0.1 Weber-Hecke $L$-Functions for Imaginary Quadratic Fields

1 Imaginary Quadratic Fields

Def. (0.1.1.1) [$\mathcal{O}_D$]. For $D \in \mathbb{Z}^\times$ s.t. $D \equiv 0, 1 \pmod{4}$, denote $\mathcal{O}_D = \mathbb{Z}[\frac{D+\sqrt{D}}{2}]$.

Prop. (0.1.1.2) [Ring of Integers in Quadratic Fields]. Let $n \in \mathbb{Z}^\times$ be a square-free, and let $\mathcal{K} = \mathbb{Q}(\sqrt{n})$.

- If $n \equiv 2, 3 \pmod{4}$, then $\mathcal{O}_K = \mathbb{Z}[\sqrt{n}]$, and $\text{disc}(\mathcal{K}) = 4n$.
- If $n \equiv 1 \pmod{4}$, then $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{n}}{2}]$, and $\text{disc}(\mathcal{K}) = n$.

In particular, $\mathcal{O}_K = \mathcal{O}_{\text{disc}(\mathcal{K})} = \mathbb{Z}[\frac{\text{disc}(\mathcal{K})+\sqrt{\text{disc}(\mathcal{K})}}{2}]$.

Proof: 1: the minimal polynomial of $\sqrt{n}$ is $X^2 - n$, whose discriminant is $4n$, which doesn’t have a proper divisor $\beta$ that $4n/\beta$ is a square and $\beta \equiv 0, 1 \pmod{4}$, so $\mathbb{Z}[\sqrt{n}]$ is the ring of integers.

2: the minimal polynomial of $\frac{1+\sqrt{n}}{2}$ is $X^2 - X + \frac{1-n}{4}$, whose discriminant is $n$, which doesn’t have a proper divisor $\beta$ that $4n/\beta$ is a square, so $\mathbb{Z}[\frac{1+\sqrt{n}}{2}]$ is the ring of integers.

For the last assertion, given the basis for the ring of integers, we can easily calculate the discriminant. It equals $4n$ in the first case and $n$ in the second case. Thus the assertion follows.

Def. (0.1.1.3) [Fundamental Discriminants]. A fundamental discriminant is an element $d \in \mathbb{Z}^\times$ s.t. $d$ is the discriminant of quadratic field $\mathcal{K} \in \text{NF}_{\text{field}}$.

Then any fundamental discriminant is a product of distinct elements in $\{-4, 8, -8\} \cup \left\{\frac{-1}{p}\right\}_{p \in \text{Prime}_{\geq 3}}$

by (0.1.1.2).

Def. (0.1.1.4) [Imaginary Quadratic Orders].

- $\mathcal{O}_{-1} = \mathbb{Z}[i]$ is called the ring of Gaussian integers.
- $\mathcal{O}_{-3} = \mathbb{Z}[\frac{1+\sqrt{3}}{2}]$ is called the ring of Eisenstein integers.
- $\mathcal{O}_{-7} = \mathbb{Z}[\frac{1+\sqrt{7}}{2}]$ is called the ring of Kleinian Integers.

These are all PIDs.

Thm. (0.1.1.5) [Primes in Quadratic Fields]. Let $\mathcal{K}$ be a quadratic field with discriminant $\text{disc}(\mathcal{K})$.

Then for $p \in \text{Prime}$,

- If $p | \text{disc}(\mathcal{K})$, then $p\mathcal{O}_K = p^2$, where $p$ is a prime in $\mathcal{O}_K$, and $p = (p, \sqrt{\text{disc}(\mathcal{K})})$ if $p$ is odd.
- If $\left(\frac{\text{disc}(\mathcal{K})}{p}\right) = 1$, then $p\mathcal{O}_K = pp'$, where $p, p' \neq p'$ are primes in $\mathcal{O}_K$.
- If $\left(\frac{\text{disc}(\mathcal{K})}{p}\right) = -1$, then $p\mathcal{O}_K$ is a prime in $\mathcal{O}_K$.

