0.1 Weber-Hecke L-Functions for Imaginary Quadratic Fields

1 Imaginary Quadratic Fields

Def. (0.1.1.1) [$\mathcal{O}_D$]. For $D \in \mathbb{Z}^\times$ s.t. $D \equiv 0, 1 \pmod 4$, denote $\mathcal{O}_D = \mathbb{Z}[\frac{D+\sqrt{D}}{2}]$. ⌟

Prop. (0.1.1.2) [Ring of Integers in Quadratic Fields]. Let $n \in \mathbb{Z}^\times$ be a square-free, and let $\mathcal{K} = \mathbb{Q}(\sqrt{n})$.

- If $n \equiv 2, 3 \mod 4$, then $\mathcal{O}_\mathcal{K} = \mathbb{Z}[\sqrt{n}]$, and $\mathrm{disc}(\mathcal{K}) = 4n$.
- If $n \equiv 1 \mod 4$, then $\mathcal{O}_\mathcal{K} = \mathbb{Z}[\frac{1+\sqrt{n}}{2}]$, and $\mathrm{disc}(\mathcal{K}) = n$.

In particular, $\mathcal{O}_\mathcal{K} = \mathcal{O}_{\mathrm{disc}(\mathcal{K})} = \mathbb{Z}[\frac{\mathrm{disc}(\mathcal{K})+\sqrt{\mathrm{disc}(\mathcal{K})}}{2}]$. ⌟

Proof: 1: the minimal polynomial of $\sqrt{n}$ is $X^2 - n$, whose discriminant is $4n$, which doesn't have a proper divisor β that $4n/\beta$ is a square and $\beta \equiv 0, 1 \pmod 4$, so $\mathbb{Z}[\sqrt{n}]$ is the ring of integers.

2: the minimal polynomial of $\frac{1+\sqrt{n}}{2}$ is $X^2 - X + \frac{1-n}{4}$, whose discriminant is n, which doesn't have a proper divisor β that $4n/\beta$ is a square, so $\mathbb{Z}[\frac{1+\sqrt{n}}{2}]$ is the ring of integers.

For the last assertion, given the basis for the ring of integers, we can easily calculate the discriminant. It equals $4n$ in the first case and n in the second case. Thus the assertion follows. □

Def. (0.1.1.3) [Fundamental Discriminants]. A **fundamental discriminant** is an element $d \in \mathbb{Z}^\times$ s.t. d is the discriminant of quadratic field $\mathcal{K} \in \mathtt{NField}$.

Then any fundamental discriminant is a product of distinct elements in $\{-4, 8, -8\} \cup \{\left(\frac{-1}{p}\right)p\}_{p \in \mathtt{Prime}_{\geq 3}}$, by(0.1.1.2). ⌟

Def. (0.1.1.4) [Imaginary Quadratic Orders].

- $\mathcal{O}_{-1} = \mathbb{Z}[\mathrm{i}]$ is called the ring of **Gaussian integers**.
- $\mathcal{O}_{-3} = \mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ is called the ring of **Eisenstein integers**.
- $\mathcal{O}_{-7} = \mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$ is called the ring of **Kleinian Integers**.

These are all PIDs. ⌟

Thm. (0.1.1.5) [Primes in Quadratic Fields]. Let $\mathcal{K}$ be a quadratic field with discriminant $\mathrm{disc}(\mathcal{K})$. Then for $p \in \mathtt{Prime}$,

- If $p | \mathrm{disc}(\mathcal{K})$, then $p\mathcal{O}_K = \mathfrak{p}^2$, where $\mathfrak{p}$ is a prime in $\mathcal{O}_\mathcal{K}$, and $\mathfrak{p} = (p, \sqrt{\mathrm{disc}(\mathcal{K})})$ if p is odd.
- If $\left(\frac{\mathrm{disc}(\mathcal{K})}{p}\right) = 1$, then $p\mathcal{O}_K = \mathfrak{p}\mathfrak{p}'$, where $p, \mathfrak{p} \neq \mathfrak{p}'$ are primes in $\mathcal{O}_\mathcal{K}$.
- If $\left(\frac{\mathrm{disc}(\mathcal{K})}{p}\right) = -1$, then $p\mathcal{O}_\mathcal{K}$ is a prime in $\mathcal{O}_\mathcal{K}$.

And every maximal prime in $\mathcal{O}_\mathcal{K}$ are of the form.

