FARGUES–SCHOLZE PARAMETERS AND TORSION VANISHING FOR SPECIAL ORTHOGONAL AND UNITARY GROUPS

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ABSTRACT. We show that if p is an odd prime, K is an unramified finite extension of $\mathbb{Q}_p$ and G is a special orthogonal group or a unitary group over K that splits over an unramified extension, then the Fargues–Scholze local Langlands correspondence for G agrees with the semi-simplification of the classical local Langlands correspondence for G developed through the work of Arthur and others. As applications, we construct an unambiguous local Langlands correspondence for even special orthogonal groups, deduce the eigensheaf conjecture of Fargues, and establish new torsion vanishing results for orthogonal and unitary Shimura varieties.

CONTENTS

1. INTRODUCTION

For a connected reductive group G over a finite extension K of $\mathbb{Q}_p$ for some rational prime p, the conjectural local Langlands correspondence is a map from the set $\Pi(G)$ of isomorphism classes of irreducible admissible representations of $G(K)$ to the set $\Phi(G)$ of conjugacy classes of L-parameters

$$\phi : W_K \times \mathrm{SL}(2, \mathbb{C}) \to {}^L G, {}^{1}$$

which should have finite fibers called L-packets and satisfy various properties; see [Bor79]. When G is a special orthogonal group or a unitary group over K, such a map is constructed by Arthur [Art13], Chen–Zou [CZ21], and Ishimoto [Ish24] in the special orthogonal case, and by Mok [Mok15] and Kaletha–Minguez–Shin–White [KMSW14] in the unitary case. Note that when G is a special orthogonal group associated to an quadratic space of even dimension $2n$, the correspondence is only well-defined up to $\mathrm{O}(2n, \mathbb{C})$-conjugation. Also, when $G = \mathrm{GSpin}(V)$ where V is a quadratic space over K, the map is constructed for representations π whose central character is the square of another character, by [GT19, Theorem 2.6.1]. These constructions ultimately rely on the theory of endoscopy and trace formula techniques. We denote the map by

$$\mathrm{rec}_G : \Pi(G) \to \Phi(G).$$

On the other hand, for any rational prime ℓ different from p with a fixed isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$, using excursion operators on the moduli stack of G-bundles on the Fargues–Fontaine curve, Fargues and Scholze [FS24] have constructed a candidate for semisimplified version of the local Langlands correspondence for all connected reductive groups G, i.e., they constructed a map

$$\mathrm{rec}_G^{\mathrm{FS}} : \Pi(G) \to \Phi^{\mathrm{ss}}(G) : \pi \mapsto \iota_\ell^{-1} \phi_{\iota_\ell \pi}^{\mathrm{FS}},$$

where $\Phi^{\mathrm{ss}}(G)$ is the set of conjugacy classes of continuous semisimple maps

$$\phi : W_K \to {}^L G$$

commuting with the projection ${}^L G \to W_K$. Moreover, $\mathrm{rec}_G^{\mathrm{FS}}$ satisfies some desired properties listed in [FS24, Theorem 1.9.6]. It is known that $\mathrm{rec}_G^{\mathrm{FS}}$ is independent of ℓ chosen; see [Sch25].

It is both natural and nontrivial to ask whether rec_G and $\mathrm{rec}_G^{\mathrm{FS}}$ are compatible when they both exist, in the sense that there exists a commutative diagram

(1.1)
$$\begin{array}{ccc} \Pi(G) & \xrightarrow{\mathrm{rec}_G} & \Phi(G) \\ & {}_{\mathrm{rec}_G^{\mathrm{FS}}} \searrow & \downarrow {}_{(-)^{\mathrm{ss}}} \\ & & \Phi^{\mathrm{ss}}(G) \end{array},$$

where $(-)^{\mathrm{ss}}$ precomposes a parameter $\phi \in \Phi(G)$ with the map

$$W_K \to W_K \times \mathrm{SL}(2, \mathbb{C}) : g \mapsto \left(g, \begin{bmatrix} |g|_K^{1/2} & 0 \\ 0 & |g|_K^{-1/2} \end{bmatrix} \right).$$

Here $|-|_K$ is defined to be the composition $W_K \to W_K^{\mathrm{ab}} \xrightarrow{\mathrm{Art}_K^{-1}} K^\times \xrightarrow{|-|_K} \mathbb{R}_+$.

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[1]Here and throughout the paper we take the Langlands L-group ${}^L G$ in the Weil form.

Our main result is the following theorem on compatibility of Fargues–Scholze local Langlands correspondence with "classical local Langlands correspondence" defined in [Art13], [Mok15], [KMSW14], [CZ21], [CZ21b], and [Ish24]:

Theorem A. *Suppose $p > 2$ and $K/\mathbb{Q}_p$ is unramified.*

  (1) If $G = \mathrm{U}(V)$ where V is a Hermitian space with respect to the unramified quadratic extension K_1/K, then the diagram (1.1) is commutative.

  (2) If $G = \mathrm{SO}(V)$ where V is a quadratic space over K with $\dim(V) = 2n + 1$ for some positive integer n, then the diagram (1.1) is commutative.

  (3) If $G = \mathrm{SO}(V)$ where V is a quadratic space over K of dimension $2n$ for some positive integer n such that G splits over an unramified quadratic extension of K (equivalently, $\mathrm{ord}_K(\mathrm{disc}(V)) \equiv 0 (\mathrm{mod}\ 2)$; see §2.1), then the diagram (1.1) is commutative up to conjugation by $\mathrm{O}(2n, \mathbb{C})$.

Remark.

  (1) When $G = \mathrm{U}(V)$ or $\mathrm{GU}(V)$ where V is an odd dimensional Hermitian space with respect to the unramified quadratic extension $\mathbb{Q}_{p^2}/\mathbb{Q}_p$, the compatibility is established in [MHN24, Theorem 1.1], and their proof is different from ours. In fact, they established the Kottwitz conjecture first by proving a Shin's averaging formula for $\mathrm{GU}(V)$, and they restricted to the case $K = \mathbb{Q}_p$ because the Hasse principle holds for unitary similitude groups over $\mathbb{Q}$.

  (2) The assumptions that $p > 2$ and that $K_1/\mathbb{Q}_p$ is unramified, as well as the assumption that
  $$\mathrm{ord}_K(\mathrm{disc}(V)) \equiv 0(\mathrm{mod}\ 2)$$
  in the third case are necessary in order to apply Shen's result [She20] that the relevant local shtuka spaces uniformize Shimura varieties of Abelian type. Both assumptions can be lifted for such G once the main result of [She20] is established for a Shimura datum $(\mathbb{G}, \{\mu\})$ such that $\mathbb{G} \otimes \mathbb{Q}_p \cong \mathrm{Res}_{K/\mathbb{Q}_p} G$.

Theorem A is proved in §6. Moreover, in the third case of Theorem A (i.e., special even orthogonal groups), we use the compatibility property to construct an unambiguous version of the local Langlands correspondence for G, eliminating the ambiguity up to outer automorphisms by requiring compatibility with the Fargues–Scholze local Langlands correspondence, which is defined canonically without outer automorphisms.

Theorem B. *In the third case of* Theorem A *(i.e., special even orthogonal groups), there exists a map*
$$\mathrm{rec}_G^\flat : \Pi(G) \to \Phi(G^*)$$
lifting the correspondence defined in Arthur [Art13] and Chen–Zou [CZ21]. As usual, this $\mathrm{rec}_G^\flat$ matches discreteness and temperedness properties on both sides, has fine characterization of the fibers (i.e., the L-packets), and satisfies compatibility with Langlands quotients, local intertwining relations, and more. It is also compatible with the Fargues–Scholze local Langlands correspondence in the sense that the diagram (1.1) is commutative. In particular, Vogan's version of the local Langlands conjecture [Vog93] holds for unramified special even orthogonal groups.

Theorem B is proved in Theorem 7.1.1.

Using the unambiguous local Langlands correspondence, we verify in §7.2 the naturality property of the Fargues–Scholze local Langlands correspondence for those G appearing in Theorem A, therefore confirming [Ham22, Assumption 7.5]. We also establish a weaker result for a central extension of $\mathrm{Res}_{K/\mathbb{Q}_p} G$, which will be used to deduce a torsion vanishing result for suitable Shimura varieties.

We next show that the classical Langlands correspondence, together with geometric techniques, provides sufficient input to verify part of the categorical local Langlands conjecture of Fargues–Scholze [FS24, Conjecture X.1.4].

Theorem C. *Suppose $p > 2$, $K/\mathbb{Q}_p$ is unramified, and $\phi \in \Phi(G^*)$ is supercuspidal.*

  (1) The sheaf
  $$\mathcal{G}_\phi = \prod_{b \in B(G)_{\mathrm{bas}}} \bigoplus_{\pi_b \in \Pi_\phi(G_b)} i_{b!}(\pi_b) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{G^*}, \overline{\mathbb{Q}_\ell})$$
  admits an action of $\mathfrak{S}_\phi := Z_{\widehat{G}}(\phi)$ satisfying conditions (i)(iv) of Fargues' conjecture [Far16, Conjecture 4.4] for G^*. In particular, $\mathcal{G}_\phi$ is a Hecke eigensheaf for ϕ.

(2) *The strong Kottwitz conjecture* [HKW22, Conjecture 1.0.1] *holds for G and any conjugacy class of geometric cocharacters $\{\mu\}$ for $G^*_{\overline{K}}$, up to a reparametrization of elements of the L-packet of ϕ that is independent of the choice of $\{\mu\}$.*

Remark. The difference between the original Kottwitz conjecture and Theorem C is that we do not establish whether the reparametrization of the elements of the L-packet of ϕ by irreducible representations of $Z_{\widehat{G}}(\phi)$ coincides with that given by the theory of endoscopy. When ϕ is simple (as defined in §2.2), no nontrivial reparametrizations arises. In general, verifying the coincidence of these two parametrizations requires a more refined analysis of the cohomology of global Shimura varieties. More specifically, when ϕ is not simple, it must be globalized to an endoscopic global parameter rather than a stable one. For instance, this has been established when G is an unramified odd unitary group over $\mathbb{Q}_p$, see [MHN24].

Theorem C is proved in Theorem 7.3.4 and Theorem 7.3.6.

1.1. **Torsion vanishing for special orthogonal and unitary Shimura varieties.** We use the compatibility result to establish new torsion vanishing results for Shimura varieties of orthogonal or unitary type. We now introduce the necessary background and notation. Let $\mathbb{G}$ be a connected reductive group over $\mathbb{Q}$ with a Shimura datum $(\mathbb{G}, \mathbb{X})$, and let $E \subset \mathbb{C}$ be the associated reflex field. Fix an odd rational prime p that is coprime to $\#\pi_1([\mathbb{G}, \mathbb{G}])$, together with an isomorphism $\iota_p : \mathbb{C} \to \overline{\mathbb{Q}_p}$ inducing an embedding $E \to \overline{\mathbb{Q}_p}$. We denote by G the base change of $\mathbb{G}$ to $\mathbb{Q}_p$. We assume that G is unramified and equipped with a Borel pair (B, T) and a hyperspecial subgroup $\mathcal{K}_p$ of $G(\mathbb{Q}_p)$. Let $\mathcal{K}^p \leq \mathbb{G}(\mathbf{A}_f^p)$ be a compact open subgroup such that $\mathcal{K} := \mathcal{K}_p \mathcal{K}^p \leq \mathbb{G}(\mathbf{A}_f)$ is neat. Let ℓ be a rational prime that is coprime to $p \cdot \#\pi_0(Z(\mathsf{G}))$, and let the coefficient field Λ be either $\overline{\mathbb{Q}_\ell}$ or $\overline{\mathbb{F}_\ell}$.

Definition ([HL24, Definition 6.2]). Suppose G is an arbitrary quasi-split reductive group over a finite extension $K/\mathbb{Q}_p$ with a Borel pair (B, T), and $\phi_T \in \Phi^{\mathrm{ss}}(T, \Lambda)$ is a semisimple L-parameter. Let $\phi_T^\vee$ denote the Chevalley dual of ϕ_T. Then ϕ_T is said to be *generic* (or of Langlands–Shahidi type) if for every dominant coroot $\mu \in \Phi^\vee(G, T)^+ \subset X_\bullet(T)$, the following Galois cohomologies vanish:

$$R\Gamma(W_K, {}^L\mathcal{T}_\mu \circ \phi_T), \quad R\Gamma(W_K, {}^L\mathcal{T}_\mu \circ \phi_T^\vee).$$

Here ${}^L\mathcal{T}_\mu$ denotes the extended highest weight tilting module ${}^L\mathcal{T}_\mu$ of ${}^L T$ with Λ-coefficients associated to μ; see (3.6).

We now state the following torsion-vanishing conjecture for Shimura varieties.

Conjecture ([Car23, HL24]). Suppose $\phi \in \Phi^{\mathrm{ss}}(G; \overline{\mathbb{F}_\ell})$ is an unramified, semisimple, toral generic L-parameter, corresponding via the Satake isomorphism to a maximal ideal $\mathfrak{m} \subset \overline{\mathbb{F}_\ell}[\mathcal{K}_p \backslash G(\mathbb{Q}_p)/\mathcal{K}_p]$. Then the complex $R\Gamma_c(\mathrm{Sh}_\mathcal{K}(\mathbb{G}, \mathbb{X})_{\overline{E}}, \overline{\mathbb{F}_\ell})_\mathfrak{m}$ (resp. $R\Gamma(\mathrm{Sh}_\mathcal{K}(\mathbb{G}, \mathbb{X})_{\overline{E}}, \overline{\mathbb{F}_\ell})_\mathfrak{m}$) is concentrated in degrees $0 \leq i \leq \dim_\mathbb{C}(\mathbb{X})$ (resp. $\dim_\mathbb{C}(\mathbb{X}) \leq i \leq 2\dim_\mathbb{C}(\mathbb{X})$).

This torsion vanishing conjecture has been established in the works of [CS17], [CS24], [Kos21], and [HL24] in the case where $(\mathbb{G}, \mathbb{X})$ is a PEL-type Shimura datum of type A or C_2 and G is a product of certain groups that is related to either GL_n over an unramified extension of $\mathbb{Q}_p$, or U_{2k+1} with respect to $\mathbb{Q}_{p^2}/\mathbb{Q}_p$, or U_2 with respect to a quadratic extension of unramified extensions of $\mathbb{Q}_p$, with p and ℓ satisfying certain properties. We remark that when G is not split, one has to impose an extra condition on ϕ, known as weakly normalized regularity; see [HL24, Definition 4.12].

In this work, we extend the list of known cases, particularly when $(\mathbb{G}, \mathbb{X})$ is an orthogonal or unitary Shimura datum associated to a quadratic or Hermitian space over a totally real number field F with standard indefinite signature, with G among the unramified groups listed in Theorem A, and ℓ sufficiently large.

Remark. The above torsion vanishing conjecture was established in [DvHKZ24] in the case where $G = \mathbb{G} \otimes \mathbb{Q}_p$ is split and the Shimura variety is compact of Hodge type, under the hypothesis that the Fargues–Scholze correspondence for G is compatible with the so-called classical local Langlands correspondence; see [Ham22, Assumption 7.5]. However, their result does not apply directly to the orthogonal Shimura variety $\mathrm{Sh}(\mathrm{Res}_{F/\mathbb{Q}} \mathrm{SO}(\mathbf{V}), \mathbf{X})$ when the quadratic space $\mathbf{V}$ over a totally real field F has large rank, because it is not of Hodge type. A natural approach is to consider a Hodge type Shimura datum $(\mathbf{G}^\sharp, \mathbf{X}^\sharp)$ with a map of Shimura data $(\mathbf{G}^\sharp, \mathbf{X}^\sharp) \to (\mathrm{Res}_{F/\mathbb{Q}} \mathrm{SO}(\mathbf{V}), \mathbf{X})$ such that the morphism $\mathbf{G}^\sharp \to \mathrm{Res}_{F/\mathbb{Q}} \mathrm{SO}(\mathbf{V})$ is a central extension. However, $\mathbf{G}^\sharp$ has derived subgroup $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{Spin}(\mathbf{V})$, for which the so-called classical

local Langlands correspondence has not yet been constructed. In this work, we modify the argument of [DvHKZ24] and weaken the hypothesis [Ham24, Assumption 7.5] to our Axiom D below. We then establish the torsion vanishing for $\mathrm{Sh}(\mathbf{G}^\sharp, \mathbf{X}^\sharp)$ and use the Hochschild–Serre spectral sequence to deduce the torsion vanishing for $\mathrm{Sh}(\mathrm{Res}_{F/\mathbb{Q}} \mathrm{SO}(\mathbf{V}), \mathbf{X})$. We also treat certain cases where p is not split in F.

If G_{ad} is not a product of Weil restrictions of split simple groups, one still needs to impose the weakly normalized regularity condition on ϕ. We refer to Theorem 8.2.6 for a precise statement. On the other hand, when G_{ad} is a product of Weil restrictions of split simple groups, there exists a simpler proof without use of geometric Eisenstein series. In this introduction, we state a more general result, which applies to the above-mentioned orthogonal or unitary case by constructing a central extension in §4.1. We first need an axiom on the Fargues–Scholze local Langlands correspondence:

Axiom D. Suppose G is a quasi-split connected reductive group over a p-adic number field K with a Borel pair (B, T) and $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{Q}_\ell})$ is a semisimple generic toral L-parameter. Then for any $\mathsf{b} \in B(\mathsf{G})$ and any $\rho \in \Pi(\mathsf{G}_\mathsf{b}, \overline{\mathbb{Q}_\ell})$, if the composition of $\phi_\rho^{\mathrm{FS}} : W_K \to {}^L\mathsf{G}_\mathsf{b}(\overline{\mathbb{Q}_\ell})$ with the twisted embedding ${}^L\mathsf{G}_\mathsf{b}(\overline{\mathbb{Q}_\ell}) \to {}^L\mathsf{G}(\overline{\mathbb{Q}_\ell})$ (as defined in [FS24, §IX.7.1]) equals ϕ, then b is unramified.

By our main theorem and [Ham24, Lemma 3.17], this axiom holds for those groups G/K appearing in Theorem A that are quasi-split.

We now state our main theorem on torsion vanishing for certain Shimura varieties of Abelian type. This theorem is proved in Theorem 8.2.10.

Theorem E. *Suppose that the following assumptions hold:*

(1) *$\mathrm{Sh}_{\mathscr{K}}(\mathbb{G}, \mathbb{X})$ is proper, and there exists a Shimura datum of Hodge type $(\mathbb{G}^\sharp, \mathbb{X}^\sharp)$, and a morphism of Shimura data $(\mathbb{G}^\sharp, \mathbb{X}^\sharp) \to (\mathbb{G}, \mathbb{X})$ such that $\mathbb{G}^\sharp_{\mathrm{ad}} \to \mathbb{G}_{\mathrm{ad}}$ is an isomorphism.*

(2) *G_{ad} is a product of unramified Weil restrictions of split simple groups $\prod_{i=1}^k \mathrm{Res}_{L_i/\mathbb{Q}_p} \mathsf{H}_i$, and the conjugacy class of Hodge cocharacters $\{\mu\}$ associated to $\mathbb{X}^\sharp$ induces a dominant cocharacter $\mu_{\mathrm{ad}} = (\mu_1, \ldots, \mu_k)$ of $\mathsf{G}_{\overline{\mathbb{Q}_p}}$ via ι_p, such that each μ_i is trivial on all but possibly one simple factor of $(\mathsf{H}_i)_{\overline{\mathbb{Q}_p}} \cong \prod_{\mathrm{Hom}(L_i, \overline{\mathbb{Q}_p})} (\mathsf{H}_i)_{\overline{L_i}}$.*

(3) *ℓ is a rational prime that is coprime to $p \cdot \#\pi_0(Z(\mathsf{G}^\sharp)) \cdot \#\pi_0(Z(\mathsf{G}))$, and $\mathfrak{m}$ is a maximal ideal of the ℓ-torsion Hecke algebra $\mathcal{H}_{\mathscr{K}_p} := \overline{\mathbb{F}_\ell}[\mathscr{K}_p \backslash \mathsf{G}(\mathbb{Q}_p) / \mathscr{K}_p]$.*

If the semisimple toral L-parameter $\phi_\mathfrak{m}$ corresponding to $\mathfrak{m}$ is generic and Axiom D holds for $\phi_\mathfrak{m}$, then $\mathrm{H}^i_{\text{ét}}(\mathrm{Sh}_{\mathscr{K}}(\mathbb{G}, \mathbb{X})_{\overline{E}}, \overline{\mathbb{F}_\ell})_\mathfrak{m}$ vanishes unless $i = \dim_{\mathbb{C}}(\mathbb{X})$.

Remark. The hypothesis that $\mathrm{Sh}_{\mathscr{K}}(\mathbb{G}, \mathbb{X})$ is proper is expected to be unnecessary, once we have constructed the minimally compactified Igusa stack for $\mathrm{Sh}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)$ in the sense of [Zha23] and compare the fibers of the Hodge–Tate map on it with the minimally compactified Igusa varieties. This has been done when $\mathrm{Sh}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)$ is of PEL type A/C in [HL24].

In particular, this theorem generalizes previous results of [CS17, Kos21, CS24, HL24] to compact orthogonal and unitary Shimura varieties. This is because we may construct a central extension $\mathbf{G}^\sharp$ of $\mathbf{G} := \mathrm{Res}_{F/\mathbb{Q}} \mathrm{U}(\mathbf{V})^\circ$ with a morphism of Shimura data $(\mathbf{G}^\sharp, \mathbf{X}^\sharp) \to (\mathbf{G}, \mathbf{X})$, so that $(\mathbf{G}^\sharp, \mathbf{X}^\sharp)$ defines a Shimura datum of Hodge type. In the unitary case, we take

$$\mathbf{G}^\sharp = \mathbf{G} \times \mathbf{Z}^\mathbb{Q}.$$

Here

$$\mathbf{Z}^\mathbb{Q} = \{z \in \mathrm{Res}_{F_1/\mathbb{Q}} \mathrm{GL}(1) : \mathrm{Nm}_{F_1/F}(z) \in \mathbb{Q}^\times\}$$

and F_1/F is the CM-extension associated to the Hermitian space $\mathbf{V}$. The desired map of Shimura data $(\mathbf{G}^\sharp, \mathbf{X}^\sharp) \to (\mathbf{G}, \mathbf{X})$ is constructed by Rapoport, Smithling, and Zhang in [RSZ20]. In the orthogonal case, following Carayol [Car86, p. 163], we construct, for each imaginary quadratic element $\daleth \in \mathbb{R}_+ i$, a group $\mathbf{G}^\sharp$ fitting into a exact sequence

$$1 \to \mathbf{Z}^\mathbb{Q} \to \mathbf{G}^\sharp \to \mathrm{Res}_{F/\mathbb{Q}} \mathbf{G} \to 1,$$

where

$$\mathbf{Z}^\mathbb{Q} = \{z \in \mathrm{Res}_{F(\daleth)/\mathbb{Q}} \mathrm{GL}(1) : \mathrm{Nm}_{F(\daleth)/F}(z) \in \mathbb{Q}^\times\}.$$

If moreover we assume that p is unramified in F and $\mathbb{Q}(\daleth)/\mathbb{Q}$ is split at p, then there exists an isomorphism

$$\mathbf{G}^\sharp \otimes \mathbb{Q}_p \cong \mathrm{GL}(1) \times \mathrm{Res}_{F \otimes \mathbb{Q}_p/\mathbb{Q}_p} \mathrm{GSpin}(\mathbf{V} \otimes \mathbb{Q}_p).$$

In fact, proving this torsion vanishing result for orthogonal Shimura varieties is one of the main motivations for this paper. In Euler system arguments via "level-raising congruences" in higher dimensional Shimura varieties, as pioneered by Bertolini and Darmon [BD05] for Shimura curves, we need to construct elements in the cohomology of Shimura varieties via the Jacquet–Langlands correspondence. The natural way to do so is to take the Abel–Jacobi map of a globally defined cycle that is cohomologically trivial. The point is that certain Hecke translate of a global cycle class becomes cohomologically trivial if the target cohomology group is itself trivial. For example, this strategy was applied in [Liu16, Liu19, LT20, LTX^{+}22] and will be used in the author's upcoming work [Pen26] on higher dimensional analogues of Kolyvagin theorems on the product of Shimura varieties of orthogonal type in the arithmetic Gross–Prasad setting. For further applications of torsion vanishing results, we refer the reader to Caraiani's ICM report [Car23].

1.2. **An overview of the proof.** We summarize the proof of Theorem A, adapting the method of Hamann [Ham22]. We exclude the case of even orthogonal groups in this introduction, as the additional outer automorphism complicates the notation. The proof proceeds by induction on the geometric rank of G, with low-rank cases verified by direct inspection. For higher ranks, if $\pi \in \Pi(G)$ is non-supercuspidal, we invoke the induction hypothesis along with the compatibility of rec_G (resp. $\mathrm{rec}_G^{\mathrm{FS}}$) with parabolic induction, since any proper Levi subgroup of G is a product of (Weil restrictions of) general linear groups and a group of the same type as G, but with smaller geometric rank. We may therefore assume that π is supercuspidal. We establish the compatibility for pure inner forms of G simultaneously. If ϕ is the classical L-parameter of π and $\Pi_\phi(G^*)$ contains a non-supercuspidal representation ρ_{nsc}, then the compatibility is already known for ρ_{nsc}. We then propagate this property to other representations in the L-packets of pure inner forms of G with classical parameter ϕ. The crucial input is a description of the cohomology of the local shtuka spaces $\mathrm{Sht}_{G,b,\{\mu\}}$ defined in [SW20], where $\{\mu\}$ is a geometric conjugacy class of G related to the Hodge cocharacter of suitable global Shimura variety of orthogonal or unitary type, and $b \in B(G, \{\mu\})$ is the unique nontrivial basic element. This local shtuka space carries an action of $G_b(K) \times G(K) \times W_{E_{\{\mu\}}}$, where $E_{\{\mu\}}/K$ is the reflex field of $\{\mu\}$. For any $\rho \in \Pi_\phi(G_b)$, the complex $\mathrm{R}\Gamma^\flat(G, b, \{\mu\})[\rho]$ is isomorphic to the result of applying a Hecke operator to ρ. Since Hecke operators and excursion operators commute, it follows that any representation of $G(K)$ occurring in $\mathrm{R}\Gamma^\flat(G, b, \{\mu\})[\rho]$ has Fargues–Scholze parameter equal to that of ρ.

To analyze which representations of G appear, we use the weak Kottwitz conjecture established by Hansen, Kaletha and Weinstein [HKW22].[2] In fact, π doesn't necessarily appear in the complex $\mathrm{R}\Gamma_c^\flat(G, b, \{\mu\})[\rho_{\mathrm{nsc}}]$, but we may iterate this process, replacing ρ_{nsc} by those representations that appear, until π eventually appears. This ultimately depends on a detailed analysis of the combinatorics of the centralizer of the L-parameter ϕ in $\widehat{G}$, see §2.6.

We are now left with the case in which $\Pi_\phi(G^*)$ consists entirely of supercuspidal representations. In this case, it follows from a result of Moeglin and Tadić (see Proposition Proposition 2.5.1) that ϕ is supercuspidal; that is, it is discrete and trivial on the $\mathrm{SL}(2, \mathbb{C})$-component. Since the Hecke operators and the excursion algebra are commutative, we may reduce to the case when G is quasi-split, and it suffices to prove the compatibility for each $\rho \in \Pi_\phi(G_b)$, see [Ham22, Lemma 3.15]. Let $\widetilde{\mathrm{Std}}$ denote the standard representation of $\widehat{G} \rtimes \mathrm{Gal}(K'/K_1)$, where K' is the splitting field of G, and write

$$\widetilde{\mathrm{Std}} \circ \phi|_{W_{K_1}} = \phi_1 + \ldots + \phi_r$$

as a decomposition into irreducible representations of W_{K_1}, where K_1 denotes the unramified quadratic extension of K in the unitary case and $K_1 = K$ in the orthogonal case. Hamann [Ham22] and Koshikawa [Kos21] established that if each ϕ_i appears as a subquotient of the complex

$$\bigoplus_{\rho' \in \Pi_\phi(G_b)} \mathrm{R}\Gamma_c^\flat(G, b, \{\mu\})[\rho']$$

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[2]We remark that when G is an even special orthogonal group, the hypothesis in [HKW22] remains unproven, since only a version of local Langlands correspondence up to conjugacy by the full orthogonal group is available. Instead, we use the weak endoscopic character identity established in [Pen25] and modify the arguments therein to establish a weaker version of the weak Kottwitz conjecture Theorem 3.4.1, valid up to conjugacy by the full orthogonal group. This weaker version is enough for the argument to work.

as a W_{K_1}-representation, then each ϕ_i also appears in $\widehat{\mathrm{Std}} \circ \phi_\rho^{\mathrm{FS}}|_{W_{K_1}}$. As both $\widehat{\mathrm{Std}} \circ \phi_\rho^{\mathrm{FS}}|_{W_{K_1}}$ and $\widehat{\mathrm{Std}} \circ \phi|_{W_{K_1}}$ are semisimple, they must be equal. It then follows from the results of [GGP12] that $\phi_\rho^{\mathrm{FS}} = \phi$.

Thus it suffices to prove that

$$\bigoplus_{\rho' \in \Pi_\phi(G_b)} \mathrm{R\Gamma}_c^\flat(G, b, \{\mu\})[\rho']$$

admits a subquotient isomorphic to $\widehat{\mathrm{Std}} \circ \phi_{K_1}$ as a W_{K_1}-representation. This is where global inputs become necessary, i.e., by relating this complex to the cohomology of a relevant global Shimura variety. The cohomology of the relevant global Shimura variety is studied via the Langlands–Kottwitz method and related to automorphic forms, whose local components are governed by classical L-parameters via Arthur's multiplicity formula. To be more precise, we elaborate on the case when G is a special orthogonal group of odd rank as an example. According to a result of Shen [She20], the local shtuka space uniformizes the basic Newton stratum of the generic fiber of a relevant Shimura variety, as defined in [CS17]. A relevant Shimura variety is given by $(\mathbf{G}, \mathbf{X})$, where $\mathbf{G}$ is a standard indefinite special orthogonal group over a totally real field F with p inert and $F_p \cong K$, such that $\mathbf{G} \otimes_F K \cong G$. Let $\mathbf{A}_F$ and $\mathbf{A}_{F,f}$ denote the ring of adeles and finite adeles of F, respectively, and let $\mathcal{K}_p \le \mathbf{G}(K)$ and $\mathcal{K}^p \le \mathbf{G}(\mathbf{A}_{F,f}^p)$ be sufficiently small level subgroups, thereby yielding the adic Shimura variety $\mathcal{S}_{\mathcal{K}_p \mathcal{K}^p}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})$ defined over $\mathbb{C}_p$. For an algebraic representation ξ of $(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}) \otimes \overline{\mathbb{Q}_\ell}$ with sufficiently regular highest weight, let $\mathcal{L}_\xi$ be the associated $\overline{\mathbb{Q}_\ell}$-local system on $\mathcal{S}_{\mathcal{K}_p \mathcal{K}^p}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})$, and consider the cohomology

$$\mathrm{R\Gamma}_c(\mathcal{S}_{\mathcal{K}_p \mathcal{K}^p}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X}), \mathcal{L}_\xi).$$

The basic uniformization result of Shen implies a $G(K) \times W_K$-invariant map

$$\Theta : \mathrm{R\Gamma}_c(G, b, \mathbf{1}, \{\mu\}) \otimes \iota_\ell \left|-\right|_{K_1}^{\frac{-\dim_{\mathbb{C}}(\mathbf{X})}{2}} [\dim_{\mathbb{C}}(\mathbf{X})] \otimes_{J(K)}^{\mathrm{L}} \mathcal{A}\left(\mathbf{G}'(F) \backslash \mathbf{G}'(\mathbf{A}_{F,f})/\mathcal{K}^p, \mathscr{L}_\xi\right)$$
$$\to \mathrm{R\Gamma}_c\left(\mathcal{S}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})_{\mathcal{K}^p}, \mathcal{L}_\xi\right),$$

where $\mathbf{G}'$ is an inner form of $\mathbf{G}$ with $\mathbf{G}' \otimes_F K \cong G_b^*$ and $\mathbf{G}'(F \otimes \mathbb{R})$ compact, and

$$\mathcal{A}(\mathbf{G}'(F) \backslash \mathbf{G}'(\mathbf{A}_f)/\mathcal{K}^p, \mathscr{L}_\xi)$$

is the space of $\mathcal{K}^p$-invariant algebraic automorphic forms valued in ξ. Next, we note that the pair $(\mathrm{Res}_{F/\mathbb{Q}}G, \{\mu_{\mathrm{Hdg}}\})$ is totally Hodge-Newton reducible as defined in [GHN19], where μ_{Hdg} is the Hodge cocharacter associated to $\mathbf{X}$. This implies that the flag variety $\mathrm{Gr}_{\mathrm{Res}_{K/\mathbb{Q}_p}G, \{\mu_{\mathrm{Hdg}}\}}$ is parabolically induced as a $G(K)$-space. This is called the "Boyer's trick". Using the Hodge–Tate period map

$$\pi_{\mathrm{HT}} : \mathcal{S}_{\mathcal{K}^p}(\mathbf{G}, \mathbf{X}) \to \mathrm{Gr}_{\mathrm{Res}_{K/\mathbb{Q}_p}G, \{\mu_{\mathrm{Hdg}}\}},$$

defined in [CS17], when we restrict to the summands on both sides of Θ where $G(K)$ acts by a supercuspidal representation, we obtain a $G(K) \times W_K$-equivariant isomorphism, which is also functorial with respect to $\mathcal{K}^p$.

