then:

- $f, g$ are transversal iff $f \times g$ are transversal to $\Delta_Y$.
- $I(f, g) = (-1)^{\dim Z} (f \times g, \Delta_Y)$.

Proof: This a simple local tangent vector calculation.

Cor. (IX.2.4.19). If $f' \sim f, g' \sim g$, then $I(f, g) = I(f', g')$ if they are definable. This is because $f \times g \sim f' \times g'$.

Prop. (IX.2.4.20). $I(f, g) = (-1)^{\dim X - \dim Z} I(g, f)$. This is obvious from the definition.

Cor. (IX.2.4.21). This shows that the intersection number of a odd-dimensional orientable submanifold of an orientable submanifold with itself is 0. If this fails, then the ambient space is not orientable, for example the Möbius band with the central circle.

Prop. (IX.2.4.22). The Euler character of an orientable compact manifold $Y$ equals the intersection of the diagonals $I(\Delta, \Delta)$. 

Proof: For this, we use the Poincare-Hopf theorem (IX.2.4.24). It is clear that on a triangulation, we can place a source on the center of each face/edge/..., thus producing a smooth vector fields, thus it is clear the sum of their indices equals both the combinatorial Euler character and the defined character.

Cor. (IX.2.4.23). The Euler character of an odd dimensional compact manifold $Y$ is $0$.

Prop. (IX.2.4.24) (Poincare-Hopf Index theorem). In a compact manifold $M$, any vector field $V$ with isolated zeros has sum of its index equal to $\chi(M)$. Where the index of a singularity is the mapping degree of $V$ on a surrounding sphere.

Proof: Should use Euler character defined in (IX.2.4.22), Cf.[Pollack].

5 Flow

Def. (IX.2.5.1) (Integral Curves). Let $V$ be a vector field over a smooth manifold $M$, then an integral curve of $V$ is a smooth curve $\gamma : J \rightarrow M$ that $\gamma'(t) = V_{\gamma(t)}$ for any $t \in J$.

Def. (IX.2.5.2) (Flow). Let $M$ be a manifold, then a flow on $M$ is a continuous map $\theta : D \rightarrow M$, where
- $D \in \mathbb{R} \times M$ is an open subset.
- for any $p \in M$, $D^p = \{ t | (t,p) \in D \}$ is an open interval containing 0.
- When it is defined, $\theta(s, \theta(t,p)) = \theta(s + t, p)$.

If $\theta$ is a flow, we define $\theta_t(p) = \theta^{(p)}(t) = \theta(t, p)$.

If $\theta$ is smooth, then we can define the infinitesimal generator of $\theta$ to be the vector field $V_p = \theta^{(p)}(0)$.

Def. (IX.2.5.3) (Complete Vector Fields). A complete vector field on a smooth manifold is a vector field that generates a global flow.

Prop. (IX.2.5.4). If $\theta : D \rightarrow M$ is a smooth flow, then the infinitesimal generator $V$ of $\theta$ is a smooth vector field, and each $\theta^{(p)}$ is an integral curve of $V$.


Prop. (IX.2.5.5) (Isotopy Extension Theorem). Let $M$ be a manifold and $A$ be a compact subset. Then an isotopy $F : A \times I \rightarrow M$ can be extended to a diffeotopy of $M$.

Proof: Consider $F(A \times I) \subset M \times I$ is a compact set, and $TM \times I \rightarrow M \times I$ is a vector bundle. The time lines generate a section $F(A \times I) \rightarrow TM \times I$, so it guarantees an extension $M \times I \rightarrow TM \times I$, and because manifolds are locally compact, this section can be chosen to be compactly supported, then the flow it generates is a diffeotopy.

6 Distributions and Foliations

7 Spin Structure

Prop. (IX.2.7.1) (Spin Structure Obstruction). For a oriented real bundle, its transformation map can be chosen to be in $SO(n)$, and constitute a Čech Cohomology $H^1(X, SO(n))$, and by exact sequence of

$$0 \rightarrow \pm 1 \rightarrow \text{Spin}(n) \rightarrow SO(n),$$

this can be lifted to a $H^1(X, \text{Spin}(n))$ iff its image $w$ in $H^2(X, \mathbb{Z}/2\mathbb{Z})$ is 0. and then its inverse image will be parametrized by $H^1(X, \mathbb{Z}/2\mathbb{Z})$ (By the non-commutative spectral sequence of Čech).

We have $w = w_2$, the Whitney class. (Just need to reduce to $sk_2 X$ and in this case, check they both equivalent to the bundle can be lift). Cf. [XieYi 几何学专题]. Or we can use the Postnikov system of $BO(n)$ (IX.4.8.2).

Proof: First prove that if $E \oplus R^n$ is spin, then $E$ is spin, and then pull $H^2(X, \mathbb{Z}/2\mathbb{Z})$ into $H^2(\text{sk}_2(X), \mathbb{Z}/2\mathbb{Z})$, this in a injection, and the homology is natural, so we only have to prove this for $\text{sk}_2(X)$. But $E$ on $\text{sk}_2(X)$ can decompose into a $E'$ of dimension on more than 2, and for this, we see $E$ is Spin iff it is the square of another bundle, so $w$ and $w_2$ are the same.

Prop. (IX.2.7.2). For a bundle $E$, the Spin-principal bundle with the Spinor representation (IX.4.8.10) will generate a bundle $S$ called the Spinor bundle. And the Ad action of $\text{Spin}(n)$ on $Cl_{n,0}$ will generate a Clifford bundle $Cl(E)$. The Spin($n$) actions are compatible, so the Clifford bundle can act on the spinor bundle. bundle. The act of the chirality operator on the Spinor bundle will generate two half spinor bundles $S^\pm$. Then $TM$ will maps $S^\pm \rightarrow S^\mp$ for $n$ even, (because of anti-commutative with $\Gamma$).

Prop. (IX.2.7.3) (Spin$^c$-structure). The group Spin$^c$ is the covering space of $SO(n) \times S^1$ ($n \geq 2$) that corresponds to the group of elements mod 0 mod 2 in $\mathbb{Z}_2 \times \mathbb{Z}$, i.e. $\text{Spin}(n) \times S^1 / \{ \pm 1 \}$.

For example, $\text{Spin}^c(4) = \{(A_1, A_2) \in U(2) \times U(2) | \det A_1 = \det A_2\}$, and $\text{Spin}^c(3) = U(2)$.

Then a $SO(n)$ bundle can be lift to be a Spin$^c$-bundle if the line bundle determined by $S^1$ is determine the same $w_2$ as it, i.e. $w_2 = c_1(L) \mod 2$. This is equivalent to $w_2$ is in the image of $H^2(X, \mathbb{Z})$, and this is equivalent to the Bockstein image of it is zero.

Use a variant of Wu’s formula: $w_2(TM)[\alpha] = \alpha \cdot \alpha (\mod 2)$ for $M$ orientable of dimension 4, we have any orientable manifold of dimension 4 has a Spin$^c$-structure. Cf. [XieYi 几何学专题 Homework3].

There is a connection on the Clifford bundle and on the Spinor bundle induced by the Levi-Civita connection of $M$ (IX.3.3.2). This is compatible with the Clifford action. and it is also metric because the connection 1-form is in $\mathfrak{so}(n)$ because the action of $SO(n)$ preserves metric.

8 Young-Mills Equation & Seiberg-Witten Equation

Def. (IX.2.8.1) (Yong-Mills). The Young-Mills functional on connections $A$ on a bundle $E$ on a compact oriented space:

$$YM(A)^2 = ||F_A||^2 = - \int_X \text{tr}(F_A \wedge \ast F_A)$$

it is a critical point when $d_A \ast F_A = 0$ and $d_A F_A = 0$.

Prop. (IX.2.8.2) (2-dim Case). $\ast F \in \Omega^0(su(E))$ is parallel thus its characteristic spaces is orthogonal and a stable under parallel transport. So an irreducible YM $SU(2)$-connection must by flat, thus correspond to irreducible $SU(2)$ representation of $\pi_1(X)$. 
**Prop. (IX.2.8.3) (4-dim Case).** \(** = (-1)^{2g_2} = \text{id} \) on \(\Omega^2(E)\) on \(E\) a \(SU(n)\)-bundle, so \(\Omega^2(E) = \Omega^+ \oplus \Omega^-\). We have

\[
||F_A^-||^2 + ||F_A^+||^2 \geq ||F_A^-||^2 - ||F_A^+||^2 = \int_X \text{tr}(F_A \wedge F_A) = 8\pi^2 c_2(E)
\]

Cf. [Geometric Analysis Jost P143,153]. So it attains minimum at the connection that \(*F_A = \pm F_A\) and \(d_AF_A = 0\).((Anti)self-dual((anti)instanton)) depending on the sign of \(c_2(E)\).

**Prop. (IX.2.8.4) (Anti-Instanton Connection on Complex Line Bundle).** For a \((1)\)-bundle, \(d_AF_A = dF_A\), so \(F_A\) is harmonic, thus \(c_1(L) = \left[\frac{1}{2\pi} F_A\right] \in \mathcal{H}^2(X,\mathbb{Z}) \cap \mathcal{H}^2(X,\mathbb{R})\). In fact, this is equivalent to the existence of a anti-self-dual connection on this bundle.

If this is the case, then we have the ASD-connections module Gauge equivalence is isomorphic to \(H^1(X,\mathbb{R})/H^1(X,\mathbb{Z}) = T^{\text{bi}}(X)\).

**Proof:** Because a gauge is just a \(X \rightarrow S^1\), and its connected component thus equals \([X, S^1] = H^1(X,\mathbb{Z})\) (MacLane space), and its identity is just the map that is homotopic to \(\text{id}\). and \(d(gA) = dA - g^{-1}dg = da - idu\), for \(g = exp(iu)\), so \(\Omega^1/G = H^1(X,\mathbb{R})/H^1(X,\mathbb{Z}) = T^{\text{bi}}(X)\).

**Lemma (IX.2.8.5) (Weizenbock Formula).** On a Riemannian manifold \(M\), the Laplace operator has the form:

\[
\Delta = -\nabla^2_{\xi \epsilon \xi} - \xi \wedge \iota(e_j)R(e_i, e_j)
\]

where \(\nabla^2_{\xi \epsilon \xi} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}\).

\[
\int |\mathcal{D}_A \varphi|^2 = \int |\nabla_A \varphi|^2 + \frac{1}{4}|R|^2 + \frac{1}{2}(F_A^+ \varphi, \varphi).
\]

If \(M\) is a spin manifold, then the Dirac operator \(D\) satisfies:

\[
D^2 = -\nabla^2_{\epsilon \epsilon \epsilon} + \frac{1}{4}R
\]

where \(R\) is the scalar curvature form on \(M\). If \(M\) is a \(Spin^c\) manifold with a \(Spin^c\) connection \(\nabla_A\), then the Dirac operator satisfies

\[
D_A^2 = -\nabla^2_{\epsilon \epsilon \epsilon} + \frac{1}{4}R + \frac{1}{2}F_A
\]

Cf. [Geometric Analysis Jost P143,153].

**Prop. (IX.2.8.6) (Seiberg-Witten).** The Seiberg-Witten equation functional for a unitary connection \(A\) on the determinant bundle of a \(Spin^c\) structure of \(M\) and a section of \(S^+\) is:

\[
SW(\varphi, A) = \int \left( |\nabla_A \varphi|^2 + |F_A^+|^2 + \frac{R}{4}|\varphi|^2 + \frac{1}{8}|\varphi^2| \right) Vol.
\]

\[
= \int \left( |\mathcal{D}_A \varphi|^2 + |F_A^+ - \frac{1}{4}(e_j e_k \varphi, \varphi)e_j \wedge e_k| \right) Vol.
\]

So the Seiberg-Witten equation is the lowest topological possible value of the Seiberg-Witten functional. It writes:

\[
\mathcal{D}_A \varphi = 0, \quad F_A^+ = \frac{1}{4}(e_j e_k \varphi, \varphi)e_j \wedge e_k.
\]

Cf. [Jost Chapter 7].

**Cor. (IX.2.8.7).** If a compact oriented \(Spin^c\) manifold \(M\) has nonnegative scalar curvature, then the only possible solution is \(\varphi = F_A^+ = 0\). (See from the equivalence of forms of Seiberg-Witten functional.)
9 Chern-Weil Theory

Prop. (IX.2.9.1) (Chern-Weil). An Invariant polynomial of the entries of \( M_n(k) \) is one that is invariant under the conjugation action (I.2.2.17).

For any connection on \( E \), the Chern-Weil map \( CW \) from invariant polynomial ring to \( H^*(X) : P \mapsto [P(\Omega)] \) is a ring homomorphism independent on the connection \( A \).

There are relations between \( c_i \) and \( \text{tr}(\Omega^k) \), they can be derived formally by considering diagonal elements.

Proof: To prove \( P(\Omega) \) is closed, notice by (I.2.2.17), it suffice to show \( \text{tr}(\Omega^k) \) is closed. By (IX.3.3.7), \( d\text{tr}(\Omega^k) = \text{tr}(\omega \wedge \Omega^k - \Omega^k \wedge \omega) = 0 \), which is zero because \( \Omega \) is of even dimension.

For the independence of connections, use (IX.4.7.16). For two connection \( \nabla_i, \nabla = t\nabla_0 + (1 - t)\nabla_1 \) (you can smooth it) is a connection on the vector bundle \( \pi^*E \) on \( M \times I \), and the section \( 0 \) and \( 1 \) induces the connection \( \nabla_0 \) and \( \nabla_1 \). Thus \( s_0^* \) and \( s_1^* \) are the same map, thus \( CW_M(p) = s_i^*CW_{M \times I}(p) \) are all the same map. □

Cor. (IX.2.9.2). For a complex line bundle of degree \( r \) over a complex manifold,

\[
\det(1 - \frac{1}{2\pi i}F_A) = 1 + c_1 + \ldots + c_r
\]

gives out the Chern class, because it satisfies the axioms of Chern class (IX.6.5.1). In other words, \( c_k = \text{tr}((-\frac{1}{2\pi i}F_A)^k) \).

For a real line bundle of degree \( r \),

\[
\det(1 - \frac{1}{2\pi i}F_A) = 1 + p_1 + \ldots + p_{\lfloor r/2 \rfloor}
\]

gives out the Pontryagin class, where \( p_k \in H^{4k}(X) \). (Notice the \( \omega \) thus \( \Omega \) can be chosen to be skew-symmetric, thus for odd \( k \) the classes \( \text{tr}(\Omega^k) \in H^{2k}(X) \) vanish).

For an oriented real bundle of degree \( 2r \), the \( \omega \) and thus \( \Omega \) can be chosen to be skew-symmetric and the transformation matrix in \( SO(2r) \), then

\[
Pf(\frac{1}{2\pi} \Omega) \in H^{2r}(X)
\]

is well-defined and closed and gives the Euler class \( e(E) \) (recall \( e(E)^2 = p_r(E) \)). (Use \( Pf^2 = \det \) to get that \( \frac{\partial Pf}{\partial \Omega_{ij}} \) commutes with \( \Omega \), then calculate \( dPf(\Omega) = 0 \)).

Proof: In fact, the construction is natural w.r.t the connection because connection can be pulled back and summed. Then the only task is the normality, which is direct calculation on \( \mathbb{C}P^1 \). □

Cor. (IX.2.9.3).

\[
c_1(E) = c_1(\wedge^{\text{dim}E} E).
\]

Direct from the formula.

Cor. (IX.2.9.4) (Whitney Product Formula).

\[
c(E \oplus F) = c(E)c(F), \quad p(E \oplus F) = p(E)p(F)
\]

Directly form the product connection on \( E \oplus F \).
Prop. (IX.2.9.5) (Chern Character). The Chern character

\[ ch(E) = \text{tr} \exp \left( \frac{i}{2\pi} F_A \right) \]

satisfies \( ch(E \oplus F) = ch(E) + ch(F) \) and \( ch(E \otimes F) = ch(E)ch(F) \) by simple calculation. So it defines a ring homomorphism from \( K(X) \) to \( H^*(X) \).

Prop. (IX.2.9.6) (Chern-Gauss-Bonnet). For a \( 2n \)-dimensional orientable manifold \( M \),

\[ \int_M e(TM) = \chi(M) \]

Prop. (IX.2.9.7). For a vector bundle and a flat connection \( d_A \) on a manifold, i.e. \( d_A^2 = 0 \), we have a deRham like cohomology, and there is a sheaf of flat sections.

\[ H^*(X, A) = H^*(X, E) \]

10 Index Theorems (Atiyah-Singer)

References are [Heat equation and the Index Theorem Atiyah] and [Index Theorem].

Prop. (IX.2.10.1) (Gilkey). For a natural transformation \( \omega \) from the functor \( p : M \to \) the Riemannian structure on \( M \) to the functor \( q : M \to k \)-forms on \( M \), if it is homogenous of weight 0 w.r.t to metric \( g \)(i.e. \( \omega(\lambda^2 g) = \omega(g) \)) and in local coordinates it has the coefficients of \( \omega(g) \) generated by \( g_{ij} \) and \( \det g^{-1} \) and their derivatives, then is is a polynomial of Pontryagin classes of the given dimension. (not only up to homology).

Proof: Cf.[Heat equation and the Index Theorem Atiyah P284].

Prop. (IX.2.10.2) (Gilkey Generalized). For a natural transformation \( \omega \) from the functor \( p : M \to \) Riemannian structures on \( M \) with a Hermitian bundle \( E \) with a Hermitian connection and the functor \( q : M \to k \)-forms on \( M \), if it is homogenous of weight \( (0, 0) \) w.r.t to metric \( g, h \) and the Hermitian structure(i.e. \( \omega(\mu^2 \xi) = \omega(g, \xi) \)) and in local coordinates it has the coefficients of \( \omega(g, \xi) \) generated by \( g_{ij}, h_{ij}, \det h^{-1}, \det g^{-1} \) and \( \Gamma^i_{jk} \) (the connection form) and their derivatives, then is is a polynomial of Pontryagin classes and Chern classes of \( E \) of the given dimension. (not only up to homology).

Proof: Cf.[Heat equation and the Index Theorem Atiyah P290].

Cor. (IX.2.10.3). For a natural transformation \( \omega \) from the functor \( p : M \to \) Hermitian bundle \( E \) on \( M \) with a Hermitian connection and the functor \( q : M \to k \)-forms on \( M \), if it is homogenous of weight 0 w.r.t metric \( h \) and the Hermitian structure(i.e. \( \omega(\mu^2 \xi) = \omega(g, \xi) \)) and in local coordinates it has the form \( \omega(g, \xi) \) generated by \( h_{ij}, \det h^{-1} \) and \( \Gamma^i_{jk} \) (the connection form) and their derivatives, then is is a polynomial of Chern classes of \( E \) of the given dimension. Because when composed with the forgetful functor, it gives a transformation as above. And it is obviously independent of \( g \).

Prop. (IX.2.10.4) (Hodge). For any differential operator \( A \) from a vector bundle \( E \) to a vector bundle \( F \), we form two operators \( AA^* \) and \( A^*A \), then they are both self adjoint elliptic operators, let these corresponding eigenspace be \( \Gamma_\lambda(E) \) and \( \Gamma_\lambda(F) \), then \( A \) and \( A^* \) define an isomorphism between \( \Gamma_\lambda(E) \) and \( \Gamma_\lambda(F) \).
Proof: □

Prop. (IX.2.10.5) (Hirzebruch Signature Formula). On a 4n-dimensional orientable manifold \( M \), the Poincare duality defines a bilinear pairing \( H^{2n}(M) \times H^{2n}(M) \to \mathbb{R} \), its signature \( \sigma(M) \) is given by:

\[
\sigma(M) = \int_M L_n(p_1, \ldots, p_n).
\]

Where \( L_n \) is the degree \( n \) part of the Taylor expansion of \( \prod_{i=1}^{r} \frac{x_i^{1/2}}{\tanh x_i^{1/2}} \) in terms of the symmetric polynomial.

Proof: We consider the operator \( \tau : \alpha \mapsto i^{l+p(p-1)} * \alpha \), \( \tau^2 = 1 \), thus \( \Gamma^* \) is decomposed into two eigenspaces of \( \tau \). We define the signature operator \( A \) as the restriction of \( \Delta = d - \tau d\tau \) to \( \Gamma^+ \). \( \Delta \) anti commutes with \( \tau \) thus maps \( \Omega^+ \) to \( \Omega^- \), then we have \( \text{Ker} A = \text{Ker} \Delta \cap \Omega^+ \), which is the positive harmonic forms \( H^+ \). So

\[
\text{Ind} A = \dim H^+ - \dim H^-.
\]

And we notice the positive and negative harmonic forms neutralize each other unless on the \( 2n \)-forms, so only need to consider them. In fact, if we consider \( 4n + 2 \) manifolds, then \( \tau \) is pure imaginary and the conjugation neutralize even the \( 2n+1 \) forms, so there are no signature.

Now the inner product \( \alpha \to \int \alpha \wedge \ast \alpha \) is positive definite for a real form \( \alpha \), so this index of \( A \) is just the signature of the intersection form defined by cup product. □

Cor. (IX.2.10.6). For a 4n-dimensional \( M \) which is a boundary of a manifold, its signature is 0.

Proof: By Stokes theorem, if \( M \) is a boundary of a manifold, then all its Pontryagin numbers, i.e. \( \int_M \prod x_i^{1/2} \), \( \sum n_i = n \), vanish. □

Prop. (IX.2.10.7) (Generalized Hirzebruch Signature Formula). Let \( M \) be a 2l dimensional smooth manifold and \( E \) be a Hermitian bundle over \( M \), then The index of the generalized signature operator is giving by

\[
\text{Ind} A_{\eta} = 2^l \cdot ch(E)L(p_1, \ldots, p_l).
\]

where \( L(M)(p_i) = \prod \frac{x_i/2}{\tanh x_i/2} \).

Prop. (IX.2.10.8) (Hirzebruch-Riemann-Roch). For a \( n \)-dimensional complex line bundle \( L \) over a compact Kähler manifold \( M \),

\[
\chi(M, L) = \int_M [\text{ch}(E)\text{td}(T^{1,0}M)]_n.
\]

Where \( \chi(M, L) = \sum_{q=0}^{n}(-1)^q \dim H^q(M, E) \), \( \text{ch} \) is the Chern character (IX.2.9.5) and \( \text{td}(T^{1,0}M) \) is the Todd polynomial, i.e. Taylor expansion of \( \prod_{i=1}^{r} \frac{t_i}{1 - e^{-t_i}} \) in terms of the symmetric polynomial, applied to \( c_i(T^{1,0}M) \).

Cor. (IX.2.10.9) (Riemann-Roch). For a \( n \)-dimensional complex vector bundle \( E \) over a Riemann Surface \( M \), let \( \deg E = \int_M c_1(E) \), then

\[
\chi(M, E) = H^0(M, E) - \dim H^1(M, E) = \deg E + \text{rk}(E)(1 - g).
\]

Cf.[Index Theorem P115].
Hodge Theory

Prop. (IX.2.10.10) (Hodge). By (X.8.8.12), if we investigate the Laplace operator $\Delta_d$ on a compact orientable Riemannian manifold, we get that

$$\Omega^i = \mathcal{H}^i \oplus \text{Im } \Delta_d = \mathcal{H}^i \oplus \text{Im } d \oplus \text{Im } d^*.$$  

Thus $H^i$ can be uniquely represented by elements of $\mathcal{H}^i$.

Proof: It suffice to prove $\Delta_d$ is self-adjoint elliptic.

Im $\Delta_d \subset \text{Im } d \oplus \text{Im } d^*$, and the result follows if we show $\mathcal{H}^i, \text{Im } d, \text{Im } d^*$ are orthogonal. In fact, let $\omega$ be harmonic, then $(\omega, d^*\xi) = (d\omega, \xi) = 0$, $(\omega, d\eta) = (d^*\omega, \eta) = 0$, $(d\eta, d^*\xi) = (dd\eta, \xi) = 0$. □

Cor. (IX.2.10.11) (Poincare Duality for deRham Cohomology). If $M$ is a $n$-dimensional oriented Riemannian manifold, then

$$H^p_{dR}(M) \cong H^{n-p}_{dR}(M)$$

Induced by $\ast$, because $\ast\ast = \pm 1$ and $\ast$ commutes with $\Delta_d$ (IX.2.3.10), so it induce an isomorphism $H^p \cong H^{n-p}$.

Moreover, $\ast$ in fact induces a perfect pairing:

$$H^k_{dR}(M) \times H^{n-k}_{dR}(M) \to \mathbb{R}$$

induced by the map

$$\ast : \mathcal{H}^k(M) \times \mathcal{H}^{n-k}(M) \to \mathbb{R} : (\alpha, \beta) \mapsto \int_M \alpha \wedge \ast \beta$$

As $\int_M \alpha \wedge \ast \alpha = ||\alpha||^2 \neq 0$.

Prop. (IX.2.10.12). On a compact complex manifold, the formal adjoint of $\partial$ is $\ast \partial$. (By direct calculation). Also $d^* = (-1)^{n(p+1)+1} \ast d = - \ast d^*$.

Prop. (IX.2.10.13) (Hodge). Given a compact Hermitian complex manifold $(X, J, g)$ and a holomorphic line bundle $E$ over it, there is a Hermitian metric on $A^{p,q}E$, and an operator $\partial$ on it. Then $\mathcal{D}$ has a formal adjoint $\mathcal{D}^*$, and $\Delta_{\mathcal{D}_E}$ can be defined. Let $\mathcal{H}^{p,q}_E$ be the kernel of $\Delta_{\mathcal{D}_E}$ on $A^{p,q}E$, called the $E$-valued $(p,q)$-forms, then there is an orthonormal decomposition

$$A^{p,q}_E = \mathcal{H}^{p,q}_E \oplus \text{Im } \Delta_{\mathcal{D}_E} = \mathcal{H}^{p,q}_E \oplus \text{Im } \partial_E \oplus \text{Im } \partial^*_E$$

And thus $\mathcal{H}^{p,q}(X, E) \cong H^{p,q}_\mathcal{D}(X, E)$.

Proof: It suffice to prove $\Delta_{\mathcal{D}_E}$ is self-adjoint elliptic. The rest is verbatim as the proof of (IX.2.10.10).

Cor. (IX.2.10.14) (Hodge). In case $E = O_X$, $\mathcal{H}^{p,q}(X) \cong H^{p,q}_\mathcal{D}(X)$.

Cor. (IX.2.10.15) (Kodaira-Serre Duality). For a Hermitian line bundle over a compact Hermitian complex manifold $X$, from Hodge theorem and (IX.10.5.2), we get

$$H^p(X, \Omega^q(E)) \cong H^{n-p}(X, \Omega^{n-q}(E^*))$$

induced by $\mathcal{J}_E$ and $\mathcal{J}_{E^*}$.

11 Knots and Links

Prop. (IX.2.11.1) (Linking Number). For two knots $A, B$ in $\mathbb{R}^n$, we can choose a $D \cong D^2$ with boundary $A$, then define their linking number as the intersection number of $D$ with $B$.

This can be extended to higher dimensions.
IX.3  Riemannian Geometry

Basic references are [Riemannian Geometry Do Carmo], [Geometric Analysis Jost] and [Differential Geometry Loring Tu].

1  $\mathbb{R}^3$-Geometry

Different Coordinates

Prop. (IX.3.1.1). In a polar coordinate,

$$g_{11} = 1, g_{12} = 0, g_{22} = \left| \frac{\partial f}{\partial \theta} \right|^2, \quad K = -\frac{(\sqrt{g_{22}})^{\rho\rho}}{\sqrt{g_{22}}}$$

And $\sqrt{g_{22}} \sim \rho$. (Use the formula relating Jacobi Field with curvature)

Moving Frame Method

Prop. (IX.3.1.2) (Theorema Egregium).

$$R_{1212} = K(g_{11}g_{22} - g_{12}^2)$$

Which is a special case of the definition of curvature.

Prop. (IX.3.1.3) (Gauss-Bonnet). Let $M$ be a compact oriented 2-dimensional Riemannian manifold, then

$$\chi(M) = \frac{1}{2\pi} \int_M K \text{ Vol.}$$

Proof: Should be an direct corollary of (IX.2.9.6). □

Topology and Geometry

Prop. (IX.3.1.4). Every compact orientable surface of genus $p > 1$ can be provided with a metric of constant negative curvature.

Remark (IX.3.1.5) (Hilbert Theorem). There exist complete surfaces with $K \leq 0$ in $\mathbb{R}^3$, but the hyperbolic surface cannot be immersed into $\mathbb{R}^3$.

2  Basics

Prop. (IX.3.2.1). If the metric tensor on the tangent space is $g$ in a coordinate, then it is $g^{-1}$ in the cotangent space. (Follows from ??).

3  Connections

Def. (IX.3.3.1) (Affine Connection). An affine connection on a vector bundle $E$ is a map $D : \Gamma(E) \to \Gamma(E) \otimes \Gamma(T^*M)$ that satisfies differential-like properties, it can be written as $D = d + \omega$, with $\omega \in \Omega^1(\text{End}(E))$. 
Prop. (IX.3.3.2) (Transformation Law). In two coordinates $\varphi = e^a$ for $a : U \to GL(r, \mathbb{R})$, $d_A = d + \omega, d + \varphi$, then $\varphi = a^{-1}\omega a + a^{-1}da$.

Moreover, giving any locally compatible $d + \omega, \omega \in \Omega^1(g)$ in the sense above, then for any $G$-associated bundle $E$, where $G$ has lie algebra $g$, there is a connection that locally looks like $d + \omega$,

(IX.3.3.3) (Local Nature of Connection). From the description of connection given above, it’s easy to say if the is a local connection that satisfies these transformation laws, then it generate a global connection. So by partition of unity (IX.1.7.10), connection exists in any vector bundle over a manifold.

Cor. (IX.3.3.4) (Simplification). $d \gamma ^A (s) = g d \gamma ^A (g^{-1} (s))$, So for any connection $d_A$ and any point $x_0$, there is a gauge transformation that makes $d_A = d$ at $x_0$.

Proof: Just need to have $s(x_0) = id, ds(x_0) = -A(x_0)$. This is possible because $A \in \Omega^1(AdE)$ which is the fiber of the frame bundle, use exp.

Prop. (IX.3.3.5) (Induced connections). The connection action $d_A = d + \omega$ on a vector bundle $E$ induces connection on many relevant bundles. the action on dual bundle is by

$$d_A(s^*) = ds^* - \omega^j (s^*) = ds^* - s^* \circ \omega.$$ 

And the connection on End $E$ by

$$d_A(\alpha) = d\alpha + [\omega, \alpha] = [\nabla, \alpha]$$

And they act on $\Omega^*(E)$ by Leibniz rule thus the formula looks the same. (Note that the convention is section write on the left of the differential forms, so for example, $[\omega, \omega] = 2\omega \wedge \omega$).

Proof: Cf.[Jost P110].

Cor. (IX.3.3.6). For a line bundle $L$, for a connection on it with curvature $\Omega$, the induced on the dual line bundle $L^*$ has connection $-\Omega$. (because $\Omega = d\omega$ and $\omega' = -\omega$).

Prop. (IX.3.3.7) (Second Bianchi’s Identity). A affine connection on $E$ looks locally like $d_A = d + \omega$, where $\omega \in \Omega^1(End E)$. And $F_A = d_A \circ d_A \in \Omega^2(End(E))$ satisfies

$$d_A F_A = dF_A + [\omega, F_A] = 0.$$ 

Proof: Notice $dF_A = dd\omega + d(\omega \wedge \omega) = d\omega \wedge \omega - \omega \wedge d\omega$, and $\omega(d\omega + \omega \wedge \omega) - (d\omega + \omega \wedge \omega)\omega = \omega \wedge d\omega - d\omega \wedge \omega$.

Def. (IX.3.3.8) (Christoffel Symbol). The Christoffel symbol of a connection is defined by the equations: $\nabla_{X_i} X_j = \sum_k \Gamma^k_{ij} X_k$.

The geodesic equations is $\frac{D}{dt}(\frac{dx}{dt}) = \ddot{x}_k + \sum_{i,j} \Gamma^k_{ij} \dot{x}_i \dot{x}_j = 0 \forall k$.

Def. (IX.3.3.9). The torsion tensor of a connection $\nabla$ on $TM$ is defined as $T(X,Y) = \nabla_X Y - \nabla_Y X - [X, Y]$. The connection is called torsion-free if $T = 0$. This is equivalent to $\Gamma^k_{ij} = \Gamma^k_{ji}$.

A connection is called metric if it preserves metric. i.e. $\nabla g = 0$. 

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Proof: $T$ is a tensor because it is skew-symmetric, and

$$T(fX, Y) = f \nabla_X Y - f \nabla_Y X - df(Y)X - (f[X, Y] - df(Y))X = fT(X, Y),$$

where (IX.2.3.5) is used.

□

Prop. (IX.3.3.10). If $\nabla$ is a torsion-free connection on $TM$, then its induced connection on $T^*M$ satisfies

$$(da)(v_1, \ldots, v_k) = \sum (-1)^i(D_{v_i}a)(v_1, \ldots, \hat{v}_i, \ldots, v_k).$$

Proof:

Def. (IX.3.3.11) (Curvature Tensor). The curvature of a (affine) connection $d_A$ is $F_A = d_A \circ d_A \in \Omega^2(\text{End}(E))$. The curvature tensor it induced is

$$F_A(Z)(X, Y) = R(X, Y)Z = D_Y D_X Z - D_X D_Y Z + D_{[X, Y]}Z.$$  

In particular, the curvature depends only on the point, and locally $F_A = d\omega + \omega \wedge \omega$

In two coordinates $\tilde{e} = ea$ for $a : U \to GL(r, \mathbb{R}), \tilde{F}_A = a^{-1}F_Aa$. The connection is called flat if $F_A = 0$.

Proof: To verify the equation, check first the left side is pointwise, and the third component of the right side assures it is pointwise, too, thus we can check for a local coordinate vector field $([X_i, X_j] = 0)$, then because $\nabla s = \sum_i \nabla_i s dx_i$,

$$\nabla^2 s = \nabla (\sum_i \nabla_i s dx_i) = \sum_{ij} \nabla_j \nabla_i s dx_j dx_i = \sum_{i<j} (\nabla_i \nabla_j - \nabla_j \nabla_i) s dx_i \wedge dx_j$$

□

Prop. (IX.3.3.12) (Flat coordinate). A connection on $TM$ assumes near every point a flat coordinate, i.e. $\nabla(\partial/\partial x^i) = 0$, iff it is flat and torsion-free.

Proof: One side is easy because its Christoffels vanish. On the other side, use integrability theorems (X.8.6.2). Cf.[Jost P115].

□

Prop. (IX.3.3.13).

$$\Delta \langle \varphi, \varphi \rangle = 2(\langle D^* D \varphi, \varphi \rangle - \langle D \varphi, D \varphi \rangle).$$

Proof: Cf.[Jost P118].

□

Prop. (IX.3.3.14). For a flat connection, there is a bundle isomorphism (Gauge transform) that transforms $d_A$ into natural $d$.

Proof: Because $d_{gA}(s) = gd_A(g^{-1}(s)), d_{gA} = d - dg \cdot g^{-1} + g \cdot \omega \cdot g^{-1}$. Solve this PDE directly. (Cf.[Topics in Geometry Xie Yi week3]).

□

Cor. (IX.3.3.15). For a flat connection, by (IX.3.3.14), the parallel transportation only depends on the homotopy type of the loop, thus gives an action of $\pi(X)$ on $SO(T_p(X))$ (or $SU(T_p(X))$). (because it is locally constant).

In this way, connections module gauge equivalence (preserving metric) equals representation of $\pi(X)$ module conjugations. The reverse map is giving by principal bundle.

Proof:
Levi-Civita Connection

Def. (IX.3.3.16) (Levi-Civita Connection). The Levi-Civita connection is the unique connection on $M$ that is metric and torsion-free:

$$X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle,$$

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

It satisfies:

$$\langle Z, \nabla_Y X \rangle = 1/2 \{ g_{jk,i} + g_{ki,j} - g_{ij,k} \} g^{km}.$$

Then

$$\Gamma^m_{ij} = 1/2 \sum_k \{ g_{jk,i} + g_{ki,j} - g_{ij,k} \} g^{km}.$$

Thus geodesic is a solution that only depends on the metric (IX.3.3.8), so a local isometry preserves geodesics.

Prop. (IX.3.3.17). Now the Lie derivative has the form:

$$L_X (S(Y_1, \ldots, Y_p)) = \nabla_X (S(Y_1, \ldots, Y_p)) + \sum_{i=1}^p S(Y_1, \ldots, Y_i, \nabla Y_i X, \ldots, Y_p).$$

The exterior derivative $d$ and its adjoint $d^*$ have the form:

$$d\omega(Y_i) = \sum (-1)^i \nabla Y_i \omega(Y_p), \quad d^* \omega(Y_i) = - \sum \nabla e_i \omega(e_j, Y_i)$$

where $e_i$ is an orthonormal basis. Cf. [Jost P140].

Prop. (IX.3.3.18) (Covariant Differential Symmetry). For a parametrized surface: $s : (u, v) \to M$,

$$D \frac{\partial s}{\partial u} \frac{\partial s}{\partial v} = D \frac{\partial s}{\partial v} \frac{\partial s}{\partial u}.$$

Proof:

$$\frac{D \partial s}{\partial u} \frac{\partial s}{\partial v} = \frac{D}{\partial u} \left( \sum \frac{\partial s_i}{\partial v} X_i \right) = \frac{\partial^2 s_i}{\partial u \partial v} + \sum \frac{\partial s_i}{\partial v} \left( \sum \frac{\partial s_j}{\partial u} \nabla_j X_i \right) = \frac{\partial^2 s_i}{\partial u \partial v} + \sum_{ij} \frac{\partial s_i}{\partial v} \frac{\partial s_j}{\partial u} \nabla_j X_i.$$

But now the Levi-Civita connection is symmetric, thus $\nabla_j X_i = \nabla_i X_j$, showing the symmetry in $u$ and $v$. □

Lemma (IX.3.3.19) (Gauss). Let $p \in M$ and $v \in T_p M$ s.t. $\exp_p v$ is defined, $w \in T_p M$, then

$$\langle (d \exp_p)_v (w), (d \exp_p)_v (w) \rangle = \langle v, w \rangle.$$

Proof: Cf. [Do Carmo P69]. □

Prop. (IX.3.3.20) (Geodesic Locally Minimizing). In a normal nbhd of $p$, the geodesic starting at $p$ is the minimal line.
Proof: And curve \(c(t)\) can be written as \(\exp_p\left(r(t)v(t)\right) = f(r(t), t)\), where \(f(s, t) = \exp_p(sv(t))\), so by Gauss lemma, \(\langle \frac{\partial}{\partial r}, \frac{\partial f}{\partial r} \rangle = 0\). Now \(dc/\partial t = \partial f/\partial r r'(t) + \partial f/\partial t\), so

\[
|dc/\partial t|^2 = |r'(t)|^2 + |\partial f/\partial t|^2 \geq |r'(t)|^2.
\]

Integrate this will give us the desired result. □

Prop. (IX.3.3.21) (Totally normal nbhd). For any point \(p\), there exists a nbhd \(W\) and a number \(\delta > 0\) s.t. for every \(q \in W\), \(\exp_q\) is a diffeomorphism on \(B_\delta(0)\) and \(\exp_q(B_\delta(0)) \supset W\). Thus, fine cover exists in every smooth manifold, because Riemannian metric exists on these manifolds.

Proof: Cf.[Do Carmo P72]. □

- (Geodesic Frame) In a neighborhood of every point \(p\), there exists \(n\) vector fields, orthonormal at each point, and \(\nabla_{E_i}E_j(p) = 0\). (Choose normal nbhd and parallel a orthonormal basis to every point. (WARNING: this is not a flat coordinate, it only helps when dealing with point-wise properties).

Def. (IX.3.3.22) (Killing Fields). A Killing field is a vector field which generates an infinitesimal isometry. \(X\) is killing \(\iff L_X(g) = 0 \iff \langle \nabla_Y X, Z \rangle + \langle \nabla_Z X, Y \rangle = 0\) for all \(Y, Z\), which is called the Killing equation.

Proof: Use Lie formula,

\[
L_X(g)(Y, Z) = X(Y, Z) - \langle [X, Y], Z \rangle - \langle Y, [X, Z] \rangle
\]

and Levi-Civita connection is torsion-free. □

Prop. (IX.3.3.23). Let \(M\) be a compact Riemannian manifold of even dimension with positive sectional curvatures, then every Killing field on \(M\) has a singularity.

Proof: Cf.[Do Carmo P104]. □

Def. (IX.3.3.24) (Geometric Differential Notions).

- The gradient is defined to be \(\langle \text{grad} f, X \rangle = X(f)(p)\).
- The divergence is defined to be \(\text{div} X(p) = \text{trace of the linear map } Y(p) \rightarrow \nabla_Y X(p) = \sum_i \langle \nabla_{E_i} X, E_i \rangle\). It measures the variation of the volume and it depends only on the point.
- The Hessian is defined to be \(\text{Hess} f\) is a self-adjoint operator that \((\text{Hess} f)Y = \nabla_Y \text{grad} f\) as well as a symmetric form \((\text{Hess} f)(X, Y) = \langle (\text{Hess} f)X, Y \rangle\).
- The Laplacian is defined to be \(\Delta f = \text{div} \text{grad} f = \text{trace} \text{Hess} f\).

Prop. (IX.3.3.25). In a geodesic frame,

\[
\text{grad} f(p) = \sum_{i=1}^n (E_i(f)) E_i
\]

\[
\text{div} X(p) = \sum_{i=1}^n E_i(f_i)(p), \text{where } X = \sum_i f_i E_i.
\]

\[
\Delta f = \sum_i E_i(E_i(f))(p).
\]
Cor. (IX.3.3.26). \[ \Delta(f \cdot g) = f \Delta g + g \Delta f + 2 \langle \text{grad} f, \text{grad} g \rangle, \]
because these only depends on the point.

Prop. (IX.3.3.27). \( d(\iota(X)m) = (\text{div} X)m \). where \( m \) is the volume form.

Proof: Choose a geodesic frame \( E_i, \theta_i \) is a dual form of \( E_i \), let \( X = \sum f_i E_i \), then \( \iota(X)m = \sum (-1)^{i+1} f_i \theta_i \), so \[ d(\iota(X)m) = \sum (-1)^{i+1} df_i \wedge \theta_i + \sum (-1)^{i+1} f_i \wedge d\theta, \]
Notice that \( d\theta_i = 0 \), because \( d\theta_k(E_i, E_j) = E_i \theta_k(E_j) - E_j \theta_k(E_i) - \theta_k([E_i, E_j]) = 0 \) (IX.2.3.6), as it is a geodesic frame. And \( \sum E_i(f_i) = \text{div}(X) \) (IX.3.3.25).

Prop. (IX.3.3.28) (Hopf theorem). If \( f \) is a differentiable function on a compact orientable manifold with \( \Delta f \geq 0 \), then \( f \) is constant.

Proof: Let \( \text{grad}(f) = X \), then \[ \int_M \Delta f dm = \int_M \text{div}(X) dm = \int_M d(\iota(X)dm) = 0. \]
So \( \Delta f = 0 \). Now \[ 0 = \int_M \Delta(f^2/2) dm = \int_M f \Delta f dm + \int_M |\text{grad}(f)|^2 dm \]
by (IX.3.3.26), thus \( \text{grad}(f) = 0 \), so \( f \) is constant.

Def. (IX.3.3.29) (Riemannian Curvatures).
- The sectional curvature \( K(X,Y) = \frac{\langle R(X,Y)X,Y \rangle}{|X \wedge Y|^2} \).
- The Ricci curvature \( \text{Ric}(x) = \text{Ric}(x,x) \), where \( \text{Ric}(x,y) \) is the symmetric form of \( \frac{1}{n} \) of trace of the map \( z \to R(x,z)y \).
  Thus \( \text{Ric}_p(x) = \frac{1}{n-1} \sum \langle R(x,z_i)x,z_i \rangle \), for \( x \) a unit vector, where \( z_i \) is an orthonormal basis orthogonal to \( x \).
- The scalar curvature \( K(p) = 1/n \sum \text{Ric}_p(z_i) \), where \( z_i \) is an orthonormal basis.
The curvatures only depends on the point (IX.3.3.11).

Lemma (IX.3.3.30).
\[ \frac{D}{Dt} \frac{D}{Ds} V - \frac{D}{Ds} \frac{D}{Dt} V = R(\frac{\partial f}{\partial S}, \frac{\partial f}{\partial t}) V. \] (obvious because \( \frac{\partial}{\partial S} \frac{\partial}{\partial t} \) commutes)

Proof: \[ \square \]

Prop. (IX.3.3.31) (Sectional Curvature Define Curvature). The curvature tensor is determined by its sectional curvature.
In particular, if \( M \) is isotropic at a point \( p \) (The sectional curvature depends only on the point), then
\[ R(X,Y,W,Z) = K_0(\langle X,W \rangle \langle Y,Z \rangle - \langle X,Z \rangle \langle Y,W \rangle). \]
Proof: Cf.[Do Carmo P95], should use the cyclicity of the first three terms.

Prop. (IX.3.3.32) (Bianchi Identities). Recall the covariant differential $\nabla R(Y_i, Z) = Z(R(Y_i)) - \sum_i R(\nabla Z Y_i, Y_i)$ (IX.3.3.5).
   - (Bianchi Identity) $\sum (X,Y,Z) R(X,Y)Z = 0$.
   - (Second Bianchi Identity) $\sum (Z,W,T) \nabla R(X,Y,Z,W,T) = 0$.

Proof: 1: Cf.[Do Carmo P91], should reduce to Jacobi identity.

Prop. (IX.3.3.33) (Schur’s Theorem). Let $M$ be a manifold of dimension $n \geq 3$, suppose the sectional curvature only depends on $p$, then $M$ has constant curvature.

Proof: Use the second Bianchi Identity and geodesic frame and (IX.3.3.31). Cf.[Do Carmo P106].

Def. (IX.3.3.34) (Eisenstein Curvature). A manifold $M$ is called an Eisenstein manifold if its Ricci curvature $\lambda(p)$ only depends on the point. Then
   - If $M$ is connected and Eisenstein of dimension $\geq 3$, then it has constant Ricci curvatures everywhere every direction.
   - If $M$ is connected and Eisenstein of dimension $3$, then it has constant sectional curvatures.

Proof: 1: Cf.[Do Carmo P108].

Prop. (IX.3.3.35) (Riemannian Curvature Identities).

Proof: Cf.[DO Carmo P91].

- $B(X, Y) = \nabla_X Y - \nabla_Y X$. It is bilinear and symmetric.
- $H_\eta(x, y) = \langle B(x, y), \eta \rangle$. Thus $B(x, y) = \sum H_i(x, y) E_i$ for an orthonormal frame $E_i$ in $\mathfrak{X}(U)^\perp$.
- $S_\eta(x) = -\langle \nabla_x \eta \rangle^T$. It satisfies $\langle S_\eta(x), y \rangle = H_\eta(x, y) = \langle B(x, y), \eta \rangle$. It is self-adjoint. When codimension 1, it is the derivative of the Gauss mapping.
- (Gauss Formula): let $x, y$ be orthonormal tangent vector. Then:
  $$K(x, y) - \overline{K}(x, y) = \langle B(x, x), B(y, y) \rangle - |B(x, y)|^2.$$  
- An immersion is called geodesic at $p$ if the second fundamental form $S_\eta$ is zero for all $\eta$, (which means $\nabla_X Y$ has no normal component). It is called minimal if the trace of $S_\eta$ is zero.
- An immersion is called umbilic if there exists a normal unit field $\eta$ s.t. $\langle B(X, Y), \eta \rangle(p) = \lambda(p)(X, Y)$. 
• If the ambient space has constant sectional curvature and the immersed manifold is totally umbilic, then \( \lambda \) is constant.

• mean curvature tensor of immersion \( f = 1/n \sum_i (\text{tr} S_i) E_i = 1/n \text{tr} B \). It is zero if \( f \) is minimal.

• normal connection \( \nabla^\perp_X \eta = (\nabla_X \eta)^N = \nabla_X \eta + S_\eta(X) \).

\[ \langle \mathcal{R}(X,Y)Z,T \rangle = \langle R(X,Y)Z,T \rangle - \langle B(Y,T),B(X,Z) \rangle + \langle B(X,T),B(Y,Z) \rangle. \]

\[ \langle \mathcal{R}(X,Y)\eta,\zeta \rangle - \langle R^\perp(X,Y)\eta,\zeta \rangle = \langle [S_\eta,S_\zeta]X,Y \rangle. \]

\[ \langle \mathcal{R}(X,Y)Z,\eta \rangle = (\nabla_Y B)(X,Z,\eta) - (\nabla_X B)(Y,Z,\eta). \] (Lie bracket)

**Parallel Transportation**

**Def. (IX.3.3.37) (Parallel Transportation).**

**Def. (IX.3.3.38) (Holonomy Group).** The holonomy group \( \text{Hol}_x(g) \) of a Riemannian manifold \( M \) w.r.t to the Levi-Civita connection is defined to be the subgroup of \( O(T_x(M)) \) induced by the parallel transportation along a loop. If \( M \) is connected, for different points, holonomy groups are conjugate, so holonomy group is defined up to conjugation.

**Prop. (IX.3.3.39) (Trivial Holonomy Group).** If \( M \) is a Riemannian manifold and the holonomy group is trivial, then for any \( X,Y,Z \in \mathfrak{X}(M) \), \( R(X,Y)Z = 0 \).

**Proof:** Cf.[Do Carmo P105]. \( \square \)

**Prop. (IX.3.3.40) (Berger).** in fact, the groups that can be realized as a holonomy group of a simply connected complete Riemannian manifold can be classified.

**Proof:** Cf.[Complex geometry Daniel P214]. \( \square \)

**Def. (IX.3.3.41).** The **Geodesic flow** for a connection on \( TM \) is the flow on \( TM \) whose trajectories are \( t \mapsto (\gamma(t), \gamma'(t)) \), where \( \gamma \) is a geodesic on \( M \).

**Prop. (IX.3.3.42) (The smoothness of geodesics).** For every point \( p \), there exists a nbhd \( V \) and a \( C^\infty \) mapping

\[ \gamma : (-\delta,\delta) \times V \times B(0,\epsilon) \to M, \]

s.t. \( \gamma(t,q,v) \) is the geodesic passing through \( p \) with velocity \( v \).

**Prop. (IX.3.3.43) (Curvature and Metric, Cartan).** Let \( M, \tilde{M} \) be two Riemannian manifold of dimension \( n \) and let \( p \in M, \tilde{p} \in \tilde{M} \). Choose a linear isometry \( i : T_p(M) \cong T_{\tilde{p}}(\tilde{M}) \). Let \( V \) be a normal neighbourhood of \( p \) that \( \text{exp}_p \) is defined on \( i \circ \text{exp}_p^{-1}(V) \). Define a mapping \( f : V \to \tilde{M} \) by \( f(q) = \text{exp}_{\tilde{p}}^{-1} \circ i \circ \text{exp}_p^{-1}(q) \).

For any \( q \in V \), there is a unique normalized geodesic \( \gamma : [0,t] \to M \) that \( \gamma(0) = p, \gamma(t) = q \). Denote by \( P_t \) the parallel transportation along \( \gamma \), and the map \( \varphi_t : T_q(M) \to T_{\gamma(t)}(\tilde{M}) \) by \( \varphi_t(v) = P_t \circ i \circ P_t^{-1}(v) \).
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If for all \( q \in V \) and all \( x, y, u, v \in T_q(M) \), we have
\[
\langle R(x, y)u, v \rangle = \langle \tilde{R}(\varphi_t(x), \varphi_t(y)\varphi_t(u), \varphi_t(v)) \rangle,
\]
then \( f : V \to f(V) \subset \tilde{M} \) is an isometry and \( df_p = i \).

Proof: Cf.[Do Carmo P157]. Use Jacobi fields. The point is that the hypothesis implies that the map of a Jacobi field is also a Jacobi field. \( \square \)

Cor. (IX.3.3.44). Let \( M, \tilde{M} \) be Riemannian manifolds with the same dimension \( n \) in which parallel transportation preserves sectional curvature. Let \( p \in M, \tilde{p} \in \tilde{M} \). If there is a linear isometry \( i : T_p(M) \cong T_{\tilde{p}}(\tilde{M}) \) s.t. \( K(p, E) = K(\tilde{p}, i(E)) \) for any 2-dimensional subspace \( E \subset T_p(M) \), then there exist nbhd \( V \) of \( p \), nbhd \( \tilde{V} \) of \( \tilde{p} \) and an isometry \( f : V \to \tilde{V} \) that \( df_p = i \).

Cor. (IX.3.3.45). If in situation (IX.3.3.44), \( M \) and \( \tilde{M} \) are moreover complete and simply connected, then there is a unique isometry \( f : M \to \tilde{M} \) s.t. \( f(p) = \tilde{p} \) and \( df_p = i \).

Complete manifold

Prop. (IX.3.3.46) (Hopf-Rinow theorem). The following is equivalent definition of completeness.

1. \( \exp_p \) is defined for all of \( T_p(M) \) and all \( p \in M \).
2. The closed and bounded sets of \( M \) are compact.
3. \( M \) is complete as a metric space.
4. \( M \) is \( \sigma \)-compact and if \( q_n \notin K_n, d(p, q_n) \to \infty \).
5. The length of any divergent (compact escaping) curve is unbounded.

and if \( M \) is complete, then for any \( p, q \in M \), there exists a minimizing geodesic between \( p, q \). In particular, any compact submanifold of a complete manifold is complete.

Proof: Cf.[Do Carmo P147]. \( \square \)

• For any two manifold of the same constant curvature and any two orthogonal basis, there is a local isometry (It is locally isotropic).
• Any complete manifold with a sectional curvature is like \( \tilde{M}/\Gamma \), where \( \tilde{M} \) is \( H^n, R^n \) or \( S^n \).

Prop. (IX.3.3.47) (Cartan). in any nontrivial homotopy class in a compact manifold , there exists a closed geodesic.

Proof: \( \square \)

4 Jacobi Field and Comparison Theorems

Def. (IX.3.4.1) (Jacobi Field). The Jacobi field equation along a normalized geodesic \( \gamma \) is defined to be
\[
D^2 J(t) + R(\dot{\gamma}(t), J(t))\dot{\gamma}(t) = 0.
\]
It is defined by its initial condition \( J(0) \) and \( J'(0) \). It can be used to detect the sectional curvature, the critical point of \( \exp_p \) and calculate variation of energy.
Prop. (IX.3.4.2) (Constant Curvature Case). On a manifold with constant curvature \(K\), the Jacobi field equation for a vector field \(J\) normal to \(\gamma\) is equivalent to

\[D^2 J(t) + KJ(t) = 0.\]

Proof: Use (IX.3.3.31), we have

\[\langle R(\gamma', J)\gamma', T \rangle = K \{ \langle \gamma', \gamma' \rangle \langle J, T \rangle - \langle \gamma', T \rangle \langle J, \gamma' \rangle \} = K \langle J, T \rangle\]

So \(R(\gamma', J)\gamma' = KJ\). □

Prop. (IX.3.4.3). The Jacobi field along a point with initial velocity 0 all has the form

\[J(t) = (d\exp_p)_{t\gamma(0)}(t\dot{J}(0)).\]

Proof: Cf. [Do Carmo P113]. Should use uniqueness theorem of ODE. □

Cor. (IX.3.4.4) (Conjugate Points). If two points \(p, q\) are connected by a geodesic \(\gamma\), and \(q = \exp_p(v_0)\), then \(p, q\) are called conjugate along \(\gamma\), if there is a non-trivial Jacobi field on \(\gamma\) that \(J(p) = J(q) = 0\).

Then \(q\) is conjugate to \(p\) iff \(v_0\) is the critical point of \(\exp_p\), and the multiplicity of conjugacy is equal to the kernel of \((\exp_p)_{v_0}\).

Prop. (IX.3.4.5). For a Jacobi field \(J\) along \(\gamma\), \(\langle J(t), \gamma'(t) \rangle\) is linear in \(t\).

Proof: Take second derivatives. □

- If \(J\) is a Jacobi field \(J(t) = (d\exp_p)_{t\gamma}(tw), |v| = |w| = 1\), then

\[|J(t)| = t - \frac{1}{6}K_p(v,w)t^3 + o(t^3).\]

Prop. (IX.3.4.6). There are no conjugate points on a Riemannian manifold of non-positive curvature.

Proof: Cf. [Do Carmo P119]. □

Prop. (IX.3.4.7) (Killing Field is everywhere Jacobi). A Killing field is a Jacobi field along geodesics.

And if \(X(p) = 0\), then \(X\) is tangent to the geodesic sphere near \(p\), because \(X\) preserves length.

Proof: □

Energy Analysis

Def. (IX.3.4.8) (Energy). The energy of a geodesic \(\gamma\) is defined to be

\[E(s) = \int_0^a |\frac{\partial f}{\partial t}(s,t)|^2 dt.\]

Prop. (IX.3.4.9). A minimizing geodesic must minimize energy.
• (First Variation of Energy)

\[ 1/2E'(0) = - \int_0^a \langle V(t), D\dot{c}(t) \rangle dt + \langle V(a), \dot{c}(a) \rangle - \langle V(0), \dot{c}(0) \rangle. \]

A piecewise differentiable curve is a geodesic iff every proper variation has first derivative 0.

• (Second Variation of Energy) If \( \gamma \) is a geodesic,

\[ 1/2E''(0) = \int_0^a \{ \langle DV(t), DV(t) \rangle - \langle R(\dot{\gamma}, V)\dot{\gamma}, V \rangle \} dt + \langle D_sV(a), \dot{\gamma}(a) \rangle - \langle D_sV(0), \dot{\gamma}(0) \rangle. \]

A variation is equivalent to a vector field along the curve, and a variation that \( f_s(t) \) are all piecewise geodesics corresponds to a piecewise Jacobi field (Choose a normal partition).

Prop. (IX.3.4.10) (Rauch Comparison theorem). Let \( M \) and \( \tilde{M} \) be manifolds, \( \dim \tilde{M} \geq \dim M \). If \( J \) and \( \tilde{J} \) be two normal Jacobi fields along geodesics \( \gamma \) and \( \tilde{\gamma} \) that \( |J(0)| = |	ilde{J}(0)| = 0 \) and \( |J'(0)| = |	ilde{J}'(0)| \). If \( \tilde{\gamma} \) has no conjugate point or focal point free and \( \tilde{K}(\tilde{x}, \tilde{\gamma}(t)) \geq K(x, \gamma) \) for any vector \( x, \tilde{x} \), then \( |\tilde{J}| \leq |J| \).

Cor. (IX.3.4.11) (Injectivity Radius Estimate). If the sectional curvature of \( M \) satisfies: \( 0 < L \leq K \leq H \), then the distance between any two conjugate points satisfies: \( \frac{\pi}{\sqrt{H}} \leq d \leq \frac{\pi}{\sqrt{L}} \).

Prop. (IX.3.4.12). If two manifold \( M \) and \( M' \) satisfy \( K \leq K' \), then in a normal nbhd of a point \( p \) in \( M \) and a nbhd of \( p' \) that exp is nonsingular, the transformation of a curve \( c \) shortens length.

Note that this is not Toponogov theorem, because if you try to map from a large curvature manifold to a small curvature, then you cannot guarantee that the mapped curve is the shortest.

Cor. (IX.3.4.13). In a complete simply connected manifold of non-positive curvature,

\[ A^2 + B^2 - 2AB \cos \gamma \leq C^2 \]

thus \( \alpha + \beta + \gamma \leq \pi \).

Prop. (IX.3.4.14) (Moore theorem). Let \( \overline{M} \) be a complete simply connected manifold of sectional curvature \( \overline{K} \leq -b \leq 0 \), \( M \) a compact manifold of sectional curvature satisfying \( K - \overline{K} \leq b \). If \( \dim \overline{M} < \dim M \), \( M \) cannot be immersed into \( \overline{M} \). (use Hadamard theorem to choose the furthest geodesic and calculate the second variation of energy and use Gauss formula).

Cor. (IX.3.4.15). Let \( \overline{M} \) be a complete simply connected manifold of sectional curvature \( \overline{K} \leq 0 \), \( M \) a compact manifold of sectional curvature satisfying \( K \leq \overline{K} \). If \( \dim \overline{M} \leq \dim M \), \( M \) cannot immerse into \( \overline{M} \).

Lemma (IX.3.4.16) (Klingenberg Lemma). Let \( M \) be a complete manifold of sectional curvature \( K \geq K_0 \), let \( \gamma_0, \gamma_1 \) be two homotopic geodesics from \( p \) to \( q \), then there exists a middle curve \( \gamma_s \) s.t.

\[ l(\gamma_0) + l(\gamma_s) \geq \frac{2\pi}{\sqrt{K_0}}. \]

Proof: Assume \( l(\gamma_0) < \frac{2\pi}{\sqrt{K_0}} \), otherwise we are done. Then by Rauch comparison (IX.3.4.10), the \( \exp_p : T_p M \to M \) has no critical point in the open ball \( B \) of radius \( \pi/\sqrt{K_0} \). Now we want to lift \( \gamma_s \) to \( T_p M \). It is clear that we cannot lift \( \gamma_1 \), because otherwise it is not a curve. Hence for every
\( \varepsilon > 0 \), there is a curve \( \alpha_t(\varepsilon) \) that can be lifted and has a point with distance smaller than \( \varepsilon \) to the boundary \( \partial B \), otherwise the \( s \) that can be lifted will be open and closed in \([0, 1] \), thus containing 1.

So now if we choose a sequence of lifts curves \( \gamma_s \) converging to the boundary, then \( s \) has a convergent point, then we have \( l(\gamma_0) + l(\gamma_0) \geq \frac{\pi}{\sqrt{K_0}} \).

**Prop. (IX.3.4.17) (Klingenberg).** Let \( M \) be a simply connected compact manifold of dimension \( n \geq 3 \) such that \( \frac{1}{4} < K \leq 1 \), then \( i(M) \) (The infimum of distance to the cut locus) \( \geq \pi \).

**Cor. (IX.3.4.18).** If \( M \) is a compact orientable manifold of even dimension satisfying \( 0 < K \leq 1 \), then \( i(M) \geq \pi \).

**Prop. (IX.3.4.19) (1/4-pinch Sphere Theorem).** Let \( M \) be a compact simply connected manifold satisfying \( 0 < \frac{1}{4}K_{\text{max}} < K \leq K_{\text{max}} \), then \( M \) is homeomorphic to a sphere.

(Use Klingenberg Theorem, this is a special case of diameter geodesic sphere theorem). Cf.(IX.3.4.29).

It can be shown that in this case, this sphere is even diffeomorphic to \( S^n \) using Ricci flow.

**Remark (IX.3.4.20).** \( 0 < \frac{1}{4}K_{\text{max}} < K \) cannot be changed to \( \geq \). In fact, the Funabi-Study metric on \( CP^n \) has sectional curvature \( 1 \geq K \geq 4 \). Cf. ??

\( \text{Hess}_\rho(X, Y) \), where \( \rho \) is the distance to a fixed point, is important.

**Prop. (IX.3.4.21).** \( \text{Hess}_\rho(X, Y) \) is positive definite on the tangent space of the geodesic sphere within the injective radius, and its principal value is \( |\frac{J'}{J}| \) for a Jacobi field in that direction. And it is zero on the normal direction.

So there would be a Riccati comparison theorem on the eigenvalue of \( \Pi_2 : \lambda' \leq -K - \lambda^2 \), Hess(\( \rho \)) is bounded.

**Proof:** Notice that

\[
\text{Hess}_\rho(X, Y) = (\nabla_X \text{grad}_\rho, Y) = XY\rho - (\nabla_X Y)\rho
\]

so if choose a normal geodesic \( \gamma \) of initial vector \( X \), then

\[
\text{Hess}_\rho(X, X) = X(\dot{\gamma}, d\rho) - (\nabla_X \dot{\gamma})\rho = X(\dot{\gamma}, d\rho) = (\dot{\gamma}, d(\dot{\gamma}, d\rho)) = E''(0)
\]

\[
= I_q(X, X) = ((\nabla_X X)(q), X(q)) = \frac{\langle J', J \rangle}{|J|^2}
\]

**Prop. (IX.3.4.22) (Toponogov).** Let \( M \) be a complete manifold with \( K \geq H \).

If a hinge satisfies \( \gamma_1 \) is minimal and \( \gamma_2 \geq \frac{\pi}{\sqrt{H}} \) if \( H > 0 \), then on \( M^H \) the same hinge has smaller distance of endpoints than this hinge.

**Proof:** Cf.[Cheeger Comparison Theorems in Riemannian Geometry P42]. And there is another triangle version: For a minimal geodesic triangle, the comparison triangle has smaller angles. NOTE this theorem cannot be derived from Rauch Comparison Theorem.
Critical Point for Distance Function

Prop. (IX.3.4.23). The critical point for distance function on a complete manifold is that for every direction \( v \), there is a minimal geodesic \( \gamma \) s.t. \( \langle \gamma'(l), v \rangle \leq \frac{\pi}{2} \).

The set of regular point is open and there exists a smooth gradient like vector field (i.e. acute angle with every minimal geodesic) on this open subset.

Prop. (IX.3.4.24) (Berger’s Lemma). A maximal point for the distance function is a critical point.

Proof: If not, choose a convergent point \( v \) of the minimal geodesics with endpoint in a curve of that direction, then \( \exp \) near \( v \) will generate a Jacobi field with endpoint Jacobi is the same of that direction. So the distance will increase by \( \cos \theta \) along that direction, contradiction.

Prop. (IX.3.4.25) (Soul Lemma). Let \( M \) is a Riemannian manifold and \( A \) is a closed submanifold. If \( \text{dist}(A, -) \) has no critical point on \( D(A, R) \setminus A \), then \( B(A, R) \) is diffeomorphic to the normal bundle of \( A \to M \).

Proof: \( A \) has a normal \( \exp \) radius \( \epsilon \), and we can vary the gradient-like vector field to be identical to the normal vector near \( A \), and use Morse lemma (the flow) to get a diffeomorphism.

Cor. (IX.3.4.26) (Disk Theorem). If \( A \) is a point then \( M \) is diffeomorphic to a disk.

Lemma (IX.3.4.27) (Generalized Schoenflies Theorem). Easy to do, just use the fact that \( \exp \) is continuous to find a boundary sphere depending continuously on the direction (both \( p \) and \( q \)).

Prop. (IX.3.4.28) (Sphere Theorem). If \( M \) is a closed manifold and has a distance function with only one critical point (the furthest one), then \( M \) is homeomorphic to a twisted ball.

Proof: There exists a \( \epsilon \) and \( r \) that \( B(q, \epsilon) \) and \( B(p, r) \) covering \( M \), (Use the convergent point argument). Then use the generalized Schoenflies theorem.

Prop. (IX.3.4.29) (Diameter Sphere Theorem). If a closed manifold \( M \) satisfies \( \sec M \geq K > 0 \), and \( \text{diam}(M) > \frac{\pi}{2\sqrt{K}} \), then \( M \) is homeomorphic to \( S^n \).

Proof: First, if there are two maximal distance point, then use Toponogov to show contradiction. Second, at other points \( x \),

\[
\angle pxq > \frac{\pi}{2}
\]

(Regular domain) because of Toponogov and The formula

\[
\cos \bar{a} = \frac{\cos l - \cos l_1 \cos l_2}{\sin l_1 \sin l_2}.
\]

So the geodesic direction \( \vec{pq} \) will serve as a geodesic-like vector field (might need paracompactness).

Prop. (IX.3.4.30) (Critical Principle). In a complete manifold \( M \) of sectional curvature \( > K \), if \( q \) is a critical point of \( p \), then for any point \( x \) with \( d(p, x) > d(p, q) \) and any minimal geodesic from \( p \) to \( x \), the \( \angle xpq \) is smaller than the \( \cosh^{-1}_K\left(\frac{d(p,x)}{d(p,q)}\right) \).
Proof: Use Toponogov for the hinge $xpq$. Then notice that there is a different minimal geodesic from $p \to q$ that makes the $\angle pqx < \pi/2$ by the definition of critical point, thus there is another Toponogov inequality, this two inequality contradicts. □

Cor. (IX.3.4.31). For a complete open manifold whose $K$ are lower bounded, then it is homeomorphic to the interior of a manifold with boundary. (Use Soul lemma, otherwise there will be a sequence of critical point whose angles are big).

Prop. (IX.3.4.32). ray construction and Line construction?

Prop. (IX.3.4.33) (Soul Theorem). If $M$ is an open manifold with $K \geq 0$, then there is a totally geodesic submanifold $S$ that $M$ is diffeomorphic to the normal bundle over $S$.

Proof: Use the ray construction to get a totally convex compact subset, hence it is a manifold or with boundary, if it has boundary, then find to set of maximal distance to the distance to boundary, the distance to the boundary is a convex function, so it is a smaller totally geodesic manifold. So a $S$ without boundary must exist and this constitutes a stratification, all the level set is strongly convex. Thus all point outside $S$ is not critical, hence the soul lemma applies. Cf.[GeJian Comparison theorems in Riemannian Geometry Lecture7]. □

Prop. (IX.3.4.34) (Perelman). There is a distance non-increasing contraction unto the soul, and it must be just the projection along the normal bundle. Moreover, for any geodesic on the soul and a parallel vector field in the normal bundle along it, it spans a flat surface (by Rauch comparison).

Cor. (IX.3.4.35) (Soul Conjecture). For an open(non-compact) complete manifold $M$ with $K \geq 0$, if it has a point $p$ s.t. sectional curvature at $p$ are all positive, then $M$ is diffeomorphic to $\mathbb{R}^n$. (It’s enough to show that its soul is a point, otherwise for any point, it must has a direction that is flat, $K = 0$).

5 Curvature Inequalities and Topology

Sectional Curvature

Prop. (IX.3.5.1) (Hadamard theorem). $M$ a complete simply connected Riemann manifold of sectional curvature $\leq 0$, then $\exp_p : T_p M \to M$ is an isomorphism of $M$ to $\mathbb{R}^n$. (negative sectional curvature to show $\exp$ is a local isomorphism, complete to show it is a covering map)

Prop. (IX.3.5.2) (Liouville Theorem). Any conformal mapping for an open subset of $\mathbb{R}^n, n > 2$ is restriction of a composition of isometry, dilations and/or inversions, at most once.

Prop. (IX.3.5.3) (Positive Curved, Closed Geodesic not Minimal). If $M$ is an even dimensional orientable Riemannian manifold with positive sectional curvature, let $\sigma : [0, 1] \to M$ be a closed geodesic curve, then there exists an $\varepsilon > 0$ that parametrized closed curves $F; [0, 1] \times (-\varepsilon, \varepsilon) \to M$ near $\sigma$ with lengths less than that of $\sigma$.

Proof: Cf.[Solution to Yau Test Geometry Individual2013 Prob5]. □

Prop. (IX.3.5.4) (Synge). $f$ is an isometry of a compact oriented manifold $M^n$ of positive sectional curvature, $f$ alter orientation by $(-1)^n$, then $f$ has a fixed pt.
IX.3. RIEMANNIAN GEOMETRY

**Proof:** Cf. [Do Carmo P203]. □

**Cor. (IX.3.5.5).** $M$ a compact manifold of positive sectional curvature, then

1. If $M$ is orientable and $n$ is even, then $M$ is simply connected. So if $M$ is compact and even dimension, then $\pi(M) = 1$ or $\mathbb{Z}_2$.
2. If $n$ is odd, then $M$ is orientable.

(Use the universal cover and covering transformation.)

**Conjecture (IX.3.5.6) (Hopf Conjecture).** If $M$ is a compact Riemannian manifold of even dimension that $K > 0$, then it has positive Euler characteristic.

**Morse Index**

**Prop. (IX.3.5.7) (Index Lemma).** Among the piecewise differentiable vector fields along a geodesic without conjugate point or without focal point, with initial value 0 and fixed end value, the Jacobi field attain minimum of the index form:

$$I_a(V,V) = \int_0^a \{ \langle DV(t), DV(t) \rangle - \langle R(\dot{\gamma}, V)\dot{\gamma}, V \rangle \} dt.$$

**Cor. (IX.3.5.8).** $I_l(J,J) = \langle J, J' \rangle(l)$ for a Jacobi field.

**Prop. (IX.3.5.9).** a focal point is a critical value of $exp^\perp$. For an embedded manifold, the focal point equals $x + 1/\eta t$, where $\eta$ is a vertical vector and $t$ is a principal value of $S_\eta t a$.

**Prop. (IX.3.5.10) (Morse Index theorem).** The index of the index form $I_a(V,W)$ on the space of vector fields 0 at the endpoints, equal to the number of points conjugate to $\gamma(0)$ in $[0,a)$.

**Cor. (IX.3.5.11).** If $\gamma$ is minimizing, $\gamma$ has no conjugate points on $(0,a)$, $\gamma$ has a conjugate point, it is not minimizing.

**Prop. (IX.3.5.12) (Morse).** If $M$ is complete with non-negative sectional curvature, then $\pi_1(M)$ have no finite non-trivial cyclic group and $\pi_k(M) = 0$.

**Proof:** because universal cover of $M$ is contractible, so the higher homotopy group vanish and $H^k(M) = H^k(\pi_1(M))$, so if a subgroup is finite cyclic, its homology is periodic, contradiction. □

**Prop. (IX.3.5.13) (Preissman).** For a compact manifold with $K < 0$, any nontrivial abelian subgroup of $\pi_1$ is infinite cyclic.

**Prop. (IX.3.5.14).** If $M$ is compact and $K < 0$, $\pi_1(M)$ is not abelian.

Assuming $M$ complete,

- The cut point of $p$ along $\gamma$ is the maximum $\gamma(t)$ s.t. $d(p, \gamma(t)) = t$. It is either the first conjugate point of $p$ or the intersection of two minimizing geodesics.

- Conversely, if a point is a conjugate point of $p$ or is intersection of two geodesics of equal length, then there is a cut point before it. So, if intersection of two minimizing geodesics happens, it must happen before the occurrence of conjugate point.

- thus the cut point relation is reflexive, and if $q \in M \setminus C_m(p)$, then there exists a unique minimizing geodesic joining $p$ and $q$. 


• $M \setminus C_m(p)$ is homeomorphic to an open ball through $\exp$.
• the distance of $p$ to the cut locus is continuous, thus $C_m(p)$ is closed.
• If $M$ is complete and there is a $p$ which has a cut point for every geodesic, then $M$ is compact.
• for $q$ the closest of $C_m(p)$ to $p$, either there exists a minimizing geodesic and $q$ is conjugate to $p$ or there is to minimizing geodesic connecting at $q$.

Prop. (IX.3.5.15). The index of a geodesic will decrease when transferred to a manifold of smaller sectional curvature $K$.

Prop. (IX.3.5.16). In a complete manifold, if there is a sequence of points $\{p_i\}$ converging to a point $p$, choose for each point a minimal geodesic, then a subsequence of them will converge to a minimal geodesic to $p$.

Proof: The convergence is by smoothness and of $\exp$ and Hadamard. The minimality is by comparing distance. \hfill $\square$

Ricci Curvature

Prop. (IX.3.5.17) (Ricci Comparison). Volume comparison, Laplacian Comparison, Mean Curvature comparison. Cf.[葛健 Week13].

Prop. (IX.3.5.18) (Bishop-Gromov). Let $M$ be an open manifold with $\text{Ric} \geq H$, let $\bar{M}(H)$ be a complete simply connected manifold of constant sectional curvature $H$, then
\[
\text{Vol}(B_r(x)) \leq \text{Vol}(B_r(\bar{p})), \quad \frac{\text{Vol}(B_R(x))}{\text{Vol}(B_r(x))} \leq \frac{\text{Vol}(B_R(\bar{p}))}{\text{Vol}(B_r(\bar{p}))},
\]
Cf.[葛健 Week13].

Prop. (IX.3.5.19) (Bonnet-Myer). $M$ a complete manifold of Ricci curvature $\text{Ric}_p(v) \geq \frac{1}{r^2}$, then $M$ is compact and have diameter $\leq \pi r$.
And if the identity is achieved, $M \cong S^n$.

Proof: Use Laplacian comparison $\Delta r \leq (n-1) \cot r$. Cf.[葛健 week13]. \hfill $\square$

Cor. (IX.3.5.20) (Positive Ricci Finite Fundamental Groups). $M$ is a complete manifold of Ricci curvature $\geq \delta > 0$, then the universal cover is compact thus $\pi_1(M)$ is finite. This can be seen as an obstruction for a compact manifold to have positive Ricci curvature.

Cor. (IX.3.5.21) (Calabi-Yau). For an open manifold with non-negative Ricci curvature, for any point, $\text{Vol}(B(p, r)) \geq c_p r$.

Prop. (IX.3.5.22) (Milnor). Let $M$ be an open manifold of non-negative Ricci curvature of dimension $n$, then any f.g. subgroup of $\pi_1(M)$ has polynomial growth $\leq n$. Milnor conjectured that $\pi_1(M)$ in fact is f.g..

Prop. (IX.3.5.23) (First Betti Number Theorem). There is a number $f(n, \lambda, D)$, $f(n, 0, D) = n$, $f(n, \lambda, D) = 0$ for $\lambda > 0$ that for a manifold of diameter $\leq D$ and Ricci curvature $\geq \lambda$, $b_1(M) \leq f(n, \lambda, D)$.

Cor. (IX.3.5.24) (Splitting Theorem). The universal cover of a compact Riemannian manifold with non-negative Ricci curvature splits isometrically as a product $\tilde{M} = N \times \mathbb{R}^k$ where $N$ is a compact manifold manifold.
Scalar Curvature
IX.4 Algebraic Topology

Main references are [Hatcher P52], [AGP02] and [同调论, 姜伯驹]. [https://ncatlab.org/nlab/show/Introduction+to+Homotopy+Theory#HomotopyGroupsOfTopologicalSpaces].

1 Homotopy Types

Def. (IX.4.1.1) (Retraction). A retraction of a space $X$ to a subspace $A$ is a map $r : X \to A$ that $r|_A = \text{id}_A$.

Def. (IX.4.1.2) (Homotopy). A homotopy $f_t : X \to Y$ is a family of maps $f_t$ for every $t \in I$ that $f : X \times I \to Y$ is continuous.

Two maps $f_0, f_1 : X \to Y$ are called homotopic if there is a homotopy $f_t : X \to Y$ connecting them. Homotopy relations are denoted by $f_0 \simeq f_1$.

Let $A$ be a subspace of $X$, then a homotopy relative to $A$ is a homotopy $f_t : X \to Y$ whose restriction to $A$ is fixed.

Let $E_1, E_2$ be spaces over $B$, then maps $f_0, f_1 : E_1 \to E_2$ over $B$ are called fiber homotopic if there is a homotopy $f_t : E_1 \to Y$ connecting them that each $f_t$ are maps over $B$. Homotopy relations over $B$ are denoted by $f_0 \simeq_B f_1$.

Def. (IX.4.1.3) (Homotopy Equivalence). A map $f : X \to Y$ is called a homotopy equivalence if there is a map $g : Y \to X$ that $f \circ g \simeq \text{id}$ and $g \circ f \simeq \text{id}$.

A space having the homotopy type of a point is called contractible.

A map $f : X \to Y$ over $B$ is called a fiber homotopy equivalence if there is a map $g : Y \to X$ over $B$ that $f \circ g \simeq_B \text{id}$ and $g \circ f \simeq_B \text{id}$.

Def. (IX.4.1.4) (Deformation Retraction). A deformation retraction of a space $X$ onto a subspace $A$ is a homotopy $f_t : X \to X, t \in I$ that $f_0 = \text{id}, f_1(X) = A$ and $f_t|_A = \text{id}_A$ for all $t$.

2 Fundamental Groups

Def. (IX.4.2.1) (Simply Connected). A space is called simply connected if it is connected and $\pi_1(X, x) = 0$ for some point $x \in X$.

Def. (IX.4.2.2) (Semilocally Simply Connected). A space is called semilocally simply connected if for any point $x \in X$, there is a nbhd $U$ of $x$ that the image of the inclusion-induced map $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.

Prop. (IX.4.2.3) (Van Kampen). If $X$ is a union of path-connected subsets $A_\alpha$ all containing $x_0$ that $A_\alpha \cap A_\beta$ and $A_\alpha \cap A_\gamma$ are all path-connected, then $*\pi_1(A_\alpha)/\sim$ where $\sim$ is generated by $\iota_* (\pi_1(A_\alpha \cap A_\beta)) \in \pi_1(A_\alpha) \sim \iota_* (\pi_1(A_\alpha \cap A_\beta)) \in A_\beta$ for every $\alpha, \beta$.

Proof: Cf.[Hatcher P52].

Prop. (IX.4.2.4) (Wedge Sums).

Prop. (IX.4.2.5) (Graphs). For a connected graph $X$ with a maximal tree $T$, $\pi_1(X)$ is a free group with basis the classes $[f_\alpha]$ corresponding to the edges $e_\alpha \in X \setminus T$.

Proof: The quotient map $X \to X/T$ is a homotopy equivalence by (IX.4.6.4). And the quotient space $X/T$ has only one vertex, thus is a wedge sum of circles corresponding to $e_\alpha \in X \setminus T$. So we are done by (IX.4.2.4).
3 Applications

Prop. (IX.4.3.1). For a compact connected manifold $M$ with boundary, there doesn’t exist a retraction of $M$ onto $\partial M$.

Proof: We may assume $\partial M$ is connected, otherwise clearly there is no retraction. Let $M$ be of dimension $n$, it suffices to show that $H_{n-1}(\partial M, \mathbb{Z}_2) \to H_{n-1}(M, \mathbb{Z}_2)$ is 0. So it suffices to show that $H_n(M, \partial M, \mathbb{Z}_2) \to H_{n-1}(\partial M, \mathbb{Z}_2)$ is surjective. For this, use Lefschetz Duality, both these two homology group is isomorphic to $\mathbb{Z}_2$. But $H_n(M, \mathbb{Z}_2) = 0$, thus it is an isomorphism. □

4 CW Complex

Def. (IX.4.4.1) (Simplicial Complexes). A simplicial complex is the geometrization of a simplicial set.

Def. (IX.4.4.2) (CW Complexes). A CW complex is a space constructed as follows:

• $X^0$ is a discrete set, whose elements are regarded as 0-cells.

• Inductively, form the $n$-skeleton $X^n$ from $X^{n-1}$ by adding $n$-cell $e^n_\alpha$ via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$.

• Let $X = \bigcup X^n$ be given the weak topology (quotient topology from $\bigsqcup_n X^n$).

Prop. (IX.4.4.3) (CW Complex has Homotopy Extension Property). If $(X, A)$ is a CW pair, then $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$, thus $(X, A)$ has the homotopy extension property (IX.4.6.1).

Proof: Cf.[Hat02]P15. □

Prop. (IX.4.4.4) (Pathspace is CW Complex). The homotopy fibers (IX.1.3.11) of any map $f : A \to B$ between CW complexes are homotopic to a CW complex.

Proof: [Milnor, 1959]. □

Cor. (IX.4.4.5) (Loop Space). The loop space (IX.1.3.12) $\Omega X$ for $X$ a CW complex is homotopic to a CW complex. In particular, if it has only finitely many cells for a given dimension, then so does $\Omega X$.

Proof: This is a special case of (IX.4.4.4) applied to $A = \{x_0\}$, by (IX.1.3.13). □

Prop. (IX.4.4.6) (Compression Theorem). If $(X, A)$ is a CW pair that $(Y, B)$ be a pair that $\pi_n(Y, B, y_0) = 0$, for any $n$, then every map $(X, A)$ to $(Y \subset B)$ is homotopic rel $A$ to a map $X \to B$. (Use extension property to extend one dimension a time). This shows that the homotopy doesn’t depend on higher dimensional cells, (but might on lower one).

Proof: □

Cor. (IX.4.4.7) (Whitehead Combinatorial Homotopy Theorem I). If $M$ and $K$ is dominated by CW complexes, then any weak homotopy equivalence is an homotopy equivalence. If the map is an inclusion, then it is a deformation retract. In particular, if $M$ is manifold, then it is dominated by its tubular nbhd, so this theorem is applied.

Proof: For inclusion, use compression, and in general use mapping cylinder and cellular approximation. □
Cor. (IX.4.4.8). If $\pi_n(X) = 0$ for all $n$ and a CW complex $X$, then $X$ is contractible.

Def. (IX.4.4.9) (Weak Homotopy Equivalence). A morphism is called a weak homotopy equivalence iff it induces isomorphism on homotopy groups on every dimension w.r.t any basepoint $x$.

Prop. (IX.4.4.10) (Weak Homotopy Equivalence Isomorphic on (Co)Homology). A weak homotopy equivalence induce isomorphism on all homology and cohomology. And also $[K, A] \cong [K, B]$ and $\langle K, A \rangle = \langle K, B \rangle$ for every finite CW complex $K$.

Proof: Pass to the mapping cylinder, the homotopy case follows easily from the compression lemma (IX.4.4.6), and the cohomology follows from universal coefficient theorem (IX.4.7.5).

We may use (reduced) mapping cylinder to assume $A \to B$ is an injection, then compression shows surjectivity, and the relative case for homotopy also show injectivity. □

Cor. (IX.4.4.11) (Whitehead theorem). A $f$ between two simply connected CW complexes that induce isomorphism on homology groups is a homotopy equivalence. (using mapping cylinder, we can assume it’s an inclusion, and $\pi_1(Y, X) = 0$, so the theorem shows that $\pi_n(Y, X) = 0$, and use Whitehead (IX.4.4.7)).

Prop. (IX.4.4.12). A closed manifold or the interior of a manifold with boundary has a homotopy type of a CW complex of finite type.

Remark (IX.4.4.13). The use of mapping cylinder and relative mapping cylinder is important.

Cellular and CW Approximations

Prop. (IX.4.4.14) (Cellular Approximation Theorem). Every map $f : X \to Y$ of CW complexes is homotopic to a cellular map. This makes calculation of homotopy easy. (It suffice to show a map cannot be surjective on a higher dim cell).

Moreover, Any map of pairs of CW complexes can be deformed to a cellular map. (first deform the small complex, then deform the big by dimension).

Proof: Cf.[Hatcher P349]. □

Cor. (IX.4.4.15). The cellular approximation makes the computation of homotopy theoretically easier, but the difficulty comes from the complexity of the homotopy group of the sphere. If a CW complex has only cells of dim $> n$, then it’s homotopy group vanishes for $i < n$. In particular, $\pi_n(S^k) = 0$ for $n < k$.

Prop. (IX.4.4.16) ($n$-Connected CW Models). For a pair $(A, X)$, if $A$ is CW complex, then there is a $n$-connected (IX.4.5.4) CW pairs $(Z, A) \to (X, A)$ that is identity on $A$, and $\pi_i(Z) \to \pi_i(X)$ is isomorphism for $i > n$ and injection for $i = n$.

Such a pair $(Z, A)$ is called a $n$-connected CW model of $(X, A)$, and moreover it can be constructed from $A$ by attaching cells of dimension greater than $n$.

Proof: Cf.[Hatcher P353]. □

Cor. (IX.4.4.17). For an $n$-connected CW pair $(X, A)$, there exist a homotopic $(Z, A) \cong (X, A)$ rel $A$ that $Z \setminus A$ has only cells of dimension greater than $n$. 
Proof: Choose the $n$-connected approximation as above. The map induce and isomorphism on $\pi_{>n}$ by definition and on $\pi_{\leq n}$ because $\pi_i(A) \to \pi_i(Z)$ and $\pi_i(A) \to \pi_i(X)$ are isomorphisms. And on $\pi_n$, it is injective by definition and surjective because $\pi_n(A) \to \pi_n(Z) \to \pi_n(X)$ is isomorphism. Hence we know that the collapsed mapping cylinder at is weak-homotopy equivalent to $Z$, thus it deforms into $Z$ by (IX.4.4.7), thus $Z \to X$ rel $A$ by (IX.4.5.5).

□

Cor. (IX.4.4.18) (CW Approximations). A CW approximation of a space $X$ is a CW complex $Z$ and a weak homotopy equivalence $Z \to X$. A CW approximation of a pair $(X, A)$ is pair of CW complexes $(Z, Z_0)$ and a morphism $(Z, Z_0) \to (X, A)$ that induces isomorphism on relative and absolute homotopy groups.

Thus there exists a CW approximation for any space $A$, and also there exists a CW approximation for any pair $(X, X_0)$.

Proof: Just choose $A$ to be a set containing a point for each connected component of $X$, then $\pi_0(Z) \to \pi_0(X)$ is surjective hence injective.

For pairs, first approximate $X_0$ and use the mapping cylinder to get a embedding.

□

Prop. (IX.4.4.19) (Functoriality of CW Models). Given $n$-connected CW model $f : (Z, A) \to (X, A)$ and $f' : (Z', A') \to (X', A')$, then any map of pairs $g : (X, A) \to (X', A')$ can be extended to a map of pairs $h : (Z, A) \to (Z', A')$ that $gf \cong f'h$ rel $A$. And such a map $h$ is unique up to homotopy rel $A$.

Proof: Cf. [Hatcher, P355].

□

Cor. (IX.4.4.20) (Uniqueness of CW approximation). The CW approximation is unique up to homotopy.

Cor. (IX.4.4.21) (Localizing Category). Together with Whitehead combinatorial homotopy theorem (IX.4.4.7) the homotopy category of spaces defined in (II.6.2.3) is the category of spaces $\mathcal{CG}$ localized by the class of weak homotopy equivalence classes.

Constructing CW complexes

Def. (IX.4.4.22) (Infinite Vector Spaces). Let $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, then the infinite vector space $K^\infty$ is the CW complex $\varinjlim_n K^n$.

Def. (IX.4.4.23) (Infinite Sphere). The infinite sphere $S^\infty$ is defined to be the CW complex $\varinjlim_n S^n$. It is isomorphic to

$$S^\infty = \varinjlim_{k} S^{2k+1} = \varinjlim_{k} \mathbb{C}^{2k+1}\setminus \{0\}/\mathbb{R}^+ = (\mathbb{C}^\infty\setminus \{0\})/\mathbb{R}^+.$$

Def. (IX.4.4.24) (Infinite Grassmanian). Let $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, then the infinite Grassmanian $G_k(K^\infty)$ is the CW complex $\varinjlim_n G_k(K^n)$ (IX.8.2.10). In particular, $\mathbb{C}P^\infty = \varinjlim_n \mathbb{C}P^n$.

Def. (IX.4.4.25) (Infinite Stiefel Manifold). Let $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, then the infinite Stiefel Manifold $V_k(K^\infty)$ is the CW complex $\varinjlim_n V_k(K^n)$ (IX.8.2.10).

Prop. (IX.4.4.26). $\mathbb{R}P^\infty = S^\infty/\{\pm 1\} = (\mathbb{C}^\infty\setminus \{0\})/\mathbb{R}^*$. 

□
Def. (IX.4.4.27) (James Reduced Product). Let $X$ be a space with a basepoint $e$, let the James reduced product space $J(X)$ be the quotient of $\prod_n X^n$ under the identification $(x_1,\ldots,x_j,\ldots,x_n) \sim (x_1,\ldots,\bar{x}_j,\ldots,x_n)$ if $x_j = e$.

$J(X)$ is a union of subspaces $J_m(X)$, where $J_m(X)$ is the quotient space of $X^m$ under the identification $(x_1,\ldots,x_j,e,\ldots,x_n) \sim (x_1,\ldots,e,x_j,\ldots,x_n)$. If $(X,e)$ is a CW-pair, then $J_m$ is obtained from $X^m$ by gluing together its $m$ subcomplexes where one of the coordinates is $e$. It is then clear $J$ is a CW complex.

Def. (IX.4.4.28) (Infinite Symmetric Product). Let $X$ be a space with a basepoint $e$, define the infinite symmetric product as the quotient of $J(X)$ by permutations. If $X$ is a simplicial complex, then $SP(X)$ is a CW-complex.

Proof: Cf. [Hat02]P482. □

5 Homotopy Groups

Def. (IX.4.5.1) (Relative Homotopy Groups).

Prop. (IX.4.5.2) (Long Exact Sequence of Relative Homotopy Groups).

Prop. (IX.4.5.3). The homotopy group defines a long exact sequence for triples $(X,A,B)$, in particular for $B = pt$.

Def. (IX.4.5.4) ($n$-Connectedness). A pair $(X,A)$ of spaces is called $n$-connected if $\pi_i(X,A,x_0) = 0$ for any $i \leq n$ and $x_0 \in A$. A space $X$ is called $n$-connected if $\pi_i(X,x_0) = 0$ for any $i \leq n$ and $x_0 \in X$.

Prop. (IX.4.5.5). A map $X \rightarrow Y$ is a homotopy equivalence iff the mapping cylinder deformation retracts onto $X$.

Prop. (IX.4.5.6) ($\pi_1$ of Universal Cover). The universal cover have the same homotopy group $\pi_{>1}$, by lifting property.

Prop. (IX.4.5.7) (Excision Theorem). If $A,B$ are CW-complexes, then if $(A,A \cap B)$ are $m$-connected and $(B,A \cap B)$ are $n$-connected, then $\pi_i(A,A \cap B) \rightarrow \pi_i(A \cup B,A)$ is isomorphism for $i < m + n$, and surjective for $i = m + n$. Cf. [Hatcher P360].

Moreover, if $(X,A)$ is r-connected and $A$ is s-connected, then $\pi_i(X,A) \rightarrow \pi_i(X/A)$ is isomorphism for $i \leq r + s$ and surjection for $i = r + s + 1$.

Cor. (IX.4.5.8). For $n > 1$, $\pi_n(\vee_\alpha S^n)$ is free Abelian with $\pi_n(S^n)$ as generators. This is because $(\prod_\alpha S^n,\vee_\alpha S^n)$ is $(2n - 1)$-connected thus use excision, because $\pi_n \prod_\alpha S^n$ is easy to calculate.

Cor. (IX.4.5.9) (Freudenthal Suspension Theorem). For $i < 2n - 1$, $\pi_i(S^n) \cong \pi_{i+1}(S^{n+1})$ and . (Can also be derived considering antipodal point point of $S^n$ by (IX.4.9.11)) and surjective for $i = 2n - 1$. In general, this holds when $X$ is $(n - 1)$-connected. Thus we have $\pi_n(S^n) = \mathbb{Z}$.

Proof: Use the suspension, $\pi_i(X) \cong \pi_{i+1}(C_+X,X) \rightarrow \pi_{i+1}(SX,C_-X) \cong \pi_{i+1}(SX)$. for $n = 1$ for the homotopy of sphere, we can use Hopf bundle. □
IX.4. ALGEBRAIC TOPOLOGY

Prop. (IX.4.5.10) (Generalized Hurewicz theorem). If \((X,A)\) is a \((n-1)\)-connected pair of spaces, \(n \geq 2\), then the Hurewicz map induces isomorphisms

\[ \pi_k(X,A)/\pi_1(A) \cong H_k(X,A), \]

and \(H_k(X,A) = 0, k < n\). And for \(k = n\), the corresponding map is surjective for \(n > 1\)

\[\text{Proof:}\] Cf.[Hatcher P390Ex23] for surjectiveness. \(\square\)

Prop. (IX.4.5.11). \(\pi_{i+1}(M) \cong \pi_i(\Omega(M))\), where \(\Omega\) is the loop space. More generally,

\[ \langle \Sigma X, K \rangle = \langle X, \Omega K \rangle. \]

Prop. (IX.4.5.12). The homotopic direct limit of a family of homotopy equivalence is a homotopy equivalence. Cf.[Morse Theory Milnor].

Examples of Calculations

Prop. (IX.4.5.13). By (IX.6.1.5) and (IX.4.6.12), there is a long exact sequence

\[ \pi_i(S^1) \rightarrow \pi_i(S^3) \rightarrow \pi_i(S^2) \rightarrow \pi_{i-1}(S^1) \rightarrow \ldots \rightarrow \pi_0(S^2) \rightarrow 0. \]

Thus

Prop. (IX.4.5.14). for \(i \leq 2m\), \(\pi_iG_m(C^{2m}) \cong \pi_{i-1}U(m)\), and

\[ \pi_{i-1}U(m) \cong \pi_{i-1}U(m+1) \cong \ldots. \]

and for \(j \neq 1\), \(\pi_jU(m) \cong \pi_jSU(m)\).

Similarly, \(\pi_i\Omega(2m) \cong \pi_{i+1}O(2m)\) for \(i \leq n - 4\). (IX.4.9.12), Cf[Morse Theory Milnor Prop23.4].

Cor. (IX.4.5.15) (Bott Periodicity theorem for Unitary Groups). The stable homotopy group \(\pi_iU\) has period 2. \(\pi_{2k+1}U \cong 0\) and \(\pi_{2k}U \cong \mathbb{Z}\).

\[\text{Proof:}\] Use the last proposition and long exact sequence to show that for \(1 \leq i \leq 2m\),

\[ \pi_{i-1}U = \pi_{i-1}U(m) \cong \pi_iG_m(C^{2m}) \cong \pi_{i+1}SU(2m) \cong \pi_{i+1}U. \]

Notice that \(U(m) \rightarrow U(2m)/U(m) \rightarrow G_m(C^{2m})\) \(\square\)

Prop. (IX.4.5.16) (Bott Periodicity for \(O\)). For the infinite dimensional orthogonal space \(O\), \(\Omega_8(16r) \cong O(r), \Omega_4(8r) \cong Sp(2r)\). So \(\Omega_8 \cong O\) and \(\Omega_4O \cong Sp\). Thus by (IX.4.5.11),

\[ \pi_i(O) = \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}, \ldots, \]

\[ \pi_i(Sp) = 0, 0, 0, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, \ldots \]

respectively. (Use (IX.4.9.13)) Cf.[Morse Theory Prop24.7].

Prop. (IX.4.5.17). There is a covering space \(\mathbb{R} \rightarrow S\) with fiber \(\mathbb{Z}\), thus we can calculate the homotopy group of \(S^1\), Cf.[AGP02]P132.
6 Fibrations and Cofibrations

Cofibrations

Def. (IX.4.6.1) (Homotopy Extension Property). Let $A$ be a subspace of a space $X$, then the pair $(X, A)$ is said to satisfy the homotopy extension property if every map $X \times \{0\} \coprod_{A \times \{0\}} A \times I \to Y$ can be extended to a map $X \times I \to Y$.

This implies $X \times \{0\} \cup A \times I$ is a retraction of $X \times I$, and the converse is also true if $A$ is closed. (The closedness of $A$ is used to show that the quotient topology and the subspace topology are the same on $X \times \{0\} \cup A \times I$). Notice if $X$ is Hausdorff, then $A$ is automatically closed, because in this case, $X \times \{0\} \cup A \times I$ is closed in $X \times I$ because it is a retraction, and thus $A \times \{1\}$ is closed in $X \times \{1\}$.

Def. (IX.4.6.2) (Cofibrations). A cofibration is a map $j : A \to X$ that for every map $X \times \{0\} \coprod_{A \times \{0\}} A \times I \to Y$ can be extended to a map $X \times I \to Y$.

Then a cofibration is just a topological embedding that $(X, j(A))$ has the homotopy extension property.

Proof: ?

Def. (IX.4.6.3) (Hurewicz Cofibration). A closed Hurewicz cofibration $i : A \subset B$ is a closed inclusion of spaces that $B \times \{0\} \coprod A \times [0, 1]$ has left extension property w.r.t any map $Y \to pt$.

Prop. (IX.4.6.4). If $A \to X$ is a cofibration and $A$ is contractible, then the quotient map $X \to X/A$ is a homotopy equivalence. In particular, this applies to CW pairs $(X, A)$, by (IX.4.4.3).

Proof: Let $f_t : X \to X$ be a homotopy extending a contraction of $A$ to a point, with $f_0 = \text{id}$. Since $f_t(A) \subset A$, they descend to a homotopy $\tilde{f}_t : X/A \to X/A$. Because $f_t(A)$ is a point, there is a map $g$ in the following diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f_t} & X \\
\downarrow{\pi} & & \downarrow{\pi} \\
X/A & \xrightarrow{\tilde{f}_t} & X/A \\
\end{array}
$$

So $g$ and $\pi$ are inverse homotopy equivalences, because $f_1 \cong f_0 = \text{id}$ and $\tilde{f}_1 \cong \tilde{f}_0 = \text{id}$.

Prop. (IX.4.6.5). Let $X$ be a normal space, then an inclusion $j : A \hookrightarrow X$ is a cofibration iff $A \hookrightarrow V$ is a cofibration for some open nbhd $V$ of $j(A) \subset X$.

Proof: Cf.[AGP02]P92.

Prop. (IX.4.6.6) (Homotopic Glueing Functions). Let $A \to X_1$ be a cofibration with $X$ Hausdorff, and we have attaching maps $f, g : A \to X_0$ that is homotopic, then $X_0 \coprod_f X_1 \cong X_0 \coprod_g X_1$ rel $X_0$.

Proof: Now choose a homotopy $H : A \times I \to X_0$ connecting $f$ and $g$, then $H$ induces a quotient space $Z = X_0 \coprod_H (X_1 \times I)$. Let $X = X_0 \coprod_f X_1$, $Y = X_0 \coprod_g X_1$, then there are natural inclusion maps $i : X_1 \to Z$, $j : Y \to Z$, and there are also deformation retractions $Z \to X, Z \to Y$ constructed as follows:
Choose a deformation retraction $r$ of $X_1 \times I$ onto $X_1 \times \{0\} \amalg A \times I$ (IX.4.6.1), and $H$ induces a map $\overline{r} : D^n \times \{0\} \amalg S^n \times I \to S^n \amalg f_! D^n$, making the following diagram commutative

$$
\begin{array}{ccc}
A \times I & \longrightarrow & X_1 \times I \\
\downarrow^H & & \downarrow^r \\
X_0 & \longrightarrow & X_0 \amalg f_! X_1
\end{array}
$$

which by definition defines a deformation retraction $r_1 : X_0 \amalg_H (X_1 \times I) \to X_0 \amalg f_! X_1$. Similarly we have deformation retraction $r_2 : Z \to Y$. So $r_2 \circ i$ and $r_1 \circ j$ induce an homotopy equivalence between $X$ and $Y$. \hfill $\square$

**Prop.** (IX.4.6.7). If $A \to X, A \to Y$ are cofibrations, and $f : X \to Y$ is a homotopy equivalence that $f|_A = \text{id}_A$, then $f$ is a homotopy equivalence rel $A$.

*Proof:* Cf. [Hatcher] P16. \hfill $\square$

**Cor.** (IX.4.6.8). If $j : A \to X$ is a cofibration which is also a homotopy equivalence, and $X$ is Hausdorff, then $A$ is a deformation retraction of $X$.

**Cor.** (IX.4.6.9) (Homotopy equivalence and mapping cylinder). A map $f : X \to Y$ is a homotopy equivalence iff $X$ is a deformation retraction of the mapping cylinder $M_f$. In particular, two spaces are homotopically equivalent iff there is a third space containing both of them as deformation retractions.

*Proof:* Cf. [Hatcher] P16. \hfill $\square$

**Fibrations**

**Def.** (IX.4.6.10) (Serre Fibration). A **Serre fibration** is the right lifting class of $D^n \to D^n \times I$ for every $n$. This is equivalent to: for any homotopy of $\partial D^n$ and a image $D^n$, there is a homotopy of $D^n$.

In particular, Serre fibrations are stable under base change, by (II.5.0.1).

**Prop.** (IX.4.6.11). Being a Serre fibration is local on the target.

*Proof:* Cf. [Homotopical Point of View] P127. \hfill $\square$

**Prop.** (IX.4.6.12) (Long Exact Sequence of Serre Fibration). Let $\pi : E \to X$ be a Serre fibration, let $b_0 \in B$ and $x_0 \in F = \pi^{-1}(b_0)$, then the map $\pi_* : \pi_*(E, F, x_0) \to \pi_*(B, b_0)$ is an isomorphism for all $n \geq 1$. In particular, there is a long exact sequence

$$
\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0.
$$

by (IX.4.5.2).

*Proof:* Cf. [Hat02] P376. [nLab]. \hfill $\square$

**Def.** (IX.4.6.13) (Hurewicz Fibration). A **Hurewicz fibration** is a map $p : X \to Y$ that has right lifting property w.r.t maps $A \times \{0\} \to A \times [0,1]$ for any space $A$. 

Prop. (IX.4.6.14) (Fibers of Hurewicz Fibration are Homotopic). If $E \to B$ is a Hurewicz fibration, then the fibers $f^{-1}(b)$ are homotopy equivalent over each path components of $B$.

Proof: Cf.[Hat02]P405. □

Cor. (IX.4.6.15). For a Hurewicz fibration $E \to B$, if $B$ is contractible, then it is fiber homotopy equivalent to a trivial fiber bundle.

Prop. (IX.4.6.16) (Homotopy Fiber of Contractible Fibration). Let $f : E \to B$ be a Hurewicz fibration with $E$ contractible, then the homotopy fibers(IX.1.3.11) $F$ over $b_0$ is weak homotopy equivalent to $\Omega(B, b_0)$.

Proof: Let $x_0 \in \pi^{-1}(b_0)$. If we compose the contraction of $E$ to $x_0$ with $\pi$, then we get for each $x \in E$ a path $\gamma_x$ from $\pi(x)$ to $b_0$. Then these give a map $E \to PB : x \mapsto \pi \gamma_x^{-1}$, which is a lift of $\pi$. Then this map gives a commutative diagram

\[
\begin{array}{c}
\Omega(B, b_0) \\
\downarrow \\
PB \\
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
\Omega(B, b_0) \\
\downarrow \\
PB \\
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
E \\
\downarrow \\
B \\
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
E \\
\downarrow \\
B \\
\end{array}
\]

that gives a map of their corresponding long exact sequences (IX.4.6.12). Thus $F \to \Omega(B, b_0)$ is a weak homotopy equivalence because $E, PB$ are both contractible (IX.1.3.13). □

Prop. (IX.4.6.17) (Pathspace is a Hurewicz Fibration). For any map $f : A \to B$, the map $\pi : E_f \to B : (a, \gamma) \mapsto \gamma(1)$ from the pathspace (IX.1.3.11) is a Hurewicz fibration.

In particular, take $A = \{x_0\}$, then the path space $PB \to B$ is a Hurewicz fibration.

Proof: Firstly this map is continuous by (IX.1.3.8). To verify the homotopy lifting property, let $g_t : X \to B$ be a homotopy and $g_0 : X \to E_f$ be a lifting, let $g_0(x) = (h(x), \gamma_x)$. Define a lift $g_t : X \to E_f$ by $g_t(x) = (h(x), \gamma_x \cdot g_0(t)(x))$. The second term is concatenation, which can be defined because $g_0(x) = \pi g_0(x) = \gamma_x(1)$.

To check this is a continuous homotopy, by (IX.1.3.6), it suffices to show $A \times I \times I \to B : (x, s, t) \mapsto \gamma_x \cdot g_{0,0}(x)(t) = \begin{cases} 
\gamma(x, (1 + t)s) & s \leq \frac{1}{1+t} \\
g((1+t)s-1)(x) & s \geq \frac{1}{1+t}
\end{cases}$ is continuous. □

Cor. (IX.4.6.18) (Homotopy Fibers). We can embed $A$ into $E_f$ by mapping $x$ to $(x, \gamma_x)$, where $\gamma_x$ is the trivial loop at $x$. Then $A$ is a deformation contraction of $E_f$, by restricting to shorter and shorter initial segments: $H_t : E_f \to E_f : (a, \gamma) \mapsto (a, \gamma_t)$, where $\gamma_t(x) = \gamma(tx)$.

Then we factored $f$ as a homotopy equivalence followed by a fibration: $A \to E_f \to B$.

If $x_0 \in X$ and $F_I$ is the fiber of $E_f$ over $x_0$, then a map $(I^{i+1}, \partial I^{i+1}, I^i) \to (A, B, x_0)$ is the same as a map $(I^i, \partial I^i) \to (F_I, \gamma_0)$, where $\gamma_0$ is the trivial loop at $x_0$. Thus $\pi_{i+1}(A, B, x_0) = \pi_i(F_I, \gamma_0)$.

Prop. (IX.4.6.19) (Pathspace of a Hurewicz Fibration). If $\pi : E \to B$ is a fibration, then the inclusion $E \to E_\pi$ (IX.4.6.18) is a fiber homotopy equivalence. In particular, the homotopy fibers of $\pi$ are homotopy equivalent to the actual fibers.

Proof: □

Prop. (IX.4.6.20) (Fibration Sequence). Given a fibration $\pi : E \to B$ with $F = \pi^{-1}(b_0), x_0 \in F$, there is a sequence

\[
\ldots \to \Omega^2(B, b_0) \to \Omega(F, x_0) \to \Omega(E, x_0) \to \Omega(B, b_0) \to F \to E \to B \to 0
\]

where any two consecutive maps form a fibration, up to homotopy.
Proof: By (IX.4.6.19), the inclusion \( i \) of \( F \) to the homotopy fiber \( F_\pi \) over \( \pi \) over \( p \) is a homotopy equivalence, and it extends to a map \( i : F_p \to E : (x, \gamma) \mapsto x \), which is also a Hurewicz fibration, because it is the pullback of the fibration \( PB \to B \) (IX.4.6.17).

Thus we can take the homotopy fiber \( F_i \) of \( i : F_p \to E \) over \( x_0 \), and similarly there is a fibration \( j : F_i \to F \), and \( F \) is naturally homotopic to the actual fiber of \( i \), which is just \( \Omega(B, b_0) \).

**Cohomology of Fiber Bundles**

**Prop. (IX.4.6.21) (Leray-Hirsch).** For a fiber bundle \( F \to E \to B \) and a ring \( R \) s.t. \( H^n(F, R) \) is f.g free for all \( n \), and there exist classes \( c_j \) of \( H^*(E) \) that constitute a basis for each fiber \( F \), then

\[
H^*(B, R) \otimes H^*(F, R) \to H^*(E, R)
\]

is an isomorphism of \( H^*(B, R) \)-modules.

**Cor. (IX.4.6.22).**
- \( H^*(U(n); \mathbb{Z}) = \Lambda[ x_1, x_3, \ldots, x_{2n-1} ] \).
- \( H^*(SU(n); \mathbb{Z}) = \Lambda[ x_3, \ldots, x_{2n-1} ] \).
- \( H^*(Sp(n); \mathbb{Z}) = \Lambda[ x_3, x_7, \ldots, x_{4n-1} ] \).

**Prop. (IX.4.6.23).** \( H^*(G_n(\mathbb{K}^\infty); \mathbb{Z}) \) where \( \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H} \) is generated by the symmetric polynomials, where for \( \mathbb{R} \) the coefficient is \( \mathbb{Z}_2 \).

**Proof:** Use the flag variety and first calculate for \( \infty \). Then use Poincare duality to show it is mapped onto the symmetric polynomials. Cf. [Hatcher P435].

**Prop. (IX.4.6.24) (Leray-Serre).** For a Serre fibration, e.g. fiber bundle, \( F \to E \to B \), that \( B \) is simply connected, then there is a spectral sequence

\[
E_2^{pq} = H_p(B, H_q(F)) \Rightarrow H_{p+q}(E) \quad E_2^{pq} = H^p(B, H^q(F)) \Rightarrow H^{p+q}(E)
\]

**Cor. (IX.4.6.25) (Wang Sequence).** When \( B = S^n \), there is a long exact sequence:

\[
\cdots \to H_q(F) \to H_q(E) \to H_{q-n}(F) \to H_{q-1}(F) \to H_{q-1}(E) \to \cdots
\]

**Cor. (IX.4.6.26) (Gysin Sequence).** When \( F = S^n \), there is a long exact sequence:

\[
\cdots \to H_{p-n}(B) \to H_p(E) \to H_p(B) \to H_{p-n-1}(B) \to H_{p-1}(E) \to \cdots
\]

**Model Categories**

**Prop. (IX.4.6.27) (Serre-Quillen).** The category \( \text{Top} \) can be given a **Serre-Quillen model structure** with

- Weak equivalences: weak homotopy equivalence,
- Fibrations: Serre fibrations,
- Cofibrations: Retracts of morphisms \( X \to Y \) where \( Y \) is obtained from \( X \) by attaching cells.

**Proof:** Cf. [Homotopy Theories and Model Categories, Chap8].
Prop. (IX.4.6.28). The homotopy category $\text{hTop}$ of the Serre-Quillen model structure is equivalent to the homotopy category of spaces $\mathcal{H}$.

Lemma (IX.4.6.29). Every map can be decomposed as a homotopy equivalence followed by a fibration, by the construction of homotopy fibers.

Proof:

Prop. (IX.4.6.30) (Hurewicz-Strøm). The category $\text{Top}$ can be given a Hurewicz-Strøm model structure with

- Weak equivalences: homotopy equivalences.
- Cofibrations: closed Hurewicz cofibrations.
- Fibrations: Hurewicz fibrations.

Proof: Cf.[strum paper].

Prop. (IX.4.6.31). The homotopy category of the Hurewicz-Strøm model structure is equivalent to the usual category of homotopy types.

7 Homology and Cohomology

Def. (IX.4.7.1) (Singular Cohomology). The singular cohomology of a topological space with coefficients $R$ is the cohomology groups of the Moore complex (XI.1.2.1) of $R[\text{Sing}X]$ (II.6.3.7).

Prop. (IX.4.7.2) (Homotopy Axiom for Singular Cohomology). For two homotopic maps between two topological spaces, they induce the same map on singular (co)homology.

Proof: For singular homology, the combinatorial 'pillarization' can be constructed that $f - g = k^{n-1} \circ d + d \circ k^n$.

Prop. (IX.4.7.3). The cellular (co)homology coincides with the singular (co)homology for CW-complex.

Prop. (IX.4.7.4) (Morse Inequality). for any field $F$,

$$\sum_{i=0}^{k} (-1)^i \dim H_i(X,F) \leq \sum_{i=0}^{k} (-1)^i c_i,$$

where $c_i$ is the number of $i$-dimensional cells. (Use the dimension counting of the long exact sequence).

Prop. (IX.4.7.5) (Universal Coefficient Theorem). See (I.11.4.8).

Cor. (IX.4.7.6). A map between topological spaces that induce isomorphism on arbitrary homology group induce isomorphisms on cohomology groups.

Prop. (IX.4.7.7) (Poincare Duality). For $X$ a closed manifold, if $X$ is oriented or $\text{char} k = 2$, then there is an isomorphism

$$H_i(X,k) \cong H^{n-i}(X,k)$$

which follows immediately from (V.7.4.15) and (V.6.8.12). (Should also attain the compact cohomology case if know the relation of compact sheaf cohomology better).
Cor. (IX.4.7.8). If $X$ is a compact manifold of odd dimension, then $\chi(X) = 0$, using mod 2 Euler characteristic.

Cor. (IX.4.7.9). $$H^*(\mathbb{RP}^n, \mathbb{Z}_2) = \mathbb{Z}_2[X]/X^n, \quad H^*(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}[X]/X^n$$

Proof: Use induction and Poincaré duality to find that $\alpha \ast \alpha^{n-1} = \alpha^n \neq 0$. □

Prop. (IX.4.7.10) (Brouwer Fixed Point Theorem). If $f$ is a continuous map from $D^n$ to $D^n$, then $f$ has a fixed pt in $D^n$.

Proof: If $f$ does not has a fixed pt, consider the intersection of the ray $f(x)x$ with $S^{n-1}$, then it depends continuously on $x$, and this defines a function from $D^n$ to $S^n$ that is identity on $S^n$, but then this will induce a map $H^*(S^{n-1}) \to H^*(D^n) \to H^*(S^{n-1})$ which is impossible. □

Prop. (IX.4.7.11) (Alexander Duality).

Prop. (IX.4.7.12) (Thom isomorphism). Cf.[姜伯驹同调论].

Prop. (IX.4.7.13) (Gysin Sequence). Cf.[姜伯驹同调论].

Prop. (IX.4.7.14) (Lefschetz Fixed Point Theorem).

**deRham Cohomology**

Prop. (IX.4.7.15) (De Rham). For a smooth manifold and an Abelian group $G$,

$$H^0_{dR}(X, G) \cong H^0(X, G)$$

Where the right is constant sheaf cohomology. (V.6.8.11).