In particular, p ramifies in $\mathcal{K}$ iff $p | \mathrm{disc}(\mathcal{K})$, and p splits completely in $\mathcal{K}$ iff $\left(\frac{\mathrm{disc}(\mathcal{K})}{p}\right) = 1$. ⌟

Prop. (0.1.1.6) [Primitive Ideals]. For a quadratic field $\mathcal{K}$ with $D = \mathrm{disc}(\mathcal{K})$, an ideal $\mathfrak{a} \in \mathrm{Ideal}^\times(\mathcal{O}_\mathcal{K})$ is called a **primitive ideal** if $\mathfrak{a} | (m)$ implies $m = 1$ for any $m \in \mathbb{Z}_+$. Then every primitive ideal is of the form

$$\mathfrak{a} = \mathbb{Z}\left\{a, \frac{b+\sqrt{D}}{2}\right\},$$

where $a = \|\mathfrak{a}\|$ and $b \in \mathbb{Z}_+$ is determined by $-a < b \leq a, b^2 \equiv D \pmod{4a}$. ⌟

Hao Peng, March 17, 2024

Proof: Clearly $a = \|\mathfrak{a}\|$ is contained in $\mathfrak{a}$, and it is the smallest positive integer contained in $\mathfrak{a}$. Thus there must be another element $\frac{b+\sqrt{D}}{2}$ s.t. $\left\{a, \frac{b+\sqrt{D}}{2}\right\}$ is a basis of $\mathfrak{a}$. But then $b^2 \equiv D(\text{mod } 4a)$, and we may assume $-a < b \leq a$. And conversely, if $(b')^2 \equiv D(\text{mod } 4a)$, then it is easy to verify $\mathbb{Z}\left\{a, \frac{b+\sqrt{D}}{2}\right\}$ is a primitive ideal with norm $\mathfrak{a}$. Which implies that $b' \equiv b(\text{mod } 2a)$. So the solution of b is unique. □

2 Hecke Characters

Let $\mathcal{K} = \mathbb{Q}(\sqrt{D})$ be a imaginary quadratic field with discriminant $D < 0$.

Def. (0.1.2.1) [Places]. Let $F = \mathbb{Q}$ or $\mathcal{K}$, a **place** of F is a valuation on F: i.e. a function

$$v = |-| : F \to \mathbb{R}_{\geq 0}$$

such that

- $|0| = 0$.
- $|ab| = |a||b|$ for any $a, b \in F$.
- $|a+b| \leq |a| + |b|$ for any $a, b \in F$.

each prime ideal $\mathfrak{p} \subset \mathcal{O}_F$ defines a valuation $v_{\mathfrak{p}}$ on F as follows: If $a \in F^\times$, let k be the maximal integer s.t. $a \in \mathfrak{p}^k$, then define

$$|a|_v = \frac{1}{N(\mathfrak{p})^k}.$$

Remember the absolute value on $\mathbb{C}$ and $\mathbb{R}$ are also valuations.

The set of equivalence classes of valuations on F is called the set of places of F, denoted by Σ_F.

⌟

Def. (0.1.2.2) [Completion Fields]. Let $F = \mathbb{Q}$ or $\mathcal{K}$, for a prime ideal $\mathfrak{p} \subset \mathcal{O}_F$, the completion field $F_{\mathfrak{p}}$ is the field of elements of the form

$$a = a_k + a_{k+1} + \ldots + a_{k+2} + \ldots$$

where $k \in \mathbb{Z}_+$, $a_k \in \mathfrak{p}^k, a_{k+1} \in \mathfrak{p}^{k+1}, \ldots$. It is clearly an additive group and has multiplications, and is a field because we can take inversion: Suppose

$$a = a_k + a_{k+1} + \ldots$$

where $a_k \neq 0$, we can take

$$a^{-1} = \frac{1}{a_k + a_{k+1} + \ldots} = a_k^{-1}\left(1 - \frac{a_{k+1} + a_{k+2} + \ldots}{a_k} + \left(\frac{a_{k+1} + a_{k+2} + \ldots}{a_k}\right)^2 - \left(\frac{a_{k+1} + a_{k+2} + \ldots}{a_k}\right)^3 + \ldots\right)$$

For example, if $F = \mathbb{Q}$,

$$\frac{1}{1-p} = 1 + p + p^2 + \ldots, \quad \frac{1}{1+p+p^2+\ldots+} = 1 - p.$$

$$\frac{1}{1-p+p^2} = 1 + (p - p^2) + (p - p^2)^2 + \ldots = 1 + p - p^3 + \ldots$$

The set of elements of the form

$$a = a_0 + a_1 + \dots$$

is denoted by $\mathcal{O}_\mathfrak{p}$, or when $F = \mathbb{Q}$, also denoted by $\mathbb{Z}_p$.