And every maximal prime in $\mathcal{O}_K$ are of the form.

In particular, $p$ ramifies in $\mathcal{K}$ iff $p | \text{disc}(\mathcal{K})$, and $p$ splits completely in $\mathcal{K}$ iff $\left(\frac{\text{disc}(\mathcal{K})}{p}\right) = 1$.

Prop. (0.1.1.6) [Primitive Ideals]. For a quadratic field $\mathcal{K}$ with $D = \text{disc}(\mathcal{K})$, an ideal $a \in \text{Ideal}_{\text{times}}(\mathcal{O}_K)$ is called a primitive ideal if $a | (m)$ implies $m = 1$ for any $m \in \mathbb{Z}_+$. Then every primitive ideal is of the form

$$a = \mathbb{Z}\left\{a, \frac{b+\sqrt{D}}{2}\right\},$$

where $a = \|a\|$ and $b \in \mathbb{Z}_+$ is determined by $-a < b \leq a, b^2 \equiv D \pmod{4a}$.
Proof: Clearly $a = \|a\|$ is contained in $a$, and it is the smallest positive integer contained in $a$. Thus there must be another element $b + \sqrt{D}/2$ s.t. $\{a, b + \sqrt{D}/2\}$ is a basis of $a$. But then $b^2 \equiv D(\text{mod } 4a)$, and we may assume $-a < b \leq a$. And conversely, if $(b')^2 \equiv D(\text{mod } 4a)$, then it is easy to verify $\mathbb{Z}\{a, b + \sqrt{D}/2\}$ is a primitive ideal with norm $a$. Which implies that $b' \equiv b(\text{mod } 2a)$. So the solution of $b$ is unique. \qed

2 Hecke Characters

Let $K = \mathbb{Q}(\sqrt{D})$ be a imaginary quadratic field with discriminant $D < 0$.

Def. (0.1.2.1) [Places]. Let $F = \mathbb{Q}$ or $K$, a place of $F$ is a valuation on $F$: i.e. a function

$$v = |-|: F \to \mathbb{R}_{\geq 0}$$

such that

- $|0| = 0$.
- $|ab| = |a||b|$ for any $a, b \in F$.
- $|a + b| \leq |a| + |b|$ for any $a, b \in F$.

each prime ideal $p \subset \mathcal{O}_F$ defines a valuation $v_p$ on $F$ as follows: If $a \in F^\times$, let $k$ be the maximal integer s.t. $a \in p^k$, then define

$$|a|_{v_p} = \frac{1}{N(p)^k}.$$  

Remember the absolute value on $\mathbb{C}$ and $\mathbb{R}$ are also valuations.

The set of equivalence classes of valuations on $F$ is called the set of places of $F$, denoted by $\Sigma_F$.

Def. (0.1.2.2) [Completion Fields]. Let $F = \mathbb{Q}$ or $K$, for a prime ideal $p \subset \mathcal{O}_F$, the completion field $F_p$ is the field of elements of the form

$$a = a_k + a_{k+1} + \ldots + a_{k+2} + \ldots$$

where $k \in \mathbb{Z}_+$, $a_k \in p^k, a_{k+1} \in p^{k+1}, \ldots$. It is clearly an additive group and has multiplications, and is a field because we can take inversion: Suppose

$$a = a_k + a_{k+1} + \ldots$$

where $a_k \neq 0$, we can take

$$a^{-1} = \frac{1}{a_k + a_{k+1} + \ldots} = a_k^{-1} \left(1 - \frac{a_{k+1} + a_{k+2} + \ldots}{a_k} \right) - \left(\frac{a_{k+1} + a_{k+2} + \ldots}{a_k} \right)^2 + \ldots$$

