# Simple Shimura Varieties

$p$ prime #

$E$ imag quadratic field $p = u\bar{u}$.

$c \in \mathrm{Gal}(E/\mathbb{Q})$.

$F^+/\mathbb{Q}$ totally real field deg $d$.

$F = E \cdot F^+$ CM $/F^+$.

$w = w_1 .. w_r$ places of $F$ above $u$

$v = v_1 .. v_r$ places of $F^+$ (restriction)

$B/F$ central division alg of dim $n^2$

... $B$ split at $w$.

$*: B \to B$ pos involution of 2nd kind.

$V = B$ realized as $B^{op}$-mod, $\rightsquigarrow$ $B \otimes B^{op}$-mod.

$p \equiv 1 \bmod 4$, $(a+bi)(a-bi)$, $u$ $\bar{u}$

$E = \mathbb{Q}(i)$

$F^+ = \mathbb{Q}$ $d=1$

$F = \mathbb{Q}(i)$

$w = u$ $r = 1$

$v = p$

$B = \left(\frac{2,3}{\mathbb{Q}(i)}\right)$ $n = 2$.

$B \cong B^{op}$, $B \otimes_{\mathbb{Q}(i)} B^{op} = M_4(\mathbb{Q}(i))$.

**Prop:** $*$-Herm. alt pairings $V \times V \to \mathbb{Q}$ all come from $\beta \in B^{*=-1}$

i.e. $\langle x_1, x_2 \rangle = \mathrm{tr}_{B/\mathbb{Q}}(x_1 \beta x_2^*)$

Using $\beta$, construct $\#_\beta : B \to B$ inv. of 2nd kind

$x \mapsto \beta x^* \beta^{-1}$.

**Defn** (Unitary similitude gp) $G_\beta / \mathbb{Q}$ whose $R$-pts are

$$\{ g \in (B^{op} \otimes_{\mathbb{Q}} R)^\times : g g^{\#_\beta} \in R^\times \}.$$

has obvious map $G_\beta \to G_m$ whose kernel is $G_{\beta,1}$.
↑ unitary gp.

Fix an embedding $\tau: F^+ \hookrightarrow \mathbb{R}$.

(Lemma) $\exists \beta \in B^{*=-1}$ nonzero such that

(i) $x$ rational prime not split in $E$ then $G_{\beta,1}, G_\beta$ q. split at $x$

(ii) Pairing $(\cdot)_{\mathbb{R}}$ has unitary inv. $(1, n-1)$ at $\tau$
$(0, n)$ at $\tau' \neq \tau$.

$\rightarrow \quad G_1(\mathbb{R}) \cong U(1, n-1) \times U(0,n)^{[F^+:\mathbb{Q}]-1}$.

(Simple ex): $n=2$. $F^+=\mathbb{Q}$: $G_1(\mathbb{R}) \cong U(1,1)$.

similarly adelic versions of unitary (sim.) gps. //

Next construct lattices in $B_{w_i}$:

Pick max order $\Lambda_i = \mathcal{O}_{B_{w_i}} \subseteq B_{w_i}$ for $i=1 \ldots r$.

$(\cdot)_{\mathbb{Q}_p}$ extends perf. pairing $V_{w_i} \simeq V_{\bar{w}_i}^\vee$

Let $\Lambda_i^\vee$ corresp lattice in $V_{\bar{w}_i}$.

Define $\Lambda := \bigoplus_{i=1}^r \Lambda_i \oplus \bigoplus_{i=1}^r \Lambda_i^\vee \subseteq V \otimes_{\mathbb{Q}} \mathbb{Q}_p = \bigoplus_{i=1}^r B_{w_i}$

① $\Lambda$ is $\mathbb{Z}_p$-lattice in $V \otimes_{\mathbb{Q}} \mathbb{Q}_p$

② $(\cdot)_{\mathbb{Q}_p}$ restricts to perf. pairing $\Lambda \times \Lambda \to \mathbb{Z}_p$.