There is a valuation on $F_\mathfrak{p}$ given by $|a|_\mathfrak{p} = \mathrm{Nm}(\mathfrak{p}_v)^{-k}$, where

$$a = a_k + a_{k+1} + \dots$$

and $a_k \in \mathfrak{p}^k \setminus \mathfrak{p}^{k+1}$.

For the infinite place ∞, we define the completion in the same way, and get

$$\mathbb{Q}_\infty = \mathbb{R}, \quad \mathcal{K}_\infty = \mathbb{C}$$

⌟

Def. (0.1.2.3) [Ideles]. Let $F = \mathbb{Q}$ or $\mathcal{K}$, the **Idele group** is defined to be

$$\mathbf{I}_F = \prod_{v \in \Sigma_F}' F_v^\times.$$

where the weird $\prod'$ means the following: an element of $\mathbf{I}_F$ is a tuple

$$\mathfrak{a} = (a_v)_{v \in \Sigma_F}$$

s.t. only finitely many a_v(including the infinite place) satisfies $a_v \notin \mathcal{O}_v^*$.

And $\mathbf{I}_F$ is endowed with some topology. There is an embedding of groups

$$F^\times \subset \mathbf{I}_F : a \mapsto (a, a, \dots, a, \dots)$$

⌟

Def. (0.1.2.4) [Hecke Characters]. A **Hecke character** is a continuous character of $\mathbf{I}_F$(i.e. a continuous group homomorphism $\mathbf{I}_F \to \{z \in \mathbb{C} : |z| = 1\}$) that is trivial on $F^\times$.

A strange topology is defined on $\mathbf{I}_F$ s.t. that each Hecke character χ of $\mathbf{I}_F$ is of the form

$$\chi((a_v)) = \prod_{v \in \Sigma_F} \chi_v(a_v)$$

where each χ_v is a character of F_v, and only finitely many of them are non-trivial(i.e. not always take value 1). ⌟

Prop. (0.1.2.5) [Dirichlet Characters as Hecke Characters]. We have a decomposition of groups:

$$\mathbf{I}_\mathbb{Q} = \mathbb{Q}^\times \times \prod_p \mathbb{Z}_p^* \times \mathbb{R}_+,$$

The proof goes as follows: Given an element

$$\mathfrak{a} = (a_p) \in \mathbf{I}_\mathbb{Q},$$

only finitely many $a_v \notin \mathbb{Z}_p^*$, so by weak approximation, we can take some $a \in \mathbb{Q}^\times$ s.t. $a_p a^{-1} \in \mathbb{Z}_p^*$ for each prime p. We can also guarantee that $a_\infty a^{-1} \in \mathbb{R}_+$ by multiplying -1 if necessary. Then

$$\mathfrak{a} = a \times \mathfrak{a}a^{-1}, \quad \mathfrak{a}a^{-1} \in \prod_p \mathbb{Z}_p^* \times \mathbb{R}_+.$$

For any $m \in \mathbb{Z}_+$, there is a natural map

$$\prod_p \mathbb{Z}_p^* \to (\mathbb{Z}/m)^*.$$

defined as follows: If $m = p_1^{e_1} \dots p_k^{e_k}$, the map is

$$\prod_p \mathbb{Z}_p^* \to \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_k} \to \frac{\mathbb{Z}_{p_1}}{1 + p_1^{e_1}\mathbb{Z}_{p_1}} \times \dots \times \frac{\mathbb{Z}_{p_k}}{1 + p_1^{e_k}\mathbb{Z}_{p_k}} = (\mathbb{Z}/m)^*.$$

So for any Dirichlet character χ_0 modulo m, we get a Hecke character

$$\chi : \mathbf{I}_{\mathbb{Q}} \to \prod_p \mathbb{Z}_p^* \to (\mathbb{Z}/m)^* \xrightarrow{\chi_0^{-1}} \mathbb{C}^\times.$$

In this way, $\chi([p]_p) = \chi_0(p)$ for any $p \in \texttt{Prime}$ s.t. $(p, m) = 1$. ⌟

