Contents

Quick recap

Rough idea of [HT]:

unitary group G + moduli problem involving $G \Rightarrow$ Shimura variety X

$H^i(X, \mathcal{L}_\xi)$ is a $G(\mathbb{A}^\infty) \times \mathrm{Gal}(F^{ac}/F)$-module and

$$H^i(X, \mathcal{L}_\xi) = \oplus_\pi (\pi \otimes R^i_\xi(\pi)),$$

where the sum is over irreducible admissible representations π of $G(\mathbb{A}^\infty)$ and R^i_ξ is a finite dimensional continuous representation of $\mathrm{Gal}(F^{ac}/F)$.

$$R_\xi(\pi) = (-1)^{n-1} \sum_i (-1)^i [R^i_\xi(\pi)].$$

Relate $[R_\xi(\pi)]$ to other things.

1 The unitary group G

- E - imaginary quadratic field in which p splits
- u - a prime of E above p
- c - complex conjugation in $\mathrm{Gal}(E/\mathbb{Q})$
- $F^+/\mathbb{Q}$ - a totally real field of degree d
- $F = EF^+$
- $w = w_1, \cdots, w_r$ places of F above u
- $\mathbb{A}^\infty = \prod_{q \nmid \infty} \mathbb{Q}_q$

Let B be a division algebra over F of dimension n^2 such that

- F is the center of B
- $B^{\mathrm{op}} \cong B \otimes_{E,c} E$
- B is split at w, i.e., $B_w \cong M_n(F_w)$ for one w or for all $w_1, \cdots, w_r$?
- at any place x of F which is not split over F^+, B_x is split
- at any place x of F which is split over F^+, B_x is either split or a division algebra
- if n is even then $1 + dn/2$ is congruent mod 2 to the number of places of F^+ above which B is ramified, i.e. not split. This condition is used in the proof of Lemma I.7.1

where are these conditions used? Pick a positive involution of second kind $*$ on B (i.e. $*|_F = c$ and $\mathrm{tr}_{B/\mathbb{Q}}(xx^*) > 0$ for all $0 \neq x \in B$).

Let V denote the $B \otimes_F B^{\mathrm{op}}$ module B. We want to look at $*$-hermitian alternating pairings $V \times V \to \mathbb{Q}$ (resp. $(V \otimes \mathbb{A}^\infty) \times (V \otimes \mathbb{A}^\infty) \to \mathbb{A}^\infty$) (i.e., $\langle bx, y \rangle = \langle x, b^* y \rangle$ for all $b \in B$). Such pairings are of the form

$$\langle x, y \rangle_\beta = \mathrm{tr}_{B/\mathbb{Q}}(x\beta y^*)$$

for some $\beta \in B^{*=-1}$ (resp. $B^{*=-1} \otimes A^\infty$). Define the action $\#_\beta$ to be $x^{\#_\beta} = \beta x^* \beta^{-1}$, then $\#_\beta$ is an involution of the second kind on B (resp. $B \otimes \mathbb{A}^\infty$). We have

$$\langle (b_1 \otimes b_2) x_1, x_2 \rangle_\beta = \langle x_1, (b_1^* \otimes b_2^{\#_\beta}) x_2 \rangle_\beta.$$

Let $G_\beta/\mathbb{Q}$ (resp. $G_\beta/\mathbb{A}^\infty$) be the algebraic group (general unitary group) whose R-points are

$$G_\beta(R) = \{(\lambda, g) \in R^\times \times (B^{\mathrm{op}} \otimes_{\mathbb{Q}} R)^\times | g g^{\#_\beta} = \lambda\}.$$

Let $G_{\beta,1}$ (unitary group) be the kernel of the map $G_\beta \to \mathbb{G}_m$ given by $(\lambda, g) \to \lambda$. The structure map $G_{\beta,1} \to \mathrm{Spec}\ \mathbb{Q}$ (resp. $\mathbb{A}^\infty$) factors through $\mathrm{Spec}\ F^+$ (resp. $\mathrm{Spec}\ (F^+ \otimes \mathbb{A}^\infty)$), so $G_{\beta,1}$ can be seen as an algebraic group over F^+. Action of $\#_\beta$ on B^{op}? Also why use B^{op} instead of B?