$\varepsilon \in M_n(\mathbb{Z})$, $\Lambda_{11} = \varepsilon \Lambda_1$ //

$= \delta_{11} = (' )$

## §1. Moduli problem.

$U \subseteq G(\mathbb{A}^\infty)$ suff small open cpt subgp of $G(\mathbb{A}^\infty)$.

$\hookrightarrow \exists\, x$ s.t. proj to $G(\mathbb{Q}_x)$ the image is torsion-free.

Define moduli functor:

$$\mathfrak{X}_U : \left\{ \begin{array}{l} \text{connected locally Noeth} \\ F\text{-sch w/} \\ \text{geom pt } \bar{s} \end{array} \right\}^{op} \longrightarrow \text{Sets}$$

Sending $(S, \bar{s})$ to equiv. classes of quad $(A, \lambda, i, \bar{\eta})$ where

- $A$ abelian sch $/S$ of (rel). dim $g := dn^2$.
- $\lambda : A \to A^\vee$ polarization
- $\underline{i} : B \hookrightarrow \mathrm{End}^\circ_S(A) = \mathrm{End}_S(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ st $(A, i)$ <u>compatible</u>

& for all $b \in B$, the diag commutes:

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & A^\vee \\ i(b) \downarrow & = & \downarrow i(b^*)^\vee \\ A & \xrightarrow{\lambda} & A^\vee \end{array}$$

<u>Defn</u>. $S = \mathbb{C}$. $\mathrm{Lie}(A) \cong H_1^{sing}(A^{an}, \mathbb{R}) \cong \mathbb{C}^{\oplus g}$

free $\mathbb{C}$-vs of dim $g$ & action by $B$

$\Rightarrow \mathrm{Lie}\,A = \mathrm{Lie}^+ A \oplus \mathrm{Lie}^- A$ as $\mathbb{C} \otimes B$-mod.

where $\mathrm{Lie}^+ A = \mathrm{Lie}\,A \otimes_{\mathbb{C} \otimes E \hookrightarrow} \mathbb{C}$, $\mathrm{Lie}^- A = \mathrm{Lie}\,A \otimes_{\mathbb{C} \otimes E \hookrightarrow_{\text{cpx conj.}}} \mathbb{C}$

We say $(A, i)$ is <u>compatible</u> if $\dim_{\mathbb{C}} (\mathrm{Lie}^+ A) = n$

and the 2 actions $F^+ \subseteq \mathbb{C}$, $F^+ \subseteq B$ coincide. //

$\pi_1(S,\bar s)$

- $\bar\eta \in \left( \left\{ \text{Isoms of } B\otimes\mathbb{A}^\infty\text{-mod } \eta : V \otimes_{\mathbb{Q}} \mathbb{A}^\infty \xrightarrow{\sim} VA_{\bar s} : \text{take } \langle\cdot\rangle_{\mathbb{A}^\infty}\text{-pairing to } (\mathbb{A}^\infty)^\times\text{-multiple of } \lambda\text{-Weil pairing} \right\} \Big/ U\text{-orbits} \right)$

  (level structure)

$V = B/F/F^+/\mathbb{Q}$, with $n^2 \cdot 2 \cdot d$

$VA_{\bar s} = \varprojlim_N A[N](\bar s) \otimes \mathbb{Q} = (\mathbb{A}^\infty)^{\oplus 2g}$, $g = dn^2$.

How does $U$ act on set of isoms?

$$U \subseteq G(\mathbb{A}^\infty) \subseteq (\mathbb{A}^\infty)^\times \times (B \otimes_{\mathbb{Q}} \mathbb{A}^\infty)^\times \circlearrowright V \otimes \mathbb{A}^\infty.$$

What's the $\pi_1(S,\bar s)$-action?