Prop. (IX.4.7.16) (Homotopy Axiom for deRham Cohomology). For two homotopic map between two smooth manifold, they induce the same map on deRham Cohomology.

Proof: We only have to prove the case of $M \times \mathbb{R} \to M$, where any constant section map induces an isomorphism $H^*_{dR}(M \times I) \cong H^*_{dR}(M)$. Because any homotopy is a morphism $M \times I \to N$ where $f$ and $g$ are the sections 0 and 1.

For the zero section, we define $K : a + b dt \mapsto f_0^t b$. This is the desired homotopy, Cf.[Differential Forms in Algebraic Topology Bott Tu]. □

**Cup Product and Cohomology Operators**

Main references are [Hat02]Chap3.2.

Prop. (IX.4.7.17). The cup product will restrict to a relative version:

$$H^*(X, A) \times H^*(X, B) \to H^*(X, A \cup B),$$

This implies that if $X$ is a union of $n$ contractible open set, then the cup product of $n$-elements vanish. In particular, the cup product in a suspension vanishes.

Prop. (IX.4.7.18) (Kunneth Formula). The cross product $H^*(X, \mathbb{R}) \otimes_R H^*(Y, \mathbb{R}) \to H^*(X \times Y, \mathbb{R})$ is an isomorphisms of rings if $X, Y$ are CW complexes and $H^*(Y, \mathbb{R})$ are a finite free $R$-modules for any $k$. 

Prop. (IX.4.7.19) (Steenrod Powers). The total Steenrod squares $\text{Sq}$ is a map from $H^n(X, \mathbb{Z}_2) \to H^{n+*}(X, \mathbb{Z}_2)$ that:
- it is natural and stable under suspension.
- it is additive.
- $\text{Sq}(\alpha \cup \beta) = \text{Sq}(\alpha) \cup \text{Sq}(\beta)$.
- $\text{Sq}^i(\alpha) = \alpha^2$ if $i = |\alpha|$, and 0 if $i > |\alpha|$.

The total Steenrod Powers $P$ is a similar map from $H^n(X, \mathbb{Z}_p) \to H^{n+*}(X, \mathbb{Z}_p)$ that $P^i(\alpha) = \alpha^p$ if $2i = |\alpha|$ and 0 if $2i > \alpha$.

The algebra of powers is generated respectively by elements $\text{Sq}^{-k}$, and for $p$ it is generated by $\beta$ and the elements $P^p$. (Because of Adem relations) Cf.[Hatcher P497].

H-Spaces

Def. (IX.4.7.20) (H-Spaces). An H-space is a unital magma object in the homotopy category of spaces with a fixed point. Equivalently, a H-group is a space $X$ with a map $\mu: X \times X \to X$ and a point $e \in X$ that $X \to \{e\} \times X \to X$ and $X \to X \times \{e\} \to X$ is homotopic to $\text{id}_X$ rel $\{e\}$. In particular, $\mu(e, e) = e$.

An H-space is called strictly associative(or a monoid space) if it comes from a unital magma object in the category of spaces.

Prop. (IX.4.7.21). The definition of a H-space structure can be modified when $X$ is a CW complex. In fact, when $(X, e)$ is a CW-pair, if there exists a map $\mu: X \times X \to X$ and a point $e \in X$ that $X \to \{e\} \times X \to X$ and $X \to X \times \{e\} \to X$ is homotopic to $\text{id}_X$, then $\mu$ can be homotoped that $e$ is a strict identity.

Prop. (IX.4.7.22). $\mathbb{C}P^\infty$ can be given a commutative strictly associative H-space structure. More generally, $B(\mathbb{Z}/n) \cong (\mathbb{C}^\infty\setminus\{0\})/(\mathbb{R}^+ \times \mu_n)$ (IX.6.4.8) can be given a commutative strictly associative H-space structure. In particular this applies to $\mathbb{R}P^\infty$.

Proof: We can regard $\mathbb{C}^\infty$ as the space of polynomials with complex coefficients, then the polynomial multiplication gives a map $(\mathbb{C}^\infty\setminus\{0\}) \times (\mathbb{C}^\infty\setminus\{0\}) \to \mathbb{C}^\infty\setminus\{0\}$ that descends to a map $B(\mathbb{Z}/n) \times B(\mathbb{Z}/n) \to B(\mathbb{Z}/n)$. And $e = (1, 0, \ldots, 0, \ldots)$ is the unit. □

Prop. (IX.4.7.23). The James reduced product $J(X)$??s a strictly associative H-space with multiplication given by $(x_1, \ldots, x_n)(y_1, \ldots, y_m) = (x_1, \ldots, x_n, y_1, \ldots, y_m)$, and the identity $e$. This H-space structure also descends to a H-space structure on $SP(X)$ (IX.4.4.28).

Prop. (IX.4.7.24). The universal cover of a H-space is a H-space.

Proof: Take an arbitrary lift that maps $(\bar{e}, \bar{e})$ to $\bar{e}$. Notice the homotopy can also be lifted. □

Prop. (IX.4.7.25) (Cohomology Ring). The cohomology ring of a H-space is a topologists’ Hopf algebra, by Kunneth formula and naturality.

Cor. (IX.4.7.26). $\mathbb{C}P^n$ is not a H-space.

Proof: This is because $H^*(\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}[\alpha]/\alpha^n, |\alpha| = 2$, which is a not a topologist’s Hopf algebra, by(I.15.1.14). □
**Prop. (IX.4.7.27).** The fundamental group of an H-space is Abelian.

*Proof:* This is because \( \pi_1 \) preserves products, so takes unital magma space to unital magma objects (II.1.1.51). And the unital magma objects in the category of groups is the Abelian groups (II.1.1.50). \( \square \)

**Prop. (IX.4.7.28) (Adam).** \( S^0, S^1, S^3, S^7 \) are the only spheres that have H-structures.

*Proof:* Firstly \( S^1 \subset \mathbb{C}, S^3 \subset \mathbb{H}, S^7 \subset \mathbb{O} \) are submonoids, so they have H-structures. \( \square \)

**Cor. (IX.4.7.29).** \( \mathbb{R}P^n \) has a H-structure iff \( n = 1, 3, 7 \).

Notice alternatively it follows from (IX.4.7.25) and (I.15.1.14) that the numbers \( n \) that \( \mathbb{R}P^n \) has a H-space structure must be \( n = 2^k - 1 \).

*Proof:* This is because the universal cover of a H-space is a H-space (IX.4.7.24). Also \( S^1, S^2, S^3, S^7 \) are monoid spaces, and \(-1\) are in their center, so the quotients are also monoid spaces. \( \square \)

### Examples of Calculations

**Prop. (IX.4.7.30).** The cohomology ring of the Klein bottle \( K \) is \( H^*(K, \mathbb{F}_2) = \mathbb{F}_2[x, y]/(xy, x^2 - y^2, x^3, y^3) \).

*Proof:* Let \( \varphi \in C^0(K, \mathbb{F}_2) \) be the dual of \( v \), \( \alpha, \beta, \gamma \in C^1(K, \mathbb{F}_2) \) be the dual of \( a, b, c \), and \( \mu, \lambda \in C^2(K, \mathbb{F}_2) \) be the dual of \( A, B \), then 

\[
\partial(\varphi)(a) = \partial(\varphi)(b) = \partial(\varphi)(c) = 0
\]

\[
\delta(\alpha)(A) = \alpha(\partial(A)) = \alpha(a + b - c) = 1, \quad \delta(\alpha)(B) = \alpha(\partial B) = \alpha(b + c - a) = -1
\]

\[
\delta(\beta)(A) = \beta(\partial(A)) = \beta(a + b - c) = 1, \quad \delta(\beta)(B) = \beta(\partial(B)) = \beta(b + c - a) = 1
\]

\[
\delta(\gamma)(A) = \gamma(\partial(A)) = \gamma(a + b - c) = -1, \quad \delta(\gamma)(B) = \gamma(\partial(B)) = \gamma(b + c - a) = 1
\]

So 

\[
H^0(K, \mathbb{F}_2) = \mathbb{F}_2 \varphi, \quad H^1(K, \mathbb{F}_2) = \mathbb{F}_2(\alpha + \beta) \oplus \mathbb{F}_2(\alpha + \gamma), \quad H^2(K, \mathbb{F}_2) = (\mathbb{F}_2 \mu \oplus \mathbb{F}_2 \lambda)/(\mu + \lambda).
\]

Now we calculate the cup product:

\[
(\alpha + \beta) \cup (\alpha + \beta)(A) = (\alpha + \beta)(a) \cdot (\alpha + \beta)(b) = 1
\]

\[
(\alpha + \beta) \cup (\alpha + \gamma)(A) = (\alpha + \beta)(a) \cdot (\alpha + \gamma)(b) = 0
\]

\[
(\alpha + \gamma) \cup (\alpha + \gamma)(A) = (\alpha + \gamma)(a) \cdot (\alpha + \gamma)(b) = 0
\]

\[
(\alpha + \beta) \cup (\alpha + \beta)(B) = (\alpha + \beta)(a) \cdot (\alpha + \beta)(c) = 0
\]

\[
(\alpha + \beta) \cup (\alpha + \gamma)(B) = (\alpha + \beta)(b) \cdot (\alpha + \gamma)(c) = 1
\]

\[
(\alpha + \gamma) \cup (\alpha + \gamma)(B) = (\alpha + \gamma)(b) \cdot (\alpha + \gamma)(c) = 0
\]

Then \( (\alpha + \beta) \cup (\alpha + \beta) = (\alpha + \beta) \cup (\alpha + \gamma) = \mu \in H^2(K, bb \mathbb{F}_2) \). Now if we set \( \alpha + \beta = x, \beta + \gamma = y \), then the cohomology ring

\[
H^*(K, \mathbb{F}_2) = \mathbb{F}_2[x, y]/(xy, x^2 - y^2, x^3, y^3).
\]

\( \square \)
8 Obstruction Theory & General Cohomology Theory

Towers

Prop. (IX.4.8.1) (Towers). There are Whitehead Towers and Postnikov Towers for a CW complex $X$.

$$\cdots \to Z_2 \to Z_1 \to Z_0 \to X \to \cdots \to X_2 \to X_1 \to X_0$$

$Z_n$ annihilate $\pi_{\leq n}(X)$, $X_n$ remains only $\pi_{\leq n}(X)$. The towers can be chosen to be fibrations, with fibers $K(\pi_n X, n)$ by (IX.4.6.29).

Prop. (IX.4.8.2). There is a Postnikov towers of:

$$B\text{String}(n) \to B\text{Spin}(n) \to B\text{SO}(n) \to BO(n)$$

with corresponding obstructions $w_1(X), w_2(X)$ and $p_1(X)/2$.

Prop. (IX.4.8.3) (Obstructions). If a connected abelian CW complex $X$ ($\pi_1(X)$ abelian and action on higher homotopy trivial) and $(W, A)$ satisfies $H^{n+1}(W, A; \pi_n X) = 0$ for all $n$, then $A \to X$ can extend to a map $M \to X$.

Proof: Cf. [Hatcher P417].

Cor. (IX.4.8.4). A map between Abelian CW complexes that induce isomorphisms on homology is a homotopy equivalence.

Proof: Notice that $\pi_1(X)$ acts trivially on $\pi_1(Y, X)$ and use Hurewicz.

Eilenberg-Maclane Space

Prop. (IX.4.8.5) (Generalized Cohomology). If $K_n$ is an $\Omega$-spectrum, i.e. $K_n \cong \Omega K_{n+1}$ weak equivalence, then the functors $X \mapsto h^n(X) = \langle X, K_n \rangle$ define a reduced cohomology theory on the category of basepointed CW complexes, i.e. it satisfies the long exact sequence for $A \to X \to X/A$ and wedge axiom. Cf. [Hatcher P397].

Proof: Use (IX.4.5.11) and there is a Cofibration sequence:

$$A \to X \to X/A \to \Sigma A \to \Sigma X \to \cdots$$

Def. (IX.4.8.6). For a discrete Abelian group $G$, an Eilenberg-Maclane spaces $K(G, n)$ is a space having only one nontrivial homotopy group $\pi_n(K(G, n)) = G$.

It can be constructed by $K(A, 0) = A$, $K(A, n + 1) = B(K(A, n))(\text{IX.6.4.4})$. Note $K(G, 1)$ is constructed the same as by (II.6.7.30).

Alternatively, it can also be constructed by first use (IX.4.5.8) and then use higher cells to kill higher homotopies.

Prop. (IX.4.8.7). The homotopy type of a CW complex $K(G, n)$ is unique, thus $\Omega(K(G, n)) \cong K(G, n - 1)$ hence $H^n(X, A) \cong [X, K(A, n)](\text{IX.4.8.5})$ and this isomorphism is generated by a distinguished class of $H^n(K(G, n), G)$.

Proof: Cf. [Hatcher P366].

Prop. (IX.4.8.8). $K(\mathbb{Z}, 1) = S^1 = U(1)$, $K(\mathbb{Z}, 2) = CP^\infty$, Because $S^\infty \to CP^\infty$ is a contractable covering.
9 Morse Theory & Floer Homology

Morse Theory (Milnor)

Def. (IX.4.9.1) (Non-Degenerate Critical Point). For a smooth map \( f : X \to \mathbb{R} \), a critical point is called a non-degenerate critical point iff the Hessian matrix is non-singular at \( x \).

The notion of non-degenerate critical is independent of the coordinate chosen.

Proof: Cf.[Pollack P42].

Prop. (IX.4.9.2) (Non-Degenerate Critical General). Non-degenerate critical points are the general situation in the following sense: For a manifold \( M \subset \mathbb{R}^n \), for any smooth function \( f \) on \( M \), consider the functions \( f_a = f + \sum a_i x_i \), then for almost all \( (a_i) \), all critical points of \( f_a \) is non-degenerate.

Proof: Cf.[Pollack P43]

Prop. (IX.4.9.3) (Morse Lemma). In a non-degenerate critical point of \( f \), there is a coordinate that
\[
f = f(p) + x_1^2 + \cdots + x_{n-\lambda}^2 - y_1^2 - \cdots - y_{\lambda}^2.
\]

Proof: Just extract the first order part out and reform the bilinear form one-by-one. Cf.[Milnor Morse Theory lemma 2.2].

Prop. (IX.4.9.4). If \( f \) is a smooth function that \( f^{-1}([a, b]) \) is compact and have no critical points, then \( M^a \) is a deformation retracts of \( M^b \) using \( |\nabla f| / |\nabla f|^2 \).

Prop. (IX.4.9.5) (Morse Main Lemma). If \( f \) is a smooth function with \( p \) a non-degenerate critical point and \( \lambda \) downward pointing direction. If for some \( f^{-1}([c - \epsilon, c + \epsilon]) \) is compact, then \( M^{c+\epsilon} \) is homotopic to \( M^{c-\epsilon} \) gluing a \( \lambda \) dimensional cell.

Proof: Cf.[Milnor Prop3.2].

Prop. (IX.4.9.6). For an embedded manifold and almost all point \( p \), the distance to \( p \) is a morse function. (Use Sard theorem and degenerate \( \iff \) \( p \) is a focal point.

Cor. (IX.4.9.7). smooth manifold has CW type; on a compact manifold any vector field with discrete singular points has its index sum equal to \( \chi(M) \) (Hopf-Rinow), and there exists one.

Prop. (IX.4.9.8). for \( \Omega(p, q)^c \) the path space of energy \(< c \), the piecewise geodesic path space \( B \) (piece fixed), the energy function is smooth and \( B^a \) is compact and is the deformation contraction of \( \text{int}\Omega^a \) for \( a < c \). \( E \) has the same critical point and same index and nullity on \( B \) and \( \Omega^c \). (Just geodesicrize any path in \( \Omega \).

So for two point not conjugate in \( B^a \), \( \Omega^a \) has a finite CW complex type and a \( \lambda \)-dimensional cell for every geodesic of index \( \lambda \) in \( B^a \).

Prop. (IX.4.9.9) (Morse Main Theorem). If \( p \) and \( q \) are not conjugate along any geodesic, then \( \Omega(p, q) \) has a countable CW complex type and has a \( \lambda \)-cell for every geodesic of index \( \lambda \).

If \( M \) has nonnegative Ricci curvature, then \( M \) has only finite cell for every dimension.

Proof: Cf.[Milnor Morse Theory Prop17.3].
Cor. (IX.4.9.10). The path space homotopy type only depend on the homotopy type of $M$ (use the two homotopy to id to get a composition of homotopy of the two path space), so one can get the information of path space of $M$ by looking at the homotopy type of $M$.

Prop. (IX.4.9.11) (Minimal Geodesics). If $p, q$ in a complete manifold $M$ has distance $\sqrt{d}$ and the minimal geodesics form a topological manifold, and if all non-minimal geodesic has index $\geq \lambda$, then for $0 \leq i < \lambda$, $\pi_i(\Omega, \Omega^d) = 0$.

Lemma (IX.4.9.12). In $SU(2m)$, the minimal geodesic from $I$ to $-I$ is homeomorphic to Grassmannian $G_m(\mathbb{C}^{2m})$ and and non-minimal geodesic has index $\geq 2m + 2$.

Similarly, The space of minimal geodesic from $I$ to $-I$ in $O(2m)$ is homeomorphic to the space of complex structures in $\mathbb{R}^{2m}$, and any non-minimal geodesic has index $\geq 2m - 2$.

Proof: Cf.[Milnor Morse Theory Lemma23.1 Lemma24.4].

Lemma (IX.4.9.13). $\Omega_{k+1}$ is homotopic to the space of minimal geodesics in $\Omega_k$ from $J$ to $-J$. (The same way, calculate the index of geodesics from $J$ to $-J$ and use (IX.4.9.11)). Cf.[Milnor Morse Theory Prop24.5] for definition of $\Omega_{k+1}$.
IX.5  Differential Forms in Algebraic Topology(Bott-Tu)

This section is dedicated to the analysis of algebraic geometry, using the tool of differential forms.

Basic references are [Differential Forms in Algebraic Geometry Bott-Tu].

1 Basics

Prop. (IX.5.1.1) (Cohomological Generator of Sphere). Let \( v : x \mapsto x/|x| \) be the outward-pointing vector field on \( \mathbb{R}^n - \{0\} \), then the differential form

\[
i(v)(dm) = \frac{1}{|x|} \sum x_i(-1)^{i-1} dx_1 \wedge dx_2 \wedge \ldots \widehat{dx_i} \wedge \ldots dx_n
\]

restricts to a differential form on \( S^{n-1} \) that is a generator of \( H^{n-1}(S^{n-1}) \).

**Proof:** First we calculate \( d(i(v)(dm)) = \text{div}(v)dm = \frac{n-1}{|r|} dm \). Consider using Stoke’s formula:

\[
\int_{\partial B_1} i(v)(dm) - \int_{\partial B_2} i(v)(dm) = \int_{B(0,1)-B(0,\varepsilon)} \frac{n-1}{|r|} dm = 0
\]

Lleting \( \varepsilon \to 0 \), \( \int_{\partial B_2} i(v)(dm) \) converges to 0 as \( |i(v)(dm)| \) is bounded and \( V(B_\varepsilon) \) converges to 0. And using the polar coordinate \( dm = r^{n-1} drd\omega \), the right hand side is just

\[
\int_{S^{n-1}} \int_0^1 (n-1)r^{n-2} drd\omega = V(S^{n-1}).
\]

\( \square \)

Prop. (IX.5.1.2) (Degree Formula). If \( f : X \to Y \) is an arbitrary map of two compact oriented manifolds of dimension \( k \), then for any \( k \)-form \( \omega \) on \( Y \),

\[
\int_X f^*\omega = \text{deg}(f) \int_Y \omega.
\]

**Proof:** \( \text{deg} \) is defined in (IX.2.4.16). Cf.[Pollack P188]. \( \square \)

Prop. (IX.5.1.3) (Hopf Invariant). Let \( n > 1 \), given a map \( S^{2n-1} \to S^n \), let \( \alpha \) be a generator of \( H^nS^n \), then \( f^*\alpha = d\omega \) on \( S^{2n-1} \) for some \( \omega \). Define the **Hopf invariant** of \( f \) to be \( H(f) = \int_{S^{2n-1}} \omega \wedge d\omega \), then:

- The definition of Hopf invariant is independent of \( \omega \) chosen.
- For odd \( n \), the Hopf invariant is 0.
- Homotopic maps \( f, g \) have the same Hopf invariant.

**Proof:** 1: If \( d\omega = d\omega' \), then

\[
\int_{S^{2n-1}} \omega' \wedge d\omega' - \int_{S^{2n-1}} \omega \wedge d\omega = \int_{S^{2n-1}} (\omega' - \omega) \wedge d\omega = \pm \int_{S^{2n-1}} d((\omega - \omega') \wedge \omega) = 0.
\]

2: If \( n \) is odd, then \( \omega \) is of even dimensional, thus \( \omega \wedge d\omega = \frac{1}{2} d(\omega \wedge \omega) \), so \( H(f) = 0 \) by Stokes.

3: If \( F : S^{2n-1} \times I \to S^n \) is a homotopy of \( f, g \), then \( F^*\alpha = d\omega \) for some \( \omega \) on \( S^{2n-1} \times I \). Thus consider

\[
H(f) - H(g) = \int_{S^{2n-1}} \omega_1 \wedge d\omega_1 - \int_{S^{2n-1}} \omega_0 \wedge d\omega_0 = \int_{\partial(S^{2n-1} \times I)} \omega \wedge d\omega = \int_{S^{2n-1} \times I} d\omega \wedge d\omega,
\]

But \( d\omega \wedge d\omega = F^*(\alpha \wedge \alpha) \), and \( \alpha \wedge \alpha = 0 \). \( \square \)
IX.6 Fiber Bundles & K-Theory

Main references are [Ati64], [AGP02].

Remark (IX.6.0.1). Any base space $X$ in this section is assumed to be paracompact and Hausdorff.

1 Fiber Bundles

Def. (IX.6.1.1) (Fiber Bundles). A map $\pi : E \to X$ is called a fiber bundle with fiber $F$ if every fiber $\pi^{-1}(x)$ is homeomorphic to $F$. If any $x \in X$ has a nbhd $U$ together with a homeomorphism $\pi^{-1}(U) \cong U \times F$ over $U$, then it is called a locally trivial fiber bundle.

Prop. (IX.6.1.2) (Fibrations are Serre Fibrations). Every trivial bundle is a Hurewicz fibration. Every fibration is a Serre fibration, by (IX.4.6.11).

Prop. (IX.6.1.3) (Pullback Bundle). Let $\pi : E \to X$ be a fiber bundle with fiber $F$ and $f : Y \to X$ a map, then the pullback space $f^*E \to Y$ (IX.1.1.5) is also a fiber bundle over $Y$ with fiber $F$, called the pullback bundle.

Prop. (IX.6.1.4). If $E \to B, E' \to B'$ are fiber bundles both with compact Hausdorff fibers/or both with discrete fibers/, and $f : E/B \to E'/B'$ is a bundle map that induces isomorphisms on the fibers, then $E \cong f^*E'$ over $B$.

Prop. (IX.6.1.5) (Hopf Fibration). There is a Hopf fibration $S^3 \to S^2$ with fiber $S^1$.

Lemma (IX.6.1.6). Suppose $\pi : E \to B \times I$ is a fiber bundle whose restriction to $B \times [0, a]$ and $B \times [a, 1]$ are all trivial for some $a \in I$, then $E \to B \times I$ is a trivial bundle.

Proof: Choose trivializations $\varphi_1 : B \times [0, a] \times B \times F \to \pi^{-1}(B \times [0, a])$, and $\varphi_2 : B \times [a, 1] \times B \times F \to \pi^{-1}(B \times [a, 1])$, then these induces a map

$$B \times \{a\} \times F \xrightarrow{\varphi_1} \pi^{-1}(B \times \{a\}) \xrightarrow{\varphi_2^{-1}} B \times \{a\} \times F,$$

of the form $(b, a, v) \mapsto (b, a, g(b)v)$, where $g : B \to \text{Homeo}(F)$ is continuous. Then we get a trivialization

$$B \times I \times F \to E : \varphi(b, t, v) = \begin{cases} \varphi_1(b, t, v) & t \leq a \\ \varphi_2(b, t, g(b)v) & t \geq a. \end{cases}$$

Lemma (IX.6.1.7). Let $E \to B \times I$ be a fiber bundle, then there exists a covering $\{U_i\}$ of $B$ that $E$ is trivial on each $U_i \times I$.

Proof: For each $b \in B$, we can find a nbhd $U_b$ and a division $0 = s_0 < s_1 < \ldots < s_n = 1$ that $E$ is trivial on each of $U_b \times [s_i, s_{i+1}]$. Then by (IX.6.1.6), $E$ is trivial on $U_b \times I$. Then these $\{U_b\}$ is a covering of $X$ that $E$ is trivial on each $U_b \times I$.  

□
Lemma (IX.6.1.8). Let \( \pi : E \to B \times I \) be a fiber bundle, where \( B \) is a paracompact space, and \( r : B \times I \to B \times I \) defined by \( r(b,x) = (b,1) \), then there exists a bundle morphism \( f \) over \( r \) that induces an isomorphism \( r^*E \cong E \) over \( B \times I \).

Proof: By (IX.6.1.7), we can choose a covering \( \{ U_\alpha \} \) of \( B \) that is \( E \) is trivial over each \( U_\alpha \times I \), and let \( \psi_\alpha \) be a partition of unity \( 1 = \sum \psi_\alpha \) that \( \text{Supp}(\psi_\alpha) \subset U_\alpha \) and \( \{ \text{Supp}(\psi_\alpha) \} \) is locally finite. We also define \( \mu_\alpha(x) = \frac{\psi_\alpha(x)}{\max(\psi_\alpha(x))} \), then \( \mu_\alpha \) are all continuous and subordinate to \( \{ U_\alpha \} \), and for each \( x \in B, \max\{\mu_\alpha(x)\} = 1 \).

Let \( \varphi_\alpha : U_\alpha \times I \to \pi^{-1}(U_\alpha \times I) \) be the local trivializations. We define a bundle map \( f_\alpha : E/B \times I \to E/B \times I \) by identity outside \( \pi^{-1}(U_\alpha \times I) \) and \( f_\alpha(\varphi_\alpha(b,t,v)) = \varphi_\alpha(b,\max(\mu_\alpha(x),t),v) \), then \( f_\alpha \) is continuous and induces an isomorphism \( f_\alpha^*E \cong E \) over \( B \times I \). Now choose a well-ordering on \( \alpha \), by local finiteness, for each \( v \in E \), there is a nhhd \( W_v \times I \) of \( \pi(v) \in B \times I \) that \( W_v \cap U_\alpha \neq \emptyset \) for only \( \alpha \) in a finite set \( \{ \alpha_1, \ldots, \alpha_m \} \) with \( \alpha_1 < \alpha_2 \ldots < \alpha_m \). Then we define a bundle map \( f : E \to E \) that \( f|_{\pi^{-1}(W_v \times I)} = f_{\alpha_m} \circ \cdots \circ f_{\alpha_1} \). Then this is well-defined, and it is a bundle map over \( r \) that induces an isomorphism \( f^*E \cong E \). \( \square \)

Prop. (IX.6.1.9) (Homotopy Invariance of Fiber Bundles). Let \( E' \to B' \) be fiber bundles. If \( f, g : B \to B' \) are two homotopic maps with \( B \) paracompact, then there is a bundle isomorphism \( f^*E' \cong g^*E' \).

Proof: Let \( F : B \times I \to B' \) be a homotopy from \( f \) to \( g \), and let \( i_v : B \to B \times I : i_v(b,x) = (b,v) \) for \( v = 0, 1 \). Let \( r : B \times I \to B \times I \) be the retraction defined by \( r(b,x) = (b,1) \), then by (IX.6.1.8), there is an isomorphism of fiber bundles

\[
f^*E' \cong (F \circ i_0)^*E' \cong i_0^*F^*E' \cong i_0^*i_1^*F^*E' \cong i_1^*F^*E' \cong g^*E'.
\]

\( \square \)

Covering Space

Def. (IX.6.1.10) (Covering Space). A covering space is a fiber bundle \( E \to X \) with discrete fibers.

Prop. (IX.6.1.11). if \( X \) and \( Y \) are Hausdorff spaces, \( f : X \to Y \) is a local homeomorphism, \( X \) is compact, and \( Y \) is connected, then \( f \) a covering map.

Proof: First, \( f \) is surjective (using the connectedness), and that for each \( y \in Y \), \( f^{-1}(y) \) is finite. Because \( X \) is compact, there exists a finite open cover of \( X \) by \( \{ U_i \} \) such that \( f(U_i) \) is open and \( f|_{U_i} : U_i \to f(U_i) \) is a homeomorphism. For \( y \in Y \), let \( \{ x_1, \ldots, x_n \} = f^{-1}(y) \) (the \( x_i \) all being different points). Choose pairwise disjoint neighborhoods \( U_1, \ldots, U_n \) of \( x_1, \ldots, x_n \), respectively (using the Hausdorff property).

By shrinking the \( U_i \) further, we may assume that each one is mapped homeomorphically onto some neighborhood \( V_i \) of \( y \).

Now let \( C = X \setminus (U_1 \cup \cdots \cup U_n) \) and set

\[
V = (V_1 \cap \cdots \cap V_n) \setminus f(C)
\]

\( V \) should be an evenly covered nbhd of \( y \). \( \square \)
Prop. (IX.6.1.12). If \( \pi : \tilde{B} \to B \) is a local onto homeomorphism with the property of lifting arcs. Let \( \tilde{B} \) be arcwise connected and \( B \) simply connected, then \( \pi \) is a homomorphism.

Proof: only need to prove injective. If \( p_1 \) and \( p_2 \) map to the same point, then they can be connected, and the image is a loop thus contractable, contradiction. \( \square \)

Cor. (IX.6.1.13). If \( \tilde{B} \) is locally arcwise connected and \( B \) is locally simply connected, then \( \pi \) is a covering map.

Proof: Choose the connected components of a simply connected nbhd of a point \( p \) and use (IX.6.1.12). \( \square \)

Prop. (IX.6.1.14) (Homotopy Lifting Property). Given a covering space \( \pi : \tilde{X} \to X \), and a homotopy \( f_t : Y \to X \), and a map \( \tilde{f}_0 : Y \to \tilde{X} \) lifting \( X \), then there is a unique homotopy \( \tilde{f}_t : Y \to \tilde{X} \) of \( \tilde{f}_0 \) lifting \( f_t \).

Proof: Let \( U_\alpha \) be a covering of \( X \) that the We first construct a lift \( \tilde{F} : N \times I \to \tilde{X} \) for \( N \) a nbhd near some point \( y_0 \in Y \). Because \( f \) is continuous, there is a nbhd \( N \) of \( y_0 \) and a partition \( 0 = t_0 < t_1 < \ldots < t_m = 1 \) of \( I \) that each \( N \times [t_i, t_{i+1}] \) is mapped into some \( U_\alpha \). Then we can construct a lifting \( \tilde{F} : N \times I \to \tilde{X} \) by induction using the local homeomorphism property of covering space.

Next we show the uniqueness in the special case that \( Y \) is a point. This can also be done using a partition of \( I \) and induction.

Finally we can construct lifting near every point \( y \in Y \), and also they coincide on the overlap because of the uniqueness we just proved. So these liftings glue together to give a lifting \( \tilde{f}_t : Y \to \tilde{X} \). \( \square \)

Cor. (IX.6.1.15). The map \( \pi_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0) \) induced by the covering map is injective. And the image of this map consists of homotopy types of loops that based at \( x_0 \) whose lift starting at \( \tilde{x}_0 \) are also loops.

Proof: This is because a homotopy of a the image of a loop to trivial loop in \( \tilde{X} \) can be lifted to a homotopy of the loop itself to trivial loop. And this homotopy also fixes the endpoint, because the lifting of a trivial loop must be a trivial loop.

For the second assertion, one direction is easy, for the other, if a loop is a homotopic to the image of a loop of \( \tilde{X} \), then it is itself the image of a loop of \( \tilde{X} \). \( \square \)

Prop. (IX.6.1.16) (Sheets of a Covering). Let \( \pi : \tilde{X} \to X \) be a covering map, then the cardinality of \( \pi^{-1}(x) \) is a a locally constant function of \( x \). Thus if \( X \) is constant, this cardinality is fixed for any \( x \in X \), and it is called the number of sheets of the covering.

The number of sheets of a covering with \( \tilde{X} \) path-connected equals the index of \( \pi_* (\pi_1(\tilde{X}, \tilde{x}_0)) \) in \( \pi_1(X, x_0) \).

Proof: For a loop \( g \) in \( X \) based at \( x_0 \), let \( \tilde{g} \) be its lift to \( \tilde{X} \) starting at \( \tilde{x}_0 \). Now if \( h \in \pi_* (\pi_1(\tilde{X}, \tilde{x}_0)) \), then the loop \( h \cdot \tilde{g} \) has lift that has the same ending as \( \tilde{g} \). So we get a map from the quotient set \( \pi_1(X, x_0) / \pi_* (\pi_1(\tilde{X}, \tilde{x}_0)) \) to \( p^{-1}(x_0) \) mapping \( \pi_* (\pi_1(\tilde{X}, \tilde{x}_0))[g] \) to \( \tilde{g}(1) \). This map is injective, and it is surjective because \( \tilde{X} \) is path-connected. Then we are done. \( \square \)

Prop. (IX.6.1.17) (Unique Lifting Property). Let \( \pi : (\tilde{X}, \tilde{x}_0) \to (X, x_0) \) be a covering space and \( f : (Y, y_0) \to (X, x_0) \) be a map with \( Y \) path-connected and locally path-connected, then a lift
\(\bar{f} : (Y, y_0) \to (\bar{X}, \bar{x}_0)\) exists if \(f_*(\pi_1(Y, y_0)) \subset \pi_*(\pi_1(\bar{X}, \bar{x}))\). And when \(Y\) is connected, this lifting is unique.

In particular, a covering space has unique path lifting property.

**Proof:** One direction is clear, for the other, to construct a lifting, choose a path \(\gamma\) from \(y_0\) to \(y\), the path \(f\gamma\) has a unique lifting \(\bar{f}\gamma\) starting from \(\bar{x}_0\). Define \(\bar{f}(y) = \bar{f}\gamma\). This is a well-defined map: if \(\gamma'\) is another path from \(y_0\) to \(y\), then \(f\gamma^{-1}f\gamma'\) is a loop that is homotopic to the image of a loop at \(\bar{x}_0\). Now we can lift this homotopy, and then \(f\gamma^{-1}f\gamma'\) is also the image of a loop at \(\bar{x}_0\), which must be \(\bar{f}\gamma^{-1}\bar{f}\gamma'\) by uniqueness. So \(\bar{f}\) is well-defined.

It can be verified that \(\bar{f}\) is continuous.

The uniqueness is clear, because if there are two lifts, the points that they are equal and the points that they are not are both open in \(Y\).

**Prop. (IX.6.1.18) (Galois Theory of Covers).** Let \(X\) be a path-connected and locally path-connected and semilocally simply-connected space (IX.4.2.2), then

- there is a simply-connected covering space \(\bar{X}\) of \(X\), called a **universal cover** of \(X\).

- The fundamental group acts continuously and properly on \(\bar{X}/X\).

- For any subgroup \(H\) of \(\pi_1(X, x_0)\), there is a covering space \(\pi : X_H \to X\) that \(\pi_*(\pi_1(X_H, \bar{x}_0)) = H\) for a suitably chosen base point \(\bar{x}_0\). And this covering space is unique up to isomorphism over \((X, x_0)\). Thus by (IX.6.1.17), there is an inclusion-preserving bijection between isomorphism classes of covering spaces over \(X\) and the the set of conjugacy classes of subgroups of \(\pi_1(X, x_0)\).

**Proof:** Cf.[Hat02]P64, P67.

**Def. (IX.6.1.19) (Normal Covering Spaces).** A **normal covering space** is a covering space \(\pi : \bar{X} \to X\) that for any \(x \in X\) and two elements \(\bar{x}, \bar{x}' \in \pi^{-1}(x)\), there is a covering isomorphism of \(\bar{X}/X\) taking \(\bar{x}\) to \(\bar{x}'\).

**Prop. (IX.6.1.20).** Let \(\pi : (\bar{X}, \bar{x}_0) \to (X, x_0)\) be a path-connected covering space of a path-connected, locally path-connected space \(X\), and let \(H\) be the subgroup \(\pi_*(\pi_1(\bar{X}, \bar{x}_0)) \subset \pi_1(X, x_0)\), then

- The covering space is normal iff \(H\) is normal in \(\pi_1(X, x_0)\).

- The group \(G(\bar{X})\) of covering transformations of \(\bar{X}\) is isomorphic to \(N(H)/H\).

**Proof:** 1: Let \(\bar{x}_1, \bar{x}_2 \in \pi^{-1}(x_0)\) and \(\gamma\) a path from \(\bar{x}_1\) to \(\bar{x}_2\) corresponding to an element of \(\pi_1(X, x_0)\), then \(H\) is normal is equivalent to \(\pi_*(\pi_1(\bar{X}, \bar{x}_1)) = \pi_*(\pi_1(\bar{X}, \bar{x}_2))\). Then the lifting criterion shows there is a covering transformation taking \(\bar{x}_1\) to \(\bar{x}_2\). The converse is also true.

2: From the above argument, we can define a map \(N(H) \to G(\bar{x})\) by mapping a \(\gamma \in N(H)\) to a covering transformation mapping \(\bar{x}_0\) to \(\bar{x}_1\). And the kernel of this map is exactly those \(\gamma\) lifting to a loop at \(\bar{x}_0\), which are exactly the elements of \(H\).

**Prop. (IX.6.1.21) (Covering Space Action).** If \(G\) is a discrete group and \(G \times Y \to Y\) is a covering space action (IX.1.12.14), then the quotient map \(Y \mapsto Y/G\) is a normal covering space. And if \(Y\) is path-connected, \(G\) is the group of covering transformations.

**Proof:** The condition on the action shows it is locally a homeomorphism, thus it is a covering space. And it is a normal covering space because \(g_1g_2^{-1}\) takes any \(g_1(x)\) to \(g_2(x)\). The group of covering transformations is just \(G\), because the covering transformation on a path-connected space is determined by its action on a single point.
Prop. (IX.6.1.22). A simply connected manifold is orientable. (Use the orientable double cover).

Prop. (IX.6.1.23) (Covering Spaces of Graphs). Every covering space $\pi : \tilde{X} \to X$ of a graph is also a graph, with vertices and edges lifts of vertices and edges of $X$.

Proof: Let the sets $\pi^{-1}(v)$ be vertices of $\tilde{X}$, where $v$ are vertices of $X$. And if we write $X$ as a quotient space of $X^0 \cup \alpha I_a$, each $I_a$ can be lifted to maps to $\tilde{X}$, and we let these be edges of $\tilde{X}$. The topology of $\tilde{X}$ coming from quotient topology of this is the same as the original topology, because they has the same basis open sets, because $\pi$ is a local homeomorphism. □

2 Vector Bundles

Def. (IX.6.2.1) (Vector Bundle). Let $K = \mathbb{R}$ or $\mathbb{C}$, a vector bundle of dimension $n$ over a topological space $X$ is a fiber bundle over $X$ with fiber $K^n$ that each trivialization $\varphi_\alpha$ restricts to linear isomorphisms on the fibers. The category of vector bundles over $X$ is denoted by $\text{Vect}(X)$. A vector bundle homomorphism $E \to F$ is a map of spaces over $X$ that the maps on the fibered are all linear.

Prop. (IX.6.2.2) (Constructions of Vector Bundles). Let $T : (\text{Vect}_K^n) \to \text{Vect}_K^n$ be a functor that is either covariant or contravariant for each of its factor that $T : \prod_i \text{Hom}(V_i, W_i) \to \text{Hom}(T(V_i), T(W_i))$ is continuous, then we have a functor $T : \text{Vect}(X)^n \to \text{Vect}(X)$ that is either covariant or contravariant for each of its factor.


Cor. (IX.6.2.3). In this way, given a vector bundles $E, F$ on $X$, we can construct

\[ E \oplus F, \quad E \otimes F, \quad \text{Hom}(E, F) \cong E^* \otimes F, \quad E^*, \quad T^n E, \quad \wedge^i E. \]

Prop. (IX.6.2.4) (Existence of Hermitian Metric). There exists a Hermitian(Riemannian) metric on any bundle $E$ over a paracompact space $X$.

Proof: Choose a metric on each trivialization open subset and use partition of unity to glue. □

Cor. (IX.6.2.5). A vector bundle over a paracompact space can have its transform maps $\in O(n)$ (or $U(n)$).

Proof: We can choose the metric on it compatible with the given metric. In this way, the transform map is $\in O(n)$ (or $U(n)$). □

Cor. (IX.6.2.6). Any exact sequence of vector bundles over $X$ splits.

Proof: Because we can take the orthogonal complement. □

Cor. (IX.6.2.7). If $f : X \to Y$ is a homotopy equivalence, then $f^* : \text{Vect}(Y) \to \text{Vect}(X)$ is an isomorphism.

If $X$ is contractible, then every bundle over $X$ is trivial.

Lemma (IX.6.2.8). Suppose $\pi : E \to B \times I$ is a fiber bundle whose restriction to $B \times [0, a]$ and $B \times [a, 1]$ are all trivial for some $a \in I$, then $E \to B \times I$ is a trivial bundle.

Proof: The proof is similar to that of(IX.6.1.6) □
Lemma (IX.6.2.9). Let $E \to B \times I$ be a fiber bundle, then there exists a covering $\{U_i\}$ of $B$ that $E$ is trivial on each $U_i \times I$.

Proof: The proof is similar to that of (IX.6.1.7) \hfill \square

Lemma (IX.6.2.10). Let $\pi : E \to B \times I$ be a fiber bundle, where $B$ is a paracompact space, and $r : B \times I \to B \times I$ defined by $r(b, x) = (b, 1)$, then there exists a bundle morphism $f$ over $r$ that induces an isomorphism $r^*E \cong E$ over $B \times I$.

Proof: The proof is similar to that of (IX.6.1.8) \hfill \square

Prop. (IX.6.2.11) (Homotopy Invariance of Vector Bundles). Let $E' \to B'$ be a vector bundle. If $f, g : B \to B'$ are two homotopic maps with $B$ paracompact, then there is a bundle isomorphism $f^*E' \cong g^*E'$.

Proof: The proof is similar to that of (IX.6.1.9) \hfill \square

Prop. (IX.6.2.12) (Classifying Bundles). Let $X$ be paracompact, then there is a natural bijection

$$[X, BU(k)] \to Vect_k(X) : f \mapsto f^*E_k(\mathbb{R}^\infty).(IX.6.4.8)(IX.6.3.6)$$

Proof: Because $E$ is of f.t., we can find a f.d. vector space $W$ with a metric and a vector bundle epimorphism $\varphi : X \times W \to E$ via partition of unity. Then we can take the map $\varphi : X \to G_k(W) : x \mapsto \text{Ker}(\varphi_x)^\perp$. Then $f^*E_k(\mathbb{R}^\infty) \cong E$ via restriction of $\varphi$. The last assertion follows from the definition of $E_k(\mathbb{R}^\infty)(IX.6.3.6)$ and (IX.6.2.11).

Cf. [AGP02] P284. \hfill \square

Def. (IX.6.2.13) (Bundles of Finite Type). Let $X$ be a paracompact Hausdorff space, then a vector bundle of finite type over $X$ is a vector bundle over $X$ that has a covering by f.m. trivialization maps. The category of $k$-dimensional vector bundles over $X$ of f.t. is denoted by $K_k(X)$. Trivially, any vector bundle over a compact space is of f.t..

Prop. (IX.6.2.14) (Vector Bundles on the Quotient). Let $Y$ be a closed subspace of $X$, $E$ a vector bundle over $X$, then a trivialization $\alpha : E|_Y \cong Y \times V$ defines a bundle $E/\alpha$ over $X/Y$. The isomorphism class of $E/\alpha$ only depends on the homotopy type of $\alpha$.

Proof: To show it is a vector bundle, notice the trivialization $\alpha$ extends to a \hfill \square

Prop. (IX.6.2.15) (Splitting Principle). For a vector bundle $E \to X$, there is a space $Y \to X$ that $p^*$ is injective on $H^*(-, \mathbb{Z})$ and $p^*E$ splits as a sum of line bundles. This proposition is useful when proving theorems about characteristic classes.

Proof: It suffice to find a $Y$ that $p^*E$ has a subbundle, then choose its orthogonal part, and use induction. For this, choose $Y = P(E)$, then $Y$ has a tautological bundle, which is a subbundle of $p^*E$, and $Y$ is fibered over $X$ with fiber $\mathbb{P}^n$, and we want to use Leray-Hirsch, so check the fact $H^*(\mathbb{P}^n)$ is free and generated by the first Chern class, by (IX.6.5.1) and (V.6.7.16). And Chern class is functorial, so the powers of Chern class of $f^*E$ will generate the cohomology ring of any stalks. \hfill \square

Prop. (IX.6.2.16). For any bundle $E$ over a compact Hausdorff space $X$, there is a surjective bundle map $X \otimes \mathbb{R}^m \to E$ for some $m \geq 0$. 

Proof: Choose a finite cover of trivialization of $E$, then we can glue these maps together via a partition function.

Cor. (IX.6.2.17) (Negation of Bundles). For any vector bundle $E$ on a compact space $X$, there is a vector bundle $F$ that $E \oplus F$ is a trivial bundle.

Proof: Choose a bundle map $\mathbb{R}^n \times X \to E$ that is surjective, then the kernel of this map is a bundle $F$, such that $E \oplus F \cong \mathbb{R}^n$ (By taking a Hermitian metric (IX.6.2.4) and taking the orthogonal bundle).

Cor. (IX.6.2.18) (Global Transversal Sections). For a vector bundle over a compact manifold, there exists a global section transversal to the zero section, in particular, if $\dim E > M$, then it has no zero.

Proof: Choose a bundle map $\mathbb{R}^n \times X \to E$ that is surjective, and then use parametric transversality theorem (IX.2.4.5) to prove there is a section that is transversal.

Cor. (IX.6.2.19) (Vector Field with Isolated Zeros). There is a vector field on compact manifold of only isolated zeros. And a vector bundle over a $k$ dimensional curve splits to components of dimension no bigger than $k$. Determined by its Chern class.

Prop. (IX.6.2.20) (Constructing Vector Bundles).

Cor. (IX.6.2.21). There is a natural isomorphism $\text{Vect}_n(S(X)) \cong [X, GL(n, \mathbb{C})]$.

Proof: Write $S(X) = C^+(X) \coprod C^-(X)$, and $C^\pm(X)$ are both contractible, thus $E$ are trivial restricted to them (IX.6.2.7). Let $\alpha^\pm$ be the trivialization isomorphism, then $\alpha^+ \circ \alpha^-$ is a bundle map of $X \times \mathbb{R}^n$, which is equivalent to a map $\alpha : X \to GL(n, \mathbb{C})$. The homotopy type of $\alpha$ is determined because $C^\pm(X)$ are both contractible, and vice versa.

Prop. (IX.6.2.22) (Vector Bundles as Modules). $\Gamma$ induces an equivalence between the category of vector bundles over $X$ and the category of finitely projective modules over $C(X)$.

Proof: Clearly a bundle induces a module over $C(X)$. And it is a fully faithful functor. Now the image is the subcategory of finite projective modules, because every bundle is a direct summand of a trivial bundle, and a trivial bundle corresponds to a finite free $C(X)$-modules.

3 Thom isomorphism

Prop. (IX.6.3.1) (Thom Class). For a vector bundle $E$ over base $B$, we can compactify its bundles to get a $(D^n, S^n)$-bundle, if there is a Thom class that induce a generator $H^n(D^n, S^n)$ on every fiber. Then the relative Leray-Hirsch will give that $c$ induces an isomorphism $H^i(B, R) \to H^{i+n}(E, E', R)$. For $\mathbb{Z}_2$ coefficient there exists a Thom class, and for orientable bundle there exists a $\mathbb{Z}$-Thom class. Notice that fiber bundle over a simply connected base is orientable.

Prop. (IX.6.3.2). Similarly, for an orientable fiber bundle $S^{n-1} \to E \to B$, make it a $D^n \to E' \to B$ bundle, then $E'$ is homotopy equivalent to $B$ so there is a Gysin sequence

$$\to H^{i-n}(B) \xrightarrow{\cup e} H^i(B) \to H^i(E) \to H^{i-n+1}(B) \to$$

Where the Euler class $e$ is chosen to commute with the Thom isomorphism.
Examples of Vector Bundles

Def. (IX.6.3.3) \((n\text{-Universal } k\text{-Vector Bundle})\). Let \(K = \mathbb{R}\) or \(\mathbb{C}\), endow \(\mathbb{K}^n\) with the canonical Hermitian metric, define \(E_k(\mathbb{K}^n)\) be the subspace of \(G_k(\mathbb{K}^n) \times \mathbb{K}^n\) consisting of pairs \((W, v)\) that \(v \in W\). Then this is a vector bundle over the Grassmannian \(G_k(\mathbb{K}^n)(\text{IX.8.2.10})\), called the \(n\)-universal \(k\)-vector bundles.

Proof: We construct localization maps: endow \(\mathbb{K}^n\) with the natural metric. For \(W_0 \in G_k(\mathbb{K}^n)\), then the subspace \(U\) of \(G_k(\mathbb{K}^n)\) consisting of \(W\) that \(W \cap W_0^\perp = \emptyset\) is a nbhd of \(W_0\), and it is naturally homeomorphic to \(\text{Hom}(W_0, W_0^\perp)\). There is an isomorphism

\[
\pi^{-1}(U) \cong U \times W_0 : (f, (f + \text{id})w_0) \mapsto (f, w_0).
\]

Prop. (IX.6.3.4) (Pullback the Universal Bundle). If \(V\) is a f.d. vector space, for any continuous map \(\varphi : X \to G_k(V)\), we get a subspace \(E_\varphi = \{(x, v) \in X \times V | v \in \varphi(x)\}\). This is a vector bundle over \(X\), and it is a subbundle of the trivial bundle \(X \times V \to X\). In fact, \(E_\varphi \cong f^*E_k(V)(\text{IX.6.3.3})\).

Cor. (IX.6.3.5). Let \(f : X' \to X\) and \(\varphi : X \to Pr(V)\), then \(f^*(E_\varphi) = E_{\varphi \circ f}\).

Prop. (IX.6.3.6) (Infinite Universal \(k\)-Vector Bundle). The \(n\)-universal \(k\)-vector bundles \(E_k(\mathbb{K}^n)\) for various \(n\) gets together to a bundle \(E_k(\mathbb{K}^\infty)\) on \(G_k(\mathbb{K}^\infty)\) that the pullback of \(E_k(\mathbb{K}^\infty)\) via \(i : G_k(\mathbb{K}^n) \to G_k(\mathbb{K}^\infty)\) is \(E_k(\mathbb{K}^n)\).

Def. (IX.6.3.7) (Hopf Bundle). Define a map \(\varphi : \mathbb{C}P^n \to G_1(\mathbb{C}^{n+1})\) that \(\varphi([z]) = [z]\), then this defines a vector bundle on \(\mathbb{C}P^n\) by (IX.6.3.4), called the dual of Hopf bundle. The Hopf bundle is defined to be the dual of the dual of Hopf bundle.

4 Principal Bundles

Main reference is [Principal Bundles and Classifying Space].

Def. (IX.6.4.1). A principal bundle or \(G\)-bundle is a bundle \(P\) with \(G\)-fibers that the transition function is right \(G\)-map, i.e. left multiplication by some \(g_{\alpha\beta}\). A associated bundle of a representation \(G \to \text{End}(V)\) is the total space of \(P \times V\) module the equivalence \([gg_0, v] = [g, g_0v]\). The corresponding transition function is just the left action by \(g_{\alpha\beta}\).

Prop. (IX.6.4.2) (Homogenous Space). If \(G\) is a Lie group and \(H\) is a closed subgroup, then the quotient \(H\backslash G\) can be given a structure of a \(G\)-homogenous space and \(G \to H\backslash G\) is a principal \(H\)-bundle.

Proof:

Prop. (IX.6.4.3). The projection \(S^{2n+1} \to \mathbb{C}P^n\) is a principal \(S^1\)-bundle.
Classifying Space

Def. (IX.6.4.4). The **classifying space** for a topological group $G$ is a CW complex $BG$ with a weakly contractable universal cover $EG$ that $EG$ is a $G$-fiber bundle on $BG$.

$$\pi_{n+1}(BG) = \pi_n(G) \text{ by (IX.4.6.12).}$$

Prop. (IX.6.4.5). For any topological group, there exists a fiber bundle $G \to EG \to BG$ with $EG$ contractible. In particular, classifying space exists for any topological group $G$. In particular, $G \cong \Omega(BG)$, by (IX.4.6.16).

Proof: Cf. [Milnor, Construction of Universal Fiber Bundles, I, II]. □

Prop. (IX.6.4.6). $[X, BG] \cong G$-bundles on $X$. And $BG$ is Abelian if $G$ is Abelian. Thus the classifying space $BG$ is unique up to homotopy equivalence because they all represent the functor from the CW homotopy category to the set of $G$-bundles on it.

Proof: Cf. [Principal Bundles and Classifying Space P13]. □

Cor. (IX.6.4.7). If $H \to G$ is a homomorphism of topological groups, then any $H$-principal bundle can be made into a $G$-bundle by right tensor $G$. Thus there is a map $BH \to BG$ by Yoneda lemma. In other words, there is a **classifying functor** $\theta$ from the category of topological space to the category of homotopy class of CW complexes.

Prop. (IX.6.4.8) (Examples of Classifying Spaces).

- $B(\mathbb{Z}/n\mathbb{Z}) \cong S^\infty/\{\mathbb{Z}/n\} = (C^\infty \setminus \{0\})/(\mathbb{R}^+ \times \mu_n)$ (IX.4.4.23). In particular, $B(\mathbb{Z}/2) \cong \mathbb{RP}^\infty$ (IX.4.4.26).
- $BSU(2) = \mathbb{HP}^\infty$.
- $B(\mathbb{Z}^2) = \text{torus of genus } g$ because torus has the upper half plane as universal cover, this can be seen observing only has to satisfy the sum of inner angle is $\pi$.
- $BO(n), BU(n), BSp(n)$ are respectively the infinite Grassmannians $G_n(\mathbb{R}^\infty), G_n(\mathbb{C}^n), G_n(\mathbb{H}^\infty)$, because we have $O(n) \to V_n(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty)$.

and similarly for $\mathbb{C}$ and $\mathbb{H}$, and $V_n(\mathbb{R}^\infty)$ is contractible by linear homotopy and Schmidt orthogonalization. In particular, $B(\mathbb{Z}/2) = \mathbb{RP}^\infty$ and $BS^1 = \mathbb{CP}^\infty$.

Proof: □

Def. (IX.6.4.9). A subgroup of a topological group $G$ is called **admissible** if $G \to G/H$ is a $H$-bundle.

Prop. (IX.6.4.10). If $H$ is an admissible subgroup of $G$, then there is a homotopy fiber sequence $G/H \to BH \to BG$.

Proof: Cf. [Principal Bundles and Classifying Space P22]. □

Cor. (IX.6.4.11). There are homotopy equivalences $\Omega BK \cong K$ and $B\Omega K \cong K$.

Prop. (IX.6.4.12). If $H$ is an admissible normal subgroup of $G$, then there is a homotopy fiber sequence $BH \to BG \to B(G/H)$. 
Cor. (IX.6.4.13).
- there are fiber bundles $S^0 \to BSO(n) \to BO(n)$ and similarly for $\mathbb{C}$ and $\mathbb{H}$.
- there are fiber bundles $S^n \to BO(n) \to BO(n + 1)$.
- there are fiber bundles $U(n)/T^n \to (\mathbb{CP}^\infty)^n \to BU(n)$, and where $U(n)/T^n$ is the variety of complete flags in $\mathbb{C}^n$.
- for a discrete group $H \subset G$, $BH \to BG$ is a covering map.
- there are fiber bundles $BSO(n) \to BO(n) \to \mathbb{RP}^\infty$ and similarly for $\mathbb{C}$ and $\mathbb{H}$.
- there are fiber bundles $\mathbb{RP}^\infty \to BSpin(n) \to BSO(n)$.

Prop. (IX.6.4.14). $H_*(BG, \mathbb{Z}) \cong H_*(G, \mathbb{Z})$ and $H^*(BG, \mathbb{Z}) \cong H^*(G, \mathbb{Z})$.

Proof: Because $EG$ is weakly contractible, $S_*(EG)$ is a free $\mathbb{Z}[G]$-module resolution of $\mathbb{Z}$ and $S_*(EG)_G$ is identified with $S_*(BG)$. The rest is easy.

5 Characteristic Classes

References are [Cohomology of Classifying Space Toda] and [Characteristic Classes Milnor].

Def. (IX.6.5.1). Axioms for Chern classes for complex bundles:
- $c(E) = 1 + c_1(E) + \ldots + c_n(E) \in H^*(X, \mathbb{Z})$, $n = \deg(E)$.
- $f^*(c(E)) = c(f^*(E))$.
- $c(E \oplus F) = c(E)c(F)$.
- On the tautological bundle over $\mathbb{CP}^1$, $c(\eta) = 1 + c_1(\eta)$ and $\int_{\mathbb{CP}^1} c_1(\eta) = -1$. There is an affine connection definition of Chern class.

Prop. (IX.6.5.2). There exists uniquely a natural transformation $c : Vect_\mathbb{C}(X) \to H^*(X, \mathbb{Z})$ satisfying these axioms. (For this, it suffice to calculate the cohomology ring of $BGL_n(\mathbb{C})$, Cf.[Cohomology of Classifying Space Toda]).

Prop. (IX.6.5.3) (First Chern Class Map). A complex line bundle can be seen as an element of $H^1(X, \mathbb{C}^*)$, by (V.6.2.12), by the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\exp(1 \pi \cdot -)} \mathbb{C}^* \to 0$$

($\mathbb{C}$ is sheaf of smooth functions from $X$ to $\mathbb{C}$)which gives a map $H^1(X, \mathbb{C}^*) \to H^2(X, \mathbb{Z})$, called the first Chern class map. It is called so because it gives the first Chern class of this complex line bundle. It is also an isomorphism because $\mathbb{C}$ is fine sheaf so acyclic.

Proof: Only have to prove they are equal in $H^2(X, \mathbb{C})$. We choose a totally convex covering $U_i$ of $X$ by (IX.3.3.21), then it is a fine cover, so by (V.6.2.13) the Čech cohomology and sheaf cohomology equal.

Use the Chern-Weil map definition of the Chern class, a connection on a line bundle satisfies $\nabla e_\alpha = \omega_\alpha e_\alpha$, and if $e_\beta = e_\alpha g_{\alpha\beta}$, then $\omega_\beta = g_{\alpha\beta}^{-1} \omega_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta} = \omega_\alpha + d(\log g_{\alpha\beta})$. So $\Omega_\alpha = d\omega_\alpha$ locally, and the first Chern class is giving by $\Omega_\alpha$ in $H^2(X, \mathbb{C})$. 
Then we need to understand the deRham isomorphism. For the exact sequence $0 \rightarrow \mathbb{C} \rightarrow A^0 \rightarrow A^1 \rightarrow \ldots$, it has a splitting: $0 \rightarrow \mathbb{C} \rightarrow K^1 \rightarrow 0$ and $0 \rightarrow K^1 \rightarrow 0 \rightarrow A^1 \rightarrow K^2 \rightarrow 0$, this gives

$$0 \rightarrow H^1(X, K^1) \overset{\delta}{\rightarrow} H^2(X, \mathbb{C}) \rightarrow 0, \quad A^1(X) \rightarrow K^2(X) \overset{\delta}{\rightarrow} H^1(X, K^1) \rightarrow 0.$$ 

because $A^k$ are fine sheaves. The composite of them is just the de Rham isomorphism (Here we are identifying $H^2(X, \mathbb{C})$ to $H^2(X, \mathbb{C})$ by (V.6.8.12)). Tracking the lifting, we notice $\Omega$ is mapped to the cocycle \{log $g_{\alpha \beta} + log g_{\beta \gamma} - log g_{\alpha - \beta}$\}, which is exactly the image of the first Chern class map. \hfill \square

**Cor. (IX.6.5.4).** Complex line bundles are characterized by the first Chern class up to smooth isomorphism, because $H^1(X, \mathbb{C}^*) \rightarrow H^2(X, \mathbb{Z})$ is an isomorphism.

**Def. (IX.6.5.5).** Axioms for **Stiefel-Whitney classes** for real bundles:
- $w(E) = 1 + w_1(E) + \ldots + w_n(E) \in H^*(X, \mathbb{Z}/2\mathbb{Z})$, $n = \deg(E)$.
- $f^*(w(E)) = w(f^*(E))$.
- $w(E \oplus F) = w(E)w(F)$.
- On the tautological bundle over $\mathbb{R}P^1$, $w(\eta) = 1 + w_1(\eta)$ and $\int_{\mathbb{C}P^k} c_1(\eta) = -1$.

**Def. (IX.6.5.6).** The Pontryagin class is defined as $p_k(E) = (-1)^k c_k(E \mathbb{C}) \in H^{4k}(X, \mathbb{Z})$.

**Def. (IX.6.5.7).** Axioms for **Euler classes** for orientable real bundles:
- if $E$ has non-vanishing section, then $e(E) = 0$.
- $f^*(w(E)) = w(f^*(E))$.
- $w(E \oplus F) = w(E)w(F)$.
- for the opposite orientation $\overline{E}$, $e(\overline{E}) = -e(E)$.

**Definition via Classifying Space**

**Prop. (IX.6.5.8).**

\[
H^*(BO(n), \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_1, w_2, \ldots, w_n].
\]

\[
H^*(BSO(n), \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_2, \ldots, w_n].
\]

Cf.[Cohomology of Classifying Space Toda P82].

**Prop. (IX.6.5.9).**

\[
H^*(BU(n), R) = R[c_1, c_2, \ldots, c_n].
\]

\[
H^*(BSU(n), R) = R[c_2, \ldots, c_n].
\]

Cf.[Cohomology of Classifying Space Toda P81].

**Prop. (IX.6.5.10).** For $R$ of characteristic $\neq 2$,

\[
H^*(BSO(2n + 1), R) = R[p_1, p_2, \ldots, p_n].
\]

\[
H^*(BSO(2n), R) = R[p_1, p_2, \ldots, p_n, e], e^2 = p_n.
\]

Cf.[Cohomology of Classifying Space Toda P81].
Prop. (IX.6.5.11).  There are maps \( t : SO(n) \to U(n) \), and it will induce a map of classifying spaces, and induce\[ p_k = (-1)^kBt^*(c_{2k}). \]

There are maps \( O(n) \xrightarrow{\cong} U(n) \xrightarrow{\cong} SO(2n) \), and it will induce a map of classifying spaces, and induce
\[ Bi^*(c_k) = w_k^2, \quad Bj^*(w_{2k}) = c_k. \]

There are maps \( k : U(n) \to SO(m) \) \( m = 2n \) or \( 2n + 1 \), then for a field \( R \) of characteristic \( \neq 2 \), \[ Bk^*(p_k) = \sum_{i+j=k} (-1)^ic_ic_j, \quad Bk^*(e) = c_n. \]

Cf.[Cohomology of Classifying Space Toda P81].

Applications

Prop. (IX.6.5.12).  Note that \( K(\mathbb{Z}, 2) = \mathbb{C}P^\infty = B(K(\mathbb{Z}, 1)) = BS^1 \) (IX.6.4.8) thus it is also the classifying space of \( U(1) \), thus we have \( H^2(X, \mathbb{Z}) \cong [X, K(\mathbb{Z}, 2)] \cong \text{complex line bundles on } X \). Similarly, we have \( H^1(X, \mathbb{Z}/2\mathbb{Z}) \cong \text{real line bundles on } X \).

6 K-Groups

Def. (IX.6.6.1).  If \( X \) is a paracompact Hausdorff space, the K-group \( K(X) \) is defined to be \( K(Vect(X)) \) (I.11.1.21), which is a ring under sum and tensor. Two vector bundles \( E, F \) are called stable equivalent if \( [E] = [F] \).

A continuous map \( f : X \to Y \) induces a group morphism \( f^* : K(Y) \to K(X) \), and by (IX.6.1.9), this map only depends on the homotopy type of \( f \).

Prop. (IX.6.6.2).  By (IX.6.2.17), over a compact Hausdorff space, \( E, F \) being stably equivalent iff \( E \oplus \mathbb{R}^n \cong F \oplus \mathbb{R}^n \) for some \( n \).

Thm. (IX.6.6.3) (Periodicity Theorem).  Let \( L \) be a line bundle over \( X \), then as a \( K(X) \)-algebra, \( K(P(L \oplus 1)) \) is generated by \( [H] \), and is subject to the single relation \( ([H] - [1])([L][H] - [1]) = 0 \).

Proof:  Cf.[K theory, Atiyah, P46]. \( \square \)

Cor. (IX.6.6.4).  \( K(S^2) \) is generated by \( [H] \) as a \( K(pt) \)-module, and is subject to the single relation \( ([H] - [1])^2 = 0 \).

Cor. (IX.6.6.5).  There is an isomorphism \( \mu : K(X) \otimes K(S^2) \to K(X \times S^2) \), where \( \mu(a \otimes b) = (\pi_1^*a)(\pi_2^*b) \).

Def. (IX.6.6.6) (Setup for Proof of Periodicity Theorem).  Given a line bundle \( E \) over \( X \), we can associate a projective bundle \( P(E) \) that \( P(E)_x = P(E_x) \). Now denote \( P^0 \) the subspace of \( P(E) \) consisting of all vectors of length \( \leq 1 \) and \( P_\infty \) the subspace consisting of all vectors of length \( \geq 1 \) together with the infinity section. There are projections \( \pi_0 : P_0 \to X \) and \( \pi_\infty : P_\infty \to X \), which are homotopy equivalences.

Now by (IX.6.2.7),

7 Adam Operators
CHAPTER IX. GEOMETRY

IX.7 Symplectic Geometry

Cf. [Methods in Classical Mechanics Arnold Chapter 8], [辛几何讲义范辉军].

1 Basics

Symplectic Forms

Def. (IX.7.1.1). A symplectic form $\omega$ is a closed 2-form that is non-degenerate on any point. A smooth manifold with a symplectic form is called a symplectic manifold. A symplectic manifold must be even dimensional and orientable.

Prop. (IX.7.1.2). A hamiltonian phase flow preserves the symplectic form. $g^* \omega = \omega$.

Proof: by Cartan’s magic formula,

$$\frac{d}{dt} (g^t)^* \omega = L_X \omega = \iota_X (d\omega) + d(\iota_X \omega) = \iota_X \omega$$

because $\omega$ is closed. And by definition, $d(\iota_X \omega)(\eta) = \omega(JdH, \eta) = \langle dH, \eta \rangle$, so $d(\iota_X \omega) = dH$. Thus the theorem.

For the following Cf. [辛几何讲义范辉军 lecture 3].

Prop. (IX.7.1.3) (Moser’s Stability). If $\omega_t$ is a smooth family of cohomologous forms on a closed manifold $M$, then there exists an isotopy $\Psi_t$ s.t.

$$\Psi_t^* (\omega_t) = \omega_0.$$  □

Prop. (IX.7.1.4) (Relative Moser Stability). If $M$ is a closed manifold and $S$ is a compact submanifold, then if two closed 2-form equals on $S$, then there is an open neighborhood $N_0, N_1$ of $S$ and a diffeomorphism $\Psi : N_0 \to N_1$ that

$$\Psi|S = id, \Psi^* \omega_1 = \omega_0.$$  □

Cor. (IX.7.1.5) (Darboux’s Theorem). Every symplectic form $\omega$ on $M$ is locally diffeomorphic to the standard form $\omega_0$ on $\mathbb{R}^{2n}$.

Proof: Choose $S = pt$ and uses relative Moser stability. □

Prop. (IX.7.1.6). For a compact symplectic manifold $M$, its even dimensional cohomology groups doesn’t vanish, because $\omega^k$ are nontrivial.

Proof: This is because $\omega^n$ is a volume form on $M$ that never vanish, so it gives $M$ an orientation and $\int_M \omega^n \neq 0$. If $\omega^k$ is exact, then $\omega^n$ is exact, so $\int_M \omega^n = 0$ by Stokes’, contradiction. □
IX.8 Lie Groups

Main references are [Eti21], [Lee13], [Kna96].

1 Basics

Def. (IX.8.1.1) (Lie Groups). Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, a Lie group is a group object in the category of smooth manifolds over $\mathbb{K}$. Notice it suffices to check that multiplication is smooth. The left and right translations $L_g, R_g$ are all smooth morphisms hence diffeomorphisms.

A homomorphism of Lie groups is a smooth morphism that is also a group homomorphism. By translation invariance, a group homomorphism always has constant ranks, so a homomorphism of Lie groups that is a bijection is an isomorphism by global rank theorem (IX.2.1.10).

The tangent space of $G$ at $e$ is denoted by $\mathfrak{g}$.

Proof: We show if a topological group $G$ that is a smooth manifold satisfies $m : G \times G \to G$ is smooth, then $G$ is a smooth manifold: consider the map $F : G \times G \to G \times G : (g, h) \mapsto (g, gh)$, it is a smooth map that is bijective.

The tangent map of $F$ at $(g, h)$ is $(X, Y) \mapsto (X, (dR_h)_g(X) + (dL_g)_h(Y))$, which is surjective because $L_g$ is a diffeomorphism by (IX.2.1.10). Then $F^{-1} : G \times G \to G \times G : (g, h) \mapsto (g, g^{-1}h)$ is smooth, and $g \mapsto g^{-1}$ is smooth.

Prop. (IX.8.1.2). A connected Lie group is automatically second countable.

Proof: This follows from the fact that a Lie group is a manifold hence locally second-countable and it is a union of products of a nbhd of $e$ (IX.1.12.3).

Prop. (IX.8.1.3). Any homomorphism of smooth manifolds has constant rank.

Proof: $F$ being a homomorphism means that $F \circ L_g = L_{F(g)} \circ F$. Taking derivative and noticing the fact $L_g, L_{F(g)}$ are diffeomorphisms shows $dF_{g_0}$ and $dF_e$ have the same rank for any $g$.

Prop. (IX.8.1.4) (Discrete Subgroup). Any discrete subgroup $\Gamma$ of a Lie group $G$ is a closed Lie subgroup of dimension 0.

Proof: Firstly $\Gamma$ is countable: Let $U$ be a nbhd of $e$ containing no other points, choose another nbhd of $e$ that $VV \subset U$, then $\{gV\}_{g \in \Gamma}$ is a family of disjoint open subsets of $G$, so there are countably many because $G$ is second countable. Secondly $\Gamma$ is closed in $G$, because $\Gamma$ is closed in $G$: Let $U$ be a nbhd of $e$ containing no other points, choose another nbhd of $e$ that $VV \subset U$, then $\{gV\}_{g \in \Gamma}$ is a family of disjoint open subsets of $G$ each containing an element of $\Gamma$. Then it is clear $\Gamma$ is closed. Then (IX.8.1.22) shows $\Gamma$ is a closed Lie subgroup of dimension 0.

Def. (IX.8.1.5) (Adjoint Representation). For a Lie group $G$, the conjugation map $C_g : G \to G : h \mapsto ghg^{-1}$ is a Lie group homomorphism. Let $\text{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ denote its derivative, then $\text{Ad} : g \mapsto \text{Ad}(g)$ is an action of $G$ on $\mathfrak{g}$, because $C_{g_1g_2} = C_{g_1}C_{g_2}$, $\text{Ad}(g_1g_2) = \text{Ad}(g_1)\text{Ad}(g_2)$.

Prop. (IX.8.1.6). Let $G$ be a connected Lie group, and $\Gamma \subset G$ be a discrete normal subgroup. Show that $\Gamma$ is in the center of $G$.

Proof: For $\gamma \in \Gamma$, consider the map $G \to G : g \mapsto g\gamma g^{-1}$, then it is a map with images in $\Gamma$. But $\Gamma$ is discrete, so its image must be a single point, which is $\gamma$ because $e\gamma e^{-1} = \gamma$. This means $\gamma$ is in the center of $G$. 

□
Thus $\gamma$ is an integral curve of the left-invariant vector field on $G$ corresponding to $X$ (IX.8.3.3), which is unique.

Now to construct such a homomorphism, we use ODE theorems to construct a $\gamma$ satisfying this for $|t| < \varepsilon$, and then check $\gamma(s + t) = \gamma(s)\gamma(t)$ because they are both integral curves for $t$ starting at $\gamma(s)$. In particular, $d(L_{\gamma(t)})(X) = d(R_{\gamma(s)})(X)$. Now we can extend this $\gamma$ to whole of $\mathbb{K}$ by defining $\gamma(2^n s) = \gamma(s)^{2^n}$, then we check by induction on $n$ that

$$
\gamma'(t) = \frac{1}{2}(d(R_{\gamma(t/2)})\gamma'(\frac{t}{2}) + d(L_{\gamma(t/2)})\gamma'(-\frac{t}{2}))(X) = d(R_{\gamma(t/2)})(R_{\gamma(t/2)})(X) = d(R_{\gamma(t)})(X).
$$

Cor. (IX.8.1.10) (One-Parameter Subgroup and Lie Algebras). Let $G$ be a Lie group, then the one-parameter subgroups of $G$ correspond to maximal integral curves of left invariant vector fields starting at $e$. In particular, the one-parameter subgroups of $G$ corresponds to $\mathfrak{g}$ and also $T_e(G)$.

Also, the flow of the right-invariant vector field $R_X$ is given by $(g,t) \mapsto \exp(tX)g$, and the flow of the left-invariant vector field $L_X$ is given by $(g,t) \mapsto g \exp(tX)$.

Def. (IX.8.1.11) (Exponential Map). Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$, then we can define an exponential map $\exp : \mathfrak{g} \to G$ that for any $X \in \mathfrak{g}$, $\exp(X) = \gamma_X(1)$, where $\gamma_X$ is the one-parameter subgroup of $G$ generated by $X$ (IX.8.1.10). It can be shown that $\gamma(sX)$ is the one-parameter subgroup of $G$ generated by $X$.

Prop. (IX.8.1.12) (Properties of Exponential Map).

1. $\exp : \mathfrak{g} \to G$ is a smooth map which is a local diffeomorphism near $0$ that $\exp(0) = e, \exp_* = \text{id}_g$.

2. $\exp(s + t) = \exp(s)\exp(t)$ for $s, t \in \mathbb{K}$.

3. For any group homomorphism $\varphi : G \to H$ and $X \in \mathfrak{g}$, $\varphi(\exp(X)) = \exp(\varphi_*(X))$. 

Proof: For the Lie algebra, it suffices to show for $A \in \text{End}(\mathfrak{g})$, $[e^{tA}X, e^{tA}Y] = e^{tA}[X,Y]$ iff $A([X,Y]) = [A(X),Y] + [X,A(Y)]$.

One direction is by taking derivative w.r.t. $t$, for the other, we can show $y_1(t) = [e^{tA}X, e^{tA}Y]$ and $y_2(t) = e^{tA}[X,Y]$ both satisfy the ordinary differential equation $y_i(t)' = Ay_i(t)$. □
4. \( g \exp(tX)g^{-1} = \exp(\Ad(g)X) \). Also, \( \Ad_* = \ad \), or equivalently by item 3, \( \Ad(\exp(X)) = \exp(\ad(x)) \) as operators.

5. If we identify \( \mathfrak{gl}_n(K) \) with \( GL_n(K) \), then we can check directly that the exponential map of \( GL_n(K) \) is

\[
\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.
\]

**Proof:**

1: The smoothness follows from the smoothness of the solution of ODE. The smooth inverse theorem shows it is a local diffeomorphism at identity.

2: trivial.

3: Both \( \varphi(\exp(tX)) \) and \( \exp(t\varphi_*(X)) \) are integral curves of the vector field \( L_{\varphi_*(X)} \) and have the same initial point.

4: The first assertion is just item 3 applied to the conjugate action \( C_g \). For the Lie algebra homomorphism,

\[
d(\Ad)(X)Y = \frac{\partial}{\partial s} \frac{\partial}{\partial t} \exp(sX) \exp(tY) \exp(-sX) = [X,Y]
\]

by (IX.8.1.14).

**Prop. (IX.8.1.13).** Let \( G \) be a Lie group with Lie algebra \( \mathfrak{g} \), and \( \mathfrak{g} = A \oplus B \) a decomposition as subspaces, then the map

\[
F : A \oplus B \to G : (X,Y) \mapsto \exp(X) \exp(Y)
\]

is a local diffeomorphism at \( 0 \in \mathfrak{g} \).

**Proof:** Identify \( \mathfrak{g} \) and the tangent space of \( G \) at \( e \), then the differential \( F \) is identity, so it is a local diffeomorphism.

**Prop. (IX.8.1.14) (Baker-Campbell-Hausdorff).** Let \( G \) be a Lie group with Lie algebra \( \mathfrak{g} \), \( X,Y \in \mathfrak{g} \), then

\[
\exp(tX) \exp(tY) = \exp(\sum_{n=1}^{\infty} \frac{t^n}{n!} \mu_n(X,Y)).
\]

where \( \mu_n(X,Y) \) can be written as \( \mathbb{Q} \)-Lie polynomials of \( X \) and \( Y \) that is invariant of \( G \).

In particular, \( \mu_1(X,Y) = X + Y \), \( \mu_2(X,Y) = \frac{1}{2} t^3 ([X,[X,Y]] + [Y,[Y,X]]) \) and so on.

**Proof:** By the Lie correspondence (IX.8.3.15), we can first assume that \( G \) is simply-connected, then there is a mapping of \( G \) onto some subgroup of \( GL(n,k) \) with discrete kernel. If we can prove the formula for \( G = GL(n,k) \), then \( \exp(\sum_{n=1}^{\infty} \frac{t^n}{n!} \mu_n(X,Y)) \exp(tY)^{-1} \exp(tX)^{-1} \) is contained the kernel, but it is a smooth function in \( t \), and its value is 1 for \( t = 0 \), thus it holds for any \( t \).

Now let \( T \mathbb{K}^2 = \mathbb{K}(x,y) \) be the free non-commutative algebra in variables \( x, y \), the series \( \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) can be viewed as an element in \( \widehat{\mathbb{K}(x,y)} \). Then we can define

\[
\mu = \log(\exp(x) \exp(y)) \in \widehat{\mathbb{K}(x,y)},
\]

where \( \log(A) = -\sum_{n=1}^{\infty} \frac{(1-A)^n}{n!} \). Then \( \mu = \sum_{n=1}^{\infty} \frac{\mu_n}{n!} \), where \( \mu \in \mathbb{K}(x,y) \) are polynomials in \( x, y \) of degree \( n \) with coefficients in \( \mathbb{Q} \).
Then it remains to show that $\mu_n$ can be written as Lie polynomials in $x, y$. Notice 
\[ \Delta(x) = x \otimes 1 + 1 \otimes x, \]
thus \[ \Delta(\exp(x)) = \exp(x) \otimes \exp(x), \]
and \[ \Delta(\log(\exp(x) \exp(y))) = \log(\Delta(\exp(x) \exp(y))) = \log((\exp(x) \exp(y) \otimes 1) (1 \otimes \exp(x) \exp(y))) \]
= \[ \log(\exp(x) \exp(y)) \otimes 1 + 1 \otimes \log(\exp(x) \exp(y)). \]
then by separating degrees, each $\mu_n$ is primitive (I.15.1.3), thus they are contained in the free Lie-algebra generated by $x, y$, by (I.12.8.15) and (I.12.8.18).

The calculation is invariant of $n$ in $GL(n, \mathbb{R})$, thus it is invariant of $G$. □

**Cor. (IX.8.1.15).** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$, $X, Y \in \mathfrak{g}$, then

\[ \lim_{n \to \infty} (\exp(\frac{t}{n}X) \exp(\frac{t}{n}Y))^n = \exp(t(X + Y)). \]

**Proof:**

\[ (\exp(\frac{t}{n}X) \exp(\frac{t}{n}Y))^n = (\exp(\frac{t}{n}(X + Y) + \frac{t^2}{n^2}O(1)))^n = \exp(t(X + Y) + \frac{t^2}{n}O(1)). \]

Taking $n \to \infty$, we get the desired result. □

**Prop. (IX.8.1.16).** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $H, G$ be Lie groups over $\mathbb{K}$, then

- A continuous homomorphism $\gamma : \mathbb{K} \to H$ is smooth.
- A continuous homomorphism between smooth Lie groups $F : G \to H$ is smooth.
- There is at most one smooth structure on a Lie group $G$ that makes it a Lie group group.

**Proof:**

1: Let $V$ be a nbhd of $0 \in \mathfrak{h}$ that exp is a diffeomorphism on $2V$ (IX.8.1.12). Choose $t_0$ small that $\gamma(t) \in \exp(V)$ for any $|t| \leq t_0$, and let $X \in V$ that $\exp(X) = \gamma(t_0)$, then we can show $\gamma(t) = \exp(tX)$ for any $t = \frac{m}{n}$, so this holds for any $t$ by continuity/analyticity, and $\gamma$ is smooth.

2: By the proof of 1, we can construct a map(not necessary continuous) $F_* : \mathfrak{g} \to \mathfrak{h}$ with

\[ \begin{array}{ccc} \mathfrak{g} & \xrightarrow{F_*} & \mathfrak{h} \\ \xrightarrow{\exp} & & \xrightarrow{\exp} \\ G & \xrightarrow{F} & H \end{array} \]

can show $F_*$ is linear. Thus $F$ is smooth at a nbhd of $G$, and smooth everywhere by translation. □

**Group Aspects**

**Def. (IX.8.1.17) (Lie Subgroup).** A Lie subgroup of a Lie group $G$ is a subgroup that is also an immersed submanifold (IX.2.1.12).

An **embedded Lie subgroup** of a Lie group $G$ is a subgroup that is also an immerge submanifold (IX.2.1.12).

**Prop. (IX.8.1.18) (Lie subgroup is Weakly Embedded).** Any Lie subgroup of a Lie group $G$ is weakly embedded.

**Proof:** Cf.[Lee13]P506.
Lemma (IX.8.1.19). Let $G$ be a Lie group and $H$ a subgroup that is also an embedded submanifold, then $H$ is an embedded Lie subgroup.

Proof: We need to show the multiplication and inverse on $H$ is smooth: $H \times H \to G$ is smooth and has image in $H$, thus $H \times H \to H$ is also smooth, by (IX.2.1.20).

Lemma (IX.8.1.20). Let $G$ be a Lie group and $H \subset G$ a Lie subgroup, if $H$ is an embedded submanifold of $G$, then $H$ is closed in $G$.

Proof: Assume $H$ is an embedded submanifold of $G$, then it is locally compact in the induced topology, so (X.6.1.6) shows $H$ is closed in $G$.

Lemma (IX.8.1.21). Let $G$ be a Lie group and $H$ a subgroup of $G$ that is also a closed subset of $G$, then $H$ is an embedded Lie subgroup.

Proof: By (IX.8.1.20), it suffices to show that $H$ is an embedded submanifold of $G$. Let $g$ be the Lie algebra of $G$, and define a subspace $h \subset g$ that $h = \{X \in g | \exp(tX) \in H, \forall t \in \mathbb{R}\}$. By (IX.8.1.15) and the fact $H$ is closed in $G$, $h$ is a linear subspace of $g$.

Next we show that there exists a nbhd $U$ of $0 \in g$ that is a diffeomorphism and also $\exp(U \cap h) = \exp(U) \cap H$: Let $U$ be any open nbhd of $0 \in g$ that is a diffeomorphism $U \to \exp(U)$, then $\exp(U \cap h) \subset \exp(U) \cap H$ by definition.

Let $b \subset g$ be chosen s.t. $h \oplus b = g$, then $F : h \oplus b \to G : (X, Y) \mapsto \exp(X) \exp(Y)$ is a local diffeomorphism. Choose nbhd $U$ of $0 \in g$ and $\tilde{U}$ of $0 \in h \oplus b$ that both $\exp|_U$ and $F|_{\tilde{U}}$ are diffeomorphisms, and choose a countable nbhd basis $\{U_i\}$ of $0 \in g$. Denote $V_i = \exp(U_i)$ and $\tilde{U}_i = F^{-1}(V_i)$, then $V_i$ is a nbhd basis of $e \in G$ and $\tilde{U}_i$ is a nbhd basis of $0 \in h \oplus b$. We may assume $U_i \subset U$ and $\tilde{U}_i \subset \tilde{U}$.

If $\exp(U_i \cap h) \subset \exp(U_i) \cap H$ for any $i$, then we can choose $h_i = \exp(Z_i) \in H$ that $h_i \notin \exp(U_i \cap h)$.

Because $\exp(U_i) = F(\tilde{U}_i)$, set $h_i = \exp(X_i) \exp(Y_i)$, where $(X_i, Y_i) \in \tilde{U}_i$. Now $Y_i \neq 0$, otherwise $\exp(Z_i) = \exp(X_i)$, which implies $Z_i = X_i$ and $h \notin \exp(U_i \cap h)$. Notice $\exp(Y_i) = \exp(X_i)^{-1}h_i \in H$.

Because $\tilde{U}_i$ is a basis of $h \oplus b$, $Y_i \to 0$, Choose an inner product on $b$, and let $c_i = |Y_i|$, then $c_i^{-1}Y_i$ lies on the unit sphere of $b$. Replacing by a subsequence, we can assume $c_i^{-1}Y_i \to Y$ for some $Y \in b$. Then $|Y| = 1$ by continuity.

For any $t \in \mathbb{R}$, let $n_i = \left[\frac{t}{c_i}\right]$, then $|nc_i - t| \leq c_i \to 0$, which means $n_iY_i \to tY$, so $\exp(n_iY_i) \to \exp(tY)$. But $\exp(n_iY_i) = \exp(Y_i)^{n_i} \in H$, so $\exp(tY) \in H$ because $H$ is closed. Thus $Y \in h$, contradiction.

Thus in this way we can construct a slice chart $\varphi$ of $H$ at $e$, and for any $h \in H$, because

$$L_h((\exp(U) \cap H)) = L_h(\exp(U)) \cap H,$$

$\varphi \circ L_{h^{-1}}$ is a slice chart of $H$ at $h$. Thus $H$ is an embedded submanifold of $G$ by (IX.2.1.15).

Prop. (IX.8.1.22) (Closed Subgroup Theorem). Let $G$ be a Lie group and $H$ a subgroup of $G$, then the following are equivalent:

- $H$ is closed in $G$.
- $H$ is an embedded submanifold.
- $H$ is an embedded Lie subgroup.

Proof: $3 \to 2$ is trivial, $3 \to 1$ is (IX.8.1.20). $1 \to 3$ is (IX.8.1.21), $2 \to 3$ is (IX.8.1.19).
Remark (IX.8.1.23). The dense line of the torus is a Lie subgroup that is not a closed Lie subgroup.

Lemma (IX.8.1.24) (One-Parameter subgroup of Subgroups). Let \( H \subset G \) be a Lie subgroup, then the one-parameter subgroups of \( H \) are exactly those of \( G \) with initial velocity in \( T_e(H) \).

Proof: This is because a one-parameter subgroup of \( H \) is naturally a one-parameter subgroup of \( G \), and two one-parameter subgroups with the same initial velocity is identical. \( \square \)

Prop. (IX.8.1.25). Let \( H \subset G \) be a Lie subgroup, then the exponential map of \( H \) is the exponential map of \( G \) restricted to \( h \), and

\[ h = \{ X \in g \mid \exp(tX) \in H, \forall t \in \mathbb{R} \}. \]

Proof: The first assertion is an immediate corollary of (IX.8.1.24). Now if \( X \in h \), then the first assertion shows \( \exp(tX) \in H \) for all \( t \). Conversely, if \( \exp(tX) \in H \) for all \( t \), then \( \exp(tX) \) is a smooth map to \( H \), by (IX.8.1.18), thus its derivative at \( e \) is in \( h \), which means \( X \in H \). \( \square \)

Prop. (IX.8.1.26) (Quotient Theorem for Lie Groups). Let \( G \) be a Lie group and \( H \) a normal closed Lie subgroup, then the quotient \( G/H \) is a Lie group, and the quotient map \( \pi \) is a surjective Lie group homomorphism with kernel \( H \).

Proof: By (IX.8.2.6), \( G/H \) is a smooth manifold that the quotient map is surjective, smooth, and is a group homomorphism with kernel \( H \). It suffices to show the multiplication of \( G/H \) is smooth, which is easy by (IX.2.1.6). \( \square \)

Prop. (IX.8.1.27) (First Isomorphism Theorem for Lie Groups). Let \( \varphi : G \to H \) be a Lie group homomorphism, then there kernel of \( F \) is an closed Lie subgroup of \( G \) with Lie algebra \( \text{Ker}(\varphi_*) \). The image of \( \varphi \) has a unique smooth structure making it a Lie subgroup of \( H \) that \( G/\text{Ker}(F) \to \text{Im}(\varphi) \) is a diffeomorphism, and it is a closed Lie subgroup when it is embedded in \( H \), e.g. when \( \varphi \) induces a proper action (IX.8.2.3).

Proof: This follows from (IX.8.2.2) and (IX.8.1.22). \( \square \)

Def. (IX.8.1.28) (Adjoint Group). For a Lie group \( G \), the center \( Z(G) \) of \( G \) is a closed Lie subgroup because it the kernel of \( \text{Ad}(\text{IX.8.1.27}) \). We call the group \( G/Z(G) \) the adjoint group of \( G \), which is an immersed subgroup of \( GL(g) \) by (IX.8.1.27).

Lie Groups and Analytic Groups

Prop. (IX.8.1.29). What condition makes a Lie group a complex Lie group?

Prop. (IX.8.1.30). Any connected Lie group has a compact subgroup as deformation contraction.

Proof: \( \square \)

Prop. (IX.8.1.31) (Gleason; Montgomery-Zippin). A real topological group \( G \) admits a (unique) Lie group structure iff the underlying topological space \( G \) is a topological manifold.

Prop. (IX.8.1.32) (Lie Groups and Analytic Groups). A real Lie group admits a unique real analytic structure, so it is not important to distinguish between a Lie group and an analytic Lie group, and call it analytic group if it is a connected Lie group. The uniqueness follows from (IX.8.1.16).

Proof: Use (IX.8.1.14) to show that in a local exp char \((U, \varphi)\) of 1, \(U \times U \rightarrow U\) is analytic, then we can choose an analytic atlas on \(G\) given by \(\{(gU, \varphi \circ L_{g^{-1}})\}\). This is an analytic atlas, because the transition function is \(\varphi L_{g^{-1}}^{-1}\) on \(U \cap h^{-1} gU\). Because \(hU \cap gU \neq \emptyset\), let \(x = hu_1 = gu_2\), then \(L_{g^{-1}} h = L_{u_2 u_1^{-1}}\), which is analytic.

To show multiplication is analytic w.r.t. this atlas: \(\Box\)

Cor. (IX.8.1.33). The proof above can be used to show that any \(C^k\)-group manifold be upgraded uniquely to a Lie group structure. So basically the study of \(C^0\)-group manifold and analytic groups are the same.

2 Homogeneous Spaces

Actions of Lie Groups

Prop. (IX.8.2.1) (Fundamental Theorem on Lie Group Actions). Let \(\theta\) be a right smooth action of a Lie group \(G\) on a smooth manifold \(M\), then we can define a complete Lie algebra homomorphism

\[
\bar{\theta} : g \rightarrow \mathfrak{X}(M) : \theta(X)_p = \frac{d}{dt}|_{t=0} p \cdot \exp(tX) = d(\theta(p))_e(X_e),
\]

called the infinitesimal generator of \(\theta\). where a Lie homomorphism \(\bar{\theta} : g \rightarrow \mathfrak{X}(M)\) is called complete iff for any \(X \in g\), \(\theta(X)\) is a complete vector field (IX.2.5.3).

Conversely, if \(G\) is simply-connected and \(\bar{\theta} : g \rightarrow \mathfrak{X}(M)\) is a complete Lie algebra homomorphism, then there exists a unique smooth right action \(\theta\) of \(G\) on \(M\) with infinitesimal generator \(\bar{\theta}\).

Proof: \(\theta(X)\) is smooth because it is the infinitesimal generator of the smooth flow \(\mathbb{R} \times M \rightarrow M\) : \((t, p) \mapsto p \exp(tX)\) (IX.2.5.4). \(\theta\) is a Lie algebra homomorphism by (IX.2.2.13). \(\Box\)

Prop. (IX.8.2.2) (Isotropy Group and Orbits). Let \(\theta\) be an action of \(G\) on a manifold \(M\), let \(\bar{\theta}_x : g \rightarrow T_x M\) be given by \(\bar{\theta}_x(X) = \theta_x(X)\). Then

- The stabilizer \(G_x\) is a closed subgroup of \(G\) with Lie subalgebra \(g_x = \text{Ker}(\bar{\theta}_x)\).
- \(Gx\) has a unique smooth structure making it an immersed subgroup of \(M\) that \(G/G_x \rightarrow Gx\) is a diffeomorphism, and \(T_x(Gx) \cong \text{Im}(\bar{\theta}_x) \cong g/g_x\).

Proof: Cf. [E621]P48. \(\Box\)

Prop. (IX.8.2.3). If a Lie group \(G\) acts properly on a manifold \(M\), then each orbit is a closed subgroup of \(M\), and each isotropy group is compact, by (IX.1.12.15) and (IX.1.12.17).

Prop. (IX.8.2.4) (Quotient Map Theorem). Let \(G\) be a Lie group that acts smoothly, freely and properly on a manifold \(M\), then the quotient space \(M/G\) is a topological manifold with dimension \(\dim M - \dim G\), and it has a unique smooth structure that \(M \rightarrow M/G\) is a smooth submersion.

Proof: Cf. [Lee13]P544. \(\Box\)
Homogeneous Spaces

Def. (IX.8.2.5) (Homogeneous Spaces). Let $G$ be a Lie group, then a homogeneous space for $G$ is a smooth manifold $M$ with a smooth transitive $G$-action.

Prop. (IX.8.2.6) (Characterizing Homogeneous Spaces). Let $G$ be a Lie group.

- if $H$ is a closed subgroup of $G$, then the left coset space $G/H$ is a topological manifold of dimension $\dim G - \dim H$, and has a unique smooth structure that the quotient map $G \to G/H$ is a smooth submersion. With this smooth structure, the left action of $G$ on $G/H$ turns it into a homogeneous space.

- If $M$ is a homogeneous $G$-space, and $p \in M$, then the isotropy group $G_p$ of $M$ is a closed subgroup of $G$, and $G/G_p \to M$ is a diffeomorphism of $G$-spaces.


Cor. (IX.8.2.7) (Homogeneous Space Structure on Sets). Suppose $X$ is a set with a transitive action of a Lie group $G$ that for some point $p \in X$, the isotropy group $G_p$ is closed in $G$. Then $X$ has a unique smooth manifold structure with respect to which the given action is smooth. With this structure, $\dim X = \dim G - \dim G_p$.

Proof: This is because $G/G_p$ is a smooth manifold that is $G$-equivariantly isomorphism to $X$, and the uniqueness also follows from the proposition.

Prop. (IX.8.2.8) (Quotients of Lie Groups by Discrete Subgroups). Let $G$ be a Lie group and $\Gamma$ a discrete subgroup of $G$, then $G/\Gamma$ is a smooth manifold, and the quotient map $G \to G/\Gamma$ is a smooth normal covering.

Proof: (IX.8.1.4) and the proof of (IX.8.2.6) shows $\Gamma$ acts smoothly, freely, and properly on $G$ on the right. Then the theorem is a consequence of (IX.2.1.8).

Prop. (IX.8.2.9) (Contractible Homogeneous Space). If $X$ is a homogeneous $G$-manifold that is contractible, $x \in X$, then $G$ is diffeomorphic to $G_x \times X$.

Proof: 

Prop. (IX.8.2.10) (Examples of Homogeneous Spaces). Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

- The Grassmannian manifold $G_k(\mathbb{K}^n)$ is defined to be the set of $k$-dimensional spaces in $\mathbb{K}^n$. $GL_k(n, \mathbb{K})$ acts transitively on $G_k(\mathbb{K}^n)$, and the stabilizer $B$ is a closed subset of $GL_k(n, \mathbb{K})$. Thus $G_k(\mathbb{K}^n)$ is a homogeneous $G_k(\mathbb{K}^n)$-space by (IX.8.2.7).

- The Stiefel manifold $V_k(\mathbb{K}^n)$ of orthogonal $k$-frames in $\mathbb{K}^n$ is defined to be the set of tuples $(v_1, \ldots, v_n)$ in $\mathbb{K}^n$ that $(v_i, v_j) = \delta_{ij}$. $U(n, \mathbb{K})$ acts transitively on $V_k(\mathbb{K}^n)$ with stabilizer $U(n-k, \mathbb{K})$, Thus $V_k(\mathbb{K}^n)$ is a homogeneous $G_k(\mathbb{K}^n)$-space by (IX.8.2.7).

- The flag Variety.

3 Lie Theory

Lie Algebras of Lie Groups

Def. (IX.8.3.1) (Invariant Vector Fields). Let $G$ be a Lie group, then a smooth vector field $X$ on $G$ is called left-invariant if $d(L_g)_{g'}(X_{g'}) = X_{gg'}$ for any $g, g' \in G$. The set of left-invariant vector fields on $G$ is denoted by $\text{Lie}(G)$. 
Prop. (IX.8.3.2). If \( X, Y \) are left-invariant vector fields over \( G \), then \([X, Y]\) is also left-invariant, by (IX.2.2.13).

Prop. (IX.8.3.3) (Invariant Vector Fields and Tangent Spaces). Let \( G \) be a Lie group, then the evaluation map \( \text{Lie}(G) \to \mathfrak{g} \) : \( X \mapsto X_e \) is a vector space isomorphism.

Proof: The inverse map is given by \( X \mapsto \frac{d}{dt}|_0 f(\exp(tX)) \). This is clearly a left-invariant vector field. It suffices to show that this \( \tilde{X} \) is smooth. By (IX.2.2.10), it suffices to show \( L_X(f) \) is smooth for any smooth function \( f \).

\[
L_X(f)(g) = d(L_g)_e(X)(f)(g) = X(L_g f)(0) = \frac{d}{dt}|_0 f(g \exp(tX))
\]

which is a differential of a smooth map \( \mathbb{R} \times G \to \text{GL} : (t, g) \mapsto f(g \exp(tX)) \), so it is smooth in \( g \).

Cor. (IX.8.3.4) (Lie Group is Parallelizable). Every Lie group admits a left-invariant smooth global frame, thus any Lie group is parallelizable.

Prop. (IX.8.3.5). If \( X \in \mathfrak{g} \) corresponds to a left-invariant vector field \( L_X \), then for any \( g \in G \),

\[
d(R_g)_p((L_X)_p) = (L_{\text{Ad}(g^{-1})X})_{pg}.
\]

Proof:

\[
d(R_g)_p((L_X)_p)(f)(pg) = (L_X)_p(R_g f) = d(L_p)_e(X)(R_g f) = X(L_p R_g(f)) = \frac{d}{dt}|_0 f(p \exp(tX)g)
\]

\[
= \frac{d}{dt}|_0 f(p \exp(t \text{Ad}(g^{-1})X)) = (\text{Ad}(g^{-1}X)(L_p(f)) = (L_{\text{Ad}(g^{-1})X})_{pg}(f)(pg)
\]

Def. (IX.8.3.6) (Lie Algebra of a Lie Group). If \( G \) is a Lie group, \( \text{Lie}(G) \) is a Lie algebra w.r.t. the Lie Bracket (IX.2.2.12). It is called the Lie algebra associated to \( G \).

Proof: This is clear from the definition \([X, Y](f) = XY(f) - YX(f)\).

Prop. (IX.8.3.7) (Induced Map of Lie Algebras). A homomorphism \( F : G \to H \) of Lie groups induces a morphism of their Lie algebras via the tangent space.

Proof: For any \( X \in \text{Lie}(G) \), define \( F_*(X)_g = (dL_g)_e(dF_e(X_e)) \), then this is a left-invariant vector field, and it clearly corresponds to the tangent map via isomorphism in (IX.8.3.3). This map is a Lie algebra map by a variant of (IX.2.2.13).

Cor. (IX.8.3.8). If \( H \subset G \) is a Lie subgroup, then there is a natural isomorphism

\[
\mathfrak{h} \cong \{ X \in \text{Lie}(G) | X_e \in T_e H \}.
\]

In particular, the tangent space \( \mathfrak{h} \) of \( H \) is a Lie subalgebra of \( \mathfrak{g} \).
Proof: There is a commutative diagram

\[
\begin{align*}
\text{Lie}(H) \xrightarrow{X \mapsto X_e} & \mathfrak{h} \\
\text{Lie}(G) \xrightarrow{X \mapsto X_e} & \mathfrak{g}
\end{align*}
\]

Prop. (IX.8.3.9) (Covering of Lie Groups). Let \( F : G \to H \) be a homomorphism of connected Lie groups, then the following are equivalent:

- \( F \) is surjective with discrete kernel.
- \( F \) is a smooth covering map.
- \( F \) is a local diffeomorphism.
- The induced homomorphism \( F_* : \mathfrak{g} \to \mathfrak{h} \) is an isomorphism.

Proof: 1 \( \to \) 2: \( F \) is surjective thus \( H \) is a homogeneous \( G \)-space, so (IX.8.2.6) shows \( H \cong G/\text{Ker}(F) \). And \( G \to G/\text{Ker}(F) \) is a smooth covering map by (IX.8.2.8).
2 \( \to \) 3 is trivial.
3 \( \to \) 1: If \( F \) is a local diffeomorphism, then \( \text{Ker}(F) \) is discrete, and \( F \) is open. Thus \( F(G) \) is an open subgroup of \( H \), thus all of \( H \) because \( H \) is connected (IX.1.12.3).
3 \( \to \) 4 is trivial. 4 \( \to \) 3 is by inverse function theorem. □

Cor. (IX.8.3.10) (Homomorphism with Discrete Kernel). Let \( G \to H \) be a homomorphism of Lie groups of the same dimension with discrete kernel and \( H \) connected, then it is a covering space map, and the kernel is in the center of \( G \), by (IX.8.1.6).

Proof: This homomorphism is locally injective at 1, so it has rank \( \dim G = \dim H \), so it is local diffeomorphism. In particular, the image contains a nbhd of 1 in \( H \), thus it contains all \( H \) by (IX.1.12.3). □

Prop. (IX.8.3.11) (Universal Covering Lie Group). If \( G \) is a connected Lie group, then its universal covering space \( \tilde{G} \) can be given a Lie group structure that the covering map is a group homomorphism. Moreover, the group structure is unique up to isomorphism over \( G \). Moreover, the kernel of \( \tilde{G} \to G \) is a discrete central subgroup of \( \tilde{G} \).

Proof: Because \( \tilde{G} \) is simply connected, so does \( \tilde{G} \times \tilde{G} \), let \( \tilde{e} \) be an element over \( e \), we can lift the map \( \tilde{G} \times \tilde{G} \to \tilde{G} \times \tilde{G} \xrightarrow{m} \tilde{G} \) to a map \( \tilde{m} : \tilde{G} \times \tilde{G} \to \tilde{G} \) that \( \tilde{m}(\tilde{e}, \tilde{e}) = \tilde{e} \). Similar we can lift an inverse map \( \tilde{i} \) that \( \tilde{i}(\tilde{e}) = \tilde{e} \). These maps are smooth because \( \pi : \tilde{G} \to G \) is a local diffeomorphism.

It’s left to show that \( (\tilde{m}, \tilde{i}) \) makes \( \tilde{G} \) into a Lie group: For example, the map \( L_G^{-1} : \tilde{G} \to \tilde{G} \) is a lift of \( \text{id} \) and it coincides with \( \text{id}_{\tilde{G}} \) on a point \( \tilde{e} \), thus it is just \( \text{id}_{\tilde{G}} \), which means \( \tilde{e} \) is a left identity. The rest is easy.

For the uniqueness: By the universal property of covering, if there are two coverings, we can lift it to a map connecting them that maps identity to identity, then show it is a group homomorphism.

The last assertion follows from (IX.8.3.10). □

Prop. (IX.8.3.12) (Lie Subalgebra Subgroup). For a Lie group \( G \), for any lie subalgebra \( \mathfrak{h} \subset \mathfrak{g} \), there exists uniquely a connected Lie subgroup \( H \) s.t. \( \mathfrak{h} \) is the lie algebra of \( H \).

Proof: By (IX.2.3.1), there is a maximal connected manifold \( H \) corresponding to \( \mathfrak{h} \), we only need to show that it is a group. But the left invariance of \( \mathfrak{h} \) shows that \( HH \subset H \) because \( H \) is maximal. Cf.[Lee13]P506. □
Cor. (IX.8.3.13). If $G_1$ is a simply connected Lie group and $G_2$ is a connected Lie group, then any Lie algebra homomorphism $	ilde{h} : g_1 \to g_2$ can be lifted to a unique Lie group homomorphism.

**Proof:** Consider the image of $\tilde{h} : \Gamma(\tilde{h}) \subset g_1 \times g_2$, which is a Lie subalgebra. First notice a Lie group homomorphism $h$ is equivalent to a Lie subgroup $G_h$ of $G_1 \times G_2$ that $\pi_1|_{G_h}$ is a diffeomorphism onto $G_1$. And this Lie homomorphism induces the desired Lie algebra homomorphism iff the Lie algebra of $G_h$ is just $\tilde{h}$.

(IX.8.3.12) shows that there exists a unique Lie group $G$ in $G_1 \times G_2$ with Lie subalgebra $\Gamma(\tilde{h})$. The projection $\pi_1|_G$ is a diffeomorphism onto $G_1$, because the tangent map at $e$ is an isomorphism, thus a local diffeomorphism and by (IX.8.3.10) a covering map, so it must be an isomorphism because $G_1$ is simply connected and $G$ is connected. □

Cor. (IX.8.3.14) (Representations of Simply-Connected Lie Groups). The category of representations of s simply-connected Lie groups is equivalent to the category of representations of its Lie algebra.

Thm. (IX.8.3.15) (The Lie Correspondence).
- The category of finite dimensional Lie algebras is equivalent to the category of simply connected Lie groups.
- For a f.d. Lie algebra $\mathfrak{g}$, the connected Lie groups with Lie algebras isomorphic to $\mathfrak{g}$ corresponds to $G/\Gamma$, where $G$ is a simply connected subgroup with Lie algebra $\mathfrak{g}$, and $\Gamma$ is a discrete central subgroup of $G$.

**Proof:**
1: By (IX.8.3.12)/(IX.8.3.13) together with Ado’s theorem.
2: (IX.8.3.11) and (IX.8.3.9) shows any Lie group is a quotient of its universal covering Lie group by a discrete central subgroup. Conversely, for any discrete central subgroup, $G/\Gamma$ is a Lie subgroup and $\pi : G \to G/\Gamma$ is a homomorphism with kernel $\Gamma$ by (IX.8.1.4) and (IX.8.1.26), thus $\text{Lie}(G) = \text{Lie}(G/\Gamma) \cong \mathfrak{g}$ (IX.8.1.26). □

Prop. (IX.8.3.16) (Ideals and Normal Subgroups). Let $G$ be a connected Lie group and $H$ a connected Lie subgroup, then $H$ is a normal subgroup of $G$ iff $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

**Proof:** Because $G,H$ are both connected, (IX.1.12.3) shows $H$ is normal in $G$ iff for any $X \in \mathfrak{g}, Y \in \mathfrak{h}, \exp(X) \exp(Y) \exp(X)^{-1} \in H$. Taking derivative w.r.t. $Y$, this is equivalent to $d(\text{Ad}(\exp(X)))(Y) = \text{ad}(X)Y \in \mathfrak{h}$ (IX.8.1.12), by (IX.8.1.25), which is equivalent to $\mathfrak{h}$ being an ideal of $\mathfrak{g}$. □

Prop. (IX.8.3.17) (Center). Let $G$ be a connected subgroup with Lie algebra $\mathfrak{g}$ and $Z$ the center of $G$, $\mathfrak{z}$ the center of $\mathfrak{g}$, then $Z$ is a closed Lie subgroup of $G$ with Lie algebra $\mathfrak{z}$.

**Proof:** Because $G,H$ are both connected, (IX.1.12.3) shows $H$ is normal in $G$ iff for any $X \in \mathfrak{g}, Y \in \mathfrak{h}, \exp(X) \exp(Y) \exp(X)^{-1} \in H$. Taking derivative w.r.t. $Y$, this is equivalent to $d(\text{Ad}(\exp(X)))(Y) = \text{ad}(X)Y \in \mathfrak{z}$ (IX.8.1.12), by (IX.8.1.25), which is equivalent to $\mathfrak{h}$ being an ideal of $\mathfrak{g}$. □

Prop. (IX.8.3.18) (Chevalley’s Theorem). Let $G$ be a complex connected Lie group and $\mathfrak{g}$ the Lie algebra of $G$, $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}$, and $W$ the Weyl group, then the restriction of functions induces a graded algebra isomorphism

$$\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{h}]^W.$$
Prop. (IX.8.3.19) (Simply-Connected Compact Lie Groups). Any simply connected compact Lie groups is a product of the following types:

- Spin$(n)$ for $n \geq 3$.
- $SU(2)$ for $n \geq 2$.
- Sp$(n)$ for $n \geq 1$.
- $E_6, E_7, E_8, F_4, G_2$.

Proof: \qed

4 Classical Groups

For more classical groups, Cf. [Classical Groups Baker].

Def. (IX.8.4.1) (Examples of Classical Groups). Let $\mathbb{K}$ be either $\mathbb{R}, \mathbb{C}$.

- For any associative algebra $\mathbb{K}$ over a field, the general linear group $GL(n, \mathbb{K})$ is the subgroup of $M_n(\mathbb{K})$ consisting of invertible matrices.

- The special linear group $SL(n, \mathbb{K})$ is the subgroup of $GL_n(\mathbb{K})$ consisting of matrices of determinant 1. $SL(n, \mathbb{H})$ is the subgroup of $GL_n(\mathbb{K})$ of matrices of determinant 1, where the determinant is defined in (I.1.11.12).

- The orthogonal group $O(n, \mathbb{K})$ is the group of matrices preserving a bilinear form. In some coordinate, $(x, y) = \sum_{i=1}^{n} x_i y_i$, then in this coordinate,

$$O(n, \mathbb{K}) = \{ M \in GL_n(\mathbb{K}) | M^t M = I \}.$$

- The unitary group $U(n)$ is the subgroup of $GL_n(\mathbb{C})$ consisting of matrices fixing a non-degenerate Hermitian form.

- The special unitary group $SU(n) = U(n) \cap SL_n(\mathbb{C})$.

- The symplectic group $Sp_{2n}(\mathbb{K})$ is the group fixing an alternating form in $2n$ variables. In some coordinate, $\omega = \sum_{i=1}^{n} x_i \wedge y_i + \sum_{i=n+1}^{n} x_i \wedge y_i$, then in this coordinate,

$$Sp_{2n}(\mathbb{K}) = \{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} | A^t C = C^t A, \quad A^t D - C^t B = I, \quad B^t D = D^t B \}.$$

- The pseudo-unitary groups $U(p, q, \mathbb{K})$: If $\mathbb{K} = \mathbb{R}$, it is the subgroup of $GL(n, \mathbb{R})$ consisting of matrices preserving a bilinear form of signature $(p, q)$. If $\mathbb{K} = \mathbb{C}$, it is the subgroup of $GL(n, \mathbb{C})$ consisting of matrices preserving a Hermitian form of signature $(p, q)$. If $\mathbb{K} = \mathbb{H}$, it is the subgroup of $GL(n, \mathbb{H})$ consisting of matrices preserving a non-degenerate quaternionic Hermitian form of signature $(p, q)$(I.1.11.13).

- The quaternionic orthogonal group $O^*(2n)$ is the subgroup of $GL(n, \mathbb{H})$ consisting of matrices preserving a non-degenerate quaternionic skew-Hermitian form(I.1.11.13).

- $PGL(n, \mathbb{K})$ is the quotient group of $GL(n, \mathbb{K})$ by scalar matrices.

- $PSL(n, \mathbb{K})$ is the quotient group of $SL(n, \mathbb{K})$ by scalar matrices.

- $PSO(2n)$ is the quotient group of $SO(2n)$ by scalar matrices.
• \( PU(n) \) is the quotient group of \( SU(n)(\text{or } U(n)) \) by scalar matrices.

**Proof:** \( GL_n(\mathbb{K}) \) has a natural smooth structure as an open subset of \( \mathbb{R}^{n^2} \), and the multiplication map is clearly smooth, so it is a Lie group by (IX.8.1.1). The other groups are closed subgroups of \( GL_n(\mathbb{K}) \), so they have unique smooth manifold structure making them Lie subgroups of \( GL_n(\mathbb{K}) \), by (IX.8.1.22)(IX.8.1.16). Finally the quotient group by normal closed subgroups have natural Lie group structure by (IX.8.1.26). □

**Prop. (IX.8.4.2) (\( SU(2) \)).**

\[
SU(2) = \{ A_{\alpha,\beta} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}, \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \}
\]

is isomorphic to the group of unit quaternions and diffeomorphic to \( S^3 \). The Lie algebra of \( SU(2) \) is isomorphic to \( \mathfrak{s}(2, \mathbb{R}) \).

**Proof:** Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SU(2) \), then

\[
aa + \overline{ab} = 1, \quad a\overline{c} + b\overline{d} = 0, \quad c\overline{e} + d\overline{d} = 1, ac - bd = 1.
\]

So \( (c, d) = \lambda \cdot (\overline{\beta}, \overline{\alpha}) \), and we can calculate \( \lambda = 1 \). So the first assertion follows.

The multiplication of \( SU(2) \) is given by \( A_{\alpha_1,\beta_1}A_{\alpha_2,\beta_2} = A_{\alpha_1\alpha_2 - \overline{\beta_2}\beta_1, \beta_1\alpha_2 + \overline{\beta_2}\overline{\alpha_1}} \), so the map

\[
SU(2) \to \mathbb{H} : A_{\alpha,\beta} \mapsto \alpha + \beta j
\]

is a group isomorphism onto the unit quaternions, which is clearly isomorphic to \( S^3 \). □

**Prop. (IX.8.4.3) (Action of \( SU(2) \)).**

• There is a double covering of Lie groups \( SU(2) \to SO(3) \) with kernel \{±1\}.

• There is a double covering of Lie groups \( SU(2) \times SU(2) \to SO(4) \) with kernel \{±1\}.

**Proof:** 1: Regard \( SU(2) \) as the unit quaternions and \( SO(3) \) the transformation group of pure unit quaternions, then \( SU(2) \to SO(3) \) is given by

\[
u \mapsto (v \mapsto uv\overline{u}).
\]

Because \( u \cdot \overline{u} \) preserves orthogonality relations, it preserves the space of pure quaternions, so it has image in \( O(3) \). But it has image in \( SO(3) \) because \( SU(2) \cong S^3 \) is connected. Its kernel is in the center of \( \mathbb{H} \), so must be ±1. Now (IX.8.3.10) shows this is a covering space map.

2: Consider the action \( SU(2) \times SU(2) \) on the section of unit vectors of \( \mathbb{H} \): \((u, v)(z) = uzv^{-1}\). if \((u, v)\) is in the kernel of this map, then \( uv^{-1} = 1 \), so \( u = v \) and \( u \) is in the center of \( \mathbb{H} \), so \( u = v = ±1 \). Because \( \dim SU(2) = 3 \) and \( \dim SO(4) = 6 \), (IX.8.3.10) shows this is a double cover. □

**Prop. (IX.8.4.4) (Action of \( SU(4) \)).** \( SU(4) \) acts on a 4-dimensional Hermitian space \( V \). Then its acts on the Hermitian space \( \wedge^2 V \). Consider the Hodge star operator \( \ast : \wedge^2 V \to \wedge^2 V \) (IX.2.3.9), it is an anti-linear operator, \( \ast^2 = \text{id} \) and \( \ast \) commutes with \( SU(4) \) action. Then \( W = \text{Ker}(\ast - \text{id}) \) is a real vector space of dimension 6 that \( SU(4) \) acts on. This kernel of this action is the same as the kernel of the action \( \wedge^2 V \), which is \{±1\}.
The Hermitian form induces a symmetric form on $V$: Take a conjugation on $V$, because $V = W \oplus iW$, $W = \text{Im}(\ast + i\text{id}V)$. So for $a, b \in W$, let $a = \ast c + c, b = \ast d + d$, then $(a, b) = ((\ast a, \ast b) + (a, \ast b) + (a, b))\omega = \ast a \wedge b + \ast b \wedge a + a \wedge b + \ast a \wedge b$ is a real form. Then this representation induces a map $SU(4) \to SO(6)$ (because $SU(4)$ is connected). Because $\dim SU(4) = \dim SO(6) = 6$, it is a double cover by (IX.8.3.10).