Lemma 1. *For any embedding $\tau : F^+ \hookrightarrow \mathbb{R}$ there exists $0 \neq \beta \in B^{*=-1}$ such that*

1. *if x is a rational prime not split in E, then $G_{\beta,1}$ and G_β are quasi-split at x,*
2. *and the pairing $\langle , \rangle_\beta$ on $V \otimes_{\mathbb{Q}} \mathbb{R}$ has invariants $(1, n-1)$ at τ and $(0, n)$ at any embedding $\tau' \neq \tau$.*

Proof. Parametrize pairings by some H^1, then use some local-global exact sequence of cohomologies. □

Fix some choice of τ, we choose some β as in the lemma and write $\langle , \rangle, \#, G, G_1$ for the corresponding objects arising from β. By part (2) of the lemma, the pairing $\langle , \rangle$ has a well-defined extension

$$\langle , \rangle : (V \otimes \mathbb{A}) \times (V \otimes \mathbb{A}) \to \mathbb{A}$$

with invariants $(1, n-1)$ at τ and $(0, n)$ at all other infinite places. Thus we get an involution $\#_\tau$ on $B^{\mathrm{op}} \otimes \mathbb{A}$ and groups $G_\tau, G_{\tau,1}$ over $\mathbb{A}$. Up to equivalence, the involution $\#_\tau$ and groups G_τ, G_{τ_1} only depends on τ and not β.

Some properties of G_τ that will be used later. For an E-algebra R,

$$G_\tau(R) \cong \{(g_1, g_2) \in (B^{\mathrm{op}} \otimes_E R) \times (B^{\mathrm{op}} \otimes_{E,c} R) | g_1 g_2^{\#_\tau}, g_2 g_1^{\#_\tau} \in R^\times\}.$$

From this we have

$$\begin{aligned} G_\tau(R) &\cong (B^{\mathrm{op}} \otimes_E R)^\times \times R^\times \\ (g_1, g_2) &\mapsto (g_1, g_1 g_2^{\#_\tau}) \\ (g, \nu g^{-\#_\tau}) &\leftarrow\!\shortmid (g, \nu). \end{aligned}$$

When a place x of $\mathbb{Q}$ splits into $x = yy^c$ in E, we can identify $\mathbb{Q}_x$ and E_y as E-algebras and identify

$$G(\mathbb{Q}_x) = (B_y^{\mathrm{op}})^\times \times \mathbb{Q}_x^\times.$$

2 Jacquet-Langlands correspondence

Let K be a p-adic field. Let $g \in \mathbb{Z}^+$ and $D_{K,g}$ be a division algebra with center K and rank g^2 over K.

Rogawski, Deligne–Kazhdan–Vigneras showed that there exists a unique bijection

JL: {irreducible admissible representations of $D_{K,g}^{\times}$} $\to$ {square integrable irreducible admissible representations of $\mathrm{GL}_g(K)$} that satisfy certain properties of characters.

For a supercuspidal representation π of $\mathrm{GL}_m(K)$, we denote by $Q(\pi, s)$ the unique irreducible quotient of n-$\mathrm{Ind}(\pi \times \cdots \times \pi \otimes |\det|^{s-1})$. cf. [Ze]p197 Denote by $Z(\pi, s)$ the unique irreducible subrepresentation of n-$\mathrm{Ind}(\pi \times \cdots \times \pi \otimes |\det|^{s-1})$ cf. [Ze]p180.