Give sheaf-theoretic interpretation of Tate mod.

$$\begin{array}{ccc} A_{\bar s} & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow \pi \\ \bar s & \longrightarrow & S \end{array}$$

$\mathcal{V} = (R^1\pi_* \hat{\mathbb{Z}})^\vee \otimes \mathbb{Q}$ is locally constant sheaf on $S$. (ét site)

$\xrightarrow{\text{ét } \pi_1}$ $\mathcal{V}_{\bar s} = H^1_{ét}(A_{\bar s}, \hat{\mathbb{Z}})^\vee \otimes \mathbb{Q} = VA_{\bar s}$ as $\pi_1(S,\bar s)$-module

Finally explain: $(A,\lambda,i,\bar\eta) \sim (A',\lambda',i',\bar\eta')$ if $\exists$ iso $A \xrightarrow{\alpha} A'$ which takes $\lambda$ to $\lambda'$ (up to $\mathbb{Q}^\times$), $i$ to $i'$, $\bar\eta$ to $\bar\eta'$.

**Rmks**: Independent of geom pt, $Sh_U(S,\bar s) \simeq Sh_U(S,\bar s')$.

**Thm** (Representability). $\mathcal{X}_U$ rep. by smooth proj sch $X_U / F$.

Consider forgetful functor $\mathcal{X}_U \to \mathcal{A}_{g,N}$ (forgetting endo.)

Mumford: GIT — q. proj sch / F.

$$\begin{array}{ccc} T \dashrightarrow \mathcal{X}_U \times_{\mathcal{A}_{g,N}} S & \longrightarrow & \mathcal{X}_U \\ \downarrow & \lrcorner & \downarrow \text{ representable} \\ S & \longrightarrow & \mathcal{A}_{g,N} \end{array}$$

Given $T \to S$, $(T \to \mathcal{X}_U)$

$$\begin{array}{c} T \to S \\ \searrow \;\downarrow \\ \mathcal{A}_{g,N} \end{array}$$

polarization ✓
endo ☐
level ✓

$A_T$ polarization is fixed, level fixed

(Hilbert sch): $A/S$, the functor $\{\text{loc Noeth schs}/S\}^{op} \to \text{Sets}$ sending $T \mapsto \mathrm{End}(A_T)$ is representable by disjoint union of proj schemes / S.

⟶ [Ko3] Points on SV over finite fields. //

**¶ Action of $G(\mathbb{A}^{\infty})$ on tower:**

Let $g \in G(\mathbb{A}^{\infty})$, $U, V \subseteq G(\mathbb{A}^{\infty})$ suff small open cpt. w/ $g^{-1} U g \subseteq U$.

then $\exists g : X_V \to X_{V'}$ sends

$(A, \lambda, i, \bar\eta) \mapsto (A, \lambda, i, \overline{\eta \circ g})$ }

2nd perspective — isolate what happens at $p$.

$U \subseteq G(\mathbb{A}^{\infty}) = G(\mathbb{A}^{\infty,p}) \times G(\mathbb{Q}_p) \subseteq G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times \prod_{i=1}^r B_{w_i}$

Suppose $U = U^p \times \mathbb{Z}_p^\times \times \prod_{i=1}^r U_{w_i}$ where

$U_{w_i} = \ker((\mathcal{O}_{B_{w_i}}^{op})^\times \to (\mathcal{O}_{B_{w_i}}^{op} / w_i^{m_i})^\times)$

for some $\underline{m} = (m_1 \dots m_r) \geq 0$.

$\mathfrak{X}_U'' : \left\{ \begin{matrix} \text{loc noeth schs} \\ \text{w/ geom pt } \bar{s} \end{matrix} \right\}^{op} \to \text{Sets}$ send $(S, \bar{s})$ to eq classes

of $(r+4)$-tuples $(A, \lambda, i, \bar\eta^p, \alpha_1 \dots \alpha_r)$.