**Prop. (IX.8.4.5) (SU(1, 1) and SL(2, $\mathbb{R}$)).**

$$SU(1, 1) = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid ||a||^2 - ||b||^2 = 1 \}.$$  

The group $SL(2, \mathbb{C})$ acts on $\mathbb{P}^1(\mathbb{C})$ by linear transformations. The stabilizer of the upper half plane $\mathcal{H}$ under this action is $SL(2, \mathbb{R})$, the stabilizer of the unit disk $\mathbb{D}$ under this action is $SU(1, 1)$.

The action of the matrix $C = \frac{1}{\sqrt{2e^{i/4}}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \in SL(2, \mathbb{C})$ induces a Cayley transformation $\mathcal{H} \cong \mathbb{D}$, thus induces an conjugacy

$$SU(1, 1) = C \cdot SL(2, \mathbb{R}) \cdot C^{-1}$$

inside $SL(2, \mathbb{C})$.

**Proof:** If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \subset SU(1, 1)$, then

$$||a||^2 - ||c||^2 = 1, \quad a\overline{b} - c\overline{d} = 0, \quad ||b||^2 - ||d||^2 = -1, \quad ad - bc = 1$$

which means $(b, d) = \lambda(\overline{c}, \overline{a})$, and $\lambda = 1$.

$C$ induces an isomorphism between $\mathcal{H}$ and $\mathbb{D}$, because $\frac{z}{z+i} \in \mathbb{D}$ iff $|z - i| < |z + i|$ iff $z$ is in the upper half plane.

To show the stabilizer of $\mathcal{H}$ is $SL(2, \mathbb{R})$, notice first $SL(2, \mathbb{R})$ preserves $\mathcal{H}$ by (X.2.2.5), and it acts transitively on it because $\sqrt{y} \begin{bmatrix} y \\ x \\ 1 \end{bmatrix}$ maps $i$ to $x + iy \in \mathcal{H}$. \qed

**Prop. (IX.8.4.6) (Center of SU($p,q$)).** $Z_{U(p,q)}(SU(p,q))$ are all scalar matrices.

**Proof:** Firstly we consider $Z_{U(2)}(SU(2))$, if $A \in Z(SU(2))$, then it commutes with diag$(i, -i)$, so $A$ is a scalar. Similarly, If $A \in Z_{U(1,1)}(SU(1,1))$, then $A$ commutes with diag$(i, -i)$, thus $A$ is a scalar.

If $X \in Z(SU(p,q))$ is not a scalar matrix, then it has two eigenvectors $s, t$ with different eigenvalues. Consider the space $V$ generated by $s, t$ and its orthogonal complement, then $X$ restricted to this space is in the center of $U(V)$. Now $SU(V) \cong SU(1, 1)$ or $SU(2)$, both have scalar matrice as centers, so $X$ cannot have different eigenvalue, contradiction. Thus $X$ is a scaler matrix. \qed

**Prop. (IX.8.4.7) (Real and Complex Matrices).** $GL_n(\mathbb{C})$ can be embedded into $GL_{2n}(\mathbb{R})$, with determinant $|\det|^2$. And in this way, $U(n)$ is mapped into $O(2n, \mathbb{R})$. Also, $O(n, \mathbb{R})$ embeds into $U(n)$ diagonally.

**Proof:**

$$X + iY \mapsto \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \sim \begin{bmatrix} X & -Y \\ iX + Y & X - iY \end{bmatrix} \sim \begin{bmatrix} X + iY & Y \\ 0 & X - iY \end{bmatrix}$$

\qed
Prop. (IX.8.4.8) (Symplectic Groups).

- \( U(p, q, \mathbb{H}) = Sp(2n, \mathbb{C}) \cap U(2p, 2q, \mathbb{C}) \). \( U(n, \mathbb{H}) = U(n, 0, \mathbb{H}) \) is also denoted by \( Sp(n) \), called the compact symplectic group.

- \( O^*(2n) = O(2n, \mathbb{C}) \cap U(n, n) \).

- \( Sp(2n, \mathbb{R}) \cap O(2n, \mathbb{R}) = Sp(2n, \mathbb{R}) \cap GL(n, \mathbb{C}) = O(2n, \mathbb{R}) \cap GL(n, \mathbb{C}) = U(n) = \{ \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}, \; X + iY \in U(n) \} \).

Proof: 1: By (I.1.11.15), notice that any \( \mathbb{C} \)-linear automorphism preserving \( B_1 \) and \( B_2 \) is \( \mathbb{H} \)-linear.

2: By (I.1.11.15), notice that any \( \mathbb{C} \)-linear automorphism preserving \( B_1 \) and \( B_2 \) is \( \mathbb{H} \)-linear.

3: If \( A \in Sp(2n, \mathbb{R}) \cap O(2n, \mathbb{R}) \), then \( AA^t = A^tA = 1, A^tJA = 1 \), then \( AJ = JA \). The rest and the other identities are easy by (IX.8.4.7).

\[ \square \]

Clifford Algebra

Prop. (IX.8.4.9). Let \( Cl_{r,s} \) denote the real Clifford algebra of signature \( r - s \), then

\[ Cl_{1,0} \cong \mathbb{C}, \quad Cl_{0,1} \cong \mathbb{R} \oplus \mathbb{R}, \quad Cl_{2,0} \cong \mathbb{H} \subset M(2, \mathbb{C}), \quad Cl_{0,2} \cong R(2) = M(2, \mathbb{R}), \]

And we have

\[ Cl_{n+2,0} \cong Cl_{0,n} \otimes Cl_{2,0}, \quad Cl_{0,n+2} \cong Cl_{n,0} \otimes Cl_{0,2}. \]

by the mapping \( e_i \rightarrow e_i \otimes e_1', e_{n+j} \rightarrow 1 \otimes e_j' \).

So we have

\[ Cl_{n+8,0} \cong Cl_n \otimes R(16), \quad Cl_{n,0} = Cl_{n+2,0} \otimes \mathbb{C} = Cl_{n,0} \otimes \mathbb{C}(2). \]

because \( \mathbb{H} \otimes \mathbb{C} = \mathbb{C}(2) \), and

\[
\begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
Cl_{0,0} & \mathbb{R} & \mathbb{C} & \mathbb{H} & \mathbb{H} \oplus \mathbb{H} & \mathbb{H}(2) & \mathbb{C}(4) & \mathbb{R}(8) & \mathbb{R}(8) \oplus \mathbb{R}(8) \\
Cl_{n,0} & \mathbb{C} & \mathbb{C} & \mathbb{C} \oplus \mathbb{C}
\end{bmatrix}
\]

The Clifford algebra is a \( \mathbb{Z}_2 \)-graded algebra, \( Cl = Cl^0 \otimes Cl^1 \) and \( Cl_{n,-1} \cong Cl_n^0 \) by the mapping \( e_i \rightarrow e_i \otimes e_{n+1} \). This is in fact the decomposition of the chirality operator \( \Gamma = (-1)^{\frac{n+1}{2}}e_1 e_2 \ldots e_n, \Gamma^2 = 1 \).

Prop. (IX.8.4.10). For \( n \) even, \( \mathbb{C}(V) \) is naturally isomorphic to \( \text{End}_\mathbb{C}(\Lambda^n W) \), where \( W = \{ \frac{1}{\sqrt{2}}(e_{2i-1} - ie_{2i}) \} \). This isomorphism is not obvious and restrict to a Spinor representation of \( \text{Spin}(n) \) and \( \rho(\Gamma)^2 = 1 \) induce two representations of \( Cl(n)^0 \), in particular \( \text{Spin}(n) \), called the (half Spinor representations). This has a unique extension to representation of Spin. \( \Lambda^n W \) comes with a Hermitian metric which is preserved by the action of \( \text{Pin}(n) \) (check). So the image is \( SO(\Lambda^n W) \). Cf. [Jost Geometric analysis P72].

Def. (IX.8.4.11) (Spin(\( n \))). denote \( \text{Pin}(n) \) as the group in \( Cl_n \) generated by \( v_i \) of norm 1. Because \( v_i \cdot v_i = -1 \), it is a group. And denote \( \text{Spin}(n) \) as the subgroup of \( \text{Pin}(n) \) generated by even number of \( v_i \)'s.
Prop. (IX.8.4.12) (Action of Spin(n)). The conjugation action $Ad(v) = v(-)v$ =reflection w.r.t $v$, maps Pin(n) to $O(n)$ and Spin(n) to SO(n). The kernel of this mapping is $\{\pm 1\}$ when $n$ is even. This is a double covering of SO(n) and O(n), it is nontrivial because 1, −1 is connected by $(\cos te_1 + \sin te_2)(\cos te_1 − \sin te_2)$.

In particular, Spin(n) is a universal covering of SO(n) and thus simply-connected for $n \geq 3$.

Proof: Let $\alpha = e_i \beta + \gamma$, then $\beta, \gamma \in CL^0$ and so $\alpha = ce_1 \cdots e_n + d$, and c can happen only when n is odd. \qed

Prop. (IX.8.4.13) (Center of Spin(n)). $Z(\text{Spin}(n)) = \begin{cases} S^1 & n = 2 \\ \mathbb{Z}/2\mathbb{Z} & n = 2k + 1, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & n = 4k, \\ \mathbb{Z}/4\mathbb{Z} & n = 4k + 2, \end{cases}$

Proof: \qed

Prop. (IX.8.4.14) (Low Dimensional Accidental Isomorphisms).
- Spin(2) $\cong$ SO(2) $\cong U(1)$.
- Spin(3) $\cong$ SU(2) because they are both universal covering of SO(3), by (IX.8.4.15) and (IX.8.4.12).
- Spin(4) $\cong$ SU(2) $\times$ SU(2) because they are both universal coverings of SO(4), by (IX.8.4.15) and (IX.8.4.12).
- Spin(5) $\cong$ Sp(2).
- Spin(6) $\cong$ SU(4).

Fundamental Groups

Prop. (IX.8.4.15).
- $SU(n − 1) \rightarrow SU(n)$ $\rightarrow$ $S^{2n−1}$ shows $SU(n)$ are simply connected.
- $SO(n − 1) \rightarrow SO(n)$ $\rightarrow$ $S^{n−1}$ shows $\pi_1(SO(n)) \cong \pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$ by (IX.8.4.3) (IX.8.4.15)and(IX.8.4.2). And $\pi_1(SO(2)) = \mathbb{Z}$.
- $U(n − 1) \rightarrow U(n)$ $\rightarrow$ $S^{2n−1}$ shows $\pi_1(U(n)) \cong \mathbb{Z}$.
- $SO(n, \mathbb{R})$ is a deformation retraction of $SL(n, \mathbb{R})$, and $SU(n)$ is a deformation retraction of $SL(n, \mathbb{C})$.

\[ \pi_1(PSO(n)) = \begin{cases} \mathbb{Z} & n = 2 \\ \mathbb{Z}/2\mathbb{Z} & n = 2k + 1, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & n = 4k, \\ \mathbb{Z}/4\mathbb{Z} & n = 4k + 2, \end{cases} \]

Spin(n), so $\pi_1(PS(n)) = Z(\text{Spin}(n))(\text{IX.8.4.13})$.
- $\pi_1(PSU(n)) = \mathbb{Z}/n\mathbb{Z}$.
- $\pi_1(PSp(2n)) = \pi_1(U(n)) = \mathbb{Z}$ and the determinant induces an isomorphism onto $\pi_1(S^1)$. In fact, this is used to define the Maslov index.
Generals

Prop. (IX.8.4.16). Every finite subgroup of $SO(3, \mathbb{R})$ is conjugate to one of the following:
- the cyclic group $C_n$ generated by rotation.
- the dihedral group $D_{2n}$ generated by adjoining a reflection to the rotation.
- the group $A_4$ of rotation of the tetrahedron.
- the group $S_4$ of rotations of the octahedron.
- the group $A_5$ of rotations of the icosahedron.

Proof: Cf.[Dornhoff, Group Representation Theory, 1971 Part A, Chap26].□

Prop. (IX.8.4.17).
- The exponential map for $GL_n(\mathbb{C})$ is surjective.
- The image of the exponential map for $GL_n(\mathbb{R})$ is $GL_n(\mathbb{R})^2$.
- The image of exponential map for $B_+$ which is the subgroup of $GL(n, \mathbb{R})$ consisting of upper-triangular matrices with positive entries, is surjective.
- The exponential for a compact Lie group is surjective, by (IX.8.10.6) and (IX.8.10.2).

Proof: 1: Use Jordan forms. Notice the logarithm of $(cI + N), c \neq 0$ is definable for $N$ nilpotent, and it is a polynomial function of the matrix itself.
2: It is clear the image is contained in $GL(n, \mathbb{R})^2$, conversely, we see from the complex case that any $B \in GL(n, \mathbb{R})$ is of the form $\exp(P(B))$ for some polynomial $P \in \mathbb{C}[X]$, then $T = B^2 = \exp(P(B) + \overline{P}(B)) \in \exp(GL(n, \mathbb{R}))$.
3: Use Jordan forms. Notice the logarithm of $(cI + N), c \neq 0$ is definable for $N$ nilpotent and $c > 0$, and it is a polynomial function of the matrix itself, so it is also upper-triangular. □

5 Compact Lie Groups and Representations

Prop. (IX.8.5.1). Any compact connected complex Lie group is Abelian. And it is a complex tori. So we only consider only compact real Lie groups.

Proof: Cf.[compact-connected-complex-Lie-group-Abelian]. □

Cor. (IX.8.5.2). $U(n)$ is not a complex Lie group, in particular not a complex algebraic variety.

Prop. (IX.8.5.3). Let $G$ be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and center $Z$. Let $G_{ss}$ be the connected subgroup of $G$ corresponding to the Lie subalgebra $[\mathfrak{g}, \mathfrak{g}]$ (IX.8.3.12), then $G_{ss}$ has finite center, and $Z^0, G_{ss}$ are closed in $G$, and $G = Z^0G_{ss}$.

Proof: Cf.[Kna96]P198. □

Prop. (IX.8.5.4) (Compact Lie Group and Representations). A compact group $G$ is a real Lie group iff it has a faithful f.d. representation.

Proof: If it has a faithful f.d. representation, then $G \subset GL(n, \mathbb{R})$ compact hence closed, thus a Lie subgroup by (IX.8.1.22).
Conversely, if $G$ is a Lie group, then we can choose a small nbhd of $e$ that contains no subgroup of $G$ (choose $\exp(\frac{1}{2}V)$ where $\exp$ is an diffeomorphism on $V$). Consider kernel $K_\pi$ for irreducible
representations $\pi$ of $G$, then $\cap_{\pi} K_{\pi} = \emptyset$ by Gelfand-Raikov (X.6.2.19), in particular $\cap_{\pi}(K_{\pi} - U) = \emptyset$. But $G - U$ is compact, hence there are f.m. $\pi_i$ that $\cap_{\pi_i} K_{\pi_i} \in U$, but by definition of $U$, $\cap_{\pi_i} K_{\pi_i} = 0$, which gives a f.d. faithful representation of $G$, by (X.6.4.5). □

**Remark (IX.8.5.5).** Theory of Representations of compact Lie groups are a special case of abstract harmonic analysis X.6.

**Examples of Representations**

**Prop. (IX.8.5.6) (Representations of $SU(2)$).** $SU(2) \cong S^3$ is simply connected with Lie algebra $\mathfrak{sl}_2(\mathbb{R})$(IX.8.4.2), thus we can use (IX.8.3.14) and (III.6.1.13) to see that the representations of $SU(2)$ are all of the form $W_n$, where $W_n$ is the representation of $SU(2)$ on the space of homogenous polynomials of degree $n$ in two variables $x, y$ induced by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$.

*Proof:* Check the central character of the induced representations of Lie algebra. □

**Maximal Compact Subgroup**

For maximal compact subgroup of general locally compact subgroups, Cf.1.

**Prop. (IX.8.5.7) (Uniqueness for Semisimple Lie group).** Maximal compact subgroup exists by (X.6.1.38), and for a semisimple Lie group $G$, the maximal compact subgroup is unique up to conjugation.

*Proof:* Cf.[Wiki]. □

**Prop. (IX.8.5.8) (Examples of Maximal Subgroups).**

- $U(n)$ is the maximal compact subgroup of $GL(n, \mathbb{C})$.
- $SU(n)$ is the maximal compact subgroup of $SL(n, \mathbb{C})$.
- $O(n)$ is the maximal compact subgroup of $GL(n, \mathbb{R})$.
- $SO(n)$ is the maximal compact subgroup of $SL(n, \mathbb{R})$.

*Proof:* It suffices to show that for any compact subgroup $K$, there is an invariant Hermitian(bilinear) form that is fixed by $K$. For this, use the averaging method, as in (X.6.4.2). □

6 Decompositions

**Prop. (IX.8.6.1) (Cartan Decomposition).** Let $G = GL(n, \mathbb{R}), K = O(n)$ or $G = GL(n, \mathbb{R})^+, K = SO(n)$, then every double coset $K \setminus G / K$ has a unique representation of diagonal matrix $D$ with decreasing positive entries.

*Proof:* For the existence, given $g$, consider $S = g^t g = k_1^{-1} \text{diag}(\lambda_1, \ldots, \lambda_n) k_1$, where $k_1 \in SO(n)$(I.1.7.2). Then consider $k_2 = g k_1^{-1} \text{diag}(\lambda_1^{-1/2}, \ldots, \lambda_n^{-1/2})$ it is orthogonal and $g = k_2 \text{diag}(\lambda_1^{1/2}, \ldots, \lambda_n^{1/2}) k_1$.

For the uniqueness, consider $g k_1^{-1} \text{diag}(\lambda_1^{-1/2}, \ldots, \lambda_n^{-1/2})$ is orthogonal, thus $k_1 S k_1^{-1}$ is diagonal with decreasing positive entries, thus uniquely defined. □
Prop. (IX.8.6.2) (Polar Decomposition). \( GL_n(\mathbb{R}) \) can be decomposed as \( P \cdot O(n) \), where \( P \) is a positive symmetric matrix and \( O(n) \) the orthogonal matrix. A positive symmetric matrix can be diagonalized, so \( GL_n(\mathbb{R}) \) have \( O(n) \) as deformation kernel.

Similarly, \( Sp(2n) \) can be decomposed as \( P \cdot U(n) \), because \( O(2n) \cap Sp(2n) = U(n) \). And it has \( U(n) \) as deformation kernel.

Prop. (IX.8.6.3) (QR-Decomposition).

- Any real matrix \( A \) has the form \( A = QR \) where \( Q \) is orthogonal and \( R \) is upper triangular with positive diagonal entries. Moreover, if \( A \) is invertible, then the decomposition is unique.

- Any complex matrix \( A \) has the form \( A = QR \) where \( Q \) is unitary and \( R \) is upper triangular with positive diagonal entries. Moreover, if \( A \) is invertible, then the decomposition is unique.

- Any Quaternion matrix \( A \) has the form \( A = QR \) where \( Q \in U(n, \mathbb{H}) \) (IX.8.4.8) and \( R \) is upper triangular with positive diagonal entries. Moreover, if \( A \) is invertible, then the decomposition is unique.

Proof: We only prove for \( GL(n, \mathbb{R}) \), the rest is similar. Use Gram-Schmidt orthogonalization: choose a basis \( v = \{v_1, \ldots, v_n\} \) of \( V \) and \( A \) acts on \( V \), then \( A \) maps \( \{v_1, \ldots, v_n\} \) to a set \( \{w_1, \ldots, w_n\} \). Then we can define an orthonormal basis \( \{e_1, \ldots, e_n\} \) that \( v_k \in \text{span}\{e_1, \ldots, e_k\} \). Now let \( \{w_1, \ldots, w_n\} = \{e_1, \ldots, e_n\}R \), then \( R \) is upper triangular, and \( Q = [e_1, \ldots, e_n]_v \) is orthogonal, and \( A = QR \). We can make diagonal entries of \( R \) positive by left multiplying a diagonal orthogonal matrix.

To show the uniqueness when \( A \) is invertible, let \( A = Q_1R_1 = Q_2R_2 \), then \( A'A = R_1'R_1 = R_2'R_2 \). Then \( (R_2)^{-1}R_1' = R_2R_1^{-1} \), where the LHS is lower-triangular and the RHS is upper-triangular, which means both of them are diagonal. Then if \( \alpha_i \) are the diagonal entries of \( R_1 \) and \( \beta_i \) are the diagonal entries of \( R_2 \), then \( \alpha_i/\beta_i = \beta_i/\alpha_i \), which means \( \alpha_i = \beta_i \), and \( R_2R_1^{-1} = 1 \).

Prop. (IX.8.6.4) (Bruhat Decomposition). Let \( K \) be a field, then

\[ GL(n, K) = BWB \]

where \( W \) is the set of permutation matrices, \( B \) is the invertible upper triangular matrices, and the decomposition is a disjoint union w.r.t. \( W \).

Proof: For any matrix \( M \in GL(n, K) \), consider the first column, then there is a lowest term \( a_{i_1} \) that are not zero, then we can left multiply an upper triangular matrix \( b_1 \) s.t. the first column of \( b_1M \) has only one non-zero entry \( a_{i_1} = 1 \), then consider the second column but ignore the \( k \)-th row, we can find a lowest term \( a_{i_2} \) that is non-zero, then left multiply an upper triangular matrix \( b_2 \) that the second column of \( b_2b_1M \) has only one non-zero entry \( a_{i_2} = 1 \). Now continuing this way, we find a permutation \( \sigma \) that only the entries \( a_{ij} \) that \( j \geq \sigma^{-1}(i) \) are non-zero. Then we can find an upper triangular matrix \( c \) that \( b_n\ldots b_1Mc \) is a permutation matrix \( M_{\sigma^{-1}} \).

So we proved \( BWB = GL(n, K) \). Now it suffices to show if \( M_{\sigma_1}^{-1}bM_{\sigma_2} \in B \) for some \( b \in B \), then \( \sigma_1 = \sigma_2 \). Because \( M_{\sigma} = \sum_{i} e_{\sigma(i)i} = \sum_{i} e_{i\sigma^{-1}(i)} \),

\[ M_{\sigma_1}^{-1}bM_{\sigma_2} = \sum_{ij} b_{\sigma_1(i)\sigma_2(j)}e_{ij}. \]

This is an element in \( B \), both its \( (\sigma_1^{-1}(k), \sigma_2^{-1}(k)) \)-entry is \( b_{kk} \neq 0 \), thus \( \sigma_1^{-1}(k) \leq \sigma_2^{-1}(k) \), which implies \( \sigma_1 = \sigma_2 \).
Prop. (IX.8.6.5) (Smith Normal Form). Let $R$ be a PID, and $K$ its fraction field. Choose a representative $P$ for associativity classes of any prime in $R$(to eliminate the distraction of units), then there is complete set of representatives for the double cosets of $GL(n, R) \setminus GL(n, K) / GL(n, R)$ consisting of diagonal matrices diag$(f_1, \ldots, f_n)$, where $f_i \in K$ are products of elements in $P$, and $f_k$ divides $f_{k+1}$. Notice $GL(n, R)$ is the matrices with unit determinants in $R$.

Proof: For the uniqueness, clearly the row operators doesn’t change the greatest common divisor of $k \times k$ minors of $M$ for any $k$(change by a scalar but the diagonal entries are monic), thus the entries are determined by the minors of $M$.

For the existence, for any $g \in GL(n, K)$ there is an $r \in R$, that $rg$ has coefficients in $R$, and also $r$ are products of elements in $P$. let $M$ be the submodule of $R^n$ generated by the rows of $rg$, then by the elementary divisor theorem(I.2.4.22), there exists a basis $\xi_i$ for $R^n$ and $d_i \in R$ that $d_i | d_{i+1}$, and $\{d_i\xi_i\}$ form a basis of $M$. we may assume $d_i$ are products of elements in $P$. Then the matrix $\xi$ with rows $\xi_i$ are in $GL(n, R)$, and the rows of the matrix diag$(d_1, \ldots, d_n)\xi$ span the same module as $rg$. Then

$$K_1 rg = \text{diag}(d_1, \ldots, d_n) \xi$$

for some $K_1 \in GL(2, R)$, so $g$ are in the same double coset as $\text{diag}(d_1, \ldots, d_n)$.

Prop. (IX.8.6.6) (Iwasawa Decomposition).

7 Semisimple Lie Groups

Def. (IX.8.7.1) (Semisimple Lie Group). A semisimple/solvable/nilpotent/simple Lie group is a Lie group with a semisimple/solvable/nilpotent/simple Lie algebra.

Def. (IX.8.7.2) (Adjoint Lie Group). An adjoint Lie group is a semisimple real Lie group(IX.8.7.1) with trivial center.

8 Real Reductive Groups

Remark (IX.8.8.1). Recall the definition of a reductive group(V.10.2.10). Let $G_\mathbb{R}$ be a reductive group.

Def. (IX.8.8.2) (Real Forms). Let $G$ be a connected complex Lie group, then a real form of $G$ is a connected Lie subgroup $K \subset G$ that $\mathfrak{k} \otimes_\mathbb{R} \mathbb{C} \cong \mathfrak{g}$.

Prop. (IX.8.8.3) (Real Lie Group and Algebraic Group). Let $H$ be a connected Lie group with Lie algebra $\mathfrak{h}$, then there needs not be an algebraic group $G$ over $\mathbb{R}$ that $G(\mathbb{R})^0 = H$. For example, the topological fundamental group of $SL_2(\mathbb{R})$ is $\mathbb{Z}$, so it has many coverings of finite degree, none of which is algebraic, because $SL_2$ as an algebraic group is simply connected?

However, if $H$ admits a f.d. representation $H \rightarrow GL(V)$, then there exists an algebraic group $G \subset GL(V)$ that $\text{Lie}(G) = [\mathfrak{h}, \mathfrak{h}]$. So if $H$ is semisimple, then there exists an algebraic group $G \subset GL(V)$ s.t. $\text{Lie}(G) = [\mathfrak{h}, \mathfrak{h}]$. When $H$ is semisimple, this means $G^0 = H$.

Compact Real Algebraic Groups

Def. (IX.8.8.4) (Compact Real Algebraic Groups). A real algebraic group $G$ over $\mathbb{R}$ is called compact iff $G(\mathbb{R})$ is a compact Lie group.
Def. (IX.8.8.5) ( Relevant Group). A compact real algebraic group is called relevant iff the map $\pi_0(G(\mathbb{R})) \to \pi_0(G)$ is surjective, where the RHS is the algebro-geometric group of connected components.

Lemma (IX.8.8.6). Let $Z$ be an affine variety over $\mathbb{R}$, let $X$ be a subset of $Z(\mathbb{R})$, and let $I_X$ be the ideal of regular functions on $Z$ that vanishes at $X$, then $X' = V(I_X)$ satisfies $X \subset X'(\mathbb{R})$. Also by construction, $X'$ is relevant and $X$ intersects real points of every connected components of $X'$.

Now if $Z$ is acted on by a compact Lie group $K$ and $X$ is a single $K$-orbit, then $X \cong X'(\mathbb{R})$.

Proof: Cf. [Gaitsgory P17]. □

Prop. (IX.8.8.7). The functor $G \mapsto G(\mathbb{R})$ is an equivalence of categories from the category of relevant real compact groups to the category of compact Lie groups.

Proof: For the fully faithfulness: given a map $\varphi : G_1(\mathbb{R}) \to G_2(\mathbb{R})$, we need to show it comes from a unique algebraic group homomorphism. Let $K \subset G_1(\mathbb{R}) \times G_2(\mathbb{R})$ be the graph of $\varphi$, then let $\Gamma$ be the subgroup of $G_1 \times G_2$ corresponding to $K$ in (IX.8.8.6), then it suffices to prove that the map $\Gamma \to G_1$ is an isomorphism. It is an isomorphism after passing to real points, so isomorphism on the level of Lie algebras. And then it is an isomorphism, because both groups are relevant? □

Cor. (IX.8.8.8). The proof actually works only if $G_1$ is relevant compact real group. So if we choose $G_2 = GL(n, \mathbb{C})_{\mathbb{R}}$, then by adjointness there is a bijection

$\text{Hom}_{\text{AlgGrp}/\mathbb{C}}(G_\mathbb{C}, GL(n, \mathbb{C})) \cong \text{Hom}_{\text{LieGrp}}(G(\mathbb{R}), GL_n(\mathbb{C}))$.

That is, their complex representations correspond.

Complex Reductive Algebraic Groups

Prop. (IX.8.8.9). If $G$ is a real reductive group, then its complexification $G_{\mathbb{C}}$ is complex reductive.

Proof: It suffices to show that $\text{Rep}(G(\mathbb{C}))$ is semisimple. For this, notice $\text{Rep}(G)$ is semisimple by definition, so it suffices to show for any representation $V$ of $G_{\mathbb{C}}$, if $W$ is a $G$-invariant subspace, then $W$ is also $W$ is $G_{\mathbb{C}}$-invariant. But the invariance condition is a vanishing of some matrix coefficients, they vanish on $G$ so also vanish on $G_{\mathbb{C}}$. □

Def. (IX.8.8.10) ( Real Form). A real form on a complex reductive Algebraic group is an anti-linear group isomorphism $\sigma : G \to G$ that $\sigma^2 = 1$. It is called compact iff $G^\sigma$ is compact real, and it is called relevant iff $G^\sigma$ is relevant compact (IX.8.8.5).

Prop. (IX.8.8.11) (Polar Decomposition). If $G$ is a complex algebraic group and $K \in G(\mathbb{C})$ is a compact Lie subgroup. Assume that

- $g \cong \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$.
- $K$ intersects non-trivially every connected components of $G(\mathbb{C})$.

Then the group $G$ contains a unique real structure $\sigma$ that $K = G(\mathbb{C})^\sigma$. And if $p \subset g$ be the subspace $\{\xi \in g | \sigma(\xi) = -\xi\}$, then the map

$k \times p \to G(\mathbb{C}) : (k, p) \mapsto k \cdot \exp(p)$

is a diffeomorphism.
Proof: Cf.[Gaitsgory P18].

Cor. (IX.8.8.12). If we denote \( P = \exp(p) \), then \( P \subset \tilde{P} = \{ g \in G(\mathbb{C}) | \sigma(g) = g^{-1} \} \), and there is a diffeomorphism:
\[
\prod_{k \in K, k^2 = 1} \{ k \} \times P \cong \tilde{P}.
\]

Cor. (IX.8.8.13). In the situation of (IX.8.8.11), \( G \) is reductive, by (IX.8.8.9).

Cor. (IX.8.8.14). \( K \to G(\mathbb{C}) \) is a homotopy equivalence.

Maximal Compact Subgroup

Cor. (IX.8.8.15). For any compact subgroup \( K' \subset G(\mathbb{R}) \), there exists an element \( g \in P^{r} \) s.t. \( Ad_g(K') \in K^{r} \).

Proof: Cf.[Gaitsgory P25].

9 Representations of (Non-Compact)Lie Groups

Remark (IX.8.9.1) (Non-compact Lie Groups). The main feature of representations of non-compact Lie groups is that it will be \( \infty \)-dimensional and involves topologies. The vector space \( V \) is assumed to be Hausdorff, second countable and complete, locally convex, and separable.

And subrepresentations are assumed to be closed.

Prop. (IX.8.9.2) (Regular Representations). For a connected Lie group \( G \), consider its left and right regular action \( \lambda, \rho \) on \( C^{\infty}(G) \) (X.6.1.1). We will write \( dX \) for \( X \in g \) as the representation of Lie algebra of \( G \) via \( \rho \), then it commutes with \( \lambda \). So it induces a map of \( U(g) \) to the ring of left \( G \)-invariant differential operators on \( G \) (I.12.8.1).

Prop. (IX.8.9.3) (Center element Bi-invariant). If \( G \) is a connected Lie group with Lie algebra \( g \) and \( D \in Z(U(g)) \), then the differential operator \( D \) defined in (IX.8.9.2) is invariant under both left and right regular representations of \( G \).

Proof: The left invariance is general from (IX.8.9.2), for the right invariance, Because \( G \) is connected, it suffices to prove invariance for a nbhd of identity of \( G \), thus suffices to prove
\[
\rho(g_t)D = D\rho(g_t), \quad g_t = \exp(tX).
\]

For this, let \( \varphi(g, t) = (\rho(g_t)D - D\rho(g_t))(g) \) and take derivative w.r.t \( t \), then it reads:
\[
\frac{\partial}{\partial t}\varphi(g, t) = (DdX\rho(g_t)f - dX\rho(g_t)Df)(g) = dX\varphi(g, t)
\]
because \( dX \) commutes with \( D \). And also \( \varphi(g, 0) = 0 \), so by lemma (IX.8.9.4), \( \varphi(g, t) = 0 \) for any \( t, g \).

Lemma (IX.8.9.4). If \( G \) is a connected Lie group with Lie algebra \( g \), and \( \varphi \in C^{\infty}(G \times \mathbb{R}) \) satisfies
\[
\frac{\partial}{\partial t}\varphi(g, t) = dX\varphi(g, t)
\]
for some \( X \in g \) and \( \varphi(g, 0) = 0 \), then \( \varphi = 0 \).
Proof: Let \( \varphi_g(u, v) = \varphi(g \exp(uX), v) \), then the condition says

\[
\frac{\partial}{\partial u} \varphi_g(u, v) = \frac{\partial}{\partial v} \varphi_g(u, v),
\]

which means \( \varphi_g(u, v) = F(u + v) \), and the fact \( \varphi_g(u, 0) = 0 \) shows \( F = 0 \).

\( \Box \)

Cor. (IX.8.9.5). If \( G = GL(2, \mathbb{R})^+ \), then \( g = \mathfrak{gl}_2(\mathbb{R}) \), and the Casimir element (I.12.8.20) \( \Delta = -1/2(1/2h^2 + ef + fe) \) corresponds to a bi-invariant differential operator on \( C^\infty(G) \), and it is called the Laplace-Beltrami operator.

**Differential Vectors**

**Def. (IX.8.9.6) (Smooth Vectors).** Let \( V \) be a continuous representation of \( G \) on a locally convex TVS. we define the space \( V^\infty \) of smooth vectors of \( V \) as \( V^\infty = \cap_n V^n \) where

\[
V^0 = \{ v \in V^{n-1}, d/dt \exp(t\xi)v \text{ exists}, \} \quad V^n = \{ v \in V^{n-1}, T_\xi(v) \in V^{n-1}, \forall \xi \in \mathfrak{g} \}.
\]

And \( V^\infty \) is given the inverse limit topology.

**Prop. (IX.8.9.7) (Action of Distribution on Smooth Vectors).** There is a continuous action of \( \text{Distr}_c(G) \) on \( V^\infty \): \( T \mapsto \pi(T) \) compatible with the convolution structure on \( \text{Distr}_c(G) \).

**Proof:** For \( v \in V^\infty \), define \( F^u(g) = g(v) \). For \( T \in \text{Distr}_c(G) \), define

\[
\pi(T)v = (T, F^v)
\]

. For example,

- \( \pi(\delta_x) = \pi(x) \).
- \( \pi(u) = u \) for \( u \in U(\mathfrak{g}) \).
- \( \pi(fdg)v = \int_G f(g)\pi(g)v \text{d}g \) for \( f \in C_c(G) \).
- \( \pi(f \ast g) = \pi(f)\pi(g) \).

\( \Box \)

**Cor. (IX.8.9.8).** If \( f \in C^\infty_c(G) \), then for any \( v \in V \), the vector \( T_{f\mu_{\text{Haar}}}v \in V^\infty \).

**Proof:** Notice \( X\pi(f)v = \pi(X \ast f)v \), and we calculate \( X \ast f \):

\[
(X \ast f, v) = \int \frac{d}{dt} \pi(e^{tX}y)v f(y) \text{d}y = \frac{d}{dt} \int f(y)\pi(e^{tX}y)v \text{d}y = \frac{d}{dt} \int f(e^{-tX}y)\pi(y)v \text{d}y = \pi(f_X)v
\]

where \( f_X(y) = \frac{d}{dt} (e^{-tX}g)|_{t=0} \) is smooth. Iterating, we can show \( \pi(f)v \in V^\infty \).

\( \Box \)

**Cor. (IX.8.9.9) (Smooth Vectors Dense).** \( V^\infty \) is dense in \( V \).

**Proof:** Choose a Dirac sequence \( \{f_n\} \), then \( T_{f_n\mu_{\text{Haar}}}v \in V^\infty \) converges to \( v \), by (IX.8.9.7).

\( \Box \)

**Cor. (IX.8.9.10).** If \( V \) is a f.d. vector space, then \( V^\infty = V \).

**Prop. (IX.8.9.11) (Smooth Vectors in \( S^1 \)).** Let \( K = S^1 \) and \( \rho \) be the regular representation on the Hilbert space \( L^2(K) = L^2[0,2\pi] \), then the smooth vectors in \( \rho \) are just precisely the elements of \( C^\infty(K) \).
Proof: Take a Fourier expansion \( f(x) = \sum a_n e^{2\pi i n x} \). Suppose \( f \) is a \( C^1 \) vector, then there is a \( g(x) = \sum b_n e^{2\pi i n x} \) that \( \lim \frac{1}{t}(f_t - f) = g \), so

\[
\lim \sum_n \left| \frac{1}{t}(e^{2\pi i n t} - 1)a_n - b_n \right|^2 = 0.
\]

which means \( b_n = 2\pi i a_n \).

So if \( f \) is a \( C^\infty \) vector, then \( |a_n| \) decay rapidly, and \( f \) and all its derivatives converge absolutely so \( f \) is smooth. Conversely, integration shows the Fourier coefficients of any smooth function \( f \) decay rapidly. \( \square \)

Examples

Prop. (IX.8.9.12) (Store-Newmann). The Heisenberg representation of \( H \) acting on \( L^2(\mathbb{R}) \) is unitary and irreducible, where \( \pi(p_a) = e^{iax} \), and \( \pi(t_b) = T_b \).

Prop. (IX.8.9.13) (\( \widehat{SL_2}(\mathbb{R}) \)). Let \( \widehat{SL_2}(\mathbb{R}) \) be the universal covering of \( SL_2(\mathbb{R}) \). Then \( \text{Rep}_f(\widehat{SL_2}(\mathbb{R})) = \text{Rep}_f(SL_2(\mathbb{R})) \). In particular, \( \widehat{SL_2}(\mathbb{R}) \) admits no f.d. faithful representations, and the only quotient groups of it that admit f.d. faithful representations are \( SL_2(\mathbb{R}) \) and \( PSL_2(\mathbb{R}) \).

Proof: Let \( \rho : \widehat{SL_2}(\mathbb{R}) \rightarrow GL(n, \mathbb{R}) \) be a representation inducing a real Lie algebra representation \( \rho_\ast \). Consider its complexification \( \rho \otimes \mathbb{C} : \widehat{SL_2(\mathbb{R})} \rightarrow GL(n, \mathbb{C}) \). Because \( SL(2, \mathbb{C}) \) is simply connected, \( \rho_\ast \otimes \mathbb{C} \) corresponds to a complex representation \( \rho' : SL(2, \mathbb{C}) \rightarrow GL(n, \mathbb{C}) \) by (IX.8.3.13). Now because \( \widehat{SL_2}(\mathbb{R}) \) is simply connected, so there is a real Lie group homomorphism \( \gamma : SL_2(\mathbb{R}) \rightarrow SL(2, \mathbb{C}) \) that \( \rho' \circ \gamma = \rho \). But because \( \rho, \rho' \) both commutes with conjugation, so does \( \gamma \). Thus the image of \( \gamma \) is in \( GL(2, \mathbb{R}) \), and \( \rho \) factors through \( SL(2, \mathbb{R}) \).

There is a quotient map \( \pi : \widehat{SL_2(\mathbb{R})} \rightarrow PSL_2(\mathbb{R}) \), and \( PSL_2(\mathbb{R}) \) has trivial center, so the center of \( \widehat{SL_2(\mathbb{R})} \) is contained in \( \text{Ker}(\pi) \), which is isomorphic to \( \pi_1(PSL_2(\mathbb{R})) = \mathbb{Z} \), and is central in \( \widehat{SL_2(\mathbb{R})} \) by (IX.8.1.6), so the center of \( \widehat{SL_2(\mathbb{R})} \) is just \( \mathbb{Z} \). By what has been proved, for any representation of other covering space of \( PSL_2(\mathbb{R}) \), the induced representation on \( \widehat{SL_2(\mathbb{R})} \) factors through \( SL_2(\mathbb{R}) \), thus trivial on the subgroup \( 2\mathbb{Z} \subset \mathbb{Z} \), and the original representation factors through \( SL_2(\mathbb{R}) \) or \( PSL_2(\mathbb{R}) \). So the only possibility of faithful representation is \( SL_2(\mathbb{R}) \) or \( PSL_2(\mathbb{R}) \). For \( PSL_2(\mathbb{R}) \), take the \( \square \)

Prop. (IX.8.9.14) (F.D. Representations of \( SL_2(\mathbb{R}) \)). Let \( \mathbb{K} = \mathbb{C} \) or \( \mathbb{R} \), then the representations of \( SL(2, \mathbb{K}) \) are isomorphic to direct representations \((\rho_n, V_n)\), where \( V_n \) = homogeneous polynomials of degree \( n \) in indeterminants \( x, y \), and

\[
\rho_n \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] (x^k y^{n-k}) = (ax + by)^k(cx + dy)^{n-k}.
\]

Proof: We can check these representations truly induce irreducible representations of their Lie algebras \( \mathfrak{sl}_2(\mathbb{K}) \). Notice \( SL(2, \mathbb{C}) \) is simply connected, so (IX.8.9.13) and (IX.8.3.13) gives the result. \( \square \)
10 Analysis

Lemma (IX.8.10.1). Let $H$ be a Lie subgroup of $G$ and $g \notin H$, then there is a smooth function $\Phi$ on $G$ that $\Phi(xh) = \Phi(x)$ for any $x \in G, h \in H$, and $\Phi(H) = 0$, yet $\Phi(g) \neq 0$.

Proof: Because $H$ is closed, there is a nbhd $U$ of $g$ disjoint from $H$ and a function $\varphi$ supported in $U$. Then $\Phi(x) = \int_H \varphi(xh)dh$ satisfies the desired condition. □

Bi-Invariant Metric

Lemma (IX.8.10.2). Bi-invariant metric exists in a compact Lie group.

Proof: Because the Haar measure on a compact metric is bi-invariant. Choose a Riemann metric and set

$$\langle V, W \rangle = \int_{G \times G} \langle L_{\sigma*}R_{\tau*}(V), L_{\sigma*}R_{\tau*}(W) \rangle d\mu(\sigma)d\mu(\tau).$$

Note that $L_*$ and $R_*$ commute. □

Prop. (IX.8.10.3). For a left-invariant metric on a connected Lie group $G$, if it is bi-invariant, then the inner product at the origin $e$ is invariant under $g(I.12.1.12)$, and the converse is also true if $G$ is connected.

Proof: If the metric is invariant, then for any $X, Y, Z \in \mathfrak{g}$, $\langle \text{Ad}(\exp(tX))Y, \text{Ad}(\exp(tX))Z \rangle = \langle X, Y \rangle$, so we take derivation w.r.t. $t$ to get

$$\langle \text{ad}(X)Y, Z \rangle + \langle Y, \text{ad}(X)Z \rangle = 0$$

by (IX.8.1.12).

Conversely, if this is invariant, then using $\exp(tX) = \exp((t - t_0)X)\exp(t_0X)$, we get $\partial(\langle \text{Ad}(\exp(tX))Y, \text{Ad}(\exp(tX))Z \rangle)/\partial t = 0$ for all $t$, thus $\langle \text{Ad}(\exp(tX))Y, \text{Ad}(\exp(tX))Z \rangle = \langle X, Y \rangle$, and it is invariant under right actions by $\exp(tX)$ also. As $G$ is generated by the elements $\exp(X)(IX.1.12.3)$, it is right-invariant under $G$. □

Prop. (IX.8.10.4). If $G$ is a Lie group with a bi-invariant metric, then

$$\nabla_X Y = 1/2[X,Y], \quad R(X,Y)Z = 1/4[[X,Y],Z], \quad K(\sigma) = 1/4\|X,Y\|^2.$$ 

So it has non-positive sectional curvature, and its curvature is non-negative, and all 1-parameter subgroups are geodesics from $e$.

Proof: It suffices to show that $\langle Z, \nabla_X Y \rangle = 1/2\langle Z, [X,Y] \rangle$ for any $Z$, and this follows from (IX.3.3.16). The second follows from the first and the definition (IX.3.3.11). □

Cor. (IX.8.10.5). A bi-invariant Lie group with $\mathfrak{g}$ having trivial center is compact and $\pi_1(G)$ finite.

Proof: The Ricci curvature has a positive lower bound, otherwise for some $X, [X,Y] = 0$ for all $Y$, thus $X$ is in the center. Hence we use Myer theorem (IX.3.5.20). □

Cor. (IX.8.10.6). If $G$ has a bi-invariant metric, then the exp map $\mathfrak{g} \rightarrow G$ is surjective.
**Proof:** Because $\exp$ is defined for any $X \in \mathfrak{g}$, and for any $g \in G$ and $v \in T_g(G)$, $\exp_p(v) = L_g(\exp(dL_{g^{-1}})_g(v))$ because $L_g$ is an isomorphism of Riemann manifolds. So by Hopf-Rinow (IX.3.3.46), $G$ is complete. Thus for any $q$, there is a geodesic connecting $e, q$, which is a 1-parameter subgroup, thus $q = \exp(X)$ for some $X \in \mathfrak{g}$. □

**Prop. (IX.8.10.7) (Structure of bi-invariant Lie groups).** A simply-connected Lie group with a bi-invariant metric is equal to $G' \times \mathbb{R}^k$, $G'$ compact.

**Proof:** Because the orthogonal complement of the center of $\mathfrak{g}$ is a Lie algebra, $G$ is like $G' \times \mathbb{R}^k$, and a simply connected abelian Lie group is $\mathbb{R}^k$. □

11 Arithmetic Subgroups

**Def. (IX.8.11.1) (Arithmetic Subgroups).** Let $G$ be an algebraic group over $\mathbb{Q}$, then a subgroup $\Gamma$ of $G(\mathbb{Q})$ is called arithmetic if it is commensurable with $G(\mathbb{Q}) \cap GL_n(\mathbb{Z})$ for some embedding $G \hookrightarrow GL_n$.

**Prop. (IX.8.11.2).** An arithmetic subgroup $\Gamma$ is commensurable with $G(\mathbb{Q}) \cap GL_n'$ for any embedding $G \hookrightarrow GL_n'$.

**Proof:** □

**Thm. (IX.8.11.3) (Margulis).** Every discrete subgroup of finite volume in a non-compact simple real Lie group $H$ is arithmetic unless $H$ is isogenous to $SO(1, n)$ or $SU(1, n)$.

Note that $SL_2(\mathbb{R})$ is isogenous to $SO(1, 2)$, so the theorem doesn’t apply to it.

**Proof:** Cf. [Mor15] 5.2. □
IX.9  Symmetric spaces

1  Symmetric Spaces

Main references are [Hel78], [Mil17b].

Def. (IX.9.1.1) (Symmetric Spaces). A Riemannian manifold is called \textbf{locally symmetric} at \( p \) if \( \nabla R(p) = 0 \). Locally symmetric is equivalent to the fact that every local reversing map is an isometry.

A \textit{symmetric space} is a Riemannian manifold that \( \nabla R = 0 \) everywhere.

A symmetric space is complete because two folding is an extension of geodesics. In particular, a symmetric space is homogenous, and to check symmetrically, it suffices to show it is homogenous and locally symmetric at a point.

\textit{Proof}:

Prop. (IX.9.1.2). A Lie group with a bi-invariant metric is a symmetric space.

\textit{Proof}:

Prop. (IX.9.1.3). The conjugate points in a symmetric space is easy to calculate, they are \( \exp\left(\frac{2k}{\sqrt{e_i}} V\right) \), counting multiplicity, where \( e_i \) is the eigenvalue of the self-adjoint operator \( K_V(W) = R(V,W)V \) at \( p \).

Prop. (IX.9.1.4) (Group of Isometries). Let \( M \) be a symmetric space, then the group of isometries \( \text{Iso}(M^0, g) \) of \( M \) has a natural structure of a Lie group.

\textit{Proof}:

Cf. [Hegalson, homogenous Spaces, 4.3.2].

Prop. (IX.9.1.5) (Symmetric Space is a Homogenous Space). Let \( (M, g) \) be a symmetric space, and \( p \in M \), then the subgroup \( K_p \subset \text{Iso}(M, g)^0 \) fixing \( p \) is compact, and the natural map

\[ \text{Iso}(M, g)^0/K_p \to M \]

is an isomorphism of smooth manifolds, where \( \text{Iso}(M, g)^0/K_p \) is given the homogenous space structure (IX.9.1.4). In particular, \( \text{Iso}(M, g)^0 \) acts transitively on \( M \).

\textit{Proof}:

Cf. [Mil17c]P12.

2  Decompositions of Symmetric Spaces

3  Non-Compact Type

4  Compact Type

5  Hermitian Symmetric Spaces

Def. (IX.9.5.1) (Hermitian Symmetric Spaces). A \textit{Hermitian symmetric space} is a Hermitian manifold that is a symmetric space (IX.9.1.1) and that the local symmetries are all holomorphic.

Prop. (IX.9.5.2) (Group of Isometries). For a Hermitian symmetric space, the group of holomorphic symmetries \( \text{Iso}(M, g) \) is closed in the group of isometries \( \text{Iso}(M^0, g) \), which is a Lie group by (IX.9.1.4), so it is also a Lie group.
Prop. (IX.9.5.3) (Basic Hermitian Symmetric Spaces). There are three different families of Hermitian symmetric spaces (but not complete):

- (Non-compact Type): These spaces are non-compact and simply connected, with negative curvatures, and $\text{Iso}(M, g)^0$ is adjoint and non-compact.
- (Compact Type): These spaces are compact and simply connected, with positive curvatures, and $\text{Iso}(M, g)^0$ is adjoint and compact.
- (Euclidean Type): These spaces have 0 curvatures.


Prop. (IX.9.5.4) (Decomposition). Any Hermitian symmetric space $M$ decomposes into a product $M^0 \times M^+ \times M^-$ of Hermitian symmetric spaces with $M^0$ Euclidean, $M^-$ of non-compact type and $M^+$ of compact type.

Proof:

Prop. (IX.9.5.5) (Examples). The upper half plane $\mathcal{H}$ is a Hermitian symmetric space with the metric $\frac{dx dy}{y^2}$, because $GL(2, \mathbb{R})$ acts transitively on $\mathcal{H}$ and preserves the metric, and the diffeomorphism $z \mapsto -1/z$ is a geodesic isomorphism at $i$.

Proof: See (IX.9.6.6).

Prop. (IX.9.5.6) (Examples). The projective space $\mathbb{P}^n(\mathbb{C})$ with the Fubini-Study metric is a Hermitian symmetric space. For any $p$, the (descent of) the rotation through $\pi$ about the axis through $p$ and its polar opposite is the geodesic isomorphism at $p$.

Proof: See (IX.11.1.6).

Prop. (IX.9.5.7). Any Euclidean Hermitian symmetric space is a quotient of $\mathbb{C}^g$ by a discrete subgroup of translations.

Proof:

6 Hermitian Symmetric Domains

Def. (IX.9.6.1) (Hermitian Symmetric Domain). A Hermitian symmetric space of non-compact type is called a Hermitian symmetric domain. A bounded symmetric domain is an open connected symmetric subset of $\mathbb{C}^n$.

Prop. (IX.9.6.2). Every Hermitian symmetric domain can be embedded into $\mathbb{C}^n$ for some $n$.

Proof:

Prop. (IX.9.6.3) (Bergman Metric). Every bounded symmetric domain has a canonical Hermitian metric called the Bergman metric, it is invariant under holomorphic automorphisms, and it has negative curvatures.

Proof: Cf. [Mil17]P11 and [Hel78]8.3.3.

Cor. (IX.9.6.4). Every Hermitian symmetric domain $D$ has a unique Hermitian metric that maps to the Bergman metric under any isomorphism of $D$ onto a bounded symmetric domain.
Def. (IX.9.6.5) (Siegel Upper Half Plane $\mathcal{H}_g$). The **Siegel upper half plane** $\mathcal{H}_g$ consists of symmetric complex $g \times g$ matrices $Z = X + iY$ with $Y$ positive definite. It is identified with an open subset of $\mathbb{C}^{g(g+1)/2}$. The symplectic group $\text{Sp}_{2g}(\mathbb{R})$ (IX.8.4.1) acts transitively on $\mathcal{H}_g$ via

$$
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}Z = (AZ + B)(CZ + D)^{-1}
$$

The matrix $\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ acts as an involution on $\mathcal{H}_g$, and has $iI_g$ has its fixed point, so $\mathcal{H}_g$ is homogenous and symmetric.

The injection into $\mathbb{C}^g$ is not holomorphic, so we cannot see from this that $\mathcal{H}_g$ is holomorphic, but we can see from $\text{Hol}(\mathcal{H}_g)$.

\textbf{Proof:} \[ \square \]

Cor. (IX.9.6.6) $(H_1)$. The group $\text{GL}(2, \mathbb{R})$ acts on $\mathbb{C}$ by $\gamma(z) = \frac{az + b}{cz + d}$ where $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. It can be checked that this is a continuous group action.

Also in the same way the group $\text{GL}(2, \mathbb{R})$ or $\text{SL}(2, \mathbb{R})$ acts on the upper plane, by (X.2.2.5). This action is transitive and the stabilizer of $i$ is $\text{SO}(2, \mathbb{R})$, thus we have $\mathcal{H} \cong \text{SL}(2, \mathbb{R})/\text{SO}(2, \mathbb{R})$, and $\text{SL}(2, \mathbb{R})$ is the group of holomorphic automorphisms of $\mathcal{H}$.

Also, the Riemannian metric $\frac{dx dy}{y^2}$ on $\mathcal{H}$ is fixed by this action.

\textbf{Proof:} \[ \square \]

Def. (IX.9.6.7) (Siegel Unit Circle $\mathcal{D}_g$). Let $\mathcal{D}_g$ be the set of symmetric complex matrixes that $I - Z^*Z$ is positive definite, then it is identified with an open subset of $\mathbb{C}^{g(g+1)/2}$, this is a holomorphic embedding.

There is an isomorphism from $\mathcal{H}_g$ (IX.9.6.5) to $\mathcal{D}_g$:

$$Z \mapsto (Z - iI_g)(Z + iI_g)^{-1}.$$ 

This is an isomorphic, so $\mathcal{D}_g$ is symmetric, and $\mathcal{H}_g$ has an invariant metric, so they are both Hermitian symmetric domains.

\textbf{Proof:} \[ \square \]

Prop. (IX.9.6.8). Let $(M, g)$ be a Hermitian symmetric domain, then the inclusions

$$\text{Iso}(\mathcal{M}^\infty, g) \supset \text{Iso}(M, g) \subset \text{Hol}(M)$$

induce equalities

$$\text{Iso}(\mathcal{M}^\infty, g)^0 = \text{Iso}(M, g)^0 = \text{Hol}(M)^0$$

Then $\text{Hol}(M)^0$ acts transitively on $M$, the stabilizer $K_p$ of $p$ in $\text{Hol}(M)^0$ is compact, and

$$\text{Hol}(M)^0/K_p \cong M^\infty.$$ 

Prop. (IX.9.6.9) (Rotation at a Point). Let $D$ be a Hermitian symmetric domain and $p \in D$, then there is a unique homomorphism $u_p : U_1 \to \text{Hol}(D)$ that $u_p(z)$ fixes $p$ and acts on $T_p(D)$ as multiplication by $z$.

Prop. (IX.9.6.10) (Classification of Hermitian Symmetric Domains). The isomorphism classes of irreducible Hermitian symmetric domains are classified by the special nodes on connected Dynkin diagrams.


7 Quotient of Hermitian Symmetric Domains

Prop. (IX.9.7.1) ($D(\Gamma)$). Let $D$ be a Hermitian symmetric domain, and let $\Gamma$ be a discrete subgroup of $\text{Hol}(D)^+$. If $\Gamma$ is torsion-free, then $\Gamma$ acts freely on $D$, and there is a unique complex structure on $\Gamma \setminus D$ that the quotient map $\pi : D \to \Gamma \setminus D$ is a local isomorphism.

In this case, we write $D(\Gamma) = \Gamma \setminus D$, and $D$ is a universal covering of $D(\Gamma)$, by(IX.9.5.3).

Proof: Cf.[Mil17b]P32.

Prop. (IX.9.7.2). $D(\Gamma)$ has only f.m. automorphisms, as a complex manifold.

Proof: Cf.[Mil17b]P41.
IX.10 Complex Geometry

Basic References are [Voisin], [Principle of Algebraic Geometry Griffith/Harris], more advanced stuffs to add from [Kähler Geometry] and [Complex Geometry Daniel]. Even more advanced is [Demailly Complex Analytic and Differential Geometry].

1 Complex Manifolds

Def. (IX.10.1.1) (Complex Manifold). A complex manifold is an even dimensional manifold that the transformation matrices are holomorphic.

Prop. (IX.10.1.2) (Andreotti-Francl). Let $M^n \subset \mathbb{C}^n$ be a complex submanifold of dimension $n$, then $M$ is homotopic to a CW complex of real dimension $\leq n$.

Prop. (IX.10.1.3) (Adjunction Formula). The normal sheaf of a submanifold $Y \subset X$ is defined the same as the case of nonsingular varieties (V.8.1.18), then the same is true of the adjunction formula:

$$K_Y \cong K_X \otimes \det \mathcal{N}_{Y/X}$$

In case $Y$ is of codimension 1, $\mathcal{N}_{Y/X} \cong \mathcal{L}(Y)|_Y = \mathcal{O}_Y(Y)$.

Prop. (IX.10.1.4) (Remmert’s Theorem). A non-compact manifold admits a proper holomorphic embedding into $\mathbb{C}^N$ for some $N$ if it is a Stein manifold.

Prop. (IX.10.1.5) (Siegel). Let $X$ be a compact complex manifold of dimension $n$, then the field $K(X)$ of meromorphic functions on $X$ has transcendence degree $\leq n$ over $\mathbb{C}$. And in case $tr.d.K(X) = \dim X$, it is a f.g. field extension of $\mathbb{C}$. Then we define the algebraic dimension of a compact connected complex manifold $X$ to be $a(X) = tr.d.K(X)$.

Proof: It suffices to show that given any meromorphic functions $f_1, \ldots, f_{n+1}$, there is an algebraic relation between them.

Now for each $x$, there is a nbd $U_x$ that any $f_i$ writes as the quotient of two holomorphic functions $\frac{g_{i,k}}{h_{i,k}}$. And assume $W_x \subset V_x \subset \overline{V_x} \subset U_x$ are the metric balls $B(x, \frac{1}{2}) \subset B(x, 1)$. As $X$ is compact, there are $N$ $x_k$ that $X = \bigcup_{x_k}$.

As on the intersections, $\frac{g_{i,k}}{h_{i,k}} = \frac{g_{i,l}}{h_{i,l}}$, any we can assume they are all prime, so $\frac{g_{i,k}}{h_{i,l}} = \varphi_{i,kl}$ is a unit. Let $\varphi_{kl} = \prod_i \varphi_{i,kl}$, as $X$ is compact, let $C = \max_{k,l} \varphi_{kl} \geq 1$.

For any homogenous polynomial $F \in \mathbb{C}[X_1, \ldots, X_{n+1}]$ of deg $m$, let $G_k = F(\frac{g_{1,k}}{h_{1,k}}, \ldots, \frac{g_{n+1,k}}{h_{n+1,k}})(\prod_i h_{ik})^m$. Then $G_k$ are holomorphic and $G_k = \varphi_{ik}^m G_l$ on the intersection. Now I claim for any $M > 0$, there is a $F$ that $G_k$ vanishes up to at least order $M$ at $x_k$.

For this, just consider the dimension of all homogenous polynomials of degree $m$ is $C_m^n$, and the number of desired equations of elements needs to be vanish is $N \cdot C_{m'-1}^{n+1}$, so this always can be achieved when $m$ is sufficiently large.

By Schwartz lemma (X.2.9.4), $|G_k(x)| \leq (\frac{1}{2})^m C'$, where $C' = \max\{|G_k(x)||k = 1, \ldots, n, x \in \overline{V_k}\}$.

If $C' = |G_k(x)|$, and $x \in W_{i_k}$, then $C' = |G_l(x)||\varphi_{ik}^m(x)| \leq \frac{C'}{2^m} \cdot C^m$. If for some $m, m'$, $C^m < 2^{m'}$, then this shows $C' = 0$ which will finish the proof.

Look back at the condition of $m, m'$, $C_m^n > N \cdot C_{m'-1}^{n+1}$ can be achieved together with $m < \lambda m'$ for any $\lambda$, because the left hand is degree $n + 1$ in $m$ and the right hand is degree $n$ in $m'$. \qed
Almost Complex Structure

Def. (IX.10.1.6) (Almost Complex Structures). For $M$ a real orientable manifold of dimension $2n$, an almost complex structure is a real bundle map $J : TC \rightarrow TC$ satisfying $J^2 = -1$. A manifold with an almost complex structure is called an almost complex manifold.

A complex manifold has an almost complex structure, just define $J(\partial/\partial x_i) = \partial/\partial y_i$ and $J(\partial/\partial y_i) = -\partial/\partial x_i$.

Def. (IX.10.1.7). $J$ will define a bundle map on $T^*M \rightarrow T^*M$, and it has two eigenvalues $\pm i$, denoted by $T^{*1,0}M$ and $T^{*0,1}M$. The formal differential forms $\wedge^k T^*M \cong \sum \wedge^{p,k-p} T^*M$. $\partial$ is defined to be $\pi_{p+1,q} \circ d$ on $\wedge^{p,q} T^*M$, and $\overline{\partial}$ is defined to be $\pi_{p,q+1} \circ d$.

Def. (IX.10.1.8) (Integrability). An almost complex structure is called integrable iff it satisfies the following equivalent conditions:

- $d\alpha = \partial \alpha + J\partial \alpha$.
- $d\alpha = \partial \alpha + \overline{\partial} \alpha$ is true for $\alpha \in A^{1,0}(X)$.
- $[T^{0,1}X, T^{0,1}X] \subset T^{0,1}X$.
- $\overline{\partial}^2 f = 0$ for functions $f$.

Proof: 1 $\iff$ 3 is because by (IX.2.3.6), if $u, v \in T^{0,1}X$,

$$d\alpha(u, v) = u(\alpha(v)) - v(\alpha(u)) - \alpha([u, v]) = -\alpha([u, v]).$$

3 $\iff$ 4 is because by (IX.2.3.6), if $\alpha = \overline{\partial} f$ and $u, v \in T^{0,1}X$, then

$$\overline{\partial}^2 f(u, v) = u(\overline{\partial} f(v)) - v(\overline{\partial} f(u)) - \overline{\partial} f([u, v]) = u(df(v)) - v(df(u)) - \overline{\partial} f([u, v]) = \partial f([u, v])$$

Prop. (IX.10.1.9) (Nirenberg-Newlander). Given an almost complex manifold $(M, J)$, it is integrable iff it comes from a complex structure.

Proof: Cf. [Foundation of Differential Geometry Kobayashi Chap9.2].

Blowing-up

Blowing-up serves as a way to magnify local properties to global ones.

Remark (IX.10.1.10). Cf. [Complex Geometry P98] for blowing up along an arbitrary subvariety.

Def. (IX.10.1.11) (Blowing-up along Point). For a nbhd $U$ of 0 in $\mathbb{C}^n$, we can define the blowing-up $\pi : \tilde{U} \rightarrow U : \tilde{U}$ is the subset of $U \times \mathbb{CP}^n$ consisting of $(z, [l])$ that $z \subset [l]$. Then $\pi^{-1}(U - \{0\}) \cong U - \{0\}$ holomorphically.

For a complex manifold $M$ and a point $x$, then choose a local coordinate centered at $x$, then we can form the blowing-up, because it is holomorphism away from $x$, so it can glue with the rest of $M$ and form a new manifold $\tilde{M}$, called the blowing-up of $M$ along $x$.

Notice this is independent of the coordinate chosen, because if $f(U)$ is a new coordinate of $U$, then $\pi^{-1}(f) \pi : \tilde{U} - \{x\} \rightarrow \tilde{U'} - \{x\}$ is a holomorphism, and it can be extended to $\tilde{U} \rightarrow \tilde{U'}$ by setting $f(x, [l]) = (x, ([\partial f]/\partial z)(0)[l])$. 
IX.10. COMPLEX GEOMETRY

Prop. (IX.10.1.12) (Exceptional Divisor). Let $E$ be $\pi^{-1}(x)$ for a blowing-up, called the exceptional divisor. Often the line bundle $\mathcal{O}_X(E)$ associated with it is called denoted by $E$.

There are canonical coordinates near $E$: let $\tilde{U}_i = \tilde{U} - \{(l_i = 0)\}$, then endow $\tilde{U}_i$ with the coordinate $z(i) = (l_j/l_i, \ldots, z_i, \ldots, l_n/l_i)$, it is holomorphic to $\mathbb{C}^n$. $\pi$ in this coordinate is written as $(z(1), \ldots, z(n)) \mapsto (z(i)z(1), \ldots, z(i), \ldots, z(n)z(i))$.

The transition function can be written, it is

$$
\varphi_j \circ \varphi_i^{-1}((z(i)_1, \ldots, z(i)_n)) = \left(\frac{z(i)_1}{z(i)_j}, \ldots, \frac{1}{z(i)_j}, \ldots, z(i)_j, \ldots, \frac{z(i)_n}{z(i)_j}\right).
$$

Notice it is somewhat tricky because it has two different coordinates.

The defining function of $E$ in this coordinate is $(z(i)) = (z_i)$. So the line bundle $\mathcal{O}_X(E)$ has transition function $g_{ij} = z(i)/z(j)$, and it can be thought of as the line bundle that has line $[l]$ at the point $(z, [l]) \in \tilde{U}$. So it is kind of tautological, in fact its restriction on $E \cong \mathbb{CP}^{n-1}$ is just the tautological line bundle.

Prop. (IX.10.1.13). The canonical line bundle $\mathcal{K}_X = \pi^*\mathcal{K}_X + (n-1)E$, where $n$ is the dimension of $X$.

Proof: Away from $E$, the $\pi$ is a holomorphism, so it suffices to compare the two transition function of the two canonical maps near $E$ using the coordinates in (IX.10.1.12), with the local section given by $dz_1 \wedge \ldots \wedge dz_n$ and $dz_1(z_1) \wedge dz_2(z_2)$ respectively. On $\tilde{U}_i$, locally $dz_1 \wedge \ldots \wedge dz_n$ is pulled by $\pi^*$ to the trivial bundle on $U_1$, and by calculation, $dz_1(z_1) \wedge dz_2(z_2) = z(i)^{-1}_1dz_1(z_1) \wedge dz_2(z_2)$, so $\mathcal{K}_X - (n-2)E$ has a global section $z(n)^{-1}dz_1 \wedge dz_2$, so it is also trivial on $\tilde{U}$, so $\mathcal{K}_X = \pi^*\mathcal{K}_X + (n-1)E$ is true.

\[\square\]

2 Deformation of Complex Structures

Cf.[Kähler Geometry] and [Complex Geometry Chap6], should be completed as soon as possible.

3 Coherent Sheaves and Analytic Spaces

Cf.[Demailly] and [GAGA Serre].

Analytic Subvarieties

Def. (IX.10.3.1) (Analytic subvariety). An analytic subvariety is a closed subset of a complex manifold that is locally defined by f.m. holomorphic functions. The regular points of an analytic subvariety locally defined by $k$ functions is the points that rank $\frac{\partial(f_1, \ldots, f_k)}{\partial(z_1, \ldots, z_n)} = k$.

Prop. (IX.10.3.2) (Proper Mapping Theorem). If $U, M$ are complex manifolds and $M \subset U$ is an analytic subvariety, then if $f : U \to N$ is a holomorphic mapping whose restriction on $M$ is proper, then $f(M)$ is an analytic subvariety of $N$.

Proof: Cf.[Griffith/harris P395].

\[\square\]

Def. (IX.10.3.3). An analytic space of $\mathbb{C}^n$ is an analytic subvariety of $\mathbb{C}^n$. On an analytic space, there is a sheaf of holomorphic functions $\mathcal{H}_U$. So we can define holomorphic map $\varphi$ as continuous functions that maps holomorphic germs to holomorphic germs, which is equivalent to the coordinates of $\varphi$ are all holomorphic.
Def. (IX.10.3.4). An **analytic space** is a Hausdorff space \( X \) with a structure sheaf \( \mathcal{H}_X \) that is locally isomorphic to an analytic set. Morphisms are continuous maps that are locally holomorphic. Sub-analytic spaces are defined as usual.

An **analytic module** is just a module over the sheaf \( \mathcal{H}_X \). For a sub-analytic space \( Y \), we have a sheaf of ideals \( \mathcal{A}_Y \) which is the sheaf of germs vanishing at \( Y \), and \( \mathcal{H}_X/\mathcal{A}_Y \) is a sheaf of \( X \) that is zero outside \( Y \), and we identify it with \( \mathcal{H}_Y \).

The products of analytic spaces can be defined, and it has the product topology, unlike the case of schemes.

**Prop. (IX.10.3.5).** The structure sheaf of an analytic space is coherent, and the sheaf of ideals of a sub-analytic space is coherent.

**Proof:** First prove for \( X \) is an open subset of \( \mathbb{C}^n \), Cf.[GAGA Serre P4]. And by definition \( \mathcal{A}_X \) is a \( \mathcal{O}_X \)-module of f.t., and it is also coherent, so \( \mathcal{H}_X \) is coherent. \( \mathcal{A}_Y \) is coherent because it is a kernel of \( \mathcal{H}_X \to \mathcal{H}_Y \).

**4 Positive Current**

**5 Hermitian Vector Bundles**

Def. (IX.10.5.1). A **holomorphic vector bundle** is a vector bundle on a complex manifold that the transition functions are holomorphic. A **Hermitian vector bundle** is a holomorphic vector bundle endowed with a Hermitian metric. Any holomorphic vector bundle has a Hermitian structure, by partition of unity method.

**Prop. (IX.10.5.2) (Hodge Star for Hermitian bundles).** If \( E \) is a Hermitian vector bundle over a compact complex manifold of complex dimension \( n \), we define a conjugate-linear operator \( \bar{*} : A^{p,q}(X) \to A^{n-p,n-q}(X) : \eta \mapsto \bar{*} \eta \), and a conjugate-linear functor \( \tau E \to E^* \) induced by the Hermitian metric on \( E \).

Then we can define \( \bar{*}_E : A^{p,q}(E) \to A^{n-p,n-q}E : \eta \otimes s \mapsto \bar{*}(\eta) \otimes \tau(s) \). It can be checked that

\[
(\alpha, \beta) \ast 1 = \alpha \wedge \ast_E \beta,
\]

\[
\bar{\partial}^* E = -\bar{*}_E \bar{\partial}_E \bar{*}_E, \quad \bar{*}_E \Delta_{\bar{\partial} E} = \Delta_{\bar{\partial} E} \bar{*}_E, \quad \bar{*}_E v = (-1)^{p+1} \text{ on } \Omega^{p,q}(E).
\]

**Hermitian Manifold**

Def. (IX.10.5.3) (Holomorphic Tangent Bundle). Let \( M \) be a complex manifold, the complexified tangent bundle \( T_{\mathbb{C}}M \) is defined as \( TM \otimes_{\mathbb{R}} \mathbb{C} \), the **holomorphic tangent bundle** \( T^{1,0}M \) and anti-holomorphic bundle \( T^{0,1}M \) are defined to be the vectors generated resp. by \( \partial/\partial z_i \) and \( \partial/\partial \bar{z}_i \). The **holomorphic cotangent bundle** and anti-holomorphic cotangent bundle is defined to be the covectors generated by \( dz_i \) and \( d\bar{z}_i \).

Def. (IX.10.5.4) (Hermitian Metric). Let \( M \) be an almost complex manifold, a **Hermitian metric** on \( T_{\mathbb{C}}M \) is a metric that is \( J \)-invariant, that is \( g(Ju,Jv) = g(u,v) \). Notice(I.1.7.11) shows a Hermitian metric is equivalent to a non-degenerate Hermitian form on \( T_{\mathbb{C}}M \), where \( g \) appears as the real part of the Hermitian form.

Def. (IX.10.5.5) (Hermitian Manifolds). A complex manifold with a Hermitian metric is called a **Hermitian manifold**.
Def. (IX.10.5.6). Given a Hermitian manifold \( M \), define the Kahler form \( \omega_g \) as \( \omega_g(u, v) = g(Ju, v) \). Then it is a real 2-form on \( M \).

Notice \( g(u, v) = \omega_g(u, Jv) \), so \( g \) can be constructed by \( \omega_g \), iff \( \omega_g \) is positive (IX.11.6.1).

Picard Group

Def. (IX.10.5.7). The group of isomorphisms of holomorphic line bundles on a complex manifold \( X \) is denoted by \( \text{Pic}_C(X) \).

Prop. (IX.10.5.8) (Picard Group). For a connected space \( X \), there is an exact sequence
\[
0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{f \to e^{\lambda \text{vol}}} \mathcal{O}_X^* \to 0,
\]
and it induces a map \( \text{Pic}_C(X) = H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \), which is a just the first Chern class (same proof as in (IX.6.5.3)).

WARNING: in this case it is not necessarily isomorphism, not as in the case of topological line bundles.

in particular, The image of the first Chern class is trivial in \( H^2(X, \mathcal{O}_X) \).

Def. (IX.10.5.9). The dual of the universal line bundle on \( \mathbb{CP}^n \) is called the hyperplane line bundle, denoted by \( \mathcal{O}(1) \).

Prop. (IX.10.5.10). \( \text{Pic}_C(\mathbb{CP}^n) \cong \mathbb{Z} \), with \( \mathcal{O}(1) \) as a generator.

Proof: As \( \mathbb{CP}^n \) is Kähler, use (IX.11.5.2), then \( H^{0,k}(X, \mathbb{C}) \cong H^k(X, \mathcal{O}_X) = H^k(X, K_X \otimes \mathcal{O}(2)) = 0 \) for \( k \geq 1 \) by Kodaira vanishing (IX.11.7.3), and then \( NS(X) = H^{1,1}(X) = H^2(X, \mathbb{Z}) = \mathbb{Z} \) by Lefschetz (1, 1)-form theorem (IX.11.5.3). It remains to prove \( c_1(\mathcal{O}(1)) \) is the generator, for this, Cf. [Demailly P280].

Prop. (IX.10.5.11). Let \( S_d \) be the set of homogenous polynomials of degree \( d \), then
\[
H^0(\mathbb{CP}^n, \mathcal{O}(d)) = \begin{cases} S_d & d \geq 0 \\ 0 & d < 0 \end{cases}
\]

Proof: This is because it is sections that satisfy \( f_{\alpha}([z]) = (\frac{z_\alpha}{z_0})^k f_{\beta}([z]) \), which says \( f_{\alpha} \) glue together to give a holomorphic function homogenous of degree \( k \) on \( \mathbb{CP}^n - \{0\} \), which extends to a function on \( \mathbb{C}^n \) by (X.2.9.3), then it is easy to see it is a homogenous polynomial using the power series expansion.

Def. (IX.10.5.12) (Neron-Severi Group). For a compact complex manifold, the Neron-Severi group \( NS(X) \) is the image of \( \text{Pic}_C(X) \to H^2(X, \mathbb{R}) \). \( \text{rank}_\mathbb{R}(NS(X)) \) is called the Picard number of \( X \).

There is a good description of \( NS(X) \) in case \( X \) is Kähler, See Lefschetz theorem (IX.11.5.3).

Chern Connection

Prop. (IX.10.5.13) (Chern Connection). Given a Hermitian holomorphic bundle \( E \to M \) on a complex manifold, there is a unique Chern connection \( \nabla \) on \( E \), that \( \nabla \) is holomorphic (i.e. the connection matrix is holomorphic w.r.t a homomorphic frame), and it is compatible with the Hermitian metric.
Proof: Write out the requirement: if $H = h_{ij}$ is the matrix of the Hermitian metric, so $H$ is Hermitian, and we need $dh_{ij} = (\nabla e_i, e_j) + (e_i, \nabla e_j) = \sum_k \omega_{ik} h_{kj} + \sum \overline{\omega_{jk}} h_{ik} \omega$ is holomorphic, so must

$$\partial H = \theta H, \quad \bar{\partial} H = \overline{H D}.$$ 

But $H^t = \overline{H}$ so these two equations are equivalent and $\theta = \partial H H^{-1}$. \qed

Cor. (IX.10.5.14). The curvature of the Chern connection is $\Omega = \partial (\partial (h_{ij})) h_{ij}^{-1})$. In particular, it is a skew-symmetric matrix of $(1, 1)$-forms. If it is of dimension 1, then $\Omega = \partial \partial \log h$.

Proof: $\Omega$ is locally $d \omega + \omega \wedge \omega$, so if we choose a unitary basis, then $\omega$ is skew-symmetric by definition and $\omega \wedge \omega$ is also skew-symmetric, so $\Omega$ is skew-symmetric. The calculation is direct calculation. \qed

Prop. (IX.10.5.15). The transformation matrix of a complex manifold is holomorphic, so it is possible to define globally $\bar{\partial}$ operator. And locally on a nbhd, $\partial$ is defined as $d - \bar{\partial}$.

Prop. (IX.10.5.16) (Normal Coordinate). For a Hermitian vector bundle $E$ over a complex manifold $X$, given any coordinate frame $(z_j)$, there exists a holomorphic frame $(e_\lambda)$ that

$$\langle e_\lambda, e_\mu \rangle = \delta_{\lambda, \mu} - \sum c_{jk\lambda\mu} z_j z_k + O(|z|^3)$$

where $c_{ij\lambda\mu}$ is the coefficient of the Chern connection $\Omega$. Such a coordinate is called the normal coordinate frame of $E$ at $x$.

Proof: Cf.[Demailly P270]. \qed

Def. (IX.10.5.17) (Dolbeault Cohomology). The Dolbeault cohomology group $H^{p,q}_{\overline{\partial}}(X, \mathcal{E})$ of a holomorphic vector bundle $\mathcal{E}$ over a complex manifold $X$ is defined to be the $q$-th cohomology group of the complex

$$0 \rightarrow \Omega^{p,0} \overline{\partial} \rightarrow \Omega^{p,1} \overline{\partial} \rightarrow \ldots \overline{\partial} \rightarrow \Omega^{p,n-p} \rightarrow 0$$

and $H^{p,q}_{\overline{\partial}}$ is defined to be $H^{p,q}_{\overline{\partial}}(X, \mathbb{C}_X)$. By [V.6.8.13], $H^{p,q}_{\overline{\partial}}(X, \mathcal{E}) \cong H^q(M, \Omega^p_{\text{hol}} \otimes \mathcal{O}_X \mathcal{E})$.

6 GAGA

Basic Reference are [GAGA Serre].

Analytification of Algebraic Varieties and Sheaves

Prop. (IX.10.6.1) (Analytification). For any variety over $\mathbb{C}$, any open affine subset is isomorphic to an analytic space of $\mathbb{C}^n$, hence can be given an analytic structure $X^{an} \rightarrow X$, called the analytification of $X$. It is a locally ringed space. This is because algebraic isomorphisms are analytic isomorphism. Moreover, for any morphism of algebraic variety, the corresponding morphism is analytic.

$X^{an}$ is Hausdorff because analytification preserves products and morphisms, and separability of $X$ shows that $\Delta(X)$ is closed in $X \times X$, hence it is also closed in the analytification.

Notice $X^{an}$ and $X$ have in fact the same underlying sets.
**Remark (IX.10.6.2).** There is in fact a more general analytification for any scheme locally of finite type over \( \mathbb{C} \). That is, we define it as the right adjoint to the forgetful functor from analytic spaces to local ringed spaces. Where an analytic space is a local ringed space that locally has immersions into \( \mathbb{C}^n \). Should consult [Grothendieck EGA1-7].

**Proof:** Notice the schemes that have an analytification is stable under open subscheme, closed subscheme and products, and we can make a glue a large space from open subschemes by the unicity. So we only need to consider \( \text{Spec} \mathbb{C}[T] \), whose analytification is \( \mathbb{C} \). \( \square \)

**Prop. (IX.10.6.3) (Transfer of Properties).**
- \( X^{an} \) is locally compact and \( \sigma \)-compact.
- \( X \) is smooth over \( \mathbb{C} \) iff \( X^{an} \) is a complex manifold.
- A morphism \( f : X \to Y \) is proper iff \( X^{an} \to Y^{an} \) is proper. In particular, \( X \) is complete(proper) iff \( X^{an} \) is compact.
- If \( X \) is projective and connected, then \( X^{an} \) is connected iff \( X \) is connected.

**Proof:** The first is because \( X \) is qc hence covered by f.m. affine subsets hence second-countable and use (IX.1.2.23). \( X^{an}/X \) is flat because completion of Noetherian rings are flat(I.5.7.14).

For proper, Cf.[GAGA Serre P8].

If \( S \) is smooth, then by Jacobian criterion(I.7.5.12), the Jacobian of local defining function is not zero everywhere, so its analytification is clearly smooth.

The last assertion in fact follows from (IX.10.6.15), as \( H^0(X, \mathcal{O}_X) = H^0(X^{an}, \mathcal{O}_{X^{an}}) \). \( \square \)

**Prop. (IX.10.6.4).** There is a natural map from \( \mathcal{O}_X \) to \( \mathcal{H}_X \) that maps \( m_x \) to \( m_x \mathcal{H}_x \), thus inducing a map \( \hat{\theta} : \hat{\mathcal{O}}_x \to \hat{\mathcal{H}}_x \). This is an isomorphism. In particular, \( \theta : \Omega_x \to \mathcal{H}_x \) is injective.

Moreover, if \( Y \) is a locally closed subscheme of \( X \), then the local ideal of functions vanishing at \( Y \) maps to \( \mathcal{A}_x(Y) \), and \( \mathcal{A}_x(Y) \) is generated by \( \theta(I_x(Y)) \). Moreover, \( \mathcal{H}_{x,Y} = \mathcal{H}_x/I_x(Y) \).

**Proof:** [GAGA Serre P6]. \( \square \)

**Cor. (IX.10.6.5).** The inclusion \( \mathcal{O}_x \subset \mathcal{H}_x \) is flat ring extension, by(I.7.1.20) and the fact \( \hat{\mathcal{A}}/A \) is flat. And \( \dim \mathcal{O}_x = \dim \mathcal{H}_x \) because \( \dim A = \dim \hat{A} \).

**Cor. (IX.10.6.6).** Given an open and dense subscheme \( U \) of an algebraic variety \( X \) over \( \mathbb{C} \), \( U^{an} \) is dense in \( X^{an} \).

**Proof:** Consider the complement \( Y \), if \( U^{an} \) is not dense in \( X^{an} \), then there exists a \( x \) that \( \mathcal{A}_x(Y) = 0 \), so by (IX.10.6.4), \( I_x(Y) = 0 \), so \( Y \) is not dense near \( x \), contradiction. \( \square \)

**Cor. (IX.10.6.7).** For a morphism \( f \) of algebraic varieties over \( \mathbb{C} \), \( f(X)^{an} = f(X)^{an} \).

**Proof:** By Chevalley theorem(V.5.8.7), there is a open dense subscheme \( U \) of \( f(X) \) that is contained in \( f(X) \), then (IX.10.6.6) shows \( U^{an} \) is dense in \( f(X)^{an} \), so \( f(X)^{an} \subset f(X)^{an} \). The converse is obvious. \( \square \)

**Def. (IX.10.6.8) (Analytification of Sheaves).** Denote for a sheaf \( F \) over \( X \) \( F' \) the inverse image sheaf over \( X^{an} \) pulled back along \( X^{an} \to X \). Define \( F^{an} \) the **analytification of \( F \)** as the sheaf \( F' \otimes_{\mathcal{O}_X} \mathcal{H}_X \).
Prop. (IX.10.6.9). \( \mathcal{F} \to \mathcal{F}^{an} \) is exact from the category of sheaves on \( X \) to the category of analytic sheaves on \( X^{an} \), \( \mathcal{F}' \to \mathcal{F}^{an} \) is injective, and it maps coherent sheaves to coherent analytic sheaves.

Proof: The first two follows from the fact that \( \mathcal{H}_X \) is flat over \( X^{an} \to X \) (IX.10.6.5). For the last assertion, notice if \( \mathcal{O}_X^I \to \mathcal{O}_X^{an} \to \mathcal{F} \to 0 \), then \( \mathcal{H}_X^i \to \mathcal{H}_X^{an} \to 0 \), so it is coherent because \( \mathcal{H}_X \) is coherent (IX.10.3.5) (V.2.2.25). □

Prop. (IX.10.6.10). Let \( i : Y \to X \) be a closed subscheme, then for a coherent sheaf \( F \) on \( Y \),
\[
(i^{an})_* F^{an} \cong (i_* F)^{an}.
\]

Proof: These two sheaves are both 0 outside \( Y^{an} \), consider a point of \( Y \), their stalks are respectively \( \mathcal{F}_x \otimes \mathcal{O}_{x,X} \mathcal{H}_{x,X} \) and \( \mathcal{F}_x \otimes \mathcal{O}_{x,Y} \mathcal{H}_{x,Y} \). By (IX.10.6.4) we notice
\[
\mathcal{H}_{x,Y} = \mathcal{H}_{x,X} / T_x(Y) \mathcal{H}_{x,X} = \mathcal{H}_{x,X} \otimes \mathcal{O}_{x,X} \mathcal{O}_{x,Y}.
\]
So this two are equal by associativity of tensor product. □

Prop. (IX.10.6.11). By Leray Spectral sequence (V.6.1.8), for an analytic sheaf \( \mathcal{G} \), there is a boundary map \( H^k(X, \mathcal{G}) \to H^k(X^{an}, \mathcal{G}) \). So for a sheaf \( \mathcal{F} \) on \( X \), there is a map
\[
\varepsilon : H^k(X, \mathcal{F}) \to H^k(X, an_* \mathcal{F}^{an}) \to H^k(X^{an}, \mathcal{F}^{an})
\]

Equivalence between Algebraic Variety and Analytic Spaces

Remark (IX.10.6.12) (GAGA Principle). Any global analytic object on a projective variety over \( \mathbb{C} \) is algebraic.

Prop. (IX.10.6.13). Let \( X, Y \) be algebraic varieties over \( \mathbb{C} \) and \( f : X \to Y \) is morphism that is bijective, if \( f^{an} \) is an analytic isomorphism, then \( f \) is an isomorphism.

Proof: Cf.[GAGA Serre P9]. □

Cor. (IX.10.6.14). Let \( X, Y \) be algebraic varieties over \( \mathbb{C} \), iff \( f : X^{an} \to Y^{an} \) is holomorphic map and the image of \( f \) in \( X^{an} \times Y^{an} = (X \times Y)^{an} \) comes from an algebraic subscheme, then \( f \) comes from an algebraic morphism. (Because \( X^{an} \to \Gamma(X) \) is an analytic isomorphism).

Prop. (IX.10.6.15) (GAGA). Let \( X \) be a projective scheme in \( \mathbb{P}^n_{\mathbb{C}} \), then: \( \mathcal{F} \to \mathcal{F}^{an} \) defines an equivalence of categories between the coherent sheaves on \( X \) and coherent analytic sheaves on \( X^{an} \) that preserves cohomology groups.

Proof: Cf.[GAGA Serre P13]. □

Remark (IX.10.6.16). This theorem can be generalized to the case that \( X \) is proper over \( \mathbb{C} \), Cf.[SGA1, Chap12].

Cor. (IX.10.6.17) (Chow). Any analytic subvariety of \( \mathbb{C} \mathbb{P}^n \) is projective algebraic.

Proof: Cf.[GAGA Serre P20]. □

Prop. (IX.10.6.18).
- Any meromorphic function on an algebraic variety \( V \subset \mathbb{C} \mathbb{P}^n \) is rational.
IX.10. COMPLEX GEOMETRY

- Any meromorphic differential form on a smooth variety is algebraic.
- Any holomorphic map between smooth varieties can be given by rational maps.
- Any holomorphic vector bundle on a smooth variety is algebraic, i.e. transition function can be made rational.

Cf.[Griffith/Harris P168,170].

Cor. (IX.10.6.19). If the analytification of a variety $X$ is a compact complex manifold, i.e. $X$ is smooth (IX.10.6.3), then $K(X) = K(X^{an})$, as they are both morphism to $\mathbb{P}^1$.

Applications

Cf.[GAGA Serre].

Prop. (IX.10.6.20) (Generalized Riemann Existence Theorem). Let $X$ be a normal scheme of finite type over $\mathbb{C}$. Given any finite morphism of analytic spaces (i.e. proper and has finite fibers) form a normal complex analytic space $f : X' \to X^{an}$, then there is a unique normal scheme $X'$ and a finite morphism $g : X' \to X$ that $g^{an} = f$.

Proof: Cf.[SGA1, Chap12]. □

Cor. (IX.10.6.21) (Algebraic Fundamental Group).

7 Moishezon Manifolds

Def. (IX.10.7.1) (Moishezon Manifold). A compact complex manifold is called a **Moishezon manifold** iff $\text{tr.deg}.K(X) = \dim X$, by (IX.10.1.5) this is the highest degree it can have. When $X$ is an analytification of an algebraic variety $X^{an}$, $K(X^{an}) = K(X)$ by (IX.10.6.19), so Moishezon is a necessary condition for a compact complex manifold to be algebraic.

Prop. (IX.10.7.2) (Artin). The category of smooth proper algebraic spaces over $\mathbb{C}$ is equivalent to the category of Moishezon manifolds.

Proof: □

Prop. (IX.10.7.3) (Moishezon). Every Moishezon manifold that is Kähler is projective algebraic.

Proof: Cf.[Moishezon On $n$-dimensional compact varieties with $n$ algebraically independent meromorphic functions]. □

Cor. (IX.10.7.4). Use all these above and Kodaira Embedding (IX.11.8.6), we have the following applications of compact complex manifolds:

- Hodge iff projective algebraic.
- Projective algebraic is abstract algebraic, abstract algebraic is Moishezon.
- Hodge manifold is Kähler.
- Kähler+Moishezon is projective algebraic.

Algebraic Compact Complex Manifold

Prop. (IX.10.7.5). If $X_t$ is an algebraic family of nonsingular projective varieties over $\mathbb{C}$ parametrized by a variety $T$, then the functions $h^i(X_t, O_y)$ are constant for all $i$. 
IX.11 Kähler Geometry

Basic References are [Voisin], [Principle of Algebraic Geometry Griffith/Harris], more advanced stuffs to add from [Kähler Geometry] and [Complex Geometry Daniel]. Even more advanced is [Demailly Complex Analytic and Differential Geometry].

1 Kähler Metric

Def. (IX.11.1.1). The metric $g$ is called Kähler iff $\omega_g$ is closed. In which case, it is called the Kähler class of $g$ in $H^2_{dR}(M)$. A complex manifold with a Kähler metric is called a Kähler manifold.

If $g_{ij} = g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j})$, then $\omega_g = \sum_{ij} g_{ij} dz^i \wedge d\bar{z}^j$. Then the condition of $\omega_g$ being closed can in fact be written in derivatives of $g$.

Prop. (IX.11.1.2). If $g$ is Hermitian, then $\omega_g$ is real, non-degenerate and $\frac{1}{n!} \omega^n$ is a volume form on $M$. In particular, if $\omega$ is Kähler, then it is a symplectic form.

Proof: If $g = \sum \phi_i \otimes \bar{\phi}_i$, then $\omega = i \sum \phi_i \wedge \bar{\phi}_i$, so it is clear that $\omega = \omega$. $\omega$ is non-degenerate as $g$ is. The last assertion follows from (I.1.7.6). □

Cor. (IX.11.1.3). If $M$ is a compact Kähler manifold, then its even dimensional cohomology group doesn’t vanish (IX.7.1.6).

Remark (IX.11.1.4). Notice there are notions like almost Hermitian and almost Kähler, similar to the definition of Hermitian and Kähler, but they are just defined using an almost complex structure on $M$. And a almost Kähler structure is Kähler iff $\nabla J = 0$, Cf.[Foundation of Differential Geometry Kobayashi].

Remark (IX.11.1.5) (Examples of Kähler Manifolds).

- If $M = \mathbb{R}^{2n}$, $g = \sum dx_i \wedge dx_i + \sum dy_i \wedge dy_i$, then $\omega_g = \frac{i}{2} \sum dz_i \wedge d\bar{z}_i$ is Kähler.
- The metric $\omega_g = \sum dz_i \wedge d\bar{z}_i$ on a complex tori $\mathbb{C}^n/\Lambda$ is Kähler.
- Any compact Riemann surface is Kähler, because $d\omega$ is a 3-form so vanish.
- if $M = \mathbb{B}(0, 1) \subset \mathbb{C}^n$ and $\omega_g = i\partial\bar{\partial} \log \frac{1}{1-|z|^2}$, then it is Kähler.
- The product metric on the product space $M \times N$ of two Kähler manifold is Kähler.
- A submanifold of a Kähler manifold is Kähler, as the Kähler form is the pullback of the Kähler form of the large manifold.

Prop. (IX.11.1.6) (Fubini-Study Metric). The Fubini-Study metric form on $\mathbb{CP}^n$ is defined locally to be $i\partial\bar{\partial}|s|^2$, for any local lifting of the projection $\mathbb{C}^n - \{0\} \to \mathbb{CP}^n$. This doesn’t depend on the lifting, as $\partial\bar{\partial}(\log f + \log \bar{f}) = 0$, so they glue together to be a global form on $\mathbb{CP}^n$. It can be checked, $\omega$ is translation invariant and on the coordinate $(1, w_1, \ldots, w_n) \to (w_1, \ldots, w_n)$, $\omega|_{(0, \ldots, 0)} = \sum dw_i \wedge dw_i$, so it is positive definite.

Cor. (IX.11.1.7). Any projective manifold is Kähler.

Prop. (IX.11.1.8). the Fubini-Study metric on $\mathbb{CP}^n$ has sectional curvature $1 \leq K \leq 4$.

Proof: Cf.[Do Carmo P188]. □
Prop. (IX.11.1.9) (Kähler Normal Coordinate). For a Hermitian metric $g$ on $M$, $g$ is Kähler iff for any point $p$ of $M$, there is a holomorphic coordinate centered at $p$, $\omega_g = \sum g_{ij} dz_i \wedge d\bar{z}_j$ satisfying $g_{ij}(p) = 0$ and $dg_{ij}(p) = 0$. This coordinate is called Kähler normal coordinate. (Notice this is different from Darboux theorem, because this coordinate should be holomorphic).

Proof: Cf.[Complex Geometry P210].

2 Geometry of Kähler Manifolds

Prop. (IX.11.2.1). Let $(M, J, g)$ be a Kähler manifold, then the complexification of the Levi-Civita connection of $g$ restricts to the Chern connection on $T^{1,0} M$.

Proof: Cf.[Complex Geometry note 石亚龙 48] and [Complex geometry Daniel Chap4.A].

Prop. (IX.11.2.2). For a Kähler manifold, $\nabla J = 0$.

Proof: The problem depends only on first derivative, so choosing a Kähler normal nbhd (IX.11.1.9), we may choose $J$ to be constant, so obviously $\nabla J(p) = 0$, $P$ is arbitrary, so $\nabla J = 0$.

Cor. (IX.11.2.3). $\nabla(JX) = J \nabla X$, so $R(X, Y)JZ = JR(X, Y)Z$, thus

$$\langle R(JX, JY)Z, W \rangle = \langle R(Z, W)JX, JY \rangle = \langle R(Z, W)X, Y \rangle = \langle R(X, Y)Z, W \rangle,$$

so $R(JX, JY)Z = R(X, Y)Z$.

Prop. (IX.11.2.4). The curvature tensor of the complexified Levi-Civita connection on a Kähler manifold can be calculated in terms of $\partial_i, \bar{\partial}_j$, Cf.[Complex Geometry note 石亚龙 50].

3 Kähler Identities

Let $X$ be a compact complex Kähler manifold.

Def. (IX.11.3.1). Introduce some operators:

- $d^c = i(\bar{\partial} - \partial)$, then $dd^c = 2i\partial \bar{\partial}$.
- The Lefschetz operator $L(\eta) = \omega \wedge \eta$. $\Lambda$ is defined as the formal adjoint of $L$ as $A^{p,q}$ is an inner space. In fact, $\Lambda = \pm \ast L \ast$.
- $h = (k - n)$ on $A^k(X)$.

Prop. (IX.11.3.2). $[L, \Lambda] = p + q - n$ on $(p, q)$-forms.

Proof: The problem doesn’t depend on the derivatives, so using the Kähler normal coordinate (IX.11.1.9), it suffice to prove for $\mathbb{C}^n$, for this, Cf.[Griffith/Harris P120] or [Complex Geometry P34].

Prop. (IX.11.3.3) (Kähler Identities).

$$[\Lambda, \partial] = i\partial^\ast, \quad [\Lambda, \partial] = -i\partial^\ast.$$  

Proof: The second one follows from the first because $\omega$ is a real form. For the first, notice only first derivation are involved, so by using the Kähler normal coordinate, it suffice to prove for $\mathbb{C}^n$, and this is by [Complex Geometry 石亚龙 P61].
Cor. (IX.11.3.4). 
\[ [\Lambda, d^c] = d^*, \quad [\Lambda, d] = -d^{c*}. \]

Prop. (IX.11.3.5). \( \Delta_d \) commutes with both \( L \) and \( \Lambda \).

Proof: \( L \) commutes with \( d \) because \( \omega \) is closed, so taking adjoints, \( \Lambda \) commutes with \( d^* \). Now by Kähler identities,
\[ \Lambda \Delta_d = \Lambda (dd^* + d^* d) = -d^{c*}d^* + dd^* \Lambda - dd^{c*} + d^* d \Lambda = \Delta_d \Lambda. \]
So taking adjoints, \( \Delta_d \) also commutes with \( L \). \( \square \)

Prop. (IX.11.3.6). In the Kähler case, \( \Delta_{\partial} = \Delta_{\bar{\partial}} = \frac{1}{2} \Delta_d \).

Proof:
\[ \Delta_d = (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial}) = \Delta_{\partial} + \Delta_{\bar{\partial}} + (\partial\bar{\partial}^* + \bar{\partial}^* \partial) + (\bar{\partial}\partial^* + \partial^* \bar{\partial}) \]
So it suffice to prove \( \partial\bar{\partial}^* + \bar{\partial}^* \partial = 0 \) (so \( \partial\partial^* + \partial^* \partial = 0 \) by conjugation), and \( \Delta_{\partial} = \Delta_{\bar{\partial}} \). For the first, use Kähler identities, then
\[ i(\partial\bar{\partial}^* + \bar{\partial}^* \partial) = \partial[\Lambda, \partial] + [\Lambda, \partial] \partial = 0 \]
For the second, using Kähler identities,
\[ i\Delta_{\bar{\partial}} = \bar{\partial}[\Lambda, \partial] + [\Lambda, \partial] \bar{\partial} = \bar{\partial}\Lambda\partial + \partial\bar{\partial} - \Lambda\bar{\partial} \partial - \partial\Lambda\bar{\partial} \]
and the same is miraculous true for \( \Delta_{\partial} \), so the result is true. \( \square \)

4 Hodge Theory

Prop. (IX.11.4.1) (Hodge Decomposition of compact Kähler Manifold). For a compact Kähler manifold \( X \),
\[ H^d_{dR}(X, \mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}_d(X) \cong \bigoplus_{p+q=r} H^q(X, \Omega^p) \]
and \( \overline{H^{p,q}_d(x)} \cong H^{q,p}(X) \). Moreover, this decomposition doesn’t depend on the Kähler metric.

Proof: (IX.11.3.6) shows that \( \Delta_d \) maps \( A^{p,q} \) to \( A^{p,q} \), so \( H^d_{dR} \cap A^{p,q} = H_{dR}^{p,q}(X) \). The last assertion is seen using the \( \Delta_d \) definition.

If chosen two different Kähler metric \( g, g' \), there \( H^{p,q}(X, g) \cong H^{p,q}(X, g') \). If \( \alpha, \alpha' \) be \( g, g' \)-harmonic respectively, so by definition \( \alpha - \alpha' = \bar{\gamma} \) for some \( \gamma \), and they are both \( d \)-harmonic, so \( d\gamma = 0 \), and \( \bar{\gamma} \) is \( g \)-orthogonal to \( H^k(X, g) \) by Hodge decomposition for \( \bar{\partial} \) with metric \( g \), so by Hodge theorem for \( d \) with metric \( g \), \( d\gamma \) is \( d \)-exact, so \( [\alpha] = [\alpha'] \). \( \square \)

Cor. (IX.11.4.2). Betti number \( b_r = \sum_{p+q=r} h^{p,q}, h^{p,q} = h^{q,p} \). In particular, \( b_{2k+1} \) is always even.

Cor. (IX.11.4.3) (Holomorphic Form on Kähler Manifold is Closed). \( \mathcal{H}^{p,0}_d(X) = H^0(X, \Omega^p) \). Now a \((p, 0)\)-form is automatically \( \bar{\partial}^* \)-closed, so it is \( \bar{\partial} \)-harmonic iff it is holomorphic. So we conclude any holomorphic \( p \)-form on a Kähler manifold is \( d \)-closed, even \( d \)-harmonic.
Lemma (IX.11.4.4) \((\partial\bar{\partial}\text{-lemma})\). A closed differential form \(\eta\) on a compact Kähler manifold \(M\) is \(d\)-exact iff it is \(\partial\)-exact iff it is \(\partial\bar{\partial}\)-exact.

Proof: Now \(\Delta_d, \Delta_{\bar{\partial}}, \Delta_{\partial}\) are all the same, By Hodge theorem, it suffice to prove, if a form is orthogonal to \(\mathcal{H}^{p,q}(X)\), then it is \(\partial\bar{\partial}\)-exact(this implies other exactness).

Now \(\eta\) is \(d\)-closed hence \(\partial\) and \(\partial\bar{\partial}\)-closed, then \(\eta = \partial \gamma + \partial \bar{\partial} \beta'\), and then \(\partial \bar{\partial} \partial \gamma = 0\), so \(\eta = \partial \partial \gamma\).

\[\square\]

Cor. (IX.11.4.5) (Kodaira-Serre Duality). By (IX.2.10.15), For a Hermitian line bundle over a compact Hermitian complex manifold \(X\), from Hodge theorem and (IX.10.5.2), we get

\[H^p(X, \Omega^q(E)) \cong H^{n-p}(X, \Omega^{n-q}(E^*))\]

induced by \(\bar{\pi}_E\) and \(\bar{\pi}_{E^*}\). Moreover, there is a perfect pairing

\[H^p(X, \Omega^q(E)) \times H^{n-p}(X, \Omega^{n-q}(E^*)) \to \mathbb{C}\]

induced by

\[\mathcal{H}^{p,q}(X, E) \times \mathcal{H}^{n-p,n-q}(X, E^*) \to \mathbb{C}: (\alpha, \beta) \mapsto \int_X \alpha \wedge \bar{\pi}_E \beta\]

In fact, \(\int_X \alpha \wedge \bar{\pi}_E \alpha = ||\alpha||^2 \neq 0\).

Prop. (IX.11.4.6). Holomorphic 1-forms on a compact complex surface is closed. ?

Prop. (IX.11.4.7) (Hard Lefschetz Theorem). For a compact Kähler manifold \(M\), the map

\[L^k : H^{n-k}(M) \to H^{n+k}(M)\]

is an isomorphism.(notice it is defined because \(L\) commutes with \(d\)).

Define the primitive cohomology class \(P^{n-k}(M) = \text{Ker} L^{k+1}\) on \(H^{n-k}\), then

\[H^m(M) = \oplus_k L^k P^{m-2k}(M).\]

Proof: Cf.[Griffith/Harris P122], using representation theory of \(\mathfrak{sl}_2\). \(\square\)

Prop. (IX.11.4.8) (Hodge-Riemann Bilinear Relation). Let \((X, \omega)\) be a Kähler manifold, if \(\alpha \neq 0 \in H^{p,q}(X)\) is a primitive cohomology class, then

\[i^{p-q}(-1)^{(p+q)(p+q-1)/2} \int_X \alpha \wedge \bar{\pi} \wedge \omega^{n-p-q} > 0\]

Proof: Cf.[Griffith/Harris] or [Complex Geometry Daniel P138]. \(\square\)

Cor. (IX.11.4.9). For a compact Kähler manifold of complex dimension \(2m\),

\[\text{sgn}(X) = \sum_{p,q=0}^{m} (-1)^p h^{p,q}(m)\]

Proof: Cf.[Complex Geometry Daniel P140]. \(\square\)
Prop. (IX.11.4.10) (Hirzebruch-Riemann-Roch). By (IX.2.10.8), for a $n$-dimensional complex line bundle $L$ over a compact Kähler manifold $M$, 

$$\chi(M, L) = \int_M [\text{ch}(E)\text{td}(T^{1,0}M)]_n.$$ 

Where $\chi(M, L) = \sum_{q=0}^n (-1)^q \dim H^q(M, E)$, $\text{ch}$ is the Chern character (IX.2.9.5) and $\text{td}(T^{1,0}M)$ is the Todd polynomial, i.e. Taylor expansion of $\prod_{i=1}^n \frac{t_i}{1 - e^{-t_i}}$ in terms of the symmetric polynomial, applied to $c_i(T^{1,0}M)$.

Cor. (IX.11.4.11) (Riemann-Roch). By (IX.2.10.9), for a complex vector bundle $E$ over a Riemann Surface $M$, let $\deg E = \int_M c_1(E)$, then 

$$\chi(M, E) = H^0(M, E) - \dim H^1(M, E) = \deg E + \text{rk}(E)(1 - g).$$

Cor. (IX.11.4.12). For other examples of corollaries of Hirzebruch-Riemann-Roch theorem, Cf. [Complex Geometry Daniel Chap3.A].

Formality of Complex Kähler Geometry

Cf. [Complex Geometry Daniel Chap3.A].

5 Jacobian and Abanese

Prop. (IX.11.5.1) (Complex Tori). The complex tori is defined to be $X = \mathbb{V}/\Gamma$, where $\mathbb{V} \cong \mathbb{C}^n$ and $\Gamma$ is a complete lattice. It is a Kähler manifold. And $H^2(X, \mathbb{R}) \cong \mathbb{V}^* \wedge \mathbb{V}^*$.

Proof: For the last assertion, use the fact that the cotangent bundle is trivial (VII.1.1.4). \qed

Lemma (IX.11.5.2). if $X$ is compact Kahler, then the natural map $H^k(X, \mathbb{C}) \rightarrow H^k(X, \mathcal{O}_X)$ is just the projection onto the $(0, k)$-part. In particular, the image is in $H^{0,k}(X)$.

Proof: By Hodge decomposition, the definition of Dolbeault cohomology and the commutative diagram

$$\begin{array}{ccc}
\mathbb{C} & \rightarrow & A^0(X) \\
& \overset{d}{\rightarrow} & A^1(X) \\
& \overset{\pi_{0,1}}{\rightarrow} & A^2(X) \\
\mathcal{O}_X & \rightarrow & A^0(X) \\
& \overset{\overline{\partial}}{\rightarrow} & A^1(X) \\
& \overset{\pi_{0,2}}{\rightarrow} & A^2(X)
\end{array}$$

\qed

Prop. (IX.11.5.3) (Lefschetz theorem on $(1,1)$-forms). By (IX.10.5.8), we know that the image of $\text{Pic}_\mathbb{C}(X) \rightarrow H^2(X, \mathbb{Z})$ is trivial in $H^2(X, \mathcal{O}_X)$. If $X$ is compact Kähler, there is Hodge decomposition (IX.11.4.1) $H^2(X, \mathcal{O}_X) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$.

So if we define $H^{1,1}(X, \mathbb{Z}) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$, then the image of $\text{Pic}_\mathbb{C}(X)$ is contained in $H^{1,1}(X, \mathbb{Z})$ by (IX.11.5.2), and it is also surjective, this is to say, $NS(X) = H^{1,1}(X)$

Proof: Because by the long exact sequence of (IX.10.5.8) and (IX.11.5.2) again, an $\alpha \in H^2(X, \mathbb{C})$ is in $H^{1,1}(X, \mathbb{Z})$ iff $\alpha$ is in the image of $\text{Pic}_\mathbb{C}(X) \rightarrow H^{1,1}(X, \mathbb{Z})$. \qed
Cor. (IX.11.5.4). The image of $H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X)$ is a lattice. In particular, it is isomorphic to $\mathbb{Z}^{b_1(X)}$.

**Proof:** $H^1(X, \mathbb{Z})$ is a lattice in $H^1(X, \mathbb{R})$, and $H^1(X, \mathbb{R}) \to H^1(X, \mathcal{O}_X) = H^{0,1}(X, \mathbb{C})$ is an isomorphism, because $H^{0,1}(X, \mathbb{C})$ are conjugate to $H^{1,0}(X, \mathbb{C})$ and $H^1(X, \mathbb{R})$ is real. □

**Def. (IX.11.5.5) (Jacobian).** The **Jacobian** $\text{Jac}(X)$ of a compact Kähler manifold $X$ is defined to be $H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$, so it is a complex torus of dimension $b_1(X)$ by (IX.11.5.4), it is also the kernel of the first Chern class map by the long exact sequence (IX.10.5.8), i.e.

$$0 \to \text{Jac}(X) \to \text{Pic}(X) \xrightarrow{\zeta} \text{NS}(X) \to 0$$

**Def. (IX.11.5.6) (Albanese).** The **Albanese** $\text{Alb}(X)$ of a compact Kähler manifold $X$ is defined to be the complex torus $H^0(X, \Omega_X^1)^*/H_1(X, \mathbb{Z})$, where

$$H^1(X, \mathbb{Z}) \to H^0(X, \Omega_X^1)^*: [\gamma] \mapsto \left( u \mapsto \int_\gamma u \right)$$

(Notice this is well-defined because by (IX.11.4.3) any $u \in H^0(X, \Omega_X^1)$ is closed).

Fix a base point $x_0$ of $X$, the **Albanese map** $\text{Alb}: X \to \text{Alb}(X)$ is defined to be

$$x \mapsto \left( u \mapsto \int_{x_0}^x u \right)$$

It is holomorphic and functorial in $(X, x_0)$. It is just the so called **Abel-Jacobi map** in case when $X$ is a Riemann surface.

6 **Positivity**

**Def. (IX.11.6.1) (Positive Line Bundle).** A 2-form $\omega$ on a Hermitian complex manifold $M$ is called **positive** iff $\omega(u, Ju) \geq 0$ for $u \neq 0 \in TM$, which is equivalent to $-i\omega(v, \overline{v}) > 0$ for all $v \in T^{1,0}X$.

A holomorphic vector bundle is called **(Griffith-)positive** iff there exists a Hermitian metric on it that the curvature form $\Omega$ for the Chern connection (IX.10.5.13) satisfies $h(\Omega(s), s)(v, \overline{v}) > 0$ for all $s \in E$ and $v \in T^{1,0}X$.

The pullback of a positive line bundle along an immersion is positive.

**Prop. (IX.11.6.2) (Positivity on Kähler Manifolds).** On a compact Kähler manifold, being positive is a topological property for line bundles. It is equivalent to the first Chern class of $L$ can be represented by a positive form in $H^2_{dR}(M)$.

**Proof:** $c_1(L) = \left[ \frac{1}{2\pi} \Omega \right]$, so one direction is trivial, and if $c_1(L) = \left[ \frac{1}{2\pi} \theta \right]$, choose an arbitrary Hermitian metric $h$ on $L$, then by $\partial\overline{\partial}$-lemma (IX.11.4.4), $\theta = \Omega + \overline{\partial}\rho$ for some smooth function $\rho$. Then $e^{\rho}h$ has $\Omega = \theta$ by formula (IX.10.5.14). □

**Cor. (IX.11.6.3).** On a compact Kähler manifold, if $L$ is positive, then for any other Hermitian line bundle $L', kL + L'$ is positive.

**Prop. (IX.11.6.4).** The hyperplane line bundle $\mathcal{O}(1)(\text{IX.10.5.9})$ is positive.
Proof: The hyperplane line bundle is dual to the tautological line bundle. The metric on the tautological line bundle is given by locally 
\[ g_i = \frac{1}{|z_i|^2} \sum |z_i|^2. \] It is compatible with the transition map, and then by (IX.10.5.14), the Chern curvature is 
\[ \overline{\partial}\partial\left( \frac{1}{|z_i|^2} \sum |z_i|^2 \right) = \overline{\partial}\partial(\sum |z_i|^2). \]
So by (IX.3.3.6) the curvature of the hyperplane line bundle times \( i \) is just the Fubini-Study metric form (IX.11.1.6), so it is positive. □

Prop. (IX.11.6.5). For \( \tilde{X} \to X \) the blowing-up of \( X \) at a point \( x \), If \( L \) is a positive line bundle on \( X \), then for any integer \( n \), there exists a \( k > 0 \) that \( \pi^* L^k - nE \) is a positive line bundle on \( \tilde{X} \), where \( E \) is the exceptional divisor.

Proof: Involves explicit metric calculation, Cf. [Kodaira Embedding Theorem P11] and [Complex Geometry P249]. □

7 Kodaira Vanishing Theorem

Prop. (IX.11.7.1) (Nakano Identities). For a holomorphic vector bundle over a compact Kähler manifold \( (M, \omega) \) with Hermitian metric \( h \), introduce operators \( L \) and \( \Lambda \) as before. If we denote the \((1, 0)\) and \((0, 1)\)-part of the Chern connection on \( E \) by \( D' \) and \( D'' = \overline{\partial} \), then
\[ [\Lambda, \overline{\partial}] = -iD'^* \quad [\Lambda, D'] = i\overline{\partial}^* \]

Proof: The question is local, choose normal coordinate frame at \( x \)(IX.10.5.16), then by the formula of Chern connection(IX.10.5.14), \( \nabla_E = d + A, A(x) = 0 \), and \( \nabla_{E^*} = d + B, B(x) = 0 \). so
\[ [\Lambda, \overline{\partial}_E] + iD'^* = [\Lambda, \partial] + i\overline{\partial}^* + [\Lambda, A^{0,1}] + iB^{0,1} \]
where the usual Kähler identities (IX.11.3.3) are used. Then it is zero when evaluated at \( x \), Cf. [Demailly Complex Analytic and Differential Geometry P329]. □

Cor. (IX.11.7.2) (Bochner-Kodaira-Nakano Identity).
\[ \Delta_{\overline{\partial}, E} - \Delta_{D', E} = i[\Omega, \Lambda] \]

Proof: \( -i\Delta_{D', E} = D'[\Lambda, \overline{\partial}] + [\Lambda, \overline{\partial}]D' = D'\Lambda\overline{\partial} - D'\overline{\partial}\Lambda + \Lambda\overline{\partial}D' - \overline{\partial}\Lambda D' \)
and similar calculation for \( i\Delta_{\overline{\partial}, E} \), so
\[ i\Delta_{\overline{\partial}, E} - i\Delta_{D', E} = \Lambda(\overline{\partial}D' + D'\overline{\partial}) - (\overline{\partial}D' + D'\overline{\partial})\Lambda = -[\Omega, \Lambda]. \]

Prop. (IX.11.7.3) (Kodaira-Akizuki-Nakano Vanishing Theorem). If \( L \) is a positive line bundle on a compact Kähler manifold \( M \), then
\[ H^p(M, \Omega^q(L)) = 0 \]
for \( p + q > n \). In particular, \( H^q(M, K_X \otimes L) = 0 \) for \( q > 0 \).
Proof: By Hodge theorem (IX.2.10.13), it suffice to prove there are no harmonic \((p, q)\)-forms \(E\in H^{p,q}(X, L)\) on \(L\).