$Q(\pi, s)$ (resp. $Z(\pi, s)$) is $Sp_s(\pi)$ (resp. $\pi \boxplus \cdots \boxplus (\pi \otimes |\det|^{s-1})$) in the notation of [HT]

Let $S(B)$ be the set of places of F at which B ramifies (which we assumed to be division algebra).

Theorem 2. *(a) If ρ is an irreducible automorphic representation of $(B^{op} \otimes \mathbb{A})^{\times}$, then there exists a unique irreducible automorphic representation $JL(\rho)$ of $GL_n(\mathbb{A}_F)$, which occurs in the discrete spectrum and for which*

$$JL(\rho)^{S(B)} \cong \rho^{S(B)}.$$

(b) If $x \in S(B)$ and $JL(\rho_x) = Q(\pi_x, s_x)$, then

- *either $JL(\rho)_x \cong Q(\pi_x, s_x)$*
- *or $JL(\rho)_x \cong Z(\pi_x, s_x)$*

The image of JL is the set of irreducible automorphic representations π of $GL_n(\mathbb{A}_F)$ such that

- *π occurs in the discrete spectrum*
- *and for every $x \in S(B)$ there exist $s_x|n$ and an irreducible supercuspidal representations π'_x of $GL_{n/s_x}(F_x)$ such that $\pi_x \cong Sp_{s_x}(\pi'_x)$ or $\pi_x \cong \pi'_x \boxplus \cdots \boxplus (\pi'_x \otimes |\det|^{s_x-1})$*

If ρ_1 and ρ_2 are two irreducible automorphic representations of $(B^{op} \otimes \mathbb{A})^{\times}$ such that $\rho_{1x} = \rho_{2x}$ for all but finitely many places x of F, then $\rho_1 = \rho_2$. (strong multiplicity one)

Corollary 3. *Suppose that ρ is an irreducible automorphic representation of $(B^{op} \otimes \mathbb{A})^{\times}$. Then the following are equivalent:*

1. *$JL(\rho)$ is cuspidal.*
2. *For one place $x \notin S(B)$ the component ρ_x is generic.*
3. *For all places $x \notin S(B)$ the component ρ_x is generic.*

3 Clozel's base change

Goal: give a surjective (Theorem 8) map (Theorem 5)

$$\left\{\begin{array}{l}\text{irreducible automorphic representations}\\ \pi \text{ of } G_\tau(\mathbb{A}) \text{ such that } \pi_\infty \text{ is cohomological}\\ \text{for } \xi'\end{array}\right\} \xrightarrow{BC} \left\{\begin{array}{l}(\Pi, \psi),\ \Pi \text{ irreducible automorphic representation}\\ \text{of } (B^{\mathrm{op}} \otimes \mathbb{A})^{\times},\ \psi \text{ character of } \mathbb{A}_E^{\times}/E^{\times},\\ \text{satisfying conditions (4)-(7) in Theorem 5}\end{array}\right\}$$

that is compatible with local base change.

Some notations:

- Fix an embedding $\tau : F \to \mathbb{C}$.
- ξ - an irreducible representation of G on a $\mathbb{Q}_l^{ac}$-vector space ($l \neq p$).
- Fix an embedding $\iota : \mathbb{Q}_l^{ac} \to \mathbb{C}$. Let $\xi' = \iota(\xi)$, then ξ' is an irreducible algebraic representation of G_τ over $\mathbb{C}$.
- For a representation π, let ψ_π denote its central character.