We globalize the given $\rho \in \Pi(G_b)$ to a cuspidal automorphic representation Π' of $\mathbf{G}'$, such that the $\mathcal{K}^p$-fixed subspace of Π' occurs as a $G_b(K)$-stable direct summand of

$$\mathcal{A}(\mathbf{G}'(F) \backslash \mathbf{G}'(\mathbf{A}_{F,f})/\mathcal{K}^p, \mathscr{L}_\xi)$$

for some ξ with sufficiently regular highest weight, where

- Π' is an unramified twist of the Steinberg representation at some non-empty subset Σ^{St} of places of F,
- Π' is supercuspidal at some non-empty subset Σ^{sc} of finite places of F disjoint with Σ^{St}, and there exists $v \in \Sigma^{\mathrm{sc}}$ such that Π'_v has a simple supercuspidal L-parameter ϕ_v, meaning that $\widehat{\mathrm{Std}} \circ \phi_v$ is irreducible as a representation of W_{F_v}.
- Π' is unramified outside some non-empty subset Σ of places of F containing $\Sigma^{\mathrm{sc}} \cup \Sigma^{\mathrm{St}} \cup \Sigma_F^\infty$, and $\mathcal{K}^p$ decomposes as $\mathcal{K}^p = \mathcal{K}_{\Sigma \setminus \{p\}} \mathcal{K}^\Sigma$.

These conditions ensure that Arthur's multiplicity formula can be applied to analyze the cuspidal automorphic representations $\dot{\Pi}'$ in the near equivalence class of Π'. In particular, for each such $\dot{\Pi}'$, $\dot{\Pi}'_p$ has classical L-parameter ϕ. If we consider the maximal ideal $\mathfrak{m} \subset \mathbb{T}^\Sigma$ corresponding to $(\Pi')^\Sigma$, where $\mathbb{T}^\Sigma$ is the Hecke algebra of $\mathbf{G}'$ away from Σ, then $\mathfrak{m}$ is non-Eisenstein in the usual sense. Moreover, after localizing

at $\mathfrak{m}$ and restricting to the summand on which $G(K)$ acts via a supercuspidal representation, we obtain an isomorphism

$$\Theta_{\mathfrak{m},\mathrm{sc}} : \mathrm{R}\Gamma_c(G, b, \mathbf{1}, \{\mu\})_{\mathrm{sc}} \otimes \iota_\ell \, |-|_{K_1}^{\frac{-\dim_{\mathbb{C}}(\mathbf{X})}{2}} \, [\dim_{\mathbb{C}}(\mathbf{X})] \otimes_{J(K)}^{\mathrm{L}} \mathcal{A}\left(\mathbf{G}'(F)\backslash\mathbf{G}'(\mathbf{A}_{F,f})/\mathcal{K}^p, \mathscr{L}_\xi\right)_{\mathfrak{m}}$$
$$\xrightarrow{\sim} \mathrm{R}\Gamma_c\left(\mathcal{S}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})_{\mathcal{K}^p}, \mathcal{L}_{\iota_\ell\xi}\right)_{\mathfrak{m}}.$$

The assertion then follows if we can prove that the right-hand side is concentrated in the middle degree $\dim_{\mathbb{C}}(\mathbf{X})$ and carries a W_{K_1}-action given by $\widehat{\mathrm{Std}} \circ \phi_{K_1} \otimes \iota_\ell \, |-|_{K_1}^{\frac{-\dim_{\mathbb{C}}(\mathbf{X})}{2}}$. To prove this, we apply the Langlands–Kottwitz method in §4.5 to compute (a power of) the traces of Frobenius elements for all sufficiently large finite place v of F, and we obtain information at the place p from a local-global compatibility result associated to the Galois representation attached to the functorial transfer of π, which is a self-dual cuspidal automorphic representation of $\mathrm{GL}(2\,\mathrm{rank}(\mathbf{G}_{\overline{\mathbb{Q}}}); \mathbf{A}_F)$.

Remark. A natural question is whether the same method can be applied to prove compatibility for other reductive groups, for example $\mathrm{GSpin}(n), \mathrm{GSp}(2n), \mathrm{Sp}(2n)$ and G_2. For inner forms of $\mathrm{GSp}(4)$ and $\mathrm{Sp}(4)$, this is known by [Ham22]. For $\mathrm{GSpin}(n)$, it is possible to extend the method to prove compatibility of Fargues–Scholze's construction with those constructed by Moeglin [Moe14] in the quasi-split case, once the endoscopic character identities, as formulated in [Kal16], is proved for all of their inner twists. On the other hand, new ideas are needed to treat the cases of $\mathrm{GSp}(2n)$ ($n \geq 3$) and G_2, because a crucial step of the proof is to use the compatibility between local and global Shimura varieties to connect the construction of Fargues and Scholze with the so-called classical local Langlands correspondence through the cohomology of global Shimura varieties. The latter is studied via the Langlands–Kottwitz method, which can only give information about the (conjectural) global Galois representation $\rho : \mathrm{Gal}_F \to {}^L G$ associated to cohomological automorphic forms after composition with the extended highest weight module ${}^L \mathcal{T}_{\{\mu\}}$ of ${}^L G$, where $\{\mu\}$ is the conjugacy class of Hodge cocharacters of the Shimura datum. However, G_2 admits no Shimura variety, and in the case of $G = \mathrm{GSp}(2n)$, the extended highest weight module $\mathcal{T}_{\{\mu\}}$ is the spin representation of $\mathrm{GSpin}(2n+1, \mathbb{C})$, so it is hard to recover the Galois representation and its local components.

In §§2.1–2.3, we review the classical local Langlands correspondence for special orthogonal and unitary groups and the statement of the endoscopic character identities. In §§2.4–2.6, we analyze more properties of the local Langlands correspondence. In §§3.1–3.3, we review the Fargues–Scholze local Langlands correspondence and the spectral action, and recall the related objects. In §3.4, we prove a weaker version of the Kottwitz conjecture. In §§4.1–4.3, we review the endoscopic classification of automorphic representations of relevant groups, and define a class of cohomological cuspidal automorphic representations with local constraints. In §4.5, we apply the Langlands–Kottwitz method to compute the Galois cohomology of relevant global Shimura varieties. In §5, we apply basic uniformization and Boyer's trick to prove a key property of the cohomology of relevant local Shimura varieties, see Corollary 5.2.3. In §6, we combine previous results together to prove the compatibility Theorem A. In §7.1, we use the compatibility property to construct an unambiguous local Langlands correspondence for even orthogonal groups. In §7.2, we prove the naturality property of Fargues–Scholze local Langlands correspondence. In §7.3, we prove Theorem C by combining the compatibility result with the spectral actions. In §8.1, we study certain properties of generic toral L-parameters. In §8.2, we use the naturality of Fargues–Scholze local Langlands correspondence to prove the torsion vanishing result for Shimura varieties of orthogonal or unitary type. In §A, we review the endoscopy theory used in the main body.

1.3. **Notation and conventions.** We fix the following general notation.

Notation 1.3.1.
- Let $\mathbb{Z}_+$ denote the set of positive integers and $\mathbb{N}$ denote the set of non-negative integers.
- For each $n \in \mathbb{Z}_+$, we define $[n]_+ := \{1, 2, \ldots, n\}$. For each $n \in \mathbb{N}$, we define $[n] := \{0, 1, \ldots, n\}$.
- For each $n \in \mathbb{Z}_+$, let Sym_n denote the n-th symmetric group acting on $[n]_+$.
- Suppose X is a set.
  - Let $\#X$ denote the cardinality of X and let $\mathscr{P}(X)$ denote the power set of X.
  - Let $\mathbf{1} \in X$ denote the distinguished trivial element (this notation is only used when the notion of triviality is clear from context).

- For two elements a, b in a set X, we define the Kronecker symbol

$$\delta_{a,b} := \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}.$$

- Let $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ denote the set of rational, real, and complex numbers, respectively. We fix a choice of square root i of -1 in $\mathbb{C}$.
- When A is a (topological/algebraic) group, we write $B \leq A$ to mean that B is a (closed) subgroup of A.
- For a finite group A, let $\mathrm{Irr}(A)$ denote the set of isomorphism classes of irreducible complex representations of A.
- All rings are assumed to be commutative and unital, and ring homomorphisms preserve units. Algebras, however, may be non-commutative and non-unital.
- The transpose of a matrix M is denoted by $M^\top$. When M is invertible, we write $M^{-\top}$ for $(M^{-1})^\top$.
- Let $J_n = (a_{ij})$ denote the anti-diagonal $n \times n$ matrix such that $a_{i,j} = \delta_{i,n+1-j}$ and $J'_n = (b_{ij})$ denote the anti-diagonal $n \times n$ matrix such that $b_{i,j} = (-1)^{i+1}\delta_{i,n+1-j}$.
- If S is a scheme over a commutative ring R and R' is a ring over R, we define $S_{R'} := S \otimes_R R'$.
- For a locally algebraic group G over a field K, let $Z(G)$ denote the center of G and G° denote the identity component of G.
- Reductive groups are assumed to be connected.
- For a reductive group G over a field K, let W_G denote the relative Weyl group and G^* denote the unique quasi-split inner form of G. A Borel pair for G^* is defined to be a pair (B^*, T^*) consisting of a Borel subgroup B^* and a maximal torus T^* contained in B^*.

We fix the following notation for a connected reductive group over a non-Archimedean local field of characteristic zero.

Notation 1.3.2. Suppose $K/\mathbb{Q}_p$ is a finite extension and G is a connected reductive group over K.
- Let κ denote the residue field of K with a fixed algebraic closure $\overline{\kappa}$, and we fix a uniformizer $\varpi_K \in K^\times$.
- Let $\mathrm{ord}_K : K^\times \to \mathbb{Z}$ denote the additive valuation map that sends a uniformizer ϖ_K to 1, and let $|-|_K : K^\times \to p^{\mathbb{Z}}$ denote the multiplicative valuation map such that $|x|_K = (\#\kappa)^{-\mathrm{ord}_K(x)}$.
- We fix an algebraic closure $\overline{K}$ of K, and for each subfield $K' \subset \overline{K}$, we define $\mathrm{Gal}_K := \mathrm{Gal}(\overline{K}/K')$.
- Denote by W_K the Weil group of K and by I_K the inertia group of K. Let $\mathrm{Art}_K : K^\times \to W_K^{\mathrm{ab}}$ denote the Artin map. Fix an arithmetic Frobenius element $\varphi_K \in W_K$. Set $\sigma_K := \varphi_K^{-1}$, and we use the same symbol $|-|_K$ to denote the composition $W_K \to W_K^{\mathrm{ab}} \xrightarrow{\mathrm{Art}_K^{-1}} K^\times \xrightarrow{|-|_K} p^{\mathbb{Z}}$.
- Let $\breve{K}$ denote the completion of the maximal unramified extension of K.
- We use the geometric normalization of the local class field theory, i.e., Artin maps are normalized so that it maps uniformizers to geometric Frobenius classes.
- Let $\widehat{\mathsf{G}}$ denote the Langlands dual group of G, which is a Chevalley group with a duality $\Phi(\widehat{\mathsf{G}}) \cong \Phi(\mathsf{G})^\vee$ between based root data. It is equipped with an action of Gal_K. Denote by ${}^L\mathsf{G} := \widehat{\mathsf{G}} \rtimes W_K$ the Langlands L-group of G in the Weil form. We usually conflate ${}^L\mathsf{G}$ (respectively, $\widehat{\mathsf{G}}$) with their $\mathbb{C}$-valued points, unless we write ${}^L\mathsf{G}(\Lambda)$ (respectively, $\widehat{\mathsf{G}}(\Lambda)$), which denotes its Λ-valued points for some ring Λ.
- Let $\mathcal{H}(\mathsf{G})$ denote the set of compactly supported locally constant $\mathbb{C}$-valued functions on $\mathsf{G}(K)$ that is bi-$\mathscr{K}$-finite for some compact open subgroup $\mathscr{K} \leq \mathsf{G}(K)$.
- If G, G' are reductive groups over K and π, π' are irreducible admissible representations of $\mathsf{G}(K)$ and $\mathsf{G}'(K)$, respectively, let $\pi \boxtimes \pi'$ denote the irreducible admissible representation of $\mathsf{G}(K) \times \mathsf{G}'(K)$ such that $(\pi \boxtimes \pi')((g, g')) = \pi(g) \otimes \pi'(g')$.

- Let $\Pi(\mathsf{G})$ denote the set of isomorphism classes of irreducible admissible representations of $\mathsf{G}(K)$, and let $\Pi_{\mathrm{temp}}(\mathsf{G})$ (resp. $\Pi_2(\mathsf{G})$, resp. $\Pi_{\mathrm{sc}}(\mathsf{G})$) denote the subset of $\Pi(\mathsf{G})$ consisting of tempered (resp. essentially square-integrable, resp. supercuspidal) representations. Set $\Pi_{2,\mathrm{temp}}(\mathsf{G}) := \Pi_2(G) \cap \Pi_{\mathrm{temp}}(G)$.

- If $\mathsf{P} \leq \mathsf{G}$ is a parabolic subgroup with a Levi factor M and $\sigma \in \Pi(\mathsf{M})$, we let $\delta_{\mathsf{P}} : \mathsf{P}(K) \to p^{\mathbb{Z}}$ denote the modulus quasi-character of P describing the change of a right Haar measure under left translation, and let $\mathsf{I}_{\mathsf{P}}^{\mathsf{G}}(\sigma \otimes \delta_{\mathsf{P}}^{1/2})$ denote the normalized parabolic induction.

- If $\mathsf{G} = \mathrm{GL}(n)$ is a general linear group, we define a character $\nu := |-|_K \circ \det : \mathsf{G}(K) \to K^{\times}$.

- Suppose ℓ is a rational prime different from p and $\Lambda \in \{\overline{\mathbb{Q}_{\ell}}, \overline{\mathbb{F}_p}\}$. Let $\mathrm{D}(\mathsf{G}, \Lambda)$ denote the derived category of smooth representations of $\mathsf{G}(K)$ with coefficients in Λ, equipped with the natural t-structure. Let $\mathrm{D}^{\mathrm{adm}}(\mathsf{G}, \Lambda)$ denote the full subcategory of admissible complexes, i.e., those complexes whose invariants under any compact open subgroup $\mathcal{K} \leq \mathsf{G}(K)$ form a perfect complex.

- Suppose ℓ is a rational prime different from p and $\Lambda \in \{\overline{\mathbb{Q}_{\ell}}, \overline{\mathbb{F}_p}\}$. For each conjugacy class of cocharacters $\{\mu\}$ for $\mathsf{G}_{\overline{K}}$, there exists an indecomposable highest weight tilting module $\mathcal{T}_{\{\mu\}} \in \mathrm{Rep}_{\Lambda}(\widehat{\mathsf{G}})$ as defined in [Rin91, Don93]; cf. [Ham24, §10.1].

- We define
$$X_*(\mathsf{G}) := \mathrm{Hom}_K(\mathrm{GL}(1)_K, \mathsf{G}), \quad X_{\bullet}(\mathsf{G}) := \mathrm{Hom}_{\overline{K}}(\mathrm{GL}(1)_{\overline{K}}, \mathsf{G}_{\overline{K}})$$
for the set of cocharacters and geometric cocharacters of G, respectively, and define
$$X^*(\mathsf{G}) := \mathrm{Hom}_K(\mathsf{G}, \mathrm{GL}(1)_K), \quad X^{\bullet}(\mathsf{G}) := \mathrm{Hom}_{\overline{K}}(\mathsf{G}_{\overline{K}}, \mathrm{GL}(1)_{\overline{K}})$$
for the set of characters and geometric characters of G, respectively.

- For any condensed ∞-category $\mathcal{C}$ and any finite index set I, let $\mathcal{C}^{BW_K^I}$ denote the category of objects with continuous W_K^I-actions, as defined in [FS24, §IX.1].

- For any subfield $\kappa' \subset \overline{\kappa}$, let $\mathcal{P}\mathrm{erfd}_{\kappa'}$ denote the category of affinoid perfectoid spaces over κ'.

- The six functor formalism of [Sch22] and [FS24] on ℓ-adic cohomology of diamonds and small Artin v-stacks is freely used. In particular, suppose ℓ is a rational prime different from p and $\Lambda \in \{\overline{\mathbb{Q}_{\ell}}, \overline{\mathbb{F}_p}\}$, then for any small Artin v-stack X, let $\mathrm{D}_{\blacksquare}(X, \Lambda)$ denote the condensed ∞-category of solid Λ-sheaves on X [FS24, §VII.1], and let $\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \subset \mathrm{D}_{\blacksquare}(X, \Lambda)$ denote the full subcategory of Λ-lisse-étale sheaves as defined in [FS24, §VII.6].

We fix the following notation for a connected reductive group over a number field F.

Notation 1.3.3. Suppose F is a number field with a fixed embedding $\tau_0 : F \to \mathbb{C}$ and $\mathbb{G}$ is an arbitrary connected reductive group over F.
- Let Σ_F^{fin} denote the set of finite places of F, and define $\Sigma_F^{\infty} :== \mathrm{Hom}(F, \mathbb{C})$. Set $\Sigma_F := \Sigma_F^{\mathrm{fin}} \cup \Sigma_F^{\infty}$.
- For each finite set S of rational primes, let $\Sigma_F(S) \subset \Sigma_F^{\mathrm{fin}}$ denote the subset of all finite places of F with residue characteristic in S.
- Let $\overline{F}$ denote the algebraic closure of F in $\mathbb{C}$,
- For each finite place v of F, let κ_v denote the residue field of F_v and let $\sigma_v \in \mathrm{Gal}_F$ denote a geometric Frobenius element at v. We also define $\|v\| := \#\kappa_v$.
- Let $\mathbf{A}_F$ denote the ring of adeles of F, and let $\mathbf{A}_{F,f}$ denote the ring of finite adeles of F. We also write $\mathbf{A} := \mathbf{A}_{\mathbb{Q}}$ and $\mathbf{A}_f := \mathbf{A}_{\mathbb{Q},f}$.
- If $\Sigma \subset \Sigma_F^{\mathrm{fin}}$ is a finite subset, set $\mathbf{A}_{F,f}^{\Sigma} := \prod_{v \in \Sigma_F \setminus \Sigma}' F_v$.
- For each discrete automorphic representation Π of $\mathbb{G}(\mathbf{A}_F)$, let $m(\Pi)$ denote its multiplicity in the discrete automorphic spectrum of $\mathbb{G}$.

2. Local Langlands correspondence via endoscopy

We begin by recalling the local Langlands correspondence defined via theory of endoscopy. Let K be a non-Archimedean local field of characteristic zero, and fix a nontrivial additive character ψ_K of K, which extends to an additive character of any finite extension K'/K by defining $\psi_{K'} := \psi_K \circ \mathrm{tr}_{K'/K}$.

2.1. The groups. Let K_1/K be an unramified extension of degree at most two, and let $\mathsf{c} \in \mathrm{Gal}(K_1/K)$ be the element with fixed field K. Let $\chi_{K_1/K} : K^\times \to \{\pm 1\}$ denote the quadratic character associated to K_1/K via local class field theory. Let V be a vector space of dimension $n \in \mathbb{Z}_+$ equipped with a non-degenerate Hermitian c-sesquilinear form $\langle -, - \rangle$, that is,

$$\langle au + bv, w \rangle = a \langle u, w \rangle + b \langle v, w \rangle, \quad \text{and} \quad \langle v, w \rangle = \langle w, v \rangle^{\mathsf{c}}$$

for all $a, b \in K_1$ and $u, v, w \in V$.

Fix an arbitrary orthogonal basis $\{v_1, \ldots, v_n\}$ of V such that $\langle v_i, v_i \rangle = a_i \in K^\times$. Define the discriminant of V as

$$\mathrm{disc}(V) = (-1)^{\binom{n}{2}} \prod_{i=1}^{n} a_i$$

whose class in $K^\times/(K^\times)^2$ (resp. in $K^\times/\mathrm{Nm}_{K_1/K}(K_1^\times)$) when $K_1 = K$ (resp. when $K_1 \neq K$) is independent of the choice of orthogonal basis.

The (normalized) Hasse–Witt invariant of V is defined as

$$\epsilon(V) = \begin{cases} \left(-1, (-1)^{\binom{n}{4}} \cdot \mathrm{disc}(V)^{\binom{n-1}{2}} \right)_K \cdot \displaystyle\prod_{i < j \in [n]_+} (a_i, a_j)_K & \text{if } K_1 = K, \\[2ex] \chi_{K_1/K}(\mathrm{disc}(V)) & \text{if } K_1 \neq K. \end{cases}$$

where

$$(-, -)_K : \left(K^\times/(K^\times)^2 \right) \times \left(K^\times/(K^\times)^2 \right) \to \mathrm{Br}(K)[2] \cong \{\pm 1\}$$

denotes the Hilbert symbol.

Recall from [Ser73, Theorem 2.3.7] that if $K_1 = K$, the isometry class of $(V, \langle -, - \rangle)$ is fully determined by the triple

$$(\dim V, \mathrm{disc}(V), \epsilon(V)) \in \mathbb{Z}_+ \times \left(K^\times/(K^\times)^2 \right) \times \{\pm 1\},$$

and, moreover, it follows from [Ser73, Proposition 2.3.6] that all triples except $(1, d, -1)$ and $(2, 1, -1)$ can happen. If $K_1 \neq K$, the isometry class of $(V, \langle -, - \rangle)$ is completely characterized by the pair

$$(\dim(V), \epsilon(V)) \in \mathbb{Z}_+ \times \left(K^\times/\mathrm{Nm}_{K_1/K}(K_1^\times) \right),$$

and all pairs can happen; see [MH73].

Let $G(V)$ denote the algebraic subgroup of $\mathrm{GL}(V)$ such that

$$G(V) = \{g \in \mathrm{GL}(V) : \langle gv, gw \rangle = \langle v, w \rangle \, \forall v, w \in V\},$$

and let $G = G(V)^\circ$ denote its identity component. Let G^* denote the unique quasi-split inner form of G over K. Exactly one of the following three cases holds:

O1 $K_1 = K$ and $\dim(V) = 2n + 1$ is odd. Then $G^* = \mathrm{SO}(2n + 1)$, the split orthogonal group in $2n + 1$-variables.

O2 $K_1 = K$ and $\dim(V) = 2n$ is even. Then $G^* = \mathrm{SO}(2n)^{\mathrm{disc}(V)}$, the quasi-split special orthogonal group associated to the quadratic space V^* over K of dimension $2n$, discriminant $\mathrm{disc}(V)$ and Hasse–Witt invariant 1.

U $K_1 \neq K$ and $\dim(V) = n$. Then $G^* = \mathrm{U}(n)$, the quasi-split unitary group associated to the Hermitian space of dimension n with respect to the unramified quadratic extension K_1/K, with Hasse–Witt invariant 1.

We collectively refer to Cases O1 and O2 together as Case O. Note that

- In Case O1, G is split if $\epsilon(V) = 1$ and non-quasi-split if $\epsilon(V) = -1$. In either case, G splits over the unramified quadratic extension of K.

- In Case O2, G is split if $\mathrm{disc}(V) = 1, \epsilon(V) = 1$; non-quasi-split if $\mathrm{disc}(V) = 1, \epsilon(V) = -1$; and quasi-split but non-split if $\mathrm{disc}(V) \neq 1$. Moreover, G splits over the unramified quadratic extension of K if and only if $\mathrm{ord}_p(\mathrm{disc}(V)) \equiv 0 (\mathrm{mod}\ 2)$.

- In Case U, G is non-quasi-split if n is even and $\epsilon(V) = -1$; otherwise it is quasi-split but non-split. In all cases, G splits over K_1.

To unify notation, let $n(G) = n(G^*)$ denote the geometric rank of G. Thus $n(\mathrm{SO}(2n+1)) = n(\mathrm{SO}(2n)^{\mathrm{disc}(V)}) = n(\mathrm{U}(n)) = n$. We define the following invariants associated to G:

$$N(G) := \begin{cases} 2n(G) & \text{in Case O}, \\ n(G) & \text{in Case U}, \end{cases}$$

(2.1)
$$d(G) := \begin{cases} 2n(G)+1 & \text{in Case O1}, \\ 2n(G) & \text{in Case O2}, \\ n(G) & \text{in Case U}, \end{cases}$$

$$b(G) := \begin{cases} -1 & \text{in Case O1}, \\ 1 & \text{in Case O2}, \\ (-1)^{n(G)-1} & \text{in Case U}. \end{cases}$$

Here $N(G)$ is the rank of the general linear group associated with the Langlands dual group of G; $d(G)$ is the dimension of the c-Hermitians space V defining G, and $b(G)$ is the sign associated with G. In Case O2, we also define $\mathrm{disc}(G) := \mathrm{disc}(V)$.

Let $\mathcal{F} = \{0 = X_0' \subset X_1' \subset X_2' \subset \ldots \subset X_r'\}$ be a flag of isotropic K_1-subspaces of V. Then there exists an orthogonal direct sum decomposition

$$V = (X_r' \oplus Y_r') \perp V',$$

where Y_r' is an isotropic subspace. The stabilizer $P \le G$ of this flag $\mathcal{F}$ is a parabolic subgroup, and every parabolic subgroup of G arises in this way. Moreover, if X_i is a complement of X_{i-1}' in X_i' for each $i \in [r]_+$, then

$$M = \mathrm{GL}(X_1) \times \cdots \times \mathrm{GL}(X_r) \times G(V')^\circ$$

is a Levi subgroup of G (Here $G(V')^\circ$ is trivial when $\dim V' = 0$). Every Levi subgroup of G arises in this way, and any two such Levi subgroups that are isomorphic are conjugate under $G(V)$.

We fix a pinning of G^* by identifying it with $G(V^*)^\circ$ for a suitable c-Hermitian space V^* over K_1, and choosing a complete flag of totally isotropic subspaces in V^*. Recall that a Whittaker datum for G^* is a $T^*(K)$-conjugacy class of generic characters of $N^*(K)$, where N^* is the unipotent radical of B^*. Whittaker data for G^* form a principal homogeneous space over the finite Abelian group

$$E = \mathrm{Coker}\left(G^*(K) \to G_{\mathrm{ad}}^*(K)\right) = \ker(H^1(K, Z(G^*)) \to H^1(K, G^*));$$

see [GGP12, §9]. The fixed pinning $(B^*, T^*, \{X_\alpha^*\}_{\alpha \in \Delta})$ of G^*, together with the additive character ψ_K of $K^\times$, determines a Whittaker datum $\mathfrak{m}$ for G^*,; see [KS99, §5.3]. When G is unramified, there exists a unique $G(K)$-conjugacy class of hyperspecial maximal compact open subgroups compatible with $\mathfrak{m}$, in the sense of [CS80]. In this case, "unramified representations of $G(K)$" refers to those unramified with respect to such a hyperspecial subgroup.

We define the Witt tower associated to G: For each $n_0 \in [n]$, let $G(n_0)$ denote the reductive group (it can be the trivial group $\mathbf{1}$) of geometric rank n_0, such that there exists a Levi subgroup of G isomorphic to

$$\mathrm{Res}_{K_1/K}\mathrm{GL}\left(\frac{n-n_0}{[K_1:K]}\right) \times G(n_0).$$

By [Tit79, §4.4],

- In Case O1, $G(n_0)$ exists if and only if $n_0 \ge \frac{1-\epsilon(V)}{2}$,
- In Case O2, $G(n_0)$ exists if any only if $n_0 \ge 1 + \delta_{\mathrm{disc}(V),1} \cdot \epsilon(V)$,
- In Case U, $G(n_0)$ exists if any only if $n - n_0$ is even and moreover $n_0 \ne 0$ when G is non-quasi-split(i.e., when $n(G)$ is even and $\epsilon(V) = -1$).

We fix an isomorphism

$$\widehat{G} \cong \begin{cases} \mathrm{Sp}(N(G), \mathbb{C}) & \text{in Case O1} \\ \mathrm{SO}(N(G), \mathbb{C}) & \text{in Case O2}, \\ \mathrm{GL}(N(G), \mathbb{C}) & \text{in Case U} \end{cases}$$

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and fix a pinning $(\widehat{T}, \widehat{B}, \{X_\alpha\})$ where $\widehat{T}$ is the diagonal torus, $\widehat{B}$ is the group of upper triangular matrices, and $\{X_\alpha\}$ is the set of standard root vectors. Let ${}^L G = \widehat{G} \rtimes W_K$ denote the Langlands L-group in the Weil form, where W_K acts on $\widehat{G}$ preserving the pinning, with the action as follows:

- In Case O1, W_K acts trivially on $\widehat{G}$.

- In Case O2, W_K acts via the quotient $\mathrm{Gal}(K(\sqrt{\mathrm{disc}(G)})/K)$. If $\mathrm{disc}(G) \neq 1$ and $\widehat{G}$ is identified with the subgroup of $\mathrm{SL}(N(G), \mathbb{C})$ preserving the non-degenerate bilinear form on $\mathbb{C}\{v_1, \ldots, v_{N(G)}\}$ defined by

$$\langle v_i, v_j \rangle = \delta_{i, N(G)+1-j},$$

  then the nontrivial element acts by conjugation via the element in $\mathrm{O}(N(G), \mathbb{C})$ that exchanges $v_{n(G)}$ and $v_{n(G)+1}$ and fixes the others.