Prop. (0.1.2.6)[Hecke Characters as Dirichlet Characters]. For an finite modulus $\mathfrak{m} \in \mathrm{Ideal}^\times(\mathcal{O}_F)$, we may define a **Dirichlet character** as a character

$$\chi_0 : (\mathcal{O}_F/\mathfrak{m})^* \to \mathbb{C}^\times.$$

There is an exact sequence

$$1 \to \mathcal{O}_F^* \to \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^* \times F_\infty^\times \to \mathbf{C}_F \to \mathrm{Cl}(F) \to 1.$$

For any Hecke character χ of F, χ_f factors through $\prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^* \to (\mathcal{O}_F/\mathfrak{m})^*$ for some finite modulus $\mathfrak{m} = \mathfrak{p}_1^{e_1} \dots \mathfrak{p}_k^{e_k}$ by continuity constraint, thus inducing a Dirichlet character χ_0 modulo $\mathfrak{m}$, called the **Dirichlet component of** χ.

However, not every naive Dirichlet character χ_0 an an infinite character χ_∞ can form a Hecke character. For example, they must satisfy $\chi_0(\varepsilon)\chi_\infty(\varepsilon) = 1$ for any unit $\varepsilon \in \mathcal{O}_F^*$. And if $\chi_0(\varepsilon)\chi_\infty(\varepsilon) = 1$, then there exists exactly $\mathrm{cl}(F)$-many Hecke character χ extending χ_0 and χ_∞, because the surjection

$$F^\times \backslash \mathbf{I}_F / \prod_{\mathfrak{p}} (1 + \mathfrak{p}^{e_{\mathfrak{p}}}) \to \mathrm{Cl}(F) \to 1$$

has sections(take case of the infinite places). ⌟

Def. (0.1.2.7) [Hecke Characters and Ideal Characters]. Any Hecke character χ of F**??** is of the form $\chi = \chi_f \times \chi_\infty$, where $\chi_f : \mathbf{I}_{F,f} \to \mathbb{C}$ and $\chi_\infty : (F \otimes \mathbb{R})^\times \to \mathbb{C}$. Then by continuity, there exists a smallest modulus $\mathfrak{m} = \prod_{v \in \Sigma_F^{\mathrm{fin}}} \mathfrak{p}_v^{e_v}$ s.t. χ_f is trivial on

$$\prod_{v \in \Sigma_F^{\mathrm{fin}}} (1 + \mathfrak{p}_v^{e_v}).$$

Thus the restriction of χ to $\mathbf{I}^{\mathfrak{m}\infty}$ gives us a ideal class map $J^{\mathfrak{m}} \to \mathbb{C}$ which we also denote by χ. And the restriction of χ to $\prod_{v \in \Sigma_F^{\mathrm{fin}}} \mathcal{O}_{F_v}^*$ gives a Dirichlet character $\chi_0^{-1} : (\mathcal{O}_F/\mathfrak{m})^* \to \mathbb{C}$.

Then we have $\chi((\alpha))\chi_\infty(\alpha) = \chi_0(\alpha)$ for any $\alpha \in \mathcal{O}$ that is prime to $\mathfrak{m}$. ⌟

3 Hecke L-Functions

Def. (0.1.3.1) [Gauss Sum for Hecke Characters]. For $F \in \texttt{NField}$ and a Hecke character χ for F with Dirichlet component χ_0 of conductor $\mathfrak{m}$(0.1.2.6), the **Gauss sum** of χ is defined to be

$$\tau(\chi) = \frac{1}{\chi_\infty(\gamma)\chi(\mathfrak{c})} \sum_{\alpha \in \mathfrak{c}/\mathfrak{c}\mathfrak{m}} \chi_0(\alpha) e^{2\pi \mathrm{i}\, \mathrm{tr}_{F/\mathbb{Q}}(\alpha/\gamma)} (0.1.2.6),$$

where $\gamma \in \mathcal{O}_F^\times$ and $\mathfrak{c} \in \mathrm{Ideal}^\times(\mathcal{O}_F)$ are chosen s.t. $(\mathfrak{c}, \mathfrak{m}) = 1$ and $\mathfrak{c}\mathfrak{d}_F\mathfrak{m} = (\gamma)$. Where

$$\mathfrak{d}_F^{-1} = \{\alpha \in F^\times : \mathrm{tr}_{F/\mathbb{Q}}(\alpha \mathcal{O}_F) \in \mathbb{Z}\}.$$

The definition of $\tau(\chi)$ is independent of $(\mathfrak{c}, \gamma)$ chosen.