As \(i\Omega = \omega\) is positive, we may endow \(M\) with the metric \(\omega\), then by (IX.11.7.2) and (IX.11.3.2),

\[\Delta_{\bar{\partial}} - \Delta_{\bar{\partial}'} = [L, \Lambda] = p + q - n \text{ on } A^{p,q}.
\]

So if \(s \in H^{p,q}(X, L)\), then \((\Delta_{\bar{\partial}}s - \Delta_{\bar{\partial}'}s, s) = (p + q - n)|s|^2 \geq 0\), but \((\Delta_{\bar{\partial}}s - \Delta_{\bar{\partial}'}s, s) = -(\Delta_{\bar{\partial}'}s, s) = -||D's||^2 - ||D'^*s||^2 \leq 0\), so \(s = 0\). \(\square\)

Cor. (IX.11.7.4) (Serre’s Theorem). Let \(L\) be a positive line bundle on a compact complex Kähler manifold \(X\), then for any holomorphic vector bundle \(E\), for \(m\) large, \(H^q(X, L^m \otimes E) = 0\).

Proof: Same notation as in the proof of (IX.11.7.3), choose Hermitian structure on \(E\) and \(L\) and their Chern connections by \(\nabla_E, \nabla_L\), the corresponding Chern connection on \(E \otimes L^m\) is denoted by \(\nabla\), and make sure \(\frac{i}{2\pi} F_{\nabla_L}\) is the Kähler form \(\omega\), then for any harmonic form \(\alpha \in H^{p,q}(X, E \otimes L^m)\), by (IX.11.7.2), \(\frac{i}{2\pi} ([\Lambda, F_{\nabla}](\alpha), \alpha) \geq 0\), but \(\frac{i}{2\pi} F_{\nabla} = \frac{i}{2\pi} F_{\nabla_L} + m\omega\), so

\[0 \leq \frac{i}{2\pi} ([\Lambda, F_{\nabla_L}](\alpha), \alpha) + m(n - p - q)||\alpha||^2\]

Notice \(||([\Lambda, F_{\nabla_L}](\alpha), \alpha)||\) has a bound by Schwartz inequality, then if \(p + q > n\) and \(m\) sufficiently large, \(\alpha\) must by 0. In this case \(H^{p,q}(X, E \otimes L^m) = 0\), but \(H^{0,q}(X, K_X \otimes E \otimes L^m) \subset H^{0,q}(X, E \otimes L^m)\), so it is 0. Now we’ve proved \(H^q(X, K_X \otimes E \otimes L^m) = 0\) for any \(E\) if \(m\) is large. But \(E\) is arbitrary, so the conclusion is true. \(\square\)

Cor. (IX.11.7.5) (Grothendieck’s Lemma). Every holomorphic line bundle \(E\) over \(\mathbb{C}P^1\) is uniquely isomorphic to a finite direct sum of \(\mathcal{O}(a_i)\).

Proof: If \(E\) has rank 1, this is the content of (IX.10.5.10), so use induction on rank of \(E\). Choose a maximal \(a\) that \(\text{Hom}(\mathcal{O}(a), E) = H^0(\mathbb{C}P^1, E(-a)) \neq 0\). This \(a\) exists because Serre’s Theorem (IX.11.7.4) shows that \(H^1(\mathbb{C}P^1, E(-a)) = 0\) for \(a\) sufficiently small, and Riemann-Roch (IX.2.10.9) shows that \(\chi(\mathbb{C}P^1, E(-a)) = \deg E + \text{rk}(E)(1 - a)\) is positive for \(a\) sufficiently small, so \(H^0(\mathbb{C}P^1, E(-a)) \neq 0\). Conversely, if \(a\) is sufficiently large, then \(H^0(\mathbb{C}P^1, E(-a)) \cong H^1(\mathbb{C}P^1, E^*(a - 2)) = 0\) (Notice \(K_{\mathbb{C}P^1} = \mathcal{O}(n - 1)\)).

So now there is an exact sequence of sheaves

\[0 \to \mathcal{O}(a) \xrightarrow{s} E \to E_1 \to 0\]

I claim \(E_1\) is also a vector bundle, because \(s\) never vanishes, otherwise if it vanish at some \(x\), then we can divide by a linear factor \(s_x \in H^0(\mathbb{C}P^1, \mathcal{O}(1))\) to get a map \(\mathcal{O}(a + 1) \to E\), contradicting the maximality. So by induction \(E_1 = \oplus \mathcal{O}(a_i)\), then I claim \(a_i \leq a\), because otherwise \(H^0(\mathbb{C}P^1, E_1(-a - 1)) \neq 0\), and by the exact sequence \(0 \to \mathcal{O}(a) \to E(-a - 1) \to E_1(-a - 1) \to 0\), \(H^0(\mathbb{C}P^1, E(-a - 1)) \neq 0\), contradiction.

Then we want to show the above sequence splits, this is equivalent to

\[0 \to E_1^*(a) \to E^*(a) \to \mathcal{O} \to 0\]

splits, and his follows from the fact \(H^1(\mathbb{C}P^1, E_1^*(a)) = H^1(\mathbb{C}P^1, \oplus \mathcal{O}(a - a_i)) = 0\), by Serre duality. So there is a section lifting \(\mathcal{O} \to E^*(a)\), which splits the sequence. \(\square\)

Prop. (IX.11.7.6) (Weak Lefschetz Theorem). Let \(X\) be a compact Kähler manifold and \(Y\) be a submanifold that the line bundle \(\mathcal{L}(Y)\) is positive, then the canonical restriction map \(H^k(X, \mathbb{C}) \to H^k(Y, \mathbb{C})\) is isomorphism for \(k \leq n - 2\) and injective for \(k = n - 1\).
**Proof:** In fact, using Hodge decomposition, it suffices to prove on the level of $H^q(X, \Omega^p_X)$. Tensoring the exact sequence

$$0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to \mathcal{O}_X \to 0$$

with $\Omega^p_X$ and taking the cohomology. By Serre duality and Kodaira vanishing $(IX.11.7.3)$, the map $H^q(X, \Omega^p_X) \to H^q(X, \Omega^p_iY_i^*O_X)$ is isomorphism for $p + q < n - 1$ and injection for $p + q = n - 1$.

Next consider the exact sequence $0 \to TY \to TX \to N_{Y/X} \to 0$. By $(V.3.1.11)$ there is an exact sequence

$$0 \to \wedge^p TY \to \wedge^p TX |_Y \to \wedge^{q-1} TY | N_{Y/X} \to 0$$

Taking dual and applying adjunction formula $(IX.10.1.3)$, it becomes:

$$0 \to \Omega^{q-1}_Y \otimes O(-N) \to \Omega^q_X |_Y \to \Omega^q_Y \to 0$$

Taking cohomology and use Serre duality and Kodaira vanishing as before, the result follows, and the composition is also true. □

**Remark (IX.11.7.7).** There is a topological proof of weak Lefschetz theorem in [Bott On a theorem of Lefschetz].

### 8 Kodaira Embedding Theorem

**Prop. (IX.11.8.1) (Kodaira map).** For a holomorphic line bundle $L$ on a compact complex manifold $M$, if $s_0, \ldots, s_n$ be a basis of $H^0(X, L)$, we try to define a map from $M$ to $\mathbb{CP}^n : x \to [s_0(x), \ldots, s_n(x)]$. This is independent of the change of coordinates because $g_{\alpha\beta}$ is invertible, and it is definable iff $L$ is basepoint-free. This map is holomorphic where it is definable.

**Def. (IX.11.8.2).** For a holomorphic vector bundle $L$ on a compact complex manifold $X$, $L$ is called

- **semi-ample** iff for $m$ large, $L^m$ is basepoint-free.
- **very ample** iff $L$ is basepoint-free and the Kodaira map $\iota_L : X \to \mathbb{CP}^N$ is a holomorphic embedding.
- **ample** iff for $m$ large, $L^m$ is very ample.

**Lemma (IX.11.8.3) (Cohomological Method for Very Ampleness).** For the above Kodaira map to be a holomorphic embedding, it suffice to show that the map is definable, injective and surjective on cotangent space. For these, it is equivalent to $H^0(X, L) \to L_x$ surjective, $H^0(X, L) \to L_x \oplus L_y$ surjective, and $L \otimes \mathcal{I}_x \to L_x \otimes T^{1,0,x}(X) \to 0$ surjective. And they are true if

$$H^1(X, L \otimes \mathcal{I}_x) = 0, \quad H^1(X, L \otimes \mathcal{I}_{x,y}) = 0, \quad H^1(X, L \otimes \mathcal{I}^2_x) = 0.$$

respectively.

**Proof:** Basepoint-free at $x$ is easily seen to be equivalent to $H^0(X, L) \to L_x$ surjective. And there is an exact sequence of sheaves:

$$0 \to L \otimes \mathcal{I}_x \to L \to L_x \to 0$$

where $L_x$ means the skyscraper sheaf. So $H^1(X, L \otimes \mathcal{I}^2_x) = 0$ induces the result.
Injective is easily seen to be equivalent to $H^0(X, L) \to L_x \oplus L_y$ surjective. And there is an exact sequence of sheaves:

$$0 \to L \otimes \mathcal{I}_{x,y} \to L \to L_x \oplus L_y \to 0$$

where $\mathcal{I}_{x,y}$ is the sheaf of functions vanishing at $x$ and $y$, and $L_x \oplus L_y$ means the skyscraper sheaf. So $H^1(X, L \otimes \mathcal{I}_{x,y}) = 0$ induces the result.

For the surjection on cotangent spaces, given any point $x$, choose a basis $s_1, \ldots, s_n$ of sections in $H^0(X, L)$ vanishing at $x$, and by basepoint-free, there is a $s_0$ not vanishing at $x$, then on a coordinate, the Kodaira map is given by $x \to (s_1/s_0, \ldots, s_n/s_0)$, then it need to be checked $d_x(s_i/s_0) = d_x(x_i)/s_0$ span $T^{1,0*}(X)_x$. But there are exact sequences of sheaves:

$$0 \to L \otimes \mathcal{T}^2_x \to L \otimes \mathcal{T}_x \xrightarrow{d_x} L_x \otimes T_x^{1,0*} \to 0$$

where $d_x$ is given by $d_x(s \otimes f) = s(x) \otimes d_x(f)$ (by the universal property of skyscraper sheaf), it suffice to give a map $(L \otimes \mathcal{I}_x \to L_x \otimes T_x^{1,0*})$, notice this is independent of the coordinate because $d_x(s_0) = d_x(g_{\alpha \beta} s_\beta) = g_{\alpha \beta} d_x(s_\beta)$, as $s_\alpha$ vanishes at $x$, so this is truly a sheaf map, and its kernel is $L \otimes \mathcal{T}^2_x$. So $H^1(X, L \otimes \mathcal{T}_{x}^2) = 0$ induces the result. □

Prop. (IX.11.8.4). A holomorphic line bundle $L$ on a compact Kähler manifold is ample iff it is positive.

Proof: If $L$ is ample, then $L^m$ is the pullback of the hyperplane bundle by the Kodaira map. The hyperplane line bundle is positive by (IX.11.6.4), so $L^m$ is positive with the induced metric, so $L$ is also positive given the $m$-th roots of the induced metric (notice the metric of line bundle is just locally a number compatible with transition map).

Conversely, using (IX.11.8.3), we want to find a $L^k$ that $H^1(X, L^k \otimes \mathcal{I}_x) = 0$, $H^1(X, L^k \otimes \mathcal{I}_{x,y}) = 0$, $H^1(X, L^k \otimes \mathcal{T}_x^2) = 0$. First notice it suffice to prove for single points when $k$ is sufficiently large, because the holomorphic embedding is an open property and $X$ is compact so a sufficiently large $k$ will suffice.

Consider the blowing-up $\tilde{X}$ at a point $x$, there is a commutative diagram

$$H^0(X, L^k) \xrightarrow{\pi^*} L^k_x \xrightarrow{\pi_*} \pi^* H^0(\tilde{X}, \pi^* L^k \otimes \mathcal{L}_x^k) \xrightarrow{\sim} H^0(E, \mathcal{O}_E) \otimes L^k_x$$

The right vertical map is isomorphism as $E \cong \mathbb{CP}^n$, so $H^0(E, \mathcal{O}_E) = \mathbb{C}$. The left exact sequence is also isomorphism: it is injective because $\pi$ is surjective, and it is surjective because: if dim $X = 1$, then $\pi = \text{id}$ so trivially true, and if dim $X \geq 2$, then because $\pi : \tilde{X} - E \cong X - \{x\}$, any holomorphic function on $\tilde{X}$ induces a holomorphic function on $X - \{x\}$ and by Hartog’s theorem (X.2.9.3), it comes from a holomorphic function on $X$.

Now the second horizontal line is part of the cohomology exact sequence of (V.7.1.17)

$$0 \to \pi^* L^k \otimes \mathcal{O}_{\tilde{X}}(-E) \to \pi^* L^k \to \pi^* L^k |_E \to 0$$

So it is reduced to prove $H^1(\tilde{X}, \pi^* L^k \otimes \mathcal{O}(-E)) = 0$, but by (IX.10.1.13), $\pi^* L^k - E = \pi^* L^k - E + \mathcal{K}_{\tilde{X}} - \pi^* \mathcal{K}_X = -(n - 1)E = \mathcal{K}_{\tilde{X}} + (\pi^* L^k - E) + \pi^* (L^k - \mathcal{K}_X)$, and by (IX.11.6.5)(IX.11.6.3) the last two are positive when $k$ is large, so the conclusion follows from Kodaira vanishing (IX.11.7.3).
The proof of $H^1(X, L^k \otimes \mathcal{I}_{x,y}) = 0$ is verbatim, just use blowing-up at two different points.

To prove $H^1(X, L^k \otimes \mathcal{I}_x^2) = 0$, consider the blowing-up $\tilde{X}$ at a point $x$, notice there is a commutative diagram

$$
\begin{array}{ccc}
H^0(X, L^k \otimes \mathcal{I}_x) & \xrightarrow{d_x} & L^k_x \otimes T^{1,0} X_x \\
\downarrow^{\pi^*} & & \downarrow^{\cong} \\
H^0(\tilde{X}, \pi^* L^k - E) & \longrightarrow & L^k_x \otimes H^0(E, -E)
\end{array}
$$

In fact this comes from the two commuting exact sequences twisted with $\pi^* L^k$:

$$
\begin{array}{ccc}
0 & \longrightarrow & \pi^* \mathcal{I}_x^2 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{O}(-2E) \\
\end{array}
\quad
\begin{array}{ccc}
\pi^* \mathcal{I}_x & \xrightarrow{d_x} & \pi^* T^{1,0} X_x \\
\downarrow & & \downarrow \\
\mathcal{O}(-E) & \longrightarrow & \mathcal{O}_E(-E) \longrightarrow 0
\end{array}
$$

The second line is (V.7.1.17) and the fact a section vanishing at $x$ lifts to a section vanishing at $E$ thus equivalent to a section in the twisted sheaf $- \otimes \mathcal{O}(-E)$. These two exact sequences commutes because

Back to the commutative diagram, the above argument also shows that the first vertical map is isomorphism. To show the second vertical map is isomorphism, notice by (IX.10.1.12) $\mathcal{O}(-E)$ is just the hyperplane line bundle on $E$, so $H^0(E, -E) \cong T^{1,0} X_x$, we need to know the vertical map is the natural map $V^* \rightarrow H^0(\mathbb{P}(V), \mathcal{O}(1))$. This in fact need some careful calculation using coordinates in (IX.10.1.12).?

Now the map $d_x$ is surjective iff the second horizontal map is surjective, with is part of the cohomology exact sequence of

$$
0 \rightarrow \pi^* L^k \otimes \mathcal{O}_{\tilde{X}}(-2E) \rightarrow \pi^* L^k \mathcal{O}_{\tilde{X}}(-E) \rightarrow \pi^* L^k|_E \rightarrow 0
$$

So it is reduced to prove $H^1(\tilde{X}, \pi^* L^k \otimes \mathcal{O}(-wE)) = 0$, which is by Kodaira vanishing theorem the same reason as before. \hfill \square

Cor. (IX.11.8.5) (Kodaira Embedding Theorem). If a compact complex manifold $M$ has a positive line bundle, then it is projective.

Cor. (IX.11.8.6) (Hodge Manifold). A compact Kähler manifold $X$ is a projective submanifold iff it has a closed positive $(1, 1)$-form $\omega$ whose cohomology class $[\omega]$ is rational/integral (i.e. in $H^2(X, \mathbb{Q})$). In fact, a Kähler manifold with a Hodge metric is called a Hodge manifold. So Hodge manifolds are just those Kähler manifolds that are projective.

Proof: if $\omega$ is rational, then a multiple of it is integral, then there is a $L$ that $c_1(L) = k[\omega]$ by Lefschetz theorem on $(1, 1)$-forms (IX.11.5.3), so $L$ is positive by (IX.11.6.2), so $X$ is projective. Conversely, the Chern class of the pullback of the hyperplane line bundle is positive rational (IX.11.6.4)/(IX.11.5.3). \hfill \square

Cor. (IX.11.8.7). if $\tilde{X}$ is the blowing-up of a Kähler manifold $X$ at a point $x$, then if $X$ is projective, then $\tilde{X}$ is also projective, because by (IX.11.5.5) $\pi^* L^k - E$ is positive for $k$ large.

Cor. (IX.11.8.8). For a finite unbranched cover of compact Kähler manifolds $\tilde{X} \rightarrow X$, $\tilde{X}$ is projective iff $X$ is projective.
Proof: A positive rational closed \((1,1)\)-form on \(X\) pull backs to a positive rational closed \((1,1)\)-form on \(\tilde{X}\), and it can even be pulled forward: \(\omega' = \sum_{y \in \pi^{-1}(x)} (\pi^{-1})^* \omega(y)\), then it is also positive closed. It is rational because \(\int_X \omega' \wedge \eta = \frac{1}{d} \int_{\tilde{X}} \omega \wedge \pi^* \eta\), where \(\tilde{X} \to X\) is branched of degree \(d\). □

Cor. (IX.11.8.9). If \(X\) is projective, then the map \(\text{Div}(X) \to \text{Pic}(X) : D \to \mathcal{L}(D)\) is surjective.

Proof: In fact, it suffice to show any line bundle \(E\) has a meromorphic section \(s\), thus \(L = \mathcal{L}(\text{div}(s))\). But \(X\) has a positive line bundle, so \(L^k + E\) and \(L^k\) are very ample thus clearly effective, with sections \(s_1\) and \(s_2\), so \(s_1/s_2\) is a section of \(E\). □

Cor. (IX.11.8.10) (Riemann Bilinear Form). For a complex variety \(V/\Lambda\), it is projective iff there is a **Riemann form** on \(V\), that is, an alternating bilinear form \(\omega : V \times V \to \mathbb{R}\) that:

- \(\omega(iu, iv) = \omega(u, v)\).
- \(\omega(v, iv) > 0\) for \(v \neq 0\).
- \(\omega(u, v) \in \mathbb{Z}\) for \(u, v \in \Gamma\).

Proof: Use (IX.11.5.1). The conditions are just equivalent to \(\omega\) is an integral positive Kähler form. □

Def. (IX.11.8.11) (Kähler Cone). For a Kähler manifold \(X\), the **Kähler cone** \(K_X\) is defined to be the set of closed real positive \((1,1)\)-forms. Then \(K_X\) is an open convex cone in \(H^{1,1}(X) \cap H^2(X, \mathbb{R})\). Then (IX.11.8.6) says \(X\) is projective iff \(K_X \cap H^2(X, \mathbb{Z}) \neq 0\).

9 Fujiki manifolds
IX.12 Riemann Surfaces and Algebraic Curves

Main references are [黎曼曲面, 伍鸿熙], and [G-H78], [Hartshorne Chap4] and [Sta]Chap53.

Notice a criterion for a theorem to be put into this subsection is that it has condition $\dim X \leq 1$.

1 Basics

Def. (IX.12.1.1). A (pre)curve is a (pre)variety over a field $k$ of dimension 1.

Def. (IX.12.1.2) (Rational Curve). A curve over a field $k$ is called rational if it is birational to $\mathbb{P}^1_k$.

Prop. (IX.12.1.3). A Noetherian separated scheme of dimension 1 has an ample invertible sheaf.

Proof: First reduce to the case when $X$ is reduced. This is because (IX.12.1.26) shows this invertible sheaf is a pullback of a sheaf of $X$ and (V.3.3.17) shows this sheaf is ample.

Second we reduce to the case $X$ is integral. Cf. [Sta]09NX. □

Cor. (IX.12.1.4) (Complete Curve Projective). A separated algebraic scheme $X$ of dimension 1 over a field $k$ is $H$-(quasi)projective, by (IX.12.1.3) and (V.3.3.21). If $X$ is proper, then it is projective.

Prop. (IX.12.1.5) (Completion of Curves). For a separated algebraic scheme $X$ of dimension $\leq 1$ over a field $k$, there is an open immersion $j : X \to \overline{X}$ that

- $\overline{X}$ is $H$-projective over $k$.
- $j(X) \subset \overline{X}$ is dense and schematically dense open subscheme.
- $\overline{X}\setminus X$ consists of f.m. closed points $\{x_i\}$ of $\overline{X}$, called the completion curve of $X$. And when $X$ is reduced, the stalk are discrete valuation rings at $x_i$ are DVRs.

Proof: By (IX.12.1.3), we can assume $X$ is a locally closed subscheme of $\mathbb{P}^n_k$. Let $\overline{X}$ be the scheme theoretic image (V.4.4.46) of the inclusion, then 1,2 holds by (V.4.4.53). 3 holds because $\overline{X}\setminus X$ is Noetherian of dimension 0.

For the last assertion, Cf. [Sta]0BXW. □

Cor. (IX.12.1.6). A morphism of prevarieties $X \to Y$ with $X$ a precurve(thus reduced) and $Y$ proper over a field $k$ factors through the completion $\overline{X}$ of $X$ by (V.4.5.8).

Prop. (IX.12.1.7) (Affine or Projective). A precurve over a field $k$ is either affine(not proper) or $H$-projective(proper).

Proof: Cf. [Sta]0A27. □

Cor. (IX.12.1.8). Let $X$ be a separated scheme algebraic over a field $k$. If $\dim X \leq 1$ and no irreducible component of $X$ is proper of dimension 1, then $X$ is affine.

Proof: Let $X_i$ be f.m. irreducible components of $X$, then they are precurves in the induced reduced structure, so they are affine by (IX.12.1.7). Now $\bigsqcup X_i \to X$ is a finite surjective morphism, so $X$ is affine by (V.4.4.34). □

Cor. (IX.12.1.9). Two birationally equivalent complete curve are isomorphic. Thus if a curve is birationally equivalent to another complete curve, then it is an open immersion, by (IX.12.1.5).
Prop. (IX.12.10) (Non-constant Morphism Finite). Let \( f : X \to Y \) be a morphism of schemes over a field \( k \) that \( Y \) is separated and \( X \) is proper of dimension \( \leq 1 \). If the image of every irreducible component of \( X \) is not a pt, then \( f \) is finite.

Proof: Cf.[Sta]0CCL.

Lemma (IX.12.11). For an Noetherian integral scheme of dimension 1, there is an isomorphism \( \mathcal{K}/\mathcal{O}_X \to \sum_p i_p(\mathcal{K}/\mathcal{O}_p) \).

Proof: Check on stalks, this is because closed subsets are finite.

Non-Singular Curves

Prop. (IX.12.12) (Extension of Rational Maps). Rational map from a normal precurve to a proper prevariety is the same as a morphism. Cf.[Sta]0BXZ.

Cor. (IX.12.13). Two birational equivalent normal proper precurves over a field is isomorphic.

Prop. (IX.12.14) (Category of Non-Singular Projective Curves). Let \( k \) be a field, the following categories are equivalent:
1. The opposite category of f.g. field extension of \( k \) of trans.deg 1.
2. The category of precurves and dominant rational maps.
3. The category of Normal proper precurves over \( k \) with non-constant morphisms.
4. The category of non-singular projective precurves over \( k \) with non-constant morphisms.

Proof: 1 and 2 are equivalent by(V.8.3.2).
3 and 4 are equivalent because normal and regular are the same(V.4.2.7).
For the rest, Cf.[Sta]0BY1.

Cor. (IX.12.15) (Non-Singular Projective Model). Comparing this and(V.8.3.2), we see that every curve over \( k \) correspond to a unique non-singular proper curve over \( k \) with the same function field, which is called the non-singular projective model.

Prop. (IX.12.16) (Flatness and Associated Points). \( f : X \to Y \) with \( Y \) integral and regular of dimension 1. Then \( f \) is flat iff every associated prime of \( X \) is mapped to the generic point of \( Y \).

In particular when \( X \) is reduced, this is equivalent to every irreducible component of \( X \) dominants \( Y \), by(I.5.4.15).

Proof: If \( x \) is mapped to a closed pt of \( Y \), then \( \mathcal{O}_{y,Y} \) is a DVR, let \( t \) be a uniformizer, then \( t \) is not a zero-divisor, and \( f^\sharp(t) \in \mathfrak{m}_x \) is also not a zero-divisor. So \( x \) is not an associated point.

Conversely, to show \( f \) is flat, if \( y \) is the generic point, then \( \mathcal{O}_{y,Y} \) is a field, so it is flat. When \( y \) is a closed pt, \( \mathcal{O}_{y,Y} \) is a DVR, so by(I.2.3.8), we need to show that it is torsion free. If it is not, then \( f^\sharp(t) \) must be a zero-divisor for a uniformizer \( t \) of \( \mathcal{O}_{y,Y} \). But then it is contained in some associated prime \( p \) of \( \mathcal{O}_{x,X} \) (I.5.4.9). Now \( p \) is mapped to \( y \), which is a contradiction.

Cor. (IX.12.17). If \( f : X \to Y \) is a dominant morphism from a variety to a nonsingular curve over \( k \), then \( f \) is flat.
Cor. (IX.12.1.18). Let $Y$ be integral and regular of dimension 1 and $P$ a closed pt. $X$ is a closed subscheme in $\mathbb{P}_{Y-P}^n$ that is flat over $Y - P$, then there is a unique closed subscheme $\overline{X}$ closed in $\mathbb{P}_Y^n$ that is flat over $Y$ and restrict to $X$ on $\mathbb{P}_{Y-P}^n$.

Proof: Choose the scheme-theoretic closure of $X$ in $\mathbb{P}_Y^n$. Cf.[Hartshorne P258].

Cor. (IX.12.1.19). Combining this with (IX.12.1.10), we say that a morphism between two non-singular curves are finite flat.

Prop. (IX.12.1.20). A projective non-singular curve of degree $d$ in $\mathbb{P}_k^n$ where $d \leq n$ not contained in any $\mathbb{P}_{k-1}^n$ is isomorphic to the $n$-tuple embedding, and $d = n$.

This has easy generalization to surfaces and higher dimensions.

Proof: (V.7.1.20) shows $\mathcal{O}_X(1) \cong \mathcal{O}(d)$ over $\mathbb{P}_k^1$, and the restriction of global sections is injective. So the global section is an isomorphism, and it defines the embedding up to a linear automorphism.

Cor. (IX.12.1.21) (Genera Equal). For a complete curve over a field $k$,

$$p_a(X) = p_g(X) = \dim_k H^1(X, \mathcal{O}_X)$$

by Serre duality (V.8.1.19) and (V.6.7.22).

So whenever talking about a complete curve, I will just call it genus without discriminant.

Prop. (IX.12.1.22). Any complete regular curve is rational iff it has genus 0.

Proof: Cf.[Hartshorne P297].

Divisors on Curves

Def. (IX.12.1.23). If $X$ is a locally Noetherian integral scheme of dimension 1, then the Weil divisors of $X$ are just locally finite formal sums of closed pts of $X$.

If $X$ is Noetherian integral algebraic over a field $k$ of dimension 1, then the sum is in fact finite, we can define the degree of a divisor $D = \sum n_P P$ as $\deg(D) = \sum n_P[k(P); k]$.

The canonical divisor $K$ of $X$ is the Weil divisor associated to the canonical sheaf $\mathcal{K}_X$, up to equivalence.

If $X$ is a regular, then $\deg(D) = \deg(\mathcal{L}(D))$, by (IX.12.1.32).

Prop. (IX.12.1.24). For a finite morphism $f$ between two non-singular curves over alg.closed field, e.g. dominant morphism between complete non-singular curves, $\deg f^* D = \deg f \cdot \deg D$.

This is because $f$ is finite locally free (IX.12.1.19), thus this follows from [[Sta]02RH].

Prop. (IX.12.1.25). An element $\notin k$ in the function fields of a projective non-singular curve over an alg.closed $k$ defines an inclusion $k(f) \subset K(X)$ thus a morphism from $X$ to $\mathbb{P}_k^1$ (V.8.3.2), and $(f) = \varphi^*(\{0\} - \{\infty\})$.

Prop. (IX.12.1.26). If $Z \to X$ is a closed immersion and $\dim X \leq 1$, then $\text{Pic } X \to \text{Pic } Z$ is a surjection.

Proof: Use the exact sequence $0 \to (1 + \mathcal{I})\mathcal{O}_X^* \to \mathcal{O}_X^* \to i_\ast \mathcal{O}_Z^* \to 0$, $\dim X \leq 1$ and the Grothendieck vanishing theorem gives the desired result, also notice $i$ is affine.

Prop. (IX.12.1.27). For a 1-dimensional integral scheme proper over $k$ and a function $f \in K(X)^*$,

$$\sum_{x \text{ closed}} [k(x) : k] \text{ord}_{\mathcal{O}_x}(f) = 0.$$
Riemann-Roch

Def. (IX.12.1.28) (Degree). The degree of a locally free sheaf $\mathcal{E}$ of rank $n$ on a proper scheme $X$ of dimension $\leq 1$ over a field $k$ is defined to be $\deg(\mathcal{E}) = \chi(X, \mathcal{E}) - n\chi(X, \mathcal{O}_X)$, where $\chi$ is the Euler characteristic (V.6.7.18).

If $X$ is integral (hence complete precurve), then this definition can extend to any coherent sheaves $\mathcal{F}$, if we define $\text{rank}(\mathcal{F}) = \text{dim}_{k^n} \mathcal{F}_n$.

Prop. (IX.12.1.29). The degree function is stable under base change of fields, additive, and stable under birational equivalence of proper scheme $X$ of dimension $\leq 1$ over a field $k$.

Proof: The base change follows from flat base change (V.6.7.34), the additivity follows from that of rank and Euler characteristic (V.6.7.18).

For the birational equivalence, Cf. [Sta]0AYU]. □

Prop. (IX.12.1.30) (Riemann-Roch). Let $D$ be a Weil divisor on a complete regular precurve $X$ of genus $g$, if $l(D) = H^0(X, \mathcal{L}(D))$, $l(D)$ is finite by (V.6.7.30).

\[ l(D) - l(\omega_X - D) = \deg D + 1 - g. \]

Proof: As $X$ is $H$-projective by (IX.12.1.7), so using Serre duality (V.7.4.9), it suffices to show $\chi(\mathcal{L}(D)) = \deg D + 1 - g$. If $D = 0$, then this follows by (V.8.1.12). Any divisor is a sum of closed pts of $X$, so we can use induction. By (V.7.1.17), if $P$ is a closed pt and $D$ is a Weil divisor, then

\[ 0 \to \mathcal{L}(D) \to (D + P) \to k(P) \to 0 \]

so by additivity, $\chi(\mathcal{L}(D + P)) = \chi(\mathcal{L}(D)) + [k(P) : k]$, so the induction is finished. □

Cor. (IX.12.1.31). If $D$ is an effective Cartier divisor on a proper scheme of dimension $\leq 1$ on a field $k$, then for any locally free sheaf $\mathcal{E}$ of rank $n$, $\deg(\mathcal{E}(D)) = n \deg(D) + \deg(\mathcal{E})$.

Proof: Cf. [Sta]0AYY]. □

Cor. (IX.12.1.32). $\deg(D) = \deg(\mathcal{L}(D))$, in fact, this is just equivalent to Riemann-Roch.

Cor. (IX.12.1.33). It is clear that if $l(D) > 0$, then $\deg D > 0$. So for any $D$ with $\deg X > 0$, for $n$ large, $l(nD) = n \deg D - g + 1$.

Cor. (IX.12.1.34). $\deg K = 2g - 2$.

Proof: Apply Riemann-Roch with $D = K$, then $l(K) = g$ and $l(0) = 1$, so $g - 1 = \deg K + 1 - g$, thus the result. □

Prop. (IX.12.1.35) (Ampleness and Degree). For an invertible $\mathcal{O}_X$-module $\mathcal{L}$ over a precurve proper over a field $k$, $\mathcal{L}$ is ample iff $\deg(\mathcal{L}) > 0$.

Proof: Cf. [Sta]0B5X]. □

Cor. (IX.12.1.36). Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module over a proper scheme of dimension $\leq 1$ over $k$, let $C_i$ be the irreducible components of $X$ of dimension 1, then $\mathcal{L}$ is a ample iff $\deg(\mathcal{L}|_{C_i}) > 0$ for all $i$.

Proof: Consider $(\bigsqcup_i C_i) \bigsqcup (\bigsqcup_j x_j)$ where $x_i$ are closed pts of $X$, then by (V.3.3.16), $\mathcal{L}$ is ample iff the pullback is ample. And use the last proposition and the fact on a pt it is obviously ample. □

Prop. (IX.12.1.37). A non-singular curve in $\mathbb{P}^2_k$ where $\text{char } k \neq 0$ is projectively isomorphic to $xy - z^2$ if it has a rational point. (Use Riemann-Roch to show that $O(p)$ has a nontrivial section which gives a isomorphism to $P^1$). And in fact the assertion can be checked directly.
Residues

Prop. (IX.12.1.38). Let $X$ be a complete regular curve over an alg.closed field $k$, $K$ be the function field, then for any closed pt $P$, there is a unique $k$-linear map $\text{res}_P : \Omega_{K/k} \to k$ with the following properties:

- $\text{res}_P(\tau) = 0$ for $\tau \in \Omega_P$, where $\Omega_P$ is the stalk of the canonical sheaf at $P$.
- $\text{res}_P(f^n df) = 0$ for $f \in K$ and $n \neq -1$.
- $\text{res}_P(f^{-1}df) = v_P(f)$, where $v_P$ is the valuation associated to $P$.

Prop. (IX.12.1.39) (Residue Theorem). For every $\tau \in \Omega_{K/k}$, we have $\sum \text{res}_P \tau = 0$.

Picard Scheme of Curves

Main references are [[Sta]Chap43].

2 Complex Algebraic Curves

Prop. (IX.12.2.1) (Genus).

Prop. (IX.12.2.2). A compact Riemann surface is Kähler, so by Hodge decomposition (IX.11.4.1),

$$H^1(M, \mathbb{C}) \cong H^0(M, \Omega^1) \oplus \overline{H^0(M, \Omega^1)},$$

so the number of holomorphic 1-forms on $M$ is equal to $b_1(M)/2 = g$, the topological genus.

Prop. (IX.12.2.3) (Riemann Existence Theorem). Any compact Riemann Surface is a Hodge manifold, thus projective algebraic, by (IX.11.8.6) and Chow’s lemma (IX.10.6.17).

Proof: Because $H^{1,1}(X) = H^2(X, \mathbb{Z})$, so it clearly contains integral classes. And it is positive because there is a basis generated by any Hermitian metric on $X$. So the theorem follows from (IX.11.8.6).

Cor. (IX.12.2.4). In fact the same argument shows that any Kähler manifold with $H^{0,2}(X) = 0$ is projective.

Prop. (IX.12.2.5).

- For any meromorphic function $f$ on a compact Riemann surface, $(f) = (f)_0 - (f)_\infty$ has degree 0.
- Let $\omega$ be a differential form on a compact Riemann surface, then the sum of residues of $\omega$ at its poles is zero.

Proof: 2: Choose a triangularization of the Riemann surface, then use the fact for any simple region $\Omega$ be boundary $C$,

$$\int_C \omega = 2\pi i \left( \sum_{\text{poles}} \text{Res}_p \omega \right).$$

And the integrals cancel out.

1: This is a direct consequence of 2 applied to the differential form $\omega = df/f$. □
Prop. (IX.12.2.6) (Riemann Hurewicz). If \( f : M \to N \) is a non-constant holomorphic map between two compact Riemann surfaces, then
\[
2g_M - 2 = 2 \deg f(2g_N - 2) + \sum_{p \in M} (v_p - 1).
\]

Proof: Cf.[黎曼曲面导引梅加强 P106].

Prop. (IX.12.2.7) (Curves in \( \mathbb{P}^1 \times \mathbb{P}^1 \)). Let \( C \) be a curve in \( \mathbb{P}^1 \times \mathbb{P}^1 \), then \( C \) is given by a bi-homogenous polynomial of type \((a,b)\). Then \( g(C) = (a - 1)(b - 1) \).

Proof: We have \( \mathcal{K}_C = (\mathcal{K}_{\mathbb{P}^1 \times \mathbb{P}^1} + C)|_C \) (V.8.1.18). We have \( \text{Pic}^1(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z}H_1 \oplus \mathbb{Z}H_2 \), thus
\[
\deg(\mathcal{K}_C) = \deg((\mathcal{K}_{\mathbb{P}^1 \times \mathbb{P}^1} + C)|_C)
= \deg((-2H_1 - 2H_2 + aH_1 + bH_2)|_C)
= \deg(((a - 2)H_1 + (b - 2)H_2)|_C)
= (a - 2)b + (b - 2)a = 2ab - 2a - 2b
= 2g - 2.
\]
Thus \( g = (a - 1)(b - 1) \).

Prop. (IX.12.2.8) (Genus Formula). Let \( C \) be a smooth curve of genus \( d \) in \( \mathbb{P}^2 \), then by adjunction formula, \( \mathcal{K}_C = \mathcal{K}_{\mathbb{P}^2}(C)|_C = \mathcal{O}_C(d - 3) \) which has degree \( d(d - 3) \). But this also equals \( 2g - 2 \) (IX.12.3.4), thus \( g = (d - 1)(d - 2)/2 \).

Prop. (IX.12.2.9) (Cubic Curves in \( \mathbb{P}^3 \)). A twisted cubic(rational) \( C \) in \( \mathbb{P}^3 \) is contained in a quadric.

Proof: Calculate that \( h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = C_5^2 = 10 \) and \( \deg(\mathcal{O}_C(2)) = 6 \), thus \( h^0(\mathcal{O}_C(2)) = 6 + 1 = 7 \) by Riemann-Roch. Then \( C \) is contained in 3 quadrics.

Abel’s Theorem and Reciprocity Law

Prop. (IX.12.2.10) (Reciprocity Law I). Cf.[Griffith/Harris P230].

Prop. (IX.12.2.11) (Weil). \( f, g \) are meromorphic functions on a compact Riemann surface that \((f), (g)\) are disjoint, then
\[
\prod f(p)^{v_p(g)} = \prod g(p)^{v_p(f)}.
\]

Proof: Cf.[Griffith/Harris, P242].

Prop. (IX.12.2.12) (Abel Theorem). Let \( C \) be a curve and \( p_0 \) a point, then the map
\[
\mu : C_d \to J : \sum p_i \mapsto \int_{p_0}^{p_i}
\]
induces an isomorphism \( \text{Pic}^d(C) \cong J \).

Prop. (IX.12.2.13) (Differentials on Plane Curves).
3 Linear Series

Prop. (IX.12.3.1) (Riemann-Roch). Let \( D \) be a divisor on a Riemann surface of degree \( d \), and \( L(D) \) be the line bundle corresponding to \( D \), \( l(D) = \dim(H^0(L(D))) \), \( r(D) = l(D) - 1 \), \( K_X \) is the canonical divisor, then there is an equality:

\[
l(D) = d - g + 1 + l(K - D).
\]

**Proof:** \[\square\]

Cor. (IX.12.3.2) (Riemann-Roch for High Degree \( D \)). If \( \deg(D) \geq 2g - 1 \), then \( \deg(K - D) < 0 \), so by (IX.12.2.5), \( l(K - D) = 0 \), thus \( l(D) = d - g + 1 \).

Prop. (IX.12.3.3) (Holomorphic 1-Forms). A Riemann curve of genus \( g \) has exactly \( g \) holomorphic 1-forms.

**Proof:** \[\square\]

Cor. (IX.12.3.4). \( \deg(K_X) = 2g - 2 \).

**Proof:** Plug in \( D = K \) into the Riemann-Roch formula. \[\square\]

Cor. (IX.12.3.5). For a divisor \( D \) of degree \( d > 0 \) on a curve \( C \) of genus \( g > 0 \), \( l(D) \leq d \).

**Proof:** If \( l(D) > d \), we can assume \( D \) is effective, \( l(D) = d - g + 1 + l(K - D) \). So if \( l(D) \geq d + 1 \), then \( l(K - D) \geq g \). But \( l(K - D) \leq l(K - p) \leq l(K) = g \) for some \( p \in D \), so \( l(p) = 2 - g + l(K - p) = 2 \), thus there is a rational map with a simple pole, which implies \( C \cong \mathbb{P}^1 \), by (IX.12.7.1), contradiction. \[\square\]

Def. (IX.12.3.6) (Special Divisors). A special divisor on a curve is a divisor \( D \) that \( h^0(K - D) > 0 \).

Def. (IX.12.3.7) (Moduli Spaces). Define the Picard variety \( \text{Pic}^d(X) \) to be the set of divisors of degree \( d \) on \( X \) modulo the equivalence relations.

Define the Hilbert scheme \( \mathcal{H}_{d,g,r} = \{ \text{Curves } C \text{ of degree } d \text{ and genus } g \} \).

Define the moduli space of curves \( \mathcal{M}_g = \{ \text{isomorphism classes of smooth projective curves of genus } g \} \).

Define \( W^r_d(C) = \{ L \in \text{Pic}^d(C), h^0(L) \geq r + 1 \} \).

Prop. (IX.12.3.8) (Linear Series and Maps). A linear series is a line bundle \( L \) together with a vector space \( V \subset H^0(L) \).

A \( g^r_d \) is a line bundle \( L \) together with a vector space \( V \subset H^0(L) \) of dimension \( r + 1 \).

Let \( L \) be a line bundle, then we let \( |L| \) denotes the linear series \( (L, H^0(L)) \).

For any base-free \( g^d(L, V) \), we get a map \( \varphi : C \to \mathbb{P}^r = PV^\vee \) by mapping \( p \) to the hypersurface of \( PV \) vanishing on \( p \). This map is injective iff for any \( p \neq q \), \( V_{p+q} \) has codimension 2. This map is an immersion iff for all \( p \), \( V_{2p} \) has codimension 2. So in particular, this map is an embedding iff for any effective divisor \( D \) of degree 2, \( V_D \) has codimension 2.

Cor. (IX.12.3.9). If \( \deg(L) \geq 2g + 1 \), then \( \varphi|_L \) is an embedding, where \( |L| = (L, H^0(L)) \).

**Proof:** This is because \( \dim H^0(L) = \deg(L) - g + 1 \) and \( \dim H^0(L_D) = \deg(L) - g - 1 \) by (IX.12.3.2), so \( \dim H^0(L_D) \) has codimension 2. \[\square\]
Lemma (IX.12.3.10). Let $L \in \text{Pic}^d$ be general, then $h^0(L) = \max\{1, d - g + 1\}$.

Proof: If $D = \sum p_i$ is a general effective divisor, notice $h^0(K) = g, h^0(K - p_1) = g - 1, h^0(K - p_1 - p_2) = g - 2$, and repeating this, we get $h^0(K - D) = \max\{0, g - d\}$, when we choose $p_i$ that are as independent as possible (any section has f.m. zeros). Then by Riemann-Roch, $l(D) = d - g + 1 + \max\{0, g - d\} = \max\{1, d - g + 1\}$. As every divisor of degree $d \geq g$ is effective, this settles the $d \geq g$ case.

If $D$ is non-effective and $d \leq g - 1$, we need to show $W^0_d$ is not dominant in $\text{Pic}^d$. But $J$ has dimension $g$ by $\text{??nd}$ $W^0_d$ has dimension at most $d$, so this is true. \qed

Prop. (IX.12.3.11). Suppose $D$ is a divisor of degree $g + 3$, then for $D$ general, $\varphi_D = \varphi_{|O(D)|}$ is an embedding.

Proof: By (IX.12.3.10), for a general $D$, $l(D) = 4$. Thus $\varphi_D$ being not an embedding is equivalent to the existence of a divisor $D_0 = p + q$ that $l(D - D_0) \geq 3$. This means $D - D_0 \in W^2_{g+1} = K - W^2_{g-3}$. So the divisor $D = D_0 + (D - D_0) \in W^2_0 + (K - W^2_{g-3})$ which has dimension at most $2 + (g - 3) = g - 1$. But a general divisor $D$ doesn’t lie on this $g - 1$-dimensional subvariety by (IX.12.2.12), so a general $D$ defines an embedding $\varphi_D$. \qed

Prop. (IX.12.3.12) (Base-Free Pencil Trick). Let $\mathcal{L}, \mathcal{M}$ be line bundles on $C$. Let $s_1, s_2$ be sections of $H^0(\mathcal{L})$ without common zero, then the kernel of the map

$$s_1H^0(\mathcal{M}) \oplus s_2H^0(\mathcal{M}) \rightarrow H^0(\mathcal{L} \otimes \mathcal{M})$$

is $H^0(\mathcal{M} \otimes \mathcal{L}^{-1})$.

Proof: Indeed, there is an exact sequence of sheaves:

$$\mathcal{M} \otimes \mathcal{L}^{-1} \rightarrow \mathcal{M} \oplus \mathcal{M} \xrightarrow{(s_1, s_2)} \mathcal{M} \otimes \mathcal{L} \rightarrow 0.$$

Where the first map maps a section $t$ to the pair $(ts_2, -ts_1)$. This is a Koszul regular sequence (on a common trivialization $U$ of $\mathcal{M}, \mathcal{L}$, $s_1, s_2 \in C(U)$, thus this is just $0 \rightarrow A \rightarrow A \oplus A \xrightarrow{s_1, s_2} A \rightarrow 0$). Taking the global section functor gives the desired result. \qed

Prop. (IX.12.3.13) (Geometric Riemann-Roch). Let $C$ be a non-hyperelliptic curve of genus $g \geq 2$. Then the canonical map $\varphi_K$ is an embedding, so we can assume $C \subset \mathbb{P}^{g-1}$. Then for a divisor $D = \sum p_i$ with $p_i$ distinct.

$$r(D) = d - 1 - \dim \mathcal{D},$$

where $\mathcal{D}$ is the linear subspace generated by $p_i$. Thus $r(D)$ can be interpreted as the number of linear relations between $p_i$.

Proof: By the definition of the canonical embedding, $l(K - D)$ is just the dimension of hypersurfaces containing $p_i$, thus it is equal to $g - 1 - \dim \mathcal{D}$. Now by Riemann-Roch, $r(D) = d - g + l(K - D) = d - g + g - 1 - \dim \mathcal{D} = d - 1 - \dim \mathcal{D}$. \qed

Prop. (IX.12.3.14) (Clifford Theorem). For a divisor $D$ of degree $0 \leq d \leq 2g - 2$ on a curve of genus $g$,

$$r(D) \leq \frac{d}{2},$$

with equality iff one of the following holds:
\[ r = d = 0, D = 0. \]
\[ d = 2g - 2, r = g - 1, D = K_C. \]
\[ C \text{ is hyperelliptic and } D = mg^1. \]

**Proof:** Cf. [Algebraic Curves, Harris, P32]. □

### 4 Hyperelliptic Curves

**Def.** (IX.12.4.1) (Canonical Map). the canonical map from a curve \( X \) of genus \( g \) to \( \mathbb{P}^{g-1} \) is the map associated to the canonical sheaf \( K_X \).

**Def.** (IX.12.4.2) (Hyperelliptic Curves). A Riemann surface \( C \) is called a hyperelliptic curve if there exists a degree 2 covering map \( C \to \mathbb{P}^1 \). Such a map is called a hyperelliptic map.

**Prop.** (IX.12.4.3) (Equation of Hyperelliptic Curves). Any image of a hyperelliptic map from a hyperelliptic curve of genus \( g \) is isomorphic to the compactification of a projective curve in \( \mathbb{P}^2 \) defined in an affine chart by \( y^2 = \prod_{i=0}^{2g+2}(x - \alpha_i) \) or \( y^2 = \prod_{i=0}^{2g+1}(x - \alpha_i) \), depending on whether the \( \infty \) is branched or not. And given \( 2g + 2 \) points in \( \mathbb{P}^1 \), there is exactly one degree 2 covering of \( \mathbb{P}^1 \) branched over these points.

**Proof:** By Riemann-Hurwitz, a degree 2 covering of \( \mathbb{P}^1 \) has \( 2g + 2 \) branch points, because the branch points are all simply branched. We can use a transformation of \( \mathbb{P}^1 \) to assume that all branch points are finite, denoted by \( \alpha_1, \ldots, \alpha_{2g+2} \).

Now consider the curve \( C' \) defined by \( y^2 = \prod_{i=0}^{2g+2}(x - \alpha_i) \), at a nbhd of \( \infty \), it consists of two disjoint punctured disks, so we can complete it to a compact Riemann surface by adding two points \( q, r \). Now consider \( \pi : C' \to \mathbb{P}^1 \) by projecting to the \( z \)-coordinates and mapping the two added points to \( \infty \). This is an holomorphism, and is a 2-cover of \( \mathbb{P}^1 \) branched over these \( 2g + 2 \) points.

Now any two Riemann curve 2-branched over the same set of points are isomorphic: This can be seen by the classical cut-paste argument: the two sheets can only be pasted in a unique way along the cut as it is a 2-sheeted covering. □

**Prop.** (IX.12.4.4). Hyperelliptic curve has the image under canonical map the rational curve.

**Proof:** With a possibly change of variables, we are given a projective curve defined by \( y^2 = \prod_{i=0}^{2g+2}(x - \alpha_i) \), we consider the divisor \((dx)\). \( dx \) vanishes at \( p_i \), and has pole at the infinity, thus \((dx) = \sum p_i - 2q - 2r\). Also \((1/y) = \sum p_i + (g + 1)(q + r)\), thus \((dx/y) = (g - 1)(q + r)\) is a holomorphic 1-form. Then also
\[
\frac{dx}{y}, \quad \frac{xdx}{y}, \ldots, \frac{x^{g-1}dx}{y}
\]
are \( g \) holomorphic 1-forms, and form a basis for \( H^0(K_C) \). In particular, the canonical map is 2 to 1 onto the canonical curve in \( \mathbb{P}^{g-1} \):
\[
C \xrightarrow{\pi} \mathbb{P}^1 \xrightarrow{\nu^{g-1}} \mathbb{P}^{g-1}.
\]

**Cor.** (IX.12.4.5). For a hyperelliptic curve, the smallest degree of an embedding \( C \to \mathbb{P}^r \) is \( g + r \). Note that hyperelliptic curves cannot be embedded into \( \mathbb{P}^2 \), i.e. smooth plane curves are non-hyperelliptic.
Thus the dimension of $b$ and points. Then we obtain a map $\phi_D$. Proof:

In particular, $\phi_D$ factors through $\pi$, thus not an embedding. \(\square\)

Prop. (IX.12.4.6) (Hyperelliptic and Canonical Map). A curve is hyperelliptic iff the canonical map is not an embedding.

Proof: The canonical map is not an embedding iff there exists an effective divisor $D$ of degree 2 that $l(K-D) > l(K)-2$. Now $l(K-D) = 2g - 4 - g + 1 + l(D)$, so this is equivalent to $l(D) > 1$.

Also a degree 2 covering is equivalent to rational function $f$ on $C$ that has $\leq 2$ zeros. Then this is also equivalent to the existence of effective divisor $D$ of degree 2 that $l(D) > 1$. \(\square\)

Prop. (IX.12.4.7) (Uniqueness of Hyperelliptic Divisors). For a hyperelliptic curve of genus $\geq 2$, up to automorphic of $\mathbb{P}^1$, there are at most one hyperelliptic map, or equivalently there is exactly one $g^1_2$ on $C$.

But for (hyperelliptic) curve of genus 1, thus is not unique: any divisor of degree 2 is effective, but they are not unique, because $\text{Pic}^2(C) \cong J(C)(\text{IX.12.12})$.

Proof: If $\mathcal{L} \neq \mathcal{M}$ are two line bundles on $C$ with $h^0 = 2$, then $h^0(\mathcal{L} \otimes \mathcal{M}) \geq 4$ by base-free pencil trick(IX.12.3.12), and in fact $h^0(\mathcal{L} \otimes \mathcal{M}) = 4$ by (IX.12.3.5). Now we can turn the same argument again to $\mathcal{L}$ and $\mathcal{L} \otimes \mathcal{M}$, to show that $h^0(\mathcal{L}^2 \otimes \mathcal{M}) = 6$. And inductively $h^0(\mathcal{L}^n \otimes \mathcal{M}) = 2n + 2$. But then for $n$ large, Riemann-Roch shows $2n + 2 = 2n + 2 - g + 1$, thus $g = 1$, contradiction. \(\square\)

Gonal Curves

Def. (IX.12.4.8) (Gonal Curves). A trigonal curve is a curve $C$ with a degree 3 map $C \to \mathbb{P}^1$. Similarly, a $k$-gonal curve is a curve $C$ with a degree $k$ map $C \to \mathbb{P}^1$. Being $k$-gonal is equivalent to having a $g^1_k$.

Def. (IX.12.4.9) (Hurewitz Spaces). Let the Hurewitz space be

$$\mathcal{H}_{d,g} = \{(C, f) | C \in \mathcal{M}_g, f : C \to \mathbb{P}^1 \text{ of degree } d \text{ with simple branching}\}.$$ 

In particular, $\mathcal{H}_{2,g}$ is just the space of hyperelliptic curves together with a hyperelliptic map.

Prop. (IX.12.4.10) (Dimension of Hurewitz Spaces). $\dim \mathcal{H}_{d,g} = 2d + 2g - 2$.

Proof: Let $b = 2d + 2g - 2$, then the branch divisor will consist of an unordered $b$-tuple of distinct points. Then we obtain a map $\mathcal{H}_{d,g} \to \mathbb{P}^b \setminus \Delta$, where we regard $\mathbb{P}^b$ as the set of polynomials of degree $b$ and $\Delta$ the determinant, and the fiber is finite by cut-paste technique. \(\square\)

Cor. (IX.12.4.11) (Dimension of Moduli Space of Curves). $\dim \mathcal{M}_g = 3g - 3$.

Proof: There is a map $\mathcal{H}_{d,g} \to \mathcal{M}_g$. When $d$ is large, we can analyze the fiber of this map, that is, given a curve of genus $g$, how many simply branched maps $C \to \mathbb{P}^1$ of degree $d$ are there? Such a map is equivalent to a line bundle of degree $d$ and a pair of base free sections $\sigma_0, \sigma_1 \in H^0(L)$. The base-free condition is an open condition, thus the dimension of the fiber is $g + 2(d - g - 1) - 1 = 2d + g - 1$. Thus the dimension of $\mathcal{M}_g = 2d + 2g - 2 - (2d + g + 1) = 3g - 3$. \(\square\)
Cor. (IX.12.4.12). The space of hyperelliptic curves has dimension $2g - 1$.

Proof: $\mathcal{H}_{2,g}$ has dimension $2g + 2$, and for any hyperelliptic curve, there is exactly one hyperelliptic map up to automorphism of $\mathbb{P}^1$, by (IX.12.4.7), thus the space of hyperelliptic curves has dimension $2g + 2 - 3 = 2g - 1$. □

Cor. (IX.12.4.13). If $g \geq 3$, then not all curves of genus $g$ is hyperelliptic.

Proof: Because when $g \geq 3$, $2g - 1 < 3g - 3$. □

Lemma (IX.12.4.14). For a line bundle of degree $3$ on a curve of genus $g \geq 3$, $h^0(L) = 2$, by Clifford’s theorem (IX.12.3.14). Thus a trigonal map must be associated to a complete linear series $g_3^1$.

Prop. (IX.12.4.15) (Hyperelliptic are not Trigonal). A curve of genus $g \geq 3$ cannot be both hyperelliptic and trigonal.

Proof: Suppose there are two line bundles $L, M$ on $C$ that $h^0(L) = h^0(M) = 2$, then the base-free pencil trick (IX.12.3.12) implies that $h^0(L \otimes M) \geq 4$. If $g = 3$, then this contradicts Riemann-Roch. If $g \geq 4$, then this contradicts Clifford’s theorem (IX.12.3.14). □

Prop. (IX.12.4.16) (Genus 3 Curves are Hyperelliptic or Trigonal). A genus 3 curve $C$ is either hyperelliptic or trigonal, but not both, by (IX.12.4.15).

Proof: If $C$ is non-hyperelliptic, then the canonical map realizes $C$ as a plane curve of degree 4, thus the projection of $C$ form a point on $C$ induces a degree 3 map $C \to \mathbb{P}^1$. □

Prop. (IX.12.4.17) (Uniqueness of Trigonal Divisors). There exists at most one $g_3^1$ on a curve of genus $g \geq 5$.

Proof: If $L \neq M$ are two line bundles of degree 3 that $h^0 = 2$, then we can use base-free trick to show that $h^0(L \otimes M) = 4$. But this contradicts Clifford’s theorem (IX.12.3.14). Notice the equality cannot hold, because the degree is too low to be the canonical bundle, and also $C$ cannot be both hyperelliptic and trigonal (IX.12.4.15). □

5 Castelnuovo’s Theory

Lemma (IX.12.5.1) (Castelnuovo’s Lemma). Let $\Gamma \in \mathbb{P}^n$ be a configuration of $d \geq 2n + 3$ points in linear general position, and if $h^0(2) = 2n + 1$, then $\Gamma$ lies in a rational normal curve.

6 Plucker Formulas

Def. (IX.12.6.1) (Weierstrass Group). Let $C$ be a compact Riemann surface of genus $g \geq 2$, $p \in C$, then $S_p = \{-\text{ord}_p(f) | f \text{ holomorphic on } C \setminus p\}$ is a semigroup, called the Weierstrass semigroup of $p \in C$. And the gap sequence is $\mathbb{N} \setminus S_p$, which is the set of orders of pole of $p$ that doesn’t occur.

Lemma (IX.12.6.2). We have $|G_p| = g$.

Proof: Notice that

$$G_p = \{m : h^0(mp) = h^0((m - 1)p)\}, \quad S_p = \{m : h^0(mp) = h^0((m - 1)p) + 1\}$$

and we know by (IX.12.3.2) that $h^0(mp) = m - g + 1$ for $m$ large, thus there are exactly $g$ jumps, which shows $|G_p| = g$. □
Def. (IX.12.6.3) (Weierstrass Points). A point \( p \in C \) is called \textbf{Weierstrass point} if the gap sequence is not \( \{1, 2, \ldots, g\} \). It is called a \textbf{hyperelliptic Weierstrass point} if \( G_p = \{1, 3, \ldots, 2g-1\} \), and is called a \textbf{normal Weierstrass point} if \( G_p = \{1, 2, 3, \ldots, g-1, g+1\} \).

Define the \textbf{weight} of the \( p \in C \) to be the sum \( w(p) = \sum_{i \leq g} (a_i - i) \), where the gap sequence is numbered \( \{a_1, \ldots, a_g\} \).

Prop. (IX.12.6.4). For any compact Riemann surface \( C \), \( \sum_{p \in C} w(p) = g(g-1)(g+1) \).

Proof: Cf. [G-H78], P274.

Cor. (IX.12.6.5). For general \( p \in C \), \( G_p = \{1, 2, \ldots, g\} \).

Prop. (IX.12.6.6) (Hyperelliptic Weierstrass Points). If \( C \) is hyperelliptic defined by \( y^2 = \prod_{i=0}^{2g+1} (x - \alpha_i) \), then there are \( 2g + 2 \) branching point of \( x \), where \( S_p = \{0, 2, 4, \ldots, 2g, 2g+1, \ldots\} \). In this way, \( w(p) = g(g-1)/2 \). Thus there are \( 2g + 2 \) Weierstrass points, and these are all of them.

If \( C \) is non-hyperelliptic, then by Clifford’s theorem (IX.12.3.14),

\[ h^0(kp) < \frac{k}{2} + 1, k = 1, \ldots, g \]

Thus \( h^0((a_i - 1)p) < \frac{a_i - 1}{2} + 1, \) and \( h^0((a_i - 1)p) = 1 + (a_i - 1) - (i - 1). \) Thus we get

\[ a_i \leq 2i - 2, i = 2, \ldots, g. \]

Then \( w(p) \leq \sum_{i=2}^{g} (i - 2) = \frac{(g-1)(g-2)}{2}. \) Thus there are at least

\[ \frac{2g(g-1)(g+1)}{(g-1)(g-2)} \geq 2g + 6 \]

Weierstrass points.

To sum up, there are no less than \( 2g + 2 \) Weierstrass points, and there are exactly \( 2g + 2 \) Weierstrass points iff \( C \) is hyperelliptic.

Prop. (IX.12.6.7) (Automorphism Group of Riemann Surface is Finite). If \( C \) is a compact Riemann surface of genus \( g \geq 2 \), then \( \text{Aut}(C) \) is finite.

Proof: Any automorphism of \( C \) fixes its set of Weierstrass sets, which is finite, so we only need to consider the case that it fixes all Weierstrass points.

If \( C \) is hyperelliptic, consider a hyperelliptic map, this covering has an involution of \( C \), and this is just \( C \mapsto C/\tau \). So modulo \( \tau \), it suffices to show \( \mathbb{P}^1 \) has f.m. automorphisms fixing the branch points. But this is true, as \( 2g + 2 > 3 \).

If \( C \) is non-hyperelliptic, then there are more than \( 2g + 2 \) Weierstrass points. But we can find a function \( f \) on \( C \) with \( g + 1 \) zeros and poles, by Riemann-Roch, then for an automorphism \( \varphi \) of \( C \), \( f - \varphi^*f \) has no more than \( 2g + 2 \) poles, so also has no more than \( 2g + 2 \) zeros. But \( \varphi \) fixes all Weierstrass points, so it has more than \( 2g + 2 \) zeros, contradiction.

Prop. (IX.12.6.8). A generic Riemann surface of genus \( g \geq 3 \) has no automorphisms.
7 Curves of Low Genus

Prop. (IX.12.7.1) (Genus 0 Curve). A Riemann surface $C$ of degree 0 is isomorphic to $\mathbb{P}^0$: By Riemann-Roch, $l(p) = 2$, thus there is a non-constant meromorphic function on $C$ with $(f)_0 = p$. Then $f$ induces a degree 1 map from $C$ to $\mathbb{P}^1$, which must be an isomorphism.

Prop. (IX.12.7.2) (Genus of 1 Curve). By (IX.12.3.9), an effective divisor of degree 3 induces an embedding of $\mathbb{C}$ into $\mathbb{P}^2$, so it is a smooth plane cubic by genus formula (IX.12.2.8). Conversely, any smooth plane cubic has genus 1.

Similarly, if we take an effective divisor of degree 4, then it gives an embedding $C \to \mathbb{P}^3$. We know that $h^0(\mathcal{O}(2)) = 10$ and $h^0(\mathcal{O}(2)) = 8$, thus $C$ is contained in 2 quadrics. Then $C$ is the intersection of these two quadrics, by Bezout’s theorem.

Prop. (IX.12.7.3) (Genus 2 Curves and Degree 4 Divisors). Let $C$ be a curve of genus 2, and $D$ a divisor of degree 4. Then $l(D - K_C) = 1 + l(2K_C - D) \geq 1$, thus $K_C - D$ is effective. Let $K_C - D = p + q$. Then $\varphi_D$ maps $p, q$ to the same point.

There are two situations, firstly if $D \neq 2K_C$, then $K_C - D$ can be written uniquely as $p + q$. Then the image of $\varphi_D$ is a degree 4 curve with a node (if $p = q$) or a cusp (if $p = q$). Counting genus, this has exactly arithmetic genus 2.

If $D = 2K_C$. Then notice $\varphi_{2K_C}$ is equal to $\varphi_K$ followed by the normal curve map $\mathbb{P}^1 \to \mathbb{P}^2$. This is because if $\omega_1, \omega_2$ are a basis of $H^0(K_C)$, then $\omega_1, \omega_2, \omega_2^2$ is a basis of $H^0(K_C^2)$. So $\varphi_D$ is a 2 to 1 map to a rational normal curve in $\mathbb{P}^2$.

Prop. (IX.12.7.4) (Genus 2 Curve and degree 5 Divisors). Let $C$ be a curve of genus 2 and $D$ a divisor of degree 5, then $\varphi_D : C \to \mathbb{P}^3$ embeds $C$ as a degree 5 curve, by (IX.12.3.9).

Notice $h^0(\mathcal{O}(3)) = 10$ and $h^0(\mathcal{O}(3)) = 10 - 2 + 1 = 9$, thus $C$ lies on at least one quadric $Q$. And it can in fact lie on only one quadric, because if it lies on two quadrics, then $C$ is the intersection, and can have degree at most 4.

Next, notice $h^0(\mathcal{O}(3)) = 20$ and $h^0(\mathcal{O}(3)) = 15 - 2 + 1 = 14$, so $C$ lies on at least 6 cubics. Without the 4 cubics containing the quadric, there are still at least 2 new cubics. Let $S$ be such a cubic, then $S \cap Q$ is a curve of degree 6, so $S \cap Q = C \cup L$, where $L \cong \mathbb{P}^1$.

Prop. (IX.12.7.5) (Genus 4 Non-Hyperelliptic Curves). Let $C$ be a non-hyperelliptic curve of genus 4 and $\varphi : C \to \mathbb{P}^3$ be the canonical embedding of $C$ as a curve of degree 6. Looking at the map $H^0(\mathcal{O}(3)) \to H^0(\mathcal{O}(3))$, we see $C$ lies on a unique quadric $Q$ (uniqueness follows from Bezout’s theorem, by the same reason as (IX.12.7.4)).

Next looking at the map $H^0(\mathcal{O}(3)) \to H^0(\mathcal{O}(3))$, then the kernel has dimension at least 5, thus there is a cubic containing $C$ but not $Q$. Then $S \cap Q$ is a curve of degree 6, thus equals $C$. Conversely, any smooth curve of the form $S \cap Q$ is a canonical curve of genus 4, by adjunction formula ($K_C = \mathcal{O}(1)$ has degree 6).

Now if $Q$ is non-singular, then $C$ is of type $(3, 3)$ in $Q$, thus the two projections are two trigonal maps from $C$ to $\mathbb{P}^1$, and if $Q$ is a cone, the projection from the vertex is a trigonal map from $C$ to $\mathbb{P}^1$.

Non-hyperelliptic Curve of Genus 5

For a non-hyperelliptic curve $C$ of genus 5, we have a canonical map $\varphi : C \to \mathbb{P}^4$. Firstly we consider what quadrics $C$ lies on: $h^0(\mathcal{O}(2)) = 15$ and $h^0(\mathcal{O}(2)) = 12$, so $C$ lies on at least 3 quadrics. There are two cases:
1. $C = \cap Q_i$ where $Q_i$ are the quadrics containing $C$.

2. $C$ is a strict subset of $\cap Q_i$.

**Prop. (IX.12.7.6) (Case 1).** The case 1 does occur, because by Bertini’s theorem, for general three quadrics $Q_i$, $\cap Q_i$ is a smooth curve, and thus $\mathcal{K}_C = (\mathcal{K}_\mathbb{P}^4(-5 + 2 + 2 + 2))|_C = \mathcal{O}_C(1)$. So $d = 8$, $g(C) = 5$.

In this case, $C$ is not trigonal, because if $C$ is trigonal, then $C$ has a $g^1_3$, which means $C$ has three colinear point. Then all the quadrics $Q_i$ must contain this line, contradiction.

**Prop. (IX.12.7.7) (non-Hyperelliptic Curve of Genus 5 is Tetragonal).** Let $C$ be a canonical embedded curve which is not trigonal admits a map of degree 4 to $\mathbb{P}^1$.

Proof: Let $\mathbb{P}^2 = \{Q | C \subset Q\}$. We can ask there are singular quadrics in this set. Inside $\mathbb{P}^14$ which is the space of all quadrics in $\mathbb{P}^4$, there is a quintic hypersurface of singular quadrics. For the rest, Cf.[Algebraic Curves, Harris, P25].

**Correspondences**

**Complex Tori and Algebraic Varieties**

**Curves and Their Jacobians**

8 Castelnuovo Theory

9 Brill-Noether Theory
IX.13 Calabi-Yau Manifolds
Chapter X

Analysis

X.1 Real Analysis

Basic references are [Folland Real Analysis], [H-J99].

1 Measures

Def. (X.1.1.1) (σ-Algebra). Let \( A \) be a set, then an algebra of subsets of \( A \) is a subset of \( P(A) \) that is closed under finite intersections and finite unions. A \( \sigma \)-algebra is an algebra of subsets of \( A \) that is closed under countable unions.

Def. (X.1.1.2) (Measure). Let \( X \) be a set endowed with a \( \sigma \)-algebra \( \mathcal{M} \), then a measure on \( (X, \mathcal{M}) \) is a function \( \mathcal{M} \rightarrow [0, \infty] \) that

- \( \mu(\emptyset) = 0 \).

- If \( E_i \) is a countable family of disjoint sets in \( \mathcal{M} \), then \( \mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i) \).

A probabilistic measure is a measure \( \mu \) on \( X \) that \( \mu(X) = 1 \).

Def. (X.1.1.3) (Measurable Map). A measurable map \( f : (X, \mathcal{M}) \rightarrow (Y, \mathcal{N}) \) is a map \( f : X \rightarrow Y \) that \( f^{-1}(E) \in \mathcal{M} \) for any \( E \in \mathcal{N} \).

Def. (X.1.1.4) (Non-Singular Maps). A non-singular measurable map is a measurable map of measure spaces that the preimage of every set of measure 0 has measure 0.

Def. (X.1.1.5) (Lebesgue Space). A one-point subset with positive measure is called an atom. A Lebesgue space is a finite measure space that is isomorphic to a finite union of intervals and countably many atoms.

Borel Sets

Def. (X.1.1.6) (Radon Measure). A Borel measure on a measurable space is a measure defined on the \( \sigma \)-algebra generated by open sets.

A Borel measure \( \mu \) is called inner regular on \( E \) iff \( \mu(E) = \inf\{\mu(K)|K \subset E \text{ compact}\} \) for every Borel set \( E \). It is called outer regular iff \( \mu(E) = \sup\{\mu(U)|E \subset U \text{ open}\} \).

A Radon measure is a Borel measure that is finite on compact set, outer regular on Borel sets, inner regular on open sets.
Prop. (X.1.1.7) (Radon-Nikodym). If two \(\sigma\)-finite measures \(v, \mu\) on a measurable space satisfies \(v\) is absolutely continuous w.r.t \(\mu\), then there is a \(\mu\)-integrable function \(f\) such that

\[ dv = fd\mu.\]

Remark (X.1.1.8). This is a special case of the Freudenthal spectral theorem (X.5.4.18).

Lemma (X.1.1.9) (Positive Linear Functional is Continuous). For a LCH space \(X\), a positive linear functional \(I\) on \(C_c(X)\) is automatically continuous, where \(C_c(X)\) is given compact convergence topology as in (X.4.2.1).

Proof: We need to prove that for any compact subset \(K\) of \(X\), there is a constant \(C_K\) that for any \(f \in C(G)\) with support in \(K\), we have \(|I(f)| \leq C_K \|f\|_\infty\).

Given any \(K\), choose by Urysohn lemma a \(\varphi \in C_c(X, [0, 1])\) that \(\varphi = 1\) on \(K\), so if \(\text{Supp} f \subset K\), then \(|f| \leq \|f\|_\infty \varphi\), thus the positivity of \(I\) shows that \(|I(f)| \leq I(\varphi) \|f\|_\infty\).

\(\square\)

Prop. (X.1.1.10) (Riesz-Markov-Kakutani Representation Theorem). For a LCH space \(X\), on \(C_c(X)\),

- If \(I\) is a positive linear functional, there is a unique regular (both inner and outer) Radon measure \(\mu\) on \(X\) such that \(I(f) = \int f d\mu\). Moreover,

\[ \mu(U) = \sup \{I(f) : f < U\} \text{ for } U \text{ open}, \]

\[ \mu(K) = \inf \{I(f) : f > \chi(K)\} \text{ for } K \text{ compact}. \]

- If \(I\) is a continuous linear functional, there is a unique regular countably additive complex Borel measure \(\mu\) on \(X\) that \(I(f) = \int f d\mu\).

In particular if \(X\) is compact, \(M(X)\) the space of Borel measures on \(X\) is the dual space of \(C(X)\).

Proof: Cf.[Real Analysis Folland P212].

Measurable Functions

Prop. (X.1.1.11) (convergences). There are three different kinds of convergences:

- almost everywhere convergence iff \(f_n(x) \to f(x)\) a.e.

- almost uniform convergence iff for any \(\delta > 0\), there is a measurable subset \(E_\delta\) that \(f_n\) convergent to \(f\) uniformly on \(E - E_\delta\).

- convergence in measure iff \(\lim_{k \to \infty} m(\{x \in E \mid |f_n(x) - f(x)| > \varepsilon\}) = 0\).

Prop. (X.1.1.12) (Relations between Convergences).

- (Egoroff) If \(m(E) < \infty\) and \(f_k\) converges to \(f\) a.e. then \(f_k\) converges to \(f\) almost uniformly.

- If \(f_k\) converges to \(f\) almost uniformly, then \(f_k\) converges to \(f\) in measure.

- (Riesz) If \(f_k\) converges to \(f\) in measure, then there is a subsequence \(f_{n_k}\) that converges to \(f\) a.e..

Proof: 1: Cf.[实变函数周明强 P113].

2: Trivial.

3: Cf.[实变函数周明强 P118].
2 Differentiation

**Lemma (X.1.2.1) (Vitali Covering Theorem).** Let $C$ be a collection of balls in $\mathbb{R}^n$, and let $U = \cup_{B \in C} B$. Then if $c > m(U)$, then there exists disjoint $B_1, \ldots, B_k \in C$ that $\sum_{i=1}^{k} m(B_k) > 3^{-n}c$.

**Lemma (X.1.2.2).** If $f \in L^1_{loc}$ and $A_r f(x) = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)dy$, then $A_r f$ is continuous in both $r$ and $x$.

**Prop. (X.1.2.3).** If $f \in L^1_{loc}$, then $\lim_{r \to 0} A_r f(x) = f(x)$ for a.e. $x \in \mathbb{R}^n$.

**Def. (X.1.3.1) (Simple Functions).**

**Def. (X.1.3.2).** A measurable function $f : \mathbb{R}^n \to \mathbb{C}$ is called **locally integrable** if $\int_K |f(x)|dx < \infty$ for every bounded measurable set $K$ of $\mathbb{R}$. The set of locally integrable function is denoted by $L^1_{loc}(\mathbb{R}^n)$.

**Prop. (X.1.3.3).** A function $f$ is real analytic on an open set of $\mathbb{R}$ iff there is an extension to a complex analytic function to an open set of $\mathbb{C}$. And this is equivalent to: For every compact subset, there is a constant $C$ that for every positive integer $k$, $|\frac{d^k f}{dx^k}(x)| \leq C^{k+1}k!$.

**Prop. (X.1.3.4) (Monotone-convergence-theorem).**

**Prop. (X.1.3.5) (Dominant Convergence Theorem).**

**Prop. (X.1.3.6).** The set $E$ of nowhere differentiable functions are of second category in $C[0,1]$, and its complement set is of first category.

**Prop. (X.1.3.7) (Fubini-Tonelli).** For two $\sigma$-finite measure spaces $X, Y$, if $f \in L^+(X \times Y)$, then $f_x \in L^+(Y)$ and $f^y \in L^+(X)$, and $\int_{X \times Y} f(x,y)dx\,dy = \int_Y \int_X f(x,y)dx\,dy = \int_X \int_Y f(x,y)dy\,dx$.

**Prop. (X.1.3.8).** If $f \in L^1(X \times Y)$, then $f_x \in L^1(Y)$ and $f^y \in L^1(X)$, a.e. and the product formula is definable and holds.

**Proof:** Cf. [Folland P67].
4 Power Series

5 \(L^p\)-space

Lemma (X.1.5.1) (Höder’s Lemma). If \(\sum x_i = 1, x_i \geq 0\), then for any \(a_i \geq 0\)
\[
\prod a_i^{x_i} \leq \sum a_i x_i.
\]

Proof: \(\square\)

Prop. (X.1.5.2) (Höder’s Inequality). Let \((S, \Omega, \mu)\) be a measure space, and \(1 \leq p, q \leq \infty\) satisfies \(1/p + 1/q = 1\), then
\[
\|fg\|_1 \leq \|f\|_p \|g\|_q
\]
More generally, if \(\sum_{i=1}^n 1/p_i = 1\), then
\[
\|\prod f_i\|_1 \leq \prod \|f_i\|_{p_i}
\]
Proof: The both sides are homogenous for \(f_i\), so we may assume \(\|f_i\|_{p_i} = 1\), then use Höder’s Lemma (X.1.5.1) for \(x_i = q/p_i\). \(\square\)

Prop. (X.1.5.3). For a \(\sigma\)-finite measure \(\mu\) on a measurable space \(X\), for \(1 \leq p < \infty\)
\[
L^p(X, \Omega, \mu)^* = L^q(X, \Omega, \mu).
\]

Proof: Firstly, Hölder inequality (X.1.5.2) shows that \(a \in L^q(X, \Omega, \mu)\) is truly defines a functional by \(f \mapsto \int f g d\mu\). Conversely, if given a functional \(F\), define a measure \(v(E) = F(\chi_E)\) for all measurable set \(E \in \Omega\). It is countably additive: first it is finitely additive, and if \(E_n\) is a descending sequence of measurable sets that \(\cap E_n = \emptyset\), then
\[
v(E_n) \leq \|F\|\|\chi_{E_n}\|_{L^q} = \|F\|\|\mu(E_n)\|^\frac{1}{q} \to 0.
\]
(where we used the fact \(p < \infty\)). And it is clearly absolutely continuous w.r.t. \(\mu\).

So by Radon-Nikodym (X.1.1.7), there is a measurable function \(g\) that \(v(E) = \int_E g d\mu\). Now for all simple function \(f\), \(F(f) = \int f g d\mu\). Next we want to prove \(\|g\|_{L^q} \leq \|F\|\), because in this way, because any measurable function \(f\) can be approximated by simple functions \(f_i\) in \(L^p\) norm (X.1.6.4), so
\[
|\int (f(x) - f_i(x))g(x) dx|d\mu| \leq \|f - f_i\|_p \|g\|_q \leq \|f - f_i\|_p \|g\|_q
\]
So \(F(f) = \lim F(f_i) = \lim \int f_i g d\mu = \int f g d\mu\).

To prove this, if \(1 < p\), then let \(E_t = \{x||g(x)| \leq t\}\), and \(f = \chi_{E_t}|g|^{q-2}g\), then
\[
\int_{E_t} |g|^q d\mu = \int f g d\mu = F(f) \leq \|F\|\|f\|_{L^p} = \|F\|\left(\int_{E_t} |g|^q d\mu\right)^\frac{1}{q},
\]
which is equivalent to \(\|g\chi_{E_t}\|_{L^q} \leq \|F\|\). Let \(t \to \infty\), then the monotone convergence theorem (X.1.3.4) gives us the result.

If \(p = 1\), then \(q = \infty\). For any \(\varepsilon > 0\), let \(A = \{x||g(x)| > \|F\| + \varepsilon\}\), \(E_t = \{x||g(x)| \leq t\}\), and let \(f = \chi_{E_t \cap A} \operatorname{sign}(g)\), then \(\|f\|_{L^1} = \mu(E_t \cap A)\), and
\[
\mu(E_t \cap A)(\|F\| + \varepsilon) \leq \int_{A \cap E_t} |g| d\mu = \int f g d\mu \leq \|F\|\mu(E_t \cap A)
\]
If \(\mu(A) \neq 0\), then let \(t \to \infty\), this is a contradiction. So \(\|g\|_{\infty} \leq \|F\|\). \(\square\)
Prop. (X.1.5.4) (Hilbert Basis for Products). For two \( \sigma \)-finite measure spaces \( M, N \) and Hilbert basis (X.3.4.9) \( \{ e_i \} \) of \( L^2(M) \) and \( \{ f_j \} \) of \( L^2(N) \), \( \{ e_i \otimes f_j \} \) gives a Hilbert basis for \( L^2(M \times N) \).

(Use Fubini).

Proof: It is easily verified that \( e_i \otimes f_j \) are mutually orthogonal, and if some \( f \in L^2(M \times N) \) satisfies \( (f, e_i \otimes f_j) = 0 \) for all \( i, j \), then

\[
\int_M e_i(x) \int_N f(x, y) f_j(y) = 0
\]

for all \( i \). But \( \int_N f(x, y) f_j(y) \in L^2(M) \) for a.e. \( x(f(x, y) \in L^2(N) \) a.e. \( x \) by Fubini-Tonelli), thus it vanishes. So \( f(x)(y) = f(x, y) = 0 \in L^2(N) \) for a.e. \( x \), so by Fubini-Tonelli again, \( ||f||_{L^2(M \times N)} = 0 \), thus \( f = 0 \).

\( \square \)

6 Approximations

Prop. (X.1.6.1) (Stone-Weierstrass Approximation). If a unital \( C^* \)-algebra \( A \) of continuous functions on a compact Hausdorff space separates points, then it is dense in \( C(X) \).

Proof: This is a consequence of Bishop theorem (X.4.3.18) because in this case the real functions in \( A \) separate points, so all \( A \)-antisymmetric sets consists of one point.

\( \square \)

Cor. (X.1.6.2). The polynomial functions are dense in \( C[-1, 1] \).

Prop. (X.1.6.3) (Simple Function Approximation).

• If \( f(x) \) is a non-negative measurable function on \( E \), then there is an ascending sequence of simple functions \( \varphi_n(x) \) that converges to \( f \) point-wise.

• If \( f(x) \) is a measurable function on \( E \), then there is a sequence of simple functions \( \varphi_n \) that \( |\varphi_n(x)| \leq |f(x)| \), and converges to \( f \) pointwise.

• If \( f(x) \) is bounded, then the convergence can be chosen to be uniform.

Proof: Cf. [实变函数周明强 P110].

\( \square \)

\( L^p \)-Approximation

Prop. (X.1.6.4) (Simple Function Approximation). Any function in \( L^p \) can be approximated by compactly supported simple functions in \( L^p \) norm.

Proof: 

Prop. (X.1.6.5). for \( 1 \leq p < +\infty \), \( C_c(X) \) are dense in \( L^p(X) \) for a Radon measure, but not for \( p = \infty \).

Proof: Use compactly supported simple function approximation (X.1.6.4) and then use outer regular approximation (X.1.1.6) and then Tietz extension.

\( \square \)

Prop. (X.1.6.6) (Lusin). If \( f \) is almost everywhere finite on \( E \), then for any \( \delta > 0 \), there is a closed subset \( F \subset E \) that \( f \) is continuous function on \( F \).
Proof: First if \( f \) is a simple function \( f = \sum_{i=1}^{n} c_i \chi_{E_i} \), then for each \( E_i \), choose a closed subset \( F_i \subset E_i \) that \( m(E_i - F_i) < \frac{\varepsilon}{n} \), and then \( \cup F_i \) satisfies the required condition.

Now if \( f \) is arbitrary, let \( g(x) = \frac{f(x)}{1 + |f(x)|} \) to make it bounded, then by (X.1.6.3), there is a sequence of simple functions \( \varphi_k \) converging to \( f \), and for each \( k \), we choose a closed subset \( F_k \) that \( m(E - F_k) < \frac{\varepsilon}{2^k} \), so if we let \( F = \cap F_k \), then \( \varphi_k \) are all continuous on \( F \), so by the uniform convergence, \( f \) is also continuous on \( F \). \( \square \)

Cor. (X.1.6.7). If \( f \) is measurable function on \( E \) that is a.e. finite, then for any \( \delta \), there is a continuous function \( g \) that \( m(\{ x \in E | f(x) \neq g(x) \}) < \delta \). And if \( E \) is bounded, \( g \) can be chosen to be compactly supported.

Proof: Now that there is a closed subset \( F \) that \( m(E - F) < \delta \) and \( f \) is continuous on \( F \), we can use Tietze extension (IX.1.6.3), there is a function \( g \) that equals \( f \) on \( F \).

If \( E \subset B(0, R) \), the we can choose a bump function to multiply with \( g \). \( \square \)

Prop. (X.1.6.8) (Approximate Identity). A family of \( L^\infty(\mathbb{T}) \) functions \( \{ \Phi_N \} \) are called an approximate identity if:
1. \( f_0^1 \Phi_N(x)dx = 1 \).
2. \( \sup_{f_0^1} |\Phi_N(x)|dx < \infty \).
3. For any \( \delta > 0, \int_{|x|>\delta} |\Phi_N(x)|dx \to 0 \) as \( N \to +\infty \).

For any approximate identity, if \( f \in C(\mathbb{T}) \) or \( L^p(\mathbb{T}) \) for \( 1 \leq p < +\infty \), then \( \Phi_N * f \to f \).

Proof: Use uniform continuity and also use continuous approximation (X.1.6.5). \( \square \)

Cor. (X.1.6.9). for \( 1 \leq p < +\infty \), trigonometric polynomials are dense in \( L^p(\mathbb{T}) \) and \( C(\mathbb{T}) \), but not for \( p = \infty \). So \( e^{2\pi inx} \) forms an orthogonal basis in \( L^2(\mathbb{T}) \).

Thus, the Parseval’s identity holds.

Proof: Just use the fact that Fejer kernels are an approximate identity. \( \square \)

Prop. (X.1.6.10). For a integrable function \( u \) that has compact support, \( u_\delta = j_\delta \ast u \) is a smooth function of compact support that \( ||u_\delta - u||_{C^k} \to 0 \) when \( u \in C^k \). Where \( j_\delta \) is the scaling of a smooth function of compact support. So Smooth function of compact support are dense in \( C^k_0 \).

Prop. (X.1.6.11). \( D(\mathbb{R}^n) \) is dense in \( W^{m,p}(\mathbb{R}^n) \).

Proof: Use the fact that \( C_0 \) are dense in \( L^p \) by (X.1.6.10). And \( f_\delta \to f \) in \( L^p \) norm for \( f \in C_0 \). So we can use the three-part argument applied to \( D_\alpha u \) to get \( D_\alpha (u_\delta) \to D_\alpha u \) in \( L^p \) norm for \( |\alpha| \leq m \). Thus the result. \( \square \)

7 Convolution

Prop. (X.1.7.1). Convolution with a smooth function makes the function smooth, in particular, \( \frac{\partial}{\partial x} (f \ast g) = \frac{\partial f}{\partial x} \ast g \).

Prop. (X.1.7.2) (Young’s Inequality). \( ||f \ast g||_r \leq ||f||_p ||g||_q \) for all \( 1 \leq r, p, q \leq \infty \) and

\[
1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.
\]

In particular, \( ||K \ast f||_p \leq ||K||_1 ||f||_p \).
Proof: By Riesz representation (X.1.1.10), it suffices to show that: for \(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2\),

\[
\int \int f(x)g(y-x)h(y) \leq ||f||_p||g||_q||h||_r.
\]

write the LHS as

\[
\int \int (f^p(x)g(y-x)^q)^{1/2} (f^p(x)h^r(y))^{1/2} (g^q(y-x)h^r(y))^{1/2}
\]

and use Holder inequality for three functions (X.1.5.2).

\[\square\]

8 Examples of Calculation

Prop. (X.1.8.1). Assume \(\text{Re}(s) > 1/2\), then

\[
\int_{-\infty}^{\infty} (1 + x^2)^{-s} e^{ik \arctan \frac{1}{x}} \, dx = (-i)^k \sqrt{\pi} \frac{\Gamma(s) \Gamma(s - \frac{1}{2})}{\Gamma(s + \frac{k}{2}) \Gamma(s - \frac{k}{2})}.
\]

Proof: Cf. [Bump, Automorphic Forms and Representations, P230]. \[\square\]
### X.2 Complex Analysis

Basic References are [Ahl79], [复变函数简明教程 谭小江 武胜健], [Complex Analysis Stein] and [复分析导引 李忠].

#### 1 Topology

**Def. (X.2.1.1).** A region is defined to be a nonempty connected open set of \( \mathbb{C} \).

#### 2 Basics

**Def. (X.2.2.1).** We introduce the following notation:

\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad dz = dx + idy \quad d\bar{z} = dx - idy.
\]

Then \( dz \) is dual to \( \frac{\partial}{\partial z} \) and \( d\bar{z} \) is dual to \( \frac{\partial}{\partial \bar{z}} \). And for any function \( f \),

\[
df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}.
\]

**Def. (X.2.2.2) (Cross Ratio).** For any three pts \( z_1, z_2, z_3 \in \mathbb{C} \), there is a unique linear transformation that maps them to \( 1, 0, \infty \). In fact, the linear transformation is just

\[
S_z = \frac{z-z_3}{z-z_1}/\frac{z_2-z_3}{z_2-z_1}.
\]

Then for any for point \( z_1, z_2, z_3, z_4 \), the cross ratio \((z_1, z_2, z_3, z_4)\) is the image of \( z_1 \) under the linear transformation that carries \( z_2, z_3, z_4 \) to \( 1, 0, \infty \).

**Prop. (X.2.2.3).** The cross ratio is invariant under linear transformation, and it is real iff \( z_1, z_2, z_3, z_4 \) are colinear or cocycle.

**Proof:** The first is because there is only one linear transformation that maps \( z_2, z_3, z_4 \) to \( 1, 0, \infty \).

For the second, notice by (X.2.2.2), \( \arg(z_1, z_2, z_3, z_4) = \arg(z_2-z_3) - \arg(z_2-z_1) \), and this is real iff \( \angle z_4z_2z_3 = \angle z_4z_1z_3 \) or \( \pi - \angle z_4z_1z_3 \), which is equivalent to cocycle. For other degenerate cases, we need some other argument. \( \square \)

**Cor. (X.2.2.4).** A linear transformation maps colinear/cocycle points to colinear/cocycle points.

**Lemma (X.2.2.5) (Invariant Factor).** If \( a, b, c, d \in \mathbb{R} \), then

\[
\text{Im}(\frac{az+b}{cz+d}) = \frac{ad-bc}{|cz+d|^2} \text{Im}(z).
\]

#### Analytic Function

**Def. (X.2.2.6) (Analytic Function).** A function on \( \mathbb{C} \) are called analytic or holomorphic if \( \frac{\partial f}{\partial \bar{z}} = 0 \). Equivalently, \( \frac{\partial f}{\partial \bar{z}} = -i \frac{\partial f}{\partial y} \), i.e. \( f \) has the same derivative vertically and horizontally, hence in every direction.

**Prop. (X.2.2.7) (Regularity Problems).** For the equation \((\frac{\partial}{\partial x} + i \frac{\partial}{\partial y})f = 0 \) or \((\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})u = 0 \), the function is elliptic, so by (X.8.8.4), the solution of the equation is automatically smooth, so no smoothness condition need to be added.
Lemma (X.2.2.8). An analytic function in a region $\Omega$ whose derivative vanishes must be a constant function.

Proof:

Prop. (X.2.2.9) (Uniqueness). If the zeros of a holomorphic function $f$ has a convergent point in the domain of definition, then $f = 0$.

Proof:

Conformal Mapping

Prop. (X.2.2.10). A 1st-differentiable conformal map in $\mathbb{C}$ is holomorphic or anti-holomorphic. In higher dimension, conformal is equivalent to $\langle df_p(v_1), df_p(v_2) \rangle = \lambda^2(p) \langle v_1, v_2 \rangle$.

Proof: Cf.[Ahlfors P74].

3 Complex Integration

Lemma (X.2.3.1) (Technical Lemma). If $f$ is analytic on a rectangle $R$ minus f.m. points $\zeta_i$ and if $\lim_{z \to \zeta_i} (z - \zeta_i) f(z) = 0$, then $\int_{\partial R} f(z) dz = 0$.

Proof: Cf.[Ahlfors P109-112]

Prop. (X.2.3.2) (Cauchy Theorem). if $\Omega$ is a simply connected region in $\mathbb{C}$, $f$ is holomorphic in $\Omega$ minus f.m. points $\zeta_i$, $\lim_{z \to \zeta_i} (z - \zeta_i) f(z) = 0$, and continuous on $\overline{\Omega}$, then $\int_{\partial \gamma} f(z) dz = 0$ for any curve $\gamma$ in $\Omega$.

Proof: Fix a $z_0 \in \Omega$, then for any $z \in \Omega$, choose a horizontal and vertical path $\gamma$ from $z_0$ to $z$, then let $F(z) = \int_{\gamma} f(z) dz$, then the above lemma(X.2.3.1) shows that $F(z)$ is independent of the path chosen, and it is clear that $F(z)$ has derivatives in both direction so holomorphic by definition(X.2.2.6). So clearly $\int_{\partial \Omega} f(z) dz = 0$, by an uniformly continuity argument.

Prop. (X.2.3.3) (Index of a Point w.r.t a Curve). If $\gamma$ is a piecewise $C^1$ curve that doesn’t pass a point $a$, then $\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a}$ is an integer $n(\gamma, a)$, called the index of a w.r.t $\gamma$.

And this index function is constant on each connected component of $\mathbb{C} - \gamma$, and 0 on the unbounded component.

Proof: Cf.[Ahlfors P115].

Cor. (X.2.3.4) (Cauchy Integral Formula). if $\Omega$ is a simply connected region in $\mathbb{C}$, $f$ is holomorphic in $\Omega$ and continuous on $\overline{\Omega}$, then for $a \notin \gamma$,

$$n(\gamma, a) f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-a} dw.$$ 

Proof: Consider the function $F(z) = \frac{f(z)-f(a)}{z-a}$, then it is analytic for $z \neq a$, and at $a$ it satisfies the condition of Cauchy theorem(X.2.3.2), so $\int_{\gamma} F(z) dz = 0$ which is $\int_{\gamma} \frac{f(z)dz}{z-a} = f(a) \int_{\gamma} \frac{dz}{z-a}$, and use(X.2.3.3).
Prop. (X.2.3.5) (Generating Analytic Functions). If \( \varphi(\zeta) \) is continuous on an arc \( \gamma \), then the function
\[
F_n(\zeta) = \int_\gamma \frac{\varphi(\zeta)d\zeta}{(\zeta - z)^n}
\]
is analytic in each of the regions defined by \( \gamma \), and its derivative satisfies \( F'_n(z) = nF_{n+1}(z) \).

Proof: Cf.[Ahlfors P121].

Cor. (X.2.3.6) (Higher Derivations). As any analytic function \( f \) can be written as
\[
f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)dw}{w-a}
\]
by Cauchy integral theorem (X.2.3.4), its derivatives are all analytic, and satisfies:
\[
f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}}
\]

Cor. (X.2.3.7) (Morera’s Theorem). If \( f \) is continuous on a region \( \Gamma \) and if \( \int_\gamma f(z)dz = 0 \) for any closed curve \( \gamma \) in \( \Omega \), then \( f(z) \) is analytic in \( \Omega \).

Proof: There is an analytic function \( F \) that \( F' = f \), by the same method of the proof of (X.2.3.2), so \( f \) is analytic, by (X.2.3.6).

Cor. (X.2.3.8) (Cauchy Estimate). If \( f \) is holomorphic on a disk \( B(a,r) \), and \( |f| \leq M \) on the boundary, then \( |f^{(n)}(a)| \leq Mn!r^{-n} \).

Cor. (X.2.3.9) (Liouville). Any bounded holomorphic function on \( \mathbb{C} \) is constant.

Proof: if \( |f(z)| \leq M \), then the Cauchy estimate shows that \( |f'(a)| \leq Mr^{-1} \), letting \( r \) tends to \( \infty \), then \( f'(a) = 0 \) for all \( a \), thus \( f \) is constant.

Cor. (X.2.3.10) (Mean Value Property). If \( f \) is holomorphic on the unit disk \( D \), then \( |f(0)| \leq \int_D |f(z)|dxdy \).

Proof: \( |f(0)| \leq \frac{1}{2\pi} \int f(re^{i\theta})d\theta \), so if multiplied by \( rdr \) and integrate, then
\[
|f(0)| \leq \int \int f(re^{i\theta})rdrd\theta = \int \int f(z)dxdy.
\]

Local Properties of Analytic Functions

Prop. (X.2.3.11). An analytic function is an open map from \( \mathbb{C} \) to \( \mathbb{C} \).

Prop. (X.2.3.12) (Removable Singularities). Let \( f \) be analytic on a disk removing the origin \( a \), then \( f \) can be extended to a analytic function on the whole disk iff \( \lim_{z \to a}(z-a)f(z) = 0 \).

Proof: Cf.[Ahlfors, P]

The Residue

Prop. (X.2.3.13) (Rouche’s Theorem).
Theorems

Prop. (X.2.4.1) (Uniformization Theorem). Any connected Riemann Surface is the quotient by a discrete subgroup of $\mathbb{C}, \mathcal{H}$ or $\mathbb{P}^1$.

Proof: □

Prop. (X.2.4.2) (Runge’s Theorem). Let $K$ be a compact subset of $\mathbb{C}$ and let $f$ be a function which is holomorphic on an open set containing $K$. If $A$ is a set containing at least one complex number from every bounded connected component of $\mathbb{C}\setminus K$, then there exists a sequence of rational functions which converges uniformly to $f$ on $K$ and all the poles of the functions are in $A$.

Proof: □

Prop. (X.2.4.3) (Mergelyan’s theorem). If $K$ is compact in $\mathbb{C}$ and $f$ is a continuous function on $K$ that is holomorphic in $\text{int}(K)$, then $f$ can be uniformly approximated by polynomials.

Prop. (X.2.4.4) (Weierstrass Theorem). For a ascending sequence of regions $\Omega_1 \subset \Omega_2 \subset \ldots$, $\bigcup_n \Omega_n = \Omega$, and $f_n$ is analytic on $\Omega_n$, and $f_n(z)$ converges to a function $f(z)$ in the compact-open topology, then $f(z)$ is also analytic, and moreover, $f'(z)$ converges to $f'(z)$ in the compact-open topology.

Proof: The analyticity follows from Morera’s theorem (X.2.3.7) as the integration on a closed curve commutes with uniform convergence, the same argument applied to the limit of equations

$$f_n'(z) = \frac{1}{2\pi i} \int_{\partial B(0,r)} \frac{f_n(\zeta)d\zeta}{(\zeta - z)^2}$$

shows that the derivative also converges, and uniformly on $\overline{B(0,\rho)}$ for $\rho < r$. □

Cor. (X.2.4.5) (Hurwitz). Cf.[Ahlfors P178].

Prop. (X.2.4.6) (Montel’s Theorem). Sets of holomorphic functions bounded in the topology of $H(\Omega)$, inter convex uniform convergence, is sequentially compact.

Proof: □

Prop. (X.2.4.7) (Little Picard Theorem). The image of a non-constant entire function $f : \mathbb{C} \to \mathbb{C}$ is either $\mathbb{C}$ or $\mathbb{C}$ with one point omitted.

Proof: □

Prop. (X.2.4.8) (Great Picard Theorem). If an analytic function $f$ has an essential singularity at a point $w$, then on any punctured nbhd of $w$, $f$ takes any value infinitely often, except for at most one single exception.

Prop. (X.2.4.9) (Phragmén-Lindelöf Principle). Let $f$ be a function that is holomorphic in the upper part of a strip $\sigma_1 \leq \text{Re}(s) \leq \sigma_2, \text{Im}(s) > c$, such that $f(\sigma + it) = O(e^{\alpha t})$ for some $\alpha > 0$ uniformly for any $\sigma_1 < \sigma < \sigma_2$. Suppose further that $f(\sigma + it) = O(t^b)$ for some $b$ and $\sigma = \sigma_1$ or $\sigma_2$, then $f(\sigma + it) = O(t^b)$ uniformly for $\sigma_1 \leq \sigma \leq \sigma_2$. 
Proof: First assume that \( b = 0 \), thus there exists \( M \) that \( |\varphi(s)| \leq M \) for \( \text{Re}(s) = \sigma_i \). Now let \( m \equiv 2 \pmod{4} \) and \( m > \alpha \), then \( \text{Re}((\sigma + iT)^m) \) is a polynomial of \( \sigma \) and \( \tau \) with highest term of \( \tau \) being \( -\tau^m \), so we have
\[
\text{Re}(s^m) = -\tau^m + O(|\tau|^{m-1}), |\tau| \to \infty
\]
uniformly on the strip. So \( \text{Re}(s^m) \) has an upper bound \( N \) on the strip. Thus for any \( \varepsilon > 0 \),
\[
|\varphi(s)e^{s^m}| \leq Me^{\varepsilon N}
\]
on the boundary of the strip and
\[
|\varphi(s)e^{s^m}| = O(e^{|\alpha| - \varepsilon \tau^m})
\]
uniformly on the strip, thus converges uniformly to 0 as \( |\text{Im}(s)| \to \infty \).

Then we can use maximum principle to see that
\[
|\varphi(s)e^{s^m}| \leq Me^{\varepsilon N}
\]
on the strip. Let \( \varepsilon \to 0 \), we get \( |\varepsilon(s)| \leq M \), thus the theorem.

In general, if \( b \neq 0 \), define \( \psi(s) = (s - \sigma_1 + 1)^b \), then \( |\psi(s)| = |s - \sigma_1 + 1|^b \sim |\tau|^b \) when \( |\tau| \to \infty \). Thus \( \varphi(s) = \varphi(s)/\psi(s) \) satisfies the same condition as \( \varphi \) with \( b = 0 \), so \( \varphi(s)/\psi(s) \) is bounded on the strip, and thus \( |\varphi(s)| = O(|\tau|^b) \).

\textbf{Remark (X.2.4.10).} Some condition on the growth rate of \( |f(\sigma + iT)| \) is necessary, otherwise we can consider \( e^{\varepsilon \tau} \) on the strip \( -\pi/2 \leq \text{Re}(z) \leq \pi/2 \), then it is bounded for \( \text{Re}(z) = \pm \pi/2 \), but not bounded for \( -\pi/2 < \text{Re}(z) < \pi/2 \).

\section{Series and Product Developments}

\textbf{Series}

\textbf{Prop. (X.2.5.1) (Abel).} Any power series \( a_0 + a_1 z + \ldots + a_n z^n + \ldots \) has a \textit{circle of convergence} \( R \) that:

- The series converges absolutely for every \( |z| < R \), if \( \rho < R \), then the convergence is uniform for \( |z| \leq \rho \).
- If \( |z| > R \), the terms are unbounded, and the series diverges.

Moreover, in \( |z| < R \), the sum of this series is an analytic function.

\textit{Proof:} Cf. [Complex Analysis Ahlfors P38].

\textbf{Prop. (X.2.5.2) (Hadamard's Formula).} In the last proposition (X.2.5.1), \( 1/R = \lim \sup \left| \frac{\psi}{a_n} \right| \)

\textit{Proof:} Cf. [Complex Analysis Ahlfors P38].

\textbf{Prop. (X.2.5.3) (Eulerian Identity).} The smallest positive real number \( \rho \) that \( e^{i\rho} + 1 = 0 \) is \( \pi \). In particular, Eulerian identity
\[
e^{i\pi} + 1 = 0
\]
holds.

\textit{Proof:}

\textbf{Prop. (X.2.5.4).} Any holomorphic function \( f \) defined on the punctured disk \( 0 < |z| < 1 \) is of the form
\[
f(z) = \sum a_n z^n
\]
where \( \limsup_{n \to 0} |a_n|^{1/n} \leq 1 \), and \( \lim_{n \to \infty} |a_{-n}|^{1/n} = 0 \).
Partial Fractions and Factorizations

Entire Functions

Def. (X.2.5.5) (Bernoulli Numbers). The Bernoulli numbers $B_k, k \geq 0$ are defined to be

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}.$$ 

Then $B_k$ are all rational numbers, and

$$B_0 = 1, B_1 = -\frac{1}{2}, B_{2k+1} = 0 (k \geq 1), B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}.$$ 

More numbers can be found at https://oeis.org/wiki/Bernoulli_numbers.

Prop. (X.2.5.6).

$$\sin(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2 \pi^2}\right).$$

Cor. (X.2.5.7).

$$z \cot(z) = 1 + 2 \sum_{n=1}^{\infty} \frac{z^2}{z^2 - n^2 \pi^2} = 1 - 2 \sum_{k=1}^{\infty} \zeta(2k) \frac{z^{2k}}{\pi^{2k}}.$$ 

Proof: Taking the logarithmic derivative of the equation in (X.2.5.6).

6 Harmonic Function

Def. (X.2.6.1) (Harmonic Function). A real-valued function on a region $\Omega$ is called harmonic if it is $C^1$ and has second order derivatives and $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ holds, as said before in (X.2.2.7), this is a elliptic function, so $u$ is automatically smooth.

Prop. (X.2.6.2). Analytic-function-norm-is-analyis-harmno if $f(z)$ is an analytic function and on a nbhd $\Omega f(z) \neq 0$, then $\log |f(z)|$ is a harmonic function on $\Omega$.

Proof:

Prop. (X.2.6.3) (Poisson Formula). For $u$ harmonic for $|z| < \rho$ and continuous for $|z| \leq \rho$,

$$u(z) = \frac{1}{2\pi} \int \frac{\rho^2 - |z|^2}{|\rho e^{i\theta} - z|^2} u(\rho e^{i\theta}) d\theta = \operatorname{Re} \left[ \frac{1}{2\pi i} \int_{|\zeta| = \rho} \frac{\zeta - z}{\zeta + z} u(\zeta) \frac{d\zeta}{\zeta} \right],$$

for $|z| < \rho$.

In particular, the bracketed part is a analytic function for $|z| < R$, so for any analytic function on $|z| \leq \rho$ by (X.2.3.5), for $|z| < R$:

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta| = \rho} \frac{\zeta - z}{\zeta + z} \operatorname{Re}(f(\zeta)) \frac{d\zeta}{\zeta} + iC.$$ 

Proof: Cf.[Ahlfors P168].
Prop. (X.2.6.4) (Schwarz’s Theorem). Cf.[Ahlfors P169].

Def. (X.2.6.5) (Mean-Value Property). A real valued function $u$ on a region $\Omega$ is said to have the mean-value property iff
\[
u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta\]
whenever $B(z_0, r) \subset \Omega$.

Lemma (X.2.6.6) (Harmonic Mean Value). If $u$ is a Harmonic function between two concentric circles, then the arithmetic mean of it over circles $|z| = r$ is a linear function of $\log r$:
\[
\frac{1}{2\pi} \int_{|z|=r} u \, d\theta = \alpha \log r + \beta.
\]
In particular, if $u$ is harmonic in the disk, then by continuity, $\alpha = 0$, and the mean value is a constant.

Proof: Cf.[Ahlfors P165].

Prop. (X.2.6.7) (Harmonicity and Mean-Value Property). A harmonic function satisfies the mean-value property, and conversely, and continuous function satisfying the mean-value property is harmonic.

Proof: Harmonic function satisfies the mean-value property by (X.2.6.6). Conversely, for any $z_0$, by Schwarz’s theorem (X.2.6.4), there is a harmonic function $v(z)$ that is harmonic in $B(z_0, \rho)$ and equals $u(z)$ on $\partial B(z_0, \rho)$. Now the maximal and minimal principles apply to $u - v$, thus $u = v$ is harmonic.

Cor. (X.2.6.8) (Maximum Principle). If $u$ is a harmonic function, then it attains neither maximum nor minimum at its region of definition.

Prop. (X.2.6.9) (Hadamard’s Three Circle Theorem). Let $f(z)$ be analytic in the annulus $r_1 < |z| < r_2$, and continuous on the boundary, if $M(r)$ denotes the maximum of $|f(z)|$ for $|z| = r$, then:
\[
M(r) \leq M(r_1)^\alpha M(r_2)^{1-\alpha}
\]
where $\alpha = \log(r_2/r) : \log(r_2/r_1)$.

Proof: Apply the maximum principle (X.2.6.8) for
\[
g(z) = \log |f(z)| - \log(M(r_1))(\log(r_2/|z|) : \log(r_2/r_1)) - \log(M(r_2))(1 - \log(r_2/|z|) : \log(r_2/r_1)),
\]
it is harmonic by ??, then $g(z) \leq 0$ on $|z| = r_1$ and $|z| = r_2$, so $g(z) \leq 0$ on all the annulus.

Cor. (X.2.6.10). Let $f(z)$ be analytic in the annulus $r_1 < |z| < r_2$, then the function
\[
s \mapsto \max_{z=r} |f(z)|
\]
is convex on the interval $[\log(a), \log(b)]$.

Prop. (X.2.6.11) (Reflection Principle).
Prop. (X.2.6.12) (Harnack’s Inequality). For a positive harmonic function \( u \) on \( B(0, \rho) \),

\[
\frac{\rho - |z|}{\rho + |z|} u(0) \leq u(z) \leq \frac{\rho + |z|}{\rho - |z|} u(0).
\]

Proof: By Poisson formula,

\[
u(z) = \frac{1}{2\pi} \int_{\rho^2 - |z|^2}^{\rho^2} \frac{\rho^2 - |z|^2}{|\rho e^{i\theta} - z|^2} u(\rho e^{i\theta}) d\theta\]

for \( |z| < \rho \), so the conclusion follows from the obvious inequality

\[
\frac{\rho - |z|}{\rho + |z|} \leq \frac{\rho^2 - |z|^2}{|\rho e^{i\theta} - z|^2} \leq \frac{\rho + |z|}{\rho - |z|}
\]

and Mean-value property (X.2.6.7).

Cor. (X.2.6.13). If \( E \) is a compact subset of a region \( \Omega \), there is a constant \( M \), depending on \( E, \Omega \) that for any positive harmonic function \( u(z) \) on \( \Omega \), \( u(z_2) \leq Mu(z_1) \) for any \( z_1, z_2 \in E \).

Proof: This is an easy consequence of Harnack inequality and the compactness of \( E \).

Cor. (X.2.6.14) (Harnack’s Principle). Consider a sequence of functions \( u_n(z) \), each harmonic in a region \( \Omega_n \), and there is a region \( \Omega \) that every point has a nbhd that is contained in all but f.m. \( \Omega_n \), and in this nbhd \( u_n(z) \leq u_{n+1}(z) \) for \( n \) large, then either \( u_n(z) \) tends to \( +\infty \) in the compact open topology, or they tends to a harmonic function in compact open topology.

Proof: The uniform continuity follows easily from Harnack’s inequality, and for the harmonicity of the limit function \( u(z) \) is a consequence of the Poisson formula.

Dirichlet Problem

Miscellaneous

Def. (X.2.6.15) \( (L^2(\mathcal{D}, \nu_k) \) Unit Disc). On the unit disk \( \mathcal{D} \), define a density \( \nu_k = \frac{4(1-|w|^2)^{k-2}dudv}{|1-w|^2} \) \( (k \geq 2) \) and define \( L^2(\mathcal{D}, \nu_k) \) to be the space of holomorphic functions on \( \mathcal{D} \) that is \( L^2(\nu_k) \) bounded. Then the space \( L^2(\mathcal{D}, \nu_k) \) is complete.

Proof: By (X.2.3.10), the uniform norm is bounded by the local \( L^1 \)-norm hence also the local \( L^2 \)-norm. Hence for a compact subset \( K \), the uniform norm is also bounded by \( L^2(\nu_k) \)-norm. Thus any Cauchy sequence in \( L^2(\mathcal{D}, \nu_k) \) converges to a holomorphic function by Weierstrass theorem (X.2.4.4).

Prop. (X.2.6.16). The space \( L^2(\mathcal{D}, \nu_k) \) has an orthogonal basis consisting of holomorphic functions

\[
\{ \psi_n = w^n(1-w)^k \}_{n \geq 0}.
\]

Proof: Firstly, \( \psi_n \) is convergent: In the polar coordinate, \( \nu_k = \frac{4(1-r^2)^{k-2}drd\theta}{|1-w|^2} \), so

\[
||\psi_n||^2 = 4 \int_0^{2\pi} \int_0^1 r^{2n}(1-r^2)^{k-2}drd\theta < \infty.
\]
And if $m \neq n$,
\[
\int_D \psi_m(w)\overline{\psi}_n(w)dw = 4 \int_D w^m(\overline{w})^n(1-r^2)^{k-2} drd\theta = 4 \int_0^1 r^{m+n}(1-r^2)^{k-2} \int_0^{2\pi} e^{i(m-n)\theta} d\theta = 0.
\]

**Def. (X.2.6.17) (Upper Half Plane $\mathcal{H}_k$).** On the upper half plane $\mathcal{H}$, we can define a density $\mu_k = y^k dx dy/k^2 (k \geq 2)$, and define $L^2(\mathcal{H}_k, \mu_k)$ to be the space of holomorphic functions on $\mathcal{H}$ that is $L^2(\mu_k)$ bounded. Then under the Cayley map $z \mapsto w = \frac{z-i}{z+i}$, this density is mapped to the density $\nu_k$ of $D$ and induces an isomorphism of spaces
\[
L^2(\mathcal{H}_k, \mu_k) \cong L^2(D, \nu_k).
\]
In particular, $L^2(\mathcal{H}_k, \mu_k)$ is complete, and it has an orthogonal basis
\[
\{\varphi_n = \left(\frac{z-i}{z+i}\right)^n \frac{(2i)^k}{(z+i)^k}\}.
\]
by (X.2.6.15) and (X.2.6.16).

7 **Elliptic Functions**

Main references are [Ahl79]Chap7.

**Prop. (X.2.7.1) ($\mathbb{C}/\Lambda$).** Let $\Lambda$ be a complete real lattice of $\mathbb{C}$, then we can make $\mathbb{C}/\Lambda$ into a Riemannian surface as the quotient space of $\mathbb{C}$. 

**Def. (X.2.7.2) (Doubly Periodic Function).** Let $\Lambda$ be a complex lattice of $\mathbb{C}$, then a meromorphic function $f$ on $\mathbb{C}$ is called **doubly periodic w.r.t. $\Lambda$** if it is invariant under $\Lambda$.

**Prop. (X.2.7.3) (Homomorphisms of Complex Tori).** Let $\Lambda, \Lambda'$ be two lattices of $\mathbb{C}$, then any holomorphic map $\varphi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ that maps 0 to 0 is of the form
\[
[z] \mapsto [\alpha z],
\]
where $\alpha \in \mathbb{C}$ that $\alpha \Lambda \subset \Lambda'$.

**Proof:** First notice $\varphi$ can be lifted to a function $\tilde{\varphi} : \mathbb{C} \to \mathbb{C}$ that maps 0 to 0. Now $\tilde{\varphi}'$ is a holomorphic doubly periodic function, so it is bounded and by Liouville’s theorem a constant function. Then $\tilde{\varphi}$ is a linear function $z \mapsto \alpha z$, and because it lifts $\varphi$, it must maps $\Lambda$ into $\Lambda'$. 

**Prop. (X.2.7.4).** Let $f$ be a periodic function for $\Lambda$, not identically zero, and let $D$ be a parallelogram for $\Lambda$ that $f$ has no zeros or poles on the boundary of $D$. Then
- $\sum_{P \in D} \text{ord}_P(f) = 0$.
- $\sum_{P \in D} \text{Res}_P(f) = 0$.
- $\sum_{P \in D} \text{ord}_P(f) \cdot P = 0$.

**Proof:** $f$ can be realized as meromorphic functions on the Riemann surface $\mathbb{C}/\Lambda$, so 1, 2 are direct consequences of (IX.12.2.5) 1, 2. 3 is 2 applied to the meromorphic function $zf'(z)/f(z)$. 


Def. (X.2.7.5) (Order). The order of an elliptic function $f$ is defined to be the number of poles of $f$ in $D$. Equivalently, it is the number of zeros of $f$ in $D$(X.2.7.4).

Prop. (X.2.7.6). The order of a non-constant elliptic function $\geq 2$.