For example, if $F = \mathbb{Q}$,

$$\frac{1}{1-p} = 1 + p + p^2 + \ldots, \quad \frac{1}{1+p+p^2+\ldots} = 1-p.$$

$$\frac{1}{1-p+p^2} = 1 + (p-p^2) + (p-p^2)^2 + \ldots = 1 + p - p^3 + \ldots$$
The set of elements of the form 
\[ a = a_0 + a_1 + \ldots \]
is denoted by \( \mathcal{O}_p \), or when \( F = \mathbb{Q} \), also denoted by \( \mathbb{Z}_p \).
There is a valuation on \( F_p \) given by 
\[ |a|_p = \text{Nm}(p_v)^{-k} \], where 
\[ a = a_k + a_{k+1} + \ldots \]
and \( a_k \in p^k \setminus p^{k+1} \).
For the infinite place \( \infty \), we define the completion in the same way, and get 
\[ \mathbb{Q}_\infty = \mathbb{R}, \quad \mathbb{K}_\infty = \mathbb{C} \]

**Def. (0.1.2.3) [Ideles].** Let \( F = \mathbb{Q} \) or \( \mathbb{K} \), the **Idele group** is defined to be
\[ I_F = \prod_{v \in \Sigma_F} F_v^\times. \]
where the weird \( \prod' \) means the following: an element of \( I_F \) is a tuple 
\[ a = (a_v)_{v \in \Sigma_F} \]
s.t. only finitely many \( a_v \) (including the infinite place) satisfies \( a_v \notin \mathcal{O}_v^* \).
And \( I_F \) is endowed with some topology. There is an embedding of groups 
\[ F^\times \subset I_F : a \mapsto (a, a, \ldots, a, \ldots) \]

**Def. (0.1.2.4) [Hecke Characters].** A **Hecke character** is a continuous character of \( I_F \) (i.e. a continuous group homomorphism \( I_F \rightarrow \{ z \in \mathbb{C} : |z| = 1 \} \)) that is trivial on \( F^\times \).
A strange topology is defined on \( I_F \) s.t. that each Hecke character \( \chi \) of \( I_F \) is of the form 
\[ \chi((a_v)) = \prod_{v \in \Sigma_F} \chi_v(a_v) \]
where each \( \chi_v \) is a character of \( F_v \), and only finitely many of them are non-trivial (i.e. not always take value 1).

**Prop. (0.1.2.5) [Dirichlet Characters as Hecke Characters].** We have a decomposition of groups:
\[ I_\mathbb{Q} = \mathbb{Q}^\times \times \prod_p \mathbb{Z}_p^* \times \mathbb{R}_+, \]
The proof goes as follows: Given an element 
\[ a = (a_p) \in I_\mathbb{Q}, \]
only finitely many \( a_v \notin \mathbb{Z}_p^* \), so by weak approximation, we can take some \( a \in \mathbb{Q}^\times \) s.t. \( a_p a^{-1} \in \mathbb{Z}_p^* \) for each prime \( p \). We can also guarantee that \( a_\infty a^{-1} \in \mathbb{R}_+ \) by multiplying \( -1 \) if necessary. Then 
\[ a = a \times aa^{-1}, \quad aa^{-1} \in \prod_p \mathbb{Z}_p^* \times \mathbb{R}_+. \]
For any \( m \in \mathbb{Z}_+ \), there is a natural map
\[
\prod_p \mathbb{Z}_p^* \to (\mathbb{Z}/m)^*.
\]
defined as follows: If \( m = p_1^{e_1} \cdots p_k^{e_k} \), the map is
\[
\prod_p \mathbb{Z}_p^* \to \mathbb{Z}_{p_1}^* \times \cdots \times \mathbb{Z}_{p_k}^* \to \frac{\mathbb{Z}_{p_1}}{1 + p_1^{e_1}} \times \cdots \times \frac{\mathbb{Z}_{p_k}}{1 + p_k^{e_k}} = (\mathbb{Z}/m)^*.
\]

So for any Dirichlet character \( \chi_0 \) modulo \( m \), we get a Hecke character
\[
\chi : \mathbb{I}_Q \to \prod_p \mathbb{Z}_p^* \to (\mathbb{Z}/m)^* \xrightarrow{\chi^{-1}_0} \mathbb{C}^*.
\]

In this way, \( \chi([p]_p) = \chi_0(p) \) for any \( p \in \text{Prime} \) s.t. \( (p, m) = 1 \).