- In Case U, W_K acts via the quotient $\mathrm{Gal}(K_1/K)$, where $\mathsf{c} \in \mathrm{Gal}(K_1/K)$ acts by

$$g \mapsto J'_n g^{-\top} (J'_n)^{-1}.$$

Finally, note that $\widehat{G}$ has a standard representation

$$\widehat{\mathrm{Std}} = \widehat{\mathrm{Std}}_G : \widehat{G} \to \mathrm{GL}(N(G), \mathbb{C}).$$

Given $b \in G(\breve{K})$ with image in $B(G)_{\mathrm{bas}}$, we may associate an *extended pure inner twist* (G_b, ϱ_b, z_b) as defined in [Kot97, §3.3, 3.4], where

- $G_b(K) = \left\{ g \in G(\breve{K}) | b\varphi_K(g)b^{-1} = g \right\}$, and

- $\varrho_b : G \otimes_K \overline{K} \xrightarrow{\sim} G_b \otimes_K \overline{K}$ is an isomorphism over $\overline{K}$,

- z_b is a Galois 1-cocycle in $Z^1(K, G\mathrm{ad})$ representing the class corresponding to b.

Let $\kappa_G : B(G)_{\mathrm{bas}} \xrightarrow{\sim} X^\bullet(Z(\widehat{G})^{\mathrm{Gal}_K})$ denote the Kottwitz map [Kot85, Proposition 5.6], which induces an identification of $H^1(K, G)$ with $X^\bullet\left(\pi_0\left(Z(\widehat{G})^{\mathrm{Gal}_K}\right)\right)$. There is a canonical isomorphism

$$H^1(K, G) = B(G)_{\mathrm{bas}} \cong \mathbb{Z}/2,$$

except in the case $G = \mathrm{SO}(2)^1$, where $H^1(K, G) = B(G)_{\mathrm{bas}} = 1$; see [GGP12, Lemma 2.1]. In particular, every extended pure inner twist of G^* is canonically a pure inner twists of G, and G embeds into an extended pure inner twist $(G^*_{b_0}, \varrho_{b_0}, z_{b_0})$ for some $b_0 \in G^*(\breve{K})$. Moreover, for each $b \in B(G)_{\mathrm{bas}}$, we can canonically identify $\widehat{G}_b$, $\widehat{G}$ and $\widehat{G^*}$ via the inner twistings ϱ_{b_0} and ϱ_b.

For each parabolic pair (M, P) of G, there exists a unique standard parabolic pair (M^*, P^*) of G^* corresponding to (M, P) under ϱ_{b_0}, and this determines an equivalence class of extended pure inner twist of M^*, also denoted by (ϱ_{b_0}, z_{b_0}) by abuse of notation.

For each $b_0 \in G^*(\breve{K})$, we write κ_{b_0} for the character of $\pi_0(Z(\widehat{G})^{\mathrm{Gal}_K})$ corresponding to it under the Kottwitz map. Note that κ_G and κ_{G^*} induces an isomorphism

$$(2.2) \qquad\qquad B(G)_{\mathrm{bas}} \cong X^\bullet(Z(\widehat{G})^{\mathrm{Gal}_K}) \cong B(G^*)_{\mathrm{bas}}$$

which maps b to $b + b_0$, where the addition is induced from κ_{G^*}. In particular, G_b is isomorphic to $G^*_{b_0+b}$.

There is an automorphism θ on $G^{\mathrm{GL}} := \mathrm{Res}_{K_1/K} \mathrm{GL}(N(G))$ such that

$$\theta(g) = J'_n \mathsf{c}(g)^{-\top} (J'_n)^{-1}$$

for any $g \in G^{\mathrm{GL}}(K) = \mathrm{GL}(N(G), K_1)$. We fix a standard Gal_{K_1}-invariant pinning $(B^{\mathrm{GL}}, T^{\mathrm{GL}}, \{X_\alpha^{\mathrm{GL}}\})$ of G^{GL} that is stabilized by θ.

The quasi-split inner form G^* can be extended to an element in $\mathcal{E}_{\mathrm{ell}}(G^{\mathrm{GL}} \rtimes \theta)$ as defined in §A.1; see [Mok15, p. 3-4, 7] in Case U (where we take $\kappa = 1$ and $\chi_\kappa = \mathbf{1}$, i.e., the standard base change L-embedding) and [Art13, §1.2] in Case O. Denote the group $\mathrm{OAut}_{G^{\mathrm{GL}}}(G^*)$ from (A.1) by $\mathrm{OAut}_N(G^*)$. Then $\mathrm{OAut}_N(G^*)$ is trivial in Case O1 and Case U, and $\mathrm{OAut}_N(G^*) = \mathrm{O}(2n(G^*), \mathbb{C}) / \mathrm{SO}(2n(G^*), \mathbb{C})$ in Case O2.

Here is the list of isomorphism classes of elliptic endoscopic triples $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(G^*)$ from [Rog90, §4.6] and [Wal10], where we only describe $G^{\mathfrak{e}}$ and $\mathrm{OAut}(\mathfrak{e})$:

- In Case U, $G^{\mathfrak{e}} \cong \mathrm{U}_{K_1/K}(a) \times \mathrm{U}_{K_1/K}(b)$ where $a, b \in [n(G)]$ such that $a + b = n(G)$, and $\mathrm{OAut}_{G^*}(\mathfrak{e})$ is trivial except when $a = b$, where there exists a unique nontrivial outer automorphism swapping the two factors of $\widehat{G^{\mathfrak{e}}} \cong \mathrm{GL}(a) \times \mathrm{GL}(b)$.

- In Case O1, $G^{\mathfrak{e}} \cong \mathrm{SO}(2a+1) \times \mathrm{SO}(2b+1)$ where $a, b \in [n(G)]$ such that $a+b = n(G)$, and $\mathrm{OAut}_{G^*}(\mathfrak{e})$ is trivial except when $a = b$, in which case there exists a unique nontrivial outer automorphism swapping the two factors of $\widehat{G^{\mathfrak{e}}} \cong \mathrm{Sp}(2a, \mathbb{C}) \times \mathrm{Sp}(2b, \mathbb{C})$.

- In Case O2, $G^{\mathfrak{e}} \cong \mathrm{SO}(2a)^{\beta} \times \mathrm{SO}(2b)^{\gamma}$, where $a, b \in [n(G)]$ such that $a + b = n(G), \beta\gamma = \mathrm{disc}(G)$, and moreover $\beta = 1$ if $a = 0$, $\gamma = 1$ if $b = 0$, and $(a, \beta) \neq (1,1), (b, \gamma) \neq (1,1)$. If $ab > 0$, there exists an outer automorphism acting by conjugation action of an element of $\mathrm{O}(2a, \mathbb{C}) \times \mathrm{O}(2b, \mathbb{C})$ on the two factors of $\widehat{G^{\mathfrak{e}}} \cong \mathrm{SO}(2a, \mathbb{C}) \times \mathrm{SO}(2b, \mathbb{C})$. There are no other nontrivial outer automorphisms except when $a = b$ and $\alpha = 1$, in which case there exists one swapping the two factors of $\widehat{G^{\mathfrak{e}}} \cong \mathrm{SO}(2a, \mathbb{C}) \times \mathrm{SO}(2b, \mathbb{C})$, and so $\mathrm{OAut}(\mathfrak{e}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ in this case.

For each $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(G^*)$, we fix a choice of $^L\xi^{\mathfrak{e}}$ as in [Wal10, §1.8], such that if $G^{\mathfrak{e}} = H_1 \times H_2$, then $\widehat{\mathrm{Std}}_G \circ {}^L\xi^{\mathfrak{e}}$ is conjugate to $(\widehat{\mathrm{Std}}_{H_1} \times \widehat{\mathrm{Std}}_{H_2}) \circ \iota$, where $\iota : {}^LG^{\mathfrak{e}} \hookrightarrow {}^LH_1 \times {}^LH_2$ is the natural inclusion.

Non-elliptic endoscopic triples $\mathfrak{e} \in \mathcal{E}(G^*)$ are described similarly, in which case $G^{\mathfrak{e}}$ is a product of groups of the same type as G with geometric rank smaller than $n(G)$ and restrictions of general linear groups, see for example [Ish24, §3.1.3] in Case O1.

2.2. The L-parameters.

We recall the description of L-parameters for G^* and their relations to conjugate self-dual representations of Weil groups.

For each positive integer $m \in \mathbb{Z}_+$, an L-parameter ϕ for $\mathrm{GL}(m)_{K'}$ over any finite extension field K'/K may be regarded as an isomorphism class of m-dimensional representations of $W_{K'} \times \mathrm{SL}(2, \mathbb{C})$. Every such representation is isomorphic to a finite direct sum of representations of the form $\rho \boxtimes \mathrm{sp}_a$ where ρ is a smooth representation of $W_{K'}$ and sp_a is the unique irreducible algebraic representation $\mathrm{SL}(2)_{\mathbb{C}}$ of dimension a.

An L-parameter ϕ for $\mathrm{GL}(m)$ over K_1 is called conjugate self-dual and irreducible if ϕ is isomorphic to $\phi^\theta := (\phi^s)^\vee$ as representations, where $s \subset W_K$ is an arbitrary lift of $\mathsf{c} \in W_K/W_{K_1} \cong \mathrm{Gal}(K_1/K)$ and ϕ^s is the conjugate action $\phi^s(g) = \phi(sgs^{-1})$. Following [GGP12, §3], we introduce the sign of a conjugate self-dual irreducible L-parameter ϕ: There exists an isomorphism $f : \phi \xrightarrow{\sim} \phi^\theta$ such that $(f^\vee)^s = b(\phi)f$ for some $b(\phi) \in \{\pm 1\}$. The value $b(\phi)$ is independent of the choice of f, and is called the sign of ϕ. If $\phi = \rho \boxtimes \mathrm{sp}_a$, where ρ is an irreducible representation of W_K, then

$$b(\phi) = b(\rho)(-1)^{a-1};$$

see [GGP12, Lemma 3.2], [KMSW14, §1.2.4].

Then it follows from [GGP12, Theorem 8.1] and [AG17, p. 365] that there exists a natural identification

$$\Phi(G^*) = \begin{cases} \{\text{admissible } \phi : W_{K_1} \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{Sp}(N(G), \mathbb{C})\} / \mathrm{Sp}(N(G), \mathbb{C}) & \text{in Case O1,} \\ \left\{ \begin{array}{l} \text{admissible } \psi : W_{K_1} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{O}(2n, \mathbb{C}) \\ \text{such that } \det(\psi) = (\mathrm{Art}_K^{-1}(-), \mathrm{disc}(V))_K \end{array} \right\} / \mathrm{SO}(2n, \mathbb{C}) & \text{in Case O2,} \\ \left\{ \begin{array}{l} \text{admissible } \phi : W_{K_1} \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GL}(N(G), \mathbb{C}) \\ \text{that is conjugate self-dual of sign } (-1)^{n(G)-1} \end{array} \right\} / \mathrm{GL}(N(G), \mathbb{C}) & \text{in Case U.} \end{cases}$$

Here ϕ is called admissible if

- $\phi(\sigma_{K_1})$ is semisimple,

- $\phi|_{I_{K_1}}$ is smooth; and

- $\phi|_{\mathrm{SL}(2, \mathbb{C})}$ is algebraic.

We write $\phi^{\natural} : W_K \times \mathrm{SL}(2, \mathbb{C}) \to {}^LG$ for the L-parameter corresponding to ϕ under this natural identification.

Then an L-parameter $\phi \in \Phi(G^*)$ is

- tempered (or bounded) if and only if $\phi(W_{K_1})$ is a pre-compact subset of the target.

- discrete if and only if $\mathrm{Im}(\phi)$ is not contained in any properly contained parabolic subgroup of the target.

- semisimple if and only if it is trivial on the $\mathrm{SL}(2, \mathbb{C})$-factor.

- supercuspidal if and only if it is discrete and semisimple.

The subset of tempered (resp. discrete/resp. semisimple/resp. supercuspidal) L-parameters for G^* is denoted by $\Phi_{\mathrm{temp}}(G^*)$ (resp. $\Phi_2(G^*)$/resp. $\Phi^{\mathrm{ss}}(G^*)$/resp. $\Phi^{\mathrm{sc}}(G^*)$).

An L-parameter $\phi \in \Phi(G^*)$ can be regarded as a $N(G)$-dimensional conjugate self-dual representation ϕ^{GL} of $W_{K_1} \times \mathrm{SL}(2,\mathbb{C})$ via the standard representation $\widehat{\mathrm{Std}}_G$. ϕ is determined by ϕ^{GL} in Case O1 and Case U, but only determined up to $\mathrm{O}(N(G),\mathbb{C})$-conjugation in Case O2; see [GGP12, Theorem 8.1]. We can write ϕ^{GL} as

$$\phi^{\mathrm{GL}} = \bigoplus_{i \in I_\phi^+} m_i \phi_i \oplus \bigoplus_{i \in I_\phi^-} 2m_i \phi_i \oplus \bigoplus_{i \in J_\phi} m_i(\phi_i \oplus \phi_i^\theta),$$

where m_i are positive integers, and $I_\phi^+, I_\phi^-, J_\phi$ index mutually inequivalent irreducible representations of $W_{K_1} \times \mathrm{SL}(2,\mathbb{C})$ such that

- for $i \in I_\phi^+$, ϕ_i is conjugate self-dual with sign $b(G)$.

- for $i \in I_\phi^-$, ϕ_i is conjugate self-dual with sign $-b(G)$.

- for $i \in J_\phi$, ϕ_i is not conjugate self-dual.

Then ϕ is discrete if and only if $m_i = 1$ for $i \in I_\phi^+$, and $I_\phi^- = J_\phi = \varnothing$. Moreover, we call ϕ simple if it is discrete and $\#I_\phi^+ = 1$.

For any $\phi \in \Phi(G^*)$, we define

$$S_\phi^\sharp := \prod_{i \in I_\phi^+} \mathrm{O}(m_i,\mathbb{C}) \times \prod_{i \in I_\phi^-} \mathrm{Sp}(2m_i,\mathbb{C}) \times \prod_{i \in J_\phi} \mathrm{GL}(m_i,\mathbb{C}),$$

which formally represents the centralizer of ϕ in ${}^LG/W_{K'}$ (where K' is a minimal splitting field of G), and its formal component group

$$\mathfrak{S}_\phi^\sharp := \pi_0(S_\phi^\sharp) \cong \bigoplus_{i \in I_\phi^+} (\mathbb{Z}/2)e_i,$$

where each e_i is a formal place-holder. There is an isomorphism $\mathrm{Irr}(\mathfrak{S}_\phi^\sharp) = \oplus_{i \in I_\phi^+}(\mathbb{Z}/2)e_i^\vee$, where $e_i^\vee(e_j) = \delta_{i,j}$ for $i,j \in I_\phi^+$. In Case O2, we can also define

$$S_\phi := Z_{\widehat{G}}(\phi)$$

which is naturally a subgroup of $S_\phi^\sharp$ of index at most two. There is a map

$$\det\nolimits_\phi : \mathfrak{S}_\phi^\sharp \to \mathbb{Z}/2, \quad \sum_{i \in I_\phi^+} x_i e_i \mapsto \sum_{i \in I_\phi^+} x_i \dim(\phi_i),$$

and we define the formal component group $\mathfrak{S}_\phi := \ker(\det_\phi) = \pi_0(S_\phi)$. To unify notation, in Case O1 and Case U, set $S_\phi = S_\phi^\sharp$ and $\mathfrak{S}_\phi = \mathfrak{S}_\phi^\sharp$.

We define the central element

$$z_\psi := \sum_{i \in I_\psi^+} m_i e_i \in \mathfrak{S}_\psi,$$

and define the reduced component group

$$\overline{\mathfrak{S}}_\psi := \mathfrak{S}_\psi / \langle z_\psi \rangle.$$

When $(G, \varrho_{b_0}, z_{b_0})$ is a pure inner form of G^*, an L-parameter for G^* is called (G, ϱ)-relevant if every Levi subgroup ${}^LM$ of ${}^LG$ such that $\mathrm{Im}(\phi) \subset {}^LM$ is relevant, i.e., ${}^LM$ is a Levi component of a (G, ϱ)-relevant parabolic subgroup of ${}^LG$; see [KMSW14, §0.4.2].

Let M be a standard Levi subgroup of G isomorphic to

$$\mathrm{Res}_{K_1/K}\mathrm{GL}(n_1) \times \cdots \mathrm{Res}_{K_1/K}\mathrm{GL}(n_k) \times G(n_0),$$

where $[K_1 : K](n_1 + \ldots + n_k) + n_0 = n(G)$, and we define

$$M^{\mathrm{GL}} := \mathrm{GL}(d_1)_{K_1} \times \cdots \times \mathrm{GL}(d_r)_{K_1} \times \mathrm{GL}(\frac{N(G)}{n(G)} n_0)_{K_1},$$

then there exists a canonical identification of $\Phi(M^{\mathrm{GL}})$ as tuples $(\phi_1, \ldots, \phi_k, \phi_0)$ where $\phi_i \in \Phi(\mathrm{GL}(n_i)_{K_1})$ and $\phi_0 \in \Phi(\mathrm{GL}(N(G))_{K_1})$. Then the canonical map $\Phi(M) \to \Phi(G)$ fits into the following commutative diagram

(2.3)
$$
\begin{array}{ccc}
\Phi(M) & \xrightarrow{\;(-)^{\mathrm{GL}}\;} & \Phi(M^{\mathrm{GL}}) \\
\downarrow & & \downarrow \\
\Phi(G) & \xrightarrow{\;(-)^{\mathrm{GL}}\;} & \Phi(G^{\mathrm{GL}})
\end{array}
\;,
$$

where the right vertical map is given by

(2.4) $$(\phi_1, \ldots, \phi_k, \phi_0) \mapsto \phi^{\mathrm{GL}} = \phi_1 + \ldots + \ldots \phi_k + \phi_0 + \phi_1^\theta + \ldots + \phi_k^\theta,$$

2.3. **The correspondence.** We state the local Langlands correspondence for pure inner twist (G, ϱ, z) of G^*. In Case O1, it is established by Arthur [Art13] when G is quasi-split, and by Ishimoto [Ish24, Theorem 3.15] when G is not quasi-split. In Case U it is established by Mok [Mok15, Theorem 2.5.1, Theorem 3.2.1] when G is quasi-split, and by Kaletha, Minguez, Shin and White [KMSW14, Theorem 1.6.1] when G is not quasi-split. In Case U, more properties of this correspondence are established in [CZ21b].

In Case O2, only a weak version of the local Langlands correspondence is established. This is due to the intrinsic nature of the endoscopy method, because when G^* is regarded as a twisted endoscopic group of $\mathrm{GL}(N(G))$, ϕ can only be recovered by ϕ^{GL} up to $\mathrm{O}(N(G), \mathbb{C})$-conjugation [GGP12, Theorem 8.1]. When G is quasi-split, the weak LLC is established by Arthur [Art13] (see also [AG17, Theorem 3.6]), and when G is not quasi-split, it is established by Chen and Zou [CZ21, Theorem A.1].

To state the weak version in Case O2, for each pure inner twist $(G, \varrho_{b_0}, z_{b_0})$ of G^*, we introduce an equivalence relation $\sim_\varsigma$ on $\Pi(G)$. Note that there exists an outer automorphism ς of G^* which preserves $\mathfrak{m}$: in fact ς can be realized as an element of the corresponding orthogonal group of determinant -1; see [Taï19, p. 847]. Via ϱ_{b_0}, the element ς acts by a rational outer automorphism on G; see [Art13, Lemma 9.1.1]. For each $\pi \in \Pi(G)$, its conjugate π is defined by $\pi^\varsigma(h) = \pi(h^\varsigma)$, and the equivalence relation $\sim_\varsigma$ is defined by

$$\pi \sim_\varsigma \pi^\varsigma.$$

For each $\pi \in \Pi(G)$, we write $\tilde{\pi}$ for the image of π under the quotient map $\Pi(G^*) \to \tilde{\Pi}(G^*)$, where $\tilde{\Pi}(G^*) = \Pi(G^*)/\sim_\varsigma$. It is clear that the sets $\Pi_{\mathrm{temp}}(G), \Pi_2(G)$ and $\Pi_{\mathrm{cusp}}(G)$ are preserved under this equivalence relation, so temperedness and discreteness are well-defined for equivalence classes $\tilde{\pi} \in \tilde{\Pi}(G^*)$.

Similarly, we define

$$\tilde{\Phi}(G^*) = \{\text{admissible } \phi : W_{K_1} \times \mathrm{SL}(2, \mathbb{C}) \to \mathrm{O}(N(G), \mathbb{C}) | \det(\phi) = (-, \mathrm{disc}(V))_K\}/\mathrm{O}(N(G), \mathbb{C}),$$

together with a natural map $\Phi(G^*) \to \tilde{\Phi}(G^*)$. Note that if ϕ_1 and ϕ_2 are conjugate up to $\mathrm{O}(N(G), \mathbb{C})$, then $\phi_1^{\mathrm{GL}} = \phi_2^{\mathrm{GL}}$. In particular, $\phi^{\mathrm{ss}}, \phi^{\mathrm{GL}}, I_\phi^+, I_\phi^-, J_\phi, S_\phi^\sharp, S_\phi, \mathfrak{S}_\phi, \overline{\mathfrak{S}}_\phi, z_\phi$ are well-defined functions for $\tilde{\phi} \in \tilde{\Phi}(G^*)$.

For uniformity of notation, in Case O1 and Case U set $\varsigma = \mathrm{id}_{G^*}$ and $\tilde{\Pi}(G^*) = \Pi(G^*)$, $\tilde{\Phi}(G^*) = \Phi(G^*)$. Define a subspace $\tilde{\mathcal{H}}(G) \subset \mathcal{H}(G)$ of test functions on $G(K)$ as follows. In Case O2, following Arthur [Art13], let $\tilde{\mathcal{H}}(G)$ denote the subspace of $\mathcal{H}(G)$ consisting of ς-invariant distributions on $G(K)$; so that irreducible smooth representations of $\tilde{\mathcal{H}}(G)$ correspond to $\mathrm{O}(V)$-conjugacy classes of irreducible admissible representations of $G(K)$. Similarly, for each $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(G)$, set

$$\tilde{\mathcal{H}}(G^\mathfrak{e}) = \tilde{\mathcal{H}}(G_1) \times \tilde{\mathcal{H}}(G_2),$$

since $G^\mathfrak{e}$ is a product of two (possibly trivial) even special orthogonal groups over K. In Case O1 and Case U, simply take $\tilde{\mathcal{H}}(G) = \mathcal{H}(G)$ and $\tilde{\mathcal{H}}(G^\mathfrak{e}) = \mathcal{H}(G^\mathfrak{e})$.

We can now reformulate the local Langlands correspondence as follows.

Theorem 2.3.1 ([Art13, Mok15, KMSW14, CZ21, CZ21b, Ish24]). *Suppose $(G, \varrho_{b_0}, z_{b_0})$ is a pure inner twist of G^*, then there exists a map*

$$\mathrm{rec}_G : \tilde{\Pi}(G) \to \tilde{\Phi}(G^*)$$

with finite fibers. For any $\tilde{\phi} \in \tilde{\Phi}(G^*)$, we write $\tilde{\Pi}_{\tilde{\phi}}(G)$ for $\mathrm{rec}_G^{-1}(\tilde{\phi})$, called the (ambiguous) L-packet for $\tilde{\phi}$. This map satisfies the following properties:

(1) If $\tilde{\phi} \in \tilde{\Phi}(G^*)$ is not relevant for G in the sense of [KMSW14, Definition 0.4.14], then $\Pi_{\tilde{\phi}}(G) = \varnothing$.

(2) For each $\tilde{\phi} \in \tilde{\Phi}(G^*)$ and $\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)$, $\tilde{\pi}$ is tempered if and only if $\tilde{\phi}$ is tempered, and $\tilde{\pi}$ is a discrete series representation if and only if $\tilde{\phi}$ is discrete.

(3) rec_G only depends on G but not on ϱ_{b_0} and z_{b_0}. For the fixed Whittaker datum $\mathfrak{m}$ of G^* which induces a Whittaker datum for each standard Levi factor of G, there exists a canonical bijection

$$\iota_{\mathfrak{m},b_0} : \tilde{\Pi}_{\tilde{\phi}}(G) \cong \mathrm{Irr}(\mathfrak{S}_{\tilde{\phi}}; \kappa_{b_0})$$

for each $\tilde{\phi} \in \tilde{\Phi}(G^*)$, where $\mathrm{Irr}(\mathfrak{S}_{\tilde{\phi}}; \kappa_{b_0})$ is the set of characters η of $\mathfrak{S}_{\tilde{\phi}}$ such that $\eta(z_{\tilde{\phi}}) = \kappa_{b_0}(-1)$. We write $\tilde{\pi} = \tilde{\pi}_{\mathfrak{m},b_0}(\tilde{\phi}, \eta)$ if $\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)$ corresponds to $\eta \in \mathrm{Irr}(\mathfrak{S}_{\tilde{\phi}})$ via $\iota_{\mathfrak{m},b_0}$.

(4) (Compatibility with Langlands quotient) For $\tilde{\phi} \in \tilde{\Phi}(G^*)$, suppose

$$\tilde{\phi}^{\mathrm{GL}} = \phi_1 \otimes |-|_{K_1}^{s_1} + \ldots + \phi_r \otimes |-|_{K_1}^{s_r} + \tilde{\phi}_0^{\mathrm{GL}} + \phi_r^\theta \otimes |-|_K^{-s_r} + \ldots + \phi_1^\theta \otimes |-|_K^{-s_1},$$

where

  - $\phi_i \in \Phi_{2,\mathrm{temp}}(\mathrm{GL}(d_i)_{K_1})$ for each $i \in [r]_+$, where $d_i > 0$,
  - $\tilde{\phi}_0 \in \tilde{\Phi}_{\mathrm{temp}}(G^*(n_0))$,
  - $s_1 \geq s_2 \geq \ldots \geq s_r > 0$,
  - $[K_1 : K](d_1 + \ldots + d_r) + n_0 = n$.

Let $\tau_i \in \Pi_{2,\mathrm{temp}}(\mathrm{GL}(d_i)_{K_1})$ be corresponding to ϕ_i for each $i \in [r]_+$, then there is a canonical identification

$$\tilde{\Pi}_{\tilde{\phi}}(G, \varrho) = \left\{ \mathrm{J}_P^G \left(\tau_1 \otimes \nu^{s_1} \boxtimes \cdots \boxtimes \tau_r \otimes \nu^{s_r} \boxtimes \tilde{\pi}_0 \right) \middle| \tilde{\pi}_0 \in \tilde{\Pi}_{\tilde{\phi}_0}(G(n_0)) \right\}.$$

Here P is a parabolic subgroup of G with a Levi factor

$$M \cong \mathrm{Res}_{K_1/K}\mathrm{GL}(d_1) \times \cdots \times \mathrm{Res}_{K_1/K}\mathrm{GL}(d_r) \times G(n_0),$$

such that $M = \varrho_{b_0}(M^*)$ where M^* is a standard Levi subgroup of G^*, and

$$\mathrm{J}_P^G \left(\tau_1 \otimes \nu^{s_1} \boxtimes \cdots \boxtimes \tau_r \otimes \nu^{s_r} \boxtimes \tilde{\pi}_0 \right)$$

is the unique irreducible quotient of

$$\mathrm{I}_P^G \left(\tau_1 \otimes \nu^{s_1} \boxtimes \cdots \boxtimes \tau_r \otimes \nu^{s_r} \boxtimes \tilde{\pi}_0 \right),$$

whose ς-equivalence class is well-defined in $\tilde{\Pi}(G)$. Moreover, if $\mathfrak{m}_0$ is the induced Whittaker datum on M^*, there exists a natural identification $\mathfrak{S}_{\tilde{\phi}_0} \cong \mathfrak{S}_{\tilde{\phi}}$, and $\iota_{\mathfrak{m},b_0}(\tilde{\pi}) = \iota_{\mathfrak{m}_0,b_0}(\tilde{\pi}_0)$ if $\tilde{\pi} = \mathrm{J}_P^G \left(\tau_1 \otimes \nu^{s_1} \boxtimes \cdots \boxtimes \tau_r \otimes \nu^{s_r} \boxtimes \tilde{\pi}_0 \right)$.

(5) (Compatibility with standard γ-factors) Suppose $\pi \in \Pi(G)$ with $\phi := \mathrm{rec}_G(\pi)$, then for any quasi-character χ of $K^\times$,

$$\gamma(\tilde{\pi}, \chi, \psi_K; s) = \gamma(\tilde{\phi}^{\mathrm{GL}} \otimes \chi, \psi_{K_1}; s).$$

Here the left-hand side is the standard γ-factor defined by Lapid–Rallis using the doubling zeta integral [LR05] but modified in [GI14], and the right-hand side is the γ-factor defined in [Tat79].

(6) (Compatibility with Plancherel measures) Suppose $\tilde{\pi} \in \tilde{\Pi}(G)$ with $\tilde{\phi} := \mathrm{rec}_G(\tilde{\pi})$, then for any $\tau \in \Pi(\mathrm{GL}(n)_{K_1})$ with L-parameter ϕ_τ,

$$\mu_{\psi_K}(\tau \otimes \nu^s \boxtimes \tilde{\pi}) = \gamma(\phi_\tau \otimes (\tilde{\phi}^{\mathrm{GL}})^\vee, \psi_{K_1}; s) \cdot \gamma(\psi_\tau^\vee \otimes \tilde{\phi}^{\mathrm{GL}}, \psi_{K_1}^{-1}, -s)$$
$$\cdot \gamma(\mathrm{R}_{G^*} \circ \phi_\tau, \psi_K; 2s) \cdot \gamma(\mathrm{R}_{G^*} \circ \phi_\tau^\vee, \psi_K^{-1}; -2s).$$

Here the left-hand side is the Plancherel measure defined in [GI14, §12] (cf. [GI16, §A.7]), and in the right-hand side R is the representation

$$\mathrm{R}_{G^*} = \begin{cases} \mathrm{Sym}^2 & \text{in Case O1} \\ \wedge^2 & \text{in Case O2} \\ \mathrm{As}^{(-1)^n} & \text{in Case U} \end{cases}$$

where As^+ and As^- are the two Asai representations of $\mathrm{Res}_{K_1/K}\mathrm{GL}(n)$.

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(7) (*Local intertwining relations*) *Suppose* $\tilde{\phi} \in \tilde{\Phi}(G^*)$ *such that* $\tilde{\phi}^{\mathrm{GL}} = \phi_\tau + \tilde{\phi}_0^{\mathrm{GL}} + \phi_\tau^\theta$, *where* $\phi_\tau \in \Phi(\mathrm{GL}(d)_{K_1})$ *is an L-parameter corresponding to* $\tau \in \Pi_{2,\mathrm{temp}}(\mathrm{GL}(d)_{K_1})$, $\tilde{\phi}_0 \in \tilde{\Phi}_{\mathrm{temp}}(G(n - [K_1 : K]d))$ *is tempered, and* $P \leq G$ *is a maximal parabolic subgroup with a Levi factor*

$$M \cong \mathrm{Res}_{K_1/K}\mathrm{GL}(d) \times G(n - [K_1 : K]d).$$

Assume that $M = \varrho_{b_0}(M^*)$ *where* M^* *is a standard Levi subgroup of* G^*, *and* $\mathfrak{m}_0$ *is the induced Whittaker datum on* M^*, *then for each* $\eta_0 \in \mathrm{Irr}(\mathfrak{S}_{\tilde{\phi}_0})$,

$$\mathrm{I}_P^G(\tau \boxtimes \tilde{\pi}_{\mathfrak{m}_0,b_0}(\eta_0)) = \bigoplus_\eta \tilde{\pi}_{\mathfrak{m},b_0}(\tilde{\phi},\eta)$$

as a $\tilde{\mathcal{H}}(G)$-*module, where* η *runs through characters of* $\mathfrak{S}_{\tilde{\phi}}$ *that restricts to* η_0 *under the natural embedding* $\mathfrak{S}_{\tilde{\phi}_0} \hookrightarrow \mathfrak{S}_{\tilde{\phi}}$. *Moreover, if* ϕ_τ *is conjugate self-dual of sign* $b(G)$, *let*

$$\mathrm{R}_{\mathfrak{m}}(w, \tau \boxtimes \tilde{\pi}_0) \in \mathrm{End}_{G(K)}\left(\mathrm{I}_P^G(\tau \boxtimes \tilde{\pi}_0)\right)$$

be the normalized intertwining operator defined in [CZ21, §7.1] *in Case* O2 *and in* [CZ21b, §5.2] *in Case* U, *where* w *is the unique nontrivial element in the relative Weyl group for* M, *then the restriction of* $\mathrm{R}_{\mathfrak{m}}(w, \tau \boxtimes \tilde{\pi}_0)$ *to* $\tilde{\pi}_{\mathfrak{m},b_0}(\tilde{\phi},\eta)$ *is the scalar multiplication by*

$$\begin{cases} \eta(e_\tau) & \text{in Case O} \\ \eta(e_\tau)\kappa_{b_0}^d(-1) & \text{in Case U} \end{cases}$$

where e_τ *is the element of* $\mathfrak{S}_{\tilde{\phi}}$ *corresponding to* ϕ_τ.