- $A/S$ ab. sch of dim $dn^2 =: g$
- $\lambda : A \to A^\vee$ prime-to-$p$ deg pol. (dag commutes)
- $i : B \hookrightarrow \text{End}^\circ A$ which is compatible w/ $\dagger$, and w/ $\lambda$.
- $\bar\eta^p \in \left( \left\{ \begin{matrix} \text{(similitude) symplectic isoms of } B \otimes \mathbb{A}^{\infty,p}\text{-mod} \\ \eta^p : V \otimes \mathbb{A}^{\infty,p} \xrightarrow{\sim} V^p A_{\bar{s}} \end{matrix} \right\} / U^p\text{-orbits} \right)^{\pi_1(S, \bar{s})}$
- $\alpha_1 : (w_1^{-m_1} \Lambda_{11} / \Lambda_{11})_S \xrightarrow{\sim} \varepsilon A[w_1^{m_1}]$

isom of $S$-sch equipped w/ $\mathcal{O}_F$-action.

- for $i > 1$ : $\alpha_i : (w_i^{-m_i} \Lambda_i / \Lambda_i)_S \xrightarrow{\sim} A[w_i^{m_i}]$

isom of $S$-sch w/ $\mathcal{O}_B$-actions.

Similar notion of equivalence . . .

Try to make sense of $A[w_1]$:

Given $A/S$ $\leadsto$ $A[p^\infty] = \varinjlim_n A[p^n] / S$ $\circlearrowleft$ $\mathcal{O}_B \otimes \mathbb{Z}_p \cong M_n(\mathbb{Z}_p)^{op} \times M_n(\mathbb{Z}_p)^{op}$.

$\leadsto$ $A[p^\infty] = X_w^n \times \cdots \times X_{w_r}^n \times X_{\bar{w}}^n \times \cdots \times X_{\bar{w}_r}^n$. "dual p-div grp"

dim $n^2$, ht $2n^2$ ; dim 1, ht $n$ ; dim $n-1$, ht $n$

dim $n$, ht $n^2$ ; dim $n^2-n$, ht $n^2$

Simplification: $d=1$, $r=1$.

$= A[w^\infty] \times A[w^\infty]^\vee$

guess: $A[w_1^m] = X_{w_1}[p^m]$. ?

## §2. Integral Models.

Have to understand Drinfeld level structure.

**Defn.** Fix $m \geq 1$. A Drinfeld level structure is a collection of sections $P_1, \ldots, P_n : S \to X_w$ such that

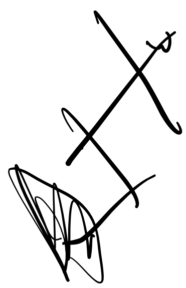
$$X_w[p^m] = \sum_{(i_1 \ldots i_n) \in (\mathbb{Z}_p/p^m\mathbb{Z})^{\oplus n}} [i_1 P_1 + \cdots + i_n P_n]$$

Get proj scheme $\mathfrak{X}_{U(p^m)} / \mathcal{O}_{F,w}$.

Give $H$ direct summand of $(\mathbb{Z}/p^m\mathbb{Z})^{\oplus n}$, define $\mathfrak{X}^{H}_{U,P,m}$

(red) closed subscheme of $\mathfrak{X}_{U,P,m}$ where

$$\sum_{\vec{i}\in H\subseteq(\mathbb{Z}/p^m\mathbb{Z})^{\oplus n}} [i_1P_1+\dots+i_nP_n] = |H|\,[e].$$



$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \mathfrak{X}_{U,P,m} \\ \downarrow & \lrcorner & \downarrow \\ k & \longrightarrow & \mathcal{O}_{F_w} \end{array}$$

**Thm** (3.14 [Sch13a]). $\mathfrak{X}^{H}_{U,P,m}$ is regular,

the special fiber $\tilde{X} = \bigcup_{H\neq 0} \mathfrak{X}^{H}_{-}$,

$\exists$ stratification
$$\mathfrak{X}_{U,P,m} = \bigcup_{H} \mathring{\mathfrak{X}}^{H}_{U,P,m}$$

where
$$\mathring{\mathfrak{X}}^{H}_{U,P,m} := \mathfrak{X}^{H}_{\cdot} \setminus \bigcup_{H\subsetneq H'} \mathfrak{X}^{H'}$$ //.