In particular, if $F = \mathbb{Q}$ and $\mathfrak{m} = (N)$ for $N \in \mathbb{Z}_+$, then

$$\tau_a(\chi) \triangleq \sum_{n \in \mathbb{Z}/(N)} \chi(n) e^{2\pi \mathrm{i} \frac{an}{N}}, \quad \tau(\chi) \triangleq \tau_1(\chi).$$

In particular, $\tau(\mathbf{1}_p)$ is called the **Ramanujan sum**, and by Möbius inversion on N, we have

$$\mathrm{Kloos}(a, 0, \mod N) = \sum_{r \in (\mathbb{Z}/(N))^*} e^{2\pi \mathrm{i} \frac{an}{N}} = \sum_{d | (N,n)} = \mu\left(\frac{N}{(N,n)}\right) \frac{\phi(N)}{\phi(N/(N,n))}.$$

⌟

Prop. (0.1.3.2). For $N_1, N_2 \in \mathbb{Z}_+$ s.t. $(N_1, N_2) = 1$ and $\chi_i \in \mathrm{Diri}(N_i)$, we have

$$\tau(\chi_1\chi_2) = \chi_1(N_2)\chi_2(N_1)\tau(\chi_1)\tau(\chi_2).$$

This also holds for Hecke characters. ⌟

Proof:

$$\tau(\chi_1\chi_2) = \sum_{n_1 (\mathrm{mod}\ N_1)} \sum_{n_2 (\mathrm{mod}\ N_2)} \chi_1(N_2 n_1)\chi_2(N_1 n_2) e^{2\pi \mathrm{i} \frac{N_1 n_2 + N_2 n_1}{N_1 N_2}} = \chi_1(N_2)\chi_2(N_1)\tau(\chi_1)\tau(\chi_2).$$

□

Def. (0.1.3.3) [Weber-Hecke L-Functions]. For a Hecke character χ of $\mathcal{K}$ of conductor $\mathfrak{m}$, define

$$L_{\mathcal{K}}(\chi; s) = \prod_{\mathfrak{p} \nmid \mathfrak{m}} \frac{1}{1 - \chi(\mathfrak{p}) \|\mathfrak{p}\|^{-s}} = \sum_{\mathfrak{a} \in \mathrm{Ideal}^\times(\mathcal{O}_F), (\mathfrak{a}, \mathfrak{m}) = 1} \frac{\chi(\mathfrak{a})}{\|\mathfrak{a}\|^s}$$

Then for $\mathrm{Re}(s)$ sufficiently large, this converges absolutely. ⌟

Thm. (0.1.3.4) [Hecke L-Functions of Imaginary Quadratic Fields]. Let $\mathfrak{m}$ be a modulus for $\mathcal{K}$, and χ a Hecke character of $\mathcal{K}$ with χ_0 of conductor $\mathfrak{m}$(0.1.2.6) and associated ideal class map $\psi : J^{\mathfrak{m}} \to \mathbb{C}^\times$. Suppose $\chi_\infty(z) = e^{k \arg z}$, then

$$\Lambda(\chi; s) = L_{\mathbb{C}}(s + \frac{|k|}{2}) L(\chi; s)$$

is the Hecke L-function attached to χ. Thus $L(\chi;s)$ can be extended to a meromorphic function for all $s\in\mathbb{C}$, and it has simple poles at $s=1$ when $\chi=\mathbf{1}$, and analytic otherwise. When $\chi=\mathbf{1}$, $L(\mathbf{1};s)=\zeta_{\mathcal{K}}(s)=\zeta(s)L(\chi_{\mathcal{K}},s)$, and

$$\operatorname{res}_{s=1}\zeta_{\mathcal{K}}(s)=\frac{2\operatorname{cl}(\mathcal{K})}{\#\mathcal{K}^{\times}_{\mathrm{tor}}\sqrt{|\operatorname{disc}(\mathcal{K})|}}\pi.$$

Moreover, there are functional equations

$$\Lambda(\chi;s)=\mathrm{i}^{-k}\frac{\tau(\chi)}{\|\mathfrak{m}\|^{1/2}}\left(|\operatorname{disc}(\mathcal{K})|\cdot\|\mathfrak{m}\|\right)^{1/2-s}\Lambda(\overline{\chi};1-s)(0.1.3.1).$$

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