Remark 2.3.2.
   (1) The independence of rec_G with respect to ϱ and z is established in Case O by [CZ21, Remark 4.6(2)] and in Case U by the argument before [CZ21b, Theorem 2.5.5].

   (2) The compatibility of rec_G with standard γ-factors and Plancherel measures can be used to characterize rec_G and show compatibility with LLC constructed via exceptional isomorphisms in low dimensions, via [GI16, Lemma A.6]. For example, in Case O2, when G is quasi-split, it is shown that the construction of Chen-Zou using theta correspondence is compatible with that defined by Arthur [CZ21, Theorem 9.1], and in Case U, when G is quasi-split, it is shown that construction of Chen-Zou using theta correspondence is compatible with that defined by Mok [CZ21b, Theorem 7.1.1].

   (3) The local intertwining relation can be used to characterize the finer structure of L-packets, i.e., $\iota_{\mathfrak{m},b_0}$. In Case O1, it follow from [Art13, Proposition 2.3.1 and Theorems 2.2.1, 2.2.4, 2.4.1 and 2.4.4] when G is quasi-split and follows from the corresponding propositions in [Ish24, §4] when G is not quasi-split. In Case U, it follows from [Mok15, Theorem 3.2.1, 3.4.3] when G is quasi-split and [CZ21b, Theorem 2.5.1] when G is not quasi-split. In Case O2, it follows from [Art13, Proposition 2.3.1 and Theorems 2.2.1, 2.2.4, 2.4.1 and 2.4.4] when G is quasi-split and follows from [CZ21, Theorem A.1] when G is not quasi-split. For the details, see, for example, [Ato17, Theorem 2.2].

   (4) Note that in Case U, the LLC stated in [Mok15] and [KMSW14] are compatible with the arithmetic normalization of the local class field theory instead of the geometric normalization of the local class field theory, i.e., the Artin map is normalized so that it maps an arithmetic Frobenius morphism to a uniformizer instead of sending a geometric Frobenius morphism to a uniformizer. But this choice can be switched by using the compatibility of local Langlands correspondence with contragredients [Kal13]; cf. [BMN23, Theorem 2.5].

We will always call this correspondence the *classical local Langlands correspondence (LLC)*, as opposed to the Fargues–Scholze local Langlands correspondence to be defined in §3.

For later use, we define what is a local functorial transfer from G to G^{GL}:

Definition 2.3.3. Let $\pi \in \Pi(G)$ be an irreducible admissible representation with classical L-parameter $\tilde{\phi}$, the unique irreducible admissible representation $\pi^{\mathrm{GL}} \in \Pi(G^{\mathrm{GL}})$ with classical L-parameter $\phi_{\pi^{\mathrm{GL}}} = \tilde{\phi}_\pi^{\mathrm{GL}}$ is called a *local functorial transfer* of π.

We also need certain endoscopic character identities for the local Langlands correspondence. In Case O1, they are established by Arthur [Art13, Theorem 2.2.1] when G is quasi-split, and by Ishimoto [Ish24, Theorem 3.15] when G is not quasi-split. In Case U, they are established by Mok [Mok15] when G is quasi-split, and by Kaletha, Minguez, Shin and White [KMSW14, Theorem 1.6.1] when G is not quasi-split. In Case O2, they are established by Arthur [Art13, Theorem 2.2.1] when G is quasi-split, and established by [Pen25, Theorem A] when G is not quasi-split. We adopt the notation for endoscopy from §A.

Theorem 2.3.4. *Fix a pure inner twist $(G, \varrho_{b_0}, z_{b_0})$ of G^*.*

(1) *Suppose $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(G)$ and $\phi^{\mathfrak{e}} \in \Phi_{2,\mathrm{temp}}(G^{\mathfrak{e}})$ with $\phi = {}^L\xi^{\mathfrak{e}} \circ \phi^{\mathfrak{e}}$ and $s^{\mathfrak{e}} \in S_\phi$. Then each $f \in \tilde{\mathcal{H}}(G)$ has a transfer $f^{G^{\mathfrak{e}}}$ contained in $\tilde{\mathcal{H}}(G^{\mathfrak{e}})$, and*

$$\sum_{\tilde{\pi}^{\mathfrak{e}} \in \tilde{\Pi}_{\tilde{\phi}^{\mathfrak{e}}}(G^{\mathfrak{e}})} \mathrm{tr}\left(f^{G^{\mathfrak{e}}} | \tilde{\pi}^{\mathfrak{e}}\right) = e(G) \sum_{\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)} \iota_{\mathfrak{m}, b_0}(\tilde{\pi})(s^{\mathfrak{e}}) \mathrm{tr}\left(f | \tilde{\pi}\right).$$

Here $e(G)$ is the Kottwitz sign of G as defined in [Kot83], $\Delta[\mathfrak{m}, \mathfrak{e}, z_{b_0}]$ is the transfer factor normalized in §A.2, and $\tilde{\Pi}_{\tilde{\phi}^{\mathfrak{e}}}(G^{\mathfrak{e}})$ is defined to be the (ambiguous) L-packet associated to $\tilde{\phi}^{\mathfrak{e}}$ as before.[3]

In particular, it follows from [AK24, Lemma 2.11.2] that

$$\sum_{h \in G^{\mathfrak{e}}(K)_{\mathrm{s.reg}}/G^{\mathfrak{e}}(\overline{K})\text{-conj}} \Delta[\mathfrak{m}, \mathfrak{e}, z_{b_0}](h, g) \mathrm{S}\Theta_{\tilde{\phi}^{\mathfrak{e}}}(h) = e(G) \sum_{\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)} \iota_{\mathfrak{m}, b_0}(\tilde{\pi})(s^{\mathfrak{e}}) \Theta_{\tilde{\pi}}(g)$$

for any strongly regular semisimple element $g \in G(K)_{\mathrm{s.reg}}$, where $\mathrm{S}\Theta_{\tilde{\phi}^{\mathfrak{e}}} = \sum_{\tilde{\pi} \in \Pi_{\tilde{\phi}^{\mathfrak{e}}}(G^{\mathfrak{e}})} \Theta_{\tilde{\pi}}$ and $\Theta_{\tilde{\pi}}$ is the average of Θ_π where π runs through the preimage of $\tilde{\pi}$ under the map $\Pi(G^{\mathfrak{e}}) \to \tilde{\Pi}(G^{\mathfrak{e}})$.

(2) *For any tempered L-parameter $\tilde{\phi} \in \tilde{\Phi}_{\mathrm{temp}}(G^*)$, if $f^{\mathrm{GL}} \in \mathcal{H}(G^{\mathrm{GL}})$, then f^{GL} has a transfer f^* to G^* contained in $\tilde{\mathcal{H}}(G^*)$, and*

$$\sum_{\tilde{\rho} \in \tilde{\Pi}_{\tilde{\phi}}(G)} \mathrm{tr}(f^* | \tilde{\rho}) = \mathrm{tr}_\theta(\tilde{f} | \pi_{\tilde{\phi}^{\mathrm{GL}}}).$$

Here the right-hand side is the θ-twisted trace, and $\pi_{\tilde{\phi}^{\mathrm{GL}}} \in \Pi(G^{\mathrm{GL}})$ is associated to the L-parameter $\tilde{\phi}^{\mathrm{GL}}$ via LLC, and the left-hand side is a stable distributions of f, i.e., it vanishes when all stable orbital integrals of f vanish.

Finally, we remark that the classical local Langlands correspondence for inner twists (or rather K-groups) of special orthogonal or unitary groups over $\mathbb{R}$ and (ordinary) endoscopic character identities are known in complete generality [ABV92, Vog93, She08, She10, Art13], where we replace irreducible admissible representations by smooth Fréchet representations of moderate growth with associated Harish-Chandra modules being admissible, as introduced by Casselman [Cas89] and Wallach [Wal92]. Moreover, the twisted endoscopic character identities are known by Arthur [Art13]. The real case is analogous to the p-adic case, except that there are more inner twists. In fact, we will only use results concerning discrete series L-packets. For a modern exposition in the discrete series case, see [AK24].

2.4. **Compatibility with parabolic inductions.** In this subsection, we recall the definition of extended cuspidal support of a discrete series representation of $G(K)$, and deduce a compatibility property of the classical semisimplified L-parameters with parabolic induction. In Case U, the assertion is established in [MHN24, Proposition 2.11], but our proof is slightly simpler than the proof given there.

Firstly we recall that it is a theorem of Bernstein and Zelevinski [BZ77, Theorem 2.5, Theorem 2.9] that all irreducible smooth representations of $G(K)$ can be constructed by parabolic induction from supercuspidal representations:

Theorem 2.4.1. *For any irreducible smooth representation $\pi \in \Pi(G)$, there exists a unique pair (M, σ) up to conjugacy by $G(K)$, where M is a Levi component of some parabolic subgroup $P \le G$ and $\sigma \in \Pi_{\mathrm{sc}}(M)$ is a supercuspidal representation, such that π is a subrepresentation of $\mathrm{I}_P^G(\sigma)$. Such pairs (M, σ) are called* cuspidal supports *of π.*

Suppose $\pi \in \Pi_{2,\mathrm{temp}}(G)$ is a discrete series representation, we write $\pi^{\mathrm{GL}} \in \Pi(\mathrm{GL}(N(G))_{K_1})$ for the representation corresponding to the L-parameter $\tilde{\phi}_\pi^{\mathrm{GL}}$, and write $\mathrm{SuppCusp}^+(\pi)$ for the cuspidal support of π^{GL}, called the *extended cuspidal support* of π.

---

[3] This L-packet is well defined because $G^{\mathfrak{e}}$ is a product of special orthogonal or unitary groups.

Lemma 2.4.2. *Suppose $\pi \in \Pi_{2,\mathrm{temp}}(G)$ is a discrete series representation, and (M, π_M) is a cuspidal support of π, then $\mathrm{SuppCusp}^+(\pi_M) = \mathrm{SuppCusp}^+(\pi)$.*

Proof. If G is quasi-split, this is established in [Moe14, p. 309]. We note that by the endoscopic character identities Theorem 2.3.4, the classical L-packets are the same as the L-packets defined in [Moe14, §6.4]. For details, the reader is referred to [MHN24, Proposition 2.10].

When G is not quasi-split, in view of the endoscopic character identities Theorem 2.3.4, we can reduce the theorem to the quasi-split case using the alternative definition of $\mathrm{SuppCusp}^+(\pi)$ via Jacquet modules [Moe14] and the fact that transfers are compatible with parabolic induction and the formation of Jacquet modules; see [She82, Proposition 3.4.2] and [Moe14, §2.6]. $\qquad\square$

Proposition 2.4.3. *If $P \leq G$ is a properly contained parabolic subgroup with Levi factor M, $\pi_M \in \Pi(M)$ with semisimplified L-parameter $\tilde{\phi}_M^{\mathrm{ss}} : W_{K_1} \to {}^L M$,[4] and $\pi \in \Pi(G)$ is a subquotient of the normalized parabolic induction $\mathrm{I}_P^G(\pi_M)$, then the semisimplified L-parameter $\tilde{\phi}_\pi^{\mathrm{ss}}$ is given by*

$$\tilde{\phi}_\pi^{\mathrm{ss}} : W_{K_1} \xrightarrow{\tilde{\phi}_M^{\mathrm{ss}}} {}^L M \to {}^L G.$$

Proof. Firstly, the same statement about compatibility of semisimplified L-parameters with parabolic induction is true when G is a general linear group, by the result of Bernstein and Zelevinski, cf. [Rod82, Théorème 4].

We argue by induction on $n(G)$. Let $\pi \in \Pi(G)$ occur as a subquotient of $\mathrm{I}_P^M(\pi_M)$. It suffices to prove the theorem when (M, π_M) is a cuspidal support of π. Indeed, assume the statement is known for all groups with smaller rank. If $(M', \pi_{M'})$ is a cuspidal support of π_M, then by transitivity of parabolic inductions it is also a cuspidal support of π. By induction hypothesis (applied to M, which is of the form $G(n_0) \times \mathrm{Res}_{K_1/K}(H)$ for some integer $0 < n_0 < n(G)$ and H is a product of general linear groups), $\tilde{\phi}_{\pi_M}$ is the composition of $\tilde{\phi}_{\pi_{M'}}$ and the canonical L-homomorphism ${}^L M' \to {}^L M$. On the other hand, $\tilde{\phi}_\pi$ is the composition of $\tilde{\phi}_{\pi_{M'}}$ and the canonical L-homomorphism ${}^L M' \to {}^L G$. Therefore $\tilde{\phi}_\pi$ is the composition of $\tilde{\phi}_{\pi_M}$ and the canonical L-homomorphism ${}^L M \to {}^L G$, as desired.

So we assume that (M, π_M) is a cuspidal support of π. By the Langlands classification of irreducible representations in terms of irreducible tempered representations (see [Sil78] and [Kon03, Theorem 3.5]), there exists a parabolic subgroup $P \leq G$ with a Levi factor

$$M \cong \mathrm{Res}_{K_1/K}\mathrm{GL}(d_1) \times \cdots \times \mathrm{Res}_{K_1/K}\mathrm{GL}(d_r) \times G(n_0),$$

such that π is the unique irreducible quotient of

$$\mathrm{I}_P^G\left(\tau_1 \otimes \nu^{s_1} \boxtimes \cdots \boxtimes \tau_r \otimes \nu^{s_r} \boxtimes \pi_0\right)$$

where $\tau_i \in \Pi_{\mathrm{temp}}(\mathrm{GL}(d_i)_{K_1})$ and $\pi_0 \in \Pi_{\mathrm{temp}}(G(n_0))$ are tempered representations, and $s_1 \geq s_2 \geq \ldots \geq s_r > 0$ are real numbers. Then it follows from the compatibility of classical LLC with Langlands quotient Theorem 2.3.1 that

$$\tilde{\phi}_\pi^{\mathrm{GL}} = \phi_{\tau_1}\,|-|_{K_1}^{s_1} + \ldots + \phi_{\tau_r}\,|-|_{K_1}^{s_r} + \tilde{\phi}_{\pi_0}^{\mathrm{GL}} + \phi_{\tau_r}^\theta\,|-|_{K_1}^{-s_r} + \ldots + \phi_{\tau_1}^\theta\,|-|_{K_1}^{-s_1}.$$

Thus $\tilde{\phi}_\pi$ is exactly the classical L-parameter of $\tau_1 \otimes \nu^{s_1} \boxtimes \cdots \boxtimes \tau_r \otimes \nu^{s_r} \boxtimes \pi_0$ composed with the canonical embedding ${}^L M \to {}^L G$. By the same argument using cuspidal support and the transitivity of parabolic inductions as above (and using that classical LLC for general linear groups are compatible with twisting by characters), we can reduce the theorem to the case when π is tempered.

We assume henceforth that π is tempered, then there exists a parabolic subgroup $P \leq G$ with a Levi factor $M \leq P$, and a discrete series representation $\sigma \in \Pi_{2,\mathrm{temp}}(M)$ such that π is a subrepresentation of $\mathrm{I}_P^G(\sigma)$, and $\tilde{\phi}_\pi$ is the classical L-parameter of $\tilde{\phi}_\sigma$ composed with the canonical embedding ${}^L M \to {}^L G$. This follows from the classification of tempered representations of classical groups [Jan14]. More explicitly, if π is not a discrete series representation and we write

$$\phi^{\mathrm{GL}} = \bigoplus_{i \in I_{\tilde{\phi}}^+} m_i \phi_i \oplus \bigoplus_{i \in I_{\tilde{\phi}}^-} 2m_i \phi_i \oplus \bigoplus_{i \in J_{\tilde{\phi}}} m_i(\phi_i \oplus \phi_i^\theta),$$

---

[4]The L-parameter $\tilde{\phi}_M^{\mathrm{ss}}$ is well-defined because M is the product of a special orthogonal or unitary group with restrictions of general linear groups

as in §2.2, then either $m_i > 1$ for some $i \in I_{\tilde\phi}^+$ or $m_i > 0$ for some $i \in I_{\tilde\phi}^- \cup J_{\tilde\phi}$. In either case we can write

$$\tilde\phi^{\mathrm{GL}} = \phi_\tau + \tilde\phi_{\pi_0}^{\mathrm{GL}} + \phi_\tau^\theta,$$

where $\tau \in \Pi_{2,\mathrm{temp}}(\mathrm{GL}(d)_{K_1})$ is a discrete series representation and $\pi_0 \in \Pi_{\mathrm{temp}}(G(n-[K_1 : K]d))$ is tempered. Then it follows from the local intertwining relations Theorem 2.3.1 that π is a subrepresentation of $\mathrm{I}_P^G(\tau \otimes \pi_0)$, where $P \leq G$ is a maximal parabolic subgroup of with Levi factor $M \cong \mathrm{Res}_{K_1/K}\mathrm{GL}(d) \times G(n - [K_1 : K]d)$. Hence the claim follows from from an easy induction on $\sum_{i \in I_{\tilde\phi}^+ \cup I_{\tilde\phi}^- \cup J_{\tilde\phi}} m_i$. By the same argument using cuspidal support and the transitivity of parabolic inductions as above, we can reduce the theorem to the case when π is a discrete series representation.

We assume henceforth that π is a discrete series representation. Write

$$M \cong \mathrm{Res}_{K_1/K}\mathrm{GL}(d_1) \times \cdots \times \mathrm{Res}_{K_1/K}\mathrm{GL}(d_r) \times G(n_0)$$

and

$$\pi_M = \tau_1 \otimes \ldots \otimes \tau_r \otimes \pi_0 \in \Pi_{\mathrm{sc}}(M),$$

and we define

$$M_\diamond^{\mathrm{GL}} := \mathrm{Res}_{K_1/K}\mathrm{GL}(d_1) \times \cdots \times \mathrm{Res}_{K_1/K}\mathrm{GL}(d_r) \times \mathrm{Res}_{K_1/K}\mathrm{GL}(\frac{N(G)}{n(G)}n_0)$$
$$\times \mathrm{Res}_{K_1/K}\mathrm{GL}(d_r) \times \cdots \times \mathrm{Res}_{K_1/K}\mathrm{GL}(d_1),$$

which is a Levi subgroup of G^{GL}. We then have the commutative diagram

$$\begin{array}{ccccc}
\tilde\Phi(M) & \xrightarrow{(-)^{\mathrm{GL}}} & \Phi(M^{\mathrm{GL}}) & \xrightarrow{(-)_\diamond} & \Phi(M_\diamond^{\mathrm{GL}}) \\
\downarrow & & & \searrow & \downarrow \\
\tilde\Phi(G) & \xrightarrow{\qquad (-)^{\mathrm{GL}} \qquad} & & & \Phi(G^{\mathrm{GL}})
\end{array}$$

from (2.3), where the vertical maps are the canonical embedding, the slanted map is given by (2.4), and $(-)_\diamond$ is given by

$$(\phi_1, \ldots, \phi_r, \phi_0) \mapsto (\phi_1, \ldots, \phi_r, \phi_0, \phi_r^\theta, \ldots, \phi_1^\theta).$$

All the maps are compatible with $(-)^{\mathrm{ss}}$.

Let $\tau_{M_\diamond} \in \Pi_{\mathrm{temp}}(M_\diamond^{\mathrm{GL}})$ (resp. $\tau_G \in \Pi_{\mathrm{temp}}(G^{\mathrm{GL}})$) be the tempered representation with classical L-parameter $(\tilde\phi_{\pi_M})_\diamond^{\mathrm{ss},\mathrm{GL}}$ (resp. $\tilde\phi_\pi^{\mathrm{ss},\mathrm{GL}}$). It follows that if the tempered representation of $\mathrm{GL}(\frac{N(G)}{n(G)}n_0)_{K_1}$ with classical L-parameter $\tilde\phi_{\pi_0}^{\mathrm{ss},\mathrm{GL}}$ has a cuspidal support (M_0, σ_0), then (M', σ') is a cuspidal support for both $\tau_{M_\diamond}$ and $\tau_{G^{\mathrm{GL}}}$ by Lemma 2.4.2, where

$$M' \cong \mathrm{Res}_{K_1/K}\mathrm{GL}(d_1) \times \cdots \times \mathrm{Res}_{K_1/K}\mathrm{GL}(d_r) \times M_0 \times \mathrm{Res}_{K_1/K}\mathrm{GL}(d_r) \times \cdots \times \mathrm{Res}_{K_1/K}\mathrm{GL}(d_1) \leq M_\diamond^{\mathrm{GL}}$$

and σ_{τ_G} corresponds to

$$\sigma' = \tau_1 \otimes \ldots \tau_r \otimes \sigma_0 \otimes \tau_r^\theta \otimes \ldots \otimes \tau_1^\theta$$

under this isomorphism. So the theorem follows from the compatibility of semisimplified L-parameters for G^{GL} and $M_\diamond^{\mathrm{GL}}$. $\qquad\square$

2.5. Supercuspidal L-packets.

We recall a result of Moeglin and Tadic that characterizes supercuspidal L-parameters in terms of their corresponding L-packets.

Let $\tilde\phi \in \tilde\Phi_2(G^*)$ be a discrete L-parameter. We write $\mathrm{Jord}(\tilde\phi)$ for the set of irreducible subrepresentations of $W_{K_1} \times \mathrm{SL}(2,\mathbb{C})$ contained in $\tilde\phi^{\mathrm{GL}}$.

An L-parameter $\tilde\phi$ is said to be *without gaps* if for every $\rho \boxtimes \mathrm{sp}_a \in \mathrm{Jord}(\tilde\phi)$ with $a > 2$, one also has $\rho \boxtimes \mathrm{sp}_{a-2} \in \mathrm{Jord}(\tilde\phi)$.

We recall the following characterization of supercuspidal representations of $G(K)$:

Proposition 2.5.1. *Suppose $\tilde\phi \in \tilde\Phi_2(G^*)$ is a discrete L-parameter and $\eta \in \mathrm{Irr}(\overline{\mathfrak{S}}_{\tilde\phi})$, then $\tilde\pi_{\mathfrak{m},\mathbf{1}}(\tilde\phi, \eta) \in \tilde\Pi_{\tilde\phi}(G^*)$ is supercuspidal if and only if the following two conditions hold:*

- *$\tilde\phi$ is without gaps,*
- *$\eta(e_{\rho \boxtimes \mathrm{sp}_a}) = -\eta(e_{\rho \boxtimes \mathrm{sp}_{a-2}})$ for each $\rho \boxtimes \mathrm{sp}_a \in \mathrm{Jord}(\tilde\phi)$ with $a \geq 2$, where we assume that $\eta(e_{\rho \boxtimes \mathrm{sp}_0}) = 1$.*

Proof. This is established in Case O in [Moe02, MT02], see also [Moe11, Theorem 1.5.1] and [Xu17, Theorem 3.3], and in Case U in [Moe07, Theorem 8.4.4]. Note that by endoscopic character identities Theorem 2.3.4, the LLC defined in [Mok15] is the same as the LLC defined in [Moe07]; cf. [MHN24, Proposition 2.10]. □

We then obtain the following corollary characterizing supercuspidal L-packets:

Corollary 2.5.2. *For a discrete L-parameter $\tilde{\phi} \in \tilde{\Phi}_2(G^*)$, $\tilde{\Pi}_{\tilde{\phi}}(G^*)$ consists of supercuspidal representations if and only if $\tilde{\phi}$ is a supercuspidal L-parameter.*

Proof. If $\tilde{\phi}$ is supercuspidal, then clearly it is without gaps, and it follows from Proposition 2.5.1 that $\tilde{\Phi}_{\tilde{\phi}}(G)$ consists of supercuspidal representations.

Conversely, suppose $\tilde{\Phi}_{\tilde{\phi}}(G)$ consists of supercuspidal representations, then $\tilde{\phi}$ is without gaps by Proposition 2.5.1. If $\rho \boxtimes \mathrm{sp}_a \in \mathrm{Jord}(\tilde{\phi})$ and $a \geq 2$, then there exists a character η of $\mathfrak{S}_{\tilde{\phi}}$ such that $\eta(z_{\tilde{\phi}}) = 1$ and $\eta(\rho \boxtimes \mathrm{sp}_a) = \eta(\rho \boxtimes \mathrm{sp}_{a-2})$. So it follows from Proposition 2.5.1 that $\tilde{\pi}_{\mathfrak{m},\mathbf{1}}(\tilde{\phi}, \eta) \in \tilde{\Pi}_{\tilde{\phi}}(G^*)$ is not supercuspidal, contradiction. Thus we conclude that $a = 1$ for each $\rho \boxtimes \mathrm{sp}_a \in \mathrm{Jord}(\tilde{\phi})$, which means that $\tilde{\phi}$ is supercuspidal. □

2.6. **Combinatorics on L-parameters.** Following [MHN24, §2.2.4], we give a combinatorial description of discrete L-packets that will be used in the explicit computations when proving Kottwitz conjecture later.

For $b \in G(\breve{K})$, if $\tilde{\phi} \in \tilde{\Phi}_2(G^*)$ is a discrete L-parameter and $\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G), \tilde{\rho} \in \tilde{\Pi}_{\tilde{\phi}}(G_b)$, we can define a character
$$\delta[\tilde{\pi}, \tilde{\rho}] = \iota_{\mathfrak{m},b_0}(\tilde{\pi})^\vee \otimes \iota_{\mathfrak{m},b_0+b}(\tilde{\rho}) \in \mathrm{Irr}(\mathfrak{S}_{\tilde{\phi}}),$$
which can be thought of as measuring the relative position of $\tilde{\pi}$ and $\tilde{\rho}$. This character is independent of the Whittaker datum $\mathfrak{m}$ and also b, because changing those shifts $\iota_{\mathfrak{m},b_0}$ by a character of $\mathfrak{S}_{\tilde{\phi}}$ [HKW22, Lemma 2.3.3].

Let $b_1 \in B(G)_{\mathrm{bas}}$ be the unique nontrivial basic element. Because $\tilde{\phi}$ is discrete, the packet $\tilde{\Pi}_{\tilde{\phi}}(G)$ (resp. via $\tilde{\Pi}_{\tilde{\phi}}(G_b)$) has size $\#\mathfrak{S}_{\tilde{\phi}}/2$ and corresponds via $\iota_{\mathfrak{m},b_0}$ (resp. $\iota_{\mathfrak{m},b_0+b}$) to those characters η of $\mathfrak{S}_{\tilde{\phi}}$ such that $\eta(z_{\tilde{\phi}}) = \kappa_{b_0}(-1)$ (resp. $\eta(z_{\tilde{\phi}}) = -\kappa_{b_0}(-1)$).

Write $\tilde{\phi}^{\mathrm{GL}} = \phi_1 + \ldots + \phi_k + \phi_{k+1} + \ldots + \phi_r$, where ϕ_i are irreducible representations of $W_{K_1} \times \mathrm{SL}(2, \mathbb{C})$ such that $\dim \phi_i$ is odd if and only if $i \leq k$. In Case O2, for subsets $I, J \subset [r]_+$, we define the symmetric different $I \oplus J = (I \setminus J) \cup (J \setminus I)$. We say two subsets $I_1, I_2 \subset [r]_+$ are equivalent (denoted by $I \sim_k J$) if $I_1 = I_2$ or $I_1 = I_2 \oplus [k]_+$. Note that we can talk about parity of cardinality of equivalences classes of $\mathscr{P}([r]_+)$, because k is always an even number in Case O2.

To unify notation, in Case O1 or Case U, we define the equivalence relation to be trivial. Then $\mathrm{Irr}(\mathfrak{S}_{\tilde{\phi}})$ is in bijection with the set of equivalence classes of $\mathscr{P}([r]_+)$ via the map
$$[I] \in \mathscr{P}([r]_+)/\sim_k \mapsto \eta_{[I]} := \sum_{i \in I} e_i^\vee \in \mathrm{Irr}(\mathfrak{S}_{\tilde{\phi}}),$$
where $I \in \mathscr{P}([r]_+)$ is an arbitrary representative for $[I]$ on the right-hand side. If $[I] \in \mathscr{P}([r]_+)/\sim_k$ has cardinality $\#[I] \equiv \frac{\kappa_{b_0}(-1)-1}{2} \pmod{2}$, then we write $\tilde{\pi}_{[I]} = \tilde{\pi}_{\mathfrak{m},b_0}(\tilde{\phi}, \eta_{[I]})$. On the other hand, if $[I] \in \mathscr{P}([r]_+)/\sim_k$ has cardinality $\#[I] \equiv \frac{\kappa_{b_0}(-1)+1}{2} \pmod{2}$, then we write $\tilde{\pi}_{[I]} = \tilde{\pi}_{\mathfrak{m},b_0+b_1}(\tilde{\phi}, \eta_{[I]})$. In particular, for any $[I], [J] \in \mathscr{P}([r]_+)/\sim_k$,
$$\delta[\tilde{\pi}_{[I]}, \tilde{\pi}_{[J]}] = \sum_{I \oplus J} e_i^\vee \in \mathrm{Irr}(\mathfrak{S}_{\tilde{\phi}}),$$
where $I, J \in \mathscr{P}([r]_+)$ are representatives for $[I], [J]$, respectively.

Let $\mu_1 \in X_\bullet(G)$ be the dominant cocharacter given by

(2.5)
$$z \mapsto \begin{cases} \mathrm{diag}(z, \underbrace{1, \ldots, 1}_{d(G)-1\text{-many}}) & \text{in Case U} \\ \mathrm{diag}(z, \underbrace{1, \ldots, 1}_{d(G)-2\text{-many}}, z^{-1}) & \text{in Case O} \end{cases},$$

(i.e., $\mu_1 = \omega_1^\vee$ in the standard notation), where in Case U we use the standard realization of $G_{\overline{K}} \cong \mathrm{GL}(n(G))_{\overline{K}}$, and in Case O we use the standard realization of $G_{\overline{K}}$ as the subgroup of $\mathrm{SL}(d(G))_{\overline{K}}$ preserving the non-degenerate bilinear form on $\overline{K}\{v_1, \ldots, v_{d(G)}\}$ defined by $\langle v_i, v_j \rangle = (1 + \delta_{i,j})\delta_{i,d(G)+1-j}$.

Then the highest weight tilting module $\mathcal{T}_{\mu_1}$ of $\widehat{G}$ is the the representation $\widehat{\mathrm{Std}}_G$ of $\widehat{G}$.