General theory: $X/\mathcal{O}$ regular flat rel dim $n$, $\tilde{X}/k$ has

stratification $\tilde{X} = \bigcup_i \mathring{Z}_i$, $Z_i = \overline{(\mathring{Z}_i)}$ regular

↑ special fiber components

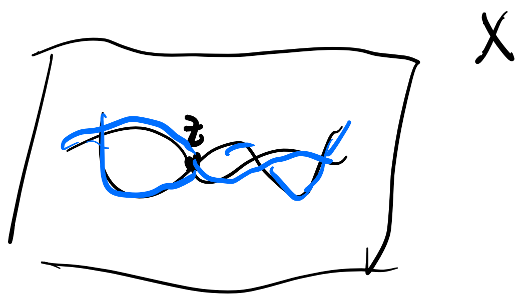
$c(Z) = \operatorname{codim}_X(Z)$.

For each special fiber comp $Z \subseteq X$, give $\mathbb{Q}_\ell$-vector space $W_Z$, w/ the prop that for all $Z'$ w/ $c(Z) = c(Z') + 1$.

If $c(Z) = 1$, $W_Z = \overline{\mathbb{Q}}_\ell$. In general,

$$\bigoplus_{\substack{Z \subseteq Z' \\ c(Z') = c(Z) - 1}} W_{Z'} \longrightarrow \bigoplus_{\substack{Z \subseteq Z' \\ c(Z') = c(Z) - 2}} W_{Z'}$$

In this case $\overline{\mathbb{Q}}_\ell^{\oplus 2} \to \overline{\mathbb{Q}}_\ell$




(*) The sequence (for any fixed $Z$)

$$0 \to W_Z \to \cdots \to \bigoplus_{\substack{Z \subseteq Z' \\ c(Z') = i}} W_{Z'} \to \cdots \to \overline{\mathbb{Q}}_\ell \to 0.$$

is exact.

Thm A (Sch Ber) If * holds then for any $x \in X_s(\overline{\mathbb{F}}_q)$ w/ gen pt $\tilde{x}$ over $x$, then canonical isom

$$(i^* R^k j_* \overline{\mathbb{Q}}_\ell)_{\tilde{x}} \cong \bigoplus_{\substack{c(Z) = k \\ x \in Z}} W_Z(-k)$$

for all $k$.

$$\begin{array}{ccccc} \overline{\mathbb{Q}}_\ell \quad X_{K^{ur}} & \xrightarrow{j} & X & \xleftarrow{i} & \tilde{X} \\ \downarrow & & \downarrow & & \downarrow \\ K^{ur} & \longrightarrow & \mathcal{O}^{ur} & \longleftarrow & \overline{\mathbb{F}}_q \end{array}$$

sketch

pf of (*) key relate $Z \longleftrightarrow H \subseteq (\mathbb{Z}/p^{m}\mathbb{Z})^{\oplus n}$

Consider rep $\rho_H : GL(H) \to GL(V)$,

$V =$ functions $\left\{ \begin{matrix} \text{complete flags of } H \\ 0 \subseteq H_1 \subseteq \dots \subseteq H_k = H \end{matrix} \right\} \to \bar{\mathbb{Q}}_\ell$

such that $\sum_{\substack{(H_i' \text{ direct summand of } H, ) \\ H_0 \subseteq \dots \subseteq H_{i-1} \subseteq H_i' \subseteq H_{i+1} \subseteq \dots = H}} f(=) = 0$

is Steinberg rep $GL(H)$ by

$St_H = \ker(\mathrm{Ind}_B^{GL(H)} 1 \to \bigoplus_{B \subsetneq P} \mathrm{Ind}_P^{GL(H)} 1)$

$B$ Borel subgp. $P \subseteq GL(H)$ parabolic.

Statement: $0 \to W_H \to \dots \bigoplus_{\substack{H' \subseteq H \\ rk(H') = 1}} W_{H'} \to \dots \to \bar{\mathbb{Q}}_\ell \to 0$

is exact

↑ √.