Note that

$$\mathrm{Res}^E_{\mathbb{Q}}(G_\tau \times E) \times \mathbb{C} \cong (G_\tau \times \mathbb{C}) \times_{\mathbb{C}} (G_\tau \times \mathbb{C}),$$

where the first factor corresponds to $\tau : E \hookrightarrow \mathbb{C}$ and the second to $\tau \circ c$. Let ξ'_E denote the representation $\xi' \otimes \xi'$ of $\mathrm{Res}^E_{\mathbb{Q}}(G_\tau \times E)$ over $\mathbb{C}$. Note that $G_\tau(E_\infty) = \mathrm{Res}^E_{\mathbb{Q}}(G_\tau \times E)(\mathbb{R})$. We will denote also by ξ'_E the restriction of the representation to $\mathrm{GL}_n(F_\infty) \subset E^\times_\infty \times \mathrm{GL}_n(F_\infty) \cong G_\tau(E_\infty)$. I don't understand details in this paragraph

Definition 4. *We say an irreducible admissible representation π_∞ of $G_\tau(\mathbb{R})$ (resp. Π_∞ of $GL_n(F_\infty)$) is cohomological for ξ' (resp. ξ'_E) if for some i,*

$$H^i((Lie\ G_\tau(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}, U_\tau, \pi_\infty \otimes \xi') \neq 0$$

(resp.

$$H^i(M_n(F_\infty) \otimes_{\mathbb{R}} \mathbb{C}, U(0,n)^{[F^+:\mathbb{Q}]}, \Pi_\infty \otimes \xi'_E) \neq 0.)$$

What are U_τ and $U(0,n)^{[F^+:\mathbb{Q}]}$?

Let x be a place of $\mathbb{Q}$. Next we define (under some conditions) the local base change from representations of $G(\mathbb{Q}_x)$ to representations of $G(E_x)$.

First suppose x **splits** into $x = yy^c$ in E. Recall that we have $G(\mathbb{Q}_x) = (B_y^{\mathrm{op}})^\times \times \mathbb{Q}_x^\times$. So we can decompose irreducible admissible representations π of $G(\mathbb{Q}_x)$ into

$$\pi \cong \pi_y \otimes \psi_{\pi, y^c}.$$

Replacing y by y^c, we get $\pi_{y^c} = \pi_y^\#$ and $\psi_{\pi,y} = \psi_{\pi_y}\psi_{\pi,y^c}$, where $\pi_y^\#$ is defined by $\pi_y^\#(g) = \pi_y(g^{-\#})$. what does $-\#$ mean? We define $BC(\pi)$ to be the representation

$$\pi_y \otimes \pi_{y^c} \otimes (\psi_{\pi,y^c} \circ c) \otimes (\psi_{\pi,y} \circ c)$$

of

$$G(E_\tau) \cong (B_x^{\mathrm{op}})^\times \times E_x^\times \cong (B_y^{\mathrm{op}})^\times \times (B_{y^c}^{\mathrm{op}})^\times \times E_y^\times \times E_{y^c}^\times.$$

Now suppose that x is **inert** in E and

- x is unramified in F,
- $(B_x^{\mathrm{op}}, \#) \cong (M_n(F_x), \dagger)$, where $g^\dagger = w(g^c)^t w^{-1}$ with $w = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$.

These two conditions only exclude finitely many places of $\mathbb{Q}$. Then $G(\mathbb{Q}_x)$ is quasi-split and split over an unramified extension. Let B_x be a Borel subgroup of $G \times F_x$ so that $B_x(F_x)$ corresponds to the set of upper triangular matrices in $M_n(F_x)$. Let T_x be a maximal torus in B_x such that $T_x(F_x)$ corresponds to the set of diagonal elements in $M_n(F_x)$. Then why?

$$T_x(\mathbb{Q}_x) \cong \{(d_0; d_1, \cdots, d_n)\} \in \mathbb{Q}_x^\times \times (F_x^\times)^n | d_0 = d_i d_{n+1-i^c}, \forall i = 1, \cdots, n.$$

For a character ψ of $T_x(\mathbb{Q}_x)$ we define a character $BC(\psi)$ of $E_x^\times \times (F_x^\times)^n$ by

$$BC(\psi)(d_0; d_1, \cdots, d_n) = \psi(d_0 d_0^c, d_0 d_1/d_n^c, \cdots, d_0 d_n/d_1^c).$$

Let B denote the Borel subgroup of upper triangular elements of GL_n. If π is an unramified representation of $G(\mathbb{Q}_x)$ which is a subquotient of n-$\mathrm{Ind}_{B_x(\mathbb{Q}_x)}^{G(\mathbb{Q}_x)} \psi$, we define $BC(\pi)$ to be the unique unramified subquotient of n-$\mathrm{Ind}_{E_x^\times \times B(F_x)}^{E_x^\times \times \mathrm{GL}_n(F_x)} BC(\psi)$.