Theorem 2.6.1. *Let $\tilde{\phi} \in \tilde{\Phi}_2(G^*)$ is a discrete L-parameter with a decomposition*

$$\tilde{\phi}^{\mathrm{GL}} = \phi_1 + \ldots + \phi_k + \phi_{k+1} + \ldots + \phi_r,$$

where ϕ_i are distinct irreducible representations of $W_{K_1} \times \mathrm{SL}(2,\mathbb{C})$ of dimension d_i, such that d_i is odd if and only $i \leq k$. Let $b_1 \leq B(G)_{\mathrm{bas}}$ be the unique nontrivial basic element. If $\tilde{\rho} = \tilde{\pi}_{[I]} \in \tilde{\Pi}_{\tilde{\phi}}(G_{b_1})$ where $I \subset [r]_+$ such that $\#I \equiv \frac{\kappa_{b_0}(-1)+1}{2} (\mathrm{mod}\ 2)$, then

$$\mathrm{Hom}_{\mathfrak{S}_{\tilde{\phi}}}(\delta[\tilde{\pi},\tilde{\rho}], \tilde{\phi}^{\mathrm{GL}}) \cong \begin{cases} \phi_i & \text{if } k \neq 2 \ \& \ \tilde{\pi} = \tilde{\pi}_{[I \oplus \{i\}]} \text{ for some } i \in [r]_+ \\ \phi_i & \text{if } k = 2 \ \& \ \tilde{\pi} = \tilde{\pi}_{[I \oplus \{i\}]} \text{ for some } 3 \leq i \leq r \\ \phi_1 + \phi_2 & \text{if } k = 2 \ \& \ \tilde{\pi} = \tilde{\pi}_{[I \oplus \{i\}]} \text{ for some } i \in [2]_+ \\ 0 & \text{otherwise} \end{cases}$$

as representations of W_{K_1}.

Proof. Because $\mathfrak{S}_{\tilde{\phi}}$ acts on ϕ_i by the character $e_i^\vee$, the assertion follows from combinatorial considerations. $\square$

3. Local Langlands correspondence via moduli of local shtukas

In this section, we review the construction of local Langlands correspondence of Fargues–Scholze and related constructions.

Let p be a rational prime and K be a finite extension of $\mathbb{Q}_p$. Let ℓ be a rational primes different from p, and $\Lambda \in \{\overline{\mathbb{Q}_\ell}, \overline{\mathbb{F}_\ell}\}$ such that $\pi_0(Z(\mathsf{G}))$ is invertible in Λ.[5] Let $\iota_\ell : \mathbb{C} \to \overline{\mathbb{Q}_\ell}$ be a fixed isomorphism, which fixes a square root $\sqrt{p} \in \overline{\mathbb{Z}_\ell}$ thus also $\sqrt{p} \in \overline{\mathbb{F}_\ell}$.

We will use L-groups and L-parameters with Λ-coefficients. Let $\Phi(\mathsf{G}, \Lambda)$ denote the set of L-parameters $W_K \times \mathrm{SL}(2,\Lambda) \to {}^L\mathsf{G}(\Lambda)$, and $\Pi(\mathsf{G}, \Lambda)$ denote the set of irreducible smooth representations of $\mathsf{G}(K)$ with Λ-coefficients.

3.1. The correspondence.

We briefly recall the construction of the Fargues–Scholze local Langlands correspondence. Let G be a connected reductive group over K. The Kottwitz set $B(\mathsf{G})$, as defined in [Kot85], consists of φ_K-equivalence classes of $\mathsf{G}(\breve{K})$, i.e.,

$$\mathsf{b} \sim \mathsf{b}' \iff \mathsf{b}' = \mathsf{g}^{-1}\mathsf{b}\varphi_K(\mathsf{g}) \text{ for some } \mathsf{g} \in \mathsf{G}(\breve{K}).$$

Each element $\mathsf{b} \in B(\mathsf{G})$ is determined by two invariants: The Kottwitz invariant $\kappa_\mathsf{G}(\mathsf{b})$ and the slope homomorphism (or Newton map) $\nu_\mathsf{b} : \mathbf{D}(\mathbb{Q}) \to \mathsf{G}_{\overline{K}}$, where $\mathbf{D}(\mathbb{Q})$ is the pro-algebraic diagonalizable group with character group $\mathbb{Q}$.

An element $\mathsf{b} \in B(\mathsf{G})$ is called a *basic* if ν_b is central in G; the set of basic elements in $B(\mathsf{G})$ is denoted $B(\mathsf{G})_{\mathrm{bas}}$. An element $\mathsf{b} \in B(\mathsf{G})$ is called *unramified* if it lies in the image of the map $B(\mathsf{T}) \to B(\mathsf{G})$ for some maximal torus $\mathsf{T} \leq G$. Denote by $B(\mathsf{G})_{\mathrm{un}} \leq B(\mathsf{G})$ the subset of unramified elements. These are precisely those $\mathsf{b} \in B(\mathsf{G})$ for which the twisted centralizer G_b is quasi-split; see[Ham24, Lemma 2.12].

We recall some material from [SW20] and [Far16] regarding the relative Fargues–Fontaine curve. For any $S \in \mathcal{P}\mathrm{erfd}_\kappa$, the associated Fargues–Fontaine curve X_S is defined as in [FF18]. When $S = \mathrm{Spa}(R, R^+)$ is affinoid with pseudo-uniformizer ϖ, the adic space X_S is defined as follow:

$$Y_S = \mathrm{Spa}(W_{\mathcal{O}_K}(R^+)) \smallsetminus \{p[\varpi] = 0\},$$

$$X_S = Y_S / \varphi_K^{\mathbb{Z}},$$

For any affinoid perfectoid space S over κ, the following sets are canonically in bijection:
(1) $\mathrm{Spd}(K)(S)$,
(2) Untilts $S^\sharp$ of S over K,
(3) Cartier divisors of Y_S of degree 1.

For any untilt $S^\sharp$ of S, we denote by $D_{S^\sharp} \subset Y_S$ the corresponding divisor.

By [FS24, Theorem III.0.2], the presheaf $\mathrm{Bun}_{\mathsf{G}}$ on $\mathcal{P}\mathrm{erfd}_{\overline{\kappa}}$, which assigns to each S the groupoid of G-torsors on X_S, is a small Artin v-stack. For any $S \in \mathcal{P}\mathrm{erfd}_{\overline{\kappa}}$, there exists a functor $\mathsf{b} \mapsto \mathcal{E}^{\mathsf{b}}$ from the category of isocrystals with G-structure to $\mathrm{Bun}_{\mathsf{G}}(S)$. When $S = \mathrm{Spd}(\overline{\kappa})$, this map induces a bijection from $B(\mathsf{G})$ to the set of isomorphism classes of G^*-bundles on X_S. More precisely, there exists a homeomorphism $|\mathrm{Bun}_{\mathsf{G}}| \to B(\mathsf{G})$ sending $\mathcal{E}^{\mathsf{b}}$ to b; see [Far20, Ans23, Vie24].

For any $\mathsf{b} \in B(\mathsf{G})$, the sub-functor

$$\mathrm{Bun}_{\mathsf{G}}^{\mathsf{b}} := \mathrm{Bun}_{\mathsf{G}} \times_{|\mathrm{Bun}_{\mathsf{G}}|} \{\mathsf{b}\} \subset \mathrm{Bun}_{\mathsf{G}}$$

is locally closed. It can be identified with the classifying stack $[\mathrm{Spd}(\overline{\kappa})/\tilde{\mathsf{G}}_{\mathsf{b}}]$, where $\tilde{\mathsf{G}}_{\mathsf{b}}$ denotes the v-sheaf of groups given by $S \mapsto \mathrm{Aut}_{X_S}(\mathcal{E}^{\mathsf{b}})$; see [FS24, Proposition III.5.3]. The natural morphism $\mathrm{Bun}_{\mathsf{G}}^{\mathsf{b}} \to [\mathrm{Spd}(\overline{\kappa})/\underline{\mathsf{G}_{\mathsf{b}}(K)}]$ induces equivalences of categories

$$\mathrm{D}(\mathsf{G}_{\mathsf{b}}, \Lambda) \cong \mathrm{D}_{\mathrm{lis}}([\mathrm{Spd}(\overline{\kappa})/\underline{\mathsf{G}_{\mathsf{b}}(K)}], \Lambda) \cong \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}}^{\mathsf{b}}, \Lambda),$$

see [FS24, Theorem VII.7.1].

Writing i_{b} for the inclusion $\mathrm{Bun}_{\mathsf{G}}^{\mathsf{b}} \subset \mathrm{Bun}_{\mathsf{G}}$, any $\pi \in \Pi(\mathsf{G}_{\mathsf{b}}, \Lambda)$ may be regarded as an object in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}}, \Lambda)$ via the extension by zero $(i_{\mathsf{b}})_!$, which is well-defined by [FS24, Proposition VII.7.3, Proposition VII.6.7]. Moreover, when b is basic, the map $\mathrm{Bun}_{\mathsf{G}}^{\mathsf{b}} \to [\mathrm{Spd}(\overline{\kappa})/\mathsf{G}_{\mathsf{b}}(K)]$ is an isomorphism; see [FS24, Proposition III.4.5].

We now introduce the Hecke operators. For each finite index set I, let $\mathrm{Rep}_{\Lambda}({}^L\mathsf{G}^I)$ denote the category of algebraic representations of I copies of ${}^L\mathsf{G}(\Lambda)$ over Λ, and let Div^I be the I-fold product of the mirror curve $\mathrm{Div}^1 := \mathrm{Spd}(\check{K})/\varphi_K^{\mathbb{Z}}$. The diamond Div^1 represents the functor that sends $S \in \mathcal{P}\mathrm{erfd}_{\overline{\kappa}}$ to the set of Cartier divisors on X_S,

We then have the Hecke correspondence diagram

(3.1)

$$
\begin{array}{ccc}
 & \mathrm{Hk}_{\mathsf{G},I} & \\
{}^{h^{\leftarrow}} \swarrow & & \searrow {}^{h^{\rightarrow} \times \mathrm{Supp}} \\
\mathrm{Bun}_{\mathsf{G}} & & \mathrm{Bun}_{\mathsf{G}} \times \mathrm{Div}^I
\end{array}
\quad,
$$

where $\mathrm{Hk}_{\mathsf{G},I}$ represents the functor sending $S \in \mathcal{P}\mathrm{erfd}_{\overline{\kappa}}$ to isomorphism classes of tuples

$$(\mathcal{E}_1, \mathcal{E}_2, \beta, (D_i)_{i \in I}),$$

where each $D_i \subset X_S$ is a Cartier divisor, $\mathcal{E}_1, \mathcal{E}_2$ are G-torsors on X_S, together with an isomorphism

$$\beta : \mathcal{E}_1|_{X_S \setminus \cup_{i \in I} D_i} \xrightarrow{\sim} \mathcal{E}_2|_{X_S \setminus \cup_{i \in I} D_i},$$

The morphism $h^{\leftarrow}$ sends the tuple to $\mathcal{E}_1$, while $h^{\rightarrow} \times \mathrm{Supp}$ sends it to $(\mathcal{E}_2, (D_i)_{i \in I})$.

For each $W \in \mathrm{Rep}_{\Lambda}({}^L\mathsf{G}^I)$, Fargues and Scholze define a solid Λ-sheaf $\mathcal{S}'_W \in \mathrm{D}_{\blacksquare}(\mathrm{Hk}_{\mathsf{G},I}, \Lambda)$ via geometric Satake correspondence, see [FS24, Theorem I.6.3]. can define the Hecke operator

(3.2) $\quad \mathrm{T}_W : \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}}, \Lambda) \to \mathrm{D}_{\blacksquare}(\mathrm{Bun}_{\mathsf{G}} \times \mathrm{Div}^I, \Lambda) : A \mapsto \mathrm{R}(h^{\rightarrow} \times \mathrm{Supp})_{\natural}(h^{\leftarrow *}(A) \otimes^{\mathrm{L}} \mathcal{S}'_W),$

where the functor $\mathrm{R}(h^{\rightarrow} \times \mathrm{Supp})_{\natural}$ is the natural push-forward that is the left adjoint to the restriction functor [FS24, Proposition VII.3.1]. In fact T_W is valued in

$$\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}}, \Lambda)^{BW_K^I};$$

see [FS24, Theorem I.7.2, Proposition IX.2.1, Corollary IX.2.3].

For any $\pi \in \Pi(\mathsf{G}, \Lambda)$, the Fargues–Scholze L-parameter ρ_{π}^{FS} comes from action of Hecke operators on π: For any tuple $(I, W, (\gamma_i)_{i \in I}, \alpha, \beta)$, where

- I is a finite index set, and we write $\Delta_I : {}^L\mathsf{G} \to {}^L\mathsf{G}^I$ for the diagonal embedding,

- $(r_W, W) \in \mathrm{Rep}_{\Lambda}({}^L\mathsf{G}^I)$ is a smooth representation,

- $\gamma_i \in W_K$ for each $i \in I$,

- $\alpha : \mathbf{1} \to \Delta_I^*(r_W)$ and $\beta : \Delta_I^*(r_W) \to \mathbf{1}$ are maps between smooth representations of ${}^L\mathsf{G}(\Lambda)$,

then we define the excursion operator on $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}}, \Lambda)$ to be the natural transformation of the identity functor given by

$$\mathrm{id} = \mathrm{T}_{\mathbf{1}} \xrightarrow{\alpha} \mathrm{T}_{\Delta_I^* W} = \mathrm{T}_W \xrightarrow{(\gamma_i)_{i \in I}} \mathrm{T}_W = \mathrm{T}_{\Delta_I^* W} \xrightarrow{\beta} \mathrm{T}_{\mathbf{1}} = \mathrm{id}.$$

Then this natural transformation valued at $\pi \in \mathrm{D}(\mathsf{G}, \Lambda) \cong \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}}^{\mathbf{1}}, \Lambda) \subset \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}}, \Lambda)$ which is a scalar element by Schur's lemma. Then the natural relations between the Hecke operators can be used to give natural relations between these scalars, which can in turn by used via Lafforgue's reconstruction theorem [Laf18, Proposition 11.7] to get the Fargues–Scholze L-parameter $\phi_\pi^{\mathrm{FS}} : W_K \to {}^L G(\Lambda)$ such that the above natural transformation valued at π equals the composition of maps

$$\Lambda \xrightarrow{\alpha} W \xrightarrow{r_W\left(\left(\phi_\pi^{\mathrm{FS}}(\gamma_i)\right)_{i \in I}\right)} W \xrightarrow{\beta} \Lambda$$

between Λ-vector spaces, which is also a scalar; see [FS24, Definition/Proposition IX.4.1].

Then Fargues and Scholze [FS24, Theorem I.9.6] showed that their construction has various desirable properties:

Theorem 3.1.1. *The Fargues–Scholze LLC* $\mathrm{rec}_{\mathsf{G}}^{\mathrm{FS}} : \Pi(\mathsf{G}, \Lambda) \to \Phi^{\mathrm{ss}}(\mathsf{G}, \Lambda)$ *satisfies the following compatibility properties:*

(1) *(Compatibility with local class field theory) If* G *is a torus, then* $\mathrm{rec}_{\mathsf{G}}^{\mathrm{FS}}$ *is the usual local Langlands correspondence constructed from local class field theory,*

(2) *(Compatible with natural operations)* $\mathrm{rec}_{\mathsf{G}}^{\mathrm{FS}}$ *is compatible with character twists, central characters, and taking contragredient,*

(3) *(Compatible with products) If* $\mathsf{G} = \mathsf{G}_1 \times \mathsf{G}_2$ *is a product of two groups and* $\pi_i \in \Pi(\mathsf{G}_i)$ *for each* $i \in \{1, 2\}$, *then* $\mathrm{rec}_{\mathsf{G}}^{\mathrm{FS}}(\pi_1 \boxtimes \pi_2) = \mathrm{rec}_{\mathsf{G}_1}^{\mathrm{FS}}(\pi_1) \times \mathrm{rec}_{\mathsf{G}_2}^{\mathrm{FS}}(\pi_2)$,

(4) *(Compatibility with central extensions) If* $\mathsf{G}' \to \mathsf{G}$ *is a map of reductive groups inducing an isomorphism on adjoint groups,* $\pi \in \Pi(\mathsf{G}, \Lambda)$, *and* $\pi' \in \Pi(\mathsf{G}', \Lambda)$ *is an irreducible admissible subquotient of* $\pi|_{\mathsf{G}'(K)}$, *then* $\mathrm{rec}_{\mathsf{G}'}^{\mathrm{FS}}(\pi')$ *is given by* $\mathrm{rec}_{\mathsf{G}}(\pi)$ *composed with the natural map* ${}^L G(\Lambda) \to {}^L G'(\Lambda)$. *In particular,* $\mathrm{rec}_{\mathsf{G}'}^{\mathrm{FS}}$ *commutes with contragredients and Chevalley involutions.*

(5) *(Compatibility with parabolic induction) If* $\mathsf{P} \leq \mathsf{G}$ *is a parabolic subgroup with Levi factor* M *and* $\pi_{\mathsf{M}} \in \Pi(\mathsf{M})$, *then for any subquotient* π *of the normalized parabolic induction* $\mathrm{I}_{\mathsf{P}}^{\mathsf{G}}(\pi_{\mathsf{M}})$, $\mathrm{rec}_{\mathsf{G}}^{\mathrm{FS}}(\pi)$ *is given by the composition*

$$W_K \xrightarrow{\mathrm{rec}_{\mathsf{G}}^{\mathrm{FS}}(\pi_{\mathsf{M}})} {}^L M(\Lambda) \to {}^L G(\Lambda),$$

where ${}^L M(\Lambda) \to {}^L G(\Lambda)$ *is the canonical embedding,*

(6) *(Compatibility with Harris–Taylor/Henniart LLC) If* $\Lambda = \overline{\mathbb{Q}}_\ell$ *and* $\mathsf{G} = \mathrm{GL}(n)$, *then* $\mathrm{rec}_{\mathrm{GL}(n)}^{\mathrm{FS}}$ *coincides the (semisimplified) local Langlands correspondence given by Harris–Taylor and Henniart in the sense of* Theorem A.

(7) *(Compatibility with restriction of scalars) If* $\mathsf{G} = \mathrm{Res}_{K'/K} \mathsf{G}'$ *for some reductive group* G' *over a finite extension* K'/K, *then* $\mathrm{rec}_{\mathsf{G}'}^{\mathrm{FS}}(\pi) = \mathrm{rec}_{\mathsf{G}}^{\mathrm{FS}}(\pi)|_{W_{K'}}$ *for any* $\pi \in \Pi(\mathsf{G}') = \Pi(\mathsf{G})$.

(8) *(Compatibility with contragredients) For* $\pi \in \Pi(\mathsf{G}, \Lambda)$, *we have* $\mathrm{rec}_{\mathsf{G}}^{\mathrm{FS}}(\pi^\vee) = \vartheta_{\widehat{\mathsf{G}}}\left(\mathrm{rec}_{\mathsf{G}}^{\mathrm{FS}}(\pi)\right)$; *see* [FS24, Proposition IX.5.3]. *Here we recall that for each Borel pair* $(\widehat{\mathsf{B}}, \widehat{\mathsf{T}})$ *of* $\widehat{G}$, *there exists a Cartan involution (also called the Chevalley involution)* $\vartheta_{\widehat{\mathsf{G}}}$ *of* $\widehat{\mathsf{G}}$ *preserving the fixed pinning and acts as* $t \mapsto w_0(t^{-1})$ *on* $\widehat{\mathsf{T}}$, *where* w_0 *is the longest-length Weyl element taking* $\widehat{\mathsf{B}}$ *to the opposite Borel group.* $\vartheta_{\widehat{\mathsf{G}}}$ *extends to an involution of* ${}^L \mathsf{G}$ *because the action of* W_K *on* $\widehat{\mathsf{G}}$ *preserves the pinning.*

(9) *(Compatibility with reduction modulo* ℓ) *If* $\pi \in \Pi(\mathsf{G}, \overline{\mathbb{Q}}_\ell)$ *admits a* $\mathsf{G}(K)$-*stable* $\overline{\mathbb{Z}}_\ell$-*lattice, and the reduction modulo* ℓ *representation of* π *has an irreducible subquotient* $\overline{\pi}'$, *then* $\mathrm{rec}_{\mathsf{G}}^{\mathrm{FS}}(\pi) : W_K \to {}^L \mathsf{G}(\overline{\mathbb{Q}}_\ell)$ *factors through* ${}^L \mathsf{G}(\overline{\mathbb{Z}}_\ell)$, *and its reduction modulo* ℓ *equals* $\mathrm{rec}_{\mathsf{G}}^{\mathrm{FS}}(\overline{\pi}')$; *see* [FS24, §IX.5.2].

Moreover, Hansen, Kaletha and Weinstein established in [HKW22, Theorem 6.6.1] that the Fargues–Scholze LLC coincides with the usual (semisimplified) LLC for inner forms of general linear groups given in [DKV84, Rog83] via Jacquet–Langlands correspondence.

When $\mathsf{G}^\sharp$ is an inner form of $\mathrm{GL}(n)$ and $\mathsf{G} = [\mathsf{G}^\sharp, \mathsf{G}^\sharp]$, the local Langlands correspondence constructed in [GK82, Tad92, HS12, ABPS16] assigns to each $\pi^\sharp \in \Pi(\mathsf{G}^\sharp)$ a parameter $\phi_{\pi^\sharp}$ with the property that, for

any irreducible admissible subrepresentation $\pi \subset \pi^\sharp|_{\mathsf{G}(K)}$, ϕ_π equals $\phi_{\pi^\sharp}$ composed with the natural map $^L\mathsf{G}^\sharp \to {}^L\mathsf{G}$. This prescription uniquely determines the local Langlands correspondence for G, since every $\pi \in \Pi(\mathsf{G})$ occurs as a subrepresentation of an irreducible admissible representation of $\pi^\sharp|_{\mathsf{G}(K)}$ for some $\pi^\sharp \in \Pi(\mathsf{G}^\sharp)$ by [GK82, Lemma 2.3].

It then follows from the compatibility of Fargues–Scholze LLC with central extensions (see Theorem 3.1.1) that:

Proposition 3.1.2. *If G is an inner form of a special linear group and $\Lambda = \overline{\mathbb{Q}_\ell}$, then the Fargues–Scholze LLC are compatible with the LLC constructed in* [GK82, Tad92, HS12, ABPS16] *in the sense of* Theorem A.

Proof. We only need to note that two parameters of G are conjugate by an element of $^L\mathsf{G}^\sharp \cong \mathrm{GL}(N, \mathbb{C})$ if and only if they are conjugate by an element of $^L\mathsf{G} \cong \mathrm{SL}(N, \mathbb{C})$. $\square$

3.2. **Local shtuka spaces.** In this subsection we introduce the local shtuka spaces whose cohomologies will be one of the main object of study of this paper.

Let G^* be a quasi-split reductive group over a non-Archimedean local field K of characteristic zero with a fixed Borel subgroup B^* containing a maximal torus T^*, and μ be a dominant cocharacter of $\mathsf{G}^*_{\overline{K}}$ with reflex field E_μ. We first introduce the notion of neutral μ-acceptable elements [RV14, Definition 2.3]: Let $\mu^\sharp$ be the image of μ in $X_\bullet\left(Z(\widehat{\mathsf{G}^*})^{\mathrm{Gal}_K}\right)$, and let

$$\overline{\mu} := \frac{1}{[E_\mu : K]} \sum_{\gamma \in \mathrm{Gal}_K / \mathrm{Gal}_{E_\mu}} \gamma(\mu) \in X_\bullet(\mathsf{T}^*)^{+,\mathrm{Gal}_K}_{\mathbb{Q}}.$$

For $\mathsf{b}_0 \in B(\mathsf{G}^*)$, we define the set of neutral μ-acceptable elements to be the set

$$(3.3) \qquad B(\mathsf{G}^*, \mathsf{b}_0, \mu) := \{\mathsf{b} \in B(\mathsf{G}^*) | \kappa_{\mathsf{G}^*}(\mathsf{b}) - \kappa_{\mathsf{G}^*}(\mathsf{b}_0) = \mu^\sharp, \overline{\mu} - (\nu_\mathsf{b} - \nu_{\mathsf{b}_0}) \in \mathbb{R}_+ \Phi^+_{\mathsf{G}^*}\}$$

where we regard $\nu_\mathsf{b}, \nu_{\mathsf{b}_0} : \mathbf{D}(\mathbb{Q}) \to \mathsf{G}^*_{\overline{K}}$ as an elements in $X_\bullet(\mathsf{T}^*)^{+,\mathrm{Gal}_K}_{\mathbb{Q}}$.

For example, for the quasi-split group G^* defined in §2.1 with a pure inner twist $(G, \varrho_{b_0}, z_{b_0})$ and the geometric cocharacter $\mu_1 \in X_\bullet(G^*)$ defined in §2.6, the unique basic element $b_1 \in B(G^*, b_0, \mu_1)_{\mathrm{bas}}$ is just the unique nontrivial basic element b_1 of $B(G)_{\mathrm{bas}}$ under the isomorphism (2.2).

We then recall Scholze's definition [SW20, §23] of the local shtuka space in the basic case: For each $\mathsf{b}_0 \in B(\mathsf{G}^*)_{\mathrm{bas}}$ and $\mathsf{b} \in B(\mathsf{G}^*, \mathsf{b}_0, \mu)_{\mathrm{bas}}$, the local shtuka space $\mathrm{Sht}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)$ is a local spatial diamond over $\mathrm{Spd}(\check{K}E_\mu)$ that represents the functor that maps $S \in \mathcal{P}\mathrm{erfd}_{\check{K}E_\mu}$ to the set of isomorphisms

$$\gamma : \mathcal{E}^\mathsf{b}|_{X_{S^\flat} \setminus D_S} \xrightarrow{\sim} \mathcal{E}^{\mathsf{b}_0}|_{X_{S^\flat} \setminus D_S},$$

of G^*-bundles that are meromorphic along the divisor $D_S \subset X_{S^\flat}$ and bounded by μ pointwise on $\mathrm{Spa}(S)$, as defined in [HKW22, p. 11].[6]

The automorphism groups of $\mathcal{E}^\mathsf{b}$ and $\mathcal{E}^{\mathsf{b}_0}$ are the constant group diamonds $\underline{\mathsf{G}^*_\mathsf{b}(K)}$ and $\underline{\mathsf{G}^*_{\mathsf{b}_0}(K)}$, respectively, so $\mathrm{Sht}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)$ is equipped with a commuting action of $\mathsf{G}^*_\mathsf{b}(K)$ and $\mathsf{G}^*_{\mathsf{b}_0}(K)$ by pre-composing and post-composing on γ.

We remark on the compatibility of the local shtuka space under Weil restrictions: If K/K_0 is an unramified extension and $\mathsf{G}^*_0 = \mathrm{Res}_{K/K_0}\mathsf{G}^*$, and μ_0 is a cocharacter of $(\mathsf{G}^*_0)_{\overline{K_0}} \cong \prod_{\mathrm{Hom}(K,\overline{K_0})} \mathsf{G}^*_{\overline{K}}$ that is central on each component corresponding to $v \in \mathrm{Hom}(K, \overline{K_0})$ except for one, where it is a cocharacter μ of $\mathsf{G}^*_{\overline{K}}$. Then the reflex fields of μ and μ_0 are equal, and for each $\mathsf{b}_0 \in B(\mathsf{G}^*_0)_{\mathrm{bas}}$ and $\mathsf{b} \in B(\mathsf{G}^*_0, \mathsf{b}_0, \mu)$, we can regard b_0, b as basic elements of $B(\mathsf{G}^*)$ via the natural isomorphism $B(\mathsf{G}^*_0) \cong B(\mathsf{G}^*)$. Then it follows from the definition that there exists a natural isomorphism of diamonds

$$\mathrm{Sht}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu) \cong \mathrm{Sht}(\mathsf{G}^*_0, \mathsf{b}, \mathsf{b}_0, \mu_0).$$

For each compact open subgroup $\mathcal{K} \leq \mathsf{G}^*_{\mathsf{b}_0}(K)$, we define $\mathrm{Sht}_\mathcal{K}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu) = \mathrm{Sht}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)/\mathcal{K}$, which is also a locally spatial diamond defined over $\mathrm{Spd}(\check{K}E_\mu)$ [SW20, Theorem 23.1.4]. Let $\mathcal{S}'_\mu$ be the Λ-sheaf

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[6]We note that our definition of $\mathrm{Sht}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)$ agrees with the definition in [Ham22, p. 23], [MHN24, p. 4757], [Ham24, p. 90], but differ with the definitions in [SW20] and [HKW22, p. 11] by changing μ to μ^{-1}. Our definition simplifies certain presentations

corresponding to the highest weight Tilting module $\mathcal{T}_\mu \in \mathrm{Rep}_\Lambda(\widehat{\mathsf{G}})$ via the geometric Satake equivalence [FS24, Theorem I.6.3]. There is a natural map

$$\mathrm{pr} : \mathrm{Sht}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu) \to \mathrm{Hk}_{\mathsf{G}^*, 1},$$

and we denote the pullback of $\mathcal{S}'_\mu$ along pr by $\mathcal{S}_\mu$. Note that pr factors through the quotient of $\mathrm{Sht}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)$ by the actions of $\mathsf{G}^*_\mathsf{b}(K)$ and $\mathsf{G}^*_{\mathsf{b}_0}(K)$, so $\mathcal{S}_\mu$ is equivariant with respect to these actions. We define

$$\mathrm{R}\Gamma_c(\mathrm{Sht}_{\mathscr{K}}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu), \mathcal{S}_\mu) := \varinjlim_U \mathrm{R}\Gamma_c(U, \mathcal{S}_\mu)$$

where U runs through quasi-compact open subsets of $\mathrm{Sht}_{\mathscr{K}}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)$, and also

$$\mathrm{R}\Gamma_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu) := \varinjlim_{\mathscr{K}} \mathrm{R}\Gamma_c(\mathrm{Sht}_{\mathscr{K}}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu), \mathcal{S}_\mu)$$

where $\mathscr{K}$ runs through open compact subgroups of $\mathsf{G}^*_{\mathsf{b}_0}(K)$.[7]

Following [Shi11], we now define a map $\mathrm{Mant}_{\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu} : \mathrm{K}_0(\mathsf{G}^*_\mathsf{b}, \Lambda) \to \mathrm{K}_0(\mathsf{G}^*_{\mathsf{b}_0}(K) \times W_{E_\mu}, \Lambda)$ describing the cohomology of $\mathrm{Sht}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)$, where for each $\mathsf{b} \in B(\mathsf{G}^*)_{\mathrm{bas}}$, we denote by $\mathrm{K}_0(\mathsf{G}^*_\mathsf{b}, \Lambda)$ the Grothendieck group of the category of finite-length admissible representations of $\mathsf{G}^*_\mathsf{b}(K)$ with Λ-coefficients, and denote by $\mathrm{K}_0(\mathsf{G}^*_\mathsf{b}(K) \times W_{E_\mu}, \Lambda)$ the Grothendieck group of the category of finite-length admissible representations of $\mathsf{G}^*_\mathsf{b}(K)$ with Λ-coefficients equipped with a continuous action of W_{E_μ} commuting with the $\mathsf{G}^*_\mathsf{b}(K)$-action.

Definition 3.2.1. For $\rho \in \Pi(\mathsf{G}^*_\mathsf{b}, \Lambda)$, we define

(3.4) $$\mathrm{R}^\flat\Gamma_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)[\rho] := \mathrm{RHom}_{\mathsf{G}^*_\mathsf{b}(K)}\left(\mathrm{R}\Gamma_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu), \rho\right),$$

$$\mathrm{R}\Gamma_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)[\rho] := \mathrm{R}\Gamma_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu) \otimes_{\mathsf{G}^*_\mathsf{b}(K)} \rho.$$

It follows from [FS24, Corollary I.7.3 and p. 317] that this defines a finite-length W_{E_μ}-equivariant object in $\mathrm{D}(\mathsf{G}^*_{\mathsf{b}_0}, \Lambda)$, and we write $\mathrm{Mant}_{\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu}(\rho)$ for the image of $\mathrm{R}\Gamma^\flat(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)[\rho]$ in $\mathrm{K}_0(\mathsf{G}^*_\mathsf{b}(K) \times W_{E_\mu}; \Lambda)$.