Proof: If its order is 0, then it is bounded on $D$ and also on $\mathbb{C}$ thus constant by Liouville’s theorem. If its order is 1, then it has a simple pole $z_0$, but then $\text{Res}_{z_0}(f) \neq 0$, contradicting(X.2.7.4). □

Def. (X.2.7.7) (Weierstrass Function $\wp$). For a lattice $\Lambda \subset \mathbb{C}$, consider the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right)$$

Then it is a doubly periodic meromorphic function that has a double pole at 0, and so does its derivative $\wp'$, having a triple pole at 0. Hence they descend to a rational function on $\mathbb{C}/\Lambda$.

Prop. (X.2.7.8). The field of doubly periodic functions for $\Lambda$ is just $\mathbb{C}(\wp, \wp')$. And $\wp, \wp'$ also satisfies the following equation:

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3\wp(z)$$

where $g_2 = 60G_4(\Lambda), g_3 = 140G_6(\Lambda)$(IV.5.4.5).

Proof: For the equational function, just notice we can calculate directly that the difference of the two sides is a holomorphic function in $z$ without constant terms, and they are both doubly periodic, thus is zero.

For the first statement, notice that any function is a sum of an odd function and an even function. $\wp'$ is odd, thus an odd function is $\wp'$ times an even function. Thus it reduces to show that any even doubly periodic function is a rational function of $\wp$.

For any even periodic function $f$, $f$ has the same order at $z_0$ and $-z_0$, also if $z_0 \equiv -z_0 \text{ mod } \Lambda$, the order of $f$ at $z_0$ must be odd. Now consider $\wp(z) - \wp(z_0)$, then it is a function with two poles at 0, so it has two zeros. If $z_0 \equiv -z_0$, then $\wp(z) - \wp(z_0)$ has a zero of order 2 at $z_0$, and otherwise it has simple zeros at $z_0$ and $-z_0$.

So for any even doubly periodic function $f$, we can use the product $\prod(\wp(z) - \wp(z_0))^{m_i}$ to get a function with the same order of poles and zeros as $f$, which implies it equals $f$. □

8 Special Functions

Gamma Function

Def. (X.2.8.1) (Gamma Function). For a complex number $s$ that $\text{Re}(s) > 0$, the Gamma function is defined to be

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx.$$ 

which is convergent for any $\text{Re}(s) > 0$ thus an analytic function on the right half plane.

Prop. (X.2.8.2). $\Gamma(s + 1) = s\Gamma(s)$.

Prop. (X.2.8.3) (Special Values).

$$\Gamma(1) = 1, \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$
Prop. (X.2.8.4) (Duplication Formula).
\[ \Gamma(2s - 1) = \frac{4^{s-1}}{\sqrt{\pi}} \Gamma(s - \frac{1}{2})\Gamma(s). \]

Prop. (X.2.8.5).
\[ \Gamma(u)\Gamma(1-u) = \frac{\pi}{\sin(\pi u)}. \]

Cor. (X.2.8.6) (Analytic Extension). This formula extends \( \Gamma \) to a meromorphic function at whole plane. It has simple poles at \( s = -n \), where it has residues \( \frac{(-1)^n}{n} \), for \( n \geq 0 \).

Prop. (X.2.8.7) (Mellin Inversion Formula). By (X.7.2.16) applied to \( f(x) = e^{-x} \), for any real \( c > 0 \),
\[ e^{-x} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)x^{-s}ds, x > 0. \]

Prop. (X.2.8.8) (Stirling’s Formula). \[ |\Gamma(\sigma + it)| \sim \sqrt{2\pi} |t|^{\sigma - \frac{1}{2}} e^{-\pi|t|/2}. \]

Bessel Function

Def. (X.2.8.9) (Bessel Function). The Bessel function is defined to be
\[ B(r, s) = \int_0^1 (1 - y)^{r-1} y^{s-1} dy. \]

Prop. (X.2.8.10).
\[ B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}. \]

9 Multi-Variable case

Basics

Should cover the part from [Complex Analytic and Differential Geometry Demailly], [Principle of Algebraic Geometry Griffith/Harris] and [Complex Geometry Daniel].

Def. (X.2.9.1). A function is called holomorphic in several variables iff it is holomorphic for each indeterminate.

Def. (X.2.9.2). The polydisc \( B_\varepsilon(a) \) in \( \mathbb{C}^n \) is defined to be the set \( \{ z \mid |z_i - a_i| < \varepsilon_i \} \).

Prop. (X.2.9.3) (Hartog’s Extension Theorem). If \( K \) is a compact subset in an open domain \( \Omega \) of \( \mathbb{C}^n (n \geq 2) \) and \( \Omega - K \) is connected, then any holomorphic function on \( \Omega - K \) extends to a holomorphic function on \( \Omega \).

Proof: \[ \square \]

Prop. (X.2.9.4). Let \( \varepsilon = (\delta, \ldots, \delta) \) and \( f \) be a holomorphic function on the polydisc \( B_\varepsilon(0) \). Then if \( f \) vanishes of order \( k \) at the origin and \( |f(z)| \leq C \), then
\[ f(z) \leq C \left( \frac{|z|}{\delta} \right)^k \]
for all \( z \in B_\varepsilon(0) \).
Proof: Fix \( z \in \overline{B_\varepsilon(0)} \neq 0 \), consider the one-variable function \( g_z(w) = w^{-k}f(w \cdot \frac{z}{|z|}) \), then \( g_z \) is holomorphic and \( |g_z(w)| \leq \delta^{-k}C \) for \( |w| = \delta \). So maximal principle implies that \( g_z(w) \leq \delta^{-k}C \) for all \( |\omega| \leq \delta \). Hence \( |z|^{-k}|f(z)| = |g_z(|z|)| \leq \delta^{-k}C \). \( \square \)
X.3 General Functional Analysis

Basic references are [Rudin Functional Analysis],[Nonarchimedean Functional Analysis].

This section only contains theorems that are applicable to both Archimedean and non-
Archimedean valuations. For theorems specialized to non-Archimedean valuations, See IV.1, for
theorems specialized to Archimedean valuations, See X.4. Many propositions in Functional Analysis
can be transplanted in the general case, but I haven’t finish yet.

The major problem is convex is not definable, so Hahn-Banach fail, causing many to fail.

1 Topological Vector Space

Def. (X.3.1.1) (Topological Vector Spaces). A topological vector space (TVS) over a
complete valued field $k$ is a $k$-vector space that the addition and scalar multiplication is continuous.

Remark (X.3.1.2). If the field $k$ is not of char 0, then we fix a sequence of elements $\{a_n\}$ that
$\lim |a_n| = \infty$. This will be applied for example in the proof of Banach-Steinhaus theorem, but we
will just write $n$ instead of $a_n$.

Prop. (X.3.1.3). For subsets $K, C$ of a TVS $X$ that $K$ is compact and $C$ is closed, there is a
nbhd $V$ that $(K + V) \cap (C + V) = \emptyset$.

Proof: For each $x \in K$, there are symmetric nbhd $V_x$ that $(x + V_x + V_x + V_x) \cap C = \emptyset$. Then
$(x + V_x + V_x) \cap (C + V_x) = \emptyset$. Because $K$ is compact, there are f.m. $x_i$ that $K \subset \cup (x_i + V_x)$, so
let $V = \cap V_{x_i}$, then $(K + V) \cap (C + V) = \emptyset$. □

Cor. (X.3.1.4) (Closed Subbasis). Every nbhd of 0 in a TVS contains a closure of another
nbhd of 0. (Apply the above proposition for $K = \{0\}$.

Def. (X.3.1.5). A subset containing 0 is called balanced iff $kU = U$ for each $|k| = 1$.

Prop. (X.3.1.6) (Balanced Subbasis). Every nbhd $U$ of 0 in a TVS contains a balanced nbhd
of 0. By (X.3.1.4), we can even assume $V \subset U$.

Proof: Since scalar multiplication is continuous, there is a $\delta > 0$ and a nbhd $V$ that $\alpha V \subset U$, for
each $|\alpha| < \delta$. Then let $W = \cup_{|\alpha| < \delta} \alpha V$. □

Def. (X.3.1.7) (F-Space and Fréchet space). A space is called a F-space if its topology is
induced by a complete invariant metric. F-space is of second Baire category by(IX.1.9.2)

A locally convex F-space is called a Fréchet space.

A TVS is said to satisfy Heine-Borel iff every closed and bounded subset of $X$ is compact.

Def. (X.3.1.8) (Norm). A seminorm on a vector space $X$ is a real-valued function $p$ that
$p(x + y) \leq p(x) + p(y)$, and $p(\alpha x) = |\alpha| p(x)$ for $\alpha \in k$. It is called a norm if moreover $p(x) = 0 \iff x = 0$.

A family of seminorms $\{p_i\}$ on $X$ is called separating iff for each $x$, at least one $p_i$ satisfies
$p_i(x) \neq 0$.

Prop. (X.3.1.9). A TVS is metrizable by a translation-invariant metric iff it has a countable
basis.

Proof: One direction is trivial, for the other, Cf.[Rudin P18]. □
Prop. (X.3.1.10). If a subspace $Y$ of a TVS $X$ is a F-space, then it is closed in it.

Proof: Choose an invariant metric $d$ compatible with its topology, Let $U_n$ be a nbhd of $X$ that $U_n \cap Y = B(0,1/n)$, and choose a symmetric nbhd $V_n$ of $X$ that $V_n + V_n \subset U_n$, and $V_{n+1} \subset V_n$.

If $y \in Y$, then for any $y_n \in Y \cap (y + V_n) = E_n$, then $y_n - y_m \subset U_{\min\{m,n\}} \cap Y = B(0,1/n)$, so it is a Cauchy sequence in $Y$, hence all $E_n$ has a unique element $y_0$ in common. Now if we intersect each $V_n$ by a nbhd $W$ of $X$, the same argument shows that there is a unique element $y_W$ in $Y \cap (y + W \cap V_n)$, and this must by just $y_0$, but $y - y_W \subset W$, so we must have $y = y_0 \subset Y$. □

Def. (X.3.1.11). A set $E$ in a TVS is called totally bounded if for every nbhd $V$ of 0, there is a finite set $F$ that $E \subset F + V$.

2 Completeness

Prop. (X.3.2.1) (Banach-Steinhaus). If a collection of continuous linear mapping between two TVS, if the set $B$ of $x$ that $\Gamma(x)$ is bounded is a second category set in $X$, then $B = X$ and $\Gamma$ is equicontinuous (thus maps bounded sets to bounded sets).

Proof: For an open balanced nbhd $W$ of 0, choose a balanced nbhd $U$ s.t. $U + U \subset W$ (X.3.1.6), set $E = \cap_{\Lambda \Gamma} \Lambda^{-1}(U)$, then $B \subset \bigcup_{n=1}^{\infty} nE$, so by Baire theorem (IX.1.9.2), $E$ has a interior point thus has a nbhd $V$ s.t. $\Gamma(V) \subset U + U \subset W$. Thus we are done. □

Cor. (X.3.2.2) (Uniform Boundedness Theorem). If a set $\Gamma$ of continuous linear mappings from a F-space $X$ to $Y$ satisfies $\Gamma(x)$ is bounded for every $x$, then $\Gamma$ is equicontinuous.

Cor. (X.3.2.3). If $A_n$ is a sequence of continuous linear mapping from $X$ to $Y$, if $X$ is a F-space, then $\lim A_n = A$ iff $\|A_n\|$ is bounded and $\lim A_n x = Ax$ for $x$ in a dense subset of $X$.

Proof: One direction is immediate from Banach-Steinhaus, the converse is an easy $\varepsilon/3$-technique. □

Prop. (X.3.2.4) (Open Mapping theorem). If a continuous linear mapping $T$ from a F-space $X$ to $Y$ satisfies $R(T)$ is of second category, then it is a surjective open mapping and $Y$ is a F-space.

In fact, we only need $T$ be defined on a subspace $D(T)$ and it is closed in the sense the graph of it is closed.

Proof: $V_n = T(B(0,\frac{1}{n}))$ are all of second category, because $\cup_n nV_n = R(T)$, so $\overline{V_n}$ has an interior by definition. Then also it contains a nbhd of $V$ because $\overline{V_{n+1}} \subset V_n$.

Now we show $\overline{V_{n+1}} \subset V_n$, this will show $T$ is open. thus for a $y \in V_n$ since $\overline{T(V_{n+1})}$ contains a nbhd of 0, we can consecutively choose $x_i \in B(0,\frac{1}{n+i})$ s.t. $y - \sum_{i=1}^n T(x_i) \in T(B(0,\frac{1}{n+i}))$. So by completeness of $X$ and closedness, $\sum x_i$ converges to some $x \in D(T)$, and $Tx = y \in V_n$.

And an open linear mapping must be surjective. hence $Y \cong X/N(T)$, so $Y$ is also a F-space. □

Cor. (X.3.2.5) (Banach Theorem). If a continuous map $T$ between F-spaces is a bijection, then it has a continuous inverse.

Cor. (X.3.2.6). If a F-space is complete w.r.t two different topologies and one is stronger than the other, then they are equivalent.

Cor. (X.3.2.7). For every operator between F-spaces that has closed image, we have $X/N(T) \cong R(T)$. 

Cor. (X.3.2.8) (Closed Graph Theorem). If \( T \) is a closed linear mapping between two F-spaces, i.e. the graph of it is closed, then it is continuous, because the metric induced by the graph is stronger than the original one, and use Banach (X.3.2.5).

The graph is closed is equivalent to if \( x_i \to x \) and \( Tx_i \to y \), then \( y = Tx \). This is very useful when proving some map is continuous.

Cor. (X.3.2.9). If \( A, B, C \) are F-spaces, and \( f : A \to B, g : B \to C \), if \( gf, g \) is continuous and \( g \) is injective, then \( f \) is continuous.

Proof: Use closed graph theorem, if \( x_i \to x, f(x_i) \to z \), then \( gf(x_i) \to g(z) \), so \( g(x) = g(z) \), so \( f(x) = z \).  

Cor. (X.3.2.10) (Finite Codimensional Image). If the image of a continuous linear mapping \( T \) between F-spaces has finite codimensional image, then the image is closed and complemented.

Proof: It has finite codimension, so we can construct \( K^n \oplus X/N(T) \to Y \), by Banach theorem (X.3.2.5) it is a homeomorphism, and the image of \( X/N(T) \) corresponds to the image, so the image is closed.

Prop. (X.3.2.11) (Separate Continuous). If a bilinear map \( B : X \times Y \to Z \) where \( X \) is a F-space is separately closed, then \( B(x_n, y_n) \) converges to \( B(x_0, y_0) \).

Proof: Use Banach-Steinhaus to prove \( B_{y_n}(x) \) is equicontinuous, then use \( B(x_n - x_0, y_n) + B(x_0, y_n - y_0) \).

3 Dual Space

Prop. (X.3.3.1) (Operator Space). If \( X, Y \) are normed spaces then \( L(X, Y) \) is also a normed space with the metric \( ||\Lambda|| = \sup \{||\Lambda x|| | x| \leq 1 \} \). And if \( Y \) is Banach, then \( L(X, Y) \) is also Banach. The proof is easy.

In particular, if \( Y = K \), then \( X^* \) is a Banach space.

Prop. (X.3.3.2). For a bounded operator \( T \),

\[
\overline{R(T)} = N(T^*)^1, \text{ Thus } \overline{R(T^*)} = N(T)^1
\]

In particular, using Hahn Banach, \( R(T) \) is dense in \( Y \) iff \( T^* \) is injective, \( T \) is injective iff \( T^* \) is weak*-dense in \( X^* \).

Prop. (X.3.3.3). Let \( \Lambda_1, \ldots, \Lambda_n, \Lambda \) are linear functionals on a vector space \( X \), let \( N = \cap \text{Ker } f_i \), the following are equivalent:

1. \( \Lambda = \sum \alpha_i \Lambda_i \).
2. \( |\Lambda x| \leq |\alpha| \Lambda_i |x| \).
3. \( \text{Ker } \Lambda \subset N \).

Proof: Only need to show \( 3 \to 1 : \) define \( \pi : X \to k^n : x \mapsto (\Lambda_1 x, \ldots, \Lambda_n x) \), then by hypothesis \( f(\pi(x)) = \Lambda(x) \) defined a linear functional on \( \pi(X) \). This can be extended to a functional on \( k^n : F(u_1, \ldots, u_n) = \sum \alpha_i u_i \), so \( \Lambda \) is a linear combination of \( \Lambda_i \). 

\]
Weak Convergence

Def. (X.3.3.4) (Operator Topologies). There are three topologies on $L(X)$ for a normed space $X$:

- norm topology: $\|T_i - T\| \to 0$.
- strong topology: $\forall x \in X, \|(T_i - T)x\| \to 0$.
- weak topology: $\forall x \in X, f \in X^*, \lim f(T_n x) = f(T x)$.

Prop. (X.3.3.5) (Weak Convergence and Bounded). In a normed space $X$, if $x_n \to x$ weakly iff $\{x_n\}$ is bounded and $\lim f(x_n) = f(x)$ for a dense subset $f \in M^* \subset X^*$.

Proof: This follows from (X.3.2.3), as $X^*$ is a Banach space, by (X.3.3.1).

4 Banach Space

Def. (X.3.4.1) (Normed(Valued) $K$-Spaces). For a complete valued field $K$, a normed(valued) $K$-space is a TVS over $K$ with a norm that satisfies $|kv| = |k||v|$ for $k \in K$.

Def. (X.3.4.2) (Banach Spaces). For $K$ complete valued field, a complete normed(valued) $K$-vector space (X.3.4.1) is called a Banach space. A $K$-algebra with a complete $K$-algebra norm is called a Banach algebra.

Prop. (X.3.4.3). The dual space of a Banach space is a Banach space. (Immediate from (X.3.3.1)).

Prop. (X.3.4.4). if $A$ is a Banach space as well as a Topological group, then there is a norm on $A$ which induce the same topology and makes $A$ into a Banach algebra.

Proof: embed $A$ into $L(A)$ by left multiplication, which is injective, and $\|x\| = \|xe\| = \|M_e\| \leq \|M_x\|\|e\|$, so its inverse is continuous. Now if we show the image $\tilde{A}$ is closed in $L(A)$, then the open mapping theorem will show that $A \cong \tilde{A}$, and $\tilde{A}$ is clearly a Banach algebra.

To show it is closed, if $T = \lim T_i$, notice $T_i(y) = T_i(e)y$, so take a limit, $T(y) = T(e)y = M_{T(e)}y$.

Cor. (X.3.4.5). Every f.d. Banach algebra is isomorphic to an algebra of matrices. In particular, if $xy = e$, then $yx = e$.

Proof: Embed $A$ into $L(A)$.

Remark (X.3.4.6) (Inequivalent Banach Norms). There exists two complete norm on a vector space that is inequivalent. For this, just choose a banach space $X$, and notice if we can choose a discontinuous bijection $X \to X$, then the induced metric is also complete, and it cannot by equivalent by Banach theorem (X.3.2.5). For this, choose a infinite dimensional Banach space over $\mathbb{C}$, and choose a $\mathbb{C}$-basis $x_i$ for it, and choose a sequence $x_n$ and maps $x_n$ to $n x_i$, the rest are invariant, then this is not continuous.

Hilbert Space

Def. (X.3.4.7). there are different topologies in the space of operators on a Hilbert space $\mathcal{H}$.

Norm operator topology: defined by the norm $\|T\|$.

Strong operator topology: defined by the separating seminorms $T \mapsto \|Tu\|, u \in \mathcal{H}$.

Weak operator topology: defined by the separating seminorms $T \mapsto (Tu, v)$, $u, v \in \mathcal{H}$.
Prop. (X.3.4.8). The strong and weak operator topology coincides on the unitary operators on $\mathcal{H}$. The sets of unitary operators that is continuous in this two topology is denoted by $U(\mathcal{H})$.

Proof: If $T_n$ converges to $T$ in the weak operator topology, then

$$||(T_n - T)u||^2 = ||Tu||^2 + ||T_n u||^2 - 2\text{Re}(T_n u, Tu).$$

The right hand side is clearly bounded by the weak seminorms, so the two topologies coincide. $\square$

Prop. (X.3.4.9) (Hilbert Basis). If $H$ is a Hilbert space and $S = \{e_\alpha\}$ is an orthonormal basis in $H$, then the following are equivalent:

1. For any $x$, $x = \sum (x, e_\alpha)e_\alpha$, (notice the sum are in fact infinite sum).
2. There is a no nonzero element $x$ that is orthogonal to all $e_\alpha$.
3. Parseval equality holds: $||x||^2 = \sum |(x, e_\alpha)|^2$.

If these are true, then $S$ is called a Hilbert basis of $H$, a Hilbert basis always exists, by Zorn’s lemma.

Proof: $1 \rightarrow 2$: if $(x, e_\alpha) = 0$ for all $e_\alpha$, then by 1, $x = \sum (x, e_\alpha)e_\alpha = 0$.

$2 \rightarrow 3$: Notice $y = x - \sum (x, e_\alpha)e_\alpha$ is orthogonal to all $e_\alpha$, and

$$||y|| = ||x||^2 - \sum |(x, e_\alpha)|^2,$$

so Parseval equality holds.

$3 \rightarrow 1$: $||x - \sum (x, e_\alpha)e_\alpha|| = 0$. $\square$

Prop. (X.3.4.10). Any symmetric operator on a Hilbert space is continuous.

Proof: Because $x_n \to 0$ implies $Tx_n \to 0$ weakly, so we can use closed graph theorem (X.3.2.8). $\square$

**Ultrannormed Banach Spaces**

The ultranormed Banach spaces are defined in (IV.1.2.4).

**Nuclear Maps and Spaces**
X.4 Archimedean Functional Analysis

Reference: [Rudin Functional Analysis]. [Rudin Functional Analysis Chap11,13] needs to be revised.

This section contains functional analysis in characteristic 0. By Ostrowski theorem(I.9.3.14), the base field is just $R$ or $C$.

1 Topological Vector Space

Def. (X.4.1.1). A sublinear functional is a function $p$ that $p(x + y) \leq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ for $\lambda > 0$.

A seminorm is a non-negative function $p$ that $p(x + y) \leq p(x) + p(y)$ and $p(\alpha x) = |\alpha|p(x)$ for all complex $\alpha$.

Def. (X.4.1.2). A absorbing set is a convex set $A$ that $\cup_{k>0}kA = X$. A convex nbhd of 0 is clearly absorbing.

Def. (X.4.1.3) (Minkowski Functional). For an absorbing set $A$, the Minkowski functional $\mu_A$ is defined to be $\mu_A(x) = \inf\{t > 0, x/t \in A\}$. It satisfies:
- $\mu_A(x + y) \leq \mu_A(x) + \mu_A(y)$.
- $\mu_A(kx) = k\mu_A(x)$ if $k > 0$.
- $\mu_A$ is a seminorm if $A$ is balanced.
- If $B = \{x|\mu(x) < 1\}, C = \{x|\mu(x) \leq 1\}$, then $B \subset A \subset C$ and $\mu_A = \mu_B = \mu_C$.

Proof: Cf.[Rudin P27].

Cor. (X.4.1.4) (Seminorm and Absorbing set). A seminorm on $X$ is exactly the Minkowski functional of a balanced absorbing set $W$, but the set may not be unique. and it is uniformly continuous iff 0 is an interior point.

Proof: If $p$ is a seminorm, then $\{x|p(x) < 1\}$ is convex, balanced and absorbing by definition(X.3.1.8). The converse is by(X.4.1.3). The last assertion is easy.

Prop. (X.4.1.5) (Minkowski Functional and Separating Seminorms). If $\mathfrak{B}$ is a convex local base in a TVS $X$, then the Minkowski functionals of elements of $\mathfrak{B}$ forms a separating family of seminorms.

Conversely, a separation family $P$ of seminorms on a vector space defines a convex balanced local base for a topology $\tau$ that is locally convex. And in this topology, a sequence converges iff $p(x_i - x) \to 0$ for $p \in \mathfrak{P}$, a set is bounded if each $p$ is bounded on it.

Proof: For any $V \in \mathfrak{B}$, $V = \{x \in X|\mu_V(x) < 1\}$, because $V$ is open and convex. (X.4.1.3) shows each $\mu_V$ is a seminorm, and it is continuous because it is bounded on $V$. And they are separating because $\mathfrak{B}$ is a local base.

Defined $V(p,n) = \{x \in X|p(x) < 1/n\}$, and let these be a local subbasis at 0, and make it a topology by translation. This is checked to be a locally convex TVS. For the last assertion, if $E$ is bounded, then $E \subset kV(p,1)$ for $k$ large, so $p$ is bounded on $E$, and conversely, for each nbhd $U$, there are $p_i$ and $M_i$ and $\cap V(p_i,M_i) \subset U$, so $E \subset kU$ for $n$ large.

Prop. (X.4.1.6). If $\mathfrak{F}$ is a family of countable separating family of semi-norms on $X$, then the topology defined in(X.4.1.5) is in fact metrizable, by a metric $d(x,y) = \sum \frac{1}{2^k} \frac{p_k(x-y)}{1+p_k(x-y)}$. 

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Finite Dimensional Subspace

Prop. (X.4.1.7) (Finite Dimensional and Locally Compact). There is only one topological vector space structure on a finite dimensional \( \mathbb{C} \)-vector space and it is complete. A TVS is locally compact iff it is f.d.

Proof: Cf. [Rudin P17].

For the second assertion, if is locally compact, then 0 has a nbhd \( V \) that is precompact, so bounded, hence \( 2^{-n}V \) forms a local basis. the compactness of \( V \) shows there are f.m. \( x_i \) that \( V \subset \bigcup (x_i + \frac{1}{2}V) \). Let \( Y \) be the subspace spanned by \( x_i \), then it is of f.d, thus closed. Now \( V \subset Y + \frac{1}{2}V \), so \( \frac{1}{2}V \subset Y + \frac{1}{4}V \), hence continuing this way, \( V \subset \bigcap (Y + 2^{-n}V) \), so \( V \subset Y = X \). But then \( Y = X \).

Cor. (X.4.1.8) (Finite Subspace Closed). A f.d subspace in a TVS over \( \mathbb{C} \) is closed, because it must be a \( F \)-space, hence it is closed by (X.3.1.10).

Prop. (X.4.1.9) (Finite Subspace in Banach Space). For a finite dimensional space \( V \) in an Archimedean Banach space, there is a continuous projection onto it. In particular, any finite dimensional space in an Archimedean Banach space is complemented.

Also finite codimensional subspace in any Banach space is complemented by (X.3.2.10).

Proof: Choose a basis \( e_i \) for \( V \), consider the dual basis \( f_i \). Because a finite dimensional space only has one topology (X.4.1.7), these \( f_i \) are bounded on \( V \). Extend them to bounded functional on \( X \), then consider \( p(x) = \sum_{i} f_i(x) e_i \), then this is a continuous projection onto \( V \).

2 Various Spaces and Duality

For a bounded connected open set \( \Omega \),

Prop. (X.4.2.1) (Various Spaces and Duality).

- **Sobolev Space** \( W^{m,p}(\Omega) \) is the completion of a subspace of \( C^\infty(\Omega) \) with the norm

\[
||u||_{m,p} = \left( \sum_{|\alpha| \leq m} \int_{\Omega} |\partial^{\alpha} u(x)|^p \, dx \right)^{1/p}.
\]

for \( m > 0 \). And we denote \( W^{m,2}(\Omega) \) by \( H^m(\Omega) \). It is also a subspace of \( L^p(\Omega) \) that satisfies this, without completion (X.7.3.1).

- \( C^\infty_0(\Omega) \) is the subspace of \( C^\infty(\Omega) \) that have compact support in \( \Omega \). Its completion \( W^{m,p}_0(\Omega) \) is a closed subspace of \( W^{m,p}(\Omega) \). And we denote \( W^{m,2}_0(\Omega) \) by \( H^m_0(\Omega) \) and the dual space of \( H^m_0(\Omega) \) by \( H^{-m}(\Omega) \).

- \( C(\Omega) \) in the topology of compact convergence is a Fréchet space. It is not locally convex.

- \( H(\Omega) \) the space of holomorphic functions in \( \Omega \) is a closed subspace of \( C(\Omega) \) thus is a Fréchet space. Montel’s theorem says exactly that \( H(\Omega) \) is Heine-Borel.

- \( H^2(D) \) the space of holomorphic functions on the unit disk \( D \) that is also \( L^2 \). It has the \( L^2 \) norm.

- \( C^\infty(\Omega) \) in the topology defined by seminorms \( p_N(f) = \max \{|D^{\alpha} f(x)| : x \in K_N, |\alpha| \leq N\} \), is a Fréchet space thus locally convex and it has the Heine-Borel property by Arzela-Ascoli.
• $D(K)$ is the closed subspace of smooth functions on $\Omega$ with support in $K$, thus a Fréchet space with Heine-Borel property.

• $D(\Omega)$ is the space of smooth functions with support in $\Omega$. It has the topology generated by translation of basis consisting of convex balanced sets $W$ that $W \cap D(K)$ is open for every compact $K$. This makes $D(\Omega)$ into a locally convex TVS, Cf.[Rudin P152]. It has Heine-Borel property (X.7.1.1).

• (Schwartz Functions) The space of Schwartz functions $\mathcal{S}$ is defined as smooth functions on $\mathbb{R}^n$ such that

$$\sup_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^N |(D_\alpha f)(x)| < \infty$$

for any $N > 0$. And it is a Fréchet space define by these seminorms.

**Dual Spaces**

**Prop. (X.4.2.2).**

- For a σ-finite measure $\mu$ on a measurable space $X$, for $1 \leq p < \infty$

$$L^p(X, \Omega, \mu)^* = L^q(X, \Omega, \mu).$$

(X.1.5.3)

- $C[0, 1]^* = BV[0, 1]$ and $C[X]^* = M(X)$, the space of complex measure on compact $X$ with the norm of total variance, by Riesz representation theorem (X.1.1.10).

**3 Convexity**

**Prop. (X.4.3.1).** Every convex nbhd of $0$ contains a balanced convex nbhd of $0$. By (X.3.1.4), we can even assume $V \subset U$.

**Proof:** If $U$ is convex, choose $W$ as in (X.3.1.6), then $W \subset A = \cap_{|\alpha|=1} \alpha U$ because it is balanced. Then $W \subset A^\circ$ and $A^\circ$ is open and balanced, satisfying the requirement. \(\square\)

**Prop. (X.4.3.2).** For a compact convex set $K$ in a TVS $X$, if a set $\Gamma$ of continuous linear mapping is bounded for every $x \in K$, then $\Gamma$ is equicontinuous on $K$.

**Proof:** The proof is similar to that of Banach-Steinhaus (X.3.2.1). For $K$ compact convex, the same argument shows that there is a nbhd $V$ that $K \cap (x_0 + V) \subset nE$, fix $p > 1$ that $K \subset x_0 + pV$, then for any $x \in K$, consider $z = (1-p^{-1})x_0 + p^{-1}x$, then $z \in K$ as $K$ is convex and $z-x_0 = p^{-1}(x-x_0) \in V$, so $z \in nE$, and since $x = pz - (p-1)x_0$, $\Lambda x \in pnW$ for each $\Lambda \in \Gamma$, so $\Gamma$ is equicontinuous. \(\square\)

**Hahn-Banach**

**Prop. (X.4.3.3) (Real Hahn).** For a sublinear functional $p$ on a real linear space $X$ and a subspace $X_0$, if a functional $f$ satisfies $f(x) \leq p(x)$ on $X_0$, then it can be extended to a functional $\Lambda$ on $X$ that $|\Lambda(x)| \leq |p(x)|$.

**Proof:** Use Zorn’s lemma, if the maximum extension is not on the whole space but on $M$, choose $x_1 \in X - M$, we want to define $f(x_1)$. Now let $M_1 = \{x + tx_1 | x \in M\}$. Since for $x,y \in M$, $f(x) + f(y) = f(x + y) \leq p(x + y) \leq p(x-y) + p(x + y)$, so

$$f(x) - p(x - x_1) \leq p(y + x_1) - f(y)$$

Note: The proof should be completed with further details or steps.
for \(x, y \in M\). Let the maximum of the left side be \(\alpha\), and define \(f(x_1) = \alpha\), then it is clear \(f(z) \leq p(z)\) still.

**Prop. (X.4.3.4) (Complex Hahn).** For a seminorm \(p\) (i.e. it can attain 0) on a complex linear space \(X\) and a subspace \(X_0\), if a functional \(f\) satisfies \(|f(x)| \leq p(x)\) on \(X_0\), then it can be extended to a functional on \(X\) with the same condition.

**Proof:** Let \(g(x) = \text{Re } f(x)\) and extend it by Hahn and set \(f(x) = g(x) - ig(ix)\), then \(f\) is complex linear, and for any \(x\), for some \(|\alpha| = 1\), \(|f(x)| = f(\alpha x) = g(\alpha x) \leq p(\alpha x) = p(x)\).

**Cor. (X.4.3.5) (Hahn).** In a normed space \(X\), a bounded linear functional on a subspace \(X_0\) can be extended to a bounded functional on \(X\) with the same norm.

**Cor. (X.4.3.6) (Extending Functional Preserving Norm).** If \(X\) is a normed space and \(N\) is a closed subspace, if \(x_0\) satisfies \(d = d(x_0, M) > 0\), then here is a continuous functional \(f\) that \(f(x) = 0\) and \(f(x_0) = d\) and \(|f| = 1\).

**Proof:** Define \(f(m + \alpha x) = |\alpha|d\) on \(\text{span}\{M, x\}\), then \(f(m + \alpha x) = |\alpha|d = |\alpha|d(x_0, M) < |\alpha||\frac{x}{\alpha} + x_0|| = ||x' + \alpha x|| = ||x||\). So \(|f| = 1\), so we can use Hahan-Banach to extend it to a functional on \(X\).

**Prop. (X.4.3.7) (Geometric Hahn).**
- If \(E_1\) and \(E_2\) are two convex set that \(E_1 \cap E_2 = \emptyset\) and \(E_1\) has interior point, then there is a continuous linear functional that separate them, i.e \(\text{Re } f(E_1) < \text{Re } f(E_2)\). (The interior point is here to assure \(f\) is continuous).
- In a locally convex TVS, if \(E_1\) is convex compact and \(E_2\) is convex closed, then there is a real functional that separate them, i.e. \(\text{Re } f(E_1) < \gamma_1 < \gamma_2 < \text{Re } f(E_2)\). Thus for a set \(E\) and a point \(x, x \in \text{span}E \iff\) for all \(f\) that \(f(E) = 0, f(x) = 0\).

**Proof:** The complex case follows from the real case, so assume it is real. Consider \(a_0 \in E_1, b_0 \in E_2\), let \(x_0 = a_0 - b_0\) and let \(C = E_1 - E_2 + x_0\), then \(C\) is a convex nbhd of 0. Let \(p\) be the Minkowski functional of \(C\), then \(p\) is sublinear by (X.4.1.3) and \(p(x_0) \geq 1\). Let \(f(tx_0) = t\) on the subspace \(M\) generated by \(x\), then it extends to a functional \(\Lambda\) that \(\leq 1\) on \(C\), thus it is bounded by 1 on \(C \cap (-C)\), hence continuous. For any \(a \in E_1, b \in E_2\), because \(\lambda(x_0) = 1\) and \(a - b + x_0 \in C\) open, \(\Lambda a < \Lambda b\).

For the second, There is a convex nbhd \(V\) of 0 that \(E_1 + V \cap B = \varnothing\), so the above argument applied with \(E_1 + V\) and \(B\) shows that there is a \(f\) that separate them. And \(f(E_1 + V)\) is open and \(f(E_1)\) is compact, so the conclusion follows.

**Cor. (X.4.3.8) (Banach-Saks).** The weak closure of a convex set in a locally convex metric space equals the original closure.

Thus if a sequence \(\{x_n\}\) weakly converges to \(x\) in a metrizable locally convex space, then a convex combination of \(\{x_n\}\) strongly converge to \(x\), i.e. \(x \in \overline{co}(\{x_n\})\), because metric space is first countable.

**Proof:** A weak closed set is closed, and to show the closure is weakly closed, use(X.4.3.7).

**Prop. (X.4.3.9).** If \(A_i\) are compact convex sets in a TVS \(X\), then \(co(A_1 \cup \ldots \cup A_n)\) is compact.
Prop. (X.4.3.10). In $F$-space, a closed subset is compact if and only if it is totally bounded by (IX.1.8.5).

Prop. (X.4.3.11). In a locally convex space, if $E$ is totally bounded, then $\text{co}(E)$ is totally bounded. Thus in a Fréchet space, if $K$ is compact, then $\overline{\text{co}}(K)$ is compact.

Proof: For a nbhd $U$ of 0, choose a convex nbhd $V$ that $V + V \subset U$, then $E \subset F + V$ for some finite set $F$, hence $\text{co}(E) \subset \text{co}(F) + V$. But $\text{co}(F)$ is compact by (X.4.3.9). So $\text{co}(F) \subset F_1 + V$ for some finite set $F_1$, then $\text{co}(E) \subset F_1 + U$.

If $K$ is compact, then it is totally bounded, and then $\text{co}(K)$ is totally bounded and $\overline{\text{co}}(K)$ is totally bounded by (IX.1.8.3), so it is compact by (X.4.3.10).

Prop. (X.4.3.12) (Weakly Bounded and Locally Convex). In a locally convex space, bounded $\iff$ weakly bounded.

Proof: One direction is trivial, for the other, suppose $E$ is weakly bounded and $U$ is a closed nbhd of 0. Because $X$ is locally convex, there is a convex, balanced nbhd of 0 that $\bar{V} \subset U$ (X.4.3.1). Now $\bar{V} = V^{**}$ the polar (X.4.4.1) by (X.4.3.8).

Now $V^*$ is weak*-compact and $|\Lambda(x)| \leq \gamma(\Lambda)$ for each $\Lambda \in X^*$ for some $\gamma(\Lambda)$ because $E$ is weakly bounded. So we can use (X.4.3.2) to show that $|Ax| \leq \gamma$ for some $\gamma$ and all $\Lambda \in V^*$. So we have $\gamma^{-1} E \subset \bar{V} \subset U$. This proves that $E$ is bounded.

Prop. (X.4.3.13) (Markov-Kakutani Fixed Point Theorem). For a commuting family $\mathcal{F}$ of continuous affine maps from $K$ to $K$ where $K$ is a compact convex set in a TVS, then there is a fixed point in $K$ for all maps in $\mathcal{F}$.

Proof: Consider the semigroup $\mathcal{F}^*$ generated by these maps together with their average, it is also commutative because they are all affine. For any $f, g \in \mathcal{F}^*$, $f(K) \cap g(K) \subset f \circ g(K)$, so by finite intersection property, there is a point in $p \in K$ in all $f(K)$.

For this $p$, consider $p = \frac{1}{n}(I + T + t^2 + \ldots + T^{n-1})(x)$, then $p - Tp = \frac{1}{n}(x - T^nx) \in \frac{1}{n}(K - K)$. as $K$ is bounded and $n$ is arbitrary, this means that $p = Tp$ for all $T$.

Cor. (X.4.3.14) (Invariant Hahn). For a commuting family $\Gamma$ of operators on a normed space and $Y$ an invariant space, then for any $\Gamma$-invariant continuous functional $f$ on $Y$, it has a $\Gamma$-invariant Hahn extension.

Proof: We may assume $||f|| = 1$. Let $K$ be all extensions of $f$ that has norm $\leq 1$. $K$ is obviously convex, and it is weak*-compact by Banach-Alaoglu. The adjoint action of $T$ is checked to be continuous in the weak*-topology, so by (X.4.3.13), there is some $F \in K$ that is invariant under $\Gamma$.

\textbf{Krein-Milman Theorem}

Prop. (X.4.3.15) (Krein-Milman Theorem). For a compact convex set in a TVS that is weak-Hausdorff ($X^*$ separate points), then $K = \overline{\text{co}}(\text{Extreme}(K))$.

If $K$ is a compact set in a locally convex space, then $K \subset \overline{\text{co}}(E(K)) = \overline{\text{co}}(K)$. 
Proof: First show that every compact extreme set $S$ of $K$ contains an extreme point. Notice arbitrary intersection of compact extreme sets of $K$ is compact extreme, because compact is closed, because $X$ is Hausdorff. And for any functional $\Lambda \in X^*$, the maximal value point in $K$ is compact extreme. Now we use Zorn’s lemma to find a minimal compact extreme set in $S$, then it must be a point because $X^*$ separate points.

Now use the weak topology Hahn(X.4.4.2), if $\overline{co}(E(K)) \subset K$ is not $K$, then it is compact, then we can find a functional that separate $\overline{co}(E(K))$ and some point of $K$. This is a contradiction because the extreme value point for any functional on $K$ is an extreme set.

In the locally convex case, the convexity of $K$ is not needed, and we can show using geometric Hahn(X.4.3.7) instead that, $K \subset \overline{co}(K)$.

Prop. (X.4.3.16) (Milman’s Theorem). If $K$ is a compact set in a locally convex space $X$ and if $\overline{co}(K)$ is also compact(e.g in a Fréchet space(X.4.3.11)), then every extreme point of $\overline{co}(K)$ lies in $K$.

Proof: □

Def. (X.4.3.17). For a compact Hausdorff space $S$ and an algebra in $C(S)$, a subset $E$ is called A-antisymmetric iff every $f \in A$ that is real on $E$ is constant on $E$. There are in fact maximal A-antisymmetric subsets of $S$.

Prop. (X.4.3.18) (Bishop Theorem). If $A$ is a closed subalgebra of $C(S)$. If $g \in C(S)$ satisfies $g|_E \in A|_E$ for every maximal $A$-antisymmetric set $E$, then $g \in A$. This theorem is a generalization of Stone-Weierstrass approximation.

Proof: The annihilator $A^\perp$ of $A$ consists of all regular complex Borel measure $\mu$ on $S$ that $\int f d\mu = 0$ for all $f \in A$ by Riesz representation(X.1.1.10). □

Cf.[Rudin P122].

Prop. (X.4.3.19) (Schauder Fixed Point Theorem). If $C$ is a closed convex subset in a metrizable TVS and continuous $T : C \to C$ has sequentially compact image (e.g. $C$ is compact and $X$ is locally convex hence $X^*$ separate points), then $T$ has a fixed point.

Proof: As $T(C)$ is sequentially compact, for each $n$, there is a $1/n$-net $N_n = \{y_i\} \subset T(C)$, let $E_n = \text{span}\{N_n\}$.

Define a map $T(C) \to co(N_n) : I_n(y) = \sum y_i \lambda_i(y)$, where $\lambda_i(y) = \frac{m_i(y)}{\sum m_i(y)}$, and $m_i(y) = 1 - n||y - y_i||$ if $y \in B(y_i, 1/n)$, and 0 otherwise.

Now $||I_n(y) - y|| = ||\sum (y_i - y)\lambda_i(y)|| \leq \sum ||y_i - y||\lambda_i(y) \leq \frac{1}{n}$ for each $y \in T(C)$. As $C$ is convex, $co(N_n) \subset C$, if we let $T_n = I_n \circ T$, then $T_n$ has a fixed pt $x_n$ in $co(N_n)$ by Brower fixed pt theorem(IX.4.7.10).

As $T(C)$ is sequentially compact and $C$ is closed, there is a subsequence $Tx_{n_k}$ that converges to $x \in C$. And then

$$||x_{n_k} - x|| = ||I_n Tx_{n_k} - x|| \leq ||I_n Tx_{n_k} - Tx_{n_k}|| + ||Tx_{n_k} - x|| < \frac{1}{n} + ||Tx_{n_k} - x||$$

so $x_{n_k} \to x$, and then by continuity, $Tx = x$. □
**Vector-valued Integration**

**Def. (X.4.3.20) (Vector-Valued Integration).** Given a measure space \((Q, \mu)\) and \(X\) is an Archimedean TVS on which \(X^*\) separate points. If \(f\) is a function from \(M\) to \(X\) that \(\Lambda \circ f\) are integrable w.r.t \(\mu\) for any \(\Lambda \in X^*\). The integration \(\int_M f d\mu\) of \(f\) w.r.t \(Q\) is an element \(y\) that
\[
\Lambda y = \int_Q (\Lambda f) d\mu
\]
for any \(\Lambda \in X^*\).

**Prop. (X.4.3.21).** If \(X\) is an Archimedean TVS on which \(X^*\) separate points, \((Q, \mu)\) is a Radon measure on a locally compact Hausdorff space that \(\mu\) is compactly supported, and \(f\) is continuous that \(\varpi(f(Q))\) is compact (e.g. when \(X\) is Fréchet (X.4.3.11)), then the integral \(y = \int_Q f d\mu\) exists, and belongs to the closed linear span of the range of \(H\). Moreover if \(\mu\) is positive and \(\mu(Q) = 1\), then \(y \in \varpi(f(Q))\).

**Proof:** Cf. [Rudin P78]. 

**Cor. (X.4.3.22).** If \(Q\) is Hausdorff, \(d\mu\) is compactly supported, \(X\) is Archimedean Banach and \(f : Q \to X\) is continuous, then
\[
||\int_Q f d\mu|| \leq \int_Q ||f|| d\mu
\]

**Proof:** Let \(y = \int_Q f d\mu\). By (X.4.3.6), there is a \(||\Lambda|| \leq 1\) that \(\Lambda y = ||y||\), so
\[
\Lambda y = ||y|| = \int_Q \Lambda f d\mu = |\int_Q \Lambda f d\mu| \leq \int_Q |\Lambda f| d\mu \leq \int_Q ||f|| d\mu
\]

**Prop. (X.4.3.23).** If \(X\) is an Archimedean TVS on which \(X^*\) separate points, \(Q\) is a compact subset of \(X\), and \(\varpi(Q)\) is compact, then \(y \in \varpi(Q)\) iff there is a regular Borel measure \(\mu\) on \(Q\) that \(y = \int_Q \varpi(x) d\mu(x)\).

**Proof:** Cf. [Rudin P79].

**Prop. (X.4.3.24) (Continuous Action Extends to Measure).** For a fixed map \(f : Q \to X\), assume \(X\) is a Fréchet space, then the integration functor in (X.4.3.21) induces a continuous map
\[
Meas_c(Q) \to X : \mu \mapsto \int_\mu f
\]
that maps \(\delta_x\) to \(f(x)\).

**Proof:** It suffices to verify continuity: for any seminorm \(\rho\), by convexity,
\[
\rho(\int_\mu f) \leq (\mu, \rho(f)),
\]
thus for \(\mu \in U\) satisfying \((\mu, \rho(f)) < 1\), \(\rho(\int_\mu f) < 1\). This proves continuity.
Prop. (X.4.3.25) (Vector Valued Integration Stronger). If $V$ is a Banach space and $\mu$ is a Radon measure on a locally compact Hausdorff space $X$. If $g \in L^1(\mu)$ and $H : X \to V$ is bounded and continuous, then $\int gH d\mu$ exists and belongs to the closed linear span of the range of $H$, and
\[
\|\int gH d\mu\| \leq \sup_{x \in X} \|H(x)\| \int |g(x)| d\mu(x)
\]

Proof: Clearly $\varphi(gH) \in L^1(\mu)$ for all $\varphi \in V^\ast$. And $\mu$ is Radon, so there is a sequence $\{g_n\} \in C_c(X)$ that converges to $g$ in $L^1$, so $\int g_n H d\mu$ is integrable by (X.4.3.21), and
\[
\int \|g_n(x)H(x) - g_m(x)H(x)\| dx |d\mu(x) \leq \int |g_n(x) - g_m(x)| d\mu(x) \to 0
\]
thus this is a Cauchy sequence, converging to some $y$. Now for any $\varphi \in V^\ast$,
\[
\varphi(y) = \lim \varphi(\int g_n H d\mu(x)) = \lim \int \varphi (g_n H) d\mu = \int \varphi (gH) d\mu
\]
The last equality uses boundedness again.
Moreover, each $\int g_n H d\mu$ belongs to the closed range of $H$ by (X.4.3.21), hence so does $\int gH d\mu$. And last assertion is also from (X.4.3.21).

Holomorphic Functions

Def. (X.4.3.26) (Holomorphic Functions). Let $\Omega$ be an open set in $\mathbb{C}$, and $X$ be a TVS over $\mathbb{C}$, then A function $f : \Omega \to X$ is called:
- weakly holomorphic if $\Lambda f$ is holomorphic for any $\Lambda \in X^\ast$.
- strongly holomorphic if $\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$ exists for every $z \in \Omega$.
A strongly holomorphic function is clearly weakly holomorphic, and the converse is true when $X$ is Fréchet space, by the following proposition (X.4.3.27).

Prop. (X.4.3.27) (Weak and Strong Holomorphic). Let $\Omega$ be an open set in $\mathbb{C}$, and $X$ be a Fréchet space over $\mathbb{C}$, then $f$ is strongly continuous, and the Cauchy integral formula (X.2.3.4) holds for $f$, and $f$ is strongly holomorphic.

Proof: We may assume $0 \in \Omega$, then Let $B(0, 2r) \subset \Omega$ and $\Gamma$ the boundary of $B(0, 2r)$, since $\Lambda f$ is holomorphic,
\[
\frac{(\Lambda f)(z) - (\Lambda f)(0)}{z} = \frac{1}{2\pi i} \int_{\Gamma} \frac{(\Lambda f)(\zeta)}{(\zeta - z)\zeta} d\zeta \quad 0 < |z| < 2r.
\]
Therefore $\{\frac{f(w) - f(0)}{w - 0} | 0 < |z| \leq r\}$ is weakly bounded, so it is also bounded by (X.4.3.12), so $f$ is strongly continuous.
The integral exists by (X.4.3.21), so $f$ satisfies Cauchy integral formula because it satisfies this when acting with any functional $\Lambda$, and $X^\ast$ separate points.
For the last assertion, Cf. [Rudin P84].

Prop. (X.4.3.28) (Liouville’s Theorem). If $X$ is a TVS over $\mathbb{C}$ on which $X^\ast$ separate points and $f : \mathbb{C} \to X$ is weakly holomorphic and $f(\mathbb{C})$ is weakly bounded, then $f$ is constant.

Proof: Immediate from Liouville’s theorem (X.2.3.9).
4 Duality Theory

Prop. (X.4.4.1) (Banach-Alaoglu). For a nbhd $V$ of 0 in a TVS $X$, the set
$$K = \{f | |fx| \leq 1, \forall x \in V\}$$
is weak*-compact in $X^*$, which is called the polar $V^*$ of $V$.

Proof: Consider the Minkowski function $\gamma$ of $V$, then for each $\Lambda \in K$, $|\Lambda x| \leq \gamma(x)$. If we consider the space $P = \prod_{x \in X}[-\gamma(x), \gamma(x)]$, then $P$ is compact by Tychonoff (IX.1.2.5).

The point is that the weak*-topology coincides with the pointwise convergence topology on $K$, because they have the same generating subbasis. If we show $K$ is weak*-compact, for this, consider any $f_0$ in its closure, then for each $x, y \in X$, $\alpha, \beta \in K$, there is a $f \in K$ that is close to $f_0$ at $x, y$ and $\alpha x + \beta y$. So $f_0$ is linear. Similarly we can show $|f_0(x)| \leq 1$ for $x \in V$, so $f_0 \in K$. \hfill \Box

Prop. (X.4.4.2). If $X$ is a TVS that $X^*$ separate points (e.g. locally convex), then the weak topology $X_w$ is a locally convex space, and $(X_w)^* = X^*$.

Proof: If $A$ is a functional that is continuous in $X_w$-topology, then $|\lambda x| < 1$ for some set defined by elements in $X^*$, so by (X.3.3.3), $A = \sum \lambda_i \Lambda_i$ which is continuous w.r.t the original topology. \hfill \Box

Prop. (X.4.4.3) (Hahn Weak Topology case). If $X$ is a TVS that $X^*$ separate points, then if $A, B$ are disjoint nonempty, compact convex sets in $X$, then there is a $\Lambda \in X^*$ that separate $A$ and $B$, i.e. $\text{Re } f(E_1) < \gamma_1 < \gamma_2 < \text{Re } f(E_2)$.

Proof: Let $X_w$ be $X$ with the weak topology, then the sets $A$ and $B$ are compact in $X_w$ as it’s weaker. And they are also closed because $X_w$ is Hausdorff. $X_w$ is convex, so we can use geometric Hahn (X.4.3.7). Now $(X_w)^* = X^*$, so the chosen functional is also continuous in the original topology. \hfill \Box

Prop. (X.4.4.4) (Dual Banach Space). For a normed space $X$, $x \in X$ can be seen as functional on $X^*$, of norm exactly $\|x\|$. And the closed ball $B^*$ of the dual space $X^*$ is weak*-compact.

Proof: The first assertion is because of (X.4.3.6), the last assertion is because of Banach-Alaoglu (X.4.4.1). \hfill \Box

Prop. (X.4.4.5) (Adjoint Norm). For $X, Y$ normed, the adjoint of $T : X \to Y$ satisfies $\|T^*\| = \|T\|$. 

Proof: Use (X.4.4.4), $\|T\| = \sup\{\|\langle Tx, y^*\rangle\| | \|x\| \leq 1, \|y^*\| \leq 1\} = \|T^*\|$. \hfill \Box

Prop. (X.4.4.6) (Closed Range Theorem). Let $T$ be continuous mapping between Banach spaces $X$ and $Y$, let $U, V$ be open balls of $X, Y$ particularly. then the following are equivalent:
1. $\|T^*y^*\| \geq \delta \|y^*\|$ for some $\delta$.
2. $\delta V \subset T(U)$.
3. $\delta V \subset T(U)$, i.e. $T^{-1}$ is continuous.
4. $T(X) = Y$.
5. $T^*$ is one-to-one and $R(T^*)$ is closed in $X$. 

Proof: 1 → 2: If \( ||T^* y^*|| \geq \delta ||y^*|| \), first prove \( \delta V \subset T(U) \). If \( y_0 \notin T(U) \), since \( T(U) \) is convex closed and balanced, geometric Hahn shows that there is a \( y^* \) that \( |y^*(y)| \leq 1 \) for every \( y \in T(U) \), and \( |y^*(y_0)| > 1 \). Then if follows \( ||T^* y^*|| \leq 1 \). So

\[
\delta < \delta |y^*(y_0)| \leq \delta ||y_0|| |y^*|| \leq ||y_0|| ||T^* y^*|| \leq ||y_0||
\]

This shows \( \delta V \subset T(U) \).

2 → 3: may assume \( \delta = 1 \). Then \( V \subset T(U) \). Then for every \( y \in Y \) and every \( \varepsilon > 0 \), there is a \( x \) that \( ||x|| \leq ||y|| \) and \( |y - Tx|| < \varepsilon \). For any \( y_1 \in V \), pick \( \varepsilon_n > 0 \) that \( \sum \varepsilon_n < 1 - ||y_1|| \), then choose \( ||x_n|| \leq ||y_n|| \) that \( ||y_n - Tx_n|| < \varepsilon_n \), and let \( y_{n+1} = y_n - Tx_n \). Then it is verified that \( x = \sum x_n \in U \) and \( T(x) \).

3 → 1: \( ||T^* y^*|| = \sup \{||\langle x, T^* y^* \rangle|| : x \in U \} \geq \sup \{||\langle y, y^* \rangle|| : y \in V \} = \delta ||y^*|| \).

3 ⇔ 4: By Open mapping theorem.

4 → 5: \( T^* \) is injective by (X.3.3.2). By open mapping theorem, \( T^* \) is a multiple of a dilation, so \( R(T^*) \) is closed by (X.1.8.8).

5 → 4: \( R(T) \) is dense in \( Y \) by (X.3.3.2), and it is closed by the proposition (X.4.4.7) below. □

Prop. (X.4.4.7) (Closed Range Theorem). If \( X, Y \) are Banach spaces and \( T \in L(X, Y) \), the following are equivalent:

1. \( R(T) \) is closed in \( Y \).
2. \( R(T^*) \) is weak*-closed in \( X^* \).
3. \( R(T^*) \) is closed in \( X \).

Proof: 1 → 2: As \( N(T)^\perp \) is the weak*-closure of \( R(T^*) \), it suffices to prove \( N(T)^\perp \subset R(T^*) \). As \( R(T) \) is complete, the open mapping theorem applies to \( X \to R(T) \), showing that each \( y \in R(T) \) corresponds to an element \( x \in X \) that \( Tx = y \) and \( ||x|| \leq K||y|| \).

For \( x^* \in N(T)^* \), define a functional \( \Lambda \) on \( R(T) \) by \( \Lambda Tx = \langle x, x^* \rangle \), this is well-defined, and \( |\Lambda| = \Lambda Tx \leq K||y|| ||x^*|| \). So it is continuous and by Hahn-Banach some continuous functional \( y^* \in Y^* \) extends \( \Lambda \). Then \( \langle Tx, y^* \rangle = \Lambda Tx = \langle x, x^* \rangle \), so \( x^* = T^* y^* \) is in the image of \( T^* \), so we are done.

3 → 1: let \( Z = R(T^*) \). \( RT \) is dense in \( Z \), so (X.3.3.2) shows \( T^* : Z^* \to X^* \) is injective. And for each \( z^* \in Z^* \), there is an extension \( y^* \) by Hahn-Banach, and then \( \langle x, T^* y^* \rangle = \langle Tx, y^* \rangle = \langle Tx, z^* \rangle = \langle x, T^* z^* \rangle \), so \( T^*(Y^*) = T^*(Z^*) \), which is closed by hypothesis.

Now use open mapping theorem for \( Z^* \to R(T^*) \), then there is a \( c \) that \( c ||z^*|| \leq ||T^* z^*|| \). So \( T : X \to Z \) is surjective, by (X.4.4.6). So \( R(T) = Z \) is closed. □

Prop. (X.4.4.8). In a normed space, if \( x_n \to x \) weakly, then \( \lim ||x_n|| \geq ||x|| \).

Proof: Choose a functional that \( ||f|| = 1 \) and \( |f(x)| = 1 \) by (X.4.3.6), then use the definition of weak convergence. □

Prop. (X.4.4.9) (Eberlein-Smulian). For a set \( A \) in a Banach space \( X \), \( A \) is weak*-sequentially compact if its weak precompact.

Proof: ?

We prove here that the case that the closed unit ball of a reflexive Banach space is weak*-self sequentially compact.

To prove this, first we show that a bounded sequence has a subsequence that is weak*-convergent in \( X \). Let \( X_0 = \text{span}\{x_n\} \), then \( X_0 \) is reflexive by (X.4.4.13), and it is separable, so \( X_0^* \) is separable by (X.4.4.11). Then the result follows from (X.4.4.14).

Finally, the weak limit \( x \) is in the closed unit ball, by (X.4.4.8). □
Reflexive and Separable

**Def. (X.4.4.10) (Reflective Banach Space).** If $X$ is a Banach space, there is an isometric immersion of $X$ onto a closed subspace of $X^{**}$ (closed because $X$ is complete). $X$ is called **reflexive** iff $X \cong X^{**}$.

**Prop. (X.4.4.11) (Separability Banach).** For a normed space $X$, if $X^{*}$ is separable, then $X$ is separable.

*Proof:* Choose a countable dense set in $X^{*}$, then their projection to the unit sphere $S^{*}$ \{${g_n}$\} are dense in $S^{*}$ (easily checked), and choose for each of them a $x_n$ that $||x_n|| = 1$ and $g_n(x_n) > \frac{1}{2}$.

Now I claim $x_n$ are dense in $X$, i.e. $X_0 = \text{span}\{x_n\} = X$. If this is not the case, then there is a $||x|| = 1$ not in $X_0$, so by (X.4.3.6), there is a $f$ that $f(X_0) = 0$ and $||f|| = 1$ and $f(x) = 1$. Then $||g_n - f|| = \sup_{||x||=1}{|g_n(x) - f(x)|} \geq |g_n(x_n) - f(x_n)| = |g_n(x_n)| \geq 1/2$, contradicting the fact $g_n$ is dense in $S^{*}$. □

**Prop. (X.4.4.12) (Duality Exact).** If $X$ is a closed subspace of a normed space $Y$, and $Y/X$ is the quotient field, then $(Y/X)^{*}$ is a closed subspace of $Y^{*}$, and $X^{*}$ is the quotient.

*Proof:* $(Y/X)^{*} \rightarrow Y^{*}$ is clearly injective, and the $X^{*}$ are all functionals on $Y$ modulo the functionals that vanish on $X$. □

**Cor. (X.4.4.13) (Pettis).** Closed subspace and quotient space of a reflexive normed space is reflexive.

*Proof:* Use the fact that $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ induces an exact sequence $0 \rightarrow X^{**} \rightarrow Y^{**} \rightarrow Z^{**} \rightarrow 0$, and there is a map $X \rightarrow X^{**}$, so we can use snake lemma (as modules). □

**Prop. (X.4.4.14) (Separable Ball Weak*-Sequentially Compact).** If a normed space $X$ is separable, then the closed unit ball of $X^{*}$ is weak*-sequentially compact.

*Proof:* Let $x_n$ be a countable dense subset of $X$, then by diagonal method, for each bounded sequence of $f_n \in X^{*}$, there is a subsequence $f_{n_k}$ that $f_{n_k}(x_m)$ converges for each $x_m$. Then by (X.3.2.3), $f_{n_k}$ converges to some $f \in X^{*}$. So the theorem is finished. □

**Prop. (X.4.4.15) (Reflexive Ball Weak*-Sequentially Compact).** In a reflexive Banach space $X$, then a set in $X$ is bounded iff it is weak*-sequentially compact.

*Proof:* If it is reflexive, then the unit ball is weak*-compact by Alaoglu, so it is weak*-sequentially compact by Eberlein (X.4.4.9). Conversely, if it is weak*-sequentially compact, then its closure is weak*-compact, thus bounded. □

**Prop. (X.4.4.16).** A closed convex set of a reflexive Banach space attains minimal norm.

*Proof:* By Hahn, a closed convex set is weakly closed. Let $d = \inf\{|||x|||\}$, then if $d \leq ||x_n|| < d + q/n$, then $\{x_n\}$ is bounded, so by (X.4.4.15) it is weak-sequentially compact (X.4.4.9), thus some $x_n \rightarrow x$ weakly, and use (X.4.4.8), $x$ must attains minimal norm $d$. □
5 Compact Operator & Fredholm Operator

Def. (X.4.5.1) (Compact Operators). A compact operator is an operator between Banach spaces that maps bounded set to sequentially compact (equivalently precompact or totally bounded (IX.1.8.5)) set. It is necessarily continuous because the norm function is continuous thus $||Tx||$ is bounded on the unit ball. The set of compact operators between $X, Y$ is denoted by $C(X, Y)$.

Prop. (X.4.5.2) (Examples of Compact Operators).
- Let $X$ be a compact measure space and $Lu(x) = \int_X K(x, y)u(y)dy$ for $K \in C(X \times X)$. This is a compact operator on $C(X)$ by Arzela-Ascali (IX.1.8.6).
- Let $\Omega$ be a $\sigma$-finite measure space and $Lu(x) = \int_\Omega K(x, y)u(y)dy$ for $K(x, y) \in L^2(\Omega \times \Omega)$. This is a compact operator on $L^2(\Omega)$, as it is Hilbert-Schmidt (X.5.5.5) (X.5.5.3).

Prop. (X.4.5.3) (Properties of Compact Operators).
1. For a continuous operator, it has f.d. image iff it is compact and the image is closed.
2. The space of compact operator is a closed subspace of $L(X, Y)$. Thus the limit of f.d. operators is compact.
3. If one of $A$ or $B$ is compact and the other is continuous, then $AB$ is compact, because continuous maps bounded to bounded and compact to compact.

Proof:
1. A finite dimensional space is closed by (X.4.1.8), and a finite dimensional space is Heine-Borel (IX.1.5.1), so it maps closed ball to precompact set, as it is continuous. Conversely, if it is compact and the image is closed, then it is an open map to its image, by open mapping theorem, and the image is locally compact because $T$ is compact, so it has finite dimension (X.4.1.7).
2. $S + T$ is continuous because sum of precompact set is precompact. To show it is closed subspace, Use totally bounded definition, for $T$ is the closure, let $||S - T|| < r$, then if $S(x_i)$ is a $r$-net for $S(B(0, 1))$, then $T(x_i)$ is a $3r$-net for $T(B(0, 1))$.
3. Because continuous function preserves both boundedness and (pre)compactness.

Prop. (X.4.5.4) (Compact and Totally Convergence). Let $x_n \to x$ weakly, if $T$ is compact, then $Tx_n \to Tn$ strongly. The converse is true when $X$ is reflexive. In particular, this applies to Hilbert space.

Proof: Assume the contrary, if $Tx_n$ doesn’t converge to $Tx$, there is a subsequence $x_{n_k}$ that $||Tx_{n_k} - Tx|| \geq \varepsilon_0$. Now $\{x_{n}\}$ is bounded by (X.3.3.5), so by $T$ compact, there is a subsequence $Tx_{n_k} \to z$ strongly. But because $x_{n_k} \to x$ weakly, $Tx_{n_k} \to Tx$ weakly because $T$ is continuous, and thus $z = Tx$.

The converse is by Eberlein (X.4.4.9), because the bounded $x_n$ has a weak convergent subsequence, and it is mapped to convergent sequence by $T$.

Prop. (X.4.5.5). $T$ is compact $\iff$ $T^*$ is compact.

Proof: We need only to show that $T^*y_n^*$ has a uniformly convergent subsequence on the unit sphere, but for this it suffice to prove $y_n^*$ is sequentially compact in $C(T(B(0, 1)))$. And we use Arzela-Ascoli because $\overline{T(B(0, 1))}$ is compact. For the other half, use the double dual space.
Lemma (X.4.5.6). If there is a chain of closed subspaces $M_1 \subset M_2 \subset \ldots$ that $T(M_n) = M_n$ and $(T - \lambda_n I)M_n \subset M_{n-1}$ for some $\lambda_n \in \sigma(A) - B(0, r)$, then $T$ is not compact.

Proof: There are $y_n \in M_n$ that $||y_n|| \leq 1$ and $||y_n - x_n|| \geq 1/2$ for $x \in M_{n-1}$, so if $m < n$, $||Ty_m - Ty_n|| = ||\lambda y_n - (Ty_m - (T - \lambda_n)y_n)|| \geq \frac{|\lambda_n|}{2} \geq \frac{r}{2}$, so $Ty_m$ has no convergent subsequence. □

Lemma (X.4.5.7). If $A$ is compact and $T = 1 - A$, then if $T$ is not injective, then it is not surjective. And for any $r > 0$, $\sigma_p(A) - B(0, r)$ is a finite set.

Proof: We use (X.4.5.6). If $R(T) = X$, then let $M_n = N(T^n)$, then $M_0 \neq 0$ because there is a $T x_0 = 0$, and $M_n \subset M_{n+1}$ because there is a $T^n x_{n+1} = x_0$, so $x_{n+1} \in M_{n+1} - M_n$.

If $\sigma_p(A) - B(0, r)$ is infinite, then choose $M_n$ to be generated by $n$ eigenvectors, then it is clear that a chain like above will be found. □

Lemma (X.4.5.8). If $A$ is compact and $T = 1 - A$, then $R(T)$ is closed.

Proof: it suffices to show $T^{-1} : R(T) \to X/N(T)$ is continuous, if this is not the case, then there is a sequence $||x_n|| = 1$ but $Tx_n \to 0$. But $A$ is compact, so there is a subsequence that $Ax_n \to z$, so $x_n \to z$. So $Tz = 0$ so $z = 0$, but then $x_n \to 0$, contradiction. □

Prop. (X.4.5.9) (Riesz-Fredholm). For a compact operator $A \in L(X)$, let $T = I - A$. Then:

1. $0 \in \sigma(A)$ if $X$ is not f.d.
2. $T$ is Fredholm of index 0 (X.4.5.17). Equivalently, $\sigma(A) \setminus \{0\} = \sigma_p(A) \setminus \{0\}$ (because either $T$ not injective or $T$ is surjective).
3. $\sigma(A)$ has at most one convergent point 0 (it must attain 0 if $X$ is a infinite-dimensional).

Hence it has at most countable spectrum.

Proof: 1: If 0 is a regular value, then $T$ is invertible, thus $T^{-1}T = I$ is compact, thus $X$ has f.d. (X.4.1.7).

2: Firstly dim $N(T) < \infty$. This is because $T|_{N(T)} = I|_{N(T)}$, so it is compact iff it is f.d. (X.4.5.3).

by [Rudin P108]?

3: By (X.4.5.7). □

Prop. (X.4.5.10) (Lomonosov’s Invariant Subspace Theorem). If $X$ is an infinite dimensional complex Banach space, and $T \neq 0$ is a compact operator in $L(X)$, then there is a proper closed subspace $M$ of $X$ that is invariant under $S$ for any $S$ that commutes with $T$.

In particular, if $S$ commutes with some compact operator $T$, then $S$ has an invariant closed subspace.

Proof: If $\Gamma$ is the subspace of all operators that commutes with $T$, then it is a subalgebra of $L(X)$, and for each $y \in X$, let $\Gamma(y) = \{S_y | S \in \Gamma\}$, then $S(\Gamma(y)) \subset \Gamma(y)$, then do the closure of $\Gamma(y)$. So if the proposition is false, then $\Gamma(y)$ is dense in $X$ for each $y$.

Pick $x_0$ that $Tx_0 \neq 0$, then there is an open ball $B$ of $x_0$ that $||Tx|| \geq \frac{1}{2}||Tx_0||$ and $||x|| \geq \frac{1}{2}||x_0||$ for $x \in B$. Now our assumption shows that for every $y \neq 0$, there is a nbhd $W$ of it that maps into $B$ by some $S \in \Gamma$ (notice $\Gamma$ is a subspace).

Now $K = \overline{T(B)}$ is compact because $T$ is compact, so there are f.m. open sets $W_i$ whose union cover $K$, and $S_i(W_i) \in B$, where $S_i \in \Gamma$. Now let $\mu = \max\{||S_i||\}$. Consider $Tx_0 \in K$, so there is a $S_i Tx_0 \in B$, then $TS_i Tx_0 \in K$, so there is a $S_{i_2}TS_i Tx_0 \in B$. Continuing this way, we get $\frac{1}{2}||x_0|| \leq ||x_N|| \leq \mu^N ||T^N|| ||x_0||$. 

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so by Gelfand theorem (X.5.1.8), $\rho(T) > 0$, so there is an eigenvalue $\lambda$ of $T$ (by (X.4.5.9)) that $N(T - \lambda I)$ is finite dimensional, so not equal to $X$, and it is clearly invariant under $\Gamma$. \qed

**Prop. (X.4.5.11) (Jordan Decomposition for Compact Operators).** For a compact operator $A$ and all the non-zero eigenvalues $\lambda_i$, we can a find a subspace

$$\bigoplus_{i=1}^{\infty} N((\lambda_i - A)^p), \quad \lambda_i \neq 0$$

on which $A$ has a Jordan form.

**Proof:** Let $T = 1 - A$. By ??, we only have to prove there are some $m, n$ that there is a $p$ that $N(T^p) = N(T^{p+1})$ and a $q$ that $R(T^q) = R(T^{q+1})$, because then we have a decomposition $X = N((T - \lambda I)^p) \oplus R((T - \lambda I)^q)$, and all these $N((T - \lambda I)^p)$ are pairwisely disjoint.

Now $q < \infty$, because if $R(T) \supset R(T^2) \supset \ldots$, because $T^k$ is of the form $1 + \text{compact operator}$, $R(T^k)$ are all closed by (X.4.5.8), so by (X.4.5.6), this is impossible.

For $p$, use Riesz-Fredholm (X.4.5.9),

$$\dim N(T^p) = \text{codim} R(T^p) = \text{codim} R(T^{q+1}) = \dim N(T^{q+1}) < \infty$$

So $p \leq q < \infty$. \qed

**Prop. (X.4.5.12).** If $X, Y$ are Banach spaces and $T, K \in L(X, Y)$, $K$ is compact and $R(A) \subset R(K)$, then $A$ is compact.

**Proof:** Use (X.3.2.9), then we can lift the function to a map $\bar{T} : X \to X/N(K)$, which is also continuous, so $T = \bar{K} \to \bar{T}$ is also compact. \qed

**Schauder Basis**

**Def. (X.4.5.13) (Schauder Basis).** Let $X$ be a Banach space, a sequence $e_n$ is called a Schauder basis iff for any $x \in X$, there is a unique sequence $C_n(x)$ that $x = \lim \sum_{n=1}^{\infty} C_n(x) e_n$. Notice in this case $X$ is automatically separable.

**Prop. (X.4.5.14).** If $X$ has a Schauder basis, then $C_n(x)$ are continuous functional on $X$.

**Proof:** Consider the module $||x||_1 = \sup ||S_Nx||$. Firstly, it is complete, because $||x|| = \lim ||S_Nx|| \leq ||x||_1$, so if there is a Cauchy sequence $\{x_i\}$ in $|| \cdot ||_1$, then it is a Cauchy sequence in $|| \cdot ||$, then it converges to some $x$. Now $C_N(x) = S_N(x_i) - S_{N-1}(x_i)$ are all Cauchy sequence, uniform in $N$, so they converges to some sequence $e_N$.

It is left to verify that $s_N = \sum_{i=1}^{N} c_n e_n$ converges to $x$, because then it is easy to verify that $\lim ||x_i - x||_1 = 0$. For this, choose $N_1$ large that $||x_n - x|| \leq \varepsilon$ for $n \geq N_1$, and choose $N_2$ large that $||S_k(x_n) - S_k(x)|| \leq \varepsilon$ for all $k$ and $n \geq N_2$. Then for $x_{N_1+N_2}$, there is a $N_3$ that $||S_{N_1+N_2} - x_{N_1+N_2}|| \leq \varepsilon$, so $||x - s_k|| \leq ||x - x_{N_1+N_2}|| + ||S_k x_{N_1+N_2} - x_{N_1+N_2}|| + ||S_k(x_{N_1+N_2}) - s_k|| \leq 3\varepsilon$ for $k$ large.

Now by Banach (X.3.2.6), $||x||_1 \leq M||x||$ for some $M$, so $|C_n(x)e_n| \leq 2M||x||$ and $C_n$ is continuous. \qed

**Prop. (X.4.5.15).** If $X$ has a Schauder basis, then any compact operator is a limit of operators of f.d. range.
Proof: Let $S_N(x) = \sum_{n=1}^{N} C_n(x)e_n$, it is continuous by (X.4.5.14). And it converges, so $\|S_N\| \leq M$, by Banach-Steinhaus (X.3.2.1).

For any compact operator, we want to find f.d. range operator $T_i$ that $T_i \to T$. For this, given any $\varepsilon > 0$, because $\overline{T(B(0,1))}$ is compact, there are operators that is an $\varepsilon/M^2$-net $y_i$, then choose $N$ large enough that $|S_Ny_i - y_i| \leq \varepsilon/M^2$, then for any $x$, there is a $y_i$ that $|Tx_i - y_i| < \varepsilon/M^2$, so $|S_NTx_i - S_Ny_i| < \varepsilon/M$, and then $|S_NTx - Tx_i| < \varepsilon$, and notice $S_N T$ has f.d. range. \hfill\Box

Prop. (X.4.5.16) (Compact Operator as Limits of F.D. Operators). Any compact operator on a Hilbert space is a limit of f.d. operators.

Proof: Cf. [Trace Classes and Hilbert-Schmidt Operators, Thm10]. \hfill\Box

Fredholm Operator

Def. (X.4.5.17) (Fredholm Operator). A bounded operator between Banach space is called a Fredholm operator if $\dim N(T) < \infty$ and $\text{codim} R(T) < \infty$. It necessarily has closed image by (X.3.2.10), so $R(T) = N(T^*)^\perp$ (X.3.3.2).

The index of a Fredholm operator is defined as $\text{ind}(T) = \dim N(T) - \text{codim} R(T)$, thus for a compact operator $A$, $I - A$ has index 0, by (X.4.5.9).

Prop. (X.4.5.18). For a Fredholm operator between Banach space, we have

$$X = N(T) \oplus R(T) \quad Y = Y/R(T) \oplus R(T)$$

and $X/N(T) \cong R(T)$.

Proof: Because $R(T)$ and $N(T)$ are finite/cofinite hence closed and complemented by (X.4.1.9). If $X = N(T) \oplus M_1$ and $Y = R(T) \oplus M_2$, then $M_1 \cong X/N(T)$, $X/N(T) \cong R(T)$ and $M_2 \cong Y/R(T)$ by Banach theorem. \hfill\Box

Prop. (X.4.5.19) (Characterizing Fredholm Operator). $T$ is Fredholm from $X$ to $Y$ iff there exist a bounded $S_1, S_2$ from $X$ to $Y$ that $S_1T = I - A_1, TS_2 = I - A_2$, where $A_1, A_2$ is compact. If this is the case, $S_1$ and $S_2$ can be chosen the same as $S$, then $S$ is called the regulator of $T$, and $S$ is Fredholm as well.

So the Fredholm operator is the set of operators invertible 'modulo compact ones'.

Proof: By (X.4.5.18) $T : X/N(T) \cong R(T)$, and there is a projection of $\pi : Y \to R(T)$. Thus we composed them to get a $S = T^{-1} \circ \pi : Y \to X$. And $ST$ and $TS$ are both 1 minus a projection with f.d. image, hence compact (X.4.5.3).

For the converse, $R(T) \supset R(1 - A_2)$ is of finite codimension because $1 - A_2$ is Fredholm, and $N(T) \subset N(1 - A_1)$ is of finite dimension because $1 - A_1$ is Fredholm. \hfill\Box

Cor. (X.4.5.20). The set of Fredholm operators is closed under composition. Index is a locally constant function on it, and $\text{ind}(T_1 T_2) = \text{ind}(T_1) + \text{Ind}(T_2)$.

Proof: There is a long exact sequence (use (I.11.4.6) in the category of vector spaces)

$$0 \to \text{Ker} T_2 \to \text{Ker} T_1 T_2 \to \text{Ker} T_2 \to \text{Coker} T_2 \to \text{Coker} T_1 T_2 \to \text{Coker} T_1 \to 0.$$ 

which shows the composition and index is additive.
For the openness and locally constancy, use (X.4.5.19), when adding a small $R$, $S(T + R) = 1 - A_1 + SR$, and if when $||R|| < ||S||^{-1}$, $E_1 = (I + RS)^{-1}$ is bounded, so $E_1S(T + R) = I - E_1A_1$, and similarly does $(T + R)SE_2$, so $T + R$ is Fredholm. And $\text{ind } E_1 + \text{ind } S + \text{ind } (T + R) = \text{ind } (1 - E_1A_1) = 0$, and $\text{ind } E_1 = 0$ because it is invertible, and $\text{ind } S + \text{ind } T = \text{ind } (1 - A_1) = 0$, so $\text{ind } T = \text{ind } (T + R)$.

\textbf{Cor. (X.4.5.21).} If $T$ is Fredholm and $A$ is compact, then $T + A$ is Fredholm, and $\text{ind } (T + A) = \text{ind } T$, so $\text{ind}$ is in fact defined on the quotient of $L(X,Y)$ by compact operators.

\textit{Proof:} It is Fredholm by (X.4.5.19), and we notice $S(T + A)$ and $ST$ are both 1 minus compact operators, thus (X.4.5.20) and (X.4.5.9) gives the result. □

\textbf{Cor. (X.4.5.22).} If $T$ is Fredholm, then $T^*$ is Fredholm, and $\text{ind } (T^*) = - \text{ind } (T)$.

\textit{Proof:} The first follows from (X.4.5.19) and (X.4.5.5). For the second, use the fact $R(T)$ and $N(T)$ are all closed. □

6 Unbounded Operators
X.5 Archimedean Banach Algebra

1 Banach Algebra

Def. (X.5.1.1) (Spectrum). For a bounded operator $A \in L(X)$ where $X$ is Banach space, a $\lambda \in \mathbb{C}$ is called a:

- **point spectrum** if $\lambda I - A$ is not injective;
- **continuous spectrum** if it is not a point spectrum and $R(\lambda I - A) \neq X$ but $\overline{R(\lambda I - A)} = X$.
- **residue point** if it is not a point spectrum and $\overline{R(\lambda I - A)} \neq X$.
- **regular point** if $\lambda I - A$ is injective and $R(\lambda I - A) = X$, in which case $(\lambda I - X)^{-1}$ is continuous, by Banach.

Denote $\sigma(A) = K - \text{regular points of } A$ the **spectrum** of $A$, and $\rho(A) = \sup\{|\lambda||\lambda \in \sigma A\}$ is called the **spectral radius** of $A$.

Prop. (X.5.1.2). If $A$ is a Banach algebra and $x$ is invertible in $A$, and $h \in A$ satisfies $||h|| < \frac{1}{2}||x^{-1}||^{-1}$, then $x + h$ is also invertible and

$$|| (x + h)^{-1} - x^{-1} + x^{-1}hx^{-1} || \leq 2||x^{-1}||^3 ||h||^2$$

Proof: $x + h = xe + x^{-1}h$ and $||x^{-1}h|| < \frac{1}{2}$, so $x + h$ is invertible by (X.5.1.14), and $||(x + h)^{-1} - x^{-1} + x^{-1}hx^{-1}|| = ||((e + x^{-1}h)^{-1} - e + x^{-1}h)x^{-1}|| \leq 2||x^{-1}||^3 ||h||^2$ also by (X.5.1.14).

□

Cor. (X.5.1.3). If $A$ is a Banach algebra, then the invertible elements $G(A)$ is an open subset of $A$, and the mapping $x \mapsto x^{-1}$ is a homeomorphism of $G(A)$ onto $G(A)$.

Prop. (X.5.1.4). For $T \in L(X)$ where $X$ is Banach space, $\mathbb{C}\setminus\sigma(T)$ is an open set and $\lambda \mapsto (\lambda I - T)^{-1}$ is a holomorphic function on $\mathbb{C}\setminus\sigma(T)$.

Thus for every bounded operator $T$ on a Banach space, $\sigma(T)$ is not empty.

Proof: The first assertion is by (X.5.1.3), for the second, let $f(\lambda) = (\lambda e - x)^{-1}$ is defined on $\Omega = \mathbb{C} - \sigma(x)$ and (X.5.1.2) shows

$$||f(\mu) - f(\lambda) + (\mu - \lambda)f^2(\lambda)|| \leq 2||f(\lambda)||^3 |\mu - \lambda|^2$$

so $\lim_{\mu \to \lambda} \frac{f(\mu) - f(\lambda)}{\mu - \lambda} = -f^2(\lambda)$, which means that $f$ is strongly holomorphic in $\Omega$.

Now if $|\lambda| > ||x||$, then $|f(\lambda)| = |\lambda^{-1}e + \lambda^{-2}x + \ldots| \leq \frac{1}{|\lambda| - ||x||}$, so $\sigma(x)$ cannot be empty by Liouville theorem (X.4.3.28).

□

Cor. (X.5.1.5) (Gelfand-Mazur). If in a Banach algebra $A$ over $\mathbb{C}$, all the nonzero element is invertible, then it is isomorphic to $\mathbb{C}$.

Proof: Any nonzero element $x$ has a nonempty spectrum, so there is a $\lambda(x)$ that $x - \lambda(x)e$ is not invertible, so it must be 0. That is, the mapping $\mathbb{C} \to A : \lambda \mapsto \lambda e$ is bijective, so is isomorphism by Banach.

□

Prop. (X.5.1.6). Notice $(I - T)$ is invertible for $||T|| < 1$ and the inverse can be calculated by definition.

In particular, for a Banach algebra $A$ and any $x \in A$, when $\lambda > ||x||$, $e - \lambda^{-1}x$ is invertible, so the spectrum of $x$ is bounded. Now that its complement is open as the inverse image of $G(A)$ by $\lambda \mapsto \lambda e - x$, so the spectrum of $x$ is compact.
Cor. (X.5.1.7) (Spectrum is Continuous). The spectrum of an element of a Banach algebra is continuous, i.e. if $\sigma(x) \subset \Omega$ for some open subset $\Omega \subset \mathbb{C}$, then there is a $\delta > 0$ that $\sigma(x+y) \subset \Omega$ for $|y| < \delta$.

Proof: As $||(\lambda e - x)^{-1}||$ is a continuous function of $\lambda$ in the complement of $\sigma$, and since the norm tends to 0 as $\lambda \to \infty$, there is a $M$ that $||(\lambda e - x)^{-1}|| < M$ for all $\lambda \notin \Omega$. Now if $|y| < 1/M$ and $\lambda \notin \Omega$, then $\lambda e - (x+y) = (\lambda e - x)(e - (\lambda e - x)^{-1}y)$ is invertible. □

Prop. (X.5.1.8) (Gelfand). The spectrum radius $\rho(A) = \lim_{n \to \infty} ||A^n||^{1/n} = \inf ||A^n||^{1/n} \leq ||A||$.

This formula is remarkable, as the LHS depends only on the algebraic structure, and the RHS depends on the metric structure.

Proof: For $r > \rho(x)$

$$x^n = \frac{1}{2\pi i} \int_{\Gamma_r} \lambda^n f(\lambda)d\lambda.$$ 

Let $M(r) = \max ||f(re^{i\theta})||$, then $||x^n|| \leq r^{n+1}M(r)$, hence $\limsup ||x^n||^{1/n} \leq r$, so $\limsup ||x^n||^{1/n} = \rho(x)$.

For the converse, if $n \in \sigma(x)$, then $\lambda^n e - x^n = (\lambda e - x)(\lambda^{n-1}e + \ldots + x^{n-1})$ and this two commutes. So $|\lambda^n| \leq ||x^n||$, so $\rho(x) \leq \inf ||x^n||^{1/n}$. □

Prop. (X.5.1.9). $\sigma(A) = \sigma(A^*)$.

Proof: It suffices to show if $T$ is invertible iff $T^*$ is invertible. If $T$ is invertible, then $T^*$ is invertible with inverse $(T^{-1})^*$. Conversely, if $T^*$ is invertible, then $T^{**}$ is invertible, so, as the restriction of $T^{**}$, $T$ is injective and image is closed. If the image is not $X$, then there is a $f$ that vanish on the image, so $T^*f = 0$, but then $f = 0$. □

Prop. (X.5.1.10). In a Banach algebra $A$, $e - xy$ is invertible iff $e - yx$ is invertible, thus $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$.

Proof: Let $z = (e - xy)^{-1}$, then we claim $e + yxz$ is just the inverse of $e - yx$: $(e - yx)(e + yxz) = e - yx + yxz - yxycz = e$ and $(e + yxz)(e - yx) = e + yxz - yx - yxycz = e$. □

Lemma (X.5.1.11). If $A$ is a Banach algebra and $x_n \in G(A)$ converges to $x \notin G(A)$, then $||x_n^{-1}|| \to \infty$.

Proof: If $||x_n^{-1}|| < M$, choose $n$ that $||x_n - x|| < 1/M$, then $||e - x_n^{-1}|| = ||x_n^{-1}(x_n - x)|| < 1$, so $x_n^{-1}x$ is invertible, so $x$ is invertible. □

Prop. (X.5.1.12). For Banach algebra $B$ and its closed subalgebra $A$, $\sigma_A(x)$ is obtained from $\sigma_B(x)$ by filling some holes. So when $\sigma_B(x)$ doesn't separate $\mathbb{C}$ or $\sigma(A)$ has empty interior, then $\sigma_A(x) = \sigma_B(x)$.

Proof: Cf.[Rudin P256]. □

Prop. (X.5.1.13). if $A$ is a Banach algebra over $\mathbb{C}$ that $||x|| ||y|| \leq M ||xy||$ for some fixed $M$, then $A$ is isomorphic to $\mathbb{C}$.

Proof: If $y$ is a boundary pt of $G(A)$, $y = \lim y_n$, then $||y_n^{-1}|| \to \infty$. But $||y_n|| ||y_n^{-1}|| \leq M ||e||$, so $y_n \to 0$, so $y = 0$.

But any boundary point of $\sigma(x)$ gives a boundary point $\lambda e - x$ of $G(A)$, so $x = \lambda e$, so $A \cong \mathbb{C}$. □
Complex Homomorphism

**Prop. (X.5.1.14).** Suppose $A$ is a Banach algebra over $\mathbb{C}$, $x \in A$ satisfies $||x|| < 1$, then $||(e - x)^{-1} - e - x|| \leq \frac{||x||^2}{1 - ||x||}$, and $|\varphi(x)| < 1$ for any complex homomorphism $\varphi$ on $A$. In particular, any complex homomorphism is continuous.

**Proof:** $||(e - x)^{-1} - e - x|| = ||x^2 + x^3 + \ldots|| \leq \sum_{n=2}^{\infty} ||x||^n = \frac{||x||^2}{1 - ||x||}$.

For the second, notice $e - \lambda^{-1}x$ is invertible for each $|\lambda| \geq 1$, so $1 - \lambda \varphi(x) \neq 0$, so $\varphi(x) \neq \lambda$. □

**Prop. (X.5.1.15) (Gleason-Kahane-Zelazko).** If $\varphi$ is a linear functional on a Banach algebra $A$ over $\mathbb{C}$, if $\varphi(e) = 1$ and $\varphi(x) \neq 0$ for every invertible element $x \in A$, then $\varphi$ is a complex homomorphism.

**Proof:** Cf.[Rudin P251]. □

Symbolic Calculus

**Prop. (X.5.1.16) (Symbolic Calculus).** For a Banach algebra $A$. For a domain $\Omega$ in $\mathbb{C}$, define $A_{\Omega}$ as the set of $x$ that $\sigma(x) \in \Omega$, it is an open set by (X.5.1.7), then:

$$ f \mapsto \tilde{f}(x) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda e - x)^{-1} d\lambda $$

for any contour $\Gamma$ that surrounds $\sigma(x)$, is a continuous algebra isomorphism of $H(\Omega)$ into the set of $A$-valued functions on $A_{\Omega}$ with the compact-open topology.

We have $\widehat{g \circ f} = \tilde{g} \circ \tilde{f}$.

**Proof:** The nontrivial part is that this map is multiplicative, but for this we can use Runge’s theorem to approximate any function on $\sigma(x)$. This theorem makes it possible to implant complex analysis to the study of Banach Algebra.

**Cor. (X.5.1.17).** $\exp(x)$ is defined on $A$ and is continuous. If $\sigma(x)$ doesn’t separate 0 from $\infty$, then $\log(x)$ is defined but might not be continuous.

**Prop. (X.5.1.18) (Spectral Mapping Theorem).** $\tilde{f}(x)$ is invertible in $A$ iff $f(\lambda) \neq 0$ on $\sigma(x)$. Thus we have $\sigma(\tilde{f}(x)) = f(\sigma(x))$.

**Prop. (X.5.1.19).** If $f$ doesn’t vanish identically on any component of $\Omega$, then $f(\sigma_p(T)) = \sigma_p(\tilde{f}(T))$. Cf.[Rudin P266].

Commutative Banach Algebra

**Lemma (X.5.1.20).** For $A$ a commutative Banach algebra, the set of maximal ideals has codimension 1 corresponds to kernels of complex homomorphisms to $\mathbb{C}$. (Consider the quotient space and use Gelfand-Mazur). Note that a complex homomorphism is all continuous because $\lambda e - x$ maps to nonzero.

$\lambda \in \sigma(x)$ iff there is a complex homomorphism $h$ s.t. $h(x) = \lambda$. (Because $x$ is invertible iff it is not contained in any proper ideal of $A$.

**Proof:**
Prop. (X.5.1.21) (Gelfand Transform). The spectrum $\Delta_A$ of a unital commutative Banach algebra $A$ is defined to be the set $\Delta$ of maximal ideals of $A$. It is a locally compact Hausdorff space w.r.t to the weak*-topology and the Gelfand transform: $x \mapsto \hat{x}(h) = h(x)$ is a continuous map of $A$ into $C(\Delta)$. And the range of $\hat{x}$ equals $\sigma(x)$, so $||\hat{x}|| = \rho(x) \leq ||x||$.

Proof: First we prove it is compact Hausdorff: As $\sigma(A) = \{h \in \text{closed ball of } A^*|h(x) = 1, h(xy) = h(x)h(y)\}$ which is a closed subset of the closed ball of $A^*$, so it is compact Hausdorff. The rest is clear and follows from (X.5.1.20).

Prop. (X.5.1.22). For $A = C(X)$ where $X$ is compact Hausdorff, $\Delta$ is homeomorphic to $X$. (otherwise it has finite $f_i \neq 0$, then $\sum |f_i|^2$ is positive thus invertible but maps to 0). In fact, for a space $X$, $\Delta(C(X))$ is the stone-Čech compactification of $X$.

Prop. (X.5.1.23). For $A = L^\infty(m)$, the spectrum of $f$ is just the essential range of $f$.

Lemma (X.5.1.24). If $\hat{A} \subset C(\Delta)$ with a chosen topology that makes it compact, and $A$ separate points, then the topology of it is the same of the weak*-topology. (Compact to Hausdorff).

Prop. (X.5.1.25). The algebra $L^1(\mathbb{R}^n) \oplus \delta$ with the multiplication by convolution has the spectrum $\mathbb{R}^n \cap \{0\}$. (Use $(L^p)^* = L^q$ and see when will it be homomorphism).

2 Hilbert spaces

Prop. (X.5.2.1) (Optimal Approximation). A closed convex subset in a Hilbert space has a unique element that attains the minimum norm.

Proof: Assume $0 \notin C$, so let $d = \inf_{z \in C} ||z|| > 0$, then there are $x_n$ that $d \leq ||x_n|| \leq d + 1/n$. It suffices to show that $x_n$ is a Cauchy sequence, because then it has a convergent point in $C$. Now

$$||x_n - x_m||^2 = 2(||x_n||^2 + ||x_m||^2) - 4||\frac{x_n + x_m}{2}||^2 \leq 2[(d + 1/n)^2 + (d + 1/m)^2] - 4d^2 \to 0.$$ 

For the unicity, if $||x_1|| = ||x_2|| = d$, then

$$||x_1 - x_2||^2 = 2(||x_1||^2 + ||x_2||^2) - 4||\frac{x_1 + x_2}{2}||^2 \leq 4d^2 - 4d^2 = 0.$$ 

Cor. (X.5.2.2) (Orthogonal Decomposition). The orthogonal complement of a closed subspace of a Hilbert space exists. and the projection on to a closed subspace exists. This is a good trait of Hilbert space.

Proof: For any element $x$, let $y$ be the optimal approximation (X.5.2.1) of $x$, then $z = x - y$ is orthogonal to $y$.

Prop. (X.5.2.3) (Riesz). Linear functionals on a Hilbert space over $\mathbb{C}$ are all of the form $x \mapsto (x, z)$ (Choose an orthogonal of the kernel). In other words, Hilbert spaces are reflexive.

Proof: Choose a $x_0$ orthogonal to $N(f)$ by (X.5.2.2) and $||x_0|| = 1$, then any $x = \alpha x_0 + y$ where $y \in N(f)$. Inner product with $x_0$, we get $\alpha = (x, x_0)$, so $f(x) = \alpha f(x_0) = (x, f(x_0)x_0)$.

Cor. (X.5.2.4). For Hilbert spaces $\mathcal{H},\mathcal{K}$ and $T \in \mathcal{L}(\mathcal{H},\mathcal{K})$, $||T|| = \sup\{(Tx, y)||x|| \leq 1, ||y|| \leq 1\}$.
Proof: Use (X.4.3.6) to find for each \( x \) a functional \( f \) of norm 1 that \( |f(Tx)| = ||Tx|| \), then use Riesz theorem. In particular, if we define

**Cor. (X.5.2.5) (Reproducing Kernel).** For a Hilbert space \( H \), if elements of \( H \) are all complex valued functions on a set \( S \), and \( J_x : f \mapsto f(x) \) is continuous functional for \( H \), then there is a function \( K(x,y) \) on \( S \times S \) that \( K_y(x) = K(x,y) \in H \), and \( f(y) = (f,K_y)_H \), called the reproducing kernel.

And if \( e_\alpha \) is a basis for \( H \), then \( K(x,y) = \sum e_\alpha(x)e_\alpha(y) \).

Proof: For any \( y \), there is a \( K_y \in H \) that \( f(y) = (f,K_y)_H \) by Riesz representation. If we let \( K(x,y) = (K_y,K_x) = K_y(x) \), then this is the desired kernel.

If \( e_\alpha \) is a basis, then \( K_x = (K_x,e_\alpha)e_\alpha = e_n(x)e_\alpha \), so by Parseval equality, \( K(x,y) = \sum e_\alpha(x)e_\alpha(y) \).

**Prop. (X.5.2.6).** Let \( H \) be a Hilbert space, then a sequence \( x_n \) converges to \( x \) iff \( x_n \) converges to 
\( x \) weakly and \( ||x||_n \to ||x|| \).

Proof: One direction is trivial, for the other, notice that \( ||x_n - x||^2 = ||x_n||^2 + ||x||^2 - 2\text{Re}(x,x_n) \) which converges to 0.

**Prop. (X.5.2.7) (Lax-Milgram Theorem).** If \( a(x,y) \) is a sesquilinear form on a Hilbert space \( H \) over \( \mathbb{C} \) that \( |a(x,y)| \leq M||x||||y|| \), then there is a unique continuous operator \( A \in L(H) \) that \( a(x,y) = (x, Ay) \). If moreover \( |a(x,x)| \geq \delta||x||^2 \), then \( A \) is bijective and \( ||A^{-1}|| \leq \frac{1}{\delta} \).

Proof: For any \( y, x \mapsto a(x,y) \) is a continuous functional, so by Riesz theorem (X.5.2.3), there is an element \( Ay \) that \( a(x,y) = (x, Ay) \).

Now \( Ay \) depends linearly on \( y \), and \( ||Ay|| = \sup||a(x,y)||/||x|| \leq M||y|| \).

If \( |a(x,x)| \geq \delta||x||^2 \), then \( A \) is clearly injective, and \( R(A) \) is closed, because for any \( z = \lim Av_n \), it is easily verified that \( v_n \) is a Cauchy sequence. And \( R(A)^\perp = 0 \), because if \( (w, Av) = 0 \) for any \( v \in H \), then \( \delta||w||^2 \leq |a(w,w)| = 0 \). \( A^{-1} \) exists by Banach theorem (X.3.2.5), and \( \delta||x||^2 \leq |a(x,x)| = (x, Ax) \leq ||x||||Ax|| \), so \( \delta||x|| \leq ||Ax|| \).

**Cor. (X.5.2.8) (Variational Inequality).** If \( H \) is a Hilbert space that \( a(x,y) \) is an anti-symmetric bilinear function that \( \delta||x||^2 \leq a(x,x) \leq M||x||^2 \), then if \( u_0 \in H \), and \( C \) is a closed convex subset of \( X \), the function
\[
f : x \mapsto a(x,x) - \text{Re}(u_0, x)
\]
attains minimum at \( C \).

Proof: Similar to the proof of (X.5.2.1). \( f(x) \geq \delta||x||^2 - ||u_0||||x|| \) is bounded below on \( C \). If \( x_n \) is a sequence that converge to the infimum \( d \), then
\[
a(x_n - x_m, x_n - x_m) = 2(a(x_n,x_n) + a(x_m,x_m)) - 4a(x_n + x_m, x_n + x_m) = 2(f(x_n) + f(x_m)) - 4f(\frac{x_n + x_m}{2}) \leq 2[(d + 1/n)^2 + (d + 1/m)^2] - 4d^2 \to 0.
\]

So \( x_n \) is a Cauchy sequence by the condition, and it contains a unique minimum.

**Cor. (X.5.2.9) (Involutions).** For a Hilbert space \( H \) over \( \mathbb{C} \), for any \( T \in L(H) \), there is an operator \( T^* \in L(H) \) that \( (Tx, y) = (x, T^*y) \), which is called the formal adjoint or involution of \( T \). Notice it is defined on \( H \), not on \( H^* \).

Moreover, \( ||T|| = ||T^*|| = ||T^*T||^{1/2} \).
Proof: Use Lax-Milgram for $a(x, y) = (Tx, y)$. For the last assertion, $|||T|| = ||T^*||$ by (X.5.2.4). And we notice

$$||T_x||^2 = (Tx, Tx) = (T^*Tx, x) \leq ||T^*T|| ||x||^2,$$

so $||T|| \leq ||T^*T||^{1/2}$. \qed

**Remark (X.5.2.10) (Examples).** The dual operator of the integral operators (X.4.5.2) with kernel $K(x, y)$ is also an integral operator with kernel $K^*(x, y) = K(y, x)$. This follows from Fubini-Tonelli theorem.

3 $B^*$-algebra

**Def. (X.5.3.1).** A $B^*$-algebra is a Banach algebra with an involution s.t. $||xx^*|| = ||x||^2$.

Any $B^*$-algebra is isomorphic to a closed subspace of $B(H)$ for some Hilbert space.

**Proof:** Cf. [Rudin P338]. \qed

**Prop. (X.5.3.2).** For a Hilbert space, the adjoint operation serves as an involution and makes $B(H)$ into a $B^*$-algebra by (X.5.2.9).

**Prop. (X.5.3.3) (Gelfand-Naimark).** For a commutative $B^*$-algebra, the Gelfand transform $x \mapsto \hat{x}$ is an isomorphism from $A$ to $C(\Delta)$ with $||x|| = ||\hat{x}||_{\infty}$ and $\hat{x^*} = \overline{x}$.

**Proof:** First use $||xx^*|| = ||x||^2$ to prove that a Hermitian element is mapped to real function, and use Stone-Weierstrass to show that the image is dense, then let $y = xx^*$ and $||y^{2^n}|| = ||y||^{2^n}$ to prove $||\hat{x}|| = ||x||$, so its image is closed. \qed

**Cor. (X.5.3.4).** If $A$ is a commutative $B^*$-algebra that contains an element $x$ s.t. polynomials of $x, x^*$ are dense in $A$, then $\hat{x}$ is an isomorphism from $\Delta_A$ to $\sigma(x)$, in particular, the Gelfand transform is an isomorphism from $C(\sigma(x))$ to $A$.

**Proof:** Cf. [Rudin P290]. \qed

Now we want to apply commutative algebra methods in the non-commutative case, there are two ways.

**Prop. (X.5.3.5).** For a commutative set of elements $S$ in $A$, its bicommutant (X.5.3.13) $B = \Gamma(\Gamma(S))$ is commutative, closed and contains $S$. And $\sigma_B(x) = \sigma_A(x)$ for $x \in B$.

**Proof:** Because $S \subset \Gamma(S)$, $\Gamma(\Gamma(S)) \subset \Gamma(S)$, thus $\Gamma(\Gamma(S))$ is commutative. And if $xy = yx$, then $x^{-1}y = yx^{-1}$, so the inverse, if exists, are in $B$. \qed

**Cor. (X.5.3.6).** In a Banach algebra, if $x, y$ commutes, then

$$\sigma(x + y) \subset \sigma(x) + \sigma(y), \quad \sigma(xy) \subset \sigma(x)\sigma(y).$$

**Proof:** Because $\sigma(x)$ is just the range of $\hat{x}$ on $\Delta_A$ where $A = \Gamma(\Gamma(\{x, y\}))$ (X.5.3.5)(X.5.1.21). \qed

The second method applies to normal elements:

**Def. (X.5.3.7) (Normal).** In a Banach algebra with an involution, a set $S$ is called normal if it is commutative and $S^* = S$. An element $x$ is called:
• **normal** iff \( x \) commutes with \( x^* \).
• **unitary** iff \( x^* = x^{-1} \).
• **Hermitian** iff \( x^* = x \).
• **positive** iff \( x = x^* \) and \( \sigma(x) \subseteq [0, \infty) \).

**Prop. (X.5.3.8).** A maximal normal set \( B \) in \( A \) is a closed subalgebra and \( \sigma_B(x) = \sigma_A(x) \) for \( x \in B \).

**Proof:** Cf.[Rudin P294]. \( \square \)

**Cor. (X.5.3.9) (Normalness and Spectra).** In a \( B^* \)-algebra \( A \),
• Hermitian elements have real spectra.
• If \( x \) is normal, then \( \rho(x) = |x| \).
• If \( u, v \geq 0 \), then \( u + v \geq 0 \).
• \( yy^* \geq 0 \). Thus \( e + yy^* \) is invertible.

**Proof:** Cf.[Rudin P295]. \( \square \)

**Prop. (X.5.3.10).** In a Banach algebra with an involution, a **positive functional** is such that \( F(xx^*) \geq 0 \). It has the following properties.
• \( F(x^*) = \overline{F(x)} \) and \( |F(xy^*)|^2 \leq F(xx^*)F(yy^*) \). (Use Swartz like trick).
• \( |F(x)|^2 \leq F(e^*F(xx^*) \leq F(e)^2\rho(xx^*) \), because \( e = ee^* \). Thus \( |F(x)| \leq F(e)\rho(x) \) for every normal \( x \) by (X.5.3.9), so \( ||F|| = F(e) \) if \( A \) is commutative.

**Proof:** Cf.[Rudin P297]. \( \square \)

**Prop. (X.5.3.11) (Positive Functional and Measure).** If \( A \) is a commutative Banach algebra with an involution that \( h(x^*) = \overline{h(x)} \), then The map

\[
\mu \rightarrow F(x) = \int_{\Delta} \hat{x} d\mu
\]

is a one-to-one correspondence between the convex set of measures \( \mu \) that \( \mu(\Delta) \leq 1 \) to the convex set \( K \) of positive functionals on \( A \) of norm \( \leq 1 \), i.e. \( F(e) \leq 1 \), so maps the extreme points, i.e. the point mass to extremes points, thus the extreme points of \( K \) is exactly \( \Delta \). This can be used to prove Bochner’s theorem??.

**Proof:** Use the last prop to show that there is a functional on \( C(\Delta) \) and use Riesz representation. It is positive and by Stone-Weierstrass, it is unique. \( \square \)

**von Neumann Algebras**

**Def. (X.5.3.12) (von Neumann Algebra).** A von Neumann Algebra is a \( B^* \)-algebra of operators in \( L(\mathcal{H}) \) that contains the identity and is closed in the weak operator topology (X.3.3.4).

**Def. (X.5.3.13) (Bicommutant).** If \( S \) is a subset of \( L(\mathcal{H}) \), then we define the commutator \( \Gamma(S) \) be the algebra of operators that commutes with \( S \in S \), then \( \Gamma(\Gamma(S)) \) contains \( S \), it is called the bicommutant of \( SS \).

If \( A \) is a \( * \)-algebra in \( L(\mathcal{H}) \), then \( \Gamma(\Gamma(A)) \) is a von-Neumann algebra.
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Proof: If \( \mathcal{A} \) is a \( * \)-algebra, then \( \Gamma(\Gamma(\mathcal{A})) \) is clearly a \( * \)-algebra, and it is weak-closed because if \( SV_i = V_i S \) and \( V_i x \to V x \) for any \( x \), then \( SV x = V S x \) for any \( x \). □

Prop. (X.5.3.14) (von Neumann Density Theorem). Let \( \mathcal{A} \) be a non-degenerate \( * \)-subalgebra of \( L(H) \), then \( \mathcal{A} \) is dense in \( \Gamma(\Gamma(\mathcal{A})) \) in the strong operator topology.

Proof: For any \( S \in \Gamma(\Gamma(\mathcal{A})) \) and any \( x_1, \ldots, x_N \in H, \varepsilon > 0 \), we need to prove there exists \( A \in \mathcal{A} \) that
\[
\sum ||Sx_i - Ax_i|| \leq \varepsilon/2.
\]
For the case \( N = 1 \) and \( x_1 = x \), consider the closure \( X \) of \( \{Ax\} \), then the orthogonal projection \( P \) onto \( X \) is an operator in \( \Gamma(\mathcal{A}) \). This implies \( A(1 - P)x = (1 - P)Ax = 0 \), thus \( x = Px \) because of non-degeneracy. Then because \( S \) commutes with \( P \), \( Sx = SPx = PSx \in X \), thus there exists an \( A \in \mathcal{A} \) that \( Ax \) is close to \( Sx \).

For \( N > 1 \), we can just apply the result to \( H^N \). □

4 Spectral Theory on Hilbert Spaces

The most useful tool is the general symbolic calculus for normal operators.

Resolution of Identity

Def. (X.5.4.1). A resolution of identity on a Hilbert space \( H \) for a \( \sigma \)-algebra on a set \( \Omega \) is a \( E \) that:
1. \( E(\emptyset) = 0, E(\Omega) = 1 \).
2. \( E(\omega) \) is self-adjoint projection.
3. \( E(\omega \cap \omega') = E(\omega')E(\omega) \).
4. \( E(\omega \cup \omega') = E(\omega) + E(\omega') \) for disjoint \( \omega, \omega' \).
5. \( E_{x,y}(\omega) = (E(\omega)x, y) \) is a complex measure on \( E \).

Thus for any \( x, \omega \to E(\omega)x \) is a countably additive \( H \)-valued measure.

This will generate an isometric*-isomorphism \( \Psi \) of the Banach algebra \( L^\infty(E) \) onto a closed normal subalgebra \( \mathcal{A} \) of \( B(H) \). (Define on simple function first).

\[
\Psi(f) = \int_{\Omega} f dE, \quad (\Psi(f)x,y) = \int_{\Omega} f dE_{x,y}
\]

Proof: Cf.[Rudin P319]. □

Prop. (X.5.4.2) (Spectral Decomposition for Normal Algebra). For any closed normal algebra \( \mathcal{A} \) of \( B(H) \), there is a unique resolution \( E \) of identity on the Borel subsets of \( \Delta_A \) that the inverse of Gelfand transform extends to an isometric *-isomorphism \( \Phi \) of the algebra \( L^\infty(E) \) to a closed subalgebra \( B \) containing \( \mathcal{A} \).

In fact, \( B = \Gamma(\Gamma(\mathcal{A})) \) is normal by Fuglede theorem (X.5.4.10).

Proof: Cf.[Rudin P322]. □

Cor. (X.5.4.3) (Generalized Symbolic Calculus for Normal Operator). For a normal operator \( T \) and the minimal closed commutative \( B^* \)-algebra \( \mathcal{A} \) it generates, then the inverse of Gelfand transform gets us a map \( \Psi : C(\sigma(x)) \to \mathcal{A} \) that \( \Psi(z) = x, \Psi(\overline{z}) = x^* \), by (X.5.3.4).

Then the above proposition says there is a resolution of identity on the Borel set of \( \sigma(T) \) that \( \Psi \) extends to a function that maps \( L^\infty(m) \) to \( B(H) \) and \( ||\Psi(f)|| = ||f||_\infty \).
Cor. (X.5.4.4) (Normal and Invariant Subspace). Any closed normal algebra $A$ has many invariant subspaces, just choose a decomposition of Borel sets $\Delta_A = \omega \bigcup \omega'$, then $R(E(\omega)) \oplus R(E(\omega')) = H$.

In particular, any normal operator has an invariant subspace.

Normal Operators on Hilbert Space

Lemma (X.5.4.5). For a Hilbert space $H$ and $T \in L(H)$, $T$ is defined by values $(Tx,x)$.

Proof: If $(Tx,x) = 0$, then $(Tx,y) + (Ty,x) = 0$, so $-i(Tx,y) + i(Ty,x) = 0$, solving $(Tx,y) = 0$ for all $x, y$, so $T = 0$.

Prop. (X.5.4.6) (Normal Operators).

1. An operator is normal iff $||T|| = ||T^*||$. So $N(T) = N(T^*)$ thus $\sigma_p(T^*) = \overline{\sigma_p(T)}$, and $R(T)$ is dense iff $T$ is injective. And different eigenspaces are orthogonal.

2. An operator is unitary iff $R(U) = H$ and $||Ux|| = ||x||$ for every $x$. (Because an operator is defined by its value $(Tx,y)$).

Prop. (X.5.4.7). For a normal operator $T$ on a Hilbert space $T$ is invertible iff there is a $\delta$ that $||Tx|| = ||T^*x|| \geq \delta ||x||$.

Proof: $T$ is invertible iff $R(T)$ is dense, and if $||Tx|| = ||T^*x|| \geq \delta ||x||$, then $R(T)$ is closed by (IX.1.8.8), so it is invertible.

Prop. (X.5.4.8). If $T$ is normal, then

1. $||T|| = \sup \{ |(Tx,x)| \ | |x|| = 1 \}$.

2. $T$ is self-adjoint iff $\sigma(T)$ is real.

3. $T$ is unitary iff $|\sigma(T)| = 1$.

Proof: For $f = (z - z_0)i_U(z)$, then $f(T)(x_0) = Tx_0 - \lambda_0 x_0$, so

$$(Tx_0,x_0) - \lambda_0 = |(f(T)x_0,x_0)| \leq ||f(T)|| \leq \varepsilon$$

This shows that $||T|| = \sup \{ |(Tx,x)| \ | |x|| = 1 \}$.

For 2, 3, by generalized symbolic calculus (X.5.4.3), $\hat{T} = \lambda$ on $\sigma$ and $\hat{T}^* = \overline{\lambda}$ on $\sigma$, so they are equal iff $\sigma(T)$ is real, and $TT^* = I$ iff $\lambda \overline{\lambda} = 1$ on $\sigma(T)$.

Prop. (X.5.4.9) (Decomposition of Operators). Every operator $S \in L(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is a linear combination of two self-adjoint operator and a linear combination of four unitary operator.
Proof: The first assertion is easy as \( S = (S + S^*)/2 + (S - S^*)/2 \). Now any self-adjoint operator is a multiple of a self-adjoint operator of norm \( ||S|| \leq 1 \), so \( 1 - S^2 \) is positive, and we have \( S = \frac{1}{2}(f_+(S) + f_-(S)) \), where \( f_\pm(s) = s \pm i\sqrt{1 - s^2} \). □

Prop. (X.5.4.10) (Fuglede). If \( N_1 \) and \( N_2 \) are normal operators and \( A \) is a bounded linear operator on a Hilbert space such that \( N_1A = AN_2 \), then \( N_1^*A = AN_2^* \).

Proof: For any \( S \in B(H) \), \( \exp(S - S^*) \) is unitary thus \( ||\exp(S - S^*)|| = 1 \), \( \exp(N_1)A = A\exp(N_2) \). Because \( \exp(M)T = T\exp(N) \), if we let \( U_1 = \exp(M^* - M) \), \( U_2 = \exp(N - N^*) \), then
\[
||\exp(N_1^*)T\exp(-N_2^*)|| = ||U_1TU_2|| \leq ||T||
\]
because \( \lambda N_i \) is normal. Now
\[
||\exp(\lambda N_1^*)T\exp(-\lambda N_2^*)|| = ||U_1TU_2|| \leq ||T||
\]
also holds, thus by Liouville, \( \exp(\lambda N_1^*)T\exp(-\lambda N_2^*) = T \). Compare the coefficients of \( \lambda \), we get the result. □

Prop. (X.5.4.11). An operator \( T \in B(H) \) has the same spectrum w.r.t all the closed \( B^* \)-algebras of \( B(H) \) containing \( T \).

Proof: If \( T \) is invertible, because \( TT^* \) is self-adjoint thus has real spectrum (X.5.4.8) so doesn’t separate \( \mathbb{C} \) thus it is invertible in any closed \( B^* \)-algebra of \( B(H)(X.5.1.12) \). so does \( T^{-1} = T^*(TT^*)^{-1} \). □

Prop. (X.5.4.12). For \( T \) normal and \( E \) its spectral decomposition, then if \( f \in C(\sigma(T)) \) and \( \omega_0 = f^{-1}(0) \), then \( N(f(T)) = R(E(\omega_0)) \).

Proof: \( \chi_{\omega_0}f = 0 \), so \( f(T)R(E(\omega_0)) = 0 \), and if we let \( \omega_n = f^{-1}([1/(n-1), 1/n]) \), and let \( f_n(\lambda) = 1/f(\lambda)\chi_{\omega_n} \), then \( f_n(T)f(T) = E(\omega_n) \), so if \( f(T) = 0 \), then \( E(\omega_n)x = 0 \), so countable additivity shows that \( E(\sigma\setminus\omega_0)x = 0 \), so \( E(\omega_0)x = x \). These shows the desired result. □

Cor. (X.5.4.13).
1. \( N(T - \lambda I) = R(\{\lambda\}) \).
2. every isolated point of \( \sigma(T) \) is point spectrum, because this point is open thus is \( E(\{x\}) \neq 0 \) by Urysohn lemma.
3. if \( \sigma(T) \) is countable, then every \( x \in H \) has a unique orthogonal decomposition \( x = \sum E(\lambda_i)x \) and \( T(E(\lambda_i)x) = \lambda_iE(\lambda_i)x \).