If Π is an irreducible automorphic representation of $(B^{\mathrm{op}} \otimes_{\mathbb{Q}} \mathbb{A})^\times$, we define $\Pi^\#$ by $\Pi^\#(g) = \Pi(g^{-\#})$. Strong multiplicity one implies that

$$JL(\Pi^\#) = JL(\Pi)^\vee \circ c.$$

Theorem 5. *Suppose that π is an irreducible automorphic representation of $G_\tau(\mathbb{A})$ such that π_∞ is cohomological for ξ'. Then there is a unique irreducible automorphic representation $BC(\pi) = (\psi, \Pi)$ of $\mathbb{A}_E^\times \times (B^{op} \otimes_{\mathbb{Q}} \mathbb{A})^\times$ such that*

1. *$\psi = \psi_\pi|_{\mathbb{A}_E^\times}^c$, how is $\mathbb{A}_E^\times$ contained in the center of $G_\tau(\mathbb{A})$?*
2. *if x is a place of $\mathbb{Q}$ that splits in E then $BC(\pi)_x = BC(\pi_x)$,*
3. *for almost all places x of $\mathbb{Q}$ (which are inert in E) we have $BC(\pi)_x = BC(\pi_x)$,*
4. *Π_∞ is cohomological for ξ'_E,*
5. *$\psi_\infty^c = \xi'|_{E_\infty^\times}^{-1}$ (where $E_\infty^\times \subset G_\tau(\mathbb{R})$),*
6. *$\psi_\Pi|_{\mathbb{A}_E^\times} = \psi^c/\psi$,*
7. *$\Pi^\# \cong \Pi$.*

I'm not sure if the cohomological condition parts of the theorem below is correct.

Theorem 6 (Clozel-Labesse, 1999)**.** *Let π be an automorphic representation of $G_{\tau,1}(\mathbb{A})$ such that π is cohomological for ξ' and one of the following conditions hold:*

- *at a finite place v of F, π_v is the Steinberg representation*
- *at a place v of F, U_v is obtained from a division algebra.*

Then there exists an automorphic representation Π of $(B^{op} \otimes \mathbb{A})^\times$ such that $\Pi_v = BC(\pi_v)$ almost everywhere and Π is cohomological for ξ'_E.

Proof idea. Comparison of trace formulas. □

Proof sketch of Theorem 5. Let $T = \mathrm{Res}_{\mathbb{Q}}^E \mathbb{G}_m$ and $T^1 = \ker T \xrightarrow{N} \mathbb{G}_m$. We want to construct an irreducible automorphic representation π' of $(T \times G_{\tau,1})(\mathbb{A})$ such that

- $\pi'|_{T^1(\mathbb{A})} = 1$,
- if x is a place of $\mathbb{Q}$ that splits in E, then $\pi'_x = \pi_x|_{(T\times G_1)(\mathbb{Q}_x)}$,
- for almost all places x of $\mathbb{Q}$ that are inert in E, π'_x is the unique unramified subquotient of $\pi_x|_{(T\times G_1)(\mathbb{Q}_x)}$,
- $\pi'_\infty|_{E_\infty^\times} = \xi'|_{E_\infty^\times}^{-1}$,
- and for some i we have

$$H^i(\text{Lie } G_{\tau,1}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}, U_\infty, (\pi'_\infty \otimes \xi')|_{G_{\tau,1}(\mathbb{R})}) \neq 0.$$