Note that $\mathrm{R}\Gamma(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)[\rho]$ is much more natural from the point of view of geometric arguments on $\mathrm{Bun}_{\mathsf{G}^*}$ as it involves the much simpler extension by zero functor, while the complex $\mathrm{R}^\flat\Gamma(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)[\rho]$ is studied in [HKW22]. It follows from Hom-Tensor duality that

(3.5) $$\mathrm{R}^\flat\Gamma_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)[\rho^*] \cong \mathrm{RHom}\left(\mathrm{R}\Gamma_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)[\rho], \Lambda\right).$$

Moreover, we have the following result of Meli, Hamann and Nguyen [MHN24, Proposition 2.25] for representations with supercuspidal Fargues–Scholze L-parameters:

Proposition 3.2.2. *If $\rho \in \Pi(\mathsf{G}^*_\mathsf{b}, \Lambda)$ has supercuspidal Fargues–Scholze L-parameter ϕ^{FS}_ρ, then there exists an isomorphism*

$$\mathrm{R}\Gamma_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)[\rho] \cong \mathrm{R}\Gamma^\flat_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)[\rho]$$

of representations of $\mathsf{G}^*_{\mathsf{b}_0}(K) \times W_{E_\mu}$.

Finally, we recall the following crucial result relating the cohomology of local shtuka spaces and Fargues–Scholze L-parameters. Before that, we recall that for any dominant cocharacters μ for G with reflex field E_μ, the highest weight tilting module $\mathcal{T}_\mu \in \mathrm{Rep}_\Lambda(\widehat{\mathsf{G}})$ extends naturally to a representation of $\widehat{\mathsf{G}} \rtimes W_{E_\mu}$, then we define the extended highest weight tilting module

(3.6) $${}^L\mathcal{T}_\mu := \mathrm{Ind}^{{}^L\mathsf{G}}_{\widehat{\mathsf{G}} \times \mathrm{Gal}_{E_\mu}} \mathcal{T}_\mu \in \mathrm{Rep}_\Lambda({}^L\mathsf{G}).$$

The isomorphism class of ${}^L\mathcal{T}_\mu$ only depends on the Gal_K-orbits of μ; see [Ham24, p. 82]. Moreover, we recall that the tilting module $\mathcal{T}_\mu$ equals the usual highest weight module (defined by un-normalized parabolic induction) when μ is minuscule.

Proposition 3.2.3. *Suppose $\rho \in \Pi(\mathsf{G}^*_\mathsf{b}, \Lambda)$ and $\pi \in \Pi(\mathsf{G}^*_{\mathsf{b}_0}, \Lambda)$ such that π appears in $\mathrm{Mant}_{\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu}(\rho)$. Here we omit the action of W_{E_μ} on $\mathrm{Mant}_{\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu}(\rho)$. We then have $\phi^{\mathrm{FS}}_\pi = \phi^{\mathrm{FS}}_\rho \in \Phi^{\mathrm{ss}}(\mathsf{G}^*, \Lambda)$.*

---

[7]Note that our $\mathrm{R}\Gamma_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)$ agrees with $\mathrm{R}\Gamma_c(\mathcal{M}_{\mathsf{G}^*_{\mathsf{b}_0}, \mathsf{b}_0, \mu^{-1}, \infty}, \Lambda)$ defined in [DvHKZ24, p. 97], by [DvHKZ24, Proposition 8.5.9], but differ from $\mathrm{R}\Gamma_c(\mathsf{G}^*_{\mathsf{b}_0}, \mathsf{b}, \mu)$ defined in [Ham22, p. 24] by a Tate twist and a t-shift.

Proof. This is established in [Ham22, Corollary 3.15]. Strictly speaking, the cited result only proves the assertion when μ is minuscule. However, the general proof is the same and we reproduce here.

Regard ρ as an element of $D(\mathsf{G}^*_{\mathsf{b}},\Lambda) \cong D_{\mathrm{lis}}(\mathrm{Bun}^{\mathsf{b}}_{\mathsf{G}^*},\Lambda)$, then there is an isomorphism

$$R\Gamma_c(\mathsf{G}^*,\mathsf{b},\mathsf{b}_0,\mu)[\rho] \cong i^*_{\mathsf{b}_0} T_\mu i_{\mathsf{b},!}(\rho) \in D(\mathsf{G}_{\mathsf{b}_0},\Lambda)^{BW_{E_\mu}},$$

by [FS24, §IX.3]. Here T_μ is the Hecke operator associated to the extended highest weight tilting module $^L\mathcal{T}_\mu$ associated to μ (3.6). Then each Schur irreducible subquotient $A \in D_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}^*},\Lambda)$ of $T_\mu i_{\mathsf{b},!}(\rho)$ has Fargues–Scholze parameter ϕ^{FS}_ρ by definition (Here we omit the W_{E_μ}-action). Furthermore, each $\pi \in \Pi(\mathsf{G}^*_{\mathsf{b}_0},\Lambda) \subset D_{\mathrm{lis}}(\mathrm{Bun}^{\mathsf{b}_0}_{\mathsf{G}^*},\Lambda)$ appearing in $i^*_{\mathsf{b}_0}(A)$ has Fargues–Scholze parameter equaling to that of A under the identification $\Phi^{\mathrm{ss}}(\mathsf{G}^*_{\mathsf{b}_0},\Lambda) = \Phi^{\mathrm{ss}}(\mathsf{G}^*,\Lambda)$, by [FS24, §IX.7.1] (see also [Ham22, Proposition 3.14]). Thus the assertion follows. $\square$

3.3. **Spectral actions.** Let G be a general quasi-split reductive group over K. we recall the spectral actions on sheaves on Bun_G by sheaves on the stack of Langlands parameters; cf. [FS24, §X], [Ham22].

For $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G},\Lambda)$, we define $D_{\mathrm{lis}}(\mathrm{Bun}_\mathsf{G},\Lambda)_\phi \subset D_{\mathrm{lis}}(\mathrm{Bun}_\mathsf{G},\Lambda)$ to be the full subcategory of objects A such that the endomorphism induced by any $f \in \mathcal{O}_{\mathfrak{X}_{\widehat{\mathsf{G}}}} \smallsetminus \mathfrak{m}_\phi$ is an isomorphism on A.

The natural inclusion $\iota_\phi : D_{\mathrm{lis}}(\mathrm{Bun}_\mathsf{G},\Lambda)_\phi \hookrightarrow D_{\mathrm{lis}}(\mathrm{Bun}_\mathsf{G},\Lambda)$ has a left adjoint denoted by $\mathcal{L}_\phi$, and we define $(-)_\phi = \iota_\phi \circ \mathcal{L}_\phi$, which is an idempotent functor on $D_{\mathrm{lis}}(\mathrm{Bun}_\mathsf{G},\Lambda)$; see [HL24, Definition 4.1].

We introduce the Act-functors defined in [Ham22, §3.2] and [MHN24, §2.3.3] via the spectral action of the moduli stack of L-parameters: Let $\mathfrak{X}_{\widehat{\mathsf{G}}} := [Z^1(W_K,\widehat{\mathsf{G}})_{\overline{\mathbb{Q}}_\ell}/\widehat{\mathsf{G}}]$ be the moduli stack of semisimple Langlands parameters defined in [DHKM24, Zhu21] and [FS24, Theorem VIII.1.3], and let $\mathcal{P}\mathrm{erf}(\mathfrak{X}_{\widehat{\mathsf{G}}})$ be the derived category of perfect complexes on $\mathfrak{X}_{\widehat{\mathsf{G}}}$. Let $D_{\mathrm{lis}}(\mathrm{Bun}_{\widehat{\mathsf{G}}},\overline{\mathbb{Q}}_\ell)^\omega \subset D_{\mathrm{lis}}(\mathrm{Bun}_{\widehat{\mathsf{G}}},\overline{\mathbb{Q}}_\ell)$ be the triangulated sub-category consisting of compact objects. Then it follows from [FS24, Corollary X.1.3] that for each finite index set I, there exists a $\overline{\mathbb{Q}}_\ell$-linear action

$$\mathcal{P}\mathrm{erf}(\mathfrak{X}_{\widehat{\mathsf{G}}})^{BW^I_K} \to \mathrm{End}\left(D_{\mathrm{lis}}(\mathrm{Bun}_\mathsf{G},\overline{\mathbb{Q}}_\ell)^\omega\right)^{BW^I_K} : C \mapsto \{A \mapsto C \star A\},$$

which is monoidal in the sense that there exists a natural equivalence of functors:

$$(C_1 \otimes^{\mathrm{L}} C_2) \star (-) \cong C_1 \star (C_2 \star (-)).$$

Fix for the rest of this subsection a supercuspidal L-parameter $\phi \in \Phi^{\mathrm{sc}}(\mathsf{G},\overline{\mathbb{Q}}_\ell)$, then there exists a connected component C_ϕ of $X_{\widehat{\mathsf{G}}}$ consisting of unramified twists of the parameter ϕ, equipped with a natural map $C_\phi \to [\mathrm{Spd}(\overline{\mathbb{Q}}_\ell)/\mathfrak{S}_\phi]$; see [FS24, §X.2]. We then have a direct summand

$$\mathcal{P}\mathrm{erf}(C_\phi) \subset \mathcal{P}\mathrm{erf}(\mathfrak{X}_{\widehat{\mathsf{G}}}),$$

and the spectral action gives rise to a direct summand

$$D^{C_\phi}_{\mathrm{lis}}(\mathrm{Bun}_\mathsf{G},\overline{\mathbb{Q}}_\ell)^\omega \subset D_{\mathrm{lis}}(\mathrm{Bun}_\mathsf{G},\overline{\mathbb{Q}}_\ell)^\omega.$$

Then there exists a decomposition [Ham22, p. 34]

$$D^{C_\phi}_{\mathrm{lis}}(\mathrm{Bun}_\mathsf{G},\overline{\mathbb{Q}}_\ell)^\omega \cong \bigoplus_{\mathsf{b} \in B(\mathsf{G})_{\mathrm{bas}}} D^{C_\phi}(\mathsf{G}_\mathsf{b},\overline{\mathbb{Q}}_\ell)^\omega,$$

where $D^{C_\phi}(\mathsf{G}_\mathsf{b},\overline{\mathbb{Q}}_\ell)^\omega \subset D(\mathsf{G}_\mathsf{b},\overline{\mathbb{Q}}_\ell)^\omega$ is a full subcategory of the subcategory of compact objects in $D(\mathsf{G}_\mathsf{b},\overline{\mathbb{Q}}_\ell)$. Let χ be the character of $Z(\mathsf{G})(K)$ determined by ϕ as in [Bor79, §10.1], then for each $\mathsf{b} \in B(\mathsf{G})$, the subcategory

$$D^{C_\phi,\chi}(\mathsf{G}_\mathsf{b},\overline{\mathbb{Q}}_\ell)^\omega \subset D^{C_\phi}(\mathsf{G}_\mathsf{b},\overline{\mathbb{Q}}_\ell)^\omega$$

spanned by the compact objects with fixed central character χ of $Z(\mathsf{G}_\mathsf{b}) \cong Z(\mathsf{G})$ is semisimple because supercuspidal representations are both injective and projective in the category of smooth representations with fixed central character. Thus we can identify $D^{C_\phi,\chi}(\mathsf{G}_\mathsf{b},\overline{\mathbb{Q}}_\ell)^\omega$ with

$$\bigoplus_{\mathsf{b} \in B(G^*)_{\mathrm{bas}}} \bigoplus_{\pi \in \Pi_\phi(G^*_\mathsf{b})} \iota_\ell \pi_b \otimes \mathcal{P}\mathrm{erf}(\overline{\mathbb{Q}}_\ell).$$

This category of preserved under the spectral action of $\mathcal{P}\mathrm{erf}(C_\phi)$; see [Ham22, p. 34].

We now recall the Act-functors: For each $\eta \in \mathrm{Irr}(\mathfrak{S}_\phi)$, we get a line bundle $\mathcal{L}_\eta$ on C_ϕ by pulling back along the natural map $C_\phi \to [\mathrm{Spd}(\overline{\mathbb{Q}}_\ell)/\mathfrak{S}_\phi]$, and we define Act_η to be the spectral action of this line bundle

on $\mathrm{D}^{C_\phi, \chi}(\mathsf{G_b}, \overline{\mathbb{Q}_\ell})^\omega$. In particular, Act-functors are symmetric monoidal, i.e., $\mathrm{Act_1}$ is the identity functor, and for any $\eta, \eta' \in \mathrm{Irr}(\mathfrak{S}_\phi)$, there exists an natural equivalence of functors

$$\mathrm{Act}_\eta \circ \mathrm{Act}_{\eta'} \cong \mathrm{Act}_{\eta\eta'}.$$

From this it is easy to show that each Act-functor sends an irreducible admissible representation

$$\iota_\ell \pi \in \bigcup_{b \in B(G^*)_{\mathrm{bas}}} \iota_\ell \Pi_\phi(G_b^*)$$

to another irreducible admissible representation

$$\iota_\ell \pi' \in \bigcup_{b \in B(G^*)_{\mathrm{bas}}} \iota_\ell \Pi_\phi(G_b^*)$$

with a t-shift; see [MHN24, Lemma 2.28].

3.4. **Weak version of the Kottwitz conjecture.** In this subsection, suppose G^* is a special orthogonal or unitary group as defined in §2.1, $(G, \varrho_{b_0}, z_{b_0})$ is a pure inner twist of G^* associated to some $b_0 \in B(G^*)_{\mathrm{bas}}$, μ is a dominant cocharacter of $G^*_{\overline{K}}$. Let $b \in B(G^*, b_0, \mu)_{\mathrm{bas}}$ be the unique basic element, regarded as a basic element in $B(G)_{\mathrm{bas}}$ via the isomorphism (2.2), so we can adopt the notation from §3.2. In particular, we write

$$\mathrm{Mant}_{G,b,\mu} := \mathrm{Mant}_{G^*, b_0+b, b_0, \mu} : \mathrm{K}_0(G_b, \overline{\mathbb{Q}_\ell}) \to \mathrm{K}_0(G, \overline{\mathbb{Q}_\ell}).$$

forgetting the W_{E_μ}-action.

We will use the weak version of the Kottwitz conjecture from [HKW22] describing the cohomology $\mathrm{Mant}_{G,b,\mu}(\iota_\ell \rho)$ for $\rho \in \Pi_\phi(G)$. The (generalized) Kottwitz conjecture describes $\mathrm{Mant}_{G,b,\mu}(\iota_\ell \rho)$ in terms of the local Langlands correspondence, where ρ lies in a supercuspidal L-packet.

In [HKW22, Theorem 1.0.2], a weak version of Kottwitz conjecture is established for all discrete L-parameters, but disregarding the action of the Weil group, and modulo a virtual representation whose character vanishes on the locus of elliptic elements. Their proof is conditional on the refined local Langlands conjecture of [Kal16, Conjecture G] (in fact, as G is always a pure inner form of G^*, the isocrystal version [Kal16, Conjecture G] suffices), but in Case O2 (with sufficiently high rank), only the weak version of this conjecture stated in §2 is known. To remedy this, we will use weak versions of the endoscopic character identities Theorem 2.3.4 to prove a weak version of [HKW22, Theorem 1.0.2].

To state results uniformly, for each $b \in B(G)_{\mathrm{bas}}$, we write $\tilde{K}_0(G_b, \overline{\mathbb{Q}_\ell}) := K_0(G_b, \overline{\mathbb{Q}_\ell})/\sim_\varsigma$, then we obtain a well-defined map

$$\mathrm{Mant}_{G,b,\mu} : \tilde{\mathrm{K}}_0(G_b, \overline{\mathbb{Q}_\ell}) \to \tilde{\mathrm{K}}_0(G, \overline{\mathbb{Q}_\ell}),$$

because ς acts on $\mathrm{Sht}(G, b, b_0, \{\mu\})$ naturally, thus also on the cohomology groups.

The set of elliptic elements of $G_b(K)$ is invariant under action of ς, so it makes sense to talk about an object of $\tilde{K}_0(G_b, \overline{\mathbb{Q}_\ell})$ whose character vanishes on the locus of elliptic elements of $G_b(K)$, and these objects are exactly those coming from properly contained Levi subgroup of G_b [HKW22, Theorem C.1.1].

Similarly, for $\tilde{\pi} \in \tilde{\Pi}(G_b)$, we can define the Harish-Chandra character

$$\Theta_{\tilde{\pi}} = \frac{1}{2}(\Theta_\pi + \Theta_{\pi^\varsigma}) \in C(G_b(K)_{\mathrm{s.reg}} /\!\!/ G_b(K), \mathbb{C}),$$

where $\pi \in \Pi(G_b)$ is an arbitrary representative of $\tilde{\pi}$.

Also the Fargues–Scholze parameter $\phi_{\tilde{\rho}}^{\mathrm{FS}} : W_{K_1} \to {}^L G^*$ is well-defined up to $\mathrm{O}(N(G), \mathbb{C})$-conjugation in Case O2 by the compatibility of Fargues–Scholze LLC with central extensions Theorem 3.1.1, so we can talk about whether it is a supercuspidal L-parameter.

Then our theorem, which slightly generalizes the main theorem of [HKW22], is stated as follows (a stronger version will be established in §7.3):

Theorem 3.4.1. *If $\tilde{\phi} \in \tilde{\Phi}_2(G^*)$ is a discrete L-parameter, and $\tilde{\rho} \in \tilde{\Pi}_{\tilde{\phi}}(G_b)$, then*

$$\mathrm{Mant}_{G,b,\mu}(\iota_\ell \tilde{\rho}) = \sum_{\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)} \dim \mathrm{Hom}_{\mathfrak{S}_{\tilde{\phi}}}(\delta[\tilde{\pi}, \tilde{\rho}], \mathcal{T}_\mu)[\iota_\ell \tilde{\pi}] + \mathrm{Err}$$

in $\tilde{\mathrm{K}}_0(G, \overline{\mathbb{Q}_\ell})$, where $\mathrm{Err} \in \tilde{\mathrm{K}}_0(G, \overline{\mathbb{Q}_\ell})$ is a virtual representation whose character vanishes on $G(K)_{\mathrm{s.reg,ell}}$.

Moreover, if the packet $\tilde{\Pi}_{\tilde{\phi}}(G)$ consists entirely of supercuspidal representations and the Fargues–Scholze L-parameter $\tilde{\phi}_{\tilde{\rho}}^{\mathrm{FS}}$ is supercuspidal, then $\mathrm{Err} = 0$.

We can apply Theorem 3.4.1 to $(b, \mu) = (b_1, \mu_1)$ defined in §2.6. Then it follows from Theorem 2.6.1 that we will have the following corollary:

Corollary 3.4.2. *Suppose $\tilde{\phi} \in \tilde{\Phi}_2(G^*)$ is a discrete L-parameter and $\tilde{\phi}^{\mathrm{GL}} = \phi_1 + \ldots + \phi_k + \phi_{k+1} + \ldots + \phi_r$ where ϕ_i is an irreducible representations of $W_{K_1} \times \mathrm{SL}(2,\mathbb{C})$ of dimension d_i for each i such that d_i is odd if and only if $i \leq k$. Let μ_1 be the dominant cocharacter of $G^*_{\overline{K}}$ defined in (2.5), and $b_1 \in B(G^*, b_0, \mu_1)_{\mathrm{bas}}$ be the unique basic element, then for any $\tilde{\rho} = \tilde{\pi}_{[I]} \in \tilde{\Pi}_{\tilde{\phi}}(G_{b_1})$ where $I \subset [r]_+$ with $\#[I] \equiv \frac{\kappa_{b_0}(-1)+1}{2}(\mathrm{mod}\ 2)$, there exists an equality*

$$\mathrm{Mant}_{G,b_1,\mu_1}[\iota_\ell \tilde{\pi}_{[I]}] = \mathrm{Mant}_{G,b_1,\mu_1^\bullet}[\iota_\ell \tilde{\pi}_{[I]}] = \sum_{i \in [r]_+} d_i [\iota_\ell \tilde{\pi}_{[I \oplus \{i\}]}] + \mathrm{Err}$$

in $\tilde{\mathrm{K}}_0(G, \overline{\mathbb{Q}_\ell})$, where $\mu_1^\bullet = -w_0(\mu_1)$ is the dominant cocharacter conjugate to μ^{-1}, and $\mathrm{Err} \in \tilde{\mathrm{K}}_0(G, \overline{\mathbb{Q}_\ell})$ is a virtual representation whose character vanishes on $G(K)_{\mathrm{s.reg,ell}}$.

Corollary 3.4.3. *If $\tilde{\phi} \in \tilde{\Phi}_{\mathrm{sc}}(G^*)$ is a supercuspidal L-parameter, then all representations $\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)$ have the same Fargues–Scholze L-parameter $\tilde{\phi}_{\iota_\ell \tilde{\pi}}^{\mathrm{FS}}$.*

Proof. For any $I \subset [r]_+$ with $\#[I] \equiv \frac{\kappa_{b_0}(-1)+1}{2}(\mathrm{mod}\ 2)$, it follows from Corollary 3.4.2 that $\pi_{[I \oplus \{i\}]}$ appears in $\mathrm{Mant}_{G,b_1,\mu_1}[\iota_\ell \tilde{\pi}_{[I]}]$, for each $i \in [r]_+$. We then use Proposition 3.2.3 to deduce that

$$\tilde{\phi}_{\iota_\ell \tilde{\pi}_{[I \oplus \{i\}]}}^{\mathrm{FS}} = \tilde{\phi}_{\iota_\ell \tilde{\pi}_{[I]}}^{\mathrm{FS}} = \tilde{\phi}_{\iota_\ell \tilde{\pi}_{[I \oplus \{j\}]}}^{\mathrm{FS}}$$

for any $i, j \in [r]_+$. Now it follows from combinatorial consideration that for any $\tilde{\pi}_{[J]}, \tilde{\pi}'_{[J']} \in \tilde{\Pi}_{\tilde{\phi}}(G)$ with $J \equiv J' \equiv 2$, there exists a chain of equalities implying that $\tilde{\phi}_{\iota_\ell \tilde{\pi}_{[J]}}^{\mathrm{FS}} = \tilde{\phi}_{\iota_\ell \tilde{\pi}_{[J']}}^{\mathrm{FS}}$. $\square$

We first recall from [HKW22, Definition 3.2.4] the transfer map from conjugation-invariant functions on $G(K)_{\mathrm{s.reg}}$ to conjugation-invariant function on $G_b(K)_{\mathrm{s.reg}}$ when $b \in B(G)_{\mathrm{bas}}$ is basic.

Definition 3.4.4. There is a diagram of topological spaces

$$\mathrm{Rel}_b$$

$$G(K)_{\mathrm{s.reg}} /\!\!/ G(K) \qquad\qquad G_b(K)_{\mathrm{s.reg}} /\!\!/ G_b(K),$$

where Rel_b is the set of $G(K) \times G_b(K)$-conjugacy classes of triples (g, g', λ) such that
- $g \in G(K)_{\mathrm{s.reg}}$ and $g' \in G_b(K)_{\mathrm{s.reg}} \subset G(\breve{K})$ are stably conjugate, i.e., conjugate under action of $G(\breve{K})$,
- $\lambda \in X_\bullet(Z_G(g))$ such that $\kappa_G(\mathrm{inv}[b](g,g'))$ agrees with the image of λ in $X_\bullet(Z_G(g))_{\mathrm{Gal}_K}$. Here $\mathrm{inv}[b](g,g')$ is the class of $y^{-1}b\varphi_K(y)$ in $B(Z_G(g))$, where $y \in G(\breve{K})$ satisfies $g' = ygy^{-1}$ (this class is independent of y chosen; see [HKW22, Definition 3.2.2, Fact 3.2.3]).

Here $(z, z') \in G(K) \times G_b(K)$ acts by conjugation on such triples by

$$\mathrm{ad}(z,z')(g,g',\lambda) = (\mathrm{ad}(z)g, \mathrm{ad}(z')g', \mathrm{ad}(z)\lambda),$$

and Rel_b is given the subspace topology by the inclusion $\mathrm{Rel}_b \subset (G(K) \times G_b(K) \times X_\bullet(G))/(G(K) \times G_b(K))$ with $X_\bullet(G)$ being discrete.

We recall the following Heck transfer map from [HKW22, Definition 3.2.7, Definition 6.3.4].

Definition 3.4.5. We define a Hecke transfer map

$$T_{b,\mu}^{G_b \to G} : C(G_b(K)_{\mathrm{s.reg}} /\!\!/ G_b(K)) \to C(G(K)_{\mathrm{s.reg}} /\!\!/ G(K))$$

such that

$$\left[T_{b,\mu}^{G_b \to G}(f')\right](g) = (-1)^{\langle \mu, 2\rho_{G^*} \rangle} \sum_{(g,g',\lambda) \in \mathrm{Rel}_b} f'(g') \dim \mathcal{T}_\mu[\lambda],$$

Since $\dim \mathcal{T}_\mu < \infty$, this sum is finite, and $T_{b,\mu}^{G_b \to G}(f')$ has compact support when f' does.

Moreover, on the strongly regular elliptic locus, $T_{b,\mu}^{G_b \to G}$ can be extended to a Hecke map on invariant distributions

$$\mathcal{T}_{b,\mu}^{G_b \to G} : \mathrm{Dist}(G_b(K)_{\mathrm{s.reg,ell}})^{G_b(K)} \to \mathrm{Dist}(G(K)_{\mathrm{s.reg,ell}})^{G(K)}.$$

We then have the following result of Hansen, Kaletha and Weinstein [HKW22, Theorem 6.5.2]:

Proposition 3.4.6. *For any $\rho \in \Pi(G_b)$ and $f \in C_c(G(K)_{\mathrm{s.reg,ell}})$,*

$$\mathrm{tr}\left(f | \iota_\ell^{-1} \, \mathrm{Mant}_{G,b,\mu}(\iota_\ell \rho)\right) = \left[\mathcal{T}_{b,\mu}^{G_b \to G}(\Theta_\rho)\right](f).$$

In particular, the virtual character of $\mathrm{Mant}_{G,b,\mu}(\rho)$ restricted to $G(K)_{\mathrm{s.reg,ell}}$ is equal to $T_{b,\mu}^{G_b,G}(\Theta_\rho)$.

We now use the weak endoscopic character identities to prove an analogue of [HKW22, Theorem 3.2.9], which relates the Hecke transfer map $T_{b,\mu}^{G_b \to G}$ to classical LLCs of G and G_b:

Proposition 3.4.7. *Assume $b \in B(G^*, b_0, \{\mu\})_{\mathrm{bas}}$ is basic and $\phi \in \Phi_2(G^*)$ is a discrete L-parameter. Let $\tilde{\rho} \in \tilde{\Pi}_{\tilde{\phi}}(G_b)$, then for any $g \in G(K)_{\mathrm{s.reg}}$ that transfers to $G_b(K)$,*

$$\left[T_{b,\mu}^{G_b \to G}\Theta_{\tilde{\rho}}\right](g) = \sum_{\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)} \dim \mathrm{Hom}_{\mathfrak{S}_\phi}(\delta[\tilde{\pi}, \tilde{\rho}], \mathcal{T}_\mu) \Theta_{\tilde{\pi}}(g).$$

Proof. We modify the proof of [HKW22, §3.3] as follows: Notation as in loc.cit., let $s \in \mathfrak{S}_\phi$ be a semisimple element, we choose a lift $\dot{s} \in S_\phi^+$ and let $s^\natural \in \mathfrak{S}_\phi$ be the image of $\dot{s}$ under the map $S_\phi^+ \to \mathfrak{S}_\phi$ defined in [Kal18, (4.7)]. In particular s and $s^\natural$ has the same image in $\overline{\mathfrak{S}}_\phi$. We have the extended endoscopic triple $\mathfrak{e} = (G^{\mathfrak{e}}, s^\natural, {}^L \xi^{\mathfrak{e}})$ associated to $s^\natural$, and $\phi^{\mathfrak{e}} \in \Phi_2(G^{\mathfrak{e}})$ associated to ϕ. Then $\dot{\mathfrak{e}} = (G^{\mathfrak{e}}, {}^L G^{\mathfrak{e}}, \dot{s}, {}^L \xi^{\mathfrak{e}})$ is exactly the endoscopic datum associated to $\dot{s}$ as defined in [HKW22, (A.1.1)], cf. [Kal18, §4.2]. Then, for any $(g, g', \lambda) \in \mathrm{Rel}_b$,

$$e(G_b) \sum_{\tilde{\rho}' \in \tilde{\Pi}_{\tilde{\phi}}(G_b)} \iota_{\mathfrak{m}, b_0 + b}(\tilde{\rho}')(s^\natural) \Theta_{\tilde{\rho}'}(g')$$

$$\overset{\text{Theorem 2.3.4}}{=} \sum_{h \in G^{\mathfrak{e}}(K)_{\mathrm{s.reg/st.conj}}} \Delta[\mathfrak{m}, z_{b_0 + b}, \mathfrak{e}](h, g') \mathrm{S}\Theta_{\tilde{\phi}^{\mathfrak{e}}}(h)$$

$$\overset{\text{[HKW22, Lemma A.1.1]}}{=} \sum_{h \in G^{\mathfrak{e}}(K)_{\mathrm{s.reg/st.conj}}} \Delta[\mathfrak{m}, z_{b_0}, \mathfrak{e}](h, g) \left\langle \mathrm{inv}[b](g, g'), s_{h,g}^\natural \right\rangle \mathrm{S}\Theta_{\tilde{\phi}^{\mathfrak{e}}}(h)$$

$$= \sum_{h \in G^{\mathfrak{e}}(K)_{\mathrm{s.reg/st.conj}}} \Delta[\mathfrak{m}, z_{b_0}, \mathfrak{e}](h, g) \lambda(s_{h,g}^\natural) \mathrm{S}\Theta_{\tilde{\phi}^{\mathfrak{e}}}(h),$$

where $s_{h,g}^\natural$ and $\left\langle \mathrm{inv}[b](g, g'), s_{h,g}^\natural \right\rangle$ are defined in [HKW22, Lemma A.1.1].

We now multiply this expression by the kernel function $\dim \mathcal{T}_{\{\mu\}}[\lambda]$, and sum over all $g' \in G_b(K)_{\mathrm{s.reg}} /\!\!/ G_b(K)$ and $\lambda \in X_\bullet(T)$ such that $(g, g', \lambda) \in \mathrm{Rel}_b$.

$$e(G_b) \sum_{(g', \lambda)} \sum_{\tilde{\rho}' \in \tilde{\Pi}_{\tilde{\phi}}(G_b)} \iota_{\mathfrak{m}, b_0 + b}(\tilde{\rho}')(s^\natural) \Theta_{\rho'}(g') \dim \mathcal{T}_\mu[\lambda]$$

$$= \sum_{h \in G^{\mathfrak{e}}(K)_{\mathrm{s.reg/st.conj}}} \Delta[\mathfrak{m}, z_{b_0}, \mathfrak{e}](h, g) \mathrm{S}\Theta_{\tilde{\phi}^{\mathfrak{e}}}(h) \sum_{(g', \lambda)} \lambda(s_{h,g}^\natural) \dim \mathcal{T}_\mu[\lambda]$$

$$= \sum_{h \in G^{\mathfrak{e}}(K)_{\mathrm{s.reg/st.conj}}} \Delta[\mathfrak{m}, z_{b_0}, \mathfrak{e}](h, g) \mathrm{S}\Theta_{\tilde{\phi}^{\mathfrak{e}}}(h) \mathrm{tr}(\mathcal{T}_\mu(s_{h,g}^\natural))$$

$$= \mathrm{tr}(\mathcal{T}_\mu(s^\natural)) \sum_{h \in G^{\mathfrak{e}}(K)_{\mathrm{s.reg/st.conj}}} \Delta(h, g) \mathrm{S}\Theta_{\tilde{\phi}^{\mathfrak{e}}}(h)$$

$$\overset{\text{Theorem 2.3.4}}{=} \mathrm{tr}(\mathcal{T}_\mu(s^\natural)) e(G) \sum_{\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)} \iota_{\mathfrak{m}, b_0}(\tilde{\pi})(s^\natural) \Theta_{\tilde{\pi}}(g).$$

Here the second and third equations hold for the same reason as in [HKW22, p. 17].