Prop. (X.5.4.14) (Normal Compact Operator). A normal operator \( T \in B(H) \) is compact iff \( \sigma(T) \) has no limit point except possibly 0 and \( \dim N(T - \lambda I) < \infty \) for \( \lambda \neq 0 \).

In particular, a normal compact operator is a limit of f.d. operators

Proof: One direction is general, by(X.4.5.9), for the other, it is a limit of operators of finite dimensional range by general symbolic calculus(X.5.4.3). □

Cor. (X.5.4.15) (Spectral Theorem). A compact normal operator (in particular a normal operator on a f.d linear space) is unitarily diagonalizable.

Proof: it suffices to find a basis of eigenvectors, but this is easy, just by(X.5.4.13). □
Cor. (X.5.4.16) (Hilbert-Schmidt). For a self-adjoint compact operator \( A \) on a Hilbert space \( H \), there is a set of orthonormal basis that \( A \) is diagonal on it. And of course, its eigenvalues are real and can only converge to 0(X.5.4.8).

Prop. (X.5.4.17). For a normal compact operator \( T \in L(H) \), then:
1. \( T \) has an eigenvalue |\( \lambda \)| that |\( \lambda \)| = ||\( T \)||.
2. \( f(T) \) is compact if \( f \in C(\sigma(T)) \) and \( f(0) = 0 \).
3. \( f(T) \) is not compact if \( f \in C(\sigma(T)) \) and \( f(0) \neq 0 \) and \( \dim H = \infty \).

Proof: 1: The spectrum of maximal norm is isolated(X.5.4.9) hence a point spectrum by(X.5.4.13). And |\( \lambda \)| = ||\( T \)|| by symbolic calculus(X.5.4.3).
2: Cf.[Rudin P330].
3: The 2 still show that \( f(0)I - f \) is compact, If \( f \) is compact, then \( f(0)I \) is compact, so \( \dim H < \infty \)(X.5.4.3).

Prop. (X.5.4.18) (Freudenthal Spectral Theorem).

Prop. (X.5.4.19) (Positive Equivalent Definition). A \( T \in L(H) \) is positive, i.e. \( T = T^* \) and \( \sigma(T) \subset [0, \infty) \) iff \( (Tx,x) \geq 0 \).

Proof: If \( (Tx,x) \geq 0 \), then \( (Tx,x) = (x,Tx) = (T^*x,x) \), so \( T = T^* \) by(X.5.4.5), so \( \sigma(T) \) is real(X.5.4.8), and for \( \lambda > 0 \),
\[
\lambda ||x||^2 = (\lambda x,x) \leq ((T + \lambda I)x,x) \leq ||(T + \lambda I)x|| |||x||,
\]
so \( T + \lambda I \) is invertible by(X.5.4.7), so \( \sigma(T) \subset [0, \infty) \).
Conversely, if \( T \) is positive, then it is normal, so \( (Tx,x) = \int_{\sigma(T)} \lambda dE_{x,x} \geq 0 \).

Prop. (X.5.4.20) (Polar Decomposition).
1. Every positive operator \( T \) has a positive square root, which is invertible if \( T \) is.
2. Polar decomposition exists in \( B(H) \): Any \( T \in L(H) \) invertible has a unique decomposition \( T = UP \) where \( U \) is unitary and \( P \) is positive. And \( ||Px|| = ||Tx|| \) for all \( x \).
3. Any normal operator has commuting decomposition \( UP \), where \( U,P,T \) commutes.

Proof: 1: Use general symbolic calculus, then \( S = \sqrt{\lambda}(T) \) is the square root of \( T \). If \( T \) is invertible, then \( S^{-1} = T^{-1}S \).
2: \( (T^*Tx,x) = (Tx,Tx) \geq 0 \), so \( T^*T \) is positive(X.5.4.19), so let \( P = \sqrt{T} \), then it is also invertible, and \( U = TP^{-1} \) is unitary.
3: Use general symbolic calculus, let \( p(\lambda) = |\lambda|, u(\lambda) = \lambda/|\lambda| \) if \( \lambda \neq 0 \), and \( u(0) = 0 \). Then \( T = UP \), and they are commutative.

Cor. (X.5.4.21) (Similar Normal Operator). Similar normal operators are unitarily equivalent.

Proof: It suffices to show that if \( M = TNT^{-1} \), and \( T = UP \) is the polar decomposition, then \( M = UNU^{-1} \). Fuglede(X.5.4.10) shows \( M^*T = TN^* \), so \( NP^2 = NT^*T = T^*MT = T^*TN = P^2N \), so \( N \) commutes with any functions \( f(P) \), in particular \( P \). Hence \( M = (UP)N(UP)^{-1} = UNU^{-1} \).
5 Hilbert-Schmidt Operators and Trace Classes

Main references are [Trace Classes and Hilbert-Schmidt Operators].

Def. (X.5.5.1) (Hilbert-Schmidt Operator). Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and $T \in L(\mathcal{H}, \mathcal{K})$. Then for any orthonormal basis $\{e_j\}$ of $\mathcal{H}$ and $\{f_k\}$ of $\mathcal{K}$,

$$\sum_j ||Te_j||^2_{\mathcal{K}} = \sum_j ||T^*f_j||^2_{\mathcal{H}}.$$

Thus we can say $T$ is Hilbert-Schmidt iff

$$||T||_{HS} = (\sum_j ||Te_j||^2_{\mathcal{K}})^{1/2} < \infty$$

for some/all basis $e_j$ of $\mathcal{H}$. The space of all Hilbert-Schmidt between $\mathcal{H}, \mathcal{K}$ is denoted by $S_2(\mathcal{H}, \mathcal{K})$.

Proof: 

$$\sum_i ||Te_k||^2 = \sum_i \sum_j |(Te_i, f_j)|^2 = \sum_i \sum_j |(T^*f_j, e_i)|^2 = \sum_j ||T^*f_j||^2$$

\qed

Cor. (X.5.5.2) (Properties of $S_2(\mathcal{H}, \mathcal{K})$).

- If $A \in S_2(\mathcal{H}, \mathcal{K})$ then $A^* \in S_2(\mathcal{K}, \mathcal{H})$ with the same HS-norm.
- For $A \in S_2(\mathcal{H}, \mathcal{K})$, $||A|| \leq ||A||_{HS}$.
- $S_2(\mathcal{H}, \mathcal{K})$ is a Banach space in the HS-norm.
- If $\mathcal{H}_1, \mathcal{K}_1$ are separable Hilbert spaces and $T \in S_2(\mathcal{H}, \mathcal{K}), A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}), B \in \mathcal{L}(\mathcal{K}, \mathcal{K}_1)$, then $BTA \in S_2(\mathcal{H}_1, \mathcal{K}_1)$.

Proof: 1 follows from (X.5.5.1). 2 is because we can extend $u$ to a basis of $\mathcal{H}$.

3: $||-||_{HS}$ is clearly a semi-norm (X.3.1.8), and it is a norm by item 2. To show the completeness, if $A_j$ is an HS-Cauchy sequence, then it is a Cauchy sequence in the operator norm, thus converges to an operator $A$. Then for any $\varepsilon$, there is an $N$ that for any $j, k \geq N$, $||A_j - A_k||_{HS} \leq \varepsilon$. This implies

$$\sum_{\alpha \in S} ||(A_k - A_j)e_{\alpha}||^2_{\mathcal{K}} \leq \varepsilon^2$$

for any finite subset $S \subset I$. Then letting $k \to \infty$ and then letting $S$ be any subset, we get $||A - A_j||_{HS} \leq \varepsilon$. Thus $A$ is Hilbert-Schmidt and $A_j \to A$ in HS-norm.

4: it is clear that $||BT||_{HS} \leq ||B|| ||T||_{HS}$, and because of the transpose invariance of HS-norm and operator norm (X.5.2.9). \qed

Prop. (X.5.5.3) (Hilbert-Schmidt Operator is Compact). If $A \in S_2(\mathcal{H}, \mathcal{K})$ and $\{f_k\}$ is an orthonormal basis of $\mathcal{K}$, and we denote $\pi_n$ as the projection of $\mathcal{K}$ onto the span of $\{f_1, \ldots, f_n\}$, then

$$||\pi_n A - A||_{HS} \to 0.$$ 

In particular, $A$ is compact, by (X.5.5.2) and (X.4.5.3).

Proof: By (X.5.5.2), it suffices to show that $||A^*\pi_n - A^*||_{HS} \to 0$. But this norm is just $\sum_{k>N} ||A^*f_k||^2_{\mathcal{H}}$, which converges to 0. \qed
**Prop. (X.5.5.4) (Hilbert-Schmidt Inner Product).** If $\mathcal{H}$ is a Hilbert space and $A,B \in S_2(\mathcal{H})$, then $B^*A \in S_1(\mathcal{H})$ by (X.5.5.2)(X.5.5.7), then we can define an **Hilbert-Schmidt inner product** on $S_2(\mathcal{H})$: 

$$(A,B) = \text{tr}(B^*A) \text{ (X.5.5.9)}.$$ 

Then this makes $S_2(\mathcal{H})$ a Hilbert space.

**Proof:** This follows from (X.5.5.2). □

**Prop. (X.5.5.5) (Integral Operator is Hilbert-Schmidt).** Let $\Omega$ be a $\sigma$-finite measure space and $K(x,y) \in L^2(\Omega \times \Omega)$, then the operator $Lu(x) = \int_{\Omega} K(x,y)u(y)dy$ defined in (X.4.5.2) is a Hilbert-Schmidt operator on $L^2(\Omega)$. In fact, $||L||_2 = ||K||_{L^2}$.

**Proof:** Let $\mathcal{E}$ be an Hilbert basis of $L^2(\Omega)$, then we have 

$$||L||_2^2 = \sum_{f_1,f_2 \in \mathcal{E}} |(Lf_1,f_2)|^2 = \sum_{f_1,f_2 \in \mathcal{E}} \left| \int_X \int_X f_2(y)K(x,y)f_1(x)dxdy \right|^2 = \sum_{f_1,f_2 \in \mathcal{E}} (K,f_1 \otimes f_2)^2$$

But by (X.1.5.4) $\{f_i \otimes f_j\}$ form a Hilbert Basis for $L^2(\Omega \times \Omega)$, then the equation equals $||K||_2^2$. □

**Trace Classes**

**Def. (X.5.5.6) (Trace Classes).** Let $\mathcal{H}$ be a Hilbert space, $\{e_i\}, \{f_i\}$ are two orthonormal basis, $A \in B(\mathcal{H})$. Let $|A| = (A^*A)^{1/2}$ which is positive, then 

$$\sum_i (|A| e_i,e_i) = \sum_i (|A|^{1/2} e_i, |A|^{1/2} e_i) = \sum_i ||||A|^{1/2} f_i|| = \sum_i (|A| f_i, f_i)$$

by (X.5.5.1), thus we can define $||A||_1 = \sum_i (|A| e_i, e_i)$, and say $A$ is a **trace class** if $||A||_1 < \infty$. The space of trace classes is denoted by $S_1(\mathcal{H})$.

A trace-class $A$ is clearly compact as $|A|$ is a limit of f.d. range operators. (Use diagonalization, then there are only countably many eigenvectors of $|A|$).

**Prop. (X.5.5.7).** If $A \in B(\mathcal{H})$, then the following are equivalent:

- $A \in S_1(\mathcal{H})$.
- $|A|^{1/2} \in S_2(\mathcal{H})$.
- $|A|$ is a product of two elements in $S_2(\mathcal{H})$.
- $A$ is a product of two elements in $S_2(\mathcal{H})$.

**Proof:** $1 \rightarrow 2 \rightarrow 3$ is clear, for $3 \rightarrow 4$: if $|A| = TS$, then by polar decomposition $A = U|A| = (UT)S$, and $UT \in S_2(\mathcal{H})$ by (X.5.5.2), so $A$ is a product of two elements in $S_2(\mathcal{H})$.

$4 \rightarrow 3$ is similar to $3 \rightarrow 4$.

$3 \rightarrow 1$: if $A = BC$ where $B,C \in S_2(\mathcal{H})$, then $B^* \in S_2(\mathcal{H})$ also by (X.5.5.2), and 

$$||A||_1 = \sum_i (Ae_i,e_i) = \sum_i (Ce_i,B^*e_i) \leq \sum_i ||C e_i|| ||B^* e_i|| \leq ||C||_2||B^*||_2 < \infty$$

$\square$
Lemma (X.5.5.8). If $T$ is a positive trace-class and $S \in \mathcal{B}(\mathcal{H})$, then if $x_i$ is an orthonormal basis of $\mathcal{H}$, then
\[
\sum_i (STx_i, x_i) \leq ||S|| ||T||_1
\]
is absolutely convergent, and is independent of the basis chosen.

Proof: Let $e_i$ be the basis of eigenvectors of $T$ of eigenvalues $\lambda_i > 0$, then $\sum \lambda_i < \infty$ by (X.5.5.7), and
\[
(STx_i, x_i) = \sum_j (x_i, e_j)(STe_j, x_i) = \sum_j \lambda_j (x_i, e_j)(Se_j, x_i)
\]
And
\[
\sum_i \sum_j \lambda_j ||x_i, e_j)(Se_j, x_i|| \leq \sum_j \lambda_j ||e_j|| ||S|| \leq ||S|| \sum \lambda_i < \infty.
\]
Moreover:
\[
\sum_i (STx_i, x_i) = \sum_i \sum_j ((x_i, e_j)STe_j, x_i) = \sum_i \sum_j (STe_j, (e_j, x_i)x_i) = \sum_j (STe_j, e_j).
\]

Prop. (X.5.5.9) (Singular Trace). If $T \in S_1(\mathcal{H})$ and $x_\alpha$ is an orthonormal basis of $\mathcal{H}$, then $\sum(Tx_\alpha, x_\alpha)$ absolutely converges, and is independent of the basis chosen, called the singular trace $tr T$ of $T$. The singular trace is a positive definite linear functional on $S_1(\mathcal{H})$.

Proof: Use polar decomposition $T = U|T|$ (X.5.4.20) and notice $|T|$ is a positive trace class (X.5.5.7) and then use (X.5.5.8).

Prop. (X.5.5.10) (Trace of Integral Operators). Let $A, B$ be $L^2$ integral operators on a $\sigma$-finite measure space $\Omega$ with kernel $K_1(x, y), K_2(x, y) \in L^2(\Omega \times \Omega)$, then $AB$ is also an integral operator with kernel
\[
\int K_1(x, z)K_2(z, y)dz,
\]
and
\[
tr(AB) = \int \int K_1(x, y)K_2(y, x)dxdy.
\]

Proof: The formula for integral kernel is an immediate consequence of Fubini-Tonelli theorem. For the trace, observe that $A^*$ is the integral operator with kernel $\overline{K_1(y, x)}$ (X.5.2.10), thus
\[
tr(AB) = \sum_i (ABe_i, e_i) = \sum_i (Be_i, A^*e_i)
\]
\[
= \sum_i \sum_k (e_k, A^*e_i)(Be_i, e_k)
\]
\[
= \sum_i \sum_k (e_k \otimes e_i, K_1^*)(K_2, e_k \otimes e_l)
\]
\[
= (K_2, K_1^*) = \int \int K_1(x, y)K_2(y, x)dxdy
\]
as $\{e_i \otimes e_k$ is a Hilbert basis for $\Omega \times \Omega$ (X.1.5.4).
Prop. (X.5.5.11) (Properties of Trace Classes).
1. $S_1(H)$ is a two-sided $\ast$-ideal of $L(H)$.
2. $|| \cdot ||_1$ is a norm on $S_1(H)$.
3. If $T \in S_1(H)$, then $\text{tr} T^* = \overline{\text{tr} T}$.
4. For any $T \in S_1(H)$ and $S \in L(H)$, $\text{tr}(ST) = \text{tr}(TS)$, and $|\text{tr}(ST)| \leq ||S|| ||T||$. In particular, the singular trace is a bounded linear functional on $S_1(H)$.

Proof: Let $S, T \in S_1(H)$, $S = V |S|, T = W |T|$, $S + T = X |S + T|$ where $V, W, X$ are unitary, then $|S + T| = X^* (S + T)$ is positive compact, so it has an orthonormal eigenbasis $e_n$ by (X.5.4.16), so

$$
\sum (|S + T|x, x) = \sum (X^*V|x, x) + \sum (X^*W|x, x) \leq ||S|| + ||T||
$$

by (X.5.5.8). So $S + T$ is a trace class, and $||S + T||_1 \leq ||S||_1 + ||T||_1$.

Now if $U$ is unitary and $T \in S_1(H)$, then $(UT)^* UT = T^* T$, so $UT$ is a trace class, and $(TU)^* TU = U^{-1} T U$, so $TU$ is also a trace class. Moreover, $\text{tr}(TU) = \sum (TUx, x) = \sum (UTUx, x) = \text{tr}(UT)$.

Then notice very $S \in L(H)$ is a linear combination of four unitary operator (X.5.4.9), so the proposition is true, and if $T$ is a trace class, then $T = V |T|$, and $T^* = |T|V^*$ is also a trace class.

4: $|\text{tr}(ST)| \leq ||SV|| ||T||_1 = ||S|| ||T||_1$ by (X.5.5.8).

Prop. (X.5.5.12) (Trace Class as a Banach Space). Let $S_0(H)$ be the space of operators of f.d. range, then the map

$$
\rho : S_1(H) \to S_0(H) : \rho(A) : C \mapsto \text{tr}(CA)
$$

is an isometric isomorphism. In particular, $S_1(H)$ is a Banach space, by (X.3.3.1).

Proof: Clearly $\rho$ is a linear map as singular trace is. For $T \in S_1(H)$, $||\rho(T)|| \leq ||T||_1$ by (X.5.5.11).

If $\Phi \in S_0(H)^*$, $g, h \in H$, consider $g \otimes h^* \in S_0(H)$ that $g \otimes h^*(v) = (v, h)g$, then $B(g, h) = \Phi (g \otimes h^*)$ is a sesquilinear form on $H$ that is bounded by $||\Phi||$. Thus by Lax-Milgram (X.5.2.7), there is a unique $T \in B(H)$ such that $B(g, h) = (g, Th)$.

Now let $A = T^*$ and let $A = U|A|$ be the polar decomposition, $E$ an orthonormal basis of $H$ and $S \subset E$ a finite subset, define

$$
C_S = (\sum_{e \in S} e \otimes e^*) U^* = \sum_{e \in S} e \otimes (Ue)^*.
$$

Then $C_S$ is of f.d. and $||C_S|| \leq 1$. And

$$
\sum_{e \in S} (|A|e, e) = \sum_{e \in S} (U^* A e, e) = \sum_{e \in S} (e, T U e) = \sum_{e \in S} B(e, U e) = \sum_{e \in S} \Phi (e \otimes (Ue)^*) = \Phi (C_S).
$$

So $||A||_1 \leq ||\Phi||$.

If $C$ is any operator of f.d. range that $C = \oplus g_k \otimes h_k^*$, then

$$
\Phi (C) = \sum \Phi (g_k \otimes h_k^*) = \sum B(g_k, h_k) = \sum (Ag_k, h_k) = \sum \text{tr}(A(g_k \otimes h_k^*)) = \text{tr}(AC) = \rho(A)(C)
$$

so the image is $A$ is just $\Phi$. This shows that $\rho$ is surjective and moreover $||A||_1 = ||\rho(A)||$, so we are done. □
X.6 Abstract Harmonic Analysis (Folland)

basis references are [Fol15], [Automorphic Forms and Representations, Bump].
All representations in this section are assumed to be over \( \mathbb{C} \).

1 Locally Compact Groups

Def. (X.6.1.1). On a topological group \( G \), the left regular action and right regular action are defined as follows: \( L_y f(x) = f(y^{-1}x), R_y f(x) = f(xy) \).

Prop. (X.6.1.2) (Translation is Continuous). If \( f \in C_c(G) \), then \( f \) is left and right uniformly continuous. Equivalently, \( G \to C_c(G) : y \mapsto R_y(f) \) and \( y \mapsto L_y(f) \) is continuous group homomorphism from \( G \to C_c(G) \).

Proof: Cf. [Folland Abstract Harmonic Analysis P38]. □

Prop. (X.6.1.3). Locally compact Hausdorff group is normal.

Proof: Notice that by choosing a precompact symmetric open neighbourhood \( U \) of identity, there exists a \( \sigma \)-compact clopen subgroup \( H \). So \( H \) can \( \sigma \)-locally refine every open cover, thus \( G \) can, too. So by (IX.1.7.2) \( G \) is paracompact. As a topological group, \( G \) is regular, thus \( G \) is normal by (IX.1.7.6). □

Prop. (X.6.1.4) (Dirac sequence). For a locally compact Hausdorff group \( G \), a Dirac sequence is a sequence \( f_n \in C_c(G) \) that \( f_n \to \delta_1 \) in the weak topology of \( Meas_c(G) \).

Dirac sequence exists.

Proof: □

Prop. (X.6.1.5). Every locally compact group \( G \) has a subgroup \( G_0 \) that is clopen and \( \sigma \)-compact.

Proof: Let \( U \) be a symmetric precompact nbhd of 1 in \( G \), then let \( U_n = U^n \), then \( \overline{U_n} \subset U_{n+1} \), so let \( G_0 = \bigcup_n U^n = \bigcup \overline{U_n} \), then it is open because each \( U_n \) does, and compact because each \( \overline{U_n} \) does. □

Prop. (X.6.1.6). If \( G \) is locally compact Hausdorff, and \( H \) is subgroup that is locally compact in the induced topology, then \( H \) is closed in \( G \).

Proof: By hypothesis there exists an open nbhd \( U \) of \( e \in G \) that \( U \cap H \) has compact closure \( K \subset H \). But then \( K \) is also compact in \( G \) thus closed. So \( K \) is the closure of \( U \cap H \) in \( G \). Choose a symmetric open nbhd \( V \) of \( e \in G \) that \( VV \subset U \), and suppose \( x \in \overline{H} \), then \( x^{-1} \in \overline{H} \) and \( Vx^{-1} \cap H \neq \emptyset \). Let \( y \in Vx^{-1} \cap H \). For any nbhd \( U_i \) of \( e \in G \), choose \( x' \subset xU_i \cap H \), then \( yx' = yx(x^{-1}x') \in xyU_i \) and also \( yx' \in H \), \( yx' \in Vx^{-1}xV \subset U \). By arbitrariness of \( U_i \), this means \( yx \in U \cap \overline{H} = K \subset H \), thus \( x \in H \), and \( H \) is closed. □

Integration on Locally Compact Groups

Prop. (X.6.1.7) (Haar Measure). A left(right) Haar measure on a topological group \( G \) is a non-zero Radon measure(X.1.1.6) \( \mu \) on \( G \) that satisfies \( \mu(xE) = \mu(E)(\mu(Ex) = \mu(E)) \).
Every Locally compact group \( G \) possesses a unique left Haar measure \( \mu \).
Def. (X.6.1.8) (Modular Function). For a left Haar measure $\mu$ on a locally compact group $G$, $\mu_x(E) = \mu(Ex)$ is also a left Haar measure, so there is a $\Delta(x)$ that $\mu_x = \Delta(x)\mu$. Then the function $\Delta$ is a group homomorphism from $G$ to $\mathbb{R}^+$, which is called the modular function of $G$.

$G$ is called unimodular iff $\Delta = 1$, i.e. a left Haar measure is also a right Haar measure. Obviously, a locally compact Abelian group is unimodular.

Prop. (X.6.1.9). $\Delta$ is a continuous group homomorphism from $G$ to $\mathbb{R}^+$, and

\[
\int R_y f \, d\mu = \Delta(y^{-1}) \int f \, d\mu.
\]
equivalently, $d\mu(xy_0) = \Delta(y_0)d\mu(x)$.

Proof: For the continuity of $\Delta$, because $y \mapsto R_y(f)$ is continuous for each $f$ (X.6.1.2), so $y \mapsto \int R_y f \, d\lambda$ is continuous, as $\mu$ is Radon measure, so by the equation just proved, $\Delta$ is continuous.

Now for any measurable function $E$, $\chi_E(xy) = \chi_{Ey^{-1}}(x)$, thus

\[
\int \chi_E(xy) \, d\mu = \mu(Ey^{-1}) = \Delta(y^{-1})\mu(E) = \Delta(y^{-1}) \int \chi_E(x) \, d\mu(x),
\]
which proves the equation for $f = \chi_E$. Then the general case follows from approximating $f$ by simple functions (X.1.6.4).

Prop. (X.6.1.10) (Involution Measure). If $\mu$ is a left Haar measure and $\rho$ is defined by $\rho(E) = \mu(E^{-1})$, then $\rho$ is a right Haar measure, and $d\rho(x) = d\mu(x^{-1}) = \Delta(x^{-1})d\mu(x)$.

Proof: Notice

\[
\int R_y(f)(x)\Delta(x^{-1}) \, d\mu(x) = \Delta(y) \int f(xy)\Delta((xy)^{-1}) \, d\mu(x) = \int f(x)\Delta(x^{-1}) \, d\mu(x),
\]
so $\Delta(x^{-1}) \, d\mu(x)$ is a right Haar measure, hence $c d\mu(x^{-1})$ for some $c$. If $c \neq 1$, we let $U$ be a precompact symmetric nbhd of 1 that $|\Delta(x^{-1}) - 1| \leq \frac{1}{2}|c - 1|$ on $U$. But then $|c - 1|\mu(U) = |\int_U (\Delta(x^{-1}) - 1) \, d\mu(x)| \leq \frac{1}{2}|c - 1|\mu(U)$, contradiction.

Prop. (X.6.1.11). For a compact group $K$ of $G$, $\Delta$ is trivial on $K$. So compact group is unimodular, and if $G/[G,G]$ is compact, then it is also unimodular.

Proof: These all follow from (X.6.1.9) and the fact that a compact subgroup of $\mathbb{R}^+$ is $\{1\}$, and $\mathbb{R}$ is Abelian.

Prop. (X.6.1.12) (Lie Group Case). Suppose $G$ is an open subset of $K^N$ where $K$ is a local field, and the left translation is given by

\[
xy = A(x)y + b(x)
\]
then the Haar measure of $G$ is given by $|\det A(x)|^{-1} dx$, where $dx$ is the Lebesgue measure on $\mathbb{R}^N$.

Also when we want to calculate the right Haar measure, consider the right action.
Proof: Use change of variable formula, because \( A(xy) = A(x)A(y) \), and

\[
|\det A(ax)|^{-1}d(ax) = |\det A(ax)|^{-1}d(A(a)x + b(x)) = |\det A(x)|^{-1}dx.
\]

\[\square\]

Cor. (X.6.1.13) (Examples of Lie Group Measures).
- \( dx/|x| \) is the Haar measure on \( \mathbb{R}^* \).
- \( dxdy/|x|^2 + y^2 \) is the Haar measure on \( \mathbb{C}^* \).
- \( \prod_{i<j} dx_{ij} \) is the left and right Haar measure on the upper-triangular unipotent group of \( GL(n, \mathbb{R}) \).
- \( |\det T|^{-n}dT \) is the left and right Haar measure on the group \( GL(n, \mathbb{R}) \), where \( dT \) is the Lebesgue measure on \( M_n(\mathbb{R}) = \mathbb{R}^{n^2} \).
- The \( ax + b \) group \( G \) of all affine (invertible)translations of \( \mathbb{R} \) has left measure \( dadb/a^2 \) and right Haar measure \( dadb/a \).

Proof: Clear. \[\square\]

Prop. (X.6.1.14) (Modular Function of Lie Groups). If \( G \) is a Lie group and \( \text{Ad} \) is the adjoint action of \( G \) on \( \mathfrak{g} \), then \( \Delta(x) = |\det \text{Ad}(x^{-1})| \).

Proof: Let \( G \) be a Lie group of dimension \( m \), then the Haar measure on a Lie group is given by the absolute value of a left-invariant \( m \)-form \( \omega \). Now for any \( X \in \mathfrak{g} \) corresponding to a left invariant vector space \( L_X \),

\[
d(R_g)p((L_X)p) = (L_{\text{Ad}(g^{-1})}X)p_g
\]

by (IX.8.3.5), so \( R_g^*\omega = \det(\text{Ad}(g^{-1}))\omega \). So \( \Delta(g)\omega = R_g^*|\omega| = |\det(\text{Ad}(g^{-1}))||\omega| \).

\[\square\]

Cor. (X.6.1.15) (Unimodular Lie Groups). Any Abelian/compact/semisimple/reductive/nilpotent Lie group is unimodular.

Proof: The nilpotent case follows directly from (X.6.1.14), as \( \det \text{Ad}(x) = \exp(\text{tr} \text{ad}(x)) \) and \( \text{ad}(x) \) is nilpotent. The compact case is by (X.6.1.11). For the semisimple case, \( G = [G,G] \) because \( \mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \) (I.12.2.4) and it is connected, so we are done by (X.6.1.11). For the reductive case: Cf. [Kna96] P467.

\[\square\]

Convolution

Def. (X.6.1.16) (Convolution of Measures). If \( \mu, \nu \) are two complex (hence finite) Radon measures on \( G \), the map

\[
I(\varphi) = \int \int \varphi(xy)d\mu(x)d\nu(y)
\]

is clearly a linear functional on \( C_c(G) \) that satisfies \( |I(\varphi)| \leq ||\varphi||_{\text{sup}}||\mu||||\nu|| \), so it defines a measure on \( G \) by Riesz representation (X.1.1.10), called the convolution of \( \mu \) and \( \nu \), denoted by \( \mu * \nu \), that \( ||\mu * \nu|| \leq ||\mu||||\nu|| \).

Prop. (X.6.1.17) (Measure Algebra).
- The convolution of measure is associative.
• $\delta_x \ast \delta_y = \delta_{xy}$.

• The convolution of measure is commutative iff $G$ is commutative.

• The convolution makes $M(G)$ into a unital Banach algebra, called the measure algebra of $G$.

Proof: 1: If $\varphi \in C_c(G)$, then

$$\int_G \varphi d[\mu \ast (\nu \ast \sigma)] = \int \int \varphi(xy) d\mu(x) d(\nu \ast \sigma)(y)$$

$$= \int \int \int \varphi(xyz) d\mu(x) d\nu(y) d\sigma(z)$$

$$= \int \varphi(yz) d(\mu \ast \nu)(y) d\sigma(z)$$

$$= \int \varphi d[(\mu \ast \nu) \ast \sigma]$$

by Fubini theorem, which shows $\mu \ast (\nu \ast \sigma) = (\mu \ast \nu) \ast \sigma$.

2: $\int \int \varphi d(\delta_x \ast \delta_y) = \int \int \varphi(uv) d\delta_x(u) d\delta_y(v) = \varphi(xy) = \int \varphi d\delta_{xy}$.

3: If $G$ is commutative, then $\varphi(xy) = \varphi(yx)$, then the commutativity follows from Fubini theorem. The converse follows from item 2.

4 It is a Banach algebra because $||\mu \ast \nu|| \leq ||\mu|| ||\nu||$ (X.6.1.16). And the point measure $\delta_1$ is the unit:

$$\int \varphi d(\delta \ast \mu) = \int \int \varphi(xy) d\delta(x) d\mu(y) = \int \varphi(y) d\mu(y)$$

shows $\delta \ast \mu = \mu$ for any $\mu$, and similarly $\mu \ast \delta = \mu$, so $\delta$ is the identity. \hfill \Box

Prop. (X.6.1.18) (Involution of Measure). $M(G)$ has a canonical involution that preserves measure:

$$\mu \mapsto \mu^* : \mu^*(E) = \overline{\mu(E^{-1})}.$$ 

Proof: $\mu^*$ clearly satisfies $||\mu^*|| = \mu^*(G) = \mu(G) = ||\mu||$. And for any $\varphi \in C_c(G)$,

$$\varphi d(\mu \ast \nu)^* = \int \varphi(x^{-1}) d(\overline{\nu \ast \mu})(x) = \int \varphi((xy)^{-1}) d\overline{\nu}(x) d\sigma(y) = \int \varphi(yx) d\mu^*(x) \nu^*(y) = \int \varphi d(\nu^* \ast \mu^*),$$

which shows $(\mu \ast \nu)^* = \nu^* \ast \mu^*$. \hfill \Box

Def. (X.6.1.19) ($L^1$ Group Algebra). Fix a Haar measure $d\mu$ on $G$, $L^1(G)$ embeds into the $M(G)$ by identifying $f$ with the measure $f(x) d\mu(x)$, and this is an isometry.

So the convolution and involution can be defined on $L^1(G)$, and the outcome turns out to be in $L^1(G)$ too:

$$f \ast g(x) = \int f(y) g(y^{-1} x) dy$$

follows from Fubini theorem, which also shows this is a.e. defined and in $L^1(G)$. The involution

$$f^*(x) = \Delta(x^{-1}) \overline{f(x^{-1})}$$

follows from (X.6.1.10).
Prop. (X.6.1.20). The convolution \( f \ast g \) can be calculated in multiple ways by left invariance and (X.6.1.10):

\[
f \ast g(x) = \int f(y)g(y^{-1}\cdot x)dy = \int f(xy)g(y^{-1})dy = \int f(y^{-1})g(y)\Delta(y^{-1})dy = \int f(xy^{-1})g(y)\Delta(y^{-1})dy.
\]

In particular, if \( G \) is unimodular, then it can be calculated anyway you want.


Homogenous Spaces

Def. (X.6.1.22) (Notations). If \( G \) is a locally compact group with left Haar measure \( dx \) and \( H \) is a closed subgroup with left Haar measure \( d\xi \), let \( q : G \to G/H \) be the quotient map.

Prop. (X.6.1.23). If \( G \) is a \( \sigma \)-compact locally compact group and \( S \) is a transitive \( G \)-space that is locally compact and Hausdorff, then if \( s_0 \in S \) and \( \text{Stab}(s_0) = H \), then \( G/H \cong S \) as \( G \)-spaces.

Proof: Cf, [Folland P60].

Lemma (X.6.1.24). If \( E \subset G/H \) is compact, then there is a compact \( K \subset G \) that \( q(K) = E \).

Proof: Choose a precompact nbhd \( V \) of \( 1 \) in \( G \), since \( q \) is open, the set \( q(xV) \) is an open cover of \( E \), so there are f.m. \( x_i \) that \( E \subset \cup q(x_iV) \). Then let \( K = q^{-1}(E) \cap (\cup x_iV) \), this will suffice.

Def. (X.6.1.25) (Fundamental Domain). A fundamental domain for a group \( \Gamma \) acting discontinuously on a locally compact second countable Hausdorff space \( X \) is an Borel subset \( F \in X \) that:

- \( \cup_{\gamma \in \Gamma} \gamma F = \mathcal{H} \).
- if \( \gamma \neq 1 \in \Gamma \), then \( \gamma F \cap F = \emptyset \).

Then fundamental domains exist.

Proof: The quotient space \( G/\Gamma \) is locally compact second countable, thus there is a countable set of precompact open basis \( \{B_i\} \) for \( G/\Gamma \), and a countable set of precompact open basis \( \{C_i\} \) for \( X \). For each \( \varphi \in G/\Gamma \), choose \( \varphi \in B_i(\varphi) \) and choose a preimage \( x \in X \) and a nbhd \( C_i(x) \) that \( C_i(x) \cap \gamma(C_i(x)) = \emptyset \) for any \( \gamma \neq 1 \), then there is a nbhd \( B_j(x) \) contained in the image of \( C_i(x) \). Then we can take a precompact preimages of \( B_j(\varphi) \) in \( C_i(x) \) by (X.6.1.24), labeled by \( U_i \). Then \( U_i \) maps isomorphically to their images in \( G/\Gamma \) and covers \( G/\Gamma \).

Now take \( V_1 = W_1 = U_1 \), and \( W_{n+1} = U_{n+1} \setminus \cup_{\gamma \in \Gamma} \gamma(V_n) \), \( V_n+1 = V_n \cup W_{n+1} \). Then \( \cup V_i \) is a fundamental domain.

Lemma (X.6.1.26). If \( F \subset G/H \) is compact, then there is a \( f \in C_c(G) \) that \( f \geq 0 \) and \( Pf = 1 \) on \( F \).

Proof: Let \( E \) be a precompact nbhd of \( F \) in \( G/H \), choose a compact \( K \subset G \) that \( q(K) = \overline{E} \) by (X.6.1.24). Choose \( g \in C_c(G) \geq 0 \) that is positive on \( K \) and \( \varphi \in C_c(G/H) \) that is 1 on \( F \) and vanish outside \( E \), then set

\[
f = \frac{\varphi \circ q}{Pg \circ q} g
\]

then \( f \geq 0 \) and \( Pf = (\varphi/Pg)Pg = \varphi \).
Lemma (X.6.1.27). If \( \varphi \in C_c(G/H) \), then there exists a \( f \in C_c(G) \) that \( Pf = \varphi \) and \( q(\text{Supp } f) = \text{Supp } \varphi \). Also if \( \varphi \geq 0, f \geq 0 \).

Prop. (X.6.1.28) (Projection of Function). There is a map \( P : C_c(G) \to C_c(G/H) : Pf(xH) = \int_H f(x\xi) d\xi \). \( Pf \) is continuous and this map is well-defined.

Prop. (X.6.1.29) (Quotient Measure Regular Case). If \( G \) is a locally compact group and \( H \) is a closed subgroup, then there is a \( G \)-invariant positive Radon measure \( \mu \) on \( G/H \) iff \( \Delta_G|_H = \Delta_H \). And if this is the case, then this measure is unique up to constant, and if suitably chose, satisfies:

\[
\int_G f(x) dx = \int_{G/H} Pf d\mu = \int_{G/H} \int_H f(x\xi) d\xi d\mu(xH).
\]

Cor. (X.6.1.30) (Decomposition of Measure). If \( G \) is a unimodular locally compact group and \( P, K \) be closed subgroups s.t. \( P \cap K \) is compact and \( G = PK \). Let \( d_{LP}, d_{RK} \) be the left and right Haar measure on \( P, K \) respectively, then the Haar measure on \( G \) is given by

\[
\int_G f(g) dg = \int_K \int_P f(pk) dLpdRk.
\]

Def. (X.6.1.31) (Quasi-Invariant Measure). There are cases that the condition \( \Delta_G|_H = \Delta_H \) in (X.6.1.29) is not satisfied, then we can only get weaker conditions. We call a Radon measure on \( G/H \)

- **quasi-invariant** if the measure \( \mu_x(E) = \mu(xE) \) is mutually absolutely continuous w.r.t \( \mu \) for any \( x \in G \).

- **strongly quasi-invariant** if the Radon-Nikodym derivative \((d\mu_x/d\mu)(PH)\) is continuous for both \( x \) and \( p \).

Lemma (X.6.1.32). Let \( G \) be a locally compact group and \( H \) a closed subgroup, then there is a continuous function \( G \to [0, \infty) \) that

- \( f^{-1}((0,\infty)) \cap xH \neq \emptyset \) for any \( x \in G \).

- \( \text{Supp } f \cap KH \) is compact for any compact group \( K \) of \( G \).

Def. (X.6.1.33) (Rho-Function). Let \( G \) be a locally compact group and \( H \) a closed subgroup, a **rho-function** for \((G, H)\) is a continuous function \( G \to (0, \infty) \) that

\[
\rho(x\xi) = \frac{\Delta_H(\xi)}{\Delta_G(\xi)} \rho(x).
\]

Then rho-function exists for \((G, H)\).
Proof: Choose $f$ as in (X.6.1.32), set
\[ \rho(x) = \int_H \frac{\Delta_G(\eta)}{\Delta_H(\eta)} f(x\eta) d\eta \]
then the properties of $f$ easily implies that the integral converges and $\rho$ is continuous on $G$. and
\[ \rho(x\xi) = \int_H \frac{\Delta_G(\eta)}{\Delta_H(\eta)} f(x\xi\eta) d\eta = \int_H \frac{\Delta_G(\xi^{-1}\eta)}{\Delta_H(\xi^{-1}\eta)} f(x\eta) d\eta = \frac{\Delta_H(\xi)}{\Delta_G(\xi)} \rho(x). \]
\[ \square \]

Lemma (X.6.1.34). If $f \in C_c(G)$ and $Pf = 0$, then $\int f \rho = 0$ for any rho-function $\rho$. In fact, this is true if $\rho$ is allowed to take value 0.

Proof: Cf.[Folland P65]. \[ \square \]

Prop. (X.6.1.35) (Rho-Function and Strongly Quasi-Invariant Measure). Given any rho-function for the pair $(G, H)$, there is a strongly quasi-invariant measure (X.6.1.31) $\mu$ on $G/H$ that
\[ \int_{G/H} Pf d\mu = \int_G f(x)\rho(x) dx. \]
Moreover, this $\mu$ also satisfies
\[ \frac{d\mu_x}{d\mu}(yH) = \frac{\rho(xy)}{\rho(y)}. \]
Proof: Cf.[Folland, P66]. \[ \square \]

Prop. (X.6.1.36) (Rho-Function and Strongly Quasi-Invariant Measure). Every strongly quasi-invariant measure on $G/H$ arises from a rho-function as in (X.6.1.35), and all such measures are mutually absolutely continuous.

Proof: Cf.[Folland, P67]. \[ \square \]

Maximal Compact Subgroup

Def. (X.6.1.37) (Maximal Compact Subgroup). A maximal compact subgroup of a locally compact group $G$ is a maximal object in the set of all compact subgroup of $G$.


Locally Profinite Groups

Def. (X.6.1.39) (Locally Profinite Group). A locally profinite group is a locally profinite topological group. A profinite group is locally profinite, and any compact open subgroup of a locally profinite group is profinite.

Cor. (X.6.1.40). A closed subgroup of a locally profinite group is locally profinite, and a quotient group is locally profinite.
Proof: The proof is very similar to that of (I.3.12.4), as the result of (IX.1.1.20) remains true, because any connected nbhd of $e$ is contained in any compact open subgroup.

Prop. (X.6.1.41). If $G$ is a locally profinite group, then the set of compact open subgroups form a basis of the nbhd of $1$.

Proof: For any nbhd $U$ of $1$, choose a precompact nbhd $V$ of $1$ contained in $U$, then there is another compact open subgroup contained in $V$, by (IX.1.12.24).

Prop. (X.6.1.42) (Quotient of Locally Profinite Group). A quotient subspace of a locally profinite group is locally profinite.

Proof: Consider the $H$ action on $G$, then it is regular, because the graph is the preimage of $H$ in the map $G \times G \to G : (g_1, g_2) \mapsto g_1^{-1}g_2$. So by (IX.1.12.10) $G/H$ is Hausdorff. But clearly $G \to G/H$ is open and $G/H$ is locally profinite as it has a basis of locally compact subsets.

Lemma (X.6.1.43). Let $G$ be a locally profinite group and $H$ a closed subgroup, then for any open compact subspace $V \subset G/H$, there is an open compact subspace $U \subset G$ that $p(U) = V$.

Proof: Cf. [Bump, P439].

Prop. (X.6.1.44) (Homogeneous Group). Let $G$ is a locally profinite group that is $\sigma$-compact. If $G$ acts transitively on a locally profinite space $X$, let $x_0 \in X$ and $\text{Stab}(x_0) = H$, then $G/H \to X$ is a homeomorphism. Is this true for locally compact groups?

Proof: Let $N$ be a compact open subset of $N$, and $g_i$ be the left coset representatives for $N$, which is countable. Now $X = \cup_i \gamma(g_i N)x_0$. Because a locally profinite space is a Baire space, some $\gamma(g_i N)x_0$ contains a nbhd of $\gamma(g_i n)x$. Now left acts $(g_i n)^{-1}$, we see $x_0$ is an interior point of $\gamma(N)x_0$. Now $N$ is arbitrary, thus (X.6.1.41) shows $g \mapsto \gamma(g)x$ is open, thus $G/H \to X$ is open. It is clearly continuous, thus $G/H \cong X$.

2 Representations

Def. (X.6.2.1) (Representation of Locally Compact Groups). Usually we consider representations on a Hilbert space. A \textit{unitary representation} of a locally compact group on a Hilbert space $\mathcal{H}$ is defined to be a homomorphism from $G$ to the group $U(\mathcal{H})$ of unitary representations of $\mathcal{H}$ continuous in the strong operator topology (X.3.4.7). Notice by (X.3.4.8), this is equivalent to the unitary and continuous in the weak operator topology.

Def. (X.6.2.2) (Intertwining Operators). If $\pi_1, \pi_2$ are unitary representations of $G$, then the space $C(\pi_1, \pi_2)$ of \textit{intertwining operators} of $\pi_1, \pi_2$ as:

$$ C(\pi_1, \pi_2) = \{ T : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2} : T\pi_1(x) = \pi_2(x)T, \; \forall x \in G \}. $$

And denote $C(\pi_1, \pi_1)$ by $C(\pi)$.

Lemma (X.6.2.3). The adjoint operator $S \mapsto S^*$ induces a bijection between the spaces $C(\pi_1, \pi_2) \cong C(\pi_2, \pi_1)$.

Lemma (X.6.2.4). If $\mathcal{H}_\pi$ is a representation of $G$, $M$ is a closed subspace. Let $P$ be the orthogonal projection onto $M$, then $M$ is invariant under $\pi$ iff $P \in C(\pi)$. 
Proof: If \( P \in C(\pi) \) and \( v \in M \), then \( \pi(x)v = \pi(x)Pv = P\pi(x)v \in M \), so \( M \) is \( \pi \)-invariant. Conversely if \( M \) is \( \pi \)-invariant, then so does \( M^\perp \), so \( \pi(x)Pv = \pi(x)v = P\pi(x)v \), and also for \( v \in M^\perp \), so \( \pi(x)P = P\pi(x) \), for any \( x \). \( \square \)

Prop. (X.6.2.5) (Schur’s Lemma).
- A unitary representation \( \pi \) of \( G \) is irreducible iff \( C(\pi) \) consists only of scalar multiples of identity.
- If \( \pi_1, \pi_2 \) are non-equivalent irreducible unitary representations of \( G \), then \( C(\pi_1, \pi_2) = 0 \).

Proof: 1: If \( \pi \) is reducible, then it contains a non-trivial projection by lemma(X.6.2.4). Conversely, if \( T \neq cI \in C(\pi) \), then we consider \( A = \frac{1}{2}(T + T^*) \), \( B = \frac{1}{2}(T - T^*) \), then at least one of them are not \( cI \). But they are normal, thus by symbolic calculus(X.5.4.3) any \( \chi_E(A) \) for some \( E \subset \mathbb{R} \) Borel is non-trivial(because the spectrum of \( A \) is not a single point) and commutes with \( \pi \), so \( \mathcal{H}_\pi \) is reducible by(X.6.2.4) again.

2: By(X.6.2.3), for \( T \in C(\pi_1, \pi_2) \), \( T^* \in C(\pi_2, \pi_1) \), thus \( TT^* = cI, T^*T = cI \). so \( T = 0 \) or \( c^{-1/2}T \) is unitary, and it is an isomorphism between \( \pi_1, \pi_2 \). \( \square \)

Cor. (X.6.2.6). if \( G \) is Abelian, then any irreducible representation of \( G \) is 1-dimensional.

Proof: If \( \pi \) is a representation of \( G \), then any \( \pi(x) \) commutes with \( \pi \), thus \( \pi(x) = c_xI \) for some \( c_x \), so every subspace of \( \mathcal{H}_\pi \) is irreducible, thus \( \dim \mathcal{H} = 1 \). \( \square \)

Prop. (X.6.2.7) (Unitary Representation and \( L^1(G) \)-Representation). Any unitary representation \((\pi, \mathcal{H})\) of \( G \) determines a representation of \( L^1(G) \) by
\[
f \mapsto \int f(x)\pi(x)dx
\]
This is a non-degenerate *-representation of \( L^1(G) \).

And conversely, any non-degenerate *-representation of \( L^1(G) \) arises from a unitary representation of \( G \).

Proof: If \( \pi \) is a unitary representation and \( f \in L^1 \), let \( \pi(f) \) be defined as
\[
\pi(f)u = \int f(x)\pi(x)udx,
\]
where the integral is in the weak sense(X.4.3.25), and it satisfies \( \|\pi(f)\| \leq \|f\|_1 \).

For the *-algebra structure(X.6.1.19), it suffices to prove that
\[
\pi(f * g) = \pi(f)\pi(g), \quad \pi(f^*) = \pi(f)^*,
\]
which are true for formal reason:
\[
\pi(f * g) = \int \int f(y)g(y^{-1}x)\pi(x)dydx = \int \int f(y)g(x)\pi(yx)dxdy = \pi(f)\pi(g),
\]
\[
\pi(f^*) = \int \Delta(x^{-1})\overline{f(x^{-1})}\pi(x)dx = \int \overline{f(x)}\pi(x^{-1})dx = \int (f(x)\pi(x))^*dx = \pi(f)^*.
\]
and verified by supplying \( u, v \). For the non-degeneracy, for any \( u \neq 0 \in \mathcal{H} \), choose a precompact nbhd \( V \) of identity that \( \|\pi(x)u - u\| < \|u\| \) for \( x \in V \), and let \( f = |V|^{-1}\chi_V \), then it can be verified that \( \|\pi(f)u\| \neq 0 \).

For the converse, Cf.[Folland P79-81]? \( \square \)
Prop. (X.6.2.8). We want to consider the difference of the image of $L^1(G)$ and $G$ under these two representations: Let $\pi$ be a unitary representation of $G$, then

- The bicommutant (X.5.3.13) of $\pi(G)$ and $\pi(L^1(G))$ are identical.
- $T \in \mathcal{L}(\mathcal{H})$ intertwines $\pi$ iff it commutes with every $\pi(f) \in \pi(L^1(G))$.
- A closed subspace $M$ of $\mathcal{H}$ is invariant under $\pi$ iff $\pi(f)M \subseteq M$ for any $f \in L^1(G)$.

Proof: 1: Cf. [Folland, P82].

2 follows from 1 noticing the fact that $T$ commutes with an algebra iff it commutes with its von-Neumann algebra.

3 follows from 2 and (X.6.2.4). □

Prop. (X.6.2.9) (Completion of Unitary Representation). If $\mathcal{H}_0$ is a Hermitian inner product space that $G$ is a topological group acting continuously on $\mathcal{H}_0$ that preserves the inner product, then if $\mathcal{H}$ is the Hilbert completion of $\mathcal{H}_0$, then the action of $G$ extends to a continuous unitary action on $\mathcal{H}$.

Proof: The extension is clear as $||\pi(g)f|| = ||f||$. For the continuity, if $v \in \mathcal{H}, g \in G, \varepsilon > 0$, let $v_0 \in \mathcal{H}_0$ that $|v - v_0| < \varepsilon/6$, then there is a nbhd $W$ of $g$ that if $g_1 \in W$, then $||\pi(g_1)v_0 - \pi(g)v_0|| < \varepsilon/3$.

Then if $|v_1 - v| < \varepsilon/6$ and $g_1 \in W$, then

$$|\pi(g_1)v_1 - \pi(g)v| = |\pi(g_1)v_1 - \pi(g_1)v_0 + \pi(g_1)v_0 - \pi(g)v_0 + \pi(g)v_0 - \pi(g)v|$$

$$\leq |\pi(g_1)v_1 - \pi(g_1)v_0| + |\pi(g_1)v_0 - \pi(g)v_0| + |\pi(g)v_0 - \pi(g)v|$$

$$= |v_1 - v_0| + |\pi(g_1)v_0 - \pi(g)v_0| + |\pi(g)v_0 - \pi(g)v|$$

$$\leq \varepsilon$$

which shows the action is continuous. □

**Functions of Positive Type**

Def. (X.6.2.10) (Positive Type Function). A function of positive type on a closed compact group $G$ is a function $\varphi \in L^\infty(G)$ that defines a positive linear functional on the $B^*$-algebra $L^1(G)$. In other word,

$$\int f(x)\overline{f(y)}\varphi(y^{-1}x)dydx \geq 0, \quad \forall f \in L^1(G).$$

We denote by $P(G)$ the set of continuous functions of positive type on $G$.

Prop. (X.6.2.11). If $\varphi$ is of positive type, then so does $\overline{\varphi}$. (Easy calculation).

Prop. (X.6.2.12). If $\pi$ is a unitary representation of $G$ and $u \in \mathcal{H}_\pi$, then $\varphi(x) = (\pi(x)u, u) \in P$.

Proof: $\varphi$ is continuous by definition, so if $f \in L^1$, then

$$\int \int f(x)\overline{f(y)}\varphi(y^{-1}x)d\mu(x)d\mu(y) = \int \int (f(x)\pi(x)u, f(y)\pi(y)u)dxdy = ||\pi(f)u||^2 \leq 0$$

□

Prop. (X.6.2.13) (Cyclic Representations and Functions of Positive Type). Any function of positive type arises from an irreducible representation and a cyclic vector $\varepsilon$ as in (X.6.2.12)
Proof: Cf.[Folland P83-85].

Cor. (X.6.2.14). If \( \varphi \) is a function of positive type, then \( \varphi \) can be chosen to be continuous.

Cor. (X.6.2.15). If \( \varphi \in P \), then \( \| \varphi \|_\infty = \varphi(1) \), and \( \varphi(x^{-1}) = \varphi(x) \).

Proof: \( \varphi(x) = (\pi(x)u, u) \) for some representation \( \pi \) and \( u \in \mathcal{H} \), so \( |\varphi(x)| \leq ||u||^2 = \varphi(1) \) and \( \varphi(x^{-1}) = (\pi(x^{-1})u, u) = (u, \pi(x)u) = \varphi(x) \).

Def. (X.6.2.16). We set:

- \( P_0(G) = \{ \varphi \| \varphi \|_\infty \leq 1 \} = \{ \varphi(1) = 1 \} \).
- \( P_1(G) = \{ \varphi \| \varphi \|_\infty = 1 \} = \{ 0 \leq \varphi(1) \leq 1 \} \).

By Banach-Alaoglu, \( P_0(G) \) and \( P_1(G) \) are a weak*-compact set.

Prop. (X.6.2.17) (Extreme Points of \( P_1 \)). A \( \varphi \in P_1 \) is an extreme point iff the representation it corresponds is irreducible. And \( E(P_0) = E(P_1) \cup \{ 0 \} \).

Proof: Cf.[Folland P86].

Prop. (X.6.2.18) (Two Topologies Coincide). On \( P_1 \), the compact-open topology coincides with that of the weak*-topology.

Proof: Cf.[Folland Abstract Harmonic Analysis P80].

Prop. (X.6.2.19) (Gelfand-Raikov). If \( G \) is a locally compact group, then the irreducible representations of \( G \) separate points of \( G \).

Proof: Cf.[Folland Abstract Analysis P91].

3 Locally Compact Abelian Group

Dual Group

Def. (X.6.3.1) (Dual Space). If \( G \) is locally compact, denote \( \hat{G} \) the set of all irreducible unitary representations of \( G \), called the dual space of \( G \).

Def. (X.6.3.2) (Dual Group). If \( G \) is locally compact Abelian, the irreducible unitary representations of \( G \) are all 1-dimensional by (X.6.2.6), so it forms a group, called the dual group of \( G \), also denoted by \( \hat{G} \).

An element of \( \hat{G} \) is called a character of \( G \), denoted by \( \xi \). And a continuous homomorphism from \( G \) to \( \mathbb{C} \) is called a quasi-character.

The topologies on \( \hat{G} \) that makes it into a LCA group is given in (X.6.3.6).

Remark (X.6.3.3). \( \hat{\mathbb{R}} \cong \mathbb{R} \), and the quasi-characters of \( \mathbb{R} \) are all of the form \( x \rightarrow e^{sx} \) for \( s \in \mathbb{C} \).

Proof: If \( \varphi \in \hat{\mathbb{R}} \), then \( \varphi(0) = 1 \), and there is an \( a > 0 \) that \( \int_0^a \varphi(t)dt \neq 0 = A \). Now \( A\varphi(x) = \int_x^{x+a} \varphi(t)dt \), so taking derivative,

\[
\varphi'(x) = \frac{\varphi(x+a) - \varphi(x)}{A} = \frac{\varphi(a) - 1}{A} \varphi(x),
\]

which shows \( \varphi(x) = e^{sx} \) for some \( s \in \mathbb{C} \).
Prop. (X.6.3.4) (Dual Group as Spectrum of $L^1(G)$). The dual group $G^*$ can be regarded as the spectrum of $L^1(G)$, i.e. multiplicative homomorphism of $L^1(G)$:

$$\xi \mapsto (\xi(f) = \int (x,\xi)f(x)dx).$$

Proof: First, $\xi$ is multiplicative because

$$\xi(f \ast g) = \int \int f(y)g(y^{-1}x)(x,\xi)dydx = \int \int f(y)g(xy,\xi)dydx = \xi(f)\xi(g).$$

Conversely, any continuous functional on $L^1$ is like $\varphi(f) = \int f(x)\varphi(x)dx$ for some $\varphi \in L^\infty$, and it is multiplicative, so

$$\varphi(f)\int \varphi(x)g(x) = \varphi(f)\varphi(g) = \varphi(f \ast g) = \int \int \varphi(y)f(yxy^{-1})g(x)dxdy = \int \varphi(L_x(f))g(x)dx$$

So $\varphi(x) = \frac{\varphi(L_x(f))}{\varphi(f)}$, a.e., for any $f$. So $\varphi(x)$ can be chosen to be continuous, as $x \to L_x(f)$ is continuous (X.6.1.2). And clearly $\varphi$ is multiplicative. \qed

Cor. (X.6.3.5). $\hat{G} \subset P_1(G)$, because $\int (f^* \ast f)\varphi d\mu = |\Phi(f)|^2 \geq 0$.

Cor. (X.6.3.6) (Dual Group as a LCA Group). Now we can give $\hat{G}$ the compact-open topology, then the group operation is clearly continuous, and the topology coincides with that inherited by the weak*-topology of the $L^\infty$ by (X.6.2.18), so $\hat{G} \cup \{0\}$ is a compact Hausdorff space because $\hat{G} \subset P_1(G)$ and it is the subset of $L^\infty$ that $\{h(xy) = h(x)h(y)\}$ which is weak*-closed hence weak*-compact. In particular, $G$ is a locally compact topological group.

Prop. (X.6.3.7) (Duality between Discrete Groups and Compact Groups). If $G$ is discrete, then $\hat{G}$ is compact, if $G$ is compact, then $\hat{G}$ is discrete.

Proof: if $G$ is discrete, then there is a unit $\delta$ in $L^1(G)$, which is 1 on $e$ and 0 otherwise. So the spectrum of $L^1(G)$ is compact by (X.5.1.21).

If $G$ is compact, then $1 \in L^1$, so $U = \{f \in L^\infty ||f| > \frac{1}{2}\}$ is weak*-open, but $U \cap \hat{G} = \{1\}$ by (X.6.4.1), so $\hat{G}$ is discrete. \qed

Fourier Transform

Prop. (X.6.3.8) (Fourier Transform). The Fourier transform on $G$ is defined as in (X.6.3.4) to be the map on $\hat{G}$:

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int f(x)(x,\xi)$$

which is a norm-decreasing *-homomorphism form $L^1(G)$ to $C_0(\hat{G})$, and its range is a dense subspace of $C_0(\hat{G})$.

Equivalently, the Fourier transform is just the Gelfand transform of $L^1(G)$ (X.5.1.21) composed with an inverse map.

Proof: Cf.[Folland Abstract Harmonic Analysis P102]. \qed

Prop. (X.6.3.9) (Fourier Transform Interchanges Multiplications). $(\hat{f} \ast \hat{g}) = \hat{f} \cdot \hat{g}$, so if $f,g \in L^2(G), (\hat{fg}) = \hat{f} \ast \hat{g}$.
Proof: Cf.[Folland Abstract Harmonic Analysis P102]. □

Prop. (X.6.3.10). There is another map from $M(\hat{G})$ to bounded continuous functions on $G$:

$$\mu \mapsto (\varphi_\mu : x \mapsto \int (x, \xi) d\mu(\xi)).$$

This is a norm decreasing injection from $M(\hat{G})$ to $L^\infty(G)$, and if $\mu$ is positive, then $\varphi_\mu$ are all functions of positive type.

Proof: It suffices to prove injectivity, but if $\varphi_\mu = 0$, then $0 = \int \int f(x, \xi) d\mu(\xi) dx = \int \hat{f}(\xi^{-1}) d\mu(\xi)$ for all $f \in L^1(G)$, so but this shows $\mu = 0$ because of (X.6.3.8) and Riesz representation.

For the positive type, notice that

$$\int \int f(x)\overline{f(y)} v_{p,\mu}(y^{-1}x) dxdy = \int \int f(x)\overline{f(y)}(x, \xi) d\mu(\xi) dxdy = \int |\hat{f}(\xi)|^2 d\mu(\xi) \geq 0$$

□

Prop. (X.6.3.11) (Bochner’s Theorem). If $\varphi \in P(G)$, there is a unique positive $\mu \in M(\hat{G})$ s.t. $\varphi = \varphi_\mu$.

Proof: We have the map defined in (X.6.3.10) injects $M(\hat{G})$ into $P(G)$ (norm-decreasing), so it suffices to prove the existence. For this, we may assume $\varphi \in P_0(G)$. Let $M_0$ be the set of positive measure $\mu \in M(\hat{G})$ that $\mu(\hat{G}) \leq 1$, then $M_0$ is weak*-compact in $M(\hat{G})$. Now

$$\int f(x)\varphi_\mu(x) dx = \int \int f(x) d\mu(\xi) dx = \int \hat{f}(\xi^{-1}) \mu(x)$$

so the mapping $\mu \mapsto P_0$ must be continuous w.r.t their weak*-topologies, so the image is a compact convex subset of $P_0$. But the image contains all characters and 0 (by taking the point mess), which are the extreme points of $P_0$, by (X.6.2.17), so it contains all the $P_0$, by Krein-Milman (X.4.3.15). □

Cor. (X.6.3.12) (Herglotz). A numerical sequence $\{a_n\}$ is positive iff there is a positive measure $\mu \in M(T)$ s.t. $a_n = \mu(n)$.

Cor. (X.6.3.13). So (X.6.3.11) together with (X.6.3.10) says that the image $B(G)$ of all $\varphi_\mu$ are just the linear span of $P(G)$. And we denote $B^0(G) = B(G) \cap L^p(G)$.

Prop. (X.6.3.14). The set of regular Borel probability measures on a compact $X$ is weak*-compact in $C(X)^*$. (Use Alaoglu).

Prop. (X.6.3.15) (Fourier Inversion Formula). (special case of (X.6.3.20)) If $f \in B^1$, then $\hat{f} \in L^1(\hat{G})$, and if the Haar measure $d\xi$ of $\hat{G}$ is suitably normalized w.r.t. the Haar measure of $G$, then $d\mu_f(\xi) = \hat{f}(\xi) d\xi$, i.e. $f(x) = \int (x, \xi) \hat{f}(\xi) d\xi$. This measure $d\xi$ is called the dual measure of $dx$.

In particular, this is true for continuous function on a compact group.

Proof: Cf.[Folland Abstract Harmonic Analysis P105]. □

Cor. (X.6.3.16). If $f \in L^1(G) \cap P$, then $\hat{f} \geq 0$, as $d\mu_f(\xi) = \hat{f}(\xi) d\xi$ and $\mu_f$ is positive, by Bochner’s theorem (X.6.3.11).
Prop. (X.6.3.17) (Dual Measure of Discrete Group). If $\mu$ is the counting measure on a discrete group, then its dual measure satisfies $|\hat{G}| = 1$, and if $G$ is compact and $|G| = 1$, then the dual measure is the counting measure on $\hat{G}$.

Proof: First (X.6.3.7) should be noticed. If $G$ is compact and $|G| = 1$, then if $g = 1$, then $\hat{g} = \chi_{\{1\}}$, so $g(x) = \sum (x, \xi) \hat{g}(\xi)$, which shows the dual measure is counting measure by definition (X.6.3.15). □

Prop. (X.6.3.18) (Plancherel Theorem). The Fourier transform on $L^1(G) \cap L^2(G)$ extends uniquely to an isomorphism from $L^2(G)$ to $L^2(\hat{G})$.

Proof: Cf. [Folland P108]. □

Pontryagin Duality

Prop. (X.6.3.19) (Pontryagin Duality). For a locally compact Abelian group $G$, $G \rightarrow \hat{G}$ is an isomorphism of topological groups.

Proof: Cf. [Folland Abstract Harmonic Analysis P110]. □

Cor. (X.6.3.20) (Fourier Inversion Theorem). If $f \in L^1(G)$ and $\hat{f} \in L^1(\hat{G})$ and the measure are dual to each other (X.6.3.15), then $f(x) = \hat{f}(x^{-1})$, i.e. $f(x) = \int (x, \xi) \hat{f}(\xi)d\xi$ a.e.

Proof: As

$$\hat{f}(\xi) = \int (x, \xi)f(x)dx = \int (x^{-1}, \xi)f(x)dx = \int (x, \xi)f(x^{-1})dx,$$

so by definition $\hat{f} \in B^1(\hat{G})$, and $d\mu_f(x) = f(x^{-1})dx$. Then by (X.6.3.15), $f(x^{-1}) = \hat{f}(x)$. □

Cor. (X.6.3.21) (Fourier Uniqueness Theorem). If $u, v \in M(G)$ satisfy $\hat{u} = \hat{v}$, then $u = v$. In particular, if $f, g \in L^1(G)$ and $\hat{f} = \hat{g}$, then $f = g$.

Proof: By (X.6.3.10) (norm decreasing), $\mu$ is uniquely determined by $\varphi_\mu(\xi) = \hat{\mu}(\xi^{-1})$ by Fourier inversion. □

Prop. (X.6.3.22) (Duality of Subgroups). $(H^\perp)^\perp = H$ for closed subgroup $H$ of a locally compact Abelian group $G$.

Proof: Suffices to prove $(H^\perp)^\perp \subset H$. If $x_0 \notin H$, then Gelfand-Raikov shows that there is a character $\eta$ on $G/H$ that $\eta(q(x_0)) \neq 1$, so $x_0 \notin (H^\perp)^\perp$. □

Prop. (X.6.3.23). If $H$ is a closed subgroup of $G$, then there are natural isomorphisms of LCA groups:

$$\Phi : (G/H) \cong H^\perp, \quad \Psi : \hat{G}/H^\perp \cong \hat{H}.$$

Proof: $\Phi$ is clearly algebraic isomorphism. If $|\eta(q(K)) - 1| < \varepsilon$, then $|\eta(K) - 1| < \varepsilon$, so $\Phi$ is continuous in the compact-open topology. Similarly, to show $\Phi$ is open, it suffices to show a compact subset of $G/H$ has a compact inverse image in $G$, but this is just (X.6.1.24).

Now for $\Psi$, notice $\hat{G}/H^\perp \cong (H^\perp)^\perp \cong H$ by (X.6.3.22), so by Pontryagin duality theorem, $\hat{G}/H^\perp \cong \hat{H}$. □
Prop. (X.6.3.24) (Poisson Summation Formula). Suppose $H$ is a closed subgroup of $G$, if $f \in L^1(G)$, define $F(xH) = \int_H f(xy) dy$ on $G/H$, then $F \in L^1(G/H)$ by (X.6.1.29), then:

- $\hat{F} = \hat{f}|_{H^\perp}$, where $G/H$ is identified with $H^\perp$ by (X.6.3.23).
- If $\hat{f}|_{H^\perp} \in L^1(H^\perp)$, then with the dual measure of $G/H$ on $H^\perp$ (X.6.3.15), we have
  \[
  \int_H f(xy) dy = \int_{H^\perp} \hat{f}(\xi)(x,\xi) d\xi.
  \]
  In particular, take $x = e$, then
  \[
  \int_H f(y) dy = \int_{H^\perp} \hat{f}(\xi) d\xi.
  \]

Proof: Notice for $\xi \in H^\perp$,

\[
\hat{F}(\xi) = \int_{G/H} F(xH)\overline{(x,\xi)} d(xH) = \int_{G/H} \int_H f(xy)\overline{xy,\xi} dy d(xH) = \int_G f(x)\overline{x,\xi} dx = \hat{f}(\xi)
\]
by (X.6.1.29). And 2 is just (X.6.3.20) applied to $F(xH)$ on $G/H$.

Cor. (X.6.3.25). In the situation of (X.6.3.24), if $H$ is discrete in $G$ and $G/H$ is compact, then both $H, H^\perp$ are discrete, then by considering the dual measure using (X.6.3.17), the Poisson summation reads:

\[
\sum_H f(xy) = \frac{1}{\mu(G/H)} \sum_{H^\perp} \hat{f}(\xi)(x,\xi).
\]

4 Compact Group

Cf. [群表示论 notes] and [Fol15] Chap5.
In this subsection, we consider representations of a compact group over $\mathbb{C}$.

Unitary Representations

Prop. (X.6.4.1). Integration of a nontrivial character on a compact group $G$ w.r.t. the Haar measure is 0.

Proof: \[
\int f(x) d\mu(x) = \int f(yx) d\mu(yx) = f(y) \int f(x) d\mu(x).
\]
Now choose a $y$ that $f(y) \neq 1$.

Prop. (X.6.4.2) (F.d. Representation is Unitary). If $V$ is a real/complex f.d. representation $\pi$ of a compact group $G$, then there is an inner product on $V$ that the action of $G$ is orthogonal/unitary.

Proof: Choose an arbitrary inner product $(\cdot, \cdot)_0$ on $V$, then consider

\[
(u, v) = \int_G (\pi(x)u, \pi(x)v)_0 dx.
\]
where $dx$ is a Haar measure on $G$. Then

\[
(\pi(y)u, \pi(y)v) = \int_G (\pi(xy)u, \pi(xy)v)_0 dx = \Delta(y) \int_G (\pi(x)u, \pi(x)v)_0 dx = \int_G (\pi(x)u, \pi(x)v)_0 dx
\]
because $G$ is compact hence unimodular (X.6.1.11). Thus this is an inner product on $V$ that is invariant under $G$. 

□
Cor. (X.6.4.3) (F.D. Representation of Compact Groups Totally Decomposable). Any f.d. representation of a compact group is totally decomposable.

Proof: This is because we can assume this representation is unitary by (X.6.4.2), and then for any subrepresentation we can take the orthogonal complement. □

Lemma (X.6.4.4). Suppose \((\pi, \mathcal{H})\) is a continuous unitary representation of the compact group \(G\), let \(u \neq 0 \in \mathcal{H}\) be a unit vector, if the operator \(T\) on \(\mathcal{H}\) is defined by

\[
Tv = \int_G (v, \pi(x)u)\pi(x)udx,
\]

then \(T\) is a positive, non-zero compact operator in \(C(\pi)\).

Proof:

\[
(Tv, v) = \int_G (v, \pi(x)u)(\pi(x)u, v)dx = \int_G |(\pi(x)u, v)|^2 dx \geq 0,
\]

so it is positive. Moreover, if \(v = u\), then \(x \mapsto |(\pi(x)u, v)|\) is a positive on a nbhd of 1, so \(T \neq 0\).

Finally, because \(G\) is compact, \(x \mapsto \pi(x)u\) is uniformly continuous, so for any \(\varepsilon > 0\), there is a disjoint partition \(E_i\) of \(G\) and \(x_i \in E_i\) that if \(x \in E_i\), then \(|\pi(x)u - \pi(x_i)u| \leq \varepsilon/2\). Then

\[
|\|(v, \pi(x)u)\pi(x)u - (v, \pi(x_i)u)\pi(x_i)u|| \leq \|(v, [\pi(x) - \pi(x_i)]u)\pi(x)u|| + \|(v, \pi(x_i)u)[\pi(x) - \pi(x_i)]u|| < \varepsilon ||v||.
\]

So consider

\[
T_{\varepsilon}(v) = \sum |E_j| (v, \pi(x_j)u)\pi(x_j)u = \sum \int_{E_i} (v, \pi(x_j)u)\pi(x_j)udx
\]

then \(||T - T_{\varepsilon}|| < \varepsilon\), and \(T_{\varepsilon}\) has f.d image, thus \(T\) is compact, by (X.4.5.3).

Also \(T \in C(\pi)\) because

\[
\pi(y)Tv = \int_G (v, \pi(x)u)\pi(\pi(y)x)udx = \int_G (v, \pi(y^{-1}x)u)\pi(x)udx = \int_G (\pi(y)v, \pi(x)u)\pi(x)udx = T\pi(y)v.
\]

Prop. (X.6.4.5) (Unitary Representations of Compact Groups). If \(G\) is compact group, then very irreducible unitary representation of \(G\) is of f.d., and every unitary representation \((\rho, V)\) of \(G\) is an orthogonal sum of irreducible unitary subrepresentations. Moreover, the isotopic part \(V^\pi\) for any irreducible representation \(\pi\) of \(G\) is uniquely determined.

Proof: If \(\pi\) is an irreducible unitary representation, choose \(T\) as in lemma (X.6.4.4) which is non-zero compact in \(C(\pi)\), then by Schur’s lemma, \(T = cI, c \neq 0\), so \(\dim \mathcal{H}_\pi < \infty\), by (X.4.1.7).

For the direct sum, by taking orthogonal complements and Zorn’s lemma, it suffices to show any unitary representation \(\pi\) has an irreducible subrepresentation. Choose \(T\) as in (X.6.4.4), then \(T\) is compact nonzero self-adjoint, so by Riesz-Fredholm (X.4.5.9) it has a finite-dimensional eigenspace, which is \(\pi\)-invariant, and it clearly has an irreducible subrepresentation by taking orthogonal complements.

For the orthogonality, Let \(V^\pi\) be the linear span of invariant subspaces isomorphic to \(\pi\), for \(L_1, L_2\) of type \(\pi_1 \neq \pi_2\), then consider the orthogonal projection \(P\) onto \(L_2\), then \(P|_{L_1} \in C(\pi_1, \pi_2)\), which vanishes by Schur (X.6.2.5), so they are orthogonal. □

Cor. (X.6.4.6). The cardinality of irreducible constituents of \(V\) that is isomorphic to \(\pi\) is independent of the decomposition, and it is equal to \(\dim \text{Hom}_G(\pi, \rho)\), and is denoted by \(\text{mult}(\pi, \rho)\).
Proof: Cf.[Folland, P137]. □

Cor. (X.6.4.7) (Representations of Product Groups). Any irreducible representation of the product group $G \times H$ for $G, H$ is of the form $\rho \boxtimes \psi$ where $\rho$ and $\psi$ are irreducible representations of $G, H$ resp.

Proof: ?(X.6.4.5) showed that we only need to consider f.d. case, and (X.6.4.2) shows it is unitary. Then decompose this representation w.r.t $G$, then $H$ acts on each isotopic part by a scalar by Shur’s lemma (X.6.2.5), so this is clearly of the form $\rho \times \psi$ for $\psi$ an irreducible representation. □

Matrix Coefficients and Peter-Weyl Theorem

Def. (X.6.4.8) ($K$-Finite Vectors). Let $K$ be compact group. For an irreducible representation $\rho$ of $K$, denote $V^\rho = \rho \otimes \hom_K(\rho, V)$ the $\rho$-isotypic component in $V$. And let $V^{K\text{-fin}} = \oplus \rho V^\rho$ the space of $K$-finite vectors in $V$ (X.6.4.5).

Def. (X.6.4.9) (Matrix Coefficients). Firstly $C(G)$ is a representation of $G \times G$ by $((g_1, g_2)f)(g) = f(g_1^{-1}g g_2)$.

For a f.d. representation $(\pi, V)$ of a topological group $G$, we can view $\text{End}(V)$ as a representation of $G \times G$ via

$$(g_1, g_2)S = T_{g_1}ST_{g_2}^{-1}$$

There is a matrix coefficient map:

$$MC_V : \text{End}(V) \to C(G), \quad MC_V(S)(g) = MC_{S, V}(g) = \text{tr}(ST_g|V)$$

is a map of $K \times K$-representations. And denote $E_\pi$ the image of

Prop. (X.6.4.10) (Orthogonality Conditions). Let $V_1, V_2$ be f.d. continuous irreducible representations of $K$, then

$$\int_K MC_{S_1, V_1}(k)MC_{S_2, V_2}(k) = 0$$

unless $V_2 \cong V_1^*$. And in this case,

$$\int_K MC_{S_1, V}(k)MC_{S_2, V^*}(k) = \frac{1}{\dim V} \text{tr}(S_1 \circ S_2^*|V)$$

Proof: Cf.[Gaitsgory P3]. □

Def. (X.6.4.11) (Characters). Let $V$ is a f.d. continuous representation of $K$, let $\chi_V = MC_V(\text{Id}_V)$, it is called the character of $V$, and if $V$ is irreducible, define $\xi_V = \dim V \cdot \chi_V$.

Prop. (X.6.4.12). If we take an invariant inner product on $V$, and $e_i$ an orthonormal basis, then

$$\chi_V(g) = \sum_i (ge_i, e_i),$$

and

$$\chi_V^*(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}, \quad \chi_V \oplus W = \chi_V + \chi_W, \quad \chi_V \otimes W = \chi_V \cdot \chi_W.$$
Prop. (X.6.4.13). Let $V,W$ be irreducible, then
\[
\int_K \chi_V(k)\chi_W(k^{-1}) = \int_K \chi_V(k)\chi_W^*(k) = \int_K \chi_V(k)\overline{\chi_W(k)}
\]
so by (X.6.4.10) this equals 1 if $V \cong W$ and 0 otherwise.

Prop. (X.6.4.14). Then for $V,W$ irreducible, $T_{\xi_V} \in \text{End} W$ if $W \cong V$ and zero otherwise.

Proof: Cf.[gaitsgory P4]. □

Cor. (X.6.4.15). $\xi_V \ast \xi_W = 0$ unless $W \cong V$ and $\xi_V \ast \xi_V = \xi_V$.

Proof: Cf.[gaitsgory P4]. □

Prop. (X.6.4.16). If $\rho$ is an irreducible f.d representation of $K$ and $V$ is a continuous representation of $K$, then for any $S \in \text{End}(\rho^*)$, the image of
\[
MC_{S,\rho\mu_{Haar}} \in \text{meas}(K)
\]
acting on $V(X.4.3.24)$ belongs to $V^\rho$.

Proof: Cf.[Gaitsgory P7]. □

Cor. (X.6.4.17). For an irreducible representation $\rho$ of $K$, the element $\xi_{\rho\mu_{Haar}} \in \text{Meas}(K)$ acts in any continuous representation $V$ as a projection with image equal to $V^\rho$.

Proof: Directly from the proposition and (X.6.4.14). □

Prop. (X.6.4.18) (Peter-Weyl). ? i

Prop. (X.6.4.19). For any continuous representation $K$, the subset $V^{K-\text{fin}}$ is dense in $V$.

Proof: For any $v \in V$, choose a Dirac sequence $f_n$, then $T_{f_n\mu_{Haar}} v \to v$. Then by Peter-Weyl(X.6.4.18), we can choose $K$-finite functions $g_n$ that $\|g_n - f_n\|_{L^2} < \frac{1}{n}$. Then $g_n$ also converges to $\delta_1$ in the weak topology. Thus
\[
T_{g_n\mu_{Haar}} v \to v
\]
and (X.6.4.16) shows $T_{g_n\mu_{Haar}} v \in V^{K-\text{fin}}$. □

Cor. (X.6.4.20). Matrix coefficients of f.d. representations are dense in $C(K)$. (Immediate from the proposition and Peter-Weyl theorem(X.6.4.18).

Fourier Analysis on Compact Groups

Prop. (X.6.4.21). Let $\chi$ be a character of a representation of compact group $G$ of dimension $n$, then
- $\chi(1) = n$.
- $\chi(s^{-1}) = \chi(s^*)$.
- $\chi(tst^{-1}) = \chi(s)$.

Proof: Notice the eigenvalues of $\rho(g)$ all have absolute value 1, because this representation is unitarizable(X.6.4.2) thus $\rho^*(g) = \rho(g)^*$. □
**Def. (X.6.4.22) (Class Functions).** A measurable function on $G$ is called a class function iff $f(y^{-1}xy) = f(x)$ a.e. $(x, y) \in G \times G$. Denote $ZL^p(G)$ the space of class functions in $L^p(G)$ and $ZC(G)$ the space of continuous functions on $G$.

**Prop. (X.6.4.23).** The space $L^p(G)$ and $C(G)$ are Banach spaces under convolution (X.6.1.19), and $ZL^p(G)$ and $ZC(G)$ are their centers.

*Proof:* Cf.[Folland, P146].

**Prop. (X.6.4.24) (Characters Orthonormal Basis).** $\{\chi_\pi | \pi \in \hat{G}\}$ is an orthogonal basis for $ZL^2(G)$.

*Proof:* $\chi_\pi \in ZC(G) \subset ZL^2(G)$ by (X.6.4.21). For the rest Cf.[Folland, P147].

**Cor. (X.6.4.25) (Characters Determine Representations).** If two f.d. unitary representations of $G$, if $\chi_1 = \chi_2$, then they are isomorphic.

*Proof:* These two unitary representations decompose into finite sums of irreducible representations of $G$ by (X.6.4.5), and we can use (X.6.4.24) to detect the multiplicity of each irreducible representation of $G$, so they are isomorphic.

## 5 Induced Representation

**Lemma (X.6.5.1).** If $H$ is a closed subgroup of a locally compact subgroup $G$, $q : G \to G/H$, for any unitary representation $(\sigma, \mathcal{H})$ of $H$, let $\mathcal{F}_0$ be the space of continuous functions $f : G \to \mathcal{H}$ that $q(\text{Supp}(f))$ is compact, and

$$f(x\xi) = \sqrt{\frac{\Delta_H(\xi)}{\Delta_G(\xi)}} \sigma(\xi^{-1})f(x).$$

Then if $\alpha : G \to \mathcal{H}$ is continuous with compact support, then

$$f_\alpha(x) = \int_H \sqrt{\frac{\Delta_G(\eta)}{\Delta_H(\eta)}} \sigma(\eta)\alpha(x\eta)d\eta \in \mathcal{F}_0$$

and is left uniformly continuous w.r.t $G$. Moreover, every element in $\mathcal{F}_0$ arises in this way.

*Proof:* Clearly $q(\text{Supp } \alpha) \subset q(\text{Supp } \alpha)$, and

$$f_\alpha(x\xi) = \int_H \sqrt{\frac{\Delta_G(\eta)}{\Delta_H(\eta)}} \sigma(\eta)\alpha(x\xi\eta)d\eta = \int_H \sqrt{\frac{\Delta_G(\xi^{-1}\eta)}{\Delta_H(\xi^{-1}\eta)}} \sigma(\xi^{-1}\eta)\alpha(x\eta)d\eta = \sqrt{\frac{\Delta_H(\xi)}{\Delta_G(\xi)}} \sigma(\xi^{-1})f_\alpha(x).$$

For left uniformly continuity, Cf.[Folland, P164].

For the surjectivity, if $f \in \mathcal{F}_0$, by (X.6.1.26), there exists $\psi \in C_c(G)$ that $\int_H \psi(x\eta)d\eta = 1$ for $x \in \text{Supp } f$. So we can let $\alpha = \psi \cdot f$, then

$$f_\alpha(x) = \int_H \sqrt{\frac{\Delta_G(\eta)}{\Delta_H(\eta)}} \psi(x\eta)\sigma(\eta)f(x\eta)d\eta = \int_H \psi(x\eta)f(x)d\eta = f(x)$$

*Remark (X.6.5.2).** If we consider the right action, then it suffices to consider all the functions $g(x) = f(x^{-1})$. 
Def. (X.6.5.3) (Induced Representation). Consider the projection $P : C_c(G) \to C_c(G/H)$ defined in (X.6.1.28), if $f \in \mathcal{F}_0$, then $x \mapsto ||f(x)||_2^2$ is nearly a rho-function, but is may not be positive-valued. Now lemma (X.6.1.27) and (X.6.1.34) shows

$$P\varphi \mapsto \int_G \varphi ||f(x)||_2^2 dx$$

is a well-defined continuous positive linear functional on $C_c(G/H)$, so by (X.1.1.10) there is a Radon measure $\mu_f$ on $G/H$ that

$$\int_{G/H} P\varphi d\mu_f = \int_G \varphi ||f(x)||_2^2 dx$$

for any $\varphi \in C_c(G)$. Clearly $\text{Supp} \mu_f \subset q(\text{Supp} f)$ hence compact. In particular, $\mu_f(G/H) < \infty$.

Then for the measure $\mu_{f,g} = \frac{1}{4}(\mu_{f+g} - \mu_{f-g} + i\mu_{f+ig} - i\mu_{f-ig})$, we have

$$\int_{G/H} P\varphi d\mu_{f,g} = \int_G \varphi(f(x),g(x))_\sigma dx.$$ 

And we define an inner product on $\mathcal{F}_0$ as $(f, g) = \mu_{f,g}(G/H)$.

Then for the left action $\lambda$ of $G$ on $\mathcal{F}_0$, if we choose $\varphi \in C_c(G)$ that $P\varphi = 1$ on $q(\text{Supp} f)$ (X.6.1.26), for any $x \in G$,

$$\int_{G/H} P\varphi(p)d\mu_{\lambda(x)}(p) = \int_G ||f(x^{-1}y)||_2^2 \varphi(y) dy = \int_G P\varphi(xp)d\mu_f(p) = \int_G P\varphi(p)d\mu_f(x^{-1}p),$$

thus $\mu_{\lambda(x)f}(pH) = \mu_f(x^{-1}pH)$, so

$$||\lambda(x)f||^2 = \mu_{\lambda(x)f}(G/H) = \mu_f(x^{-1}G/H) = \mu_f(G/H) = ||f||^2.$$ 

So the left translation action of $G$ on $\mathcal{F}_0$ is unitary, then by (X.6.2.9), if we denote $\mathcal{F}$ the completion of $\mathcal{F}_0$, then $\mathcal{F}$ is a continuous unitary representation of $G$, called the induced representation $\text{ind}_{H}^{G}(\sigma)$.

Prop. (X.6.5.4) (Frobenius Reciprocity). If $G$ is compact group and $H$ is a closed subgroup, $\pi$ is an irreducible unitary representation of $G$, $\rho$ is an irreducible unitary representation of $H$, then

$$C(\pi, \text{ind}_{H}^{G}(\rho)) = C(\pi|_{H},\rho), \quad \text{mult}(\pi, \text{ind}_{H}^{G}(\rho)) = \text{mult}(\rho, \pi|_{H}).$$

Proof: It suffices to prove the first one, the second follows from (X.6.2.3) and (X.6.4.6).

$G/H$ admits a $G$-invariant measure as $\Delta_G = \Delta_H = 1$. For the rest, it is similar to that of (III.1.8.32), Cf.[Folland, P172].

Prop. (X.6.5.5). Let $G$ be a unimodular topological group and $G = PK$, where $P, K$ are closed subgroups and $K$ is compact. Let $\Delta = \Delta_P$ and $C(P\setminus G, \Delta)$ be the space of continuous functions on $G$ that satisfies $f(pg) = \Delta(p)f(g)$, then the linear functional on it:

$$I(f) = \int_K f(k) dk$$

is invariant under right action of $G$.

Proof: Cf.[Bump, P222].

Similarly as the proof of (X.6.5.1), we can prove that any function in $C(P\setminus G, \Delta)$ is of the form $f_\alpha$ for $\alpha \in C_c(G)$ where

$$f_\alpha(g) = \int_P \alpha(pg) dp.$$ 

Now if $\alpha \in C_c(G)$, by (X.6.1.30), $I(f_\alpha) = \int_G \alpha(g) dg$. So $\alpha \mapsto I(f_\alpha)$ is invariant under right action of $G$. Now that $\alpha \mapsto f_\alpha$ is surjective and clearly commutes with right action, so does $I$. □
X.7 Harmonic Analysis

Main reference are [Rudin, Functional Analysis]. Notice much should be rewritten in greater generality of Abstract Fourier Analysis X.6.

1 Distributions

Def. (X.7.1.1) (Test Functions). The space $D(\Omega)$ of test functions has the induced topology coincides with that of $D(K)$, and any bounded subsets are in some $D(K)$, thus it is complete and has Heine-Borel because $D(K)$ does.

The space of continuous linear functionals of $D(\Omega)$ is called the space of distributions $D'(\Omega)$. It is equivalence to the restriction to every $D(K)$ is continuous, Cf.[Rudin P155]. The order of a distribution $\Lambda$ is the minimal $N$ that $|\Lambda \varphi| \leq C_K ||\varphi||_N$ for every $\varphi \in D(K)$, it might be $\infty$.

Def. (X.7.1.2). The differentiation of a distribution $\Lambda$ is defined as $D^\alpha \Lambda(\varphi) = (-1)^{|\alpha|} \Lambda(D^\alpha \varphi)$. The multiplication by a smooth function $f$ is defined by $f \Lambda(\varphi) = \Lambda(f \varphi)$. Then $D^\alpha(f \Lambda) = \sum_{\beta \leq \alpha} C_{\alpha \beta}(D^{\alpha-\beta}f)(D^\beta \Lambda)$.

Support of a Distribution

Def. (X.7.1.3). The support $\text{Supp}(\Lambda)$ of a distribution is the complement of the open sets $U$ that $\Lambda(f) = 0$ for any $f$ with support in $U$.

If $\text{Supp}(\Lambda)$ is compact, then $\Lambda$ has finite order and $|\Lambda \varphi| \leq C ||\varphi||_N$ for some $N$, and $\Lambda$ extends uniquely to a continuous linear functional on $C^\infty(\Omega)$.

Proof: This is because its support is compact so we can choose a smooth $\psi$ that = 1 on $\text{Supp} \varphi$ and has support in $W \subset \Omega$. Then by (X.7.1.1), there is a $C$ that $|\Lambda(\psi \varphi)| < C ||\psi \varphi||_N$, and Leibniz rule will give us the result.

Prop. (X.7.1.4). If the support of a $\Lambda$ is a pt $p$ (thus has finite order $m$), then it is a linear combination of $D^\alpha \delta_p, |\alpha| \leq m$. (use approximate identity and show the kernel of $\Lambda$ is contained in the kernel of $D^\alpha \delta_p$).

Proof: Cf.[Rudin P165].

Prop. (X.7.1.5). For any distribution $\Lambda$, there exist continuous functions $g_\alpha$ in $C^\infty(\Omega)$ that each compact $K$ intersects support of f.m $g_\alpha$ and $\Lambda = \sum D^\alpha g_\alpha$. When $\Lambda$ has finite order, we can use only f.m $g_\alpha$.

Proof: use partition of unity. Then for a compact $K$, find a compact-open $W$, then find a bump function between $K \subset W$, thus reduce to the case of $D^\alpha \delta_p$. For the rest, Cf.[Rudin P169].

Convolution on $\mathbb{R}^n$

Denote $D = D(\mathbb{R}^n)$, $D' = D'(\mathbb{R}^n)$.

Def. (X.7.1.6). The translation of a distribution $u$ is defined as $(\tau_x u)(\varphi) = u(\tau_{-x} \varphi)$, where $\tau_{x} \varphi(y) = \varphi(y-x)$.

The convolution of a test function with a distribution $u$ is defined as $(u * \varphi)(x) = u(\tau_x \varphi)$, where $\varphi(y) = \varphi(-y)$.
Prop. (X.7.1.7) (Special Case of (X.7.1.10)). For \( u \in D', \varphi \in D, \psi \in D \),
- \( \tau_x (u * \varphi) = (\tau_x u) * \varphi = u * (\tau_x \varphi) \).
- \( u * \varphi \in C^\infty \) and \( D^\alpha (u * \varphi) = (D^\alpha u) * \varphi = u * (D^\alpha \varphi) \).
- \( u * (\varphi * \psi) = (u * \varphi) * \psi \).

If \( u \) has compact support, then (X.7.1.3) shows that \( u \) can extend to \( C^\infty \), thus convolution is defined for \( \varphi \in C^\infty \) and the first two formulae still hold, and when \( \psi \in D \),
\[
 u * \psi \in D, \quad u * (\varphi * \psi) = (u * \varphi) * \psi = (u * \psi) * \varphi
\]

Proof: Cf.[Rudin P171], [Rudin P174]. \( \Box \)

Cor. (X.7.1.8). \( L : \varphi \mapsto u * \varphi \) is a continuous linear map into \( C^\infty \) that commutes with \( \tau_x \). And any these map comes from a \( u * : \) let \( u = (L \vec{\varphi})(0) \).

Proof: It is continuous because of of closed graph theorem(X.3.2.8), \( \lim (u * \varphi_i)(x) = \lim u(\tau_x \vec{\varphi}_i) = u(\tau_x \vec{\varphi}) \).

Cor. (X.7.1.9). When \( u, v \in D' \) and one of them has compact support, then similar to (X.7.1.8), \( L \varphi = u * (\varphi * \psi) \) is a continuous linear map that commutes with \( \tau_x \), so there is a unique convolution distribution \( u * v \) that \( (u * v) * \varphi = u * (v * \varphi) \). This convolution is compatible with the previous one when \( v \in D \).

Prop. (X.7.1.10) (Convolution of Distributions). For \( u, v, w \in D' \),
- if one of \( u, v \) has compact support, then \( u * v = v * u \), and \( \text{Supp}(u * v) \subset \text{Supp}(u) + \text{Supp}(v) \).
- if two of three of \( u, v, w \) has compact support, then \( (u * v) * w = u * (v * w) \).
- \( D^\alpha u = (D^\alpha \delta) * u \).
- if one of \( u, v \) has compact support, then \( D^\alpha (u * v) = (D^\alpha u) * v = u * (D^\alpha v) \).

Proof: Cf.[Rudin P177]. \( \Box \)

Def. (X.7.1.11). A approximate identity here is a \( h \in D \) that \( h_k(x) = k^n h(kx) \). Then we will have \( \lim \varphi * h_j = \varphi \) for \( \varphi \in D \), \( \lim u * h_j = u \) in \( D' \).

2 Fourier Analysis on \( \mathbb{R}^n \)

Def. (X.7.2.1) (Notations). We denote the normalized notation \( \mathbb{R}^n \) as \( dm = (2\pi)^{-n/2} dx \) and
\[
 D_\alpha = \frac{1}{i|\alpha|} D^\alpha = \frac{1}{i|\alpha|} \frac{\partial}{\partial x^\alpha},
\]
which will simplify many notations compared to \( D^\alpha \). The Fourier transform here of a function \( f \in L^1(\mathbb{R}^n) \) is the function \( \hat{f} \) that \( \hat{f}(t) = \int_{\mathbb{R}^n} f e^{-it \cdot \partial} dm ). \)

See(X.7.2.12) for general Fourier transform.

Prop. (X.7.2.2). For \( f \in L^1(\mathbb{R}) \),
\[
 \hat{e_x f} = e_{-x} \hat{f}, \quad e_x \hat{f} = \tau_x \hat{f},
\]
\[
 \hat{f * g} = \hat{f} \hat{g}, \quad f(x/\lambda)(t) = \lambda^n \hat{f}(\lambda t).
\]
(Note \( ||f * g||_1 \leq ||f||_1 ||g||_1 \)).
Lemma (X.7.2.3). Let $f = e^{-1/2|x|^2}$, then $f \in S$, $\hat{f} = f$ and $f(0) = \int \hat{f}$. (reduce to the 1 dimensional case, in which case, $f' + xy = 0$, and $\hat{f}$ also satisfies this).

Lemma (X.7.2.4). For $f, g \in L^1$, Fubini theorem shows $\int \hat{f}g = \int f\hat{g}$.

Prop. (X.7.2.5) (Classical Fourier Transform).
- $S$ is a Fréchet space in the topology defined by these norms.
- multiplication by $g \in S$ and derivations are continuous linear map from $S$ to $S$ (direct calculation).
- $\hat{P(D)}f(t) = P(t)\hat{f}(t)$ and $\hat{Pf} = P(-D)\hat{f}$.
- The Fourier transform is a continuous linear one-to-one automorphism of $S$, and $\Psi^2g = \hat{g}$.

Proof: 1: 2: 3: use(X.7.1.10) for the first one, and for the second one, should use definition of derivative and dominated convergence.

4:$\Psi f \in S$ by 3, and it is continuous by closed graph theorem. By(X.7.2.4) and(X.7.2.2), $\int \hat{f}(t)g(t/\lambda) = \int f(t/\lambda)\hat{g}(y)$. If $\hat{f}, \hat{g} \in L^1$, dominant convergence shows $g(0) \int \hat{f} = f(0) \int \hat{g}$. So we only need one $f$ that $f(0) = \int \hat{f}$, $f = e^{-1/2|x|^2}$ will suffice(X.7.2.3). Hence $g(0) = \int \hat{g}$ for every such $g$, and the conclusion follows by translation(X.7.2.2), and(X.7.2.7) also follows.

Cor. (X.7.2.6). If $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C_0(\mathbb{R}^n)$, and $||\hat{f}||_\infty \leq ||f||_1$, because $S$ is dense in $L^1(\mathbb{R}^n)$.

Prop. (X.7.2.7) (Inversion Theorem). If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$, then $\hat{f} = \Psi^2f$ a.e.

Proof: In(X.7.2.4), let $g \in S$ and substitute $g = \Psi g$ and use Fubini, we get $\hat{f} - \Psi^2f$ is orthogonal to every $S$, then every continuous function with compact support by(X.1.6.10). Thus they equal a.e.

Cor. (X.7.2.8). If $f, g \in S$, then $\hat{f}g = \hat{f} \ast \hat{g}$ (apply Fourier one time and use(X.7.2.2)), and thus $f \ast g \in S$.

Prop. (X.7.2.9) (Fourier-Plancherel). If $f, g \in S$, then

$$\int fg = \int g(x)\hat{f}(t)e^{ixt} = \int \hat{f}(t)\int g(x)e^{ixt} = \int \hat{f}g$$

by inversion formula. And $S$ is dense in $L^2$, thus it extends to a linear isometry of $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. This coincides with the Fourier transform on $L^1 \cap L^2$.

Prop. (X.7.2.10). $D$ injects into $S$ and is dense.(Notice they both are complete, but the subspace topology are different)(Use scaling, Cf.[Rudin Functional Analysis P189]). So we call a distribution tempered iff it comes from a continuous functional of $S$.

From(X.7.1.3), we know any distribution with compact support is tempered. By Holder, every $f \in L^p(\mathbb{R}^n), p \geq 1$ is tempered distribution, and every polynomial or functions of polynomial growth are tempered distribution.

$$D \subset S \subset L^2 = (L^2)' \subset S' \subset D'.$$

$S, S'$ is complete(X.3.4.3).
Prop. (X.7.2.11). A \( f \in S' \) iff \( f = \sum_{|\alpha| \leq m} D_\alpha (u_\alpha (1 + |x|^2)^{m/2}) \) for some \( m \), where \( u_\alpha \in L^2(\mathbb{R}^n) \).

Proof: In fact,
\[
||\varphi||'_m = (\sum_{|\alpha| \leq m} \int (1 + |x|^2)^m |D_\alpha \varphi|^2 \, dx)^2
\]
is an equivalent set of norms of \( S' \), Cf.[泛函分析张恭庆 P182]. And each of them defines a Hilbert space. So by Riesz we get the result. \( \square \)

Prop. (X.7.2.12) (Generalized Fourier Transform). For a tempered distribution \( u \in S' \), we define the Fourier transformation as the tempered distribution \( \hat{u}(\varphi) = u(\hat{\varphi}) \). It is easily verified that it is compatible with previously defined Fourier transform when seen as tempered distributions by?? In particular, this is defined for compactly supported distribution, \( L^p(\mathbb{R}^n), p \geq 1 \) and smooth functions of polynomial growth (X.7.2.10).

Prop. (X.7.2.13). \( P(D)u = \hat{P}\hat{u} \) and \( \hat{P}u = P(-D)\hat{u} \). And The Fourier transformation is a continuous linear isometry of \( S' \) in the weak* topology.

Cor. (X.7.2.14). \( \hat{1} = \delta \), thus \( \hat{P} = P(-D)\delta \) and \( P(D)\delta = P \). Now(X.7.1.4) tells us a distribution is the Fourier transform of a polynomial iff it has support in the origin.

Prop. (X.7.2.15) (Convolution of Tempered Distributions). Let \( u \in S' \) and \( \varphi, \psi \in S \), then
- \( u * \varphi \in C^\infty \) of polynomial growth and \( D^\alpha (u * \varphi) = (D^\alpha u) * \varphi = u * (D^\alpha \varphi) \).
- \( u * (\varphi * \psi) = (u * \varphi) * \psi \).
- \( \hat{u} * \varphi = \hat{\varphi} \hat{u} \), \( \hat{u} * \psi = \varphi \hat{u} \).
- If \( P \) is a polynomial and \( g \in S \), then \( D^\alpha u, Pu \) and \( gu \) are all tempered.

Proof: Cf.[Rudin Functional Analysis P195] for the first 3. \( \square \)

Prop. (X.7.2.16) (Mellin Inversion Formula). Given a function \( f : \mathbb{R}^+ \to \mathbb{C} \) satisfying suitable conditions, its Mellin transformation is defined to be
\[
M(f)(s) = \int_0^\infty f(t) t^s \frac{dt}{t}.
\]
whenever this integral is absolutely convergent.

Notice if \( \int_0^1 f(t) t^s \frac{dt}{t} \) is convergent for some \( s \), then it converges for any bigger \( s \), and if \( \int_1^\infty f(t) t^s \frac{dt}{t} \) converges for some \( s \), then it converges for any smaller \( s \). So the domain of \( M(f) \) if nonempty, is a vertical strip \( \sigma_1 < \text{Re}(\sigma) < \sigma_2 \) for \( \sigma_1, \sigma_2 \in [-\infty, \infty] \).

Then \( f \) can be recovered from \( M(f) \): for any \( \sigma_1 < \sigma < \sigma_2 \),
\[
f(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} x^{-s} M(f)(s) \, ds.
\]

Proof: Using the isomorphism of groups \( t = e^x : \mathbb{R}^+ \to \mathbb{R} \), this is just the usual Fourier transformation on \( \mathbb{R} \). \( \square \)
Paley-Wiener Theory

**Prop. (X.7.2.17).** For $\varphi \in D(\mathbb{R}^n)$ that has support in $rB$, the You-Know-How defined $\hat{\varphi}(z)$ is an entire function of several variable and satisfies:

$$|\varphi'(z)| \leq \gamma_N (1 + |z|)^{-N} e^{r |\text{Im} z|}.$$  

For $N \geq 0$. Conversely, any such function correspond to a $\varphi \in D(\mathbb{R}^n)$ that has support in $rB$.

**Proof:** Cf. [Rudin P198].

**Prop. (X.7.2.18) (Fourier-Laplace transformation).** For $u \in D'(\mathbb{R}^n)$ that has support in $rB$, of order $N$, the $\hat{u}(z) = u(e^{-z})$ is an entire function of several variable and satisfies:

$$|f(z)| \leq \gamma (1 + |z|)^N e^{r |\text{Im} z|}.$$  

Conversely, any such function correspond to a $u \in D'(\mathbb{R}^n)$ that has support in $rB$.

**Proof:** Cf. [Rudin P199].

3 Sobolev Space

**Def. (X.7.3.1).** For $1 \leq p < \infty$, the **Sobolev space** $W^{m,p}(\Omega)$ is the space of functions $u$ that $D^\alpha u \in L^p(\Omega)$ for every $|\alpha| \leq m$, with the norm $||u|| = \sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u(x)|^p \, dx$. The **Sobolev space** $W_0^{m,p}(\Omega)$ is the completion of the subspace $C_0^\infty(\Omega)$.

**Prop. (X.7.3.2) (Meyers-Serrin).** The Sobolev space $W^{m,p}(\Omega)$ is the completion of $u \in C^\infty(\Omega)$ that $D^\alpha u \in L^p(\Omega)$ for every $|\alpha| \leq m$.

**Proof:** Choose a countable partition of unity $\psi_k$, then as in the proof of (X.1.6.11), we can choose $\delta_k$ small enough and $||\psi u - (\psi u)_{\delta_k}|| < \varepsilon/2^k$ and $\varphi = \sum(\psi u)_{\delta_k}$ is definable.

**Prop. (X.7.3.3).** We denote $H^m(\Omega) = W^{m,2}(\Omega)$ and $H_0^m(\Omega) = W_0^{m,2}(\Omega)$ and $H^{-m}(\Omega) = (H_0^m(\Omega))^*$ when $m$ is an integer. Notice derivative is not applicable for $H^{-m}(\Omega)$ unless $\Omega = \mathbb{R}^n$.

When $\Omega = \mathbb{R}^n$, $D(\mathbb{R}^n)$ is dense in $W^{m,p}(\mathbb{R}^n)$, thus $W_0^{m,p} = W^{m,p}$. Define the **Sobolev space**

$$H^s = \{u|(1 + |y|^2)^s/2\hat{u}| \in L^2\}$$

$H^s$ is a Hilbert space and $H^s \subset S'$ for every $s$ (use Holder to show $\hat{u} \in S'$). $H^m$ coincides with previously defined $H^m$ when $m$ is a positive integer thus also negative-integer. A linear operator on $H = \bigcup H^s$ is said to have order $t$ if it maps every $H^s$ continuously into $H^{s-t}$.

**Proof:** By Plancherel,

$$||\varphi||_m = \left( \sum_{|\alpha| \leq m} ||D_\alpha u||_2^2 \right)^{1/2} \quad \text{and} \quad (\int (1 + |x|^2)^m |\hat{u}|^2)^{1/2}$$

are equivalence norms on $H^m$.

**Lemma (X.7.3.4) (Poincare Inequality).** For $\Omega$ bounded, on $C_0^\infty(\Omega)$ the $W^{m,p}$ norm is controlled by $L^p$ norms of its $m$th order derivatives.
Proof: We may assume \( \Omega \subset \prod_{i=1}^{n} [0, a] \), then for any \( u \in W^{m,p} \), \( u(x) = \int_0^1 D^1 u(t, x_2, \ldots, x_n) dt \), so by Holder inequality,
\[
|u(x)| \leq a^{1/q} \left( \int_0^a |D^1 u|^p dx_1 \right)^{1/p}.
\]
so
\[
\int_{\Omega} |u(x)|^p dx \leq a^q \int_{\Omega} |D^1 u|^p dx_1.
\]
Doing the same for all other derivatives, we can see the norm is controlled by the highest \((m\text{-th})\) order norms. \(\square\)

Prop. (X.7.3.5). When \( t < s \), \( H^s \subset H^t \). And \( H^s \) are isometric to \( H^t \) by \( \hat{v} = (1 + |y|^2)^{t/2} \hat{u} \) and is of order \( t \). \( D^\alpha \) is of order \( |\alpha| \). If \( f \in S \), then \( u \rightarrow fu \) is an operator of order 0.

Proof: Cf. [Rudin P217]. \(\square\)

Prop. (X.7.3.6) (Sobolev Embedding Theorem). On a manifold of dimension \( n \) which is compact with Lipschitz boundary or complete of positive injective radius and bounded sectional curvature,

- if \( k > l \) be integers and \( \frac{1}{p} - \frac{k}{n} = \frac{1}{q} - \frac{l}{n} \)
  then \( W^{k,p}(\text{int}(M)) \subset W^{l,q}(M) \) continuously.

- if \( \frac{1}{p} - \frac{k}{n} = -\frac{r + \alpha}{n} \)
  then \( W^{k,p}(\text{int}(M)) \subset C^{r,\alpha}(M) \) continuously.

Proof: Cf. [Evans P290]. \(\square\)

Cor. (X.7.3.7) (Gagliardo–Nirenberg–Sobolev). On a manifold of dimension \( n \) which is compact with Lipschitz boundary or a complete of positive injective radius and bounded sectional curvature, if \( \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n} \) (Sobolev conjugate), then \( W^{1,p}(\text{int}(M)) \subset L^{p^*}(M) \) continuously.

Cor. (X.7.3.8). On a manifold of dimension \( n \) which is compact with Lipschitz boundary or a complete of positive injective radius and bounded sectional curvature, if \( m > n/2 \), then \( W^{m,2}(\text{int}(M)) \subset C(\Omega)(M) \) continuously. And the functions in \( W^{m,2}_0 \) are continuous and vanish at the boundary, by \( C_0 \) approximation.

Proof: The \( \mathbb{R}^n \) case can be directly proved: because we have the equivalent norm (X.7.3.3), \( \hat{u} \in L^2 \) thus \( u \in L^2 \), and
\[
\int |\hat{u}| \leq \left( \int (1 + |x|^2)^m |\hat{u}|^2 \right)^{1/2} \cdot \left( \int 1/(1 + |x|^2)^m \right)^{1/2}.
\]
We have \( \hat{u} \in L^1 \), thus inversion formula applies that \( u \) is continuous and \( ||u||_\infty \leq ||\hat{u}||_1 \leq C ||u||_{H^m} \).
\(\square\)

Cor. (X.7.3.9). \( \cap_s H^s = C^\infty(M) \).
Prop. (X.7.3.10) (Rellich-Kondrechov). On a compact manifold with $C^1$ boundary of dimension $n$, if $k > l$ and 
\[
\frac{1}{p} - \frac{k}{n} < \frac{1}{q} - \frac{l}{n}
\]
then $W^{k,p} \subset W^{l,q}$ completely continuously.

Proof: Cf. [Distributions and Operators P199], [Evans P290]. □

Cor. (X.7.3.11). On a bounded extension domain of $\mathbb{R}^n$, $W^{1,p} \subset L^p$ completely continuously.

Proof: We prove the $p = 2$ case. For a sequence $u_m$ in $W^{1,2}$, we have $||u_m - u_p||_2 = ||U_m - U_p||_2$. By (X.4.4.9), there is a subsequence that $\hat{U}_m$ pointwise converge. Notice they are uniformly bounded, Now apply two region argument, for $|x| < r$, use Lebesgue dominant convergence, and for $|x| > r$, use $\int (1 + |x|^2)|\hat{U}_m - \hat{U}_p|^2$ is bounded to conclude $||u_m - u_p||_2 \to 0$. □

Prop. (X.7.3.12). $u \in D'(\Omega)$ is a locally $H^s \iff \psi u \in H^s$ for every $\psi \in D(\Omega) \iff D_\alpha u$ is locally $L^2$ for every $|\alpha| \leq s$.

Thus every smooth function is locally $H^s$ for every $s$.

Proof: $1 \to 2$ use partition of unity, $2 \to 1$ easy, and $2, 3$ are all equivalent to $D_\alpha(\psi u) \in L^2$ for every $\psi \in D(\Omega)$. by Leibniz+Plancherel or (X.7.3.5). □

Prop. (X.7.3.13). If $r > p + n/2$, then if a function $f$ on $\Omega$ has all the distribution derivative $D^k f$ locally $L^2$, $= g_{is}$, for $0 \leq k \leq r$, then $f \in C^p(\Omega)$ a.e.

Cor. (X.7.3.14). If $u \in D'(\Omega)$ is locally $H^s$, then $u \in C^{s-n/2}(\Omega)$. Thus $\cap$locally $H^s = C^\infty(\Omega)$.

Holder Space

Def. (X.7.3.15). Holder space $C^{k,\alpha}(\Omega)$ is the subspace of $C^k(\Omega)$ with the norm
\[
||f||_{C^{k,\alpha}} = ||f||_{C^k} + \max_{|\beta|=k} \sup_{x \neq y \in \Omega} \frac{|D^\beta f(x) - D^\beta f(y)|}{||x - y||^\alpha}.
\]

4 Fourier Analysis on $\mathbb{T}^n$

Prop. (X.7.4.1). If $f \in L^1(\mathbb{T})$ is absolutely continuous, then $\hat{f}(n) = 2\pi in \cdot \hat{f}(n)$.

Prop. (X.7.4.2). $f \in L^1(\mathbb{T})$ is determined by its Fourier coefficients.
X.8 Differential Operators

1 ODE-Fundamentals

Prop. (X.8.1.1).

\[ x^{(2)} = f(x) \]

It can be solved.

Proof:

\[ x'x^{(2)} = f(x)x' \]
\[ \frac{1}{2}(x')^2 = \int^x f(t)dt \]

Prop. (X.8.1.2) (Wronsky).

2 ODE-Theorems

Prop. (X.8.2.1) (Existence and Uniqueness of ODE of Lipschitz Type). If \( F(t, x) \) defined on \([-h, h] \times [\eta - \delta, \eta + \delta] \) is a function that is locally Lipschitz: that is, \( \exists \delta, L, \text{s.t. if } |t| \leq h, |x_i - \eta| \leq \delta, \]
\[ |F(t, x_1) - F(t, x_2)| \leq L|x_1 - x_2|. \]
Then the initial value problem:
\[ (Tx)(t) = \eta + \int_0^t F(\tau, x(\tau))d\tau \]
has a unique solution on the interval \([-h, h]\) if \( h < \min\{\delta/M, 1/L\} \), where \( M \) is the maximum of \( F \) on \([-h, h] \times [\eta - \delta, \eta + \delta] \). Because \( T \) is a contraction.

Prop. (X.8.2.2) (Existence of ODE of continuous Type (Caratheodory)). If \( F(t, x) \) defined on \([-h, h] \times [\eta - \delta, \eta + \delta] \) is a continuous function, then
\[ (Tx)(t) = \eta + \int_0^t F(\tau, x(\tau))d\tau \]
has a unique solution on the interval \([-h, h]\) if \( h < \delta/M \), where \( M \) is the maximum of \( F \) on \([-h, h] \times [\eta - \delta, \eta + \delta] \). (Use Schauder fixed point theorem and Arzela-Ascali).

Prop. (X.8.2.3) (Existence Theorem for Complex Differential Equations). Let \( f(z, w) \) be a holomorphic vector function in a domain \( D \subset \mathbb{C}^{n+1} \), then the initial value problem
\[ w' = f(z, w), \quad w(z_0) = w_0 \]
has exactly on holomorphic solution locally (Thus on a simply connected domain).

Cor. (X.8.2.4). So a holomorphic high-order ODE for a complex variable can be solves. And luckily it can be solved even \( \overline{z} \) appears (just regard it as a constant). ∆

Proof: Cf.[Ordinary Differential Equations, P110].
Prop. (X.8.2.5). For the equation:
\[ \frac{dy}{dx} = Ay, \]
One solution basis is:
\[
\begin{cases}
  e^{\lambda_1 x} p^{(1)}_1(x), \ldots, e^{\lambda_1 x} p^{(1)}_{n_1}(x); \\
  \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \cdots \\
  e^{\lambda_s x} p^{(d)}_1(x), \ldots, e^{\lambda_s x} p^{(1)}_{n_s}(x);
\end{cases}
\]
Where
\[ p^{(i)}_j(x) = r^{(i)}_{j0} + \frac{x}{1!} r^{(i)}_{j1} + \cdots, \]
where \( r^{(i)}_{j0} \) is a basis of solution of \((A - \lambda i I)^n x = 0\), and \( r^{(i)}_{k+1} = (A - \lambda i I) r^{(i)}_k \).

Proof: Cf.[常微分方程丁同仁定理 6.6]. □

Cor. (X.8.2.6). For the equation:
\[ y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0, \]
If the characteristic equation has \( s \) different roots \( \lambda_1, \ldots, \lambda_s \) and corresponding multiplicities \( n_1, \ldots, n_s \), then:
\[
\begin{cases}
  e^{\lambda_1 x}, xe^{\lambda_1 x}, \ldots, x^{n_1-1} e^{\lambda_1 x}; \\
  \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \cdots \\
  e^{\lambda_s x}, xe^{\lambda_s x}, \ldots, x^{n_s-1} e^{\lambda_s x};
\end{cases}
\]
is a solution basis.

Proof: Cf.[常微分方程丁同仁 P198]. □

Prop. (X.8.2.7) (Lyapunov). Consider the Lyapunov stability of an autonomous system of the form:
\[ \frac{dx}{dt} = Ax + o(|x|), \]
Then:
1. If \( A \) has an eigenvalue whose real part is positive, then the trivial solution is not weak stable.
2. If all eigenvalues of \( A \) have negative real part, then the trivial solution is strong stable.

Stum-Liouville

Prop. (X.8.2.8) (Stum-Liouville). The eigenvalue BVP problem of L-S equation:
\[ Lu = (pu')' + qu = \lambda u, \quad a_1 u(a) + a_2 u'(a) = 0, b_1 u(b) + b_2 u'(b) = 0, \sigma(x) > 0. \]
can be solved by the method of Green’s function. For the function:
\[ G(x, s) = \begin{cases} 
  Cu_1(x)u_2(s), & x < s \\
  Cu_2(x)u_1(s), & x > s
\end{cases} \]
for some $C$, where $u_1$ is a solution of the L-S equation with boundary value at $a$, and $u_2$ with boundary value at $b$ that are linear independent (This happens when the homogenous equation has no solution). It satisfies: $LG(x, s) = \delta(x - s)$ and satisfies the boundary conditions.