Prop. (0.1.2.6) [Hecke Characters as Dirichlet Characters]. For an finite modulus \( m \in \text{Ideal}^{\times}(\mathcal{O}_F) \), we may define a Dirichlet character as a character
\[
\chi_0 : (\mathcal{O}_F/m)^* \to \mathbb{C}^*.
\]

There is an exact sequence
\[
1 \to \mathcal{O}_F^* \to \prod_p \mathcal{O}_p^* \times F_\infty^\times \to \mathcal{C}_F \to \text{Cl}(F) \to 1.
\]

For any Hecke character \( \chi \) of \( F \), \( \chi_f \) factors through \( \prod_p \mathcal{O}_p^* \to (\mathcal{O}_F/m)^* \) for some finite modulus \( m = p_1^{e_1} \cdots p_k^{e_k} \) by continuity constraint, thus inducing a Dirichlet character \( \chi_0 \) modulo \( m \), called the Dirichlet component of \( \chi \).

However, not every naive Dirichlet character \( \chi_0 \) an an infinite character \( \chi_\infty \) can form a Hecke character. For example, they must satisfy \( \chi_0(\varepsilon)\chi_\infty(\varepsilon) = 1 \) for any unit \( \varepsilon \in \mathcal{O}_F^* \). And if \( \chi_0(\varepsilon)\chi_\infty(\varepsilon) = 1 \), then there exists exactly \( \text{cl}(F) \)-many Hecke character \( \chi \) extending \( \chi_0 \) and \( \chi_\infty \), because the surjection
\[
F^\times \backslash \mathcal{I}_F / \prod_p (1 + p^{e_p}) \to \text{Cl}(F) \to 1
\]
has sections (take case of the infinite places).

Def. (0.1.2.7) [Hecke Characters and Ideal Characters]. Any Hecke character \( \chi \) of \( F \) is of the form \( \chi = \chi_f \times \chi_\infty \), where \( \chi_f : \mathcal{I}_{F,f} \to \mathbb{C} \) and \( \chi_\infty : (F \otimes \mathbb{R})^\times \to \mathbb{C} \). Then by continuity, there exists a smallest modulus \( m = \prod_{v \in \Sigma_{F^+}} p_v^{e_v} \) s.t. \( \chi_f \) is trivial on
\[
\prod_{v \in \Sigma_{F^+}} (1 + p_v^{e_v}).
\]
Thus the restriction of \( \chi \) to \( \Gamma^{\infty} \) gives us a ideal class map \( J^m \to \mathbb{C} \) which we also denote by \( \chi \). And the restriction of \( \chi \) to \( \prod_{v \in \Sigma^{\infty}_{F^+}} \mathcal{O}_v^* \) gives a Dirichlet character \( \chi_0^{-1} : (\mathcal{O}_F/m)^* \to \mathbb{C} \).

Then we have \( \chi((\alpha))\chi_\infty(\alpha) = \chi_0(\alpha) \) for any \( \alpha \in \mathcal{O} \) that is prime to \( m \).
3 Hecke $L$-Functions

Def. (0.1.3.1)[Gauss Sum for Hecke Characters]. For $F \in \mathbb{N}Field$ and a Hecke character $\chi$ for $F$ with Dirichlet component $\chi_0$ of conductor $m(0.1.2.6)$, the Gauss sum of $\chi$ is defined to be

$$\tau(\chi) = \frac{1}{\chi_\infty(\gamma)\chi(c)} \sum_{\alpha \in c/m} \chi_0(\alpha)e^{2\pi \text{tr}_F/Q(\alpha/\gamma)} (0.1.2.6),$$

where $\gamma \in \mathcal{O}_K^\times$ and $c \in \text{Ideal}^\times(\mathcal{O}_F)$ are chosen s.t. $(c, m) = 1$ and $c\mathcal{O}_F m = (\gamma)$. Where

$$\mathcal{O}_F^{-1} = \{\alpha \in F^\times : \text{tr}_F/Q(\alpha \mathcal{O}_F) \in \mathbb{Z}\}.$$

The definition of $\tau(\chi)$ is independent of $(c, \gamma)$ chosen.