We multiply the above equation by $\iota_{\mathfrak{m},b_0+b}(\tilde{\rho})(s^\natural)^{-1}$, then as a function of $s^\natural \in \mathfrak{S}_{\tilde{\phi}}$, the left-hand side is invariant under translation by z_ϕ, so the same holds for the right-hand side, and both sides become functions of $\overline{s} \in \overline{\mathfrak{S}}_\phi$. We now average over $\overline{s} \in \overline{\mathfrak{S}}_\phi$ to get

$$\frac{1}{\#\overline{\mathfrak{S}}_\phi} e(G_b) \sum_{\overline{s}\in\overline{\mathfrak{S}}_\phi} \sum_{(g',\lambda)} \sum_{\tilde{\rho}'\in\tilde{\Pi}_{\tilde{\phi}}(G_b)} \frac{\iota_{\mathfrak{m},b_0+b}(\tilde{\rho}')}{\iota_{\mathfrak{m},b_0+b}(\tilde{\rho})}(s^\natural)\Theta_{\tilde{\rho}'}(g')\dim\mathcal{T}_\mu[\lambda]$$

$$= \frac{1}{\#\overline{\mathfrak{S}}_\phi} e(G) \sum_{\overline{s}\in\overline{\mathfrak{S}}_\phi} \operatorname{tr}(\mathcal{T}_\mu(s^\natural)) \sum_{\tilde{\pi}\in\tilde{\Pi}_{\tilde{\phi}}(G)} \frac{\iota_{\mathfrak{m},b_0}(\tilde{\pi})}{\iota_{\mathfrak{m},b_0+b}(\tilde{\rho})}(s^\natural)\Theta_{\tilde{\pi}}(g),$$

where we recall that $s^\natural \in \mathfrak{S}_\phi$ is a lift of $\overline{s}$. By Fourier inversion, the left-hand side equals

$$e(G_b) \sum_{(g',\lambda)} \Theta_{\tilde{\rho}}(g')\dim\mathcal{T}_\mu[\lambda] = (-1)^{\langle\mu,2\rho_{G^*}\rangle}e(G_b)\left[T_{b,\mu}^{G_b\to G}\Theta_{\tilde{\rho}}\right](g),$$

and the right-hand side equals

$$e(G) \sum_{\tilde{\pi}\in\tilde{\Pi}_{\tilde{\phi}}(G)} \Theta_{\tilde{\pi}}(g)\frac{1}{\#\overline{\mathfrak{S}}_\phi} \sum_{\overline{s}\in\overline{\mathfrak{S}}_\phi} \operatorname{tr}(\mathcal{T}_\mu(s^\natural))\delta[\tilde{\pi},\tilde{\rho}]^{-1}(s^\natural) = e(G) \sum_{\tilde{\pi}\in\tilde{\Pi}_{\tilde{\phi}}(G)} \dim\operatorname{Hom}_{\mathfrak{S}_\phi}(\delta[\tilde{\pi},\tilde{\rho}],\mathcal{T}_\mu)\Theta_{\tilde{\pi}}(g).$$

So the assertion is reduced to the identity $e(G) = (-1)^{\langle\mu,2\rho_{G^*}\rangle}e(G_b)$, which is exactly [HKW22, (3.3.3)]. $\square$

Now Theorem 3.4.1 follows from the above propositions the same way as in the proof of [HKW22, Theorem 6.5.1]:

Proof of Theorem 3.4.1. The claimed equality in $\tilde{K}_0(G(K))$ is an immediate consequence of Proposition 3.4.6 and Proposition 3.4.7. For the last claim regarding the error term, consider the non-elliptic virtual representation

$$\operatorname{Err} = \operatorname{Mant}_{G,b,\mu}(\iota_\ell\tilde{\rho}) - \sum_{\tilde{\pi}\in\tilde{\Pi}_{\tilde{\phi}}(G)} \dim\operatorname{Hom}_{S_{\tilde{\phi}}}(\delta[\tilde{\pi},\tilde{\rho}],\mathcal{T}_\mu)[\iota_\ell\tilde{\pi}].$$

By [HKW22, Theorem C.1.1], it suffices to show that Err is a virtual sum of supercuspidal representations. Since $\tilde{\Pi}_{\tilde{\phi}}(G)$ consists of supercuspidal representations by assumption, it suffices to show that $\operatorname{Mant}_{G,b,\mu}(\iota_\ell\rho)$ is a virtual sum of supercuspidal representations, where $\rho \in \Pi(G_b)$ is an arbitrary lift of $\tilde{\rho} \in \tilde{\Pi}(G_b)$. By Proposition 3.2.3, $\iota_\ell^{-1}\tilde{\phi}_{\iota_\ell\tilde{\pi}}^{\operatorname{FS}} = \iota_\ell^{-1}\tilde{\phi}_{\iota_\ell\tilde{\rho}}^{\operatorname{FS}}$ is supercuspidal. So π is supercuspidal by the compatibility of Fargues–Scholze local Langlands correspondence with parabolic inductions (see Theorem 3.1.1). $\square$

4. Cohomology of orthogonal and unitary Shimura varieties

In this section, we compute the Π-isotypic component of the cohomology of Shimura varieties $\operatorname{Sh}(\mathbf{G},\mathbf{X})$ of orthogonal or unitary type related to the local group G defined in §2.1, where Π is a special cuspidal automorphic representation of $\mathbf{G}(\mathbf{A}_f)$. These results are related to the cohomology of local shtuka spaces via the basic uniformization theorem stated in the next section §5.

4.1. **The groups.** Let F be a totally real number field and F_1 be either F or a CM field containing F, and let $\mathsf{c} \in \operatorname{Gal}(F_1/F)$ be the element with fixed field F. Let $\tau_0 : F_1 \to \mathbb{C}$ be a fixed embedding. Denote $\chi_{F_1/F} : \mathbf{A}_F^\times/F \to \{\pm 1\}$ be the character associated to F_1/F via global class field theory. If $F_1 \neq F$, we fix a totally imaginary element $\daleth \in F_1^\times$, so each embedding $\tau : F \to \mathbb{C}$ extends to an embedding $\tau : F_1 \to \mathbb{C}$ sending $\daleth$ to $\mathbb{R}_+\mathrm{i}$.

Let $\mathbf{V}$ be a vector space of dimension $n \geq 2$ equipped with a non-degenerate Hermitian c-sesquilinear form $\langle-,-\rangle$ on $\mathbf{V}$, i.e.

$$\langle au + bv, w\rangle = a\langle u,w\rangle + b\langle v,w\rangle,$$
$$\langle v,w\rangle = \langle w,v\rangle^\mathsf{c}.$$

In Case O2, we take an arbitrary diagonal basis $\{v_1,\ldots,v_n\}$ of $\mathbf{V}$ over F_1 such that $\langle v_i,v_i\rangle = a_i \in F^\times$, we can define

$$\operatorname{disc}(\mathbf{V}) = (-1)^{\binom{n}{2}}2^{-n}\prod_{i=1}^n a_i$$

whose image in $F^\times/(F^\times)^2$ is independent of the basis chosen, and we write $\operatorname{disc}(\mathbf{G}) = \operatorname{disc}(\mathbf{V})$.

Let $U(\mathbf{V}) \leq GL(\mathbf{V})$ be the algebraic subgroup defined by

$$U(\mathbf{V}) = \{g \in GL(\mathbf{V}) : \langle gv, gw \rangle = \langle v, w \rangle \, \forall v, w \in \mathbf{V}\},$$

and let $\mathbf{G} = U(\mathbf{V})^\circ$ be the neutral component of $U(\mathbf{V})$. Let $\mathbf{G}^*$ be the unique quasi-split inner form of $\mathbf{G}$ over F, then exactly one of the following three cases occurs:

O1 $F_1 = F$ and $\dim(\mathbf{V}) = 2n+1$ is odd, then $\mathbf{G}^* = SO(2n+1)$.

O2 $F_1 = F$ and $\dim(\mathbf{V}) = 2n$ is even, then $\mathbf{G}^* = SO(2n)^{\mathrm{disc}(\mathbf{V})}$ is the special orthogonal group associated with the quadratic space over F of dimension $2n$, discriminant $\mathrm{disc}(\mathbf{V})$ such that the Hasse–Witt invariants of $\mathbf{V} \otimes F_v$ is 1 for each $v \in \Sigma_F$.

U $F_1 \neq F$ and $\dim(\mathbf{V}) = n$, then $\mathbf{G}^* = U(n)$ is the unitary group associated with the Hermitian space of dimension n with respect to the quadratic extension F_1/F such that the Hasse–Witt invariant of $\mathbf{V} \otimes F_v$ is 1 for each $v \in \Sigma_F$.

We refer to cases O1 and O2 together as Case O. We assume further that

$$\dim \mathbf{V} \geq \begin{cases} 2 & \text{in Case U} \\ 5 & \text{in Case O1} \\ 6 & \text{in Case O2} \end{cases},$$

so that $\mathbf{G}_{\mathrm{ad}}$ is always geometrically simple. As $\mathbf{G}^*_v$ is unramified for each $v \in \Sigma_F^{\mathrm{fin}}$, we can fix a reductive integral model $\mathcal{G}^*_v$ over $\mathcal{O}_{F_v}$ and a hyperspecial subgroup $\mathcal{K}^{\mathrm{hs}}_v := \mathcal{G}^*_v(\mathcal{O}_{F_v}) \leq \mathbf{G}^*(F_v)$ for each $v \in \Sigma_F^{\mathrm{fin}}$.

To unify notation, we write $n(\mathbf{G}) = n(\mathbf{G}^*)$ for the rank of $\mathbf{G}_{\overline{F}}$, and we define $N(\mathbf{G}), d(\mathbf{G}), b(\mathbf{G})$ analogous to the local case (2.1), and define $\mathrm{disc}(\mathbf{G}) := \mathrm{disc}(\mathbf{V})$ in Case O2.

It follows from the theorem of Hasse–Minkowski and Landherr [Gro21, Theorem 2.1, Theorem 3.1] and [GGP12, Lemma 2.1] that pure inner twists of $\mathbf{G}^*$ are in bijection with isometry classes of c-Hermitian spaces $\mathbf{V}$ with respect to F_1/F of dimension $d(\mathbf{G}^*)$(and also with discriminant $\mathrm{disc}(\mathbf{V})$ in Case O2), and these isometry classes are determined by isometry classes of localizations $\mathbf{V}_v$ for each $v \in \Sigma_F$. In particular, $\mathbf{G}$ can always be embedded into an pure inner twist $(\mathbf{G}, \boldsymbol{\rho}, \boldsymbol{z})$ of $\mathbf{G}^*$.

We fix an isomorphism

$$\widehat{\mathbf{G}} \cong \begin{cases} Sp(N(\mathbf{G}), \mathbb{C}) & \text{in Case O1} \\ SO(N(\mathbf{G}), \mathbb{C}) & \text{in Case O2} \\ GL(N(\mathbf{G}), \mathbb{C}) & \text{in Case U} \end{cases},$$

and fix a pinning $(\widehat{\mathbf{T}}, \widehat{\mathbf{B}}, \{X_\alpha\})$ where $\widehat{\mathbf{T}}$ is the diagonal torus, $\widehat{\mathbf{B}}$ is the group of upper triangular matrices, and $\{X_\alpha\}$ is the set of standard root vectors. We writ $^L\mathbf{G} = \widehat{\mathbf{G}} \rtimes W_F$ for the Langlands L-group in the Weil form. Note that $\widehat{\mathbf{G}}$ has a standard representation $\widehat{\mathrm{Std}} = \widehat{\mathrm{Std}}_{\mathbf{G}} : \widehat{\mathbf{G}} \to GL(N(\mathbf{G}))_{\mathbb{C}}$.

Similar to the local case §2.1, there exists an automorphism θ on $\mathbf{G}^{\mathrm{GL}} := \mathrm{Res}_{F_1/F} GL(N(\mathbf{G}))$ such that $\mathbf{G}^*$ can be extended to an element in $\mathcal{E}_{\mathrm{ell}}(\mathbf{G}^{\mathrm{GL}} \rtimes \theta)$, and the description of the isomorphism classes of elliptic endoscopic triples $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(\mathbf{G})$ is similar to the local case.

Finally, we define central extension $\mathbf{G}^\sharp$ of $\mathrm{Res}_{F/\mathbb{Q}} \mathbf{G}$ as follows:

- In Case O, we imitate [Car86, P163]. Let $\mathrm{Cl}(\mathbf{V})$ and $\mathrm{Cl}^\circ(\mathbf{V})$ be the Clifford algebra and even Clifford algebra, respectively. Note that there exists an embedding $\mathbf{V} \subset \mathrm{Cl}(\mathbf{V})$ and an anti-involution $*$ on $\mathrm{Cl}(\mathbf{V})$(the main involution) [MP16, §1.1]. Let $\mathrm{GSpin}(\mathbf{V})$ be the stabilizer in $\mathrm{Cl}^\circ(\mathbf{V})^\times$ of $\mathbf{V} \subset \mathrm{Cl}(\mathbf{V})$ with respect to the conjugation action of $\mathrm{Cl}^\circ(\mathbf{V})^\times$ on $\mathrm{Cl}(\mathbf{V})$, which is a reductive group over F. The conjugation action of $\mathrm{GSpin}(\mathbf{V})$ on $\mathbf{V}$ induces an exact sequence of reductive groups over F:

$$(4.1) \qquad\qquad 1 \to GL(1)_F \to \mathrm{GSpin}(\mathbf{V}) \to \mathbf{G} \to 1.$$

There is a similitude map $\nu : \mathrm{GSpin}(\mathbf{V}) \to GL(1)_F : g \mapsto g^* g$ whose restriction on the central torus is $z \mapsto z^2$. The kernel $\mathrm{Spin}(\mathbf{V}) := \ker(\nu)$ is called the spinor group of $\mathbf{V}$.

For a fixed quadratic imaginary element $\daleth \in \mathbb{R}_+ i$ (in particular $\daleth^2 \in \mathbb{Q}_-$), we define a reductive group $\mathbf{G}_\daleth$ over $\mathbb{Q}$:

$$\mathbf{G}_\daleth = \left((\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GSpin}(\mathbf{V})) \times \mathrm{Res}_{F(\daleth)/\mathbb{Q}} GL(1) \right) / \mathrm{Res}_{F/\mathbb{Q}} GL(1),$$

where $\mathrm{Res}_{F/\mathbb{Q}} GL(1)$ is embedded anti-diagonally. We define

$$\nu^\sharp : \mathbf{G}_\daleth \to \mathrm{Res}_{F/\mathbb{Q}} GL(1) : (g, t) \mapsto \nu(g) \mathrm{Nm}_{F(\daleth)/F}(t),$$

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and we define $\mathbf{G}^\sharp \subset \mathbf{G}\neg$ to be the inverse image under $\nu^\sharp$ of the sub-torus $\mathrm{GL}(1) \subset \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}(1)$, then $[\mathbf{G}^\sharp, \mathbf{G}^\sharp] = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{Spin}(\mathbf{V})$, and the exact sequence (4.1) induces an exact sequence of reductive groups over $\mathbb{Q}$:

$$1 \to \mathbf{Z}^\mathbb{Q} \to \mathbf{G}^\sharp \to \mathrm{Res}_{F/\mathbb{Q}}\mathbf{G} \to 1,$$

where $\mathbf{Z}^\mathbb{Q} = \{z \in \mathrm{Res}_{F(\neg)/\mathbb{Q}}\mathrm{GL}(1) : \mathrm{Nm}_{F(\neg)/F}(z) \in \mathbb{Q}^\times\}$.

- In Case U, we follow [RSZ20]. Let $\mathrm{GU}^\mathbb{Q}(\mathbf{V})$ be the reductive group over $\mathbb{Q}$ defined by

$$\mathrm{GU}^\mathbb{Q}(\mathbf{V}) = \{(g,\mu) \in \mathrm{GL}(\mathbf{V}) \times \mathrm{GL}(1) : \langle gv, gw\rangle = \mu \langle v, w\rangle\},$$

which is naturally equipped with a similitude map $\nu : \mathrm{GU}^\mathbb{Q}(\mathbf{V}) \to \mathrm{GL}(1)$. (Note that $\mathrm{GU}^\mathbb{Q}(\mathbf{V})$ is a subgroup of the restriction of the unitary similitude group $\mathrm{GU}(\mathbf{V})$). We write

$$\mathbf{Z}^\mathbb{Q} = \{z \in \mathrm{Res}_{F_1/\mathbb{Q}}\mathrm{GL}(1) : \mathrm{Nm}_{F_1/F}(z) \in \mathbb{Q}^\times\}$$

which is naturally equipped with a map to $\mathrm{GL}(1)$. We then define a reductive group $\mathbf{G}^\sharp$ over $\mathbb{Q}$:

$$\mathbf{G}^\sharp = \mathrm{GU}^\mathbb{Q}(\mathbf{V}) \times_{\mathrm{GL}(1)} \mathbf{Z}^\mathbb{Q},$$

which is isomorphic to $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G} \times \mathbf{Z}^\mathbb{Q}$ via the isomorphism

$$\mathbf{G}^\sharp \cong \mathrm{Res}_{F/\mathbb{Q}}\mathbf{G} \times \mathbf{Z}^\mathbb{Q} : (g,z) \mapsto (z^{-1}g, z).$$

4.2. Endoscopic classification of automorphic representations.
Let $\mathbf{G}^*$ be as in §4.1 and $(\mathbf{G}, \rho, \boldsymbol{z})$ be a pure inner form of $\mathbf{G}^*$. To prepare for the Langlands–Kottwitz method in §4.5, we recall some results on endoscopic classifications of automorphic representations of orthogonal and unitary groups over a totally real field, following [Art13, KMSW14, Ish24, CZ24].

The discrete spectrum $L^2_{\mathrm{disc}}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbf{A}_F))$ can be decomposed into near equivalence classes of representations where two irreducible representations $\pi = \otimes'_v \pi_v$ and $\pi' = \otimes'_v \pi'_v$ are called nearly equivalent if π_v and π'_v are equivalent for all but finitely many places $v \in \Sigma_F$. The decomposition into near equivalence classes will be expressed in terms of elliptic global A-parameters. An elliptic global A-parameter $\boldsymbol{\psi} \in \Psi_{\mathrm{ell}}(\mathbf{G})$ is a formal finite sum of pairs

$$\boldsymbol{\psi} = \sum_i (\Pi_i, d_i),$$

where each Π_i is an irreducible cuspidal automorphic representation of $\mathrm{GL}(n_i; \mathbf{A}_{F_1})$ that is conjugate self-dual of sign $(-1)^{d_i-1}b(\mathbf{G})$ (defined similarly as in §2.2), such that

- $\sum_i n_i d_i = N(\mathbf{G})$.
- $(\Pi_i, d_i) \neq (\Pi_j, d_j)$ if $i \neq j$,
- In Case O2, we assume $\prod_i \omega_i^{d_i} = \chi_{F\left(\sqrt{\mathrm{disc}(\mathbf{G})}\right)/F}$, where ω_i is the central character of Π_i, and $\chi_{F\left(\sqrt{\mathrm{disc}(\mathbf{G})}\right)/F}$ is the quadratic character of Gal_F corresponding to the extension $F(\sqrt{\mathrm{disc}(\mathbf{G})})/F$ via global class field theory.

For each $v \in \Sigma_F$, we can define the localization $\boldsymbol{\psi}_v := \sum_i (\phi_{i,v}, d_i)$: If $\Sigma_{F_1}(v)$ is a singleton, which we also write by v, then $\phi_{i,v}$ is a n_i-dimensional representation of $W_{(F_1)_v} \times \mathrm{SL}(2,\mathbb{C})$ that corresponds to $\Pi_{i,v}$ via the LLC, which is conjugate self-dual of sign $(-1)^{d_i-1}b(\mathbf{G})$. We can associate to $\boldsymbol{\psi}_v$ a formal sum

$$\tilde{\phi}^{\mathrm{GL}}_{\boldsymbol{\psi}_v} := \sum_i \left(\phi_{i,v} \otimes |-|^{\frac{d_i-1}{2}}_{(F_1)_v} + \phi_{i,v} \otimes |-|^{\frac{d_i-3}{2}}_{(F_1)_v} + \ldots + \phi_{i,v} \otimes |-|^{\frac{1-d_i}{2}}_{(F_1)_v}\right),$$

which may be regarded as an element of $\tilde{\Phi}(\mathbf{G}_v)$.

On the other hand, if $\#\Sigma_{F_1}(v) = 2$, then we are in Case U. If we write $\Sigma_{F_1}(v) = \{w, w^{\mathrm{c}}\}$, then $\Pi_{i,w} \cong \Pi_{i,w^{\mathrm{c}}}^\vee$ under the identifications $\mathrm{GL}(n_i; F_{1,w}) \cong \mathrm{GL}(n_i; (F_1)_{w^{\mathrm{c}}}) \cong \mathrm{GL}(n_i; F_v)$. We define $\phi_{i,v}$ to be the n_i-dimensional representation of $W_{F_v} \times \mathrm{SL}(2,\mathbb{C})$ that corresponds to $\Pi_{i,w}$ under the identification $w : F_{1,w} \xrightarrow{\sim} F_v$. This is independent of w chosen, and we define $\tilde{\phi}^{\mathrm{GL}}_{\boldsymbol{\psi}_v}$ as before.

We then have the following theorem, usually called the "Arthur's multiplicity formula":

Theorem 4.2.1 ([Art13, Mok15, KMSW14, Ish24, CZ24]). *There exists a decomposition*

$$L^2_{\mathrm{disc}}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbf{A}_F)) = \bigoplus_{\boldsymbol{\psi}\in\Psi_{\mathrm{ell}}(\mathbf{G})} L^2_{\boldsymbol{\psi}}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbf{A}_F)),$$

where $L^2_{\boldsymbol{\psi}}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbf{A}_F))$ is the sum of irreducible representations π in $L^2_{\mathrm{disc}}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbf{A}_F))$ such that $\tilde{\phi}^{\mathrm{GL}}_{\pi_v} \cong \phi_{\boldsymbol{\psi}_v}$ for all but finitely many places $v \in \Sigma_F$ (here if $\#\Sigma_{F_1}(v) = 2$ with $\Sigma_{F_1}(v) = \{w, w^c\}$, then we write $\tilde{\phi}^{\mathrm{GL}}_{\pi_v}$ for the classical L-parameter of $\mathbf{G}_v \cong \mathrm{GL}(N(\mathbf{G}))_{F_v}$ corresponding to π_v composed with the identification $w : F_{1,w} \xrightarrow{\sim} F_v$). A representation π in $L^2_{\boldsymbol{\psi}}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbf{A}_F))$ is said to has formal parameter $\boldsymbol{\psi}$.

Moreover, when $\boldsymbol{\psi} = \Pi$ is a single cuspidal representation of $\mathrm{GL}(N(G), \mathbf{A}_{F_1})$ (which is conjugate self-dual of sign $b(G)$), then for each π in $L^2_{\boldsymbol{\psi}}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbf{A}_F))$, $L^2_{\boldsymbol{\psi}}$ consists of admissible representations π of $\mathbf{G}(\mathbf{A}_F)$ such that Π_v is a local functorial transfer of π_v (see Definition 2.3.3) for all $v \in \Sigma_F$; conversely, if $\pi = \otimes'_v \pi_v$ is an admissible representation of $\mathbf{G}(\mathbf{A}_F)$ such that Π_v is a local functorial transfer of π_v for all $v \in \Sigma_F$, then there exists a cuspidal automorphic representation $\pi' = \otimes'_v \pi'_v$ of $\mathbf{G}(\mathbf{A}_F)$ such that $\pi'_v \sim_\varsigma \pi_v$ for all $v \in \Sigma_F$.

Proof. In Case O1, this is established in [Art13] when G is quasi-split, and established in [Ish24, Theorem 3.16, 3.17] when G is not quasi-split. In Case O2, this is established in [Art13] when G is quasi-split, and in [CZ24, Theorem 2.1, 2.6] when G is not quasi-split. In Case U, this is established in [Mok15] and in established in [KMSW14, Theorem 1.7.1] when G is not quasi-split. We briefly remark that in the above theorems, the formal parameter $\boldsymbol{\psi}$ is a generic elliptic A-parameter, so the formal component group $\mathfrak{S}_{\boldsymbol{\psi}}$ of $\boldsymbol{\psi}$ defined as in the literature above is isomorphic to $\mathbb{Z}/2$. Thus, after fixing a Whittaker datum $\mathfrak{m}$ for $\mathbf{G}^*$, the condition

$$\prod_{v\in\Sigma_F} \iota_{\mathfrak{m}_v,\varrho_v,z_v}(\pi_v) = \mathbf{1} \in \mathrm{Irr}(\mathfrak{S}_{\boldsymbol{\psi}})$$

in the decomposition of $L^2_{\boldsymbol{\psi}}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbf{A}_F))$ is automatically satisfies because of the theorem of Hasse–Minkowski and Landherr and Theorem 2.3.1. $\qquad\square$

This theorem implies the following result on strong functorial transfer and strong multiplicity one for cuspidal automorphic representations of $\mathbf{G}(\mathbf{A}_F)$:

Corollary 4.2.2. *For any cuspidal automorphic representation π of $\mathbf{G}(\mathbf{A}_F)$, suppose that the formal parameter $\boldsymbol{\psi}$ of π is a single cuspidal representation of $\mathbf{G}^{\mathrm{GL}}(\mathbf{A}_F)$, and π_v has simple supercuspidal classical L-parameter for some finite place $v \in \Sigma_F^{\mathrm{fin}}$, then $\boldsymbol{\psi}_v$ is the local functorial transfer of π_v for each finite place $v \in \Sigma_F^{\mathrm{fin}}$. We write π^{GL} for $\boldsymbol{\psi}$, and call it the strong functorial transfer of π.*

Moreover, if π' is another cuspidal automorphic representations of $\mathbf{G}(\mathbf{A}_F)$ or $\mathbf{G}^*(\mathbf{A}_F)$ with formal parameter π^{GL}, then $\tilde{\phi}_{\pi'_v} = \tilde{\phi}_{\pi_v}$ for all finite places of F, and π^{GL} is the strong functorial transfer of π'.

4.3. **Controlled cuspidal automorphic representations.** Let $\mathbf{G}^*$ be as in §4.1. To make it clear what local conditions we are imposing on the automorphic forms we are considering, we will use the following notion of a control tuple:

Definition 4.3.1. A *control tuple* over for $\mathbf{G}^*$ is a tuple $\bigstar = (\Sigma^\circ, \Sigma^{\mathrm{St}}, \Sigma^{\mathrm{sc}}, \Sigma, \xi)$ where

- Σ^{St} and Σ^{sc} are disjoint finite sets of finite places of F such that $\Sigma^{\mathrm{St}} \neq \varnothing, \Sigma^{\mathrm{sc}} \neq \varnothing$.

- $\Sigma^\circ \subset \Sigma^{\mathrm{St}} \cup \Sigma^{\mathrm{sc}}$ and $\Sigma^{\mathrm{St}} \cup \Sigma^{\mathrm{sc}} \cup \Sigma_F^\infty \subset \Sigma$ are finite sets of places of F.

- $\xi = \otimes_{\tau\in\mathrm{Hom}(F,\mathbb{C})}\xi_\tau$ is an irreducible representation of $\left(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}^*\right)\otimes_{\mathbb{Q}}\mathbb{C}$ with regular highest weight.

Definition 4.3.2. Let $\bigstar$ be a control tuple for $\mathbf{G}^*$. A pure inner twist $(\mathbf{G}, \varrho, z)$ of $\mathbf{G}^*$ over F is called a $\bigstar$-*good pure inner form* of $\mathbf{G}^*$, if (ϱ_v, z_v) is trivial for each $v \in \Sigma_F^{\mathrm{fin}} \smallsetminus \Sigma^\circ$.

If $(\mathbf{G}, \varrho, z)$ is a $\bigstar$-good pure inner form of $\mathbf{G}^*$, then for each $v \in \Sigma_F^{\mathrm{fin}} \smallsetminus \Sigma^\circ$, $\mathbf{G}_v$ has a reductive integral model $\mathcal{G}_v$ over $\mathcal{O}_{F_v}$ coming from the fixed reductive integral model $\mathcal{G}^*_v$ of $\mathbf{G}^*_v$ via ϱ. We also write $\mathscr{K}^{\mathrm{hs}}_v$ for the maximal and we can define the abstract Hecke algebra away from Σ:

$$\mathbb{T}^\Sigma := \prod_{v\in\Sigma_F^{\mathrm{fin}}\smallsetminus\Sigma} \mathcal{H}(\mathbf{G}(F_v), \mathscr{K}^{\mathrm{hs}}_v).$$

Definition 4.3.3. For a control tuple $\bigstar$ for $\mathbf{G}^*$ and a $\bigstar$-good pure inner twist $(\mathbf{G}, \varrho, z)$ of $\mathbf{G}^*$, a $\bigstar$-*good automorphic representation* of $\mathbf{G}$ is a cuspidal automorphic representation $\pi = \otimes'_v \pi_v$ of $\mathbf{G}$ such that

- π_v is unramified for all $v \in \Sigma_F^{\mathrm{fin}} \setminus \Sigma$;
- π_v is an unramified twist of the Steinberg representation for any $v \in \Sigma^{\mathrm{St}}$ (see Definition A.4.6);
- π_v has supercuspidal classical L-parameter for each $v \in \Sigma^{\mathrm{sc}}$. Moreover, π_v has simple classical L-parameter for some $v \in \Sigma^{\mathrm{sc}}$ (as defined in §2.2);
- π_∞ is cohomological for ξ, i.e., $\mathrm{H}^i(\mathrm{Lie}(\mathbf{G}(F \otimes \mathbb{R})), \mathscr{K}_\infty, \pi_\infty \otimes_{\mathbb{C}} \xi) \neq 0$ for some $i \in \mathbb{N}$;
- The formal parameter ψ of π is a single cuspidal representation of $\mathbf{G}^{\mathrm{GL}}(\mathbf{A}_F)$ (see Theorem 4.2.1).

Definition 4.3.4. For a control tuple $\bigstar$ for $\mathbf{G}^*$ and a $\bigstar$-good pure inner twist $(\mathbf{G}, \varrho, z)$ of $\mathbf{G}^*$, suppose $\Sigma' \subset \Sigma^\circ$ is a subset, then a compact open subgroup $\mathscr{K}^{\Sigma'} \leq \mathbf{G}(\mathbf{A}^{\Sigma'})$ is called a $\bigstar$-*split subgroup* if it is of the form $\mathscr{K}^{\Sigma'} = \prod_{v \in \Sigma_F^{\mathrm{fin}} \setminus \Sigma'} \mathscr{K}_v$ where $\mathscr{K}_v = \mathscr{K}_v^{\mathrm{hs}}$ for $v \notin \Sigma$.