Because $L$ is self-adjoint, we have:

$$Gf(x) = \int f(s)G(x, s)ds, LG = id, GL = id$$

thus the eigenvalues of $L$ is the reciprocal of the eigenvalues of $G$, and $G$ is a compact self-adjoint operator on $L^2(\sigma, \mathbb{R})$, so by spectral theorem, the eigenvectors are countable and form an orthonormal basis.

And when the homogenous problem do have a solution $\phi$, then we have: $Lu = f$ has a solution iff $(f, \phi) = 0$. One way is simple and the other way is because we solve the initial problem of ODE and find that it automatically satisfies the boundary condition. Cf. [Stum Liouville Theory].

Prop. (X.8.2.9). More generally, if there the boundary is mixed of $u(a), U'(a), u(b), u'(b)$, the solution of

$$Lu = (pu')' + qu = 0, B_1(u) = \alpha, B_2(u) = \beta.$$ 

has a unique solution for any $\alpha, \beta$ iff the homogenous equation has only non-trivial solution. (Because the solution space is of 2 dimensional.

Prop. (X.8.2.10) (Stum Separation Theorem).

Prop. (X.8.2.11) (Stum Comparison Theorem). If $y'' + K_1(x)y = 0$ are equations. If $y_1(0) = 0$ and $|y_1'(0)| = |y_2'(0)|$, then if $K_1(x) \geq K_2(x)$, then $y_1(x) \geq y_2(x)$ until $y_2(x)$ is zero. (directly from (IX.3.4.10)).

3 Linear PDE

Def. (X.8.3.1). For a linear PDE with constant coefficients $P(D)u = v$, the fundamental solution is a distribution $E \in D'(\mathbb{R}^n)$ that $P(D)E = \delta$. This is important because if $v$ is a distribution with compact support, $P(D)(E \ast v) = (P(D)E) \ast v = \delta \ast v = v$ (X.7.1.10), so $u = E \ast v$ is a distribution solution.

Prop. (X.8.3.2). When $v \in D'(\mathbb{R}^n)$ has compact support, $P(D)u = v$ has a solution $u$ with compact support iff $Pg = \hat{v}$ has a solution $g$ entire. In this case, $g = \hat{u}$ for some distribution $u$, and $u$ has support in the convex hull of the support of $v$.

Proof: Use (X.7.2.18), and some bound relation between $g$ and $Pg$. Cf. [Rudin Functional Analysis P212].

Prop. (X.8.3.3). The fundamental solution always exist when for PDE of constant coefficients.

Proof: For a $\varphi \in D(\mathbb{R}^n)$, there is at most one $\psi$ that $\psi = P(D)\varphi$ because $\hat{\psi} = P\hat{\varphi}$ and they are entire function. Thus the task is to verify the functional $u : P(D)\varphi \rightarrow \varphi(0)$ is continuous and extend to a distribution $u \in D'(\mathbb{R}^n)$. Cf. [Rudin Functional Analysis P215].

4 Differential Operator on Manifolds

Prop. (X.8.4.1) (Index Theorem P109). has a nice definition of symbol of a differential operator on a manifold as a map form Sym$^mT^*M \otimes \mathbb{C} \rightarrow \text{Hom}(E, F)$.
5 Pseudo-Differential Operator

Def. (X.8.5.1). Denote the Japanese bracket $[x] = (1 + |x|^2)^{1/2} \sim 1 + |x|$. Motivated by the formula $(P\hat{f})^v = P(D)f$ for $f \in \mathcal{S}$ and polynomial $P$ of $\xi$ with coefficients smooth functions of $x$ we define the symbol class $S^{\mu,\beta}$ as the space of smooth functions $a : \mathbb{R}^{2n} \to \mathbb{C}$ that

$$|D_{x,\alpha}D_{\xi,\beta}a(x,\xi)| \leq C_{\alpha,\beta}[x]^\mu|\xi|^{m-|\beta|}$$

and denote $S^m = S^{0,m}$.

We denote the symbol class $\mathcal{A}^v$ as the space of smooth functions $a : \mathbb{R}^{2n} \to C$ that $|D_\alpha a| \leq C_\alpha[x + \xi]^v$ for any $\alpha$. So $S^{\mu,m} \subset \mathcal{A}^{[\mu] + |m|}$

And we define the pseudo-differential operator of symbol $a$:

$$(a(x,D)u)(x) = \int_\xi e^{ix\xi}a(x,\xi)\hat{u}$$

Moreover, we can define the amplitude function $p(x,y,\xi)$ and define

$$Pu(x) = \int e^{i(x-y)\xi}p(x,y,\xi)u(y)dy.$$  

Def. (X.8.5.2). We define the space $S^d$ of polyhomogenous symbols of degree $d$ as the set of all symbols in $S_{0,1}^d$ that there exists a set of $p_{d-l}$ homogenous in $\xi$ of degree $d-l$ that $p = \sum p_{d-l}$ modulo an operator in $S^{-\infty}$. Note that when $p_{d-l}$ is homogenous of degree $d-l$, then it is automatically in $S_{0,1}^{d-l}$.

Def. (X.8.5.3). A pseudo operator $a$ is called elliptic if $\sigma(a) \in S^m$ and $\sigma(a) \geq |\xi|^{-m}$ for $\xi$ big enough.

Prop. (X.8.5.4) (Peetre’s Inequality). For all $v \in \mathbb{R}$, there is a constant $C$ that

$$|X + Y|^v < C|X|^v|Y|^v.$$  

Proof: For $v > 0$, just as normal. For $v < 0$, use $X = (X + Y) + (-Y)$ applied to $-v$. □

Prop. (X.8.5.5). The mapping $a(x,\xi) \times u(x) \mapsto a(x,D)u$ is continuous from $\mathcal{A}^v \times \mathcal{S} \to \mathcal{S}$, thus also continuous from $S^{\mu,m} \times \mathcal{S} \to \mathcal{S}$. Cf.[Pseudo Differential Operator P28].

Lemma (X.8.5.6) (Schur Test). For a function $K$ on $\mathbb{R}^{2n}$ and $u \in L^p(\mathbb{R}^n)$, let $\|K\|_1 = \sup_x \int |K(x,y)|dy$ and $\|K(x,y)\|_2 = \sup_y \int |K(x,y)|dx$. Let $Au(x) = \int K(x,y)u(y)dy$, then

$$\|Au\|_p \leq \|K\|_1^{1-1/p}\|K\|_2^{1/p}\|u\|_{L^p}.$$  

by Holder.

Prop. (X.8.5.7) (Calderón-Vaillancourt). There is a constant $C, N_{CV}$ that for $u \in A^0$ and $\varphi \in \mathcal{S}$,

$$\|Op(u)\varphi\|_{L^2} \leq C \max_{|\alpha| + |\beta| \leq N_{CV}} \|\partial_\xi^\alpha \partial_x^\beta u\|_{L^\infty} \|\varphi\|_{L^2}.$$  

This in particular applies to $u \in S^0$.

Proof: Cf.[Calderón-Vaillancourt]. □

Cor. (X.8.5.8). $S^m$ maps $H^s$ to $H^{s-m}$. Because by symbolic calculus(X.8.5.10), we have

$$Op([\xi]^{s-m})Op(u)Op([\xi]^{-s}) = Op(b) \in S^0,$$

thus $Op(u) = Op([\xi]^{m-s})Op(b)Op([\xi]^s)$ maps $H^s$ into $H^{s-m}$.
Symbolic Calculus

Def. (X.8.5.9) (Semiclassical Operator). For \( a \in \mathcal{S}^{\mu,m} \) and \( h \in (0,1] \), we denote \( a_h(x, \xi) = a(x, h\xi) \), it is also in \( \mathcal{S}^{\mu,m} \).

Prop. (X.8.5.10) (Composition). If \( a \in \mathcal{S}^{\mu_1,m_1} \) and \( b \in \mathcal{S}^{\mu_2,m_2} \), there is a pseudo-differential operator \( (a\#b)(h) \in \mathcal{S}^{\mu_1+\mu_2,m_1+m_2} \) for every \( h \in (0,1] \) that
\[
Op(a_h)Op(b_h) = Op((a\#b)(h)_h)
\]
and for all \( J > 0 \), \( (a\#b)(h) \) can be written as
\[
a\#b(h) = \sum_{j<J} h^j \left( \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha \partial_x \bar{a} \right) + h^jr_J^\#(a, b, h)
\]
where \( r_J^\#(a, b, h) \in \mathcal{S}^{\mu_1+\mu_2,m_1+m_2-J} \) and it is bilinear of \( a, b \) and equicontinuous independently of \( h \).

Proof: Cf. [Pseudo Differential Operator P36]. \( \square \)

Prop. (X.8.5.11) (Adjoint). If \( a \in \mathcal{S}^{\mu,m} \) and \( u, v \in \mathcal{S} \), there is a pseudo-differential operator \( a^*(h) \) for every \( h \in (0,1] \) that
\[
(u, Op(a_h)v) = (Op(a^*(h)_h)u, v)
\]
in the \( L^2 \) norm and for all \( J > 0 \), \( a^*(h) \) can be written as
\[
a^*(h) = \sum_{j<J} h^j \left( \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha \partial_x a \right) + h^jr_J^*(a, h)
\]
where \( r_J^*(a, h) \in \mathcal{S}^{\mu,m-J} \) and it is anti-linear of \( a \) and equicontinuous independently of \( h \).

Proof: Cf. [Pseudo Differential Operator P30]. \( \square \)

Def. (X.8.5.12). For \( u \in \mathcal{S}' \), we define the action of \( a(x, \xi) \) on \( u \) by
\[
(Op(a_h)u)(\overline{\varphi}) = u(Op(a^*(h)_h)\varphi).
\]
This is compatible with the definition on \( \mathcal{S} \).

6 General PDE

Direct Solution

Prop. (X.8.6.1) (Characteristic Line). Consider a 1-dimensional parabolic equation:
\[
p_t + c(p, x, t)p_x = r(p, x, t)
\]
Let \( P(t) = p(X(t), t) \), this equation is equivalent to
\[
P_t = r(X(t), t, P(t)), \quad X_t = c(X(t), t).
\]
an ODE equation.
Prop. (X.8.6.2). A set of equations:
\[
\frac{\partial}{\partial x^i} \mu = A_i \mu
\]
where \( \mu \) is an \( n \)-vector. It has a solution iff
\[
[A_i, A_j] = \frac{\partial}{\partial x^i} A_i - \frac{\partial}{\partial x^j} A_j.
\]

**Proof:**

Cor. (X.8.6.3). This seems to be able to derive Frobenius integrability theorem, but I cannot figure it out.

## 7 Analysis on Manifolds

Prop. (X.8.7.1) (Peetre’s Theorem). For a linear operator from \( C^\infty(M) \) to \( C^\infty(M) \) that \( \text{Supp}(Lu) \subseteq \text{Supp}(u) \) where \( M \) is a compact manifold, then on every compact subset of a coordinate chart \( L \) looks like a differential operator of finite order.

**Proof:** The first thing is to prove on a chart \( \Omega \), \( L \) is continuous on \( C^\infty_0(\Omega) \). In fact, it suffice to show it is continuous from \( C^\infty_0(\Omega) \) to \( C^0_0(\Omega) \) because we can apply to \( D_\alpha L \). For this, Cf.[Pseudo Differential Operator P86].

Then we have \( |Lu| \leq C \max_{|\alpha| \leq m} \sup_K |D_\alpha \varphi| \) for every \( \varphi \in C_0(K) \). And the functional \( \varphi \to (L\varphi)(x) \) is a distribution supported on \( x \), thus by(X.7.1.4), it is of the form
\[
Lu(x) = \sum_{|\alpha| \leq m} a_\alpha(x) D_\alpha \varphi(x).
\]

We need to show \( a_\alpha \) is smooth, which we choose a bump function \( \chi \) to show \( a_0 \) is smooth and then choose \( x_i \chi \) applied to \( L\varphi - a_0 \varphi \) to show \( a_i \) is smooth, etc.

Prop. (X.8.7.2). The property of pseudo of order \( d \) is preserved under diffeomorphism, Cf.[Distributions and Operators P195], giving us the possibility to define pseudo differential operator on manifolds, and the principal symbol variate in this way that it forms a function from the cotangent bundle to the \( M_n(C) \). And the Sobolev space is defined by the property that all of its restrictions on a atlas are Sobolev, using the partition of unity.

Prop. (X.8.7.3). All the norms of different are commensurable up to constant factor w.r.t. each other, so it doesn’t quite matter with different norms.

Prop. (X.8.7.4). The parametrix exists for an elliptic operator on manifolds. Cf.[Distributions and Operators P207].

## 8 Elliptic Operator

Prop. (X.8.8.1). Elliptic operator is a Fredholm operator. And the kernel and cokernel are smooth functions, so it is also a Fredholm operator on \( C^\infty(\Omega) \).

**Proof:** It suffice to find a left and right inverse modulo compact operators, and in fact we find it module \( S^{-\infty} \). Since \( S^{-\infty} \) are all compact operators, i.e. it has a parametrix. Cf.[Distributions and Operators, P184].
Prop. (X.8.8.2) (Garding Inequality). For an elliptic operator of order \( d \) on \( \Gamma(E) \),

\[
\|f\|_{H^s} \leq C(\|f\|_{H^{s-d}} + \|Pf\|_{H^{s-d}})
\]

Proof: □

Cor. (X.8.8.3) (Elliptic Regularity Theorem). The inverse image of a smooth function under an elliptic operator is a smooth function, because the intersection of \( H^s(E) \) is \( C^\infty(E) \).

Cor. (X.8.8.4) (Elliptic Regularity Theorem). For \( L = \sum_{|\alpha| \leq N} f_\alpha D_\alpha \), where \( f_\alpha \in C^\infty(\Omega) \) and the equation \( Lu = v \) for distributions \( u \) and \( v \in \mathcal{D}'(\Omega) \), when \( v \) is locally \( H^s \), \( u \) is locally \( H^{s+N} \).

Thus if \( v \in C^\infty(\Omega) \), then \( u \in C^\infty(\Omega) \) by (X.7.3.12)(X.7.3.14).

Proof: We prove the case when \( L \) has leading coefficients constant. For every \( \varphi \in D(\Omega) \) that is 1 on some open ball \( B \), \( \varphi u \) has compact support thus in some \( H^t < \) and then we use a sublemma that says if \( \psi \) is 1 on the support of \( \varphi \), then if \( \psi u \) is in \( H^t \), where \( t \leq s + N - 1 \), then \( \varphi u \in H^{t+1} \). In this way, we can shrink the nbhd to reach \( H^{s+N} \). The proof of the lemma is in [Rudin Functional Analysis P220]. □

Prop. (X.8.8.5) (Analytic Ellipticity theorem). Suppose \( L \) is an analytic elliptic differential operator on a domain \( M \subset \mathbb{R}^n \), then every solution to \( L\varphi = 0 \) is analytic.

Proof: □

Prop. (X.8.8.6). The formal adjoint of an elliptic operator is an elliptic operator.

Proof: □

Cor. (X.8.8.7). The index of an elliptic operator, regarded as an operator form \( L_s : L_{s-d} \) doesn’t depend on \( s \), because all the kernel of \( P \) and \( P^* \) are smooth.

Prop. (X.8.8.8). For an elliptic operator, \( L \) has an inverse, the Green function which is a compact operator, so it has countable eigenfunctions consisting of smooth functions on \( L^2 \) with eigenvalues converging to \( \infty \). Moreover, the eigenvalues satisfy \( |\lambda_n| \geq Cn^\delta \) for some \( \delta, C \).

Proof: We prove for \( P \) self-adjoint. Use(X.8.8.1), \( \text{Ker} P \) is all smooth, so there is a map \( P(H^{-2d}) \to P(H^{-d}) \) which is bijective thus an isomorphism by Banach. So the inverse of this isomorphism composed with the Sobolev embedding \( H^{-d} \to L^2 \) is a compact operator \( G \). We notice that this map has the same eigenfunctions as \( P \), thus the result from that of compact operators.

For the second assertion, if suffice to prove \( \dim N(\lambda) \leq C\lambda^M \). Using Garding inequality and Sobolev embedding, we have for \( f \in N(\lambda) \), \( \|f\|_{C^0} \leq C(1 + \lambda^l)\|f\|_{L^2} \) for large \( l \). So if we choose an orthonormal basis \( f_i \), then \( |a_i f_i(x)| \leq C(1 + \lambda^{l})\sqrt{\sum |a_i|^2} \). Let \( a_i = f_i(x) \) and integrate over \( M \), we get the desired result. □

Cor. (X.8.8.9). For a self-adjoint elliptic operator \( P \) which is not a constant, \( L^2(E) \) has a basis consisting of eigenfunctions of \( P \).

Cor. (X.8.8.10) (Stum-Liouville). This can be used to solve for example eigenvalue problem for Liouville’s equation:

\[
(pu')' + qu = \lambda \sigma u.
\]

where \( p \) and \( \sigma \) are positive. Cf.(X.8.2.8).
**Cor. (X.8.8.11).** The Hermite functions $C_n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2}$, as the eigenvector of $\hat{H} = x^2 - \frac{d^2}{dx^2}$, forms a complete basis for $L^2(\mathbb{R})$. Because it is $e^{-x^2}$ times the solution of the operator $(e^{-x^2} F')' - e^{-x^2} F$.

**Prop. (X.8.8.12).** For a formally self-adjoint elliptic operator $P$ of degree $d$ on $E$, $\Gamma(E) = \text{Im } P \oplus P(\Gamma(E))$.

**Proof:** We know that $L^2(E) = P(H^dE) \oplus \text{Ker } P$, and $\text{Ker } P$ are all smooth by (X.8.8.3), so $\Gamma(E) = \text{Ker } P \oplus P(H^dE) \cap \Gamma(E)$. Now use Garding’s inequality (X.8.8.2), the intersection is just $P(\Gamma(E))$, thus the result. \[\Box\]

**Prop. (X.8.8.13) (Asymptotic Heat Equation).** In this case we have the series

$$h_t(A^*A) = \sum_\lambda e^{-\lambda t} \dim \Gamma_\lambda(E)$$

converges and $h_t$ has an asymptotic expansion

$$h_t = \sum_{k \geq -n} t^{k/2m} U_k(A^*A)$$

where $n = \dim M$ and $U_k = \int_M \mu_k$ for a differential form on $M$. Cf. [Heat Equation and the Index Theorem P297].

By the proposition above, the eigenspaces of eigenvalue non-zero neutralize, so Ind $A = h_t(A^*A) - h_t(AA^*)$, so

$$\text{Ind } A = U_0(A^*A) - U_0(AA^*) = \int_M \mu_0(A^*A) - \mu_0(AA^*)$$

The proof consists of the following propositions,

**Prop. (X.8.8.14).** Using the fact that an elliptic operator has a countable basis, for an elliptic operator $P$, when $t > 0$, we let $K(t, x, y) = \sum_n e^{-t\lambda_n} \Phi_n(x) \overline{\Phi}_n(y)$, then

$$e^{-tP} f(x) = \int K(t, x, y) f(y) dy.$$ 

$K(t, x, y)$ is smooth, and the trace of $e^{-tA^*A}$ is exactly $h_t(A^*A)$ as in the last proposition. And the trace is just $\int_M K(t, x, x)$, as can be easily seen.

**Proof:** Use Garding inequality and (X.8.8.8), we can show $\|K\|_{C^k}$ is bounded. \[\Box\]
X.9 Dynamic System

Main References are [Dynamic Systems Brin-Stuck].

1 Topological Dynamic System

Def. (X.9.1.1) (Topological Dynamic System). A topological dynamic system is a topological space $X$ and either a continuous map $f : X \to X$ or a continuous (semi)flow $f^t$ on $X$.

Def. (X.9.1.2) (Notations). Let $f : X \to X$ be a topological dynamic system, then

- For $x \in X$, its $\omega$-limit points are defined to be
  \[
  \omega(x) = \cap_{n \in \mathbb{N}} \cup_{i \geq n} f^i(x)
  \]
- If $f$ is invertible, for $x \in X$, its $\alpha$-limit points are defined to be
  \[
  \alpha(x) = \cap_{n \in \mathbb{N}} \cup_{i \leq n} f^i(x)
  \]
- The set $\mathcal{R}(f)$ of (positively) recurrent points are the points $x$ that $x \in \omega(x)$.
- The set $NW(f)$ of non-wandering points are the points $x$ that for any nbhd $U$ of $x$, there is an $n > 0$ that $f^n(U) \cap U = \emptyset$.
- Denote
  \[
  \mathcal{O}(x) = \cup_{n \in \mathbb{Z}} f^n(x), \quad \mathcal{O}^+(x) = \cup_{n \geq 0} f^n(x).
  \]
- Let $X$ be compact, then a closed non-empty forward $f$-invariant $Y \subset X$ is called a minimal set for $f$ if there is no smaller such set.
- A point $x \in X$ is called almost periodic if for any nbhd $U$ of $x$, $\{i | f^i(x) \in U\}$ is relatively dense in $\mathbb{N}$, i.e. appear in every $k$ consecutive integers for some $k$.

Prop. (X.9.1.3).

- $NW(f)$ is closed, $f$-invariant, and contains $\alpha(x), \omega(x)$ for all $x$.
- Every recurrent point is non-wandering, thus $\mathcal{R}(f) \subset NW(f)$.
- Let $X$ be compact Hausdorff, then $\mathcal{O}^+(x)$ is minimal for $f$ iff $x$ is almost periodic.

Proof: item1 is easy, 2 follows from 1 as $x \in \omega(x)$.

For 3: suppose $x$ is almost periodic and $y \in \overline{\mathcal{O}^+(x)}$, we need to show that $x \in \overline{\mathcal{O}^+(y)}$. For any nbhd $x \in U$, there is a small nbhd $x \in U' \subset U$ and a nbhd $\Delta \subset V \subset X \times X$, that if $x_1 \in U'$ and $(x_1, x_2) \in V$, then $x_2 \in U$. Since $x$ is periodic, there is a $K$ that for any $j \geq 0$, there is a $f^{j+k}(x) \in U'$ for some $0 \leq k \leq K$. Let $V' = \cap_{i=0}^{K} f^{-i}(V)$, then $V'$ is open and contains the diagonal. Thus there is a nbhd $W$ of $y$ that $W \times W \subset V'$. Now choose $f^n(x) \in W$ by almost periodicity, and $f^{n+k}(x) \in U'$ for some $0 \leq k \leq K$, then we have $(f^{n+k}(x), f^k(x)) \in V$. For the definition of $V'$ and $W$, and hence $f^k(x) \in U$. This shows $x \in \overline{\mathcal{O}^+(y)}$.

Conversely, if $x$ is not almost periodic, then there is a nbhd $U$ of $x$, that there is a sequence $\{a_i\} \subset \mathbb{N}$, that $f^{a_i+k}(x) \notin U$ for $j \leq i$. By convergence theorem and passing to a subsequence, we assume $y$ is a the limit of $f^{a_i}(x)$, and $f^j(y) \notin U$ for any $j > 0$, thus $x \notin \overline{\mathcal{O}^+(y)}$, showing $\overline{\mathcal{O}^+(x)}$ is not minimal. \qed
Prop. (X.9.1.4) (Minimal Set Exists on Compact Dynamic System). If \( f \) is a topological dynamic system on a compact space, then there exists a minimal set. In particular, there exists an almost periodic point, in priori positively recurrent point, by (X.9.1.3).

Proof: Use Zorn’s lemma and the finite intersection property. \( \square \)

Def. (X.9.1.5) (Topologically Transitive). A topological dynamic system \( f : X \to X \) is called **topologically transitive** if there is a point \( x \) that \( \overline{O}(x) = X \).

Prop. (X.9.1.6). Let \( f \) be a continuous map of a locally compact Hausdorff second countable space \( X \). Suppose that for any two non-empty open set \( U, V \), there is \( n > 0 \) that \( f^n(U) \cap V \neq \emptyset \), then \( f \) is topologically transitive.

Proof: For any open subset \( V \), the hypothesis says \( \bigcup_{i>0} f^{-i}(V) \) is dense in \( X \). Let \( V_i \) be a countable basis for the topology of \( X \), and \( Y = \bigcap_i \bigcup_{n>0} f^{-n}(V_i) \), then it is non-empty, by Baire category theorem (IX.1.9.2). Now the orbit of any \( y \in Y \) enters every \( V_i \), thus its orbit is dense in \( X \). \( \square \)

Prop. (X.9.1.7). Let \( f : X \to X \) be a homeomorphism of a compact metric space, and suppose \( X \) has no isolated points, then if there is a dense orbit \( O(x) \), there will be a dense orbit \( O^+(x) \).

Proof: The hypothesis that \( X \) has no isolated points shows that \( O(x) \) meets every open subset \( U \) infinitely many times, thus we can choose \( f^{n_k}(x) \in B(x, 1/k) \) that \( |n_k| \to \infty \). Thus \( f^{n_k+1}(x) \to f^l(x) \) for any \( l \).

If there are infinitely many \( n_k > 0 \), then we have \( O(x) \subset \overline{O}^+(x) \), hence \( O^+(x) \) is dense. If there are infinitely many \( n_k < 0 \), then \( O^-(x) \) is dense. Then for any open subset \( U, V \), we can find \( i < j < 0 \) that \( f^i(x) \in U, f^j(x) \in V \), thus \( f^{j-i}(U) \cap V \neq \emptyset \). Hence we use (X.9.1.6) to conclude \( f \) is topologically transitive. \( \square \)

Def. (X.9.1.8) (Topological Mixing). A topological dynamic system \( f : X \to X \) is called **topologically mixing** if for any two non-empty open subsets \( U, V \), there is \( N > 0 \) that \( f^n(U) \cap V \neq \emptyset \) for any \( n \geq N \).

Def. (X.9.1.9). A homeomorphism \( f : X \to X \) is called **expansive** if there is a \( \delta > 0 \) that for any two distinct points \( x, y \), there is some \( n \in \mathbb{Z} \) that \( d(f^n(x), f^n(y)) \geq \delta \). Similarly, we can define **positively expansive** for a non-invertible continuous map \( f : X \to X \). This constant \( \delta \) is called a ** expansiveness constant of \( f \).**

Prop. (X.9.1.10) (Compact Metric Space not positively expansive). If \( f \) be a continuous map of an infinite compact metric space \( X \), then it is not positively expansive.

Proof: First assume \( f \) is invertible. Fix \( \varepsilon > 0 \), consider all \( m \) that there are points \( x \neq y \) that

\[
d(f^n(x), f^n(y)) < \varepsilon, \quad 0 < n \leq m, \quad d(x, y) \geq \varepsilon.
\]

If these \( m \) are infinite, then we can use convergence point theorem to find a point \( x, y \) that \( d(f^n(x), f^n(y)) < \varepsilon \) for any \( n > 0 \), thus \( f \) is not expansive.

If these \( m \) are finite, let \( M \) be a maximal, then by absolute convergence, there is a \( \delta \) that if \( d(x, y) \leq \delta \), then \( d(f^n(x), f^n(y)) < \varepsilon \) for any \( 0 \leq n \leq m \). Then by definition of \( M \),

\[
d(f^{-1}(x), f^{-1}(y)) < \varepsilon, \quad \text{and similarly } \quad d(f^{-n}(x), f^{-n}(y)) < \varepsilon \text{ for any } n < 0.
\]
Now choose a finite \( \delta/2 \)-net \( \{x_i\} \) of \( M \), then for each \( j \in \mathbb{Z} \), there are two \( f^j(x_\alpha), f^j(x_\beta) \) in the same \( B(\delta/2, x_j) \), thus \( d(f^n(x_\alpha), f^n(x_\beta)) < \varepsilon \) for \( n \leq j \). Now there are only f.m. pairs of elements in \( M \), there are some pair \( (x_\alpha, x_\beta) \) appeared infinitely many times for different \( j > 0 \), thus we have \( d(f^n(x_\alpha), f^n(x_\beta)) < \varepsilon \) for any \( n \in \mathbb{Z} \).

For the non-invertible case, the proof is the same, noticing that \( f^{-1} \) is chosen wisely when it can be defined. \( \square \)

**Cor. (X.9.1.11).** Let \( f \) be an expansive homeomorphism of an infinite compact metric space \( X \), then there are distinct points \( x_0, y_0 \) that \( \lim_{n \to \infty} d(f^n(x_0), f^n(y_0)) = 0 \).

**Proof:** Let \( \delta \) be an expansive constant of \( f \), then by (X.9.1.10), there are \( x_0 \neq y_0 \in X \) that \( d(f^n(x_0), f^n(y_0)) \leq \delta \) for all \( n > 0 \). Suppose \( d(f^n(x), f^n(y)) \to 0 \), then by compactness in \( X \times X \), there is a subsequence \( \{n_k\} \in \mathbb{N} \) that \( f^{n_k}(x) \to x', f^{n_k}(y) \to y' \) with \( x' \neq y' \). Then this will show that \( d(f^{m}(x), f^{m}(y)) < \delta \) for any \( m \in \mathbb{Z} \), contradicting expansiveness of \( f \). \( \square \)

**Def. (X.9.1.12).** Let \( f : X \to X \) be a homeomorphism of compact Hausdorff space, then
- two points \( x, y \) are called **proximal** if the closure of orbits \( \overline{O((x, y))} \) under \( f \times f \) intersect the diagonal \( \Delta \subset X \times X \). It is called **distal** otherwise. \( f \) is called **distal** if every two distinct points \( x, y \) are distal.
- \( f \) is called **equicontinuous** if the family \( \{f^n\}_{n \in \mathbb{Z}} \) is equicontinuous.

**Prop. (X.9.1.13).** A equicontinuous homeomorphism \( f \) of a compact metric space are distal.

**Proof:** Suppose \( f \) is not distal, then there is a proximal pair \( (x, y) \), thus there is a sequence \( \{n_k\} \in \mathbb{Z} \) that \( d(f^{n_k}(x), f^{n_k}(y)) \to 0 \). let \( d(x, y) = \varepsilon \), thus for any \( \delta > 0 \), there is some \( d(f^{n_k}(x), f^{n_k}(y)) < \varepsilon \), thus contradicting the equicontinuity of \( f^{-n} \). \( \square \)

**Def. (X.9.1.14) (Almost Periodic Set).** For a subset \( A \subset X \) and a homeomorphism \( f : X \to X \), denote by \( f_A \) the action of \( f \) on \( X^A \). Then \( A \) is called **almost periodic** if every \( z \in \text{Mor}(A, A) \) is almost periodic for \( f_A \). Equivalently, for any finite set of points \( \{x_i\} \in A \), and nbhds \( x_i \in U_i \), the set \( \{k \in \mathbb{Z} | f^k(x_i) \in U_i\} \) is relatively dense in \( \mathbb{Z} \). This is compatible with the previous definition of almost periodic point (X.9.1.2).

**Def. (X.9.1.15).** A homeomorphism of a compact Hausdorff space is called **pointwise almost periodic** if every point \( x \) is almost periodic. By (X.9.1.3), we can see that this is equivalent to \( X \) is a union of minimal sets.

**Lemma (X.9.1.16).** Every almost periodic set \( A \) is contained in a maximal almost periodic set in \( X \).

**Proof:** It is because of the second definition of almost periodic set (X.9.1.14) that the sum of an ordered family of almost periodic sets is also almost periodic. \( \square \)

**Prop. (X.9.1.17).** Let \( f \) be a homeomorphism of a compact Hausdorff space \( X \), then every point \( x \in X \) is proximal to an almost period point.

**Proof:** If \( x \) is almost periodic, then we are done. If not, consider a maximal almost periodic set \( A \subset X \), then \( x \notin A \). Then for \( z \in X^A \) with range \( A \), consider \( (x, z) \in X \times X^A \), and find an almost period point \( (x_0, z_0) \) of \( f \times A \) in \( \overline{O(x, z)} \), by (X.9.1.4). Because \( z \) is almost periodic, \( z \in \overline{O(z_0)} \), by (X.9.1.3). Thus we see \( (x', z) \in \overline{O(x_0, z_0)} \) for some \( x' \), by the compactness of \( X \). Then \( (x', z) \) is also almost periodic, and we can forget about \( x_0, z_0 \).

Therefore, \( \{x'\} \cup \text{Im}(z) = \{x'\} \cup A \) is almost periodic for \( f \), and since \( A \) is maximal, \( x' \in A = \text{Im}(z) \). This shows \( (x', x) \in \overline{O(x, x')} \), showing \( x \) is proximal to \( x' \). \( \square \)
Cor. (X.9.1.18) (Distal Homeomorphism is Pointwise Almost Periodic). If $f$ is a distal homeomorphism of a compact Hausdorff space, then $f$ is pointwise almost periodic.

Prop. (X.9.1.19). • A homeomorphism of a compact Hausdorff space is distal iff the product system $(X \times X, f \times f)$ is pointwise almost periodic.

• A factor system of a pointwise almost periodic homeomorphism is pointwise almost periodic.

• A factor system of a distal homeomorphism is distal.

Proof: 1: If $f$ is distal, then so is $f \times f$, hence it is pointwise almost periodic, by (X.9.1.18). Conversely, if $f \times f$ is pointwise almost periodic, then if $x, y$ is proximal, then $(z, z) \in \mathcal{O}(x, y)$, but since $\mathcal{O}(x, y)$ is minimal, by (X.9.1.3), $(x, y) \subset \mathcal{O}(z, z) \subset \Delta$, hence $x = y$.

2: This is easy.

3: if $f$ is a factor of $g$, then $f \times f$ is a factor of $g \times g$, but the latter is pointwise locally periodic, thus the first is also locally periodic, by 2, and then $f$ is distal, by 1. $\square$

Topological Entropy

Def. (X.9.1.20) (Definitions). Let $(X, d)$ be a compact metric space, and $f$ a continuous map $X \to X$, we define

• for $x, y \in X$, $d_n(x, y) = \max_{0 \leq k \leq n-1} d(f^k(x), f^k(y))$.

• a subset $A \subset X$ is called $(n, \varepsilon)$-spanning if it is a $\varepsilon$-net in $(X, d_n)$. Similarly, we can define $(n, \varepsilon)$-separated and $\text{cov}(n, \varepsilon, f)$ the minimal number of covering of $X$ of $d_n$-diameter $< \varepsilon$.

Lemma (X.9.1.21). $\text{cov}(n, 2\varepsilon, f) \leq \text{span}(n, \varepsilon, f) \leq \text{sep}(n, \varepsilon, f) \leq \text{cov}(n, \varepsilon, f)$.

Proof: Easy. $\square$

Prop. (X.9.1.22) (Topological Entropy). The number

$$\lim_{n \to \infty} \frac{1}{n} \log(\text{cov}(n, \varepsilon, f)) = h_{\varepsilon}(f)$$

is well-defined and finite. It is increasing as $\varepsilon$ decreases, so $h(f) = \lim_{\varepsilon \to 0^+} h_{\varepsilon}(f)$ exists, which lies in $[0, \infty]$. It is called the topological entropy of $f$.

Notice the entropy can be calculated by either cov, span or sep, by (X.9.1.21).

Proof: For this, we need to notice if $U$ has $d_n$-diameter $< \varepsilon$ and $V$ has $d_m$-diameter $< \varepsilon$, then $U \cap f^{-m}(V)$ has $d_{m+n}$-diameter $< \varepsilon$. Hence

$$\text{cov}(m + n, \varepsilon, f) \leq \text{cov}(m, \varepsilon, f) \cdot \text{cov}(n, \varepsilon, f).$$

Thus we can use (XIV.1.1.1) to conclude. $\square$

Prop. (X.9.1.23). The topological entropy doesn’t depend on the metric generating the topology.

Proof: This is because $d'$ is a continuous function on $(X \times X, d \times d)$, thus it is uniformly continuous as $X$ is compact, so

$$\text{cov}(n, \varepsilon, f) \leq \text{cov}(n, \delta(\varepsilon), f)$$

where $\delta(\varepsilon) \to 0$ when $\varepsilon \to 0$. This shows the topological entropies are the same. $\square$
Cor. (X.9.1.24). Two conjugate dynamic systems have the same topological entropy.

Prop. (X.9.1.25) (Properties of Entropy). Let \( f : X \to X \) be a continuous map of a compact metric space \( X \), \( g : Y \to Y \) be a continuous map of a compact metric space \( Y \), then

- \( h(f^m) = m \cdot h(f) \) for \( m > 0 \).
- If \( f \) is invertible, then \( h(f^{-1}) = h(f) \).
- Let \( A_i \) be a finite family closed forward \( f \)-invariant subsets of \( X \) whose union is \( X \), then
  \[
  h(f) = \max h(f|_{A_i}).
  \]
- \( h(f \times g) = h(f) + h(g) \), and if \( f \) is an extension of \( g \), then \( h(f) \geq h(g) \).

Proof: 1: Use two inequalities:
\[
\text{span}(n, \varepsilon, f^m) \leq \text{span}(mn, \varepsilon, f), \quad \text{span}(n, \delta(\varepsilon), f^m) \geq \text{span}(mn, \varepsilon, f)
\]
where \( \delta(\varepsilon) \to 0 \) if \( \varepsilon \to 0 \), and (X.9.1.21).

2: Because \((n, \varepsilon)\)-separated sets for \( f \) and \((n, \varepsilon)\)-separated sets for \( f^{-1} \) corresponds via \( f^n \).

3: Use two inequalities:
\[
\text{span}(n, \varepsilon, f) \leq \sum_{i=1}^{k} \text{span}_i(n, \varepsilon, f) \leq k \cdot \max \text{span}_i(n, \varepsilon, f),
\]
\[
\text{sep}(n, \varepsilon, f) \geq \max \text{sep}(n, \varepsilon, f).
\]

4: Noticing two inequalities:
\[
\text{cov}(n, \varepsilon, f \times g) \geq \text{cov}(n, \varepsilon, f) \cdot \text{cov}(n, \varepsilon, g),
\]
\[
\text{sep}(n, \varepsilon, f \times g) \geq \text{sep}(n, \varepsilon, f) \cdot \text{sep}(n, \varepsilon, g).
\]
and the proof of the last assertion is similar to that of (X.9.1.23).

Prop. (X.9.1.26). Let \( f : X \to X \) be an expansive homeomorphism of a compact metric space of expansiveness constant \( \delta \), then \( h(f) = h_\varepsilon(f) \) for \( \varepsilon < \delta \).

Proof: For \( 0 < \gamma < \varepsilon < \delta \), we show that \( h_{2\gamma}(f) = h_\varepsilon(f) \). For this, it suffices to prove \( \leq \). By expansiveness, if \( x \neq y \), then there is some \( i \in \mathbb{Z} \) that \( d(f^i(x), f^i(y)) \geq \delta > \varepsilon \). Since the set \( \{(x, y) \in X \times X | d(x, y) \geq \gamma \} \) is compact, there is a \( k = k(\gamma, \varepsilon) \) such that \( d(x, y) \geq \gamma \), then \( d(f^i(x), f^i(y)) > \varepsilon \) for some \( |i| \leq k \). Thus if \( A \) is a \((n, \gamma)\)-separated set, then \( f^{-k}(A) \) is a \((n + 2k, \varepsilon)\)-separated set. Hence by (X.9.1.21), \( h_{2\gamma}(f) \leq h_\varepsilon(f) \).

Application to Ramsey Theory

Def. (X.9.1.27) (IP-System). Let \( \mathcal{F} \) be the collection of all finite non-empty subset of \( \mathbb{N} \). For \( \alpha, \beta \in \mathcal{F} \), write \( \alpha < \beta \) if every element of \( \alpha \) is smaller than that of \( \beta \).

For a commutative group \( G \), an IP-system in \( G \) is a map \( T : \mathcal{F} \to G \) that \( T_{i_1, \ldots, i_k} = T_{i_1} \cdots T_{i_k} \).

Prop. (X.9.1.28) (Furstenberg-Weiss). Let \( G \) be a commutative group acting minimally on a compact topological space \( X \), then for any open subset \( U \) of \( X \), \( n > 0 \) and \( \alpha \in \mathcal{F} \), and any IP-systems \( T^1, \ldots, T^n \) on \( G \), there is a \( \beta \in \mathcal{F} \) that \( \alpha < \beta \), and
\[
U \cap T_{\beta}^1(U) \cap \ldots \cap T_{\beta}^n(U) \neq \emptyset.
\]
Then a partition of

Consider the product space

Proof: Just need to find a minimal closed subset of \(X\) for \(G\), but this is easy by finite intersection theorem.

Cor. (X.9.1.29) Let \(G\) be a commutative group acting homeomorphically on a compact metric space \(X\) and \(T^1, \ldots, T^m\) be IP-systems on \(G\), then for any \(\alpha \in \mathcal{F}\) and \(\varepsilon > 0\), there are \(x \in X\) and \(\beta > \alpha\) that \(d(x, T^i_{\beta}(x)) < \varepsilon\) for any \(i\).

Proof: This is because every element in \(\sigma\) occurs in \(\sigma\)-shift.

Cor. (X.9.1.30) (Multiple Recurrence Property). Let \(T\) be a homeomorphism of a compact metric space \(X\), then for any \(\varepsilon > 0\) and \(q > 0\), there are \(p > 0\) and \(x \in X\) that \(d(T^{jp}(x), x) < \varepsilon\) for \(0 \leq j \leq q\).

Proof: This is a special case of (X.9.1.29), by taking \(G = \{T^k\}\), and \(T^i_{\alpha} = T^{i|\alpha|}\), where \(|\alpha|\) is the sum of elements in \(\alpha\).

Cor. (X.9.1.31) (Generalized van der Waerden Theorem). For each finite partition \(\mathbb{Z} = \bigcup_{i=1}^{m} S_k\), one of the the subset \(S_k\) contains arbitrarily long arithmetic progressions.

More generally, let \(A\) be a finite subset of \(\mathbb{Z}^d\), then for each partition \(\mathbb{Z}^d = \bigcup_{i=1}^{m} S_k\), there are some \(k\), \(z_0 \in \mathbb{Z}^d\) and \(n > 0\) that \(z_0 + na \in S_k\) for any \(a \in A\).

Proof: Consider the product space \(\Sigma_m = \{1,2,\ldots, m\}^\mathbb{Z}\) with the 2-adic metric with shifting \(\sigma\). Then a partition of \(\mathbb{Z}\) can be viewed as an element \(\xi\) in \(\{1,2,\ldots, m\}^\mathbb{Z}\), with \(\xi_i = k\) if \(i \in S_k\). Let \(X = \bigcup_{i=-\infty}^{\infty} \sigma^i(\xi)\). Let \(A_k = \{\omega \in X|\omega_0 = k\}\), then if \(x \in A_k, y \in X\) with \(d(x, y) < 1\), then \(y \in A_k\) also. Thus for any \(q > 0\), we can use (X.9.1.30) to show that there is an \(\omega \in X\) that \(d(\sigma^{jq}(\omega), \omega) < 1\) for \(0 \leq i \leq q\), thus there is an \(r \in \mathbb{Z}\) that \(\xi_j = \omega_0\) for \(j = r, r+k, \ldots, r+pq\). And this proves the theorem.

The proof of the general case is similar.

2 Symbolic Dynamics

Def. (X.9.2.1) (Subshifts). A subshift is a closed subset \(X \subset \Sigma_m\) that is invariant under the shift \(\sigma\) and \(\sigma^{-1}\). A map between to subshifts of \(\Sigma_m\) is called a code if it commutes with \(\sigma\).

Prop. (X.9.2.2). Let \(X\) be a subshift of \(\Sigma_m\), let \(W_n(X)\) be the set of words of length \(n\) that occurs in \(X\), \(\sigma|_X\) is clearly expansive of constant 1, thus we have

\[
h(\sigma|_X) = \lim_{n \to \infty} \frac{1}{n} \log |W_n(X)|.
\]

Proof: This is because every element in \(W_n\) appears in the first \(n\) term of some \(\omega \in X\), and elements in a set of \(d_n\)-diameter\(< 1\) has first \(n\) entries the same. For the other direction, notice a \((n, 1)\)-separated set has the first \(n\)-entries not the same, thus contribute to different elements in \(W_n(X)\), thus \(\text{sep}(n, 1, \sigma|_X) < W_n(X)\).

Def. (X.9.2.3) (block code). Let \(X\) be a subshift, \(k, l \geq 0, n = k + l + 1\), and \(\alpha\) be a map from \(W_n(X)\) to \(\{0, 1, \ldots, m'-1\}\), then a \((k, l)\)-block code is a morphism from \(X\) to \(\Sigma_{m'}\) that maps \(x\) to the sequence \(a_\alpha(x)\) that \(a_\alpha(x)_i = \alpha((x_{i-k}, \ldots, x_{i+l}))\).

When \(\{0, 1, \ldots, m'-1\}\) is chosen to be \(W_n(X)\) itself, then this is called a higher block presentation of \(X\).
Prop. (X.9.2.4) (Curtis–Lyndon–Hedlund). [Every Code is a block code] Every code \( c : X \to Y \) is a block code.

Proof: Let \( A \) be the symbol set of \( Y \), define \( \hat{c} : X \to A : x \mapsto c(x)_0 \). This is a continuous map, and \( X \) is compact thus it is uniformly continuous, thus there is a \( \delta > 0 \) that \( \hat{c}(x) = \hat{c}(y) \) if \( d(x, y) < \delta \). Thus there is a large \( k \) that \( \hat{c} \) only depends on the first \( 2k + 1 \) term, and it commutes with shifting, thus it is a block code. \( \square \)

Def. (X.9.2.5) (SFT). A subshift of finite type or SFT of \( k \)-step, \( k > 0 \) is a subshift \( X \) that are defined to be the elements in \( \Sigma_m \) that doesn’t contain some set of words of length \( k + 1 \). When \( k = 1 \), this is also called a topological Markov chain.

Prop. (X.9.2.6). Every SFT is isomorphic to a vertex shift.

Proof: For this SFT \( X \) of step \( k \), construct a graph, whose vertices are \( W_k(X) \), and two element of \( W_k(X) \) are connected by an edge if they adjoint to an element in \( W_{k+1}(X) \). \( \square \)

Cor. (X.9.2.7). Every SFT is isomorphic to an edge shift. This is because every vertex shift can be 2-block isomorphic to its edge shift.

Prop. (X.9.2.8) (Perron).

Sofic Shifts and Data Transmission

Def. (X.9.2.9) (Sofic Shifts). A subshift \( X \subset \Sigma_m \) is called sofic if it is a factor of a subshift of finite type.

Prop. (X.9.2.10). A subshift \( X \subset \Sigma_m \) is sofic iff it is isomorphic to an infinite path shift for some directed graph \( \Gamma \) (Notice that different edge in \( \Gamma \) can be labeled the same).

Proof: Clearly such a path shift is a factor of the edge shift of \( \Gamma \), thus it is sofic. Conversely, A sofic shift is a factor of some edge shift \( c : \Sigma^e_A \to X \), by (X.9.2.7), and \( c \) is a block code, by (X.9.2.3). Composing with the higher block code presentation, we may assume \( c \) is a \((0,1)\)-block code. \( \square \)

3 Ergodic Theory

Prop. (X.9.3.1) (Poincaré’s Recurrence Theorem). Let \( T \) be a measure-preserving transformation of a finite measure space \( X \), if \( A \) is a measurable set, then for a.e. \( x \in A \), there is some \( n > 0 \) that \( T^n(x) \in A \).

Proof: Let \( B \) be the set of points contradicting this property, \( B = A \setminus \cup_{k>0} T^{-k}A \), thus all the preimages \( T^{-k}B \) are disjoint, measurable and have the same measure as \( B \), thus it must has measure 0 as \( X \) has finite measure. \( \square \)

Cor. (X.9.3.2) (Derivative Transformation). Given a finite measurable space and a measure-preserving map \( T : X \to X \) and a measurable subset \( A \) of finite measure, then the derivative transformation \( T_A : A \to A : x \mapsto T^k(x) \), where \( k > 0 \) is the first integer that \( T^k(x) \in A \). By Poincaré’s Recurrence theorem, the derivative transformation is defined on a subset of full measure.

Prop. (X.9.3.3). Let \( X \) be a second countable metric space and \( \mu \) a Borel probability measure on \( X \), and \( f : X \to X \) is a measure preserving continuous map, then a.e. point \( x \in X \) is recurrent, i.e. \( \text{Supp} \mu \subset \mathcal{R}(f) \).
Proof: There is a countable family of basis $\{U_i\}$ of nbhds of $X$, and for each $U_i$, elements in $U_i$ returns to $U_i$ except for a set $X_i$ of measure 0. Then $R(f) = X \setminus \bigcup_i X_i$ has full measure. \hfill \Box

Def. (X.9.3.4) (Ergodicity). A measure-preserving transformation $T$ is called ergodic if any essentially $T$-invariant measurable subset has measure 0 or full measure.

Prop. (X.9.3.5). Let $T$ be a measure-preserving transformation on a finite measure space $X$ and $p \in (0, \infty]$, then $T$ is ergodic iff every essentially $T$-invariant function $f \in L^p(X, \mu)$ is constant.

Proof: \hfill \Box

Prop. (X.9.3.6). Let $X$ be a measure space and $f$ is an essentially invariant function for a measure-preserving map or flow $T$ on $X$, then there is a strictly invariant measurable function $\hat{f}$ that $f = \hat{f}$ a.e.

Proof: [Dynamic System P74]. \hfill \Box

Def. (X.9.3.7). A measure-preserving transformation or flow on a probability space $X$ is called mixing if

$$\lim_{t \to \infty} \mu(T^{-t}A \cap B) = \mu(A)\mu(B).$$

It is called weak mixing if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n} |\mu(T^{-i}A \cap B) - \mu(A)\mu(B)| = 0.$$ 

or

$$\lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} |\mu(T^{-t}A \cap B) - \mu(A)\mu(B)| = 0.$$ 

Prop. (X.9.3.8). Mixing transformation is weak mixing, and weak mixing is ergodic.

Proof: For a weak mixing transformation, if $A$ is essentially invariant, then $\mu(A) = \mu(A)^2$, thus $\mu(A) = 1$ or 0. \hfill \Box

Prop. (X.9.3.9) (Mixing and Topological Mixing). Let $X$ be a compact metric space, $T : X \to X$ be a continuous map and $\mu$ a $T$-invariant Borel measure on $X$, then

- If $T$ is ergodic, then the orbit of a.e. point $x \in X$ is dense in $\text{Supp} \mu$.
- If $T$ is mixing, then $T$ is topologically mixing on $\text{Supp} \mu$.

Proof: 1: Let $U$ be an open subset intersecting $\text{Supp} \mu$, then the $T$-invariant subset $\bigcup_{k \geq 0} T^{-k}U$ has full measure, thus the forward orbit of a.e. $x$ intersect $U$. Then take a countable open basis of $X$, thus the forward orbit of a.e. $x$ is dense in $\text{Supp} \mu$.

2: Because $\lim_{t \to \infty} T^{-t}(A) \cap B \to \mu(A)\mu(B) > 0$, so does $\lim_{t \to \infty} A \cap T^t(B)$. \hfill \Box

Ergodic Theorems

Prop. (X.9.3.10) (von Neumann Ergodic Theorem). If $U \in L(H)$ is unitary and $x \in H$, then the average $A_n x = \frac{1}{n} (x + Ux + \ldots + U^{n-1}x)$ converges to some $P x$, where $P$ is the projection to the fixed space of $U$. 
X.9. DYNAMIC SYSTEM

**Proof:** Define \( a_n = \frac{1}{n}(1 + \lambda + \ldots + \lambda^{n-1}) \) on the unit circle, and \( b(1) = \chi_{\{1\}} \), then if \( y = b(U)x = P(x) \), then \( \|y - A_n x\|^2 = \int_{\sigma(U)} |b - a_n|^2 dE_{x,\lambda} \). But this integral converges to 0 by dominated convergence theorem.

**Prop. (X.9.3.11) (Birkhoff Ergodic Theorem).** If \( T \) is a measure-preserving transformation of a finite measure space \((X, \mu)\), and let \( f \in L^1(X, \mu) \), then the limit

\[
\bar{f}(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^k(x))
\]

exists for a.e. \( x \) and is \( \mu \)-integrable and \( T \)-invariant, and satisfies

\[
\int_X \bar{f}(x) d\mu = \int_X f(x) d\mu
\]

In addition, if \( f \in L^2(X, \mu) \), then \( \bar{f} \) is just the projection of \( f \) to the subspace of \( T \)-invariant measures.

The same thing is true for a measure-preserving flow.

**Proof:** Let

\[
A = \{ x \in X | f(x) + f(T(x)) + \ldots + f(T^k(x)) \geq 0, \exists k \geq 0 \}.
\]

Then firstly we have \( \int_A f(x) d\mu \geq 0 \): Cf.[Dynamic System, P82].

**Cor. (X.9.3.12).** A measure-preserving map \( T : X \to X \) in a finite measure space \((X, \mu)\) is ergodic iff for each \( f \in L^1(X, \mu) \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) = \frac{1}{\mu(X)} \int_X f(x) d\mu, \text{a.e.} x.
\]

I.e., the time average equals the space average for any \( L^1 \)-function.

**Proof:** If \( T \) is ergodic, then the function \( \bar{f} \) defined in (X.9.3.11) is a constant, thus the equation. Conversely, if this equation holds, then if \( f \) is \( T \)-invariant, the RHS is constant.

**Cor. (X.9.3.13).** Taking a dense subset of \( L^2(X, \mu) \) in the above corollary, we see that a measure-preserving map \( T : X \to X \) is ergodic iff for any measurable subset \( A \) and a.e. \( x \in X \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(T^i(x)) = \frac{\mu(A)}{\mu(X)}.
\]

**Invariant Measure for a Transformation**

4 Hyperbolic Dynamics

5 Complex Dynamics
Chapter XI

Derived Algebraic Geometry

XI.1 Simplicial Commutative Algebras

Main references are [Model Categories and Simplicial Methods, Paul Goerss and Kristen Schemmerhorn], [Simplicial Commutative Rings, Mathew].

1 Simplicial Groups

Prop. (XI.1.1.1). A morphism of simplicial groups, when regarded as simplicial set, is a Kan fibration iff

\[ X \to \pi_0 X \otimes_{\pi_0 Y} Y \]

is surjective. In particular, any simplicial group is a Kan complex.

Proof: Cf. [Simplicial Homology Theory Jardine P12] □

Def. (XI.1.1.2) (Simplicial Modules). A simplicial module over a simplicial ring \( R \cdot \) is a map of simplicial map \( R \cdot \times M \cdot \to M \cdot \) that is a ring action.

2 Simplicial \( R \)-Modules and Resolutions

Def. (XI.1.2.1) (Moore Complex). Giving a simplicial object in an Abelian category, we can have a Moore chain complex with Čech-like differentials. \( \partial_n = \sum_1^n (-1)^i d_i \). And we have \( \partial^2 = 0 \).

Proof: Should use \( d_i d_j = d_{j-1}d_i \) for \( i < j \). □

Def. (XI.1.2.2) (Normalization of a simplicial \( R \)-Module). The normalization of a simplicial \( R \)-module \( M \) is the chain complex

\[ NM : \cdots \to NM_n \xrightarrow{(-1)^n d_n} NM_{n-1} \to \cdots \]

where \( NM_n = \cap_{i=0}^{n-1} \ker(d_i) \in M_n \). This is a chain complex because \( d_{n-1}d_n = d_{n-1}d_{n-1} \) is 0 on \( NM_n \). In fact \( NM \) is preserved by all injections.

The homotopy groups \( \pi_\ast(M) \) of \( M \) is defined to be the homology of the normalization of \( M \). And it can be shown that as a set \( \pi_n(M) \) is just the \( n \)-th homotopy group of the geometrization of the

The degenerate complex of a Moore complex \( DM \) is the chain complex that \( D_n = \sum_{i=0}^{n-1} s_i M_{n-1} \) is a sub chain complex of \( M \) by the relation of \( d_i, s_j \).
Def. (XI.1.2.3). A morphism of simplicial Abelian groups is called a weak equivalence is it induces an isomorphism on the homotopy groups.

Prop. (XI.1.2.4) (Differential Graded Structures). For any simplicial commutative $R$-algebra $A$, the homotopy groups $\pi_*(A)$ form a graded commutative $R$-ring.

Proof: The group structure on $\pi_*(A)$ is given by smash products. Cf.[Simplicial Commutative Algebras, Mathew, P2].

Cor. (XI.1.2.5) ($\pi_0$ is an Algebra). If $Y$ is a simplicial commutative $R$-algebra, then $\pi_0(Y) = Y_0/(\text{Im}(d_0 - d_1))$, which is an algebra.

Proof: It suffices to show $\text{Im}(d_0 - d_1)$ is an ideal of $Y_0$: If $a \in Y_0$, then

$$(d_0 - d_1)(s_0(a)y) = d_0s_0(a)d_0(y) - d_1s_0(a)d_1(y) = a(d_0 - d_1)(y).$$

Prop. (XI.1.2.6). if $R_\bullet$ is a simplicial ring and $M_\bullet$ is a simplicial $R_\bullet$-ring, then $\pi_*(M)$ is a graded $\pi_*(R)$-module.

Prop. (XI.1.2.7). The simplicial homology of the Moore complex of the bar resolution $BG$ of group homology with coefficient in $R$ is just the group homology $H_n(G, R)$ for the trivial module $R$. And it has the same homology with the geometrization $|BG|$.

Lemma (XI.1.2.8). $A_\bullet \cong NA_\bullet \oplus DA_\bullet$ as a complex, $NA_\bullet, A_\bullet, (A/DA_\bullet)$ are all homotopically equivalent.

Proof: We define similarly $N_kA_\bullet$ and $D_kA_\bullet$ and induct on $k$, our conclusion is the case $k = n - 1$. When $k = 0$, $\text{Im} d_0 \oplus \text{Ker} s_0 A_n = A_n$ because $d_0s_0 = id_{n-1}$ thus $A_{n-1} \xrightarrow{s_0} A_n$ is a split injection.

There are two split exact rows by simplicial relations:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & N_{k-1}A_{n-1} & \longrightarrow & N_{k-1}A_n & \longrightarrow & N_kA_n & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{n-1}/D_{k-1}A_{n-1} & \longrightarrow & A_{n-1}/D_{k-1}A_n & \longrightarrow & A_n/D_kA_n & \longrightarrow & 0
\end{array}
$$

The first one split because it has a right section, the second one split because it has a left section. So by induction, $N_kA_n \rightarrow A_n/D_kA_n$ is an isomorphism, thus $N_kA_n \oplus D_kA_n = A_n$ because it splits.

For the homotopy equivalence, Cf.[Jardine P150].

Prop. (XI.1.2.9) (Dold-Kan Correspondence). Let $R$ be a ring, then the normalized Moore complex functor $N$ gives an equivalence of categories:

$$N : s\text{Mod}_R \cong Ch_{\geq 0}R.$$

and the inverse is given by

$$\sigma C_\bullet = \bigoplus_{[n] \in [k]} C_k$$

and a morphism $\sigma_n \rightarrow \sigma_m$ for a morphism $[m] \rightarrow [n]$ is defined as follows: For $[n] \rightarrow [k]$, write $[m] \rightarrow [n] \rightarrow [k]$ as $[m] \rightarrow [r] \xrightarrow{\varphi} [k]$ where $\varphi$ is injective, thus maps $a \in C_k$ in $\sigma C_n$ to $\psi^s(a) \in C_r$ in $\sigma C_m$, where $\psi^s$ is zero unless $\psi = d^n : \Delta[n - 1] \rightarrow \Delta[n]$. And homotopy groups and homology groups correspond via this equivalence, so does weak equivalences.
XI.1. SIMPLICIAL COMMUTATIVE ALGEBRAS

Proof: \(\sigma(C)\) defines a simplicial Abelian group because of the uniqueness of the the canonical decomposition. There is a natural map from \(\sigma(NA)\) to \(A\).

Now the task is to show that \(\sigma(NA) \cong A\) and \(\sigma(C) \cong C\). We has \(N(\sigma C)_n = C_n\) because \(d^i C_n = 0\) for \(i \neq n\) and the other components are all degeneracies thus are not in \(N(\sigma C)_n = C_n\) by (XI.1.2.8).

Then we prove \(\sigma(NA) \cong A\). It is a surjection by (XI.1.2.8) and induction. For the injectivity, if \((a_\varphi) \neq 0\) is mapped to 0, \(a_{\text{id}_n}\) is 0 by (XI.1.2.8). And we choose an ordering on the \(\varphi : [n] \rightarrow [k]\) by dominating, and suppose \(\psi\) is a minimal one. Now choose a section \(\xi\) of \(\psi\) that \(\xi\) is the maximal section, thus \(\varphi \xi\) cannot by \(\text{id}_k\) for any other \(\varphi\). Now by induction we have \(a_\psi = 0\), contradiction. □

Cor. (XI.1.2.10) (Trivial Simplicial Algebra). There is a functor form an \(R\)-algebra \(S\) to a trivial simplicial \(R\)-algebra \(s(S)\), it is a fully faithful embedding and \(\pi_0\) is left adjoint to it.

Proof: This is the adjointness of (II.3.2.7) under the equivalent \(\sigma\) (XI.1.2.9). □

Cor. (XI.1.2.11) (Model Structure on \(s\) Mod\(_R\)). By (II.5.4.2) applied to the equivalence with \(Ch_{\geq 0} R\), \(s\) Mod\(_R\) has the structure of a model category where a morphism \(X \rightarrow Y\) is

- an equivalence if it is a weak equivalence,
- a fibration if \(NX_n \rightarrow NY_n\) for any \(n \geq 1\).
- a cofibration if the the maps of the degenerate diagrams is of the form

\[
X_n \rightarrow Y_n = X_n \oplus \bigoplus_{\varphi : [n] \rightarrow [k]} \varphi^* P_k
\]

compatible with the differential, and \(P_k\) are all projectives.

Prop. (XI.1.2.12) (Fibrations). A fibration of simplicial \(R\)-modules is a fibration iff it is a fibration of simplicial sets. Moreover, by (XI.1.1.1), this is equivalent to \(X \rightarrow \pi_0 X \otimes_{\pi_0(Y)} Y\) is surjective.

Proof: □

Prop. (XI.1.2.13) (Simplicial Model Structure). \(s\) Mod\(_R\) admits a simplicial model category structure.

Proof: Firstly \(s\) Mod\(_R\) is a simplicial category tensored and cotensored over \(Set_\Delta\) (II.1.5.6) by (II.6.1.5), then it suffices to show (II.5.6.3)？.

Prop. (XI.1.2.14) (Model Structure on \(sCAlg_R\)). By (II.5.4.2) applied to the forgetful functor to \(Set_\Delta\), Let \(R\) be a commutative algebra, then the category of simplicial commutative algebras \(sCAlg_R\) has a simplicial model category structure where a morphism is

- a weak equivalence if it is a weak equivalence of simplicial sets.
- a fibration if it is fibration of simplicial sets.

Proof: □

Cor. (XI.1.2.15). For a ring map \(R \rightarrow S\), the tensor product \(S \otimes_R - \) and forgetful functor form a Quillen adjunction between \(sCAlg_R\) and \(sCAlg_S\).

Def. (XI.1.2.16) (Free Morphisms). A morphism of simplicial \(R\)-algebras is called \(\text{free}\) if it is \(s\)-free (II.6.1.4) on the a set of objects \(P_k\) where \(P_k\) are projective \(R\)-modules.
Prop. (XI.1.2.17) (Cofibrations in $sAlg_R$). A morphism in $sAlg_R$ is a cofibration iff it is a retraction of a free morphism (XI.1.2.16). In particular, a cofibrant simplicial $R$-algebra is the symmetrization of a chain of projective $R$-modules.

Proof: This follows from (II.5.4.2)(II.1.6.5) and notice the free morphisms just corresponds free commutative algebra applied to the attaching cell morphism in the category of sets. □

Simplicial Resolutions

Def. (XI.1.2.18) (Simplicial Resolutions).
- Let $M \in \text{Mod}_R$, a simplicial resolution of $M$ is a cofibrant replacement of $M$, or equivalently, it is an augmented simplicial $R$-module $X \to M$ that $NX \to M$ is a projective resolution.
- Let $M \in CAlg_R$, a free resolution of $M$ is free cofibrant replacement of $M$, or equivalently, it is an augmented simplicial commutative $R$-algebras $X \to M$ that $NX \to M$ is a resolution of $R$-modules, and $P_n$ are all projective (II.1.1.15) in $Alg_R$. Notice we can always choose a free resolution only only cofibrant resolution by (II.5.8.3) and Dold-Kan complex.

Def. (XI.1.2.19) (Bar Resolution). Let $C$ be a category and $T : C \to C$ a monad, and $X$ is an algebra over $T$ (II.2.1.2), then we can form a simplicial $T$-algebras where $B(T,X)_n = T^{n+1}X$ where the simplicial operators come from the action of $T$ on itself and the action of $T$ on $X$.

Then there is a simplicial morphism $B(T,X) \to X$, which is a simplicial homotopy.

Remark (XI.1.2.20). If $C$ is the category of sets, $R$ is an algebra and $T$ is a functor that sends a set $S$ to $R[S]$, then a $T$-algebra is just an $R$-algebra, and the bar resolution is just the canonical resolution, and it is a cofibrant replacement, by (XI.1.2.17).

Prop. (XI.1.2.21). If $f, g : A^* \to B^*$ are two homotopic maps of cosimplicial Abelian groups, then $f, g$ induces an isomorphism between their totalizations.

Proof: Cf.[Sta]019S. □

3 Properties

Def. (XI.1.3.1). If $A$ is a ring and $f_1, \ldots, f_n \in A$, the Koszul complex is defined to be

$$Kos(A, f_1, \ldots, f_n) = A \otimes \mathbb{Z}[X_1, \ldots, X_n] \mathbb{Z}.$$  

We want to extend this definition to the case of simplicial commutative rings.

Now if $A$ is a simplicial ring and $f_1, \ldots, f_n \in \pi_0(A)$, let $g_1, \ldots, g_n$ in $A_0$ lifting $f_i$, then we define

$$Kos(A, f_1, \ldots, f_n) = A \otimes \mathbb{Z}[X_1, \ldots, X_n] \mathbb{Z}.$$  

Then we need to check that this is independent of the lifting: if there is another set of lifting $h_i$, because we have identities

$$A \otimes \mathbb{Z}[X_1, \ldots, X_n] \mathbb{Z} \cong (A \otimes \mathbb{Z}[X_1, \ldots, X_n] \mathbb{Z}[X_1]) \otimes \mathbb{Z}[X] \mathbb{Z}$$

it suffices to prove for $n = 1$. Then there is a $\gamma \in A_1$ that $d_0(\gamma) = g, d_1(\gamma) = h$. Then we consider the evaluating maps

$$e_0, e_1 : \text{Hom}(\Delta_1, A) \to A$$
are weak equivalences, and then the maps
\[ e_0 : \text{Hom}(\Delta_1, A) \otimes^L_{\pi_0, \mathbb{Z}[X]} \mathbb{Z} \to A \otimes^L_{\pi_0, \mathbb{Z}[X]} \mathbb{Z} \]
\[ e_1 : \text{Hom}(\Delta_1, A) \otimes^L_{\pi_0, \mathbb{Z}[X]} \mathbb{Z} \to A \otimes^L_{\pi_0, \mathbb{Z}[X]} \mathbb{Z} \]
are also weak equivalences, so we are done.

And if \( M \) is a simplicial \( A \)-module, then we define
\[ \text{Kos}(M, f_1, \ldots, f_n) = M \otimes^L_A \text{Kos}(A, f_1, \ldots, f_n) \]

**Def. (XI.1.3.2) (Flatness).** A map \( A \to B \) of simplicial rings is called (faithfully) flat if \( \pi_0(B) \) is (faithfully) flat over \( \pi_0(A) \) and \( \pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_i(B) \) is an isomorphism for any \( i \).

**Prop. (XI.1.3.3).** Flatness is stable under base change.

**Proof:** Cf.[Emerton note, completely flatness, P12]. □

**Prop. (XI.1.3.4).** If \( \pi_0(A) \otimes^L_A M \) is (faithfully) flat over \( \pi_0(A) \), then \( M \) is (faithfully) flat over \( A \).

**Proof:** Cf.[Emerton note, completely flatness, P12]. □

**Def. (XI.1.3.5).** If \( A \) is a simplicial ring and \( M \) is a simplicial \( A \)-module, and \( I = (f_1, \ldots, f_n) \) is an ideal of \( \pi_0(A) \), then \( M \) is called \( I \)-completely flat over \( A \) if \( \text{Kos}(M, f_1, \ldots, f_n) \) is flat over \( \text{Kos}(A, f_1, \ldots, f_n) \).

Clearly if \( M \) is \( A \)-flat, then it is \( I \)-completely flat.

**Prop. (XI.1.3.6) (Flatness and Derived Completion).** If \( A \) is a simplicial ring, \( M \) is a flat simplicial \( A \)-module, and \( I = (f_1, \ldots, f_n) \) is an ideal of \( \pi_0(A) \), then its derived \( I \)-completion \( \hat{M} = \text{holim}_n \text{Kos}(M, f_1^N, \ldots, f_n^N) \) is \( I \)-completely flat. The proof is similar to that of(I.10.7.4).

**Prop. (XI.1.3.7) (Relative Regular Sequence).** If \( A \to B \) is a map of simplicial rings, then \( x_1, x_2, \ldots, x_n \in \pi_0(B) \) is called regular with respect to \( A \) if
\[ A \to \text{Kos}(B, x_1, \ldots, x_n) \]
is flat. And if \( I = (f_1, \ldots, f_m) \in \pi_0(A) \), then it is called \( I \)-completely regular if
\[ \text{Kos}(A, f_1, \ldots, f_m) \to \text{Kos}(A, f_1, \ldots, f_m) \otimes^L_A \text{Kos}(B, x_1, \ldots, x_n) = \text{Kos}(B, f_1, \ldots, f_m, x_1, \ldots, x_n) \]
is flat. In particular, regular relative to \( A_\bullet \) implies \( I \)-completely regular relative to \( A_\bullet \).

**Prop. (XI.1.3.8).** If \( (f_1, \ldots, f_{r-1}, f) \) is regular w.r.t. \( A \) and \( (f_1, \ldots, f_{r-1}, g) \) is regular w.r.t \( A \), then \( (f_1, \ldots, f_{r-1}, fg) \) is also regular w.r.t. \( A \). In particular, for any \( n_i > 0 \), \( f_1^{n_1}, \ldots, f_r^{n_r} \) is also regular w.r.t. \( A \).

**Proof:** Similarly as in(I.7.4.6), we have a distinguished triangle
\[ \text{Kos}(M, f_1, \ldots, f_{r-1}, f) \to \text{Kos}(M, f_1, \ldots, f_{r-1}, fg) \to \text{Kos}(M, f_1, \ldots, g). \]

Then we can use induction. □

**Prop. (XI.1.3.9) (Regular and Derived Completion).** If \( x_1, \ldots, x_n \in \pi_0(B) \) is regular w.r.t \( A \), then they are \( I \)-completely regular w.r.t. \( A \) in the derived \( I \)-completion \( \hat{B} \). The proof is similar to that of(XI.1.3.6).
4 Non-Abelian Derived Functors

Prop. (XI.1.4.1) (Left Derived Functor). Let $F : \text{Poly}_A \to \mathcal{C}$ be a functor where $\mathcal{C}$ is any $\infty$-category admitting all colimits (e.g. $\mathcal{D}(\text{Ab})$), then there exists a left Kan extension $LF$ of $F$ along $\text{Poly}_A \subset \text{CAlg}_A$ that

- $LF$ commutes with filtered colimit.
- $LF$ commutes with geometric realization of simplicial resolutions: given $B \in \text{CAlg}_A$ and a simplicial resolution $P_\bullet \to B$ by $A$-algebras, the geometric realization $|LF(P_\bullet)|$ is equivalent to $LF(B)$.

which is called the left derived functor of $F$.

Proof: Cf.[Bhatt, Prism, 7.1.2].
XI.2 Andre-Quillen-Cohomology

Basic references are [Andre-Quillen Cohomology of Commutative Algebras Iyenger]. See also [Quillen Cohomology of Commutative Rings] and [Quillen On the (Co-)homology of Commutative Rings]. [Smoothness, Regularity and Complete Intersections] is a must read.

1 Naive Cotangent Complex

This subsection is obsolete.

Prop. (XI.2.1.1) (Polynomial Replacement). For a morphism of ring morphisms \((R \to S) \to (R' \to S')\), let \(\alpha, \alpha'\) be two presentations, then there exists morphism of presentations, and different morphisms induce homotopic maps \(NL_{S/R} \to NL_{S'/R'}\).

Proof: Cf.[Sta]00S1. In fact, any surjective formally smooth representation will give the naive cotangent complex, up to quasi-isomorphism (XI.2.1.4).

Cor. (XI.2.1.2). If \(A = R[X_i]\) be a polynomial algebras, then \(NL_{A/R}\) is homotopic to \((0 \to \Omega_{B/A})\) because \(A \to A\) is a presentation with zero kernel.

If \(R \to A\) is surjective with kernel \(I\), then \(NL_{A/R}\) is homotopic to \((I/I^2 \to 0)\).

Lemma (XI.2.1.3) (Formally Smooth Replacement 1). If \(A \to B\) is a ring map that has two surjective presentations \(C \to B, D \to B\) with kernels \(I, J\). If there is a map \(C \to D\) commutating these two presentation, \(D\) formally smooth, and \(C \to D\) is surjective or \(C\) is formally smooth, then their corresponding naive cotangent complexes are quasi isomorphic.

Proof: Cf.[Foundations of Perfectoid Geometry P123].

Prop. (XI.2.1.4) (Formally Smooth Replacement 2). If \(B\) is an \(A\)-algebra that has two formally smooth presentation \(C \to B, D \to B\) with kernels \(I, J\). then their corresponding naive cotangent complexes are quasi isomorphic.

Proof: It suffices to prove they are both quasi isomorphic to the canonical cotangent complex.

For this, we first consider the diagram \(\begin{array}{ccc} D & \to & C \\ \downarrow & & \downarrow \\ A[B] & \to & B \end{array}\), where \(D = A[S]\) and \(S = C \coprod A[B]\) as sets.

The two map \(D \to A[B]\) and \(D \to A[B]\) can be chosen because \(C \to B\) is surjective. So the results follows from (XI.2.1.3).

Prop. (XI.2.1.5) (Jacobi-Zariski Sequence). Let \(A \to B \to C\) be a ring map. Choose a presentation \(\alpha : P \to B\) for \(B/A\) with kernel \(I\), a presentation \(\beta : Q \to C\) for \(C/B\) with kernel \(J\), a presentation \(\gamma : R \to C\) for \(C/A\) with kernel \(K\), then there is an exact sequence of complexes:

\[
\begin{align*}
I/I^2 \otimes_B C &\longrightarrow K/K^2 \longrightarrow J/J^2 \longrightarrow 0 \\
0 &\longrightarrow \Omega_{P/A} \otimes_B C \longrightarrow \Omega_{R/A} \otimes C \longrightarrow \Omega_{Q/B} \otimes C \longrightarrow 0
\end{align*}
\]

Applying snake lemma, we get

\[
H_1(NL_{B/A} \otimes_B C) \to H_1(L_{C/A}) \to H_1(L_{C/B}) \to C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0.
\]
Proof: Cf.[Sta]00S2. □

**Prop. (XI.2.1.6) (Localization).** Let $A \to B$ be a ring map, for a multiplicative set $S$ of $B$, we have $NL_{B/A} \otimes_B S^{-1} B$ is quasi-isomorphic to $NL_{S^{-1}B/A}$.

**Proof:** Because it commutes with colimit, it suffice to prove for $S = f$, and this is the content of lemma (XI.2.1.7) below. □

**Lemma (XI.2.1.7).** If $A \to B$ is a ring map and $\alpha : P \to B$ is a presentation of $B$ with kernel $I$, then $\beta : P[X] \to B_g : X \to 1/g$ is a presentation of $B_g$ with kernel $J = I + (gX - 1)$. Then we have

- $J/J^2 = (I/I^2)_g \oplus B_g(fX - 1)$.
- $\Omega_{P[X]/A} \otimes_{P[X]} B_g = \Omega_{P/A} \otimes_P B_g \oplus B_g dX$.
- $NL(\beta) \cong NL(\alpha) \otimes_B B_g \oplus (B_g \xrightarrow{g} B_g)$.

Hence $NL_{B/A} \otimes_B B_g \to NL_{B_g/A}$ is a homotopy equivalence.

**Proof:** Cf.[Sta]08JZ. □

## 2 Cotangent Complex

**Def. (XI.2.2.1) (Cotangent Complex).** The adjunction of $A \ltimes -$ and $A \otimes_- \Omega_{-/R}(I.7.3.5)$ extends to an adjunction between $(sCAg_{sR})/A$ and $sMod_A$. Theses categories are model categories by(XI.1.2.11)(XI.1.2.14) and(II.5.4.1), and $A \ltimes -$ preserves all weak equivalences and fibrations, so it is a Quillen adjunction(II.5.2.1). Then the **cotangent complex** $L_{A/R}$ as a simplicial $A$-module is defined to be the total left derived functor applied to the trivial simplicial algebra $A$. Equivalently, it is

$$L_{A/R} = A \otimes_X \Omega_{X/R}$$

where $X$ is a cofibrant replacement(II.5.1.12) of $A$.

Because of the Dold-Kan equivalence, we sometimes also call $NL_{A/R}$ the cotangent complex.

**Def. (XI.2.2.2) (André-Quillen Homology and Cohomology).** The **André-Quillen homology** is defined to be

$$D_q(A/R) = \pi_q(L_{A/R}) = H_q(N(L_{A/R}))(XI.1.2.2).$$

More generally, if $M$ is an $A$-module, then let

$$D_q(A/R, M) = \pi_q(L_{A/R} \otimes_A M).$$

The **André-Quillen cohomology** is defined to be

$$D^q(A/R, M) = \text{Ext}^n(NL_{A/R}, M).$$

**Prop. (XI.2.3) (Functoriality).** The cotangent complex is functorial in arrows $R \to A$: If there is a commutative diagram

\[
\begin{array}{ccc}
R & \longrightarrow & S \\
\downarrow & & \downarrow \\
A & \longrightarrow & B
\end{array}
\]
then there is a natural morphism \( L_{A/S} \otimes_A B \to L_{B/R} \). This is because if \( X \) is a cofibrant replacement for \( A \), then \( X \otimes_R S \) is also cofibrant object, because \( B \otimes_A - \) is a Quillen adjunction, by (XI.1.2.15), then it factors through \( X \otimes_R S \to Y \to B \) where \( Y \) is a cofibrant replacement of \( B \), thus \( Y \) is also cofibrant. Then the functor \( L_{A/S} \otimes_A B \to L_{B/R} \) is induced by

\[
B \otimes_A (A \otimes_X \Omega_{X/R}) \to B \otimes_Y \Omega_{Y/S}.
\]

The formation of Kähler differential commutes with arbitrary colimit as it is a left adjoint, so the formation of cotangent complex commutes with filtered colimits, both in \( A \) and \( B \). Especially, it commutes with taking stalks, hence the sheaf of cotangent complexes of a map between schemes can be constructed as in the case of Kähler differentials, and it is a Qco sheaf.

**Def. (XI.2.2.4) (Canonical Resolution).** By the Dold-Kan correspondence, we will say that two simplicial \( A \)-modules are quasi-isomorphic iff their normalized nerves are quasi-isomorphic. Then \( P_A(B) \to B \) is a quasi-isomorphic resolution of \( B \), where \( B \) is the trivial complex.

**Proof:** There is a homotopy \( d \) between \( \text{id} \) and \( 0 \) for \( n \leq 0 \), where

\[
d_n : F(GF)^n G(B) \to F(FG)^n \circ GFG(B)
\]

using counit map, and on degree 0, -1, it is \( A[A[B]] \xrightarrow{\partial_1 - \partial_2} A[B] \to B \to 0 \), which is clearly 0, so this is a zero map.

Thus \( \text{Tot}(P_A(B)) \cong B \), and \( N_s(A) \cong \text{Tot}(A) \) by Dold-Kan correspondence, so we are done. \( \square \)

**Cor. (XI.2.2.5).**

\[
D^0(L_{B/A}, M) = \Omega_{B/A} \otimes M, \quad D^0(A/R, M) = \text{Der}_R(A, M)
\]

by the generator-relation definition of Kähler differential.

**Prop. (XI.2.2.6) (The fundamental Distinguished Triangle).** If \( T \) is a site and \( A \to B \to C \) are morphisms of sheaves of rings over \( T \), then there is a morphism of simplicial \( A \)-modules that corresponds to distinguished triangles in \( D^{\geq 0}(C) \) via Dold-Kan correspondence:

\[
L_{B/A} \otimes_B C \to L_{C/A} \to L_{C/B}.
\]

In particular, if \( M \) is a \( B \)-module, then there are long exact sequences

\[
\ldots \to D_1(A/R, M) \to D_1(B/R, M) \to D_1(B/A, M) \to M \otimes_A \Omega_{A/R} \to M \otimes_A \Omega_{B/A} \to M \otimes_B \Omega_{B/A} \to 0
\]

\[
\ldots \to D^1(A/R, M) \to D^1(B/R, M) \to D^1(B/A, M) \to \text{Der}_R(A, M) \to \text{Der}_A(B, M) \to \text{Der}_A(B, M) \to 0
\]

**Proof:** Choose a simplicial resolution \( X \to A \) where \( X \) is free, then we factor the morphism \( X \to A \to B \) to get a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow & & \downarrow \\
A & \to & B
\end{array}
\]

where \( i \) is free, then we have an exact sequence(in \( Ch_{\geq 0}(R) \) via Dold-Kan) of simplicial modules

\[
0 \to B \otimes_X \Omega_{X/R} \to B \otimes_Y \Omega_{Y/R} \to B \otimes_Y \Omega_{Y/X} \to 0
\]
because each \( X_n \to Y_n = X_n \otimes \bigoplus_{\varphi: [n] \to [k]} \varphi^* P_k \) has a retraction, and use (I.7.3.7).

Notice we have a simplicial map \( A \otimes X Y \to B \) which is a weak equivalence and \( A \otimes X Y \) is a free simplicial \( A \)-algebra. Then it suffices to note that \( X_n \to Y_n = X_n \otimes \bigoplus_{\varphi: [n] \to [k]} \varphi^* P_k \) is projective implies
\[
B \otimes Y \Omega_{Y/X} \cong B \otimes A \otimes X Y \Omega_{A \otimes X Y/A}
\]
, and \( \Omega_{X/R} \) is termwise projective.

Prop. (XI.2.2.7) (Properties of Cotangent Complexes).
- If \( B = s(P) \) where \( P \) is a projective \( A \)-module, then \( L_{B/A} \) is weakly equivalent to \( \Omega^1_{P/A}[0] \).
- (Künneth Formula) If \( B, C \) are Tor independent over \( A \), then
\[
L_{B \otimes C/A} \cong (L_{B/A} \otimes_A C) \oplus (L_{C/A} \otimes_A B).
\]
- (Flat Base Change) If \( B, C \) are Tor independent over \( A \), \( L_{B/A} \otimes_A C \cong L_{B \otimes C/C} \).

Proof:
- \( S_R(P) \) is already cofibrant in \( (sCAlg_R)_A \).
- Let \( X \to B, Y \to C \) be cofibrant replacement of \( A, B \) respectively, then \( X \otimes_R Y \to B \otimes_A C \) is a cofibrant replacement, as \( X \otimes_A Y \) is cofibrant, and Tor independence shows
\[
\pi_*(X \otimes_A Y) = \pi_*(X) \otimes_A \pi_*(Y) = B \otimes_A C.
\]
thus the result follows from (I.7.3.6).
- The same as Künneth formula, noticing that \( X \otimes_A C \to B \otimes_A C \) is a weak equivalence by Tor independence, and \( X \otimes_A C \) is cofibrant.

3 Relations with Algebraic Properties

Cf.[Andre-Quillen Homology].

Prop. (XI.2.3.1) (Acyclicity for Smooth Algebras). If \( A \to B \) is smooth, then \( L_{B/A} \cong \Omega^1_{B/A}[0] \). In particular, if \( A \to B \) is étale, then \( L_{B/A} = 0 \), and \( L_{C/A} \cong L_{B/A} \otimes_B C \), by distinguished triangle (XI.2.2.6).

Proof: The cotangent complex is local, so we may assume it is standard smooth, so it factors as \( A \to A[X_1, \ldots, X_k] \xrightarrow{g} B \), where \( g \) is étale, so using the distinguished triangle and polynomial case, the result follows.

Prop. (XI.2.3.2) (Compatibility with \( p \)-adic Completion). If \( A \) is a \( p \)-adically complete commutative ring with bounded \( p \)-torsion and \( B \) is a flat \( A \)-module, then \( B \) also has bounded \( p^\infty \)-torsion by (I.10.1.4), let \( \hat{B} \) be the \( p \)-adic completion of \( B \), then the cotangent complex \( L_{\hat{B}/B} \) vanishes after derived \( p \)-completion.

In particular, by the distinguished triangle (XI.2.2.6), if \( B \) is a smooth algebra, then \( L_{\hat{B}/B} \cong \Omega^1_{\hat{B}/B} \) is a finite projective \( \hat{B} \)-module.

Proof: This is true after base change \( - \otimes^L_A A/p \) by flat base change (XI.2.2.7), (I.10.7.4) and derived Nakayama (I.10.6.10).
4 Deformations

Prop. (XI.2.4.1) (Topological Invariance of Étale Site). Let $A$ be a ring, consider the following category: $C_A$ of flat $A$-algebras $B$ that $L_{B/A} = 0$, then if $A \to A$ is surjective with locally nilpotent kernel, then the base change defines an isomorphism of categories $C'_A \cong C_A$.

By (XI.2.2.7), $L_{B/A}$ vanish is equivalent to being étale, thus the properties characterize the invariance of étale site under infinitesimal thickening.

Proof: ?

Prop. (XI.2.4.2) (Relative Perfect Case). If $A$ is a ring of charp and $B$ is an $A$-algebra which is relatively perfect, i.e. $B^{(1)} = B \otimes_{A, \text{Frob}} A \to B$ is an isomorphism, then $L_{B/A} = 0$.

Proof: Notice for any $A$-algebra $C$, the relative Frobenius induces zero map $L_{C^{(1)}/A} \to L_{C/A}$, because by using the canonical polynomial resolution, $d(x^p) = px^{p-1}xs = 0$. Now the relative Frobenius is an isomorphism $B^{(1)} \to B$, thus induces an isomorphism $L_{B^{(1)}/A} \to L_{B/A}$ by Functorality, thus $L_{B/A} = 0$.

Cor. (XI.2.4.3) (Witt Vector Construction). There is an equivalence of categories of $C_n = \text{flat } \mathbb{Z}/p^n$-algebras that $A/p$ is perfect and $C_1 = \text{perfect rings over } \mathbb{Z}/p$.

moreover, taking limit, this is even equivalent to the category of flat $p$-adically complete $\mathbb{Z}_p$-algebras that $A/p$ is perfect. Which is just the construction of Witt vectors.

Proof: It suffices to show that $C_n \subset C_{\mathbb{Z}/p^n}$: By (XI.2.4.2) and flat base change (XI.2.2.7), $L_{A/(\mathbb{Z}/p^n)} \otimes_{\mathbb{Z}/p^n} \mathbb{Z}/p \cong L_{(A/p)/(\mathbb{Z}/p)} = 0$, so $L_{A/(\mathbb{Z}/p^n)} \otimes_{\mathbb{Z}/p^n} \mathbb{Z}/p^k \cong 0$ by induction, and so $L_{A/(\mathbb{Z}/p^n)} \cong 0$.

For the last assertion, it is flat because it is torsion-free, which is because if $p(x_n) = 0$, then by $0 \to p^n \mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^{n+1} \xrightarrow{p} \mathbb{Z}/p^n \to 0$ and the flatness of $A_{n+1}$, $x_{n+1} \in p^nA_{n+1}$, thus $x_n = 0$, and $x = 0$.

Prop. (XI.2.4.4) (Adjointness of Witt Vectors). Using a more careful analysis of cotangent complex (embedded deformation), we can show that if $A \to B \in C_A$ and there is a infinitesimal deformation $C \to C'$ of $A$-algebra, then a map $B \to C'$ can be lifted to an $A$-algebra map $A \to C$.

In particular, taking inverse image, we get that $\text{Hom}_{\mathbb{F}_p}(A, B/p) \cong \text{Hom}_{\text{Alg}_{\mathbb{Z}_p}}(W(A), B)$.

which is the usual adjointness of the Witt vector construction.

5 Algebra Extension

Cf. [Perfectoid Geometry Appendix B].

Def. (XI.2.5.1) (Algebra Extensions). Let $A \to B$ be a ring map and $M$ be a $B$-module, then an $A$-algebra extension of $B$ by $M$ is a short exact sequence of $A$-modules $0 \to M \to B' \to B \to 0$ that $B'$ is an $A$-algebra with $M$ being an ideal of it.

The set of such extensions are denoted by $\text{Exal}_A(B, M)$.

Prop. (XI.2.5.2). $\text{Exal}_A(B, M)$ is a group under Baer sum, where the sum of two extension is the extension given by pushout, i.e. $(B_1 \oplus B_2)/\{(m, -m) : m \in M\}$. Moreover, it is a $B$-module, where the multiplication is the pushout along multiplication of $b$ on $M$. 

Prop. (XI.2.5.3). There is a trivial extension given by \( D_B(M) = B \oplus M \) (I.7.3.5), and the automorphism of \( D_B(M) \) is isomorphic to \( \text{Der}_A(B, M) \) via \( d \mapsto \text{id} \oplus d \).

Proof: Cf. [Foundations of Perfectoid Spaces Masullo P118]. \( \square \)

Prop. (XI.2.5.4). Let \( A \to B \to C \) be ring maps, then for any \( C \)-module \( M \), there is an exact sequence

\[
0 \to \text{Der}_B(C, M) \to \text{Der}_A(C, M) \to \text{Der}_A(B, M) \to \text{Exal}_B(C, M) \to \text{Exal}_A(C, M) \to \text{Exal}_A(B, M)
\]

functorial in \( M \). Where \( \partial \) is given by (XI.2.5.3).

Proof: Cf. [Foundations of Perfectoid Spaces Masullo P119]. \( \square \)

Prop. (XI.2.5.5). Let \( A \to B \) be a ring map or a map of sheaves of rings, and let \( M \) be a \( B \)-module, then there is an isomorphism of \( B \)-modules that is natural in \( M \):

\[
\text{Exal}_A(B, M) = \text{Ext}_B^1(NL_{B/A}, M).
\]

Proof: Cf. [Foundations of Perfectoid Spaces, P127]. \( \square \)

**Infinitesimal Deformation**

Def. (XI.2.5.6). An infinitesimal deformation of a f.g. \( k \)-algebra is defined as a algebra \( A' \) flat over \( D = k[t]/(t^2) \) that \( A' \otimes_D k = A \).

A f.g. \( k \)-algebra is called rigid if it has no infinitesimal deformations.

Prop. (XI.2.5.7). Let \( A \) be a f.g. \( k \)-algebra, write \( A \) as a quotient of a polynomial ring over \( k \) with kernel \( J \), then there is an exact sequence \( J/J^2 \to \Omega_{P/k} \otimes_P A \to \Omega_{A/k} \to 0 \) by (I.7.3.7), then we apply \( \text{Hom}_A(-, A) \) and let \( T^1(A) = \text{Coker}(\text{Hom}_A(\Omega_{P/k} \otimes_A A, A) \to \text{Hom}_A(J/J^2, A)) \). Then \( T^1(A) \) parametrize infinitesimal deformations of \( A \).
XI.3 Derived Algebraic Geometry (Lurie)
XI.4 Condensed Mathematics (Scholze)

Main references are [Condensed Mathematics, Scholze].

1 Condensed Objects

Def. (XI.4.1.1) (κ-Condensed Objects). The pro-étale site $\proet$ of a point is isomorphic to the category of profinite sets in the standard topology. Then if $C$ is any category of ..., then a condensed... is defined to be the category of $C$-valued sheaves on $\proet$.

Given a condensed set $T$, we call $T(\ast)$ the underlying set of $T$.

More generally, choosing an uncountable strong limit cardinal $\kappa$, the site $\proet_{\kappa}$ is defined to be the category of $\kappa$-small profinite sets $S$ in the standard topology. If $C$ is any category of ..., then a $\kappa$-condensed... is defined to be the category of $C$-valued sheaves on $\proet_{\kappa}$.

Prop. (XI.4.1.2) (Adjointness). Given an uncountable strong limit cardinal $\kappa$, the functor $X \mapsto X$ from the category of topological spaces/groups/rings to the category of $\kappa$-condensed sets/groups/rings is faithful, and fully faithful when restricted to the category of $\kappa$-small objects.

And in the set case, the functor admits a left adjoint $T \mapsto T(\ast)_{top}$, where $T(\ast)_{top}$ is the set $T(\ast)$ equipped with the quotient topology by $\coprod_{S \to T} S \to T$, where the indices run through all $\kappa$-small profinite sets $S$ with a map to $T$. In particular, $X(\ast)_{top} \cong X^{\kappa cg}$.

Prop. (XI.4.1.3) (Free Condensed Abelian Groups). By the adjoint functor theorem (II.1.1.24), the forgetful functor from $\Cond(\Ab)$ to $\Cond(\Set)$ has a left adjoint $T \mapsto \Z[T]$. Concretely, $\Z[T]$ is the sheafification of the functor that sends a compact Hausdorff space $S$ to the free Abelian group $\Z[T(S)]$. In particular, by Yoneda lemma, for any compact Hausdorff space $S$, there is a condensed Abelian group $\Z[S]$ that for any condensed Abelian group $M$, $\text{Hom}(\Z[S], M) = M(S)$.

Lemma (XI.4.1.4). Consider the site of all $\kappa$-small compact Hausdorff topological spaces, then the category of sheaves on this site is equivalent to that of $\proet_{\kappa}$.

Proof: Use (V.1.2.21), because any $\kappa$-small compact Hausdorff space is a quotient space of a $\kappa$-small profinite space $\beta S_0$ (IX.1.2.15), noticing that $|\beta S| \leq 2^{2^{|S|}} < \kappa$.

Lemma (XI.4.1.5). Consider the site of all $\kappa$-small extremally disconnected spaces, then the category of sheaves on this site is equivalent to that of $\kappa$-small condensed sets, via restriction.
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Proof: Because any $\kappa$-small compactly generated space is a quotient space of a $\kappa$-small extremally disconnected space($\beta S$(IX.1.2.17), noticing that $|\beta S| \leq 2^{2^{\aleph_0}} < \kappa$).

Cor. (XI.4.1.6) (Cond(Ab) and Extremally Disconnected Spaces). The category of $\kappa$-condensed Abelian groups is equivalent to the category of presheaves $\mathcal{F}$ on the category of $\kappa$-extremally disconnected spaces that $\mathcal{F}(\varnothing) = 0$ and $\mathcal{F}(S_1 \coprod S_2) = \mathcal{F}(S_1) \times \mathcal{F}(S_2)$.

Proof: It suffices to show that the second sheaf condition is automatic: For a surjective map of extremally disconnected spaces $f : \bar{S} \to S$, there is an isomorphism

$$\mathcal{F}(S) \xrightarrow{\mathcal{F}(f)} \{g \in \mathcal{F}(\bar{S})|\mathcal{F}(p_1)(g) = \mathcal{F}(p_2)(g)\}.$$

By(IX.1.1.26), there is a section $\sigma : S \to \bar{S}$ that $f \circ \sigma = \text{id}_S$, thus $\mathcal{F}(\sigma) \circ \mathcal{F}(f) = \text{id}$, thus $\mathcal{F}(f)$ is injective. For the surjectivity, suppose $\mathcal{F}(p_1)(g) = \mathcal{F}(p_2)(g)$, then

$$\mathcal{F}(p_2)(\mathcal{F}(f)\mathcal{F}(\sigma)(g)) = \mathcal{F}((\sigma \circ f) \times_S \text{id}_{\bar{S}})\mathcal{F}(p_1)(g) = \mathcal{F}((\sigma \circ f) \times_S \text{id}_{\bar{S}})\mathcal{F}(p_2)(g) = \mathcal{F}(p_2)(g)$$

And similarly $\mathcal{F}(p_2)$ is injective, thus $g = \mathcal{F}(f)\mathcal{F}(\sigma)(g)$ is in the image.

Prop. (XI.4.1.7) (Category of Condensed Abelian Groups). The category of $\kappa$-condensed Abelian groups satisfies Grothendieck’s Axiom $AB3, AB4, AB5, AB6, AB3^*, AB4^*$. And also it is generated by compact projective objects.

Proof: We use(XI.4.1.6). Because all limits and colimits of Abelian groups commutes with finite products, the limits and colimits in the category of condensed Abelian groups are just the pointwise limits and colimits, thus the axioms follow form that of the category Ab.

By(XI.4.1.3), the condensed Abelian group $\mathbb{Z}[S]$ for $S$ $\kappa$-extremally disconnected satisfies $\text{Hom}(\mathbb{Z}[S], M) = M(S)$, and by arguments above, $M \to M(S)$ commutes with all limits and colimits, thus $\mathbb{Z}[S]$ is compact and projective. And we show every $M$ admits a surjection from some direct sum of $\mathbb{Z}[S]$: use Zorn’s lemma, choose the maximal object $M'$ that admits a surjection, if $M/M' \neq 0$, then find a nonzero map $\mathbb{Z}[S] \to M/M'$ (because $M(S) = 0$ for any $S$ implies $M = 0$), then it lifts to a nonzero map $\mathbb{Z}[S] \to M$ by projectivity, contradiction.

Cor. (XI.4.1.8).

• We can define the tensor of two condensed Abelian groups $M, N$ as the shiffication of the presheaf $S \mapsto M(S) \otimes N(S)$.

• We can define an internal Hom, which is right adjoint to tensor operator. In particular, for any compact Hausdorff space $S$, $\text{Hom}(M, N)(S) = \text{Hom}(\mathbb{Z}[S] \otimes M, N)$.

• The derived category $D(\text{Cond}(Ab))$ is also compactly generated, and we can define $\text{Rtensor}$ and $\text{RHom}$ as in1.

Proof: Cf.[Condensed Mathematics, P13].

Prop. (XI.4.1.9). Let $\kappa' > \kappa$ be uncountable strongly limit cardinals, then there is a natural functor from the set of $\kappa$-condensed sets to the category of $\kappa'$-condensed sets by pulling back along the morphism of sites $\ast_{\kappa'}^{-\text{proétale}} \to \ast_{\kappa}^{-\text{proétale}}$. Then this functor is fully faithful and commutes with all colimits and $\lambda$-small limits, where $\lambda$ is the cofinality of $\kappa$.

Proof: This should have something to do with(V.1.2.22), thus it is left adjoint to the restriction functor and $i^*i_{\ast} \cong \text{id}$, thus it is fully faithful and commutes with all colimits. For the limits, cf.[Condensed Mathematics, P14].
Def. (XI.4.1.10) (Condensed Objects). The category of condenseds sets is defined to be the filtered colimits of the category of $\kappa$-condensed sets along the filtered poset of all uncountable limit cardinals $\kappa$.

Prop. (XI.4.1.11). If $X$ is a $T_1$ topological space, then $X$ is a condensed set that all maps from points are quasicompact. Conversely, if $T$ is a condensed set that all maps from points are quasicompact, then $T(*_{\text{top}})$ is a compactly generated $T_1$ space.

Proof: Cf.[Condensed Mathematics, P16].

Prop. (XI.4.1.12).
- The functor $X \mapsto X$ induces an equivalence between compact Hausdorff space and qcqs condensed sets.
- A compactly generated space $X$ is weak Hausdorff iff $X$ is quasi-separated. For any quasi-separated condensed set $T$, the space $T(*_{\text{top}})$ is compactly generated weak Hausdorff.

Proof: Cf.[Condensed Mathematics, P16].

2 Cohomologies

Prop. (XI.4.2.1). For any set $I$, there is an isomorphism

$$\check{H}^i(\prod_I S^1, \mathbb{Z}) \cong \bigwedge^i (\oplus I \mathbb{Z}).$$

preserving the natural cup product.

Proof: If $I$ is finite, this follows from the classical calculation of the cohomology of the tori. If $I$ is infinite, then we should use lemma (XI.4.2.2) below.

Lemma (XI.4.2.2). If $S_j, j \in J$ is a filtered system of compact Hausdorff spaces and $S = \lim_{\leftarrow j \in J} S_j$, then there are natural maps

$$\lim_{\leftarrow j} H^i_{\check{\text{Cech}}}(S_j, \mathbb{Z}) \to H^i_{\check{\text{Cech}}}(S, \mathbb{Z})$$

are isomorphisms.

Proof:

Prop. (XI.4.2.3) (Z-Cohomology). Let $S$ be a compact Hausdorff space, then there are natural functorial isomorphisms

$$H^i_{\text{sheaf}}(S, \mathbb{Z}) \cong H^i_{\text{cond}}(S, \mathbb{Z}).$$

Proof: Because the Čech and sheaf cohomology of $S$ are equal by (V.6.8.15), it suffices to calculate for Čech cohomology.

Firstly, if $S$ is a profinite set, let $S = \lim_{\leftarrow j} S_j$ where $S_j$ are finite, then

Prop. (XI.4.2.4) (R-Cohomologies Vanish). For any compact Hausdorff space $S$,

$$H^i_{\text{cond}}(S, \mathbb{R}) = 0$$

for $i > 0$, and $H^0(S, \mathbb{R}) = C(S, \mathbb{R}).$

Proof: Cf.[Condensed Mathematics, P21].
3 Locally Compact Abelian Groups

4 Solid Abelian Groups

Def. (XI.4.4.1) (Solid Abelian Group). For a profinite set $S = \varprojlim S_i$, define the condensed Abelian group

$$Z[S]\mathbf{■} = \varprojlim Z[S_i].$$

There is a natural map $S \to Z[S]\mathbf{■}$, inducing a map $Z[S] \to Z[S]\mathbf{■}$.

Then a solid Abelian group is a condensed Abelian group $A$ that for any profinite set $S$ and a morphism of Abelian groups $Z[S] \to A$ extends to a morphism $Z[S]\mathbf{■} \to A$.

A complex $C \subset D(\text{Cond}(Ab))$ is called solid if for all profinite set $S$, the natural map

$$R\text{Hom}(Z[S]\mathbf{■}, C) \to R\text{Hom}(Z[S], C)$$

is an isomorphism.

Prop. (XI.4.4.2) (Free Solid Abelian Group). For a profinite set $S$,

- Consider $Z[S]\mathbf{■} = \varprojlim Z[S_i] = \varprojlim \text{Hom}(C(S_i, Z), Z) = \text{Hom}(C(S, Z), Z)$.

This means that the underlying Abelian group of $Z[S]\mathbf{■}$ is the $\mathbb{Z}$-valued measures on $S$.

- There is some set $|I| \leq 2^{|S|}$, that there is an isomorphism $Z[S]\mathbf{■} \cong \prod_I \mathbb{Z}$.

- $Z[S]\mathbf{■}$ is solid both as a module and a complex.

Proof: 2: Take an isomorphism $C(S, \mathbb{Z}) \cong \bigoplus_I \mathbb{Z}$, then

$$Z[S]\mathbf{■} = \text{Hom}(C(S, Z), Z) \cong \text{Hom}(\bigoplus_I \mathbb{Z}, Z) \cong \prod_I \mathbb{Z}.$$

3: We need to show the extension property, but by 2, it suffices to show for $Z[S]\mathbf{■} = \mathbb{Z}$. □

Prop. (XI.4.4.3) (Category of Solid Abelian Groups). The category $\text{Solid} \subset \text{Cond}(Ab)$ is an Abelian subcategory that is stable under all limits, colimits and extensions. The objects $\prod_I \mathbb{Z}$, where $I$ is any set, form a family of compact projective generators. The inclusion $\text{Solid} \subset \text{Cond}(Ab)$ admits a left adjoint $M \mapsto M\mathbf{■}$ which is the unique colimit-preserving extension of $Z[S] \mapsto Z[S]\mathbf{■}$.

The functor $D(\text{Solid}) \to D(\text{Cond}(Ab))$ is fully faithful and its essential image is precisely the solid Abelian groups, and the inclusion admits a left adjoint $C \to C\mathbf{■}$ which is left derived functor of $M \mapsto M\mathbf{■}$.
Chapter XII

Mathematical Logic and Foundations

XII.1 the Zermelo-Fraenkel Set Theory with Choice

Main references are [H-J99], [Model Theory Marker].

Set theory is just one choice of foundation of mathematics. And many propositions need efforts to be rigorous. A new choice of foundation of mathematics is homotopy type theory.

1 Basic Axioms

Axiom (XII.1.1.1) (Axiom of Existence). There exists a set which has no elements.

Axiom (XII.1.1.2) (Axiom of Extensionality). If every element of $X$ is an element of $Y$ and every element of $Y$ is an element of $X$, then $X = Y$.

Cor. (XII.1.1.3). There is at most one set which has no elements, it is called the empty set $\emptyset$ or 0.

Def. (XII.1.1.4). Write $x \in X$ if $x$ is an element of $X$.

Def. (XII.1.1.5). Write $A \subseteq B$ iff $\forall x, x \in A \Rightarrow x \in B$.

Axiom (XII.1.1.6) (Axiom Schema of Comprehension). For any set $A$, if $P(x)$ is a property of elements of $A$, there is a set $B$ that $x \in B$ iff $x \in A$ and $P(x)$, it is denoted by $B = \{x \in A | P(x)\}$.

Axiom (XII.1.1.7) (Axiom of Pair). For each $A, B$, there is a set $C$ that $x \in C$ iff $x = A$ or $x = B$.

Axiom (XII.1.1.8) (Axiom of Union). For any set $S$, there exists a set $U$ that $x \in U$ iff $x \in A$ for some $A \in S$.

Axiom (XII.1.1.9) (Axiom of Power Set). For any set $S$, there exists a set $P$ that $x \in P$ iff $x \subseteq S$.

Def. (XII.1.1.10) (Successor). The successor of a set $x$ is $S(x) = x \cup \{x\}$.

Def. (XII.1.1.11) (Inductive Set). A set $I$ is called inductive if $0 \in I$ and if $n \in I$, then $n + 1 \in I$, where $n + 1 = S(n)$ the successor.

Axiom (XII.1.1.12) (Axiom of infinity). An inductive set (XII.1.1.11) exists.
Def. (XII.1.1.13) (Natural Numbers). The set of natural numbers \( \mathbb{N} \) is defined to be \( \{ x \in I_0 | x \in I \text{ for all inductive set } I \} \), where \( I_0 \) is an inductive set. Elements of \( \mathbb{N} \) are called natural numbers.

Cor. (XII.1.1.14) (Inductive Principle). If \( P(x) \) is a property that \( P(0) \), and \( P(n) \) implies \( P(n+1) \), then \( P(n) \) for each natural number \( n \).

Proof: By definition \( B = \{ n \in \mathbb{N} | P(n) \} \) is an inductive set, so \( \mathbb{N} \subset B \).

Prop. (XII.1.1.15). \( \mathbb{N} \) is a linearly ordered set, Cf.[Set Theory Jech P43].

Cor. (XII.1.1.16) (Inductive Principle Second Version). If \( P(x) \) is a property that \( P(0) \), and \( P(k) \) holds for all \( k < n \) implies \( P(n) \), then \( P(n) \) for each natural number \( n \).

Proof: Use induction principle (XII.1.1.14) for the property \( Q(n) : P(k) \) for all \( k < n \). Then \( Q(n) \) implies \( Q(n+1) \).

2 Cardinal

Lemma (XII.1.2.1). If \( A_1 \subset B \subset A \) with \( |A| = |A_1| \), then \( |A| = |B| \).

Proof: Let \( f \) be a bijection from \( A \) to \( A_1 \). Define inductively \( A_{n+1} = f(A_n), B_{n+1} = f(B_n) \). Then \( A_{n+1} \subset B_n \subset A_n \). Let \( C_n = A_n - B_n, C = \cup C_n \), then \( f(C_n) = C_{n+1} \), so \( f(C) = \cup_{i>0} C_i \).

Now define \( g: A \rightarrow B = f(x) \) on \( C \) and \( x \) on \( A - C \), then it is a bijection from \( A \) to \( B \).

Prop. (XII.1.2.2) (Cantor-Schröder-Bernstein Theorem). If there is an injection from \( A \) to \( B \) and an injection from \( B \) to \( A \), then there is a bijection from \( A \) to \( B \). Thus the ordering of the cardinal is well-defined.

So we can define \( |A| \leq |B| \) iff there is an injection from \( A \) to \( B \). Then if \( |A| \leq |B| \) and \( |B| \leq |A| \), then \( |A| = |B| \).

Proof: It \( f: A \rightarrow B, g: B \rightarrow A \) be injection, then use the above lemma (XII.1.2.1) for \( g \circ f(A) \subset g(B) \subset A \).

Def. (XII.1.2.3) (Cardinal Number). A cardinal number is an equivalence class of sets, where equivalence is given by bijections. It is used to describe the ‘size’ of a set.

It is by the axiom of choice that any two cardinal number can be compared.

The cardinal number of \( \mathbb{N} \) is denoted by \( \aleph_0 \).

Countable and Uncountable

Def. (XII.1.2.4). A set is called countable iff it has the cardinality of \( \aleph_0 \). It is called finite iff it has the cardinality of \( n \) for some natural number \( n \). It is called uncountable iff it is not countable or finite.

Prop. (XII.1.2.5). The subset or image of an almost countable set is at most countable.

Prop. (XII.1.2.6). The product of two countable set is countable. (Use diagonal enumerating).

Prop. (XII.1.2.7). A countable union of almost countable subsets is almost countable.
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Proof: It suffices to prove the countable case, the rest follows from (XII.1.2.5). For this, choose an enumerating \( a_n(k) \) for each \( A_n \), the \( \bigcup A_n \) is the image of \( \mathbb{N} \times \mathbb{N} : (n, k) \mapsto a_n(k) \). Then it is countable by (XII.1.2.6).

Prop. (XII.1.2.8). The set of finite sequences and hence the set of finite subsets of a countable set is countable.

Proof: The desired set equals \( \bigcup_k A^k \), which is countable by (XII.1.2.6) and (XII.1.2.7).

Cardinal Arithmetics

Def. (XII.1.2.9). The sum, multiplication and exponentiation of two ordinal is the cardinality of the set \( A \sqcup B \), \( A \times B \) or \( A^B \) respectively.

It is easily verified to be associative and commutative, just as usual operations.

Prop. (XII.1.2.10). \( \aleph_0 \times \aleph_0 = \aleph_0 \) by (XII.1.2.6). And \( \kappa \times \kappa = \kappa \) for any infinite cardinal, if one uses the axiom of choice by (XII.1.2.6).

Prop. (XII.1.2.11). The image of a set \( X \) has cardinals no more than \( X \), if axiom of choice holds.

Proof: Use axiom of choice to choose an element from each inverse image \( f^{-1}(\{x\}) \), then it is an injection from \( f(X) \) to \( X \).

Prop. (XII.1.2.12) (Cantor). \( |P(X)| = |2^{\aleph_0}| \), and \( |X| < |P(X)| \).

Proof: The first is obvious, for the second, the function \( x \mapsto \{x\} \) is an injection of \( X \) into \( P(X) \). And there are no mapping from \( X \) onto \( P(X) \), because if \( f \) is one, the consider \( S = \{x| x \notin f(x)\} \), then \( S \) is not in the range of \( f \), because if \( f(z) = S \), then \( z \in S \) iff \( z \notin S \), contradiction.

Prop. (XII.1.2.13) (Cardinality Arithmetic of \( \aleph_0 \)). For cardinality arithmetics involving \( \aleph_0 \), Cf.[Set Theory Jech P98].

Prop. (XII.1.2.14). if \( |B| = \aleph_0 \) and \( |A| \leq \aleph_0 \), then \( |B - A| = \aleph_0 \). In fact, \( |B - A| = |B| \) for any \( |A| < |B| \), if one uses the axiom of choice.

Proof: By (XII.1.2.13), we can assume \( B = \mathbb{R} \times \mathbb{R} \), then project \( A \) onto the coordinate axis, then \( \pi(A) \) has cardinality\( \leq \aleph_0 \), so there is a \( x_0 \notin \pi(A) \), so \( x_0 \times \mathbb{R} \subset B - A \), so \( |B - A| = \aleph_0 \).

For the general case, \( \aleph_0 \).

Conjecture (XII.1.2.15) (The Continuum Hypothesis). There is no cardinal \( \kappa \) that \( \aleph_0 < \kappa < 2^{\aleph_0} \).

Notice \( 2^{\aleph_0} \geq \aleph_1 \) by Cantor’s theorem(XII.1.2.12), and this hypothesis is equivalent to \( 2^{\aleph_0} = \aleph_1 \).


3 Orderings

Def. (XII.1.3.1) (Ordering). A partial ordering on a set \( A \) is a relation \( < \) on \( A \), or equivalently a subset \( C \subset A \times A \) that
- For no \( x \in A \), \( x < x \) holds.
- If \( x < y \) and \( y < z \), then \( x < z \).
It is called a total ordering if moreover it satisfies
- For every \( x, y \in A \) that \( x \neq y \), either \( x < y \) or \( y < x \).

A poset is just a partially ordered set.

Def. (XII.1.3.2) (Reverse Ordering). The reverse ordering \( A^{op} \) of an ordered set \( A \) is the same set \( A \) with the ordering reversed.

Def. (XII.1.3.3) (Cofinality). The cofinality of or a poset (i.e partially ordered set) \( \alpha \) is the is the smallest cardinality \( \delta \) of a cofinal subset of \( \alpha \).

Def. (XII.1.3.4) (\( \kappa \)-Filtered Poset). For a cardinal \( \kappa \), a poset is called \( \kappa \)-filtered if for any subset unbounded from above has cardinality \( \geq \kappa \).

Def. (XII.1.3.5) (Directed Set). A directed set is a poset that is \( 3 \)-filtered and non-empty.

Def. (XII.1.3.6). In a poset \( P \), two element \( p, q \) are called compatible if there is an \( r \in P \) that \( r < p, r < q \).

Def. (XII.1.3.7) (\( \kappa \)-Chain Condition). For a cardinal \( \kappa \), a poset \( P \) is said to satisfy the \( \kappa \)-chain condition if for any subset \( A \subset P \) that elements of \( A \) are pairwise incompatible, then \( |A| \leq \kappa \).

Total Ordering

Def. (XII.1.3.8) (Well-Ordering). A linear ordering is just a total ordering.

A linear ordering is called a well-ordering if every nonempty subset has a minimal element.

Def. (XII.1.3.9) (Lexicographical Ordering). If given a family of linearly ordered set \( A_i \) indexed by a well-ordered set \( I \), then there is a linear ordering on \( \prod_I A_i \), where \( (f_i) < (g_i) \) and for the minimal \( i_0 \) (well-ordering used) that \( f_{i_0} \neq g_{i_0} \), \( f_{i_0} < g_{i_0} \). It is called the lexicographical ordering.

Def. (XII.1.3.10). An ordered set \( X \) is called dense iff for each \( a < b \), there is a \( x \) that \( a < x < b \).

Def. (XII.1.3.11) (Least Upper Bound). An ordered set \( A \) is said to have the least upper bound property if any subset \( A_0 \subset A \) bounded above has a least upper bound. It is said to satisfy the greatest lower bound property if \( A^{op} \) satisfies the least upper bound property.

Prop. (XII.1.3.12) (Cantor). Any two ordered set that is countable, dense and has no endpoints are isomorphic. In particular, any of these is isomorphic to the set of rational numbers \( \mathbb{Q} \).

Proof: We will built the isomorphism by extending partial embeddings. Let \( a_0, \ldots, a_n, \ldots \) be an ordering of \( A \), \( b_0, \ldots, b_n, \ldots \) be an ordering of \( B \), and we can alternatively extend mapping on \( a_n \) and \( b_n \), as \( A, B \) are complete without endpoints. Then we get an isomorphism of \( A \) and \( B \).

Cor. (XII.1.3.13). Any countable linearly ordered set can be mapped isomorphically into \( \mathbb{Q} \).