In particular, if $F = \mathbb{Q}$ and $m = (N)$ for $N \in \mathbb{Z}_+$, then

$$\tau_0(\chi) \triangleq \sum_{n \in \mathbb{Z}/(N)} \chi(n)e^{2\pi \text{in}N}, \quad \tau(\chi) \triangleq \tau_1(\chi).$$

In particular, $\tau(1_p)$ is called the Ramanujan sum, and by Möbius inversion on $N$, we have

$$\text{Kloos}(a, 0, \text{mod } N) = \sum_{r \in \mathbb{Z}/(N)^*} e^{2\pi i \frac{anr}{N}} = \sum_{d|\text{gcd}(N, n)} \mu\left(\frac{N}{(N, n)}\right) \frac{\phi(N)}{\phi(N/(N, n))}. \quad \blacksquare$$

Prop. (0.1.3.2). For $N_1, N_2 \in \mathbb{Z}_+ \text{ s.t. } (N_1, N_2) = 1$ and $\chi_i \in \text{Diri}(N_i)$, we have

$$\tau(\chi_1 \chi_2) = \chi_1(N_2) \chi_2(N_1) \tau(\chi_1) \tau(\chi_2).$$

This also holds for Hecke characters. \quad \blacksquare

Proof:

$$\tau(\chi_1 \chi_2) = \sum_{n_1 \equiv a \pmod{N_1}} \sum_{n_2 \equiv b \pmod{N_2}} \chi_1(N_2 n_1) \chi_2(N_1 n_2) e^{2\pi i \frac{N_1 a n_1 + N_2 b n_2}{N_1 N_2}} = \chi_1(N_2) \chi_2(N_1) \tau(\chi_1) \tau(\chi_2).$$

\hfill \square

Def. (0.1.3.3)[Weber-Hecke $L$-Functions]. For a Hecke character $\chi$ of $\mathcal{K}$ of conductor $m$, define

$$L_{\mathcal{K}}(\chi; s) = \prod_{p|m} \frac{1}{1 - \chi(p) \|p\|^{-s}} = \sum_{a \in \text{Ideal}^\times(\mathcal{O}_F), (a, m) = 1} \chi(a) \|a\|^s.$$

Then for $\text{Re}(s)$ sufficiently large, this converges absolutely. \quad \blacksquare

Thm. (0.1.3.4)[Hecke $L$-Functions of Imaginary Quadratic Fields]. Let $m$ be a modulus for $\mathcal{K}$, and $\chi$ a Hecke character of $\mathcal{K}$ with $\chi_0$ of conductor $m(0.1.2.6)$ and associated ideal class map $\psi : J^m \to \mathbb{C}^\times$. Suppose $\chi_\infty(z) = e^{k \arg z}$, then

$$\Lambda(\chi; s) = L_C(s + \frac{|k|}{2}) L(\chi; s)$$

and for $s = |k|/2$ and $\chi = \chi_0$, we have $\Lambda(\chi_0; s) = L_C(s + |k|/2)$.
is the Hecke $L$-function attached to $\chi$. Thus $L(\chi; s)$ can be extended to a meromorphic function for all $s \in \mathbb{C}$, and it has simple poles at $s = 1$ when $\chi = 1$, and analytic otherwise. When $\chi = 1$, $L(1; s) = \zeta_K(s) = \zeta(s)L(\chi_K, s)$, and

$$\text{res}_{s=1} \zeta_K(s) = \frac{2 \text{cl}(K)}{\# K^\times_{\text{tor}} \sqrt{\text{disc}(K)\pi}}.$$  

Moreover, there are functional equations

$$\Lambda(\chi; s) = i^{-\kappa} \frac{\tau(\chi)}{\|m\|^{1/2}} (|\text{disc}(K)|\cdot\|m\|)^{1/2-s} \Lambda(\overline{\chi}; 1 - s)(0.1.3.1).$$