Such π' is of the form $\psi^c \otimes \pi'_1$ for some character ψ of $E^\times \backslash \mathbb{A}_E^\times$ and automorphic representation π'_1 of $G_{\tau,1}(\mathbb{A})$ such that $\psi|_{T^1(\mathbb{A})} = \psi_{\pi'_1}|_{T^1(\mathbb{A})}^{-1}$. Apply Theorem 6 to π'_1, we obtain a representation Π of $(B^{\text{op}} \otimes \mathbb{A})^\times$ such that $(\psi_\pi|_{\mathbb{A}_E^\times}, \Pi)$ satisfies properties (1)-(6) in the theorem. Uniqueness and property (7) follows from strong multiplicity one (Theorem 6).

We construct the π' as follows. Note that there is a natural exact sequence

$$0 \to T^1 \to T \times G_{\tau,1} \to G_\tau$$
$$t \mapsto (t, t^{-1}),$$

where the last map is surjective on geometric points. If π is an automorphic representation of $G_\tau(\mathbb{A})$, its "restriction" to $(T \times G_{\tau,1})(\mathbb{A})$ is a semisimple admissible representation. then we want to make local components irreducible and automorphic

When x is a place of $\mathbb{Q}$ that splits in E, we have an exact sequence

$$0 \to T^1(\mathbb{Q}_x) \to (T \times G_1)(\mathbb{Q}_x) \to G(\mathbb{Q}_x) \to 0.$$

If π_x is an irreducible admissible representation of $G(\mathbb{Q}_x)$, then the restriction $\pi_x|_{(T\times G_1)(\mathbb{Q}_x)}$ is also irreducible. (With the embedding $T^1 \to T \times G_{\tau,1}$ above, the restriction of π_x to $T^1(\mathbb{Q}_x)$ is trivial. So we can reconstruct subrepresentations of π_x from subrepresentations of $\pi_x|_{(T\times G_1)(\mathbb{Q}_x)}$).

Other cases can be done using similar exact sequences. (Conjugating irreducible subquotients of $\pi_x|_{(T\times G_1)(\mathbb{Q}_x)}$ by appropriate elements in $G_\tau(\mathbb{A})$ can make them into automorphic representations.) □

Corollary 7. *If π and π' are irreducible automorphic representations of $G_\tau(\mathbb{A})$ such that π_∞ and π'_∞ are cohomological for ξ' and such that $\pi_x \cong \pi'_x$ for almost all places x of $\mathbb{Q}$, then $\pi_x \cong \pi'_x$ for all places x of $\mathbb{Q}$ which split in E.*

Proof. Corollary 3 + Theorem 5. □

Theorem 8. *Suppose that Π is an irreducible automorphic representation of $(B^{op} \otimes \mathbb{A})^\times$ and ψ is a character of $\mathbb{A}_E^\times / E^\times$ satisfying conditions (4)-(7) in Theorem 5. (all ψ appearing in the image of BC are invariant under $E^\times$ by the proof of Theorem 5) Then there is an irreducible automorphic representation π of $G_\tau(\mathbb{A})$ such that*

1. *$BC(\pi) = (\psi, \Pi)$,*
2. *and π_∞ is cohomological for ξ'.*

Moreover, $\dim[R_{\iota^{-1}(\xi')}(\iota^{-1}\pi^{\infty})] \neq 0$.

Proof sketch. Clozel: there exists a representation π_1 of $G_{\tau,1}(\mathbb{A})$ that is cohomological and compatible with local base change. Then $\psi^c \otimes \pi_1$ is an irreducible automorphic representation of $(T \times G_{\tau,1})(\mathbb{A})$ which is trivial on $T^1(\mathbb{A})$. Thus $\psi^c \times \pi_1$ is a subrepresentation of the restriction of some automorphic representation π of $G_\tau(\mathbb{A})$ to $(T \times G_{\tau,1})(\mathbb{A})$. This π satisfies the required conditions. $\square$