Def. (XII.1.3.14). A initial segment of an ordered set \( W \) is the ordered set \( W[a] = \{ x \in W | x < a \} \).

Lemma (XII.1.3.15). If \( W \) is a well-ordered set, then any increasing function \( f : W \to W \) satisfies \( f(x) \geq x \).
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Proof: If the set \( \{ x \mid f(x) < x \} \) is not empty, then it has a minimal element \( a \), then \( f(f(a)) < f(a) \), contradiction. \( \square \)

Cor. (XII.1.3.16). A well-ordered set cannot be isomorphic to an initial segment of itself, and an automorphism of a well-ordered set must be identity.

Proof: Use the above lemma (XII.1.3.15), if it is isomorphic to \( W[a] \), then \( f(a) < a \), contradiction. For any automorphism, \( f(x) \geq x \), \( f^{-1}(x) \geq x \), so \( f(x) = x \). \( \square \)

Prop. (XII.1.3.17) (Comparison of Well-Orderings). A cut of a well-ordered set is well-ordered. And for any two well-ordered sets \( W_1, W_2 \), either they are isomorphic, or one of them is isomorphic to a initial segment of another.

Proof: The three cases are mutually exclusive by (XII.1.3.16), So it suffices to show one of them holds.

Define a set \( f = \{(x, y) \in W \times W \mid W_1[x] \cong W_2[y] \} \). (XII.1.3.16) shows \( f \) is injective and monotone in both coordinates. Now we want to prove that if the domain of \( f \) is not all \( W_1 \), then it is an initial segment, and the image is all \( W_2 \), this will finish the proof.

It is clearly an initial segment \( W_1[a] \) because it is well-ordered and if \( h : W_1[x] \cong W_2[y] \) and \( x' < x \), then \( h : W_1[x'] \cong W_2[h(y)] \). If the image is not all of \( W_2 \), then similarly the image of \( f \) is an initial segment of \( W_2, W_2[b] \). But this means \( W_1[a] \cong W_2[b] \), so \( a, b \) is also in the domain(image), which is a contradiction. \( \square \)

Complete Linear Ordering

Def. (XII.1.3.18) (Complete Ordering). A cut of an ordered set \( X \) consists of two disjoint nonempty subsets \( A \cup B = X \) that \( a < b \) for any \( a \in A, b \in B \).

It is called a Dedekind cut if \( A \) doesn’t have a maximal element. It is called a gap if \( A \) doesn’t have a maximal element and \( B \) doesn’t have a minimal element.

An ordered set is called complete if there are no gaps.

Prop. (XII.1.3.19). A complete ordered set has the least upper bound property and greatest lower bound property.

Proof: Consider the cut \( A = \{ x \mid x < a \text{ for some element in } T \} \), \( B = \mathbb{R} - A \), if \( T \) is bounded above, \( B \) is not empty, so this is truly a cut, and \( A \) doesn’t have a maximal element, because if \( x < a \in \mathbb{R} \), then \( x < \frac{x + a}{2} < a \). So by completeness of \( R \), \( B \) has a minimal element, that is, the supremum of \( A \) exists. Similarly for the case \( A \) bounded from below. \( \square \)

Prop. (XII.1.3.20) (Completion of Ordering). There is an obvious ordering on the set \( C \) of all Dedekind cuts of \( X \), and \( X \) embeds into \( C \) by \( b \mapsto \{ x \mid x < b \} \cup \{ x \mid x \geq b \} \).

\( C \) is complete and has no endpoints, \( P \) is dense in \( C \), which is called a completion of \( P \).

Proof: Cf. [Set Theory Jech P88]. \( \square \)

Prop. (XII.1.3.21) (Real Numbers). \( \mathbb{Q} \) has a unique completion ordering \( \mathbb{R} \), called the set of real numbers. \( \mathbb{R} \) is not countable.

Proof: \( \mathbb{R} \) is a dense linear ordering without endpoints, so by (XII.1.3.12) if it is countable then it is isomorphic to \( \mathbb{Q} \), but this is not possible because \( \mathbb{Q} \) is not complete. \( \square \)
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Prop. (XII.1.3.22). \( |P(\mathbb{N})| = |2^\mathbb{N}| = |\mathbb{R}| \), which is denoted by \( 2^{\aleph_0} \). By (XII.1.3.21), \( \aleph_0 < 2^{\aleph_0} \).

Proof: The first equality is by (XII.1.2.12). Now by the construction of \( \mathbb{R} \), it can be embedded into \( P(\mathbb{N}) \), so \( |\mathbb{R}| \leq |P(\mathbb{Q})| = |P(\mathbb{N})| \). Conversely, \( |2^\mathbb{N}| \leq |\mathbb{R}| \) by decimal representation, so they are equal by bernstein (XII.1.2.2).

4 Ordinals

Def. (XII.1.4.1) (Ordinal Numbers). A set is called transitive iff each element of \( T \) is a subset of \( T \). A set \( \alpha \) is called an ordinal number iff \( \alpha \) is transitive and well-ordered by inclusion.

Prop. (XII.1.4.2). If \( \alpha \) is an ordinal, then \( S(\alpha) = \alpha \cup \{\alpha\} \) is also an ordinal, obviously. Thus any natural number is an ordinal by definition.

An ordinal is called a successor ordinal iff \( \alpha = S(\beta) \) for some \( \beta \), and a limit ordinal otherwise.

Lemma (XII.1.4.3).
1. If \( \alpha \) is an ordinal, then \( \alpha \not\in \alpha \).
2. Every element of an ordinal is an ordinal.
3. If ordinals \( \alpha \subset \beta \), then \( \alpha \in \beta \). That is, for ordinals, \( \subset \) is the same as \( \in \).

Proof:
1. If \( \alpha \in \alpha \), then contradiction to the fact \( \in \) is a ordering (XII.1.4.1).
2. To show \( x \in \alpha \) is transitive, it suffices to show that if \( u \in v \in x \), then \( u \in x \), because then \( v \) is a subset of \( x \). But this follows from the fact \( \in \) is an ordering. And because \( x \subset \alpha \), the inclusion of \( x \) is the restriction of inclusion in \( \alpha \), so it is a well-ordering.
3. Consider \( \beta - \alpha \), it has a minimal element \( \gamma \). Notice \( \gamma \subset \alpha \), because otherwise there is an element of \( \beta - \alpha \) smaller than \( \gamma \), by definition (XII.1.4.1).

Now we show \( \gamma = \alpha \), then it will follow that \( \alpha \in \beta \). For this, if \( \delta \in \alpha \) and \( \delta \not\in \gamma \), then \( \gamma \in \delta \) or \( \gamma = \delta \). But then this implies that \( \delta \in \alpha \) because \( \alpha \) is an ordinal, contradicting the fact \( \gamma \in \beta - \alpha \).

Prop. (XII.1.4.4) (Ordinal is Well-Ordered). Define the ordering of ordinal by \( \alpha < \beta \) iff \( \alpha \in \beta \). The ordering of ordinals is a total ordering and is a well-ordering.

Proof: If \( \alpha \in \gamma \), then \( \alpha \in \gamma \) because \( \gamma \) is transitive. If \( \alpha \in \beta \in \alpha \), then \( \alpha \in \alpha \), contradicting (XII.1.4.3).

Given any two ordinals, \( \alpha \cap \beta \) is also an ordinal by definition. If \( \alpha \cap \beta = \beta \) or \( \alpha \), then \( \alpha \subset \beta \), hence \( \alpha \in \beta \) by (XII.1.4.3). If \( \alpha \cap \beta \not\subset \alpha \) and \( \alpha \cap \beta \not\subset \beta \), then \( \alpha \cap \beta \not\subset \alpha \cap \beta \), contradiction.

Well-ordering: Given a set of ordinals, take \( \alpha \in A \) and consider the set \( \alpha \cap A \). If \( \alpha \cap A = \emptyset \), then \( \alpha \) is minimal in \( A \), because otherwise some \( \beta \in \alpha \cap A \). If \( \alpha \cap A \neq \emptyset \), then it has a minimal element \( \beta \) in the inclusion because \( \alpha \) is an ordinal. Then \( \beta \) is the minimal element of \( A \).

Cor. (XII.1.4.5) (Supremum Ordinal). Any set of ordinals has a supremum ordinal, it is just \( \cup_{\alpha \in X} \alpha \).
XII.1. THE ZERMELO-FRAENKEL SET THEORY WITH CHOICE

Proof: Firstly $\cup_{\alpha \in X} \alpha$ is transitive and it is well-ordered (for each subset $A \subseteq X$, choose an $\alpha \in X$ that $\alpha \cap A \neq 0$, then the minimal element of $\alpha \cap A$ is just the minimal element of $A$.) so it is an ordinal.

Now if $\alpha \in X$, then $\alpha \subseteq \cup X$, so $\alpha \leq \cup X$ by (XII.1.4.3). And if $\alpha \in \gamma$ for some ordinal $\gamma$, then $\cup X \subseteq \gamma$. So $\cup X$ is truly the supremum. $\square$

Cor. (XII.1.4.6). For any set $X$ of ordinals, there is an ordinal $\alpha$ that is not in $X$, just choose $S(\cup X)$.

Axiom (XII.1.4.7) (Axiom Scheme of Replacement). Let $P(x, y)$ be a property that for each $x$ there exists uniquely a $y$ that $P(x, y)$, then for each set $A$ there is a set $B$ that for each $x \in A$ there is an element $y \in B$ that $P(x, y)$.

Prop. (XII.1.4.8). Every well-ordered set is isomorphic to a unique ordinal.

So we can regard an ordinal as an equivalence class of isomorphic well-ordered sets.

Proof: Cf. [Set Theory Jech P111]. $\square$

Cor. (XII.1.4.9) (Cardinal as Initial Ordinal). The axiom of choice together with (XII.1.4.4) asserts that every cardinal has a unique smallest ordinal, called the initial ordinal. So we can identify cardinal number $\alpha$ as an ordinal that is the initial ordinal $\omega_\alpha$ of $\alpha$. Anyway, cardinal number is fewer than ordinal numbers.

The first infinite cardinal number (or the first initial ordinal) is denoted by $\omega$ or $\aleph_0$.

Prop. (XII.1.4.10) (Transfinite Induction/Recursion). If a property defined for the set of ordinals satisfies:

1. $P(0)$.
2. $P(\alpha + 1)$ if $P(\alpha)$.
3. $P(\lambda)$ if $P(\beta)$ for all $\beta < \lambda$.

then $P$ is true for all ordinals.

Transfinite recursion:

Proof: $\square$

Ordinal Arithmetic

Cf. [Set Theory Jech Chap5.5].

Def. (XII.1.4.11). We use infinite recursion to define addition of ordinals as

- $\beta + 0 = \beta$
- $\beta + (\alpha + 1) = (\beta + \alpha) + 1$, where $\alpha + 1$ is the successor of $\alpha$.
- $\beta + \alpha = \sup\{\beta + \gamma | \gamma < \alpha\}$ for a limit ordinal $\alpha$.

The multiplication and exponentiation are defined similarly.

Remark (XII.1.4.12) (Cardinal and Ordinal Arithmetics). Note that the ordinal arithmetics may be smaller than the ordinal sum of the corresponding initial ordinal (XII.1.4.9), because operations of initial ordinals may not be initial, the deeper reason is that the cardinal case, we can rearrange the order to get a smaller ordinal.

Prop. (XII.1.4.13). The addition and multiplication of ordinals are of the order type of $\alpha \uplus \beta$ in adjunction order and $\alpha \times \beta$ in lexicographical order respectively, Cf. [Set Theory Jech P120,122]
Cantor Normal Form

Prop. (XII.1.4.14) (Cantor Normal Form). Any ordinal $\alpha$ can be expressed uniquely as the form $\alpha = \sum_{i<n} \omega^{\beta_i}$, where $\beta_0 \geq \beta_1 \geq \ldots \beta_{n-1}$ are ordinals.

Proof: Cf. [Jech Set Theory P124]. \hfill \Box

Prop. (XII.1.4.15) (Goodstein Sequence). The weak Goodstein sequence is a sequence that $m_2$ is any positive integer, $m_{k+1}$ is $m_k$ written in $k$-basis and replacing the base by $k+1$, and then minus 1.

The Goodstein sequence is a sequence that $m_2$ is any positive integer, $m_{k+1}$ is $m_k$ written in $k$-basis and even the exponents in $k$-basis and and replacing the base by $k+1$, and then minus 1.

Then for each Goodstein sequence and weak Goodstein sequence, it reaches 1 in a finite number of times.

Proof: Let $m_k = \sum k^{a_i}b_i$, then let the ordinal $\alpha_k = \sum \omega^{a_i}b_i$. Then it is clear that $\alpha_2 > \alpha_3 > \ldots$

But if the weak Goodstein sequence doesn’t terminate, we constructed a descending sequence of ordinals that doesn’t terminate, contradiction (choose a minimal element).

Similarly for Goodstein sequences, just replace every base $k$ by $\omega$. \hfill \Box

5 Alephs

Prop. (XII.1.5.1). For any set $A$, there is a least ordinal that is not equipotent to any subset of $A$, called the Hartogs number of $A$. This is clearly an initial ordinal.

Proof: By axiom schema of replacement, any well-ordered subsets of $A$ is equipotent to an ordinal, and also by axiom schema of replacement, there is a set $H$ that for any well-ordering of subsets of $A$ in $P(A \times A)$, this ordered sets is equipotent to a $\alpha \in H$. Then use (XII.1.4.6) to find a minimal ordinal that is not equipotent to any subset of $A$. In fact, this is just $h(A) = \{\alpha \in H | \alpha$ equipotent to some subset of $A\}$. \hfill \Box

Def. (XII.1.5.2) (Aleph). The alephs for ordinal numbers are defined recursively: $\aleph_0 = \omega$, $\aleph_{\alpha+1} = h(\aleph_\alpha)$, and $\aleph_\alpha = \sup\{\aleph_\beta | \beta < \alpha\}$ for a limit ordinal $\alpha$. By definition $\alpha_\alpha < \aleph_\beta$ when $\alpha < \beta$.

Then $\aleph_\alpha$ are all infinite initial ordinal numbers, and any infinite ordinal number is of the form $\aleph_\alpha$ for some ordinal $\alpha$. So natural numbers together with alephs are just all the cardinal numbers.

Notice: to avoid confusion, when do arithmetic of ordinal numbers, $\aleph_\alpha$ is written as $\omega_\alpha$.

Proof: Use transfinite induction on $\alpha$. The only nontrivial case is when $\alpha$ is a limit ordinal, where if $\gamma < \aleph_\alpha$ and $|\gamma| = |\aleph_\alpha|$, then there is a $\beta < \alpha$ that $\gamma \leq \aleph_\beta$ by definition, so $|\aleph_\alpha| < |\gamma| \leq |\aleph_\beta| < |\aleph_\alpha|$ as $\aleph_\alpha$ is an initial ordinal.

To prove that any infinite initial ordinal is an aleph, first notice that $\alpha < \aleph_\alpha$ by a simple transfinite induction. So we may use transfinite induction on the following assertion for $\alpha$: if $\Omega < \aleph_\alpha$, then there is a $\gamma < \alpha$ that $\Omega = \aleph_\gamma$. For this, $\alpha = 0$ is trivially true, if $\alpha = \beta + 1$, then $\Omega < h(\aleph_\alpha)$ implies that $|\Omega| < |\aleph_\alpha|$ by definition. Because $\Omega$ is initial, $\Omega = \aleph_\beta$ or $\Omega < \aleph_\beta$, so by induction hypothesis it is true. If $\alpha$ is a limit ordinal, then $\Omega < \omega_\beta$ for some $\beta < \alpha$, so also by induction hypothesis it is true. \hfill \Box
Aleph Arithmetics

Prop. (XII.1.5.3). $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$.

Proof: Cf. [Set Theory Jech P134]. □

Cor. (XII.1.5.4). $\aleph_\alpha \cdot \aleph_\beta = \aleph_\beta$ for $\alpha \leq \beta$, and $n \cdot \aleph_\beta = \aleph_\beta$.

So $\aleph_\alpha + \aleph_\beta = \aleph_\beta$ for $\alpha \leq \beta$, and $n + \aleph_\beta = \aleph_\beta$.

6 Natural Numbers and Real Numbers

Prop. (XII.1.6.1). $(\mathbb{N}, <)$ is a well-ordered set.

Prop. (XII.1.6.2). $\mathbb{Z}$ is countable.

Arithmetic of Real numbers

Prop. (XII.1.6.3). As in (XII.1.3.21), the set of real numbers is defined as an ordered set that is the completion of the ordered set of rational numbers $\mathbb{Q}$. And it can be endowed with a field structure, making it an ordered field.

Proof: Cf. [Set Theory Jech P175]. □

7 The Axiom of Choice

Def. (XII.1.7.1). Let $S$ be a system of sets, a function $g$ defined on $S$ is called a choice function iff $g(X) \in X$ for each $X \in S$.

Prop. (XII.1.7.2) (Zermelo). The following are equivalent:
1. (the axiom of choice) There exists a choice function for very system of sets.
2. (the well-ordering principle) Every set can be well-ordered.
3. (Zorn’s lemma) If every chain in a partially ordered sets has a upper bound, then the partially ordered set has a maximal element.

Proof: 2 $\rightarrow$ 1: If $A$ is well-ordered, then $P(A)$ clearly has a choice function, that is the minimal element of a set.
1 $\rightarrow$ 2: Use transfinite recursion, Cf. [Set Theory Jech P137].
2 $\rightarrow$ 3, 3 $\rightarrow$ 2: Cf. [Set Theory Jech P142]. □

Axiom (XII.1.7.3) (the Axiom of Choice). Any system of sets has a choice function.

Prop. (XII.1.7.4). Every infinite set $X$ has a countable subset, if the axiom of choice holds.

Proof: Choose a well-ordering of it (XII.1.7.2), then it is an infinite ordinal. Then the initial segment of the first ordinal $X[\omega]$ is a countable subset. □

Prop. (XII.1.7.5). For every infinite set $S$, there exists a unique aleph $\aleph_\alpha$ that $|S| = \aleph_\alpha$.

Proof: choose a well-ordering of $S$ (XII.1.7.2), then it is an infinite ordinal, and it has the same cardinality as an initial cardinal by (XII.1.4.9), thus the result. □

Cor. (XII.1.7.6). For any sets $A$ and $B$, either $|A| \leq |B|$ or $|B| \leq |A|$.

Proof: Because the ordinal is totally ordered (XII.1.4.4). □
8 Filters and Ultrafilters

Def. (XII.1.8.1) (Filter). For a poset \( P \), a filter on \( P \) is a subset \( F \) that
- If \( p < q \) and \( p \in F \), then \( q \in F \).
- If \( p, q \in F \), then there is an \( r \in F \) that \( r < p, r < q \).

Def. (XII.1.8.2) (Filter on Sets). Let \( S \) be a non-empty set, a filter on \( S \) is a filter \( F \) on \( \mathcal{P}(S) \) that \( \emptyset \notin F \).

An ideal on \( S \) is a collection \( F \) of subsets of \( S \) that:
- \( \emptyset \in F \) and \( S \notin F \).
- If \( X, Y \in F \), then \( X \cup Y \in F \).
- If \( X \in F, X \supset Y \), then \( Y \in F \).

An ideal is just the dual(complement) of a filter.

Def. (XII.1.8.3) (Finite intersection property). A family of subsets of a set is said to have the finite intersection property if any finite collection of elements of this family is non-empty.

Lemma (XII.1.8.4). Let \( G \) be a collection of subsets of \( S \) that has the finite intersection property(XII.1.8.3), then there is a smallest filter \( F \) that \( G \subset F \). It is just the collection of subsets of \( S \) that contain some finite intersection set of elements of \( G \).

Prop. (XII.1.8.5) (Ultrafilter). An ultrafilter is a filter \( F \) that for every subset \( X, X \in F \) iff \( S - X \notin F \). A prime ideal is an ideal that for every subset \( X, X \in F \) iff \( S - X \notin F \).

A ultrafilter is equivalent to a maximal filter. And it is equivalent to a \( \{0, 1\} \)-valued finitely additive measure on \( S \).

Prop. (XII.1.8.6) (Pushforward of Filters). If \( F \) is a(n) (ultra)filter on \( X \) and \( f : X \to Y \) is a function, then \( f^*(F) = \{ A \subset Y | f^{-1}(A) \in F \} \) is a(n) (ultra)filter on \( X \), called the pushforward filter of \( F \).

Prop. (XII.1.8.7). For an ultrafilter \( \mathcal{F} \) on \( X \), if \( U_i \notin \mathcal{F} \), then \( \sum_{i=1}^{n} U_i \notin \mathcal{F} \).

Prop. (XII.1.8.8). Any filter can be extended to an ultrafilter(maximal filter), if the axiom of choice is used.

Cor. (XII.1.8.9). Non-principal ultrafilter exists on any infinite set. And in fact, any non-principal ultrafilter contains all the cofinite sets.

For any non-principle ultrafilter, it cannot contains a single pt \( \{x\} \), so it contains every cofinite set.
Proof: Consider any ultrafilter containing the filter of cofinite sets of $S$, then it is non-principal. □

Def. (XII.1.8.10) ($\kappa$-Completeness). Let $\kappa$ be an uncountable cardinal, then a field $F$ on a set $S$ is called $\kappa$-complete if for every cardinal $\lambda < \kappa$, if $X_\alpha \in F$ for every $\alpha < \lambda$, then $\cap_{\alpha < \lambda} X_\alpha \in F$.

An $\aleph_1$-complete filter is also called a $\sigma$-complete filter.

Closed unbounded and Stationary Set

Silver’s Theorem

9 Combinatorial Set Theory

Def. (XII.1.9.1) (Notations). For a set $S$, let $[S]^r$ be the set of subsets of $S$ of order $r$. Let $\kappa, \lambda$ be cardinals, we write $\kappa \rightarrow (\lambda)^r_\kappa$ as a shorthand for: for any set $S$ with $|S| = \kappa$ and every partition of $[S]^r$ into $s$ classes, there exists a subset $H \subset S$ that $[H]^r$ is in the same class, and $|H| \geq \lambda$.

Prop. (XII.1.9.2) (Ramsey’s Theorem). For any positive natural number $r, s$ if we color the $r$-subsets of a set with cardinality $\aleph_0$ into $s$ groups, then there is a subset of cardinal $\aleph_0$ that all its $r$-subsets are colored the same.

Cor. (XII.1.9.3). Every infinite linearly ordered set contains a subset isomorphic to $(\mathbb{N}, <)$ or $(\mathbb{N}, >)$.

Proof: Choose a well ordering of it. Then consider this new ordering and the original ordering. Then there is an infinite set that is compatible with the original ordering, or converse. Then its initial segment of order type $\omega_0$ satisfies the requirement. □

Def. (XII.1.9.4) (Weakly Compact Cardinals). An weakly compact cardinal is an uncountable cardinal $\kappa$ that $\kappa \rightarrow (\kappa)^r_\kappa$ for any $r, s \in \mathbb{Z}_+$.

Prop. (XII.1.9.5). Weakly compact cardinals are strongly inaccessible.


Trees

Def. (XII.1.9.6). A tree is a partial ordered set $T$ that there is a minimal element $r$ and for each $x$, $\{y \in T| y < x\}$ is finite and linearly ordered.

A tree is called of finite branched for each $x$, there is a finite set $\{y_1, \ldots, y_r\}$ is $T$ that $y_i > x$ and if $z > x$, then $z \geq y_i$ for some $i$.

Def. (XII.1.9.7) (Height). For any node $x$, $\{y \in T| y < x\}$ is a well-ordered set, which is isomorphic to an ordinal by(XII.1.4.8), it is called the height of $x$. $T_\alpha$ denotes the set of all nodes of $T$ of order $\alpha$. The least $\alpha$ that $T_\alpha \neq \emptyset$ is called the height of $T$.

A branch is a maximal chain in $T$, its length is its ordinal. The length is always smaller than the height of the tree. If it equals the height of the tree, it is called cofinal.

Def. (XII.1.9.8). A subtree is a subset $T'$ of $T$ that if $x \in T'$, $y < x$, then $x \in T'$.

An antichain of a tree $T$ is a subset $A \subset T$ that any two elements in $A$ are incomparable.

Def. (XII.1.9.9). A path through $T$ is a morphism of ordering from $\omega$ to $T$. 
Lemma (XII.1.9.10) (König’s Lemma). If $T$ is an infinite finite branching tree, then there is a path through $T$.

Proof: Use recursion to choose for each $n$ an element that has infinite successors. \qed

Def. (XII.1.9.11). An Aronszajn tree is a tree of height $\kappa$ and all its level sets are at most countable, but has no branches of length $\kappa$.

Prop. (XII.1.9.12). An Aronszajn tree of height $\omega_1$ exists.

Proof: Cf. [Set Theory Jech P228]. \qed

10 The Axiom of Foundation

11 Large Cardinals

Def. (XII.1.11.1). Let $S$ be a non-empty set, then a measure on $S$ is a non-trivial probabilistic measure $\mu$ on $(S, \mathcal{P}(S))$ that $\mu(\{a\}) = 0$ for any $a \in S$.

Prop. (XII.1.11.2) (Regular Cardinal). A cardinal is called regular if it is not a sum of $\lambda$ cardinals $\kappa_i$ that $\lambda < \kappa$ and $\kappa_i < \kappa$.

Proof: Cf. [H-J99]P161. \qed

Def. (XII.1.11.3) (Strong Limit). A cardinal $\kappa$ is called a strong limit if $2^\lambda < \kappa$ for any $\lambda < \kappa$.

Def. (XII.1.11.4) (Strongly Inaccessible Cardinal). A cardinal $\kappa$ is called strongly inaccessible (SI) if it is regular and is a strongly limit.

Def. (XII.1.11.5) (Weakly Inaccessible Cardinal). A cardinal $\kappa$ is called weakly inaccessible if it is regular and is a limit cardinal.

Prop. (XII.1.11.6) (Measure and CH). If there exists a measure on $2^{\aleph_0}$, then the Continuum Hypothesis fails.

Proof: Cf. [H-J99]P242. \qed

Prop. (XII.1.11.7). If there is a measure on a set $S$, then some cardinal $\kappa \leq |S|$ is weakly inaccessible.

Proof: Cf. [H-J99]P243. \qed

Prop. (XII.1.11.8). Let $\mu$ be a $\{0, 1\}$-valued measure on $S$, then $U = \{X \subset S | \mu(X) = 1\}$ is a non-principal $\sigma$-complete ultrafilter of $S$.

Prop. (XII.1.11.9) (Stanislaw-Ulam Dichotomy). If there exists a measure on some set, then either there exists a $\{0, 1\}$-valued measure on some set, or there exists a measure on $2^{\aleph_0}$.

Proof: Cf. [H-J99]P245. \qed

Def. (XII.1.11.10) (Measurable Cardinals). A measurable cardinal is an uncountable cardinal $\kappa$ on which there exists a non-principal $\kappa$-complete ultrafilter.

Prop. (XII.1.11.11). A measurable cardinal is strongly inaccessible.

Def. (XII.1.11.12) (Small Categories). Given a regular cardinal \( \kappa \), a set \( S \) is called \( \kappa \)-small if it has cardinality smaller than \( \kappa \). A category \( \mathcal{C} \) is called \( \kappa \)-small if the set of all objects of \( \mathcal{C} \) is \( \kappa \)-small, and the set of all morphisms of \( \mathcal{C} \) is \( \kappa \)-small.

Throughout the whole book, we will fix a strongly inaccessible cardinal \( \kappa \) and call a set or category small if it is \( \kappa \)-small. And a category is called essentially small if it is equivalent to a small category.

12 Forcing Construction of Cohn

Def. (XII.1.12.1) (Dense Subset). In a poset \( P \), a subset \( D \subset P \) is called dense if for any \( p \in P \), there is a \( q \in D \) that \( q < p \). If \( \mathcal{D} \) is a collection of dense subsets of \( P \), a filter \( G \subset P \) is called \( \mathcal{D} \)-generic if \( D \cap G \neq \emptyset \) for all \( D \subset \mathcal{D} \).

Prop. (XII.1.12.2). If \( \mathcal{D} \) is a countable collection of dense subsets of \( P \), then there is a \( \mathcal{D} \)-generic filter \( G \).

Proof: Let \( \mathcal{D} = \{ D_1, \ldots, D_n, \ldots \} \). Choose \( p_0 \in P \), and consecutively choose \( p_n \leq p_{n-1} \) that \( p_n \in D_n \), and define \( G = \{ q \mid q \geq p_n \text{ for some } n \} \).

Axiom (XII.1.12.3) (Martin’s Axiom). If \( P \) is a partially ordered set satisfying the countable chain condition (XII.1.3.7), and \( D \) is a collection of dense subsets of \( P \) with \( |D| < 2^{\aleph_0} \), then there is a \( D \)-generic filter on \( P \).
XI.2 Model Theory

Basic references are [Model Theory Marker] and [Mathematical Logics Hamilton]. The exercises of [Model Theory Marker] is important.

1 Mathematical Logics

Turing Machine and Computability

Prop. (XI.2.1.1) (Turing Machine). For the ring structure of \( \mathbb{N} \), there is a \( \mathcal{L} \)-formula \( \varphi(e,x,s) \) that \( \mathbb{N} \models T(e,x,s) \) iff the Turing machine coded with \( e \) halts on input \( x \) within \( s \) steps. So the set of halting computation is definable by the formula: \( \exists s \varphi(e,x,s) \).

Proof: Cf.[Models of Peano Arithmetic Kaye]. \( \square \)

Def. (XI.2.1.2) (Recursively Enumerable Sets). A set \( S \) of the natural numbers is called recursively enumerable iff there is an algorithm that the set of input numbers that halts is exactly \( S \). Equivalently, a recursively enumerable set is a set that there is an algorithm that ‘enumerates’ the members of \( S \).

Prop. (XI.2.1.3) (Hilbert’s 10-th Problem). For any recursively enumerable set \( A \subset \mathbb{N} \), there is a polynomial \( P(X_1,\ldots,X_n,Y_1,\ldots,Y_m) \) that

\[
A = \{ x \in \mathbb{N}^n : \mathbb{N} \models \exists y_1 \exists y_2 \ldots, \exists y_m p(x,y) = 0 \}
\]


2 Structure and Theories

Basics

Def. (XI.2.2.1) (Boolean Algebra). A Boolean algebra is a set \( B \) with Boolean connectives \( \land, \lor, 0, 1 \), where:

- \( \land \) is a symmetric function \( B \times B \to B \) that \( \varphi \land \varphi = \varphi \);
- \( \lor \) is function \( B \to B \) that \( \neg \circ \neg = \text{id} \);
- 0, 1 are nullary operations that \( \neg 0 = 1 \), and \( x \land 0 = 0 \), and \( x \land \neg x = 0 \).

Sometimes \( \lor \) is also used, which is defined as \( \varphi \lor \psi = \neg (\neg \varphi \land \neg \psi) \).

Prop. (XI.2.2.2). The power set of any set is a Boolean algebra withe \( \land = \cap \) and \( \neg (A) = X - A \).

In fact, every Boolean algebra can be embedded as a subalgebra of a power set algebra of some set by(X.I.2.16).

Def. (XI.2.2.3) (Languages and Structures). A language \( \mathcal{L} \) is a set of symbols, where a symbol is indeterminants that are labeled constant symbol \( \mathcal{C} \), function symbol (of finite arity) \( \mathcal{F} \) or relation symbol (of finite arity) \( \mathcal{R} \).

For a language \( \mathcal{L} \), a \( \mathcal{L} \)-structure on a set \( M \) is an assignment for each constant symbol \( c \) an element \( c^M \in M \), for each function symbol \( f \) of arity \( n \) a function \( f^M : M^n \to M \), and for each
relation symbol $R$ of arity $m$ a subset $R^M \subset M^m$. These $c^M, f^M, R^M$ are called interpretations of $L$.

And there is a natural definition of morphisms of $L$-structures, and an injective morphism of $L$-structures is called an embedding or a structure extension.

**Def. (XII.2.2.4) (Formulae).** For a language $L$, the set of terms is the smallest set $T$ that $c \in T$ for each constant symbol $c$, $x_i \in T$ for each variable symbol, and if $t_1, \ldots, t_{n_f} \in T$ and $f \in F$, then the symbol $f(t_1, \ldots, t_{n_f}) \in F$.

An atomic formula is a symbol of the form $t_1 = t_2$ or $R(t_1, \ldots, t_n)$, where $t_i$ are terms.

The set of $L$-formulae is the smallest set $W$ that any atomic formula is in $W$, and if $\varphi, \psi \in W$, then $\lnot \varphi, \varphi \land \psi$ are in $W$, and adjunction with the qualifier symbol $\exists x_i \varphi$ is in $W$.

A free variable in a formula is a $x_i$ that is not qualified with $\forall$ or $\exists$, otherwise it is called bound.

A sentence is a formula without free variables.

A theory is a set of sentences in a language $L$.

**Remark (XII.2.2.5) (Simplifications).** $\varphi \rightarrow \psi$ is the simplification of $\lnot \varphi \lor \psi$, $\varphi \leftrightarrow \psi$ is the simplification of $(\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)$.

Even $\lor$ is the simplification of $\lnot(\lnot \varphi \land \lnot \psi)$, and $\forall x_i \varphi$ is the simplification of $\lnot(\exists x_i \lnot \varphi)$.

**Def. (XII.2.2.6) (First Order Logic).** A first order logic is a $L$-structure that only elements in $M$ are quantified with $\forall$ or $\exists$. For example, "Every bounded subsets has a least upper bound" cannot be expressed as a formula in a first order logic of $\mathbb{R}$.

**Def. (XII.2.2.7) (Truth).** Let $\varphi$ be a formula with free variables $\pi = (v_{i_1}, \ldots, v_{i_m})$, then we inductively define $M \models \varphi(\pi)$ as follows:

- If $\varphi$ is $t_1 = t_2$ where $t_1, t_2$ are terms, then $M \models \varphi(\pi)$ if $t_1^M(\pi) = t_2^M(\pi)$.
- If $\varphi$ is $R(t_1, \ldots, t_n)$, then $M \models \varphi(\pi)$ if $(t_1^M(\pi), \ldots, t_n^M(\pi)) \in R^M$.
- If $\varphi$ is $\lnot \psi$, then $M \models \varphi(\pi)$ if $M \not\models \psi(\pi)$.
- If $\varphi$ is $\psi \land \theta$, then $M \models \varphi(\pi)$ if $M \models \psi(\pi)$ and $M \models \theta(\pi)$.
- If $\varphi$ is $\exists v_i \psi(\pi, v_j)$, then $M \models \varphi(\pi)$ is there is a $b \in M$ that $M \models \psi(\pi, b)$.

and we say $M$ satisfies $\varphi(\pi)$. Notice if there is no free variables, $\varphi(\pi)$ just writes $\varphi$.

**Def. (XII.2.2.8) (Models).** If $T$ is a theory and the $L$-structure $M$ satisfies all $\varphi \in T$, then $M$ is called a model of $T$, and writes $M \models T$.

A theory $T$ is called satisfiable iff there is a model $M$ for $T$.

A set of $L$-structures $\mathcal{K}$ is called an elementary class iff there is an $L$-theory $T$ that $\mathcal{K} = \{M \mid M \models T\}$.

Given a $L$-structure on $M$, the theory of $M$ is the set of all sentences true in $M$.

Two $L$-structures $M$ and $N$ are called elementary equivalent, denoted by $M \equiv N$, if for all $L$-sentences $\varphi, M \models \varphi \iff N \models \varphi$.

**Prop. (XII.2.2.9) (Quantifier-Free Formulae).** Suppose $M$ is a substructure of $N$ and $\pi$ is a tuple in $M$. If $\varphi(\pi)$ is a quantifier-free formula, then $M \models \varphi(\pi)$ iff $N \models \varphi(\pi)$.

**Proof:** Cf. [Model Theory P11].

**Prop. (XII.2.2.10).** If $L$-structures $M \equiv N$, then $M$ is elementarily equivalent to $N$. □
Proof: This seemingly trivial proposition still needs proof, and the proof uses induction, just as that of (XII.2.2.9).

**Def. (XII.2.2.11) (Logical Consequence).** Let $T$ be an $\mathcal{L}$-theory and $\varphi$ an $\mathcal{L}$-sentence, then $\varphi$ is called a **logical consequence** of $T$, writes $T \models \varphi$, if for any $\mathcal{L}$-structure $M$ that $M \models T$, $M \models \varphi$.

**Definable Sets and Interpretability**

**Def. (XII.2.2.12).** Let $\mathcal{M}$ be an $\mathcal{L}$-structure, a subset $X$ of $\mathcal{M}^n$ is called **definable** iff there is an $\mathcal{L}$-formula $\varphi(v_1, \ldots, v_n, w_1, \ldots, w_m)$ and a tuple $\overline{b} \in \mathcal{M}^m$ that $X = \{ \pi \in \mathcal{M}^n : \mathcal{M} \models \varphi(\pi, \overline{b}) \}$. Moreover if $A \subset M$, $X$ is called **$A$-definable** iff $y_i \in A$.

**Prop. (XII.2.2.13) (Examples of Definable Sets).** The definability of some sets are often nontrivial, using many number theories. For example, Cf.[Marker P20].

**Prop. (XII.2.2.14).** There is an inductive characterization of definable sets, Cf.[Marker P22].

**Prop. (XII.2.2.15).** If $\mathcal{M}$ is an $\mathcal{L}$-structure, If $X \subset \mathcal{M}^n$ is $A$-definable, then very $\mathcal{L}$-automorphism of $\mathcal{M}$ that fixes $A$ pointwise will fixes $X$ setwise.

**Proof:** For an automorphism $\tau$ of $\mathcal{M}$, $\mathcal{M} \models \varphi(\overline{b}, \overline{a})$ iff $\mathcal{M} \models \varphi(\tau(\overline{b}), \tau(\overline{a})) = \varphi(\overline{b}, \overline{a})$. □

**Cor. (XII.2.2.16).** $\mathbb{R}$ is not definable in $\mathbb{C}$.

**Proof:** If $\mathbb{R}$ is definable, it is definable over a finite set $A \subset \mathbb{C}$, Let $r, s$ be algebraically independent over $A$ and $r \in \mathbb{R}, s \notin \mathbb{R}$. This can be done, otherwise $\mathbb{C}$ or $\mathbb{R}$ is finite transcendental over $\mathbb{Q}$, then $|\mathbb{C}| = |\mathbb{Q}|(I.2.6.3)$, which is impossible by (XII.1.3.21). Then there is an automorphism $\sigma$ of $\mathbb{C}$ fixing $A$ that $\sigma(r) = s$, so $\mathbb{R}$ is not definable by (XII.2.2.15). □

**Def. (XII.2.2.17) (Definably Interpretable).** An $\mathcal{L}_0$-structure $\mathcal{N}$ is called **definably interpreted** in an $\mathcal{L}$-structure $\mathcal{M}$ if there is a definable set $X \subset \mathcal{M}^n$ that we can interpret the symbols of $\mathcal{L}_0$ as definable subsets and functions of $X$ and the resulting $\mathcal{L}_0$-structure is isomorphic to $\mathcal{N}$.

The usual example is that the group structure of $GL_2(\mathbb{K})$ is definably interpreted in the ring structure of a field.

**Def. (XII.2.2.18) (Interpretability and Quotient Construction).** An $\mathcal{L}_0$-structure $\mathcal{N}$ is called **interpretable** in an $\mathcal{L}$-structure $\mathcal{M}$ iff there is a definable set $X \in \mathcal{M}^n$ and a definable equivalence relation $E$ on $X$, that we can interpret the symbols of $\mathcal{L}_0$ as definable subsets and functions of $X/E$ and the resulting $\mathcal{L}_0$-structure is isomorphic to $\mathcal{N}$.

The usual example is that the set structure of a projective space is interpretable in the ring structure of a field.

**Prop. (XII.2.2.19).** Any structure for a countable language can be interpreted in a graph.

**Proof:**
3 Some Theories often used

Def. (XII.2.3.1) (Languages).
- \( \mathcal{L}_r \) is defined to be the language of rings.
- \( \mathcal{L}_{or} \) is defined to be the language of ordered rings.

Def. (XII.2.3.2) (Theories).
- \( \text{ACF} \) is defined to be the theory of alg.closed fields, w.r.t \( \mathcal{L}_r \).
- \( \text{ACF}_p \) is defined to be the theory of alg.closed fields of char \( p \).
- \( \text{DAG} \) is defined to be the theory of non-trivial torsion-free divisible Abelian groups, w.r.t. \( \mathcal{L}_r \).
- \( \text{DLO} \) is the theory of dense linear orders without endpoints.
- \( \text{ODAG} \) is the theory of nontrivial divisible Abelian groups.
- \( \text{RCF} \) is the theory of real closed fields w.r.t \( \mathcal{L}_{or} \).

4 Basis Techniques

Def. (XII.2.4.1) (Definitions for Theories). A theory \( T \) is said to have the witness property iff whenever \( \varphi(v) \) is an \( \mathcal{L} \)-formula with one free variable \( v \), there is a constant symbol \( c \) that \( T \models (\exists v \varphi(v) \rightarrow \varphi(c)) \).

A theory \( T \) is called maximal iff for any \( \mathcal{L} \)-sentence, either \( \varphi \in T \) or \( \lnot \varphi \in T \).

A theory \( T \) is called complete iff for any \( \mathcal{L} \)-sentence, either \( T \models \varphi \) or \( T \models \lnot \varphi \).

Def. (XII.2.4.2) (Proof System). A proof of a \( \mathcal{L} \)-sentence \( \varphi \) from a theory \( T \), denoted by \( T \vdash \varphi \), is a is Cf.[Mathematical Logic Shoenfield]?

A theory \( T \) is called inconsistent if there is a sentence \( \varphi \) that \( T \vdash \varphi \land \lnot \varphi \), otherwise it is called consistent.

Def. (XII.2.4.3) (Recursiveness and Decidability). A language \( \mathcal{L} \) is called recursive iff there is an algorithm that decides whether a sequence of symbols is an \( \mathcal{L} \)-formula.

An \( \mathcal{L} \)-theory \( T \) is called recursive iff there is an algorithm that decides whether a given \( \mathcal{L} \)-sentence is in \( T \).

An \( \mathcal{L} \)-theory \( T \) is called decidable iff there is an algorithm that decides whether a given \( \varphi \) satisfies \( T \models \varphi \).

Prop. (XII.2.4.4). If \( \mathcal{L} \) is a recursive language and \( T \) is a recursive \( \mathcal{L} \)-structure, then \( \{ \varphi | T \vdash \varphi \} \) is recursively enumerable.

Proof: There is a computable listing \( \sigma_1, \ldots, \sigma_n, \ldots \) of all the finite sequences of \( \mathcal{L} \)-formulas, because \( \mathcal{L} \) is recursive. Then we can check at each stage iff \( \sigma_i \) is a proof of \( \varphi \). This involves checking if each formula is in \( T \) (checkable because \( T \) is recursive) or it is a simple consequences of formulae before it, and finally check the last formula is \( \varphi \). If \( \sigma_i \) is a proof of \( \varphi \), then halt, otherwise go on to check \( \sigma_{i+1} \).

Prop. (XII.2.4.5). The halting computation set is not computable.

Proof: Cf.[Mathematical Logic Shoenfield]?

Cor. (XII.2.4.6). The full theory \( Th(\mathbb{N}) \) of the ring structure of \( \mathbb{N} \) is undecidable.
Proof: If such an algorithm exists, then we can use it to compute whether the sentence
\[ \varphi(e, x) = \exists s \, T(1 + \ldots + 1, 1 + \ldots + 1, s) \]
is computable. Then this will contradict the fact that halting computation set is not computable (XII.2.4.5).

\[ \square \]

Prop. (XII.2.4.7) (Gödel’s Completeness Theorem). Let \( T \) be an \( L \)-theory and \( \varphi \) is an \( L \)-sentence, then \( T \models \varphi \) iff \( T \vdash \varphi \).

\[ \text{Proof:} \]

Cor. (XII.2.4.8) (Consistent and Satisfiable). A theory \( T \) is consistent iff it is satisfiable.

\[ \text{Proof:} \]

Cor. (XII.2.4.9) (Lemma on Constants). Suppose \( T \vdash \varphi(\bar{\pi}) \) and \( \bar{c} \) is a tuple of constants not appearing in \( T \), then \( T \vdash \forall x \varphi(x) \).

\[ \text{Proof:} \]

Ultraproducts of Theories

Def. (XII.2.4.10) (Ultraproducts of Theories). If \( M_i, i \in I \) is a collection of \( L \)-structures and \( \mathcal{F} \) is an ultrafilter on \( I \), then the ultraproduct \( \prod_I M_i/\mathcal{F} \) of \( M_i \) is a \( L \)-structure defined as:

- The underlying set \( M = \prod M_i/\sim \), where \( (a_i) \sim (b_i) \) iff \( \{ i \in I | a_i = b_i \} \in \mathcal{F} \), which is an equivalent relation.

- If \( c \) is a constant symbol, then \( c^M = (c_i)^{M_i} \).

- If \( f \) is a function symbol, then \( f^M([a_{i1}], \ldots, [a_{in}]) = [f^{M_i}(a_{i1}, \ldots, a_{in})] \).

- If \( R \) is a relation symbol, then \( ([a_{i1}], \ldots, [a_{in}]) \in R^M \) iff \( \{ i \in I | (a_{i1}, \ldots, a_{in}) \in R^{M_i} \} \in \mathcal{F} \).

Prop. (XII.2.4.11) (Los Theorem). Let \( M_i, i \in I \) be \( L \)-structures and \( \mathcal{F} \) be an ultrafilter on \( I \). Let \( \varphi(\bar{x}) \) be a first-order logic formula in the free variables \( \bar{x} \), and let \( ([a_i]) \) be a tuple of elements from \( \prod_I M_i/\mathcal{F} \), then

\[ \prod_I M_i/\mathcal{F} \models \varphi([a_i]) \iff \{ i \in I | M_i \models \varphi_i(\bar{a_i}) \} \in \mathcal{F}. \]

\[ \text{Proof:} \]

Cor. (XII.2.4.12). An ultraproduct of models for a theory \( T \) is also a model for \( T \).

Cor. (XII.2.4.13) (Non-Standard Model for \( \text{Th}(\mathbb{R}) \)). Consider \( \mathbb{R} \) in the language \( L_r \), where \( L_r \) is the language of rings, (i.e., the ring structure), let \( \mathcal{F} \) be a non-principle ultrafilter on \( \mathbb{N} \) (XII.1.8.9), and consider the ultraproduct \( \mathcal{R} = \prod_{i \in \mathbb{N}} \mathbb{R}/\mathcal{F} \), which is called an ultrapower of \( \mathbb{R} \).

Notice each factor satisfies \( \text{Th}(\mathbb{R}) \), so \( \mathcal{R} \) also satisfies \( \text{Th}(\mathbb{R}) \).
Cor. (XII.2.4.14). Using ultraproduct construction, we can find a field of characteristic 0 that has exactly one algebraic extension in each degree.

Proof: Just use the field model $\mathbb{F}_p$ for all $p$ and construct their ultraproduct w.r.t. a non-principal ultrafilter. □

Prop. (XII.2.4.15) (Keiler-Shelah Theorem). Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are elementary equivalent iff there is an index set $I$ and an ultrafilter $\mathcal{F}$ on $I$ that $\prod_I \mathcal{M} \cong \prod_I \mathcal{N} / \mathcal{F}$.

Proof: Cf.[C. C. Chang and H. J. Keisler, Model Theory 6.1.15]. □

Compactness Theorem and Henkin Construction

Lemma (XII.2.4.16). Suppose $T$ is a maximal and finitely satisfiable $\mathcal{L}$-theory with the witness property, then $T$ is satisfiable. In fact, $T$ has $\kappa$ constant symbols, then there is a model $\mathcal{M} \models T$ that $|\mathcal{M}| \leq \kappa$.

Proof: Cf.[Marker, P35]. □

Lemma (XII.2.4.17). Let $\mathcal{L}$ be a finitely satisfiable $\mathcal{L}$-theory, then there is a language $\mathcal{L} \subseteq \mathcal{L}^*$ and a $T \subseteq T^*$ a finitely satisfiable $\mathcal{L}^*$-theory that any $\mathcal{L}^*$-theory extending $T^*$ has the witness property. And we can choose $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$.

Proof: We first show there is a language $\mathcal{L} \subseteq \mathcal{L}_1$ and a finitely satisfiable $\mathcal{L}_1$-theory $T \subseteq T_1$ that witnesses all $\mathcal{L}$-formulae: We add a constant symbol $c_\varphi$ for each $\mathcal{L}$-formula $\varphi$, and add sentences $(\exists v \varphi(v)) \rightarrow \varphi(c_\varphi)$ to $T$, then this $T_1$ is finitely satisfiable, because for any finite subset $\Delta$ of $T_1$, only f.m. constant symbols appear, and we have a model $\mathcal{M}$ for $\Delta \cap T_1$, thus we can interpret $c_\varphi$ as the element $a$ that $\mathcal{M} \models \varphi(a)$, if $\mathcal{M} \models \exists v \varphi(v)$. Now we can inductively define $L_n, T_n$, and consider their union, then this satisfies the desired conditions. And the cardinality is also clear. □

Lemma (XII.2.4.18). If $T$ is a finitely satisfiable $\mathcal{L}$-theory, then there is a maximal and finitely satisfiable $\mathcal{L}$-theory $\mathcal{L} \subseteq \mathcal{L}'$.

Proof: it is easy to construct a maximal satisfiable $\mathcal{L}$-theory, and it is truly maximal in sense of (XII.2.4.1): for any sentence $\varphi$, if $T \cup \varphi$ is not finitely satisfiable, then there is a finite set $\Delta \subseteq T$ that $\Delta \models \neg \varphi$. Then we claim $T \cup \neg \varphi$ is finitely satisfiable: for another finite set $\Sigma$, because $\Delta \cup \Sigma$ is finitely satisfiable and $\Delta \models \neg \varphi$, $\Sigma \cup \neg \varphi$ is satisfiable. □

Prop. (XII.2.4.19) (Strong Compactness Theorem). If $T$ is a finitely satisfiable $\mathcal{L}$-theory and $\kappa$ is an infinite cardinal that $\kappa > |\mathcal{L}|$, then there is a model of $T$ of cardinality at most $\kappa$.

Proof: By (XII.2.4.17), we get a desired language $\mathcal{L}^*$ of cardinality $\leq \kappa$, and a theory $T^*$, and by (XII.2.4.18), we can assume $T^*$ is maximal and finitely satisfiable, and it has the witness property. Then (XII.2.4.16) says there is a model of cardinality $\leq \kappa$. □

Cor. (XII.2.4.20) (Compactness Theorem). A theory $T$ is satisfiable iff every finite subset of $T$ is satisfiable.

Remark (XII.2.4.21). Notice the compactness theorem is also a consequence of the completeness theorem (XII.2.4.7): if $T$ is not satisfiable, then it is not consistent by (XII.2.4.8). Let $\sigma$ be a proof of a contradiction in $T$, then $\sigma$ consists of f.m. sentences in $T$, which consists of a finite unsatisfiable subset $T_0$ of $T$. 

It is clear provable using ultrafilters: If there is a family of structures \( \{ M_\Delta \} \) indexed by the collection of all finite subsets of \( T \), with \( M_\Delta \models \Delta \) for all \( \Delta \in I \) where \( I \) is the set of all finite subsets of \( T \).

Then we want to find a family of structures \( \{ M_\Delta \} \) indexed by the collection of all finite subsets of \( T \), with \( M_\Delta \models \Delta \) for all \( \Delta \in I \) where \( I \) is the set of all finite subsets of \( T \).

Then we want to find an ultrafilter \( F \) on \( I \) that for all \( \varphi \in T \), \( \{ \Delta \mid M_\Delta \models \varphi \} \in F \), then we can use Leo's theorem (XII.2.4.11) to show that \( \prod_I M_\Delta / F \) is a model for all \( \varphi \in T \). Now in fact we make pick a ultrafilter over the filter generated by all the \( A_\varphi = \{ \Delta \mid \varphi \in \Delta \} \), because \( M_\Delta \models \Delta \). In fact, this is the case because \( A_\varphi \) has the finite intersection property trivially.

Cor. (XII.2.4.22). If \( T \models \varphi \), then \( \Delta \models \varphi \) for some finite \( \Delta \subset T \).

Proof: If not, then \( \Delta \cup \{ \neg \varphi \} \) is satisfiable for all finite \( \Delta \subset T \), so \( T \cup \{ \neg \varphi \} \) is finitely satisfiable, thus satisfiable by compactness theorem, but this cannot be true because \( T \models \varphi \).

Cor. (XII.2.4.23) (Torsion Elements). Let \( L \) be a language containing \( \{ , e \} \), the language of groups, and \( T \) is a theory extending the theory of groups, let \( \varphi(v) \) be an \( L \)-formula. If for any \( n \) there is a \( G_n \models T \) and \( G_n \) has an element of finite order greater than \( n \), then there is an \( L \)-structure \( G \) that \( G \models T \) and \( G \) has an element of infinite order.

In particular, there is no formula that defines the torsion elements in any models for \( T \).

Proof: Consider a new language \( L^* = L \cup \{ c \} \), and \( T^* \) an \( L^* \)-theory that

\[
T^* = T \cup \{ \varphi(c) \} \cup \{ \neg (e \cdots c \cdots c = e) \},
\]

then the theory \( T^* \) is finitely satisfiable by hypothesis, so \( T^* \) is satisfiable by compactness theorem.

Cor. (XII.2.4.24) (Larger than All Natural Number). Consider \( L \) the language of ordered rings, let \( Th(\mathbb{N}) \) be the theory of \( \mathbb{N} \), then there is an \( L \)-structure \( M \) that \( M \models Th(\mathbb{N}) \) and \( M \) has an element that is larger than every natural number.

Proof: The same proof as that of(XII.2.4.23), but use

\[
T^* = Th(\mathbb{N}) \cup \{ 1 + \cdots + 1 < c \}_{n \text{-times}}
\]

Prop. (XII.2.4.25) (Model of a Given Cardinality). If \( T \) is an \( L \)-theory with infinite models, then if \( \kappa \) is an infinite cardinal that \( \kappa \geq |L| \), then there is a model of \( T \) of cardinality \( \kappa \).

Proof: Let \( L^* = L \cup \{ c_\alpha : \alpha < \kappa \} \), where \( c_\alpha \) are pairwise different new constants, and \( T^* \) be the \( L^* \) theory \( T \cup \{ c_\alpha \neq c_\beta, \alpha < \beta < \kappa \} \), then any model for \( T^* \) must has cardinality\( \geq \kappa \). But then we use strong compactness theorem(XII.2.4.19), it suffices to show \( T^* \) is finitely satisfiable: for any finite \( \Delta \subset T^* \), there are only f.m. new constant symbols, thus we can use the infinite model \( M \) to interpret the constant symbols randomly, thus we are done.
Complete Theories

Def. (XII.2.4.26). Let \( \kappa \) be an infinite cardinal and \( T \) is a theory with models of size \( \kappa \). \( T \) is called \( \kappa \)-categorical iff any two models of \( T \) of cardinality \( \kappa \) are isomorphic.

Prop. (XII.2.4.27). The theory of torsion-free divisible Abelian groups is \( \kappa \)-categorical for all \( \kappa > \aleph_0 \).

Proof: A torsion-free Abelian group is just a vector space over \( \mathbb{Q} \). Thus the conclusion is trivial, just notice for a cardinal \( \kappa > \aleph_0 \), a vector space of dimension \( \kappa \) has cardinality \( \kappa \). \( \square \)

Prop. (XII.2.4.28) (Vaught’s Test). Let \( T \) be a satisfiable theory with no finite models, if \( L \) is \( \kappa \)-categorical for some \( \kappa \geq |L| \), then \( T \) is complete.

Proof: If there is a sentence \( \varphi \) that \( T \not\models \varphi \) and \( T \not\models \neg \varphi \), so \( T_0 = T \cup \{ \varphi \} \) and \( T_1 = T \cup \{ \neg \varphi \} \) is satisfiable. They both have infinite models by hypothesis, so by (XII.2.4.25) there are models of cardinal \( \kappa \) for \( T_0 \) and \( T_1 \), but they cannot be isomorphic, contradiction. So \( T \) is complete. \( \square \)

Prop. (XII.2.4.29) (Recurve Complete Satisfiable is Decidable). If \( T \) is a recursive complete satisfiable theory in a recursive language \( L \), then \( T \) is decidable.

Proof: Because \( T \) is satisfiable, The set of all \( \varphi \) that \( M \models \varphi \) and the set of all \( \varphi \) that \( M \models \neg \varphi \) are disjoint, and their sum is the set of all sentences by completeness. By Gödel’s completeness theorem, this is equivalent to \( M \models \varphi \) or \( M \models \neg \varphi \). Then by (XII.2.4.4), they are both enumerable, so it is decidable by definition. \( \square \)

Up and Down (of Cardinality)

Def. (XII.2.4.30) (Elementary Embedding). An \( L \)-structure embedding \( j : M \to N \) is called an elementary embedding, denoted by \( M \prec N \), if for any \( L \)-formula \( \varphi \),

\[
M \models \varphi(\overline{a}) \iff N \models \varphi(j(\overline{a})).
\]

Notice that one way implication is sufficient, because we can use negation. Isomorphisms are elementary embeddings by (XII.2.2.10).

Def. (XII.2.4.31) (Diagrams). Let \( M \) be an \( L \)-structure, then we can add to \( L \) constant symbols for each element of \( M \), and the class \( \text{Diag}(M) \) of atomic diagrams of \( M \) is sentences of the form \( \varphi(m_1, \ldots, m_n) \), where \( \varphi \) is an atomic \( L \)-formula or the negation of an atomic \( L \)-formula, and \( M \models \varphi(m_1, \ldots, m_n) \). And the class \( \text{Diag}_e(M) \) of elementary diagrams of \( M \) is sentences of the form \( \varphi(m_1, \ldots, m_n) \) where \( \varphi \) is an \( L \)-formula that \( M \models \varphi(m_1, \ldots, m_n) \).

Prop. (XII.2.4.32) (Diagrams and Embedding). The diagram and elementary diagram is very important in constructing embeddings of theories: Let \( N \) be an \( L_M \)-structure, then:

- If \( N \models \text{Diag}(M) \), there is an \( L \)-embedding \( M \subset N \).
- If \( N \models \text{Diag}_e(M) \), there is an elementary \( L \)-embedding \( M \prec N \).

Prop. (XII.2.4.33) (Upward Löwenheim-Skolem Theorem). Let \( M \) be an infinite \( L \)-structure and \( \kappa \) be an infinite cardinal that \( \kappa \geq |M| + |L| \), then there is an \( L \)-structure \( N \) of cardinality \( \kappa \) that \( M \) embeds into \( N \).
Proof: \( \text{Diag}_d(M) \) is clearly satisfiable, so by (XII.2.4.25), there is an \( \mathcal{L}^* \)-model \( \mathcal{N} \) of cardinality \( \kappa \) that \( \mathcal{N} \models \text{Diag}_d(M) \). Then clearly \( \mathcal{M} \prec \mathcal{N} \).

Prop. (XII.2.4.34) (Tarski-Vaught Test). A substructure \( \mathcal{M} \) of a structure \( \mathcal{N} \) is an elementary substructure iff for any formula \( \varphi(v, \overline{w}) \) and \( \overline{a} \in M \), if there is \( b \in N \) that \( \mathcal{N} \models \varphi(b, \overline{w}) \), then there is a \( c \in M \) that \( \mathcal{N} \models \varphi(c, \overline{w}) \).

Proof: Cf. [Marker P45].

Prop. (XII.2.4.35) ((Downward) Löwenheim-Skolem Theorem). Suppose \( \mathcal{M} \) is an \( \mathcal{L} \)-structure and \( X \subset \mathcal{M} \), then there is a elementary submodel \( \mathcal{N} \) that \( X \subset \mathcal{N} \) and \( |\mathcal{N}| \leq |X| + |\mathcal{L}| + \aleph_0 \).

Proof: Cf. [Marker P46].

Def. (XII.2.4.36) (Universal Sentences). A universal sentence is a sentence of the form \( \forall v \varphi(v) \), where \( \varphi \) is quantifier-free.

For a theory \( \mathcal{T} \), denote by \( \mathcal{T}_\forall \) the set of all of the universal sentences \( \varphi \) that \( \mathcal{T} \models \varphi \).

A \( \mathcal{L} \)-theory \( \mathcal{T} \) is said to have a universal axiomatization iff there is a set of universal \( \mathcal{L} \)-sentences \( \Gamma \) that \( \mathcal{M} \models \mathcal{T} \) iff \( \mathcal{M} \models \Gamma \).

Prop. (XII.2.4.37) (Universal Axiomation). An \( \mathcal{L} \)-theory \( \mathcal{T} \) has a universal axiomatization iff for any \( \mathcal{N} \subset \mathcal{M} \), if \( \mathcal{M} \models \mathcal{T} \), then \( \mathcal{N} \models \mathcal{T} \).

Proof: Cf. [Marker P47].

Prop. (XII.2.4.38) (Universal Consequences). If \( \mathcal{T} \) is an \( \mathcal{L} \)-theory, then \( \mathcal{A} \models \mathcal{T}_\forall \) iff there is an \( \mathcal{M} \models \mathcal{T} \) that \( \mathcal{A} \subset \mathcal{M} \).

Proof: If \( \mathcal{A} \subset \mathcal{M} \), then \( \mathcal{A} \models \mathcal{T}_\forall \), by (XII.2.4.37). Conversely, if \( \mathcal{A} \models \mathcal{T}_\forall \), then consider \( \text{Diag}(\mathcal{A}) \), then it suffices to show \( \text{Diag}(\mathcal{A}) \) is satisfiable. If it is not, then by compactness theorem (XII.2.4.20) there is a finite set that is not satisfiable, and the part coming from diagrams of \( M \) can be expressed by a single formula \( \chi(\overline{a}) \), where \( \chi \) is quantifier-free, \( \overline{a} \in M \), \( \mathcal{M} \models \chi(\overline{a}) \), and we are saying that \( \mathcal{T} \cup \{ \chi(\overline{a}) \} \) is not satisfiable, then \( \mathcal{T} \models \neg \chi(\overline{a}) \).

Now \( \mathcal{T} \models \forall \overline{a} \neg \varphi(\overline{a}) \) as \( \mathcal{L} \)-theory because \( \overline{a} \) can be designated arbitrarily, but this sentence is in \( \mathcal{T}_\forall \), thus \( \mathcal{M} \models \forall \overline{a} \neg \varphi(\overline{a}) \), in particular, \( \mathcal{M} \models \neg \varphi(\overline{a}) \), contradiction.

Prop. (XII.2.4.39) (Elementary Chain). If \( \mathcal{M}_i \) is a chain of \( \mathcal{L} \)-structures that \( \mathcal{M}_i \prec \mathcal{M}_j \) for any \( i < j \), then we can define there union \( \mathcal{M} = \bigcup \mathcal{M}_i \). Then \( \mathcal{M} \) is an elementary extension of each \( \mathcal{M}_i \).

Proof: Use induction on formulas to show that

\[
\mathcal{M}_i \models \varphi(\overline{a}_i) \iff \mathcal{M} \models \varphi(\overline{a}_i),
\]

for all \( \mathcal{L} \)-formulas \( \varphi \). This is true for quantifier-free formula by (XII.2.2.9), and if this is true for \( \varphi, \psi \), then this is true for \( \neg \varphi \) and \( \varphi \land \psi \). For the sentence \( \varphi = \exists v \psi(\overline{w}, \overline{v}) \), if \( \mathcal{M}_i \models \psi(b, \overline{w}) \) for some \( b \), then so does \( \mathcal{M} \). Conversely, if \( \mathcal{M} \models \psi(b, \overline{w}) \) for some \( b \), then \( b \in M_j \) for some \( j \), then \( \mathcal{M}_j \models \varphi \), by the condition, \( \mathcal{M}_i \models \varphi \) also.
Back and Forth Argument

Cf.[Marker Chap. 2.4]. Some deep ideas are involved.

**Prop. (XII.2.4.40) (Cantor).** The theory $DLO$ is $\aleph_0$-categorical and complete.

*Proof:* It is $\aleph_0$-categorical by (XII.1.3.12), and it is complete, by Vaught’s test (XII.2.4.28), as it has no finite models.

5 Quantifier Elimination

**Def. (XII.2.5.1) (Quantifier Elimination).** A theory $T$ is said to have quantifier elimination if for each formula $\varphi(\bar{a})$, there is a quantifier free $\psi$ that $T \models \forall \bar{v}(\varphi(\bar{v}) \leftrightarrow \psi(\bar{v}))$.

**Lemma (XII.2.5.2).** Suppose $L$ contains a constant symbol $c$, $T$ is an $L$-theory, and $\varphi(\bar{a})$ is an $L$-formula, then the following are equivalent:

- There is a quantifier-free $L$-formula $\psi(\bar{v})$ that $T \models \forall \bar{v}(\varphi(\bar{v}) \leftrightarrow \psi(\bar{v}))$.
- If $M, N$ are models of $T$, and $A$ is an $L$-structure that $A \subseteq M \cap N$, then for all $\bar{a} \in A$, $N \models \varphi(\bar{a}) \iff M \models \varphi(\bar{a})$.

*Proof:* 1 $\Rightarrow$ 2 is clear, because a quantifier-free formula $\psi(\bar{a})$ is preserved under substructure by (XII.2.9). 2 $\Rightarrow$ 1, Cf.[Marker, P74].

**Lemma (XII.2.5.3).** Let $T$ be an $L$-theory that for any quantifier-free $L$-formula $\theta(\bar{v}, w)$, there is a quantifier-free formula $\psi(\bar{v})$ that $T \models \forall \bar{v}(\exists w \theta(\bar{v}, w) \leftrightarrow \psi(\bar{v}))$, then $T$ has quantifier elimination.

*Proof:* We want to show for each formula $\varphi(\bar{a})$, there is a quantifier-free formula $\psi(\bar{a})$ that $T \models \forall \bar{v}(\varphi(\bar{v}) \leftrightarrow \psi(\bar{v}))$, and we use induction on the complexity of $\varphi$.

If $\varphi$ is quantifier-free, this is trivial. If $\varphi = \theta_0$ or $\varphi = \theta_0 \land \theta_1$, then we are easily done. If $\psi(\bar{v}) = \exists w \theta(\bar{v}, w)$, and $T \models \forall \bar{v}(\theta(\bar{v}, w) \leftrightarrow \psi_0(\bar{v}, w))$, then $T \models \forall \bar{v}(\varphi(\bar{v}) \leftrightarrow \exists w \psi_0(\bar{v}, w))$. But the assumption shows there is a quantifier-free $\psi$ that $T \models \forall \bar{v}(\exists w \psi_0(\bar{v}, w) \leftrightarrow \psi(\bar{v}))$, and then $T \models \forall \bar{v}(\varphi(\bar{v}) \leftrightarrow \psi(\bar{v}))$.

**Prop. (XII.2.5.4) (Criterion for Quantifier Elimination).** Combining (XII.2.5.2) and (XII.2.5.3), we get: If $T$ is an $L$-theory that for all quantifier-free formula $\varphi(\bar{v}, w)$ and models $M, N \models T$, and $A \subseteq M \cap N$, $\bar{a} \in A$, and if there is $b \in M$ that $M \models \varphi(\bar{a}, b)$ will imply there is $c \in N$ that $N \models \varphi(\bar{a}, c)$, then $T$ is quantifier-free.

**Def. (XII.2.5.5) (Algebraically Prime Models).** A theory $T$ is said to have algebraically prime models if for any $A \models T_\forall$ (XII.2.36), there is a $M \models T$ and an embedding $i : A \subseteq M$ that for any other $N \models T$, any embedding $j : A \rightarrow N$ factors through $i$.

**Def. (XII.2.5.6) (Simply Closed Model).** If $M, N$ are models of $T$, and $M \subseteq N$, then $M$ is called simply closed in $N$, denoted by $M \preceq_s N$, if for any quantifier-free formula $\varphi(\bar{v}, w)$ and $\bar{a} \in M^n$, if $N \models \exists w \varphi(\bar{a}, w)$, then so does $M$.

**Prop. (XII.2.5.7) (Quantifier Elimination Test).** If $T$ is an $L$-theory that has algebraically prime models, and any inclusion of models for $T$ is simply closed, then $T$ has quantifier elimination.

*Proof:* This is an immediate consequence of (XII.2.5.4).
Def. (XII.2.5.8) (Model Complete). An $\mathcal{L}$-theory $T$ is called model complete if $\mathcal{M} \prec \mathcal{N}$ whenever $\mathcal{M} \subset \mathcal{N}$. A complete theory is clearly model complete.

Prop. (XII.2.5.9). If $T$ has quantifier elimination, then $T$ is model complete.

Proof: If $\mathcal{M} \subset \mathcal{N}$, let $\phi(\bar{a})$ be an $\mathcal{L}$-formula, and $\bar{a} \in \mathcal{M}$, then there is a quantifier-free formula $\psi(\bar{v})$ that $T \models \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$. By (XII.2.2.9), the formula $\psi(\bar{a})$ passes between $\mathcal{M}$ and $\mathcal{N}$, so $\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\bar{a})$.

Prop. (XII.2.5.10) (Minimal Model and Completeness). If $T$ is model-complete and there is a minimal model $\mathcal{M}_0$ that $\mathcal{M}_0$ embeds into every model of $T$, then $T$ is complete.

Proof: Clearly any model is elementary equivalent to $\mathcal{M}_0$, thus clearly $T$ is complete.

Prop. (XII.2.5.11) (Eliminating Algorithm). Let $T$ be a decidable theory with quantifier elimination, then there is an algorithm to find the elimination $\psi$ of a given formula $\phi$.

Proof: We just need to find a quantifier-free $\psi$ that $T \models \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$. This an effective search because $T$ is decidable, and we can eventually find $\psi$ because $T$ has quantifier elimination.

Examples of Quantifier Elimination

Prop. (XII.2.5.12). The theory $DLO$(XII.2.3.2) has quantifier elimination.

Proof: Cf.[Marker P72].

Lemma (XII.2.5.13). $DAG$(XII.2.3.2) has algebraically prime models.

Proof: This is just the alg.closure of the quotient field of an integral domain.

Prop. (XII.2.5.14). $DAG$ has quantifier elimination.

Proof: We use(XII.2.5.7) By?? $DAG$ has algebraically prime models, thus it suffices to show any inclusion of models for $DAG$(XII.2.3.2) is simply closed: For any torsion-free divisible Abelian groups $H \subset G$, $\varphi(\bar{v}, w)$ quantifier-free, $\bar{v} \in H, b \in G$ that $G \models \varphi(\bar{v}, w)$. Then $\varphi$ is a disjunction of conjunctions of atomic or negated atomic formuals, and we need to prove $H \models \varphi(\bar{a}, w)$.

So we may assume $\varphi$ is conjunction of atomic formulas and negated atomic formuals: $\varphi(\bar{a}, w) \leftrightarrow \bigwedge (g_i + m_i w = 0) \land \bigwedge (h_i + m_i' w \neq 0)$. If any $m_i \neq 0$, then $b \in H$. If all $m_i = 0$, then there are only f.m. constraints, and there is $b' \in H$ that $H \models \varphi(\bar{v}, b')$ because $H$ is infinite.

Cor. (XII.2.5.15). The theory $DAG$ is complete, by(XII.2.5.10), as $(\mathbb{Q}, +, 0)$ embeds in every model of $DAG$, and(XII.2.5.14)(XII.2.5.9).

Prop. (XII.2.5.16). The theory $ODAG$ is a complete theory with quantifier elimination. In particular, any ordered divisible Abelian group is elementarily equivalent to $(\mathbb{Q}, +, <)$, by completeness.

Proof: Cf.[Marker, P80].

Prop. (XII.2.5.17) (Presburger Arithmetic). Presburger arithmetic is a complete decidable theory with quantifier elimination in the language $\mathcal{L}^*$.

Proof: Cf.[Marker, P84].
**Strongly Minimal Theory**

**Def. (XII.2.5.18) (Strongly Minimal Theory).** A theory $T$ is called strongly minimal iff for any $M \models T$, every definable subset of $M$ is either finite or cofinite.

**Prop. (XII.2.5.19).** $\text{DAG}(\text{XII.2.3.2})$ is strongly minimal.

**Proof:** Cf. [Marker P78].

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**6 Algebraically Closed Fields**

**Lemma (XII.2.6.1).** $ACF_p(\text{XII.2.3.2})$ is $\kappa$-categorical for all uncountable cardinal $\kappa$.

**Proof:** Because two alg.closed field of the same transcendental degree over the base field is isomorphic. The conclusion follows as an alg.closed field of transcendence degree $\kappa$ has cardinality $\kappa + \aleph_0$.

**Prop. (XII.2.6.2).** The theory $ACF_p$ is complete, by (XII.2.6.1) and (XII.2.4.28).

**Cor. (XII.2.6.3).** $ACF_p$ is decidable, in particular, Th($\mathbb{C}$), the first-order theory of the fields of complex numbers, is decidable.

**Proof:** $ACF_p$ is complete by (XII.2.6.1) and (XII.2.4.28), it is clearly recursive, and it is clearly satisfiable, so use (XII.2.4.29).

**Prop. (XII.2.6.4) (First order Lefschetz Principle).** Let $\varphi$ be a sentence in the language of rings, the following are equivalent:

1. $\varphi$ is true in $\mathbb{C}$.
2. $\varphi$ is true in every(some) alg.closed field of char 0.
3. For $p$ large/there exists arbitrary large $p$, $\varphi$ is true in any alg.closed field of char $p$.

**Proof:** 1, 2 are equivalent because $ACF_p$ is complete (XII.2.6.2) and use Gödel’s completeness theorem. If 2 is true, then $ACT_0 \models \varphi$, so by (XII.2.4.22), then some $\Delta \models \varphi$. So for $p$ sufficiently large, $ACF_p \models \Delta$, so $ACF_p \models \varphi$ for $p$ large.

If $ACF_0 \not\models \varphi$, then $ACF_0 \models \neg\varphi$ by completeness (XII.2.6.2), so by above argument, $ACF_p \models \neg\varphi$ for $p$ large, contradiction.

**Cor. (XII.2.6.5) (Ax).** Any injective polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^n$ is surjective.

**Proof:** By Lefschetz principle (XII.2.6.4), it suffices to show this for $(\mathbb{F}_p)^{alg}$ for $p$ large. If this is not true, then choose the coefficients of the coordinate map of $f$, and the coordinates of an element that is not in the image, then the subfield $k$ they generated is algebraic over $\mathbb{F}_p$, so it is finite, and clearly $f$ is also surjective on $k^n$, contradiction.

**Cor. (XII.2.6.6).** Let $K \subset L$ be alg.closed fields, and $V, W$ be varieties defined over $K$, and $f$ is a polynomial isomorphism of $V$ and $W$ over $L$, then there is a polynomial isomorphism of $V$ and $W$ over $K$.

**Proof:** Let $f$ have degree $d$. Then we can write a formula $\Psi$ saying that there is an embedding of $K$ into $\mathcal{M}$ and there is a polynomial bijection of $V$ and $W$ over $\mathcal{M}$, then $L \models \Psi$, and because $ACF$ is complete (XII.2.6.2), $K \models \Psi$, too. Thus there is also an isomorphism over $K$. □
Prop. (XII.2.6.7). $ACF$ is the theory of integral domains.

Proof: Clearly a ring is a subring of an alg.closed field iff it is an integral domain, so the result follows from (XII.2.4.38). □

Prop. (XII.2.6.8). $ACF$ has qualifier elimination.

Proof: Use (XII.2.5.5), clearly it has algebraically prime models (XII.2.6.7), and we need to check simply closedness.

For this, if $K \subseteq L$, notice that a quantifier-free formula $\varphi$ is just a conjunction of some polynomial functions and negation of polynomial functions, with their coefficients in $K$. If there are some polynomial, then the solution $b$ of $\varphi$ in $L$ is algebraic over $K$, thus in $K$ because $K$ is alg.closed. Now if it is just negations of polynomials, then clearly $\varphi$ is true in $K$ for some $c \in K$, because $K$ is alg.closed thus infinite. □

Cor. (XII.2.6.9). $ACF$ is model-complete, by (XII.2.5.9).

Lemma (XII.2.6.10). Let $K$ be a field, then the subsets of $K^n$ defined by atomic formulas are exactly Zariski closed subsets. And a subset of $K^n$ that is quantifier-definable iff it is a Boolean combination of Zariski closed subsets (constructible). (Clear).

Prop. (XII.2.6.11). $ACF$ is strongly minimal.

Proof: Because by quantifier elimination (XII.2.6.8), every definable set is a finite Boolean combination of sets of the form $V(p) = 0$, where $p \in K[X]$, but $V(p)$ is either finite or all of $K$. □

Elimination of Imaginaries in Alg.Closed Fields

Def. (XII.2.6.12) (Algebraicity). Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq \mathcal{M}$, then for $b \in \mathcal{M}$, $b \in acl(A)$ if there is a formula $\varphi(x, \overline{y})$ and $\overline{a} \in A$ that $\mathcal{M} \models \varphi(b, \overline{a})$, and $\{x \in \mathcal{M} | \mathcal{M} \models \varphi(x, \overline{a})\}$ is finite.

Def. (XII.2.6.13) (Algebraicity over Equivalence Class). Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $E$ a definable equivalence relation on $M^n$, then we say a element $c \in M$ is algebraic over $\overline{a}/E, b_1, \ldots, b_m$ if there is a formula $\varphi(c, \overline{y}, \overline{z})$ that

- $\mathcal{M} \models \varphi(c, \overline{a}, \overline{b})$.
- If $\overline{a}E\overline{a}'$, then $\mathcal{M} \models \varphi(c, \overline{a}, \overline{b}) \leftrightarrow \varphi(c, \overline{a}', \overline{b})$.
- $\{x | \mathcal{M} \models \varphi(x, \overline{a}, \overline{b})\}$ is finite.

And a sequence $\overline{a}$ is called algebraic over $a/E, b_1, \ldots, b_m$ iff each coordinate is.

Lemma (XII.2.6.14). Suppose $\overline{a}$ is algebraic over $\overline{a}/E, \overline{d}, \overline{b}$, and $\overline{b}$ is algebraic over $\overline{a}/E, \overline{d}$, then $\overline{c}$ is also algebraic over $\overline{a}/E, \overline{d}$.

Proof: Cf. [Marker, P92]. □

Lemma (XII.2.6.15). Suppose $K$ is an alg.closed field and $E$ is a definable equivalence relation on $K^n$, and $\psi(\overline{a}, \overline{y}, \overline{d})$ defines $E$. If $\overline{a} \in K^n$, then there is a $\overline{c} \in K^n$ algebraic over $\overline{a}/E, \overline{d}$ s.t. $\overline{a}E\overline{c}$.

Proof: Cf. [Marker, P92]. □
Prop. (XII.2.6.16) (Elimination of Imaginaries). Let $K$ be an alg.closed field, $A \subset K$, and $E$ is an $A$-definable equivalence relation on $K^n$, then for some $l$ there is an $A$-definable function $f : K^n \to K^l$ that $\overline{x}E\overline{y}$ iff $f(\overline{x}) = f(\overline{y})$.

Proof: Cf.[Marker, P92].

7 Real Closed Fields

Prop. (XII.2.7.1). The class of real closed fields is an elementary class (XII.2.8) of $\mathcal{L}_r$-structures.

Proof: The real closed fields are axiomatized by:

- Axioms for fields.
- For each $n \geq 1$, the sentence $\forall x_1 \ldots \forall x_n(x_1^2 + \ldots + x_n^2 + 1 \neq 0)$.
- $\forall x \exists y(y^2 = x \lor y^2 + x = 0)$.
- For each $n \geq 0$, the sentence $\forall x_1 \ldots \forall x_n \exists y(y^{2n+1} + \sum_{i=0}^{2n} x_iy^i = 0)$.

These truly axiomatize real closed fields, by(I.2.9.7).

Prop. (XII.2.7.2). Let $F$ be a real closed field, then the definable sets in $F^n$ w.r.t the $\mathcal{L}_{or}$ is also definable w.r.t. $\mathcal{L}_r$.

Proof: Because we can replace all instances $t_i < t_j$ by $\exists v(v \neq 0 \land t_i + v^2 = t_j)$.

Prop. (XII.2.7.3). RCF$_F$ is the theory of ordered integral domains (easy).

Cor. (XII.2.7.4). RCF has algebraically prime models.

Proof: Let $D$ be an ordered domain, and let $R$ be a real closure of the fraction field of $D$ compatible with the ordering of $D$. Let $F$ be another real closed field extension of $D$, then let $K$ be the algebraic closure of the fraction field of $D$ in $F$, then $K$ is a real closed field by(I.2.9.7), then by(I.2.9.12) there is an isomorphism of $R \cong K$ extending $D \to K$, thus embeds $R$ into $F$ extending $D \to F$.

Prop. (XII.2.7.5). RCF has quantifier elimination (w.r.t. $\mathcal{L}_{or}$).

Proof: Use(XII.2.5.5), it has algebraically prime models by(XII.2.7.4), and we need to check simply closedness: Let $F \subset K$ be RCFs, $\varphi(v, \overline{w})$ be a quantifier-free formula and let $\overline{v} \in F, b \in K$ that $K \models \varphi(b, \overline{w})$. Then there are polynomials that

$$\varphi(v, \overline{w}) \leftrightarrow (\bigwedge p_i(v) = 0 \land \bigwedge q_i(v) > 0).$$

If any $p_i \neq 0$, then $b$ is algebraic over $F$, but then $b \in F$ because $F$ is real closed. So we may assume $\varphi(v, \overline{w}) \leftrightarrow \land q_i(v) > 0$. Then because $q_i(b) > 0$ and $q_i$ has f.m. zeros, by intermediate property(I.2.9.9), we can find a nbhd of $b$ that is interval with endpoints in $F$ that $q_i > 0$, and because $F$ is dense(char0), there is an element $b'$ that $F \models \varphi(b', \overline{w})$.

Cor. (XII.2.7.6). RCF is complete and decidable. Thus it is just the theory of $(\mathbb{R}, +, \cdot, <)$, and it is decidable.

Proof: It is model-complete by(XII.2.5.9), and it has a minimal model that is the field $\mathbb{R}_{alg}$ of algebraic numbers in $\mathbb{R}$, then it is complete by(XII.2.5.10). It is decidable by(XII.2.4.29).

Prop. (XII.2.7.7) (RCF is $O$-Minimal). The theory $RCF$(XII.2.3.2) is $o$-minimal.

Proof: Because RCF has quantifier elimination, every definable set of a real closed field $R$ is semi-algebraic, thus it is clearly a disjoint union of intervals and points.
Semialgebraic Sets

Def. (XII.2.7.8) (Semialgebraic Sets). Let $F$ be an ordered field, then $X \subset F^m$ is called semialgebraic if there it is a Boolean combination of sets of the form $\{ \overline{\pi} \mid p(\overline{\pi}) > 0 \}$, where $p(X) \subset F[\overline{X}]$.

Prop. (XII.2.7.9) (Tarski-Seidenberg Theorem). If $F$ is a real closed field, the semialgebraic sets are closed under projection.

Proof: This is because quantifier elimination (XII.2.7.5) implies semialgebraic sets are just definable sets, and the projection of a definable set is also definable. □

Cor. (XII.2.7.10). The composite of two semialgebraic functions are semialgebraic.

Prop. (XII.2.7.11) (Closure of Semialgebraic Sets). Let $F$ be a real closed field and $A \subset F^n$ be semialgebraic, then the closure of $A$ in the Euclidean topology is also semialgebraic.

Proof: The closure of $A$ is defined by

$$\{ \overline{x} \mid \forall \varepsilon > 0, \exists \overline{y} \in A, d(\overline{x}, \overline{y}) < \varepsilon \}.$$ 

Thus it is definable, and thus semialgebraic by quantifier elimination (XII.2.7.5). □

Prop. (XII.2.7.12). Let $F$ be a real closed field and $X \subset F^n$ be closed and bounded, $f$ be a continuous semialgebraic function, then $f(X)$ is closed and bounded.

Proof: If $F = \mathbb{R}$, then this follows from (IX.1.5.1) that $X$ is compact hence the image is also compact hence closed and bounded. As we can formulate a sentence asserting the conclusion. Thus the theorem follows as $RCF$ is complete (XII.2.7.6). □

Lemma (XII.2.7.13). For $R$ a real closed field and $f : R \to R$ semialgebraic, then for any interval $U$ there is an element $x$ that $f$ is continuous at $x$.

Proof: It suffices to prove for $R = \mathbb{R}$ by completeness of $RCF$. If there is an interval that $f$ has finite range, then the inverse image of some point $b$ is infinite, but also definable, so by $o$-minimality contains an interval that $f$ is constant, hence continuous.

Otherwise, we inductively choose a chain of intervals: $V_0 = U$, and the image of $V_n$ is definable thus contains an interval of length at most $1/n$, by $o$-minimality and our assumption. Now for the same reason the inverse image of this interval contains an interval, called $V_{n+1}$, s.t. $\overline{V_{n+1}} \subset V_n$, then $\cap V_n = \cap V_n \neq \varphi$ because $\mathbb{R}$ is locally compact. For any $x \in V$, it is easy to see $f$ is continuous at $x$. □

Prop. (XII.2.7.14). For $R$ a real closed field and $f : R \to R$ semialgebraic, then $f$ is discontinuous only at f.m. points.

Proof: The discontinuous points of $f$ is definable by

$$D = \{ x \mid F \models \exists \varepsilon > 0 \forall \delta > 0 \exists y |x - y| < \delta \land |f(x) - f(y)| > \varepsilon \},$$

thus it is a finite union of intervals and points, by $o$-minimality (XII.2.7.7), but it must be f.m. points, by (XII.2.7.13). □

Def. (XII.2.7.15). We can naturally define the notion of definably connected and definably arcwise connected sets in $R^n$. 

Def. (XII.2.7.16) (Cells). For a real closed field $F$, we inductively define $n$-cells:
- $X \subset F^n$ is a 0-cell if it is a single point.
- If $X \subset F^n$ is an $n$-cell and $f : X \to F$ is a continuous definable function, then $Y = \{(\bar{x}, f(\bar{x})) | \bar{x} \in X\}$ is an $n$-cell.
- If $X \subset F^n$ is an $n$-cell and $f, g$ are either continuous functions from $X$ to $F$ or constants $\pm \infty$, then $Y = \{(x, y) | x \in X \land f(x) < y < g(x)\}$ is an $n + 1$-cell.

Prop. (XII.2.7.17) (Uniform Bounding). Let $X \subset F^{m+1}$ be semialgebraic, then there is a natural number $N$ that if $a \in F^n$ and $X_a = \{y | (a, y) \in X\}$ is finite, then $|X_a| < N$.

Proof: First notice $X_a$ is definable, thus by $o$-minimality, $\{\bar{a} | |X_a| < \infty\}$ is definable, thus we may assume these are all of $X$.

Now let $\Gamma$ be the theory

$$RCF + \text{Diag}(F) + \{\exists y_1, y_2, \ldots, y_m[\bigwedge_{i<j} y_i \neq y_j \land \bigwedge y_i \in X_{\bar{a}}]\}$$

where $m \in \mathbb{N}$, and $c_1, \ldots, c_n$ are new constants. (Notice $y_i \in X_{\bar{a}}$ is a sentence).

Then $\Gamma$ is not satisfiable, because otherwise there is an embedding $F \subset K$, and by model completeness $F \prec K$, thus there is no $\bar{a}$ that $X_{\bar{a}}$ is infinite, because it is definable, contradiction.

Then by compactness theorem, there is some f.m. sentences of $\Gamma$ that is unsatisfiable, thus the conclusion follows. \[\square\]

Prop. (XII.2.7.18) (Cell Decomposition). Let $F$ be a real closed field, then any semialgebraic set $X \subset F^m$ is a finite disjoint union of cells.

Proof: Cf.[Marker, P103]. \[\square\]

8 O-Minimal Structures

Cf.[O-Minimal Structures].

Def. (XII.2.8.1) (O-Minimal Structure). An ordered structure is called $o$-minimal if any definable set is a finite union of intervals with endpoints in $M \cup \{\pm \infty\}$ and points.

Def. (XII.2.8.2) (General O-Minimal Expansions). A structure expanding the real closed field $R$ is a collection $S = (S^n)$, where $S^n$ is a family of subsets of $R^n$, that
- Algebraic subsets are in $S$.
- Each $S^n$ is a Boolean subalgebra of the powerset of $R^n$.
- $S$ is stable under Cartesian product and projection.

And $S^n$ are called the definable subsets of $R^n$. And this structure is called $o$-minimal if moreover each subset in $S^1$ is a finite unions of intervals and points.
Chapter XIII

Theoretical Physics

XIII.1 Quantum Mechanics

Basic References are [Napkin Evan Chen].

1 Basics

Axiom (XIII.1.1.1) (Axioms). The Schrödinger equation can be derived from the Dirac-von Neumann axioms:

- The state of particles is a countable dimensional Hilbert space, and
- The observables of a quantum system are defined to be the (possibly unbounded) Hermitian operators $A$ on $\mathbb{H}$. Then any continuous observable is unitarily diagonalizable, with real eigenvalues by Hilbert-Schimidt (X.5.4.16).
- The state $\varphi$ of the quantum system is a unit vector of $\mathbb{H}$, up to scalar multiples.
- The expectation value of an observable $A$ for a system in a state $\varphi$ is given by the inner product $\langle \varphi, A\varphi \rangle$.
- (Unitarity) The time evolution of a quantum state according to the Schrödinger equation is mathematically represented by a unitary operator $U(t)$ (depends only on the state and relative time)(one-parameter subgroup).

Now that $\varphi(t) = \hat{U}(t)\varphi(t_0)$, so $\hat{U}(t)\varphi(t_0) = e^{-i\hat{H}t}, \hat{H}$ hermitian.

So now take derivative w.r.t $t$, we get $i\frac{d\varphi}{dt} = \hat{H}\varphi$. By quantum correspondence principle, it is possible to derive the expression of $\hat{H}$ by classical methods.

Def. (XIII.1.1.2). A qubit is a state that is complex combination of 0 and 1, i.e. $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$. Notice I very dislike the 'bra-ket' notation, I prefer to think of state just as an element in the Hilbert space, and use any notation I like.

Def. (XIII.1.1.3). The observables on a two dimensional Hilbert space, i.e. a qubit state space, are all combinations of Pauli Observables or Pauli matrixes plus $I$:

$$
\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}
$$
Their corresponding eigenvalues are denoted by
\[ \uparrow = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \downarrow = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \rightarrow = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \leftarrow = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \otimes = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}, \quad \circ = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \]

**Remark (XIII.1.1.4).** The solution of a Schrodinger equation for a non Relativistic particle is assumed to be a Schwartz function (Vanish fast enough at infinity). The coefficients is assumed smooth enough to guarantee at least uniqueness and existence locally.

**Prop. (XIII.1.1.5).** The wave function on the \((p,t)\) coordinates is the Fourier Transform of the wave function on the \((x,t)\) coordinates, because the eigenstate of the \(p\)-operator \(i\hbar \frac{\partial}{\partial x}\) is \(e^{ikx}\), the coefficients of which is the value (probability) of the wave function of the \((p,t)\) coordinates.

**Prop. (XIII.1.1.6) (Schrodinger Uncertainty Principle).** Set \(\sigma_A = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}\), then:
\[ \sigma_A^2 \sigma_B^2 \geq \frac{1}{2} |\langle \hat{A} \hat{B} + \hat{B} \hat{A} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle |^2 + \left| \frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right|^2. \]

*Proof:* Derived from definition and Schwarz inequality, Cf.[Wiki]. \(\square\)

**Cor. (XIII.1.1.7) (Heisenberg Uncertainty Principle).** \(\sigma_x \sigma_p \geq \frac{\hbar}{2}.\)

*Proof:*
\[ [x, i\hbar \frac{\partial}{\partial x}] = i\hbar. \]

\(\square\)

**Prop. (XIII.1.1.8) (Spectral Decomposition).** In Quantum physics, one need to use spectral decomposition of the Hamiltonian operator. But at most cases, there are only countably many eigenstate and the eigenvalue has a lower bound and tends to infinity. In this case, \((\hat{H} + A)^{-1}\) is a compact operator thus by spectral theorem(X.5.4.15) the eigenstate of \(\hat{H}\) forms a set of complete basis.

**Prop. (XIII.1.1.9) (No-Cloning Theorem).**

**Calculations**

**Prop. (XIII.1.1.10) (Virial Theorem).** For a system that \(V(r) \sim r^n\), the average kinetic energy and the average potential energy has the relation:
\[ 2\langle T \rangle = n\langle V \rangle. \]

**Spin**

2 Quantum Computations

**Classical Logic Gates**

**Def. (XIII.1.2.1).** A Boolean function is a function \(f : \mathbb{F}_2^n \to \mathbb{F}_2.\)

**Def. (XIII.1.2.2) (Classical Logic Gates).** There are four classical logic gates, if we let \(0, 1 \in \mathbb{F}_2\), then:
• AND gate: \((a, b) \mapsto ab\).
• OR gate: \((a, b) \mapsto ab + a + b\).
• NOT gate: \(a \mapsto a + 1\).
• COPY gate: \(a \mapsto (a, a)\).

Def. (XIII.1.2.3) (Reversible Gates). A gate is called reversible if it is a bijection from \(\mathbb{F}_2^n\) to \(\mathbb{F}_2^n\).

Def. (XIII.1.2.4) (Stimulation). A set of gates is said to be able to stimulate a boolean function \(f\) iff there is a composition of these gates that maps:

\[(x_1, \ldots, x_{m+n}) \mapsto (g_1(x_1, \ldots, x_{m+n}), \ldots, g_k(x_1, \ldots, x_{m+n}))\]

that if we let \(x_{i_1} = a_1, x_{i_2} = a_2, \ldots, x_{i_m} = a_m\) be fixed, where \(\{0, 1 \ldots, m + n\} \setminus \{i_1, \ldots, i_m\} = \{j_1, \ldots, j_n\}\) (in some order), then \(g_1(x_1, \ldots, x_{m+n}) = f(x_{j_1}, \ldots, x_{j_n})\).

A set of gates is called universal iff they can stimulates all Boolean function \(f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2\).

Prop. (XIII.1.2.5) (Classical Gates Universal). The four classical gates are universal. In fact, \(\text{AND}(x, y) = \text{OR}(\text{NOT}(x), \text{NOT}(y))\), so even AND is disposable, and COPY is not used as well.

Proof: Just use OR gate to juxtapose all possible combinations that are mapped to 1. \(\square\)

Def. (XIII.1.2.6) (Examples of Reversible Gates).
- The CNOT gate is defined to be \(\text{CNOT} : (a, b) \mapsto (a, a + b)\).
- The Toffoli gate is defined to be \(\text{CCNOT} : (a, b, c) \mapsto (a, b, c + ab)\).

Prop. (XIII.1.2.7). CNOT gate cannot stimulate AND. In particular, CNOT is not universal.

Proof: It can be shown that any Boolean function that can be stimulated by CNOT gate is of the form \((x_1, \ldots, x_n) \mapsto \sum a_i x_i + b\). But AND is of the form \((a, b) \mapsto ab\), which is not of the form, so it is not stimulated by CNOT. \(\square\)

Prop. (XIII.1.2.8). Toffoli gate is universal.

Proof: It suffices to show it can stimulate AND and NOT, then it can stimulate OR because \(\text{OR}(x, y) = \text{NOT}(\text{NOT}(x), \text{NOT}(y))\).

AND is outputted in the third bit with \(c = 0, a = x, b = y\), NOT is outputted in the third bit with \(a = 1 = c, b = x\). \(\square\)

Quantum Logic Gates

Def. (XIII.1.2.9). A quantum logic gate is a unitary matrix. So a quantum logic gate is always reversible.

Prop. (XIII.1.2.10) (Examples of Quantum Gates).
- The Hadamard gate \(H\) is a rotation on one single qubit given by the matrix \[
\begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}.
\]

- If a classical gate is reversible and its matrix is unitary, then the same matrix will give out a quantum gate with all entries 0 and 1, called the quantization of the classical gate.

- The Fredkin gate or CSWAP gate is a three-bit gate defined as the quantization of the gate given by: \((a, b, c) \mapsto (a, a(b + c) + b, a(b + c) + c)\).
3 Quantum Algorithms

Deutsch-Jozsa Algorithm

Prop. (XIII.1.3.1) (Deutsch-Jozsa Algorithm). The Deutsch-Jozsa problem is that: given a function \( f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2 \), which is either a constant function or a function that takes half value 0 and half value 1, If we have a box that maps: \( (x_1, \ldots, x_n, x) \mapsto (x_1, \ldots, x_n, x + f(x_1, \ldots, x_n)) \).

Now there is a Deutsch-Jozsa algorithm that can determine if \( f \) is a constant function just using the box one time.

Proof: Cf.[Napkin P270]. The circuit is

\[
(0, 0, \ldots, 0, 1) \mapsto (H(0), H(0), \ldots, H(0), H(1)) \mapsto (H(0), H(0), \ldots, H(0), H(1) + f(H(0), H(0), \ldots, H(0)))
\]

\[
\mapsto (H(H(0)), H(H(0)), \ldots, H(H(0)), H(1) + f(H(0), H(0), \ldots, H(0))).
\]

Then we measure all the first \( n \) bits in the \( |0\rangle/|1\rangle \)-basis.

Notice if \( f \) is constant, then the first \( n \) bits must be all \( \pm |00\ldots0\rangle \), so the measure is all 0. And if \( f \) is not constant, then the first \( n \) bits are entangled, equals the image of

\[
\frac{1}{\sqrt{2^n}} \sum_{(a_1, \ldots, a_n) \in \mathbb{F}_2^n} (-1)^{f(a_1\ldots a_n)} |a_1\ldots a_n\rangle.
\]

after the action of \( H^\otimes n \), then its coefficient of \( |0\ldots0\rangle \) is just 0, so the measure cannot be all 0. □

Quantum Fourier Transform

Def. (XIII.1.3.2) (Discrete Fourier Transform). The inverse Fourier transform is defined to be:\n\[
(y_0, \ldots, y_{n-1}) \mapsto (x_0, \ldots, x_{n-1}),
\]

where

\[
y_k = \frac{1}{N} \sum_{j=0}^{N-1} \omega_N^{jk} x_k.
\]

This is in fact represented by a van der Waerden matrix of \( \omega_N \) times \( \frac{1}{\sqrt{N}} \).

Prop. (XIII.1.3.3) (Fast Fourier Transform). There is a fast Fourier transform algorithm that can calculate the Fourier transform in \( O(N \log N) \) times.

Proof: □

Def. (XIII.1.3.4) (Quantum Fourier Transform). The quantum Fourier transform is a gate represented by a matrix \( U_{QFT} \) which is the van der Waerden matrix of \( \omega_N \) times \( \frac{1}{\sqrt{N}} \).

Proof: It suffices to prove this matrix is truly unitary. □

Prop. (XIII.1.3.5) (Tensor Representation). The trick of the quantum Fourier transform lies in its connection with the 2-adic decimal representation:

\[
U_{QFT}(|x_n x_{n-1} \ldots x_1\rangle) = \frac{1}{\sqrt{N}}(|0\rangle + \exp(2\pi i \cdot 0.x_1)|1\rangle)
\]

\[
\otimes(|0\rangle + \exp(2\pi i \cdot 0.x_2 x_1)|1\rangle)
\]

\[
\otimes \ldots
\]

\[
\otimes(|0\rangle + \exp(2\pi i \cdot 0.x_n x_{n-1} \ldots x_1)|1\rangle)
\]
**Proof:** Direct calculation. \(\square\)

**Prop. (XIII.1.3.6).** Now by the above tensor representation, the thing seems to be beautiful, and it seems the Fourier transform for \(2^n\) data can be done in \(n^2\) steps. And this is true, \(QFT_n\) is inductively define, Cf.[Napkin P273].

**Shor’s Algorithm**

**Def. (XIII.1.3.7).** For a number \(M = pq\), where \(p, q\) are different odd prime numbers. Then an \(x \mod M\) is called **good** iff: \((x, M) = 1, r = \text{ord}(x)\) is even, and neither of \(x^{r/2} \pm 1\) is divisible by \(M\).

Then at least half of \((\mathbb{Z}/M\mathbb{Z})^\ast\) is good.

**Proof:** It suffices to consider a fixed order \(2a\), and this is additive in \(\mathbb{Z}/(2a, p - 1)\mathbb{Z} \times \mathbb{Z}/(2a, q - 1)\mathbb{Z}\). \(\Box\)

**Remark (XIII.1.3.8).** If we find a good \(x\) for \(M\), then \(x^{r/2} \pm 1\) contains separately a prime \(p\) or \(q\), so we can use Euclidean algorithm to extract a prime of \(M\). This is just the ideal of Shor’s Algorithm.

**Prop. (XIII.1.3.9) (Shor’s Algorithm).** For \(M = pq\), we can factor \(p,q\) out in \(O((\log M)^2)\) time.

**Proof:** Cf.[Napkin P274]. \(\square\)

**Grover’s Algorithm**

**Prop. (XIII.1.3.10).**

**Prop. (XIII.1.3.11) (Grover’s Algorithm).** If there are \(n\) items labeled \(\{0, \ldots, n - 1\}\), and there is a marked item \(w\). then there is an algorithm that find \(w\) in \(O(\sqrt{n})\) times.

**Proof:** Cf.[Quantum Algorithm MIT P33,35]. \(\square\)
Chapter XIV

Others

XIV.1 Elementary Mathematics

1 Algebra

Prop. (XIV.1.1.1). If \( a_n \) is a series of real numbers that \( a_{m+n} \geq a_m + a_n \), then \( a_m/m \) converges to \( \lambda = \sup a_n/n \).

Notice by taking \( b_n = -a_n \), the \( \leq \) case is also true.

Proof: Let \( \lambda - a_n/n < \varepsilon \), then for any \( N \) large, \( N = kn + m \) for \( m < n \), so \( a_N \geq ka_n + a_m \), so \( \lim \inf a_N/N \geq a_n/n \) for \( N \) large. Thus the result follows. \( \square \)

2 Number Theory

Prop. (XIV.1.2.1) (\( p \)-Power in Product). \( v_p(n!) = \frac{n-c(n)}{p-1} \), where \( c(n) \) is the sum of the presentation of \( n \) in the \( p \)-adic base.

Cor. (XIV.1.2.2). \( v_p(n^a) \) equals the number of carries when adding \( a \) and \( b \) in base \( p \).

Def. (XIV.1.2.3) (Integral Part). For any number \( \alpha \in \mathbb{R} \), let \( [\alpha] \) be the maximal integer \( n \) that \( n \leq \alpha \), called the integral part of \( \alpha \) and \( \{\alpha\} = \alpha - [\alpha] \), called the fractional part of \( \alpha \).

Prop. (XIV.1.2.4) (Approximation of Irrational Numbers). Let \( \alpha \) be an irrational number, then for any integer \( N \), there exists an integer \( n \in N \) that \( \{n\alpha\} < 1/N \) or \( \{n\alpha\} > 1 - 1/N \).

Proof: Consider the sequence

\[ \{\alpha\}, \{2\alpha\}, \ldots, \{(N+1)\alpha\} \]

where \( \{x\} \) is the fractional part of \( x \), then this sequence sits in the \( N \) intervals \( \{(0,1/N),(1/N,2/N),\ldots,((N-1)/N,1)\} \), thus there are two that are in the same interval. if \( v_1 < v_2 \) satisfies \( \{v_1\alpha\} \) and \( \{v_2\alpha\} \) are in the same interval, then \( n = v_2 - v_1 \) satisfies the desired property. \( \square \)

Prop. (XIV.1.2.5) (Cauchy-Davenport). If \( A, B \) are two nonempty subsets of \( \mathbb{F}_p \), then \( |A + B| \geq \min\{p, |A| + |B| - 1\} \).
then $n$ is irreducible over $\mathbb{Z}$.

Proof: Consider the polynomial $x | \mathbb{C} - (x + y - c)$, then $f(a, b) = 0$ for all $a, b \in A, b \in B$, but the coefficient of the highest degree term $x^{n-1}|B|^{n-1}$ is $C_{|A|+|B|-2}^{-1} \neq 0$, so this contradicts combinatorial Nullstellensatz (I.2.2.7).

Prop. (XIV.1.2.6) (Power Lifting). If $a \equiv b \mod p$, then $a^n \equiv b^n \mod p^{n+1}$ for all $n > 0$.

Proof: Prove by induction, if $a^{n-1} = b^{n-1} + p^{n-1}c$ for some $c$, then $a^n = b^n + p^n + p^{n+1}d + t_p^{(n+1)}c$, so we are done.

Lemma (XIV.1.2.7). $\sum_{x=0}^{p-1} x^k \equiv -1 \mod p$ if $p - 1 | k$, and $\equiv 0 \mod p$ otherwise.

Proof: Choose a $a$ that $a^k - 1$ is not divisible by $p$, this is doable iff $k$ is not divisible by $p - 1$, then it is clear.

Prop. (XIV.1.2.8) (Quadratic Legendre Symbol Sum). For any prime $p$,

$$\sum_{x=0}^{p-1} \left(\frac{x^2 + 1}{p}\right) = -1.$$ 

Proof: The sum is equivalent modulo $p$ to $\sum_{x=0}^{p-1} (x^2 + ax + b)^{p-1}$, which by the lemma (XIV.1.2.7) above equivalent to $-1$ modulo $p$. Now it can not be $p - 1$, because otherwise there is a solution for $p|x^2 + 1$, and then it can be calculated directly.

Prop. (XIV.1.2.9). The multiplicative group of $\mathbb{Z}/2^n$ is $\mathbb{Z}/2 \times \mathbb{Z}/2^{n-2}$ for $n \geq 2$.

Proof: It suffices to show $3^{2^{n-3}} \not\equiv 1 \mod 2^n$, and $3^{2^{n-2}} \equiv 1 \mod 2^n$.

Lemma (XIV.1.2.10). If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ satisfies $a_n \geq 1, a_{n-1} \geq 0$, and $|a_i| \leq H$ for $i \leq n - 2$, then for any root $\alpha$ of $P$, either $\Re \alpha \leq 0$, or $|\alpha| < 1 + \sqrt{1 + 4H}$.

Prop. (XIV.1.2.11) (b-adic decomposition and Irreducibility). If $b > 2$ and $p$ is a prime, consider the $b$-adic expansion $p = \sum a_n b^n$, then the polynomial $\sum a_n x^n$ is irreducible over $\mathbb{Z}$.

Proof: If $p(x) = h(x)r(x)$, use the lemma (XIV.1.2.10) to show that $r(b)$ and $h(b)$ cannot be 1, thus $h(b)r(b) = p$ cannot happen.

Def. (XIV.1.2.12) (Numerical Polynomials). A function $f: \mathbb{Z} \to \mathbb{Z}$ is called a numerical polynomial if it there exists $a_1, \ldots, a_r \in \mathbb{Z}$ that

$$f(n) = \sum_{i=0}^{r} a_i \binom{n}{i}.$$ 

for $n$ sufficiently large.

Prop. (XIV.1.2.13). Suppose that $f: \mathbb{Z} \to \mathbb{Z}$ is defined for $n$ sufficiently large and $g(n) = f(n) - f(n-1)$ is a numerical polynomial, then $f$ is a numerical polynomial.

Proof: Suppose $f(n) - f(n-1) = \sum_{i=0}^{r} a_i \binom{n}{i}$ for all $n > 0$. If we set $g(n) = f(n) - \sum_{i=0}^{r} a_i \binom{n+1}{i+1}$, then $g(n) - g(n-1) = 0$ for $n$ sufficiently large, so it is eventually constant, which is equal to $a_{-1}$. Then $f(n) = \sum_{i=0}^{r} a_i \binom{n+1}{i+1} + a_{-1}$ is a numerical polynomial.

Cor. (XIV.1.2.14). Any polynomial function with coefficients in $\mathbb{Q}$ maps any integer sufficiently large into $\mathbb{Z}$ iff it is a numerical function.
XIV.1. ELEMENTARY MATHEMATICS

Algebraic Numbers

Prop. (XIV.1.2.15) (Lindemann). If \( \alpha \) is an algebraic number, then \( e^\alpha \) would be a transcendental number.

Proof: \( \square \)

Cor. (XIV.1.2.16). \( \pi \) and \( e \) are transcendental numbers.

Proof: \( e^{2\pi i} = 1 \), and \( e^1 = e \). \( \square \)

Continued Fractions

Def. (XIV.1.2.17). A finite continued fraction \([a_1, \ldots, a_n]\) is an abbreviation of

\[
\frac{1}{a_1 + \frac{1}{a_2 + \ldots + \frac{1}{a_n}}} , \quad a_i \in \mathbb{Z}_+.
\]

A continued fraction \([a_1, a_2, \ldots]\) is an abbreviation of

\[
\frac{1}{a_1 + \frac{1}{a_2 + \ldots}} , \quad a_i \in \mathbb{Z}_+.
\]

Def. (XIV.1.2.18) (Gauss Transformation). The Gauss transformation is the function \( \varphi : \mathbb{R} \to [0,1] : \)

\[
\varphi(x) = \begin{cases} 
\frac{1}{x} - [1/x] & x \neq 0 \\
0 & x = 0
\end{cases}.
\]

Prop. (XIV.1.2.19). For any sequence of positive numbers \( \{a_1, \ldots, a_n, \ldots\} \), \([a_1, \ldots, a_n]\) converges. In particular, \( \lim_{n \to \infty} [\varphi(x), \varphi^2(x), \ldots, \varphi^n(x)] = \{x\} \) for \( x \in \mathbb{R} \).

Proof: We may assume the sequence is infinite. Then we notice for any \( a \in \mathbb{Z}_+ \) and \( t \) varying in an interval in \((0,1]\) of length \( l \), the value of \( \frac{1}{a+ti} \) is in an interval of length \( l/a(a+l) \leq l/(l+1) \). So clearly the length of possible values of \([a_1, \ldots, a_n, t]\) for \( t \) in an interval of length \( l \) is in an interval of length \( l_n \) that \( \lim_{n \to \infty} l_n = 0 \).

Then we conclude \([a_1, \ldots, a_n]\) is a Cauchy sequence for \( n \), thus it converges. \( \square \)

Prop. (XIV.1.2.20). \( x \in \mathbb{R} \) is a rational number iff \( \varphi^m(x) = 0 \) for some \( m > 0 \).

Proof: If \( x \) is rational then this is Euclid division. If \( \varphi^m(x) = 0 \), then notice \( x = [x] + [\varphi(x), \varphi^2(x), \ldots, \varphi^{m-1}(x)] \) is rational. \( \square \)

Prop. (XIV.1.2.21). \( x \in \mathbb{R} \) has eventually periodic continued fractions \([\varphi(x), \varphi^2(x), \ldots, \varphi^n(x), \ldots]\) iff \( [\mathbb{Q}(x) : \mathbb{Q}] \leq 2 \).

Proof: If the fraction is finite, then \( \mathbb{Q}(x) = \mathbb{Q} \), by(XIV.1.2.20). If the fraction has periodic part \([a_1, \ldots, a_n]\), let \( t = [a_1, \ldots, a_n, a_1, \ldots, a_n, \ldots] \), then

\[
\frac{at + b}{ct + d} = t
\]

for some \( a, b, c, d \in \mathbb{Z} \), so \( [\mathbb{Q}[t] : \mathbb{Q}] = 2 \), and hence \( [\mathbb{Q}(x) : \mathbb{Q}] = 2 \).

For the converse, \( ? \). \( \square \)
3 Combinatorics

**Prop. (XIV.1.3.1) (Spencer’s Lemma).** If there is a triangulation of a plane polygon $P$, for arbitrary 3-color numbering $(0,1,2)$ of the vertices, if the number of edges on the boundary with color $(0,1)$ is odd, then there is a triangle with vertices of pairwise different colors.

**Proof:** In fact the number of those triangles with vertices of pairwise different colors is odd. In fact, the number of $(0,1)$-edges on a triangle is odd iff its vertices has pairwise different colors. But the sum of numbers of $(0,1)$-edges on the triangles are odd, by hypothesis, thus the result. □

**Prop. (XIV.1.3.2) (Monsky Theorem).** A square cannot be divided into $m$ triangles of the same area, where $m$ is odd.

**Proof:** Choose a 3-coloring on $\mathbb{R}^2$: By (IV.1.3.3) there can be a non-Archimedean valuation on $R$ that extends the $p$-adic valuation on $\mathbb{Q}$, choose the extended 2-adic valuation, then color a point $(x,y)$
- 0 if $|x|_2 < 1, |y|_2 < 1$,
- 1 if $|x|_2 \geq 1$ and $|x|_2 \geq |y|_2$.
- 2 if $|y|_2 \geq 1$ and $|y|_2 > |x|_2$.

Then there are two things:
- The coloring is invariant under translation by vectors represented by a point of color 0.
- The valuation of the area of a triangle with vertices of different color is bigger than 1, because we can assume one vertex is the origin, and then its area is $\frac{1}{2}|x_1y_2 - x_2y_1|$, which, because the coloring, must have 2-adic valuation bigger than 1.

Now back to the question, let the square be placed as a unit square, then its area is 1, and it has exactly one $(0,1)$-edge, so by Spencer’s lemma (XIV.1.3.1), it has a triangle with vertices of pairwise different colors, so its area $A$ has valuation $> 1$, but the total area is $mA = 1$ that has valuation 1, so $|m| < 1$, which means that $m$ is even. □

**Prop. (XIV.1.3.3) (Hindman’s Theorem).** Whenever the set of natural numbers are colored with f.m. colors, one can find an infinite subset $A \subset \mathbb{N}$ and a color $c$ that whenever $F \subset A$ is finite, the color of the sum of numbers in $F$ is colored $c$.


Graph Theory

**Def. (XIV.1.3.4) (Graphs).** A graph $X$ is a 1-dimensional CW-complex. the elements in the set of $X^0$ are called vertices of $X$, and the elements in the set $X^1$ are called edges of $X$.

**Def. (XIV.1.3.5) (Trees).** A tree is a contractible graph. A tree in a graph $X$ is called a maximal tree if it contains all vertices of $X$.

**Prop. (XIV.1.3.6) (Maximal Trees).** Every Connected graph contains a maximal tree, and any tree in such a graph is contained in a maximal tree.

**Proof:** Cf.[Hat02]P84. □
**Ramsey’s Theory**

**Prop. (XIV.1.3.7) (Finite Ramsey’s Theorem).** For any positive natural number $k, r, s$, there is a $n$ that if we color the $r$-subsets of a set with cardinality $n$ into $s$ groups, then there is a subset of cardinal $k$ that all its $r$-subsets are colored the same.

*Proof:* Cf.[Set Theory P218]. \[\square\]
### XIV.2 French to English Dictionary

#### 1 A
- aide: help
- apres: after
- ans: years
- avoir: have
- aux: to the

#### 2 B

#### 3 C
- cas: case
- cet: this
- chaleureusement: warmly
- classifiant: classifying
- corp: field
- courbe: curve

#### 4 D
- dans: in
- d'apres: according to
- de: of, than
- des: of
- du: of
- d'une: of a

#### 5 E
- en: in
- enonce: state
- est: is
- et: and
- etabli: established
- ètre: be

#### 6 F

#### 7 G
- générale: general
- grandes: large

#### 8 H
9  I

- il y a: there is
- introduite: introduced
- indépendamment: independence

10  J

- Je: I

11  K

- 

12  L

- 
- la: the
- le: the
- les: the
- lignes: lines
- lues: read

13  M

- 

14  N

- 
- nous: we

15  O

- 

16  P

- 
- par: by, through
- plus: more
- pour: for, to
- principal: main
- presenterons: introduce
- preuve: proof
- prolonge: extended

17  Q

- qui: who
18  R

- remercies: thank
- recemment: recently
- renvoyant: returning
- reste: rest

19  S

- schema: scheme
- strategie: strategy
- son: his
- sont: are
- suit: follows
- sur: on, about

20  T

- texte: text
- theoreme: theorem
- traite: treat
- toroidal: toroidal

21  U

- un: a
- une: a

22  V

- variante: variant
- vectoriel: vector

23  W

- 

24  X

- 

25  Y

- 

26  Z

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