We then have the following controlled strong transfer result:

Theorem 4.3.5. *For any control tuple $\bigstar$ for $\mathbf{G}^*$ and $\bigstar$-good pure inner twists $(\mathbf{G}, \varrho, z), (\mathbf{G}', \varrho', z')$ of $\mathbf{G}^*$, if π is a $\bigstar$-good automorphic representation of $\mathbf{G}$, then there exists a $\bigstar$-good automorphic representation τ of $\mathbf{G}'$ such that*
- $\tau^\Sigma \cong \pi^\Sigma$ *via the isomorphism $\varrho' \circ \varrho^{-1} : \mathbf{G}^\Sigma \xrightarrow{\sim} (\mathbf{G}')^\Sigma$.*
- *for any $v \in \Sigma_F^{\mathrm{fin}}$, τ_v has the same classical L-parameter $\tilde{\phi}_{\tau_v}$ as π_v.*

Such a τ is called a $\bigstar$-good transfer of π to $\mathbf{G}'$.

Proof. For a place $v \in \Sigma_F$, let $\tilde{\phi}_v$ be the classical L-parameter of π_v. By Theorem 4.2.1, it suffices to show that $\tilde{\Pi}_{\tilde{\phi}_v}(\mathbf{G}'_v)$ is non-empty: This is true for $v \notin \Sigma_F^{\mathrm{fin}} \setminus \Sigma^\circ$ because (ϱ_v, z_v) is trivial, and when $v \in \Sigma^\circ$, π_v is discrete, so $\tilde{\phi}_v$ is discrete, and $\tilde{\Pi}_{\tilde{\phi}_v}(\mathbf{G}'_v)$ is non-empty by Theorem 2.3.1. For $\tau \in \Sigma_F^\infty$, $\Pi_{\xi_\tau}(\mathbf{G}'_\tau(\mathbb{R}))$ is also non-empty, by Harish-Chandra's criterion. In fact, $\#\Pi_\xi(\mathbf{G}'(F \otimes \mathbb{R}))$ only depends on $\mathbf{G}'$ but not on ξ, by Harish-Chandra's classification of discrete series representations.

At any $v \in \Sigma^{\mathrm{St}}$, $\mathfrak{S}_{\tilde{\phi}_v}$ has order 2 and the central element $z_{\tilde{\phi}_v}$ is nontrivial. Consequently, the packet $\tilde{\Pi}_{\tilde{\phi}_v}(\mathbf{G}'_v)$ is a singleton, whose unique member is an unramified twist of the Steinberg representation, by Definition A.4.7. $\qquad\square$

We now recall the following result by many people, including Clozel, Kottwitz, Harris–Taylor [HT01], Shin [Shi11] and Chenevier–Harris [CH13], which allows us to construct ℓ-adic representations attached to $\bigstar$-good automorphic representations:

Theorem 4.3.6 ([Clo90, HT01, TY07, Shi11, Car12, CH13, Car14])**.** *For any control tuple $\bigstar$ for $\mathbf{G}^*$ and $\bigstar$-good pure inner twists $(\mathbf{G}, \varrho, z)$ of $\mathbf{G}^*$, if π is a $\bigstar$-good automorphic representation of $\mathbf{G}$, then Ramanujan's conjecture holds for π, i.e., π_v is tempered for each $v \in \Sigma_F$, and there exists a unique irreducible continuous representation*

$$\rho_{\pi,\ell} : \mathrm{Gal}_{F_1} \to \mathrm{GL}\left(N(\mathbf{G}); \overline{\mathbb{Q}_\ell}\right)$$

such that for any finite place w of F_1 with underlying finite place v of F,

$$\mathrm{WD}\left(\rho_{\pi,\ell}|_{W_{F_{1,w}}}\right) \cong \iota_\ell\left(\tilde{\phi}_{\pi_v}^{\mathrm{GL}} \otimes |-|_{F_{1,w}}^{\frac{1-N(\mathbf{G})}{2}}\right).$$

Here $\tilde{\phi}_{\pi_v}$ is the classical L-parameter of π_v (here if $\#\Sigma_{F_1}(v) = 2$ with $\Sigma_{F_1}(v) = \{w, w^c\}$ then we write $\tilde{\phi}_{\pi_v}^{\mathrm{GL}}$ for the classical L-parameter of $\mathbf{G}_v \cong \mathrm{GL}(N(\mathbf{G}))_{F_v}$ corresponding to π_v composed with the identification $F_v \cong F_{1,w}$).

Moreover, if π' is another cuspidal automorphic representation of $\mathbf{G}(\mathbf{A}_F)$ or $\mathbf{G}^*(\mathbf{A}_F)$ such that π_v and π'_v have the same classical L-parameter $\tilde{\phi}_{\pi_v} = \tilde{\phi}_{\pi'_v}$ for all but finitely many finite places v of F, then the above equation is satisfied for each $v \in \Sigma_F^{\mathrm{fin}}$ with π_v replaced by π'_v.

Proof. Let π^{GL} be the strong functorial transfer of π to $\mathbf{G}^{\mathrm{GL}}$ (Corollary 4.2.2), then π^{GL} is conjugate self-dual and cohomological with regular highest weight. We let $\rho_{\pi,\ell}$ be the Galois representation associated to π^{GL}. Such $\rho_{\pi,\ell}$ is constructed in [CH13, Theorem 3.2.3] and the local-global compatibility is established in [Car12, Theorem 1.1] and [Car14, Theorem 1.1]. The temperedness is established for Archimedean places by Clozel [Clo90, Lemma 4.9] and established for finite places by [HT01, TY07, Shi11, Car12, Clo13, Car14]. The irreducibility follows from the local-global compatibility and the fact that π_v has simple supercuspidal classical L-parameter for some $v \in \Sigma^{\mathrm{sc}}$.

The last assertion follows from Corollary 4.2.2. □

4.4. **The Shimura data.** We first define the relevant Shimura varieties following [MP16] and [RSZ20]. Let F_1/F, $\daleth \in F_1^\times$ and $\mathbf{V}, \mathbf{G}, \mathbf{G}^*, \mathbf{G}^\sharp$ be as in §4.1.

Definition 4.4.1. A pure inner twist $(\mathbf{V}, \mathbf{G} = \mathrm{U}(\mathbf{V})^\circ)$ of $\mathbf{G}^*$ is called
- *standard definite* if $\mathbf{V} \otimes_{F,\tau} \mathbb{R}$ is positive definite for each $\tau : F \to \mathbb{R}$.
- *standard indefinite* if $\mathbf{V} \otimes_{F,\tau} \mathbb{R}$ has signature $(N-2, 2)$ (resp. $(N-1, 1)$) in Case O (resp. in Case U) for $\tau = \tau_0$ and positive definite (i.e., signature $(N, 0)$) for each $\tau \in \Sigma_F^\infty \smallsetminus \{\tau_0\}$.

Suppose $(\mathbf{V}, \mathbf{G} = \mathrm{U}(\mathbf{V})^\circ)$ is a standard indefinite pure inner twist of $\mathbf{G}^*$, and $\mathbf{G}^\sharp$ is the central-extension of $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}$ defined in §4.1. We have the Shimura datum $(\mathbf{G}^\sharp, \mathbf{X}^\sharp)$, where $\mathbf{X}^\sharp$ is the conjugacy class of a Deligne homomorphism

$$(4.2) \qquad h_0^\sharp : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}(1) \to \mathbf{G}^\sharp \otimes \mathbb{R}$$

defined as follows:
- In Case O, let

$$h_{0,\natural} : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}(1) \to \left(\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSpin}(\mathbf{V})\right) \otimes \mathbb{R} \cong \prod_{\tau \in \mathrm{Hom}(F,\mathbb{R})} \mathrm{GSpin}(\mathbf{V} \otimes_{F,\tau} \mathbb{R})$$

be the homomorphism that is trivial on $\mathrm{GSpin}(\mathbf{V} \otimes_{F,\tau} \mathbb{R})$ for $\tau \neq \tau_0$, and on $\mathrm{GSpin}(\mathbf{V} \otimes_{F,\tau_0} \mathbb{R})$ it is induced by

$$h_{0,\tau_0}^\sharp : \mathbb{C}^\times \to \mathrm{GSpin}(\mathbf{V} \otimes_{F,\tau_0} \mathbb{R}) : a + b\mathrm{i} \mapsto a + be_{\tau_0,1}e_{\tau_0,2}$$

where $e_{\tau_0,1}, e_{\tau_0,2}$ are two orthogonal vectors in $\mathbf{V} \otimes_{F,\tau_0} \mathbb{R}$ such that $\|e_{\tau_0,1}\| = \|e_{\tau_0,2}\| = -1$. We also define

$$h_{0,\daleth} : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}(1) \to \left(\mathrm{Res}_{F(\daleth)/F}\mathrm{GL}(1)\right) \otimes \mathbb{R} \cong \prod_{\tau \in \mathrm{Hom}(F,\mathbb{R})} \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}(1)$$

to be the homomorphism that is trivial on the τ_0-factor and is the identity map on the other factors. We then define

$$h_0^\sharp = (h_{0,\natural}, h_{0,\daleth}) : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}(1) \to \mathbf{G}_\daleth \otimes \mathbb{R},$$

which factors through $\mathbf{G}^\sharp \otimes \mathbb{R} \subset \mathbf{G}_\mathrm{i} \otimes \mathbb{R}$.

- In Case U, for each $\tau \in \mathrm{Hom}(F, \mathbb{C})$, we fix a $\mathbb{C}$-basis $\mathbf{v} = (v_1, \ldots, v_{n(G)})^\top$ of $\mathbf{V} \otimes_{F_1,\tau} \mathbb{C}$ such that

$$J_\tau := (\langle v_i, v_j \rangle_{i,j}) = \begin{cases} \mathrm{diag}(\underbrace{1, \ldots, 1}_{(n(G)-1)\text{-many}}, -1) & \text{if } \tau = \tau_0 \\ \mathbf{1} & \text{if } \tau \neq \tau_0 \end{cases}.$$

Let

$$h_{0,\natural} : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}(1) \to \mathrm{GU}^\mathbb{Q}(\mathbf{V}) \otimes \mathbb{R} \cong \prod_{\tau \in \mathrm{Hom}(F,\mathbb{R})} \mathrm{GU}(\mathbf{V} \otimes_{F_1,\tau} \mathbb{C})$$

be the homomorphism such that on each τ-factor it is given by

$$a + b\mathrm{i} \mapsto a + b\mathrm{i}J_\tau.$$

Let

$$h_{0,\daleth} : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}(1) \to \mathbf{Z}^\mathbb{Q} \otimes \mathbb{R} \subset \prod_{\tau \in \mathrm{Hom}(F,\mathbb{R})} \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}(1)$$

be the diagonal embedding. We then define

$$h_0^\sharp = (h_{0,\natural}, h_{0,\daleth}) : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}(1) \to \mathbf{G}^\sharp \otimes \mathbb{R}.$$

It is routine to check that $(\mathbf{G}^\sharp, \mathbf{X}^\sharp)$ is a Shimura datum.

We define $h_0 : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}(1) \to \mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}$ which is the composition of $h_0^\sharp$ with the central extension $\mathbf{G}^\sharp \to \mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}$, and we write $\mathbf{X} := \{h_0\}$, then $(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})$ is also a Shimura datum. We then have the Hodge cocharacter

$$\mu : \mathrm{GL}(1)_\mathbb{C} \xrightarrow{z \mapsto (z,1)} \left(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}(1)\right)_\mathbb{C} \xrightarrow{(h_0)_\mathbb{C}} \left(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}\right)_\mathbb{C}$$

associated to h_0, with reflex field F_1 which is embedded in $\mathbb{C}$ via τ_0, except in Case U when $n(\mathbf{G}) = 2$, where the reflex field of F which is embedded in $\mathbb{C}$ via τ_0. Note that for any rational prime p with a fixed isomorphism $\iota_p : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_p}$, we get a cocharacter μ of $\left(\mathrm{Res}_{K/\mathbb{Q}_p}\mathbf{G}_K\right)_{\overline{\mathbb{Q}_p}} \cong \prod_{v \in \mathrm{Hom}(K,\overline{\mathbb{Q}_p})} \mathbf{G}_{\overline{K}}$, which is conjugate to the inverse of the cocharacter μ_1 of $\mathbf{G}_{\overline{K}}$ defined in (2.5) on one factor and trivial on the other factors.

We define $\mathscr{K}_\infty$ (resp. $\mathscr{K}_\infty^\sharp$) to be the centralizer of h_0 in $\mathbf{G}(F \otimes \mathbb{R})$ (resp. $h_0^\sharp$ in $\mathbf{G}^\sharp(F \otimes \mathbb{R})$), and define $\mathbf{X} := \mathbf{G}(F \otimes \mathbb{R})/\mathscr{K}_\infty$ (resp. $\mathbf{X}^\sharp := \mathbf{G}^\sharp(F \otimes \mathbb{R})/\mathscr{K}_\infty^\sharp$).

By work of Deligne, we get a projective system of Shimura varieties $\{\mathrm{Sh}_\mathscr{K}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})\}$ defined over F_1 indexed by neat open compact subgroups $\mathscr{K} \leq \mathbf{G}(\mathbf{A}_{F,f})$ (as defined in [Pin90, §0.6]), such that

$$\mathrm{Sh}_\mathscr{K}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X}) \otimes_{F_1, \tau_0} \mathbb{C} = \mathbf{G}(F)\backslash(\mathbf{X} \times \mathbf{G}(\mathbf{A}_{F,f})/\mathscr{K})$$

with $\dim \mathrm{Sh}_\mathscr{K}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}) = \dim_\mathbb{C}(\mathbf{X})$. Similarly, we get a projective system of Shimura varieties $\{\mathrm{Sh}_{\mathscr{K}^\sharp}(\mathbf{G}^\sharp, \mathbf{X}^\sharp)\}_\mathscr{K}$ defined over the reflex field E of dimension $\dim_\mathbb{C}(\mathbf{X})$ indexed by neat open compact subgroups $\mathscr{K}^\sharp \leq \mathbf{G}^\sharp(\mathbf{A}_f)$.[8] Let $\mathscr{K} \subset \mathbf{G}(\mathbf{A}_{F,f})$ be the image of $\mathscr{K}^\sharp$, then there exists a map of Shimura varieties

$$(4.3) \qquad \mathrm{Sh}_{\mathscr{K}^\sharp}(\mathbf{G}^\sharp, \mathbf{X}^\sharp) \to \mathrm{Sh}_\mathscr{K}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})$$

that is functorial in $\mathscr{K}^\sharp$.

Finally, we check that $(\mathbf{G}^\sharp, \mathbf{X})$ is a Shimura datum of Hodge type and thus $(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})$ is a Shimura datum of Abelian type: In Case U, this follows from [RSZ20, §3.2]: In fact $(\mathbf{G}^\sharp, \mathbf{X}^\sharp)$ is of PEL type. In Case O, let $\mathbf{H} = \mathrm{Cl}(\mathbf{V})$, viewed as an F-representation of $\mathrm{GSpin}(\mathbf{V})$ via left multiplication. For $\beta \in \mathbf{H}^\times$ with $\beta^* = -\beta$, we obtain an F-valued symplectic form ψ_β on $\mathbf{H}$ such that $\psi_\beta(v, w) = \mathrm{trrd}_{\mathbf{H}/F}(v\beta w^*)$, where $\mathrm{trrd}_{\mathbf{H}/F}$ is the reduced trace as defined in [MP16, 1.6]. We then define the F-valued pairing

$$\psi_{\beta, \urcorner} : \mathbf{H} \otimes F(\urcorner) \times \mathbf{H} \otimes F(\urcorner) \to F : (v, w) \mapsto \mathrm{tr}_{F(\urcorner)/F}\left(\urcorner \cdot \mathrm{trrd}_{\mathbf{H} \otimes F(\urcorner)/F(\urcorner)}(v\beta w^*)\right),$$

and the $\mathbb{Q}$-valued pairing

$$\psi_{\beta, \urcorner}^\mathbb{Q} = \mathrm{tr}_{F/\mathbb{Q}} \circ \psi_{\beta, \urcorner} : \mathbf{H} \otimes F(\urcorner) \times \mathbf{H} \otimes F(\urcorner) \to \mathbb{Q}.$$

For any $v, w \in \mathbf{H} \subset \mathbf{H} \otimes F(\urcorner)$,

$$\psi_{\beta, \urcorner}(v, w) = \mathrm{tr}_{F(\urcorner)/F}\left(\urcorner \cdot \psi_\beta(v, w)\right).$$

Hence there are natural inclusions

$$\mathrm{GSpin}(\mathbf{V}) \hookrightarrow \mathrm{GSp}_F(\mathbf{H}, \psi_\beta) \hookrightarrow \mathrm{GSp}_F(\mathbf{H} \otimes F(\mathrm{i}), \psi_{\beta, \mathrm{i}}).$$

These extend to a $\mathbb{Q}$-embedding

$$\mathbf{G}_\urcorner = \left((\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSpin}(\mathbf{V})) \times \mathrm{Res}_{F(\urcorner)/\mathbb{Q}}\mathrm{GL}(1)\right)/\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}(1) \xrightarrow{a} \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSp}_F(\mathbf{H} \otimes F(\urcorner), \psi_{\beta, \urcorner})$$
$$(g, t) \mapsto gt.$$

Let

$$\mathrm{Res}_{F/\mathbb{Q}}(\nu_{\mathrm{GSp}}) : \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSp}(\mathbf{H} \otimes F(\urcorner), \psi_{\beta, \urcorner}) \to \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}(1)$$

be the similitude map for GSp, then $\mathrm{Res}_{F/\mathbb{Q}}(\nu_{\mathrm{GSp}}) \circ a$ maps $\mathbf{G}^\sharp$ into $\mathrm{GL}(1) \subset \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}(1)$ by definition of $\mathbf{G}^\sharp$. In other words, $\mathbf{G}^\sharp$ preserves the pairing $\psi_{\beta, \urcorner}$, thus also $\psi_{\beta, \urcorner}^\mathbb{Q}$, up a factor of $\mathrm{GL}(1)$. So there exists a natural embedding

$$\mathbf{G}^\sharp \hookrightarrow \mathrm{GSp}(\mathbf{H} \otimes F(\urcorner), \psi_{\beta, \urcorner}^\mathbb{Q})$$

of reductive groups over $\mathbb{Q}$, where we emphasize the right-hand side is the group of elements with similitude factors in $\mathrm{GL}(1)_\mathbb{Q}$. We can choose appropriate β such that this embedding induces an embedding of Shimura data

$$(\mathbf{G}^\sharp, \mathbf{X}^\sharp) \hookrightarrow (\mathrm{GSp}(\mathbf{H} \otimes F(\urcorner), \psi_{\beta, \urcorner}^\mathbb{Q}), \mathcal{X}),$$

where $\mathcal{X}$ is the union of Siegel upper half-spaces attached to the symplectic space $\left(\mathbf{H} \otimes F(\urcorner), \psi_{\beta, \urcorner}^\mathbb{Q}\right)$; see [MP16, 3.5].

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[8]Note that the reflex field E may be bigger than F_1.

4.5. **Langlands–Kottwitz method.** In this subsubsection, we apply the Langlands–Kottwitz method to relate the action of Frobenius elements at primes of good reduction to the Hecke action on the compact support cohomology of orthogonal or unitary Shimura varieties. We adopt the notation from §4.1, and assume:

Notation 4.5.1.
- F has a finite place $\mathfrak{q}$ inert in F_1, with underlying rational prime odd.
- $\bigstar$ is a control tuple (see Definition 4.3.1) such that $\mathfrak{q} \in \Sigma^{\mathrm{St}}$ and Σ is of the form $\Sigma = \Sigma_F(S_{\mathrm{bad}})$, where S_{bad} is a finite set of rational primes containing 2 and all rational primes ramified in F.
- $(\mathbf{G}, \varrho, z)$ is a $\bigstar$-good pure inner form of $\mathbf{G}^*$, and π is a $\bigstar$-good automorphic representation of $\mathbf{G}$.
- $\mathcal{K} \leq \mathbf{G}(\mathbf{A}_{F,f})$ is a $\bigstar$-split compact open subgroup with $\pi^{\mathcal{K}} \neq 0$.
- ℓ be a rational prime together with an isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$.
- $(\mathfrak{X}, \chi)$ is a central character datum for $\mathbf{G}^*$ (see Definition A.4.2) with

$$\mathfrak{X} = (\mathcal{K} \cap Z_{\mathbf{G}^*}(\mathbf{A}_{F,f})) \times Z_{\mathbf{G}^*}(F \otimes \mathbb{R}),$$

and χ is the inverse of the central character of ξ (extended from $Z_{\mathbf{G}^*}(F \otimes \mathbb{R})$ to $\mathfrak{X}$ trivially on $Z_{\mathbf{G}^*}(\mathbf{A}_{F,f}) \cap \mathcal{K}$).

Definition 4.5.2. Suppose $(\mathbb{G}, \mathbb{X})$ is any Shimura datum with reflex field $E \subset \mathbb{C}$ with associated projective system of Shimura varieties $\{\mathrm{Sh}_{\mathcal{K}}(\mathbb{G}, \mathbb{X})\}_{\mathcal{K}}$ defined over E, indexed by the set of neat compact open subgroups $\mathcal{K} \leq \mathbb{G}(\mathbf{A}_f)$. Let Z_{a} be the maximal anisotropic $\mathbb{Q}$-subtorus of $Z(\mathbb{G})$, and let Z_{ac} be the smallest $\mathbb{Q}$-subgroup of Z_{a} whose base change to $\mathbb{R}$ contains the maximal $\mathbb{R}$-split sub-torus of Z_{a}; see [KSZ21, Definition 1.5.4]. For each irreducible algebraic representation ξ of $\mathbb{G}_{\mathbb{C}}$ that is trivial on Z_{ac}, there exists a compatible system of lisse $\overline{\mathbb{Q}_\ell}$-local system $\mathscr{L}_{\iota_\ell\xi}$ on this projective system of Shimura varieties associated to ξ; see [KSZ21, 1.5.8]. For each $i \in \mathbb{N}$, we define

$$\mathrm{H}^i_{\mathrm{\acute{e}t,c}}(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi}) := \varinjlim_{\mathcal{K} \to \mathbf{1}} \mathrm{H}^i_{\mathrm{\acute{e}t,c}}\left(\mathrm{Sh}_{\mathcal{K}}(\mathbb{G}, \mathbb{X})_{\overline{E}}, \mathscr{L}_{\iota_\ell\xi}\right),$$

which is a $\mathbb{G}(\mathbf{A}_f) \times \mathrm{Gal}_E$-module with admissible $\mathbb{G}(\mathbf{A}_f)$-action and continuous Gal_E-action. We write $\mathrm{H}^i(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi})^{\mathrm{ss}}$ for its semi-simplification as a $\mathbb{G}(\mathbf{A}_f) \times \mathrm{Gal}_E$-module.

For each automorphic representation Π of $\mathbb{G}(\mathbf{A})$, we define

(4.4) $$\mathrm{H}^i_{\mathrm{\acute{e}t,c}}(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi})^{\mathrm{ss}}[\Pi^\infty] := \mathrm{Hom}_{\mathbb{G}(\mathbf{A}_f)}\left(\iota_\ell\Pi^\infty, \mathrm{H}^i_c(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi})^{\mathrm{ss}}\right),$$

which is a finite dimensional representation of Gal_F unramified at all but finitely many places of F.

Similarly, if $\mathbb{G}'$ is any reductive group over $\mathbb{Q}$ such that $\mathbb{G}'(\mathbb{R})$ is compact, then for each irreducible algebraic representation (ξ, V_ξ) of $\mathbb{G}_{\mathbb{C}}$, we define the injective system of *algebraic automorphic forms* valued in $\iota_\ell V_\xi$ as

$$\{\mathcal{A}(\mathbb{G}'(\mathbb{Q})\backslash\mathbb{G}'(\mathbf{A}_f)/\mathcal{K}, \mathscr{L}_{\iota_\ell\xi})\}_{\mathcal{K}}$$

indexed by compact open subgroups $\mathcal{K} \leq \mathbb{G}'(\mathbf{A}_f)$, where $\mathcal{A}(\mathbb{G}'(\mathbb{Q})\backslash\mathbb{G}'(\mathbf{A}_f)/\mathcal{K}, \mathscr{L}_{\iota_\ell\xi})$ consists of maps $\phi : \mathbb{G}'(\mathbf{A}_f) \to \iota V_\xi$ such that $\phi(gk) = \phi(g)$ and $\phi(\gamma g) = \gamma.\phi(g)$ for any $g \in \mathbb{G}'(\mathbf{A}_f), \gamma \in \mathbb{G}'(\mathbb{Q})$ and $k \in \mathcal{K}$. For each compact open subgroup $\mathcal{K} \leq \mathbb{G}(\mathbf{A}_f^p)$, we write

$$\mathcal{A}(\mathbb{G}'(\mathbb{Q})\backslash\mathbb{G}'(\mathbf{A}_f)/\mathcal{K}^p, \mathscr{L}_{\iota_\ell\xi}) := \varinjlim_{\mathcal{K}_p} \mathcal{A}(\mathbb{G}'(\mathbb{Q})\backslash\mathbb{G}'(\mathbf{A}_f)/\mathcal{K}_p\mathcal{K}^p, \mathscr{L}_{\iota_\ell\xi})$$

where $\mathcal{K}_p$ runs through compact open subgroups of $\mathbb{G}'(\mathbb{Q}_p)$.

Regarding the Galois action on the cohomology of orthogonal or unitary Shimura varieties, the following is the main result of [KSZ21] (cf. [KS23, Theorem 7.3]):

Theorem 4.5.3. *Suppose $\mathcal{K}_p \leq \mathbf{G}(F \otimes \mathbb{Q}_p)$ is hyperspecial for some $p \geq 3$, and $\iota_p : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_p}$ is an isomorphism such that $\iota_p \circ \tau_0 : F_1 \to \overline{\mathbb{Q}_p}$ induces a finite place $\mathfrak{p} \in \Sigma_{F_1}(\{p\})$. Define a test function $f^\infty = f^{\infty,p}f_p \in \mathcal{H}\left(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}(\mathbf{A}_f), \mathcal{K}\right)$ with $f_p = \mathbf{1}_{\mathcal{K}_p}$, then there exists $j_0 \in \mathbb{Z}_+$ such that for all positive integers $j \geq j_0$,*

$$\sum_{i=0}^{2\dim_{\mathbb{C}}(\mathbf{X})} (-1)^i \iota_\ell^{-1}\mathrm{Tr}\left(\iota_\ell f^\infty \sigma_{\mathfrak{p}}^j | \mathrm{H}^i_{\mathrm{\acute{e}t,c}}(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi})\right) = \sum_{\mathfrak{e}\in\mathcal{E}_{\mathrm{ell}}(\mathbf{G})} \iota(\mathfrak{e})\mathrm{ST}_{\mathrm{ell},\chi}^{\mathbf{G}^{\mathfrak{e}}}(h_{\xi,j}^{\mathbf{G}^{\mathfrak{e}}}).$$

Here $\mathrm{ST}_{\mathrm{ell},\chi}^{\mathbf{G}^{\mathfrak{e}}}$ is the elliptic stable distribution associated to $\mathfrak{e}$ (see Definition A.4.9), $\iota(\mathfrak{e}) \in \mathbb{Q}$ is a constant depending on $\mathfrak{e}$, and $h_{\xi,j}^{\mathbf{G}^{\mathfrak{e}}} = h^{\mathbf{G}^{\mathfrak{e}},p\infty} h_{p,j}^{\mathbf{G}^{\mathfrak{e}}} h_{\infty,\xi}^{\mathbf{G}^{\mathfrak{e}}} \in \mathcal{H}(\mathbf{G}^{\mathfrak{e}}(\mathbf{A}_F), \chi^{-1})$ are defined in [Kot90, KSZ21]. In particular, $\iota(\mathfrak{e}) = 1$ if $\mathbf{G}^{\mathfrak{e}} = \mathbf{G}^*$, and

- $h^{\mathbf{G}^*,p\infty}$ is an endoscopic transfer of $f^{p\infty}$, (Note that such a transfer exists in the fixed-central character setting, by first lifting f along the averaging map $\mathcal{H}(\mathbb{G}(\mathbf{A}_F)) \to \mathcal{H}(\mathbb{G}(\mathbf{A}_F), \chi^{-1})$, and then take the transfer to $\mathcal{H}(\mathbb{G}^*(\mathbf{A}_F))$, and then take the image along the averaging map $\mathcal{H}(\mathbb{G}^*(\mathbf{A}_F)) \to \mathcal{H}(\mathbb{G}^*(\mathbf{A}_F), \chi^{-1})$),

- $h_{p,j}^{\mathbf{G}^*}$ is the base change transfer of

$$\phi_j := \mathbf{1}_{\mathcal{G}_p\left(\mathbb{Z}_{\|\mathfrak{p}\|^j}\right)\mu(p^{-1})\mathcal{G}_p\left(\mathbb{Z}_{\|\mathfrak{p}\|^j}\right)} \in \mathcal{H}\left(\mathbf{G}(F \otimes \mathbb{Z}_{\|\mathfrak{p}\|^j}), \mathcal{G}_p\left(\mathbb{Z}_{\|\mathfrak{p}\|^j}\right)\right)$$

(where we write $\mathcal{G}_p$ for $\prod_{v \in \Sigma_F(\{p\})} \mathrm{Res}_{\mathcal{O}_{F_v}/\mathbb{Z}_p} \mathcal{G}_v$ and write $\mathbb{Z}_{\|\mathfrak{p}\|^j}$ for the integer ring of the unramified representation $\mathbb{Q}_{\|\mathfrak{p}\|^j}$ of $\mathbb{Q}_p$ of degree $j \cdot \log_p(\|\mathfrak{p}\|)$) down to $\mathcal{H}(\mathbf{G}(F), \mathcal{G}(\mathbb{Z}_p))$; and

- $h_\infty := \#\Pi_\xi(\mathbf{G}^*(F \otimes \mathbb{R}))^{-1} \sum_{\tau_\infty \in \Pi_\xi(\mathbf{G}^*(F \otimes \mathbb{R}))} f_{\tau_\infty}$, that is, the average of the pseudo-coefficients for the discrete series L-packet of $\mathbf{G}^*(F \otimes \mathbb{R})$ associated to ξ.

Definition 4.5.4. Let $A(\pi)$ be the set of isomorphism classes of $\bigstar$-good automorphic representations Π of $\mathbf{G}(\mathbf{A}_F)$ such that $\Pi^\infty \cong \pi^\infty$.

We define the virtual Galois representation

$$\rho_{\mathrm{Sh}}^\pi := (-1)^{\dim_\mathbb{C}(\mathbf{X})} \sum_{i=0}^{2\dim_\mathbb{C}(\mathbf{X})} (-1)^i \mathrm{H}_c^i(\mathrm{Sh}, \mathscr{L}_{\iota_\ell \xi})^{\mathrm{ss}}[\pi^\infty] \in \mathrm{K}_0\left(\overline{\mathbb{Q}_\ell}[\mathrm{Gal}_{F_1}]\right),$$

where $\mathrm{K}_0\left(\overline{\mathbb{Q}_\ell}[\mathrm{Gal}_{F_1}]\right)$ is the Grothendieck group of finite dimensional continuous representations of Gal_{F_1} with $\overline{\mathbb{Q}_\ell}$-coefficients unramified at all but finitely many places.

We will need some more cohomology spaces to deal with non-compact Shimura varieties: Let

$$\mathrm{H}_{(2)}^i(\mathrm{Sh}, \mathscr{L}_\xi) := \varprojlim_{\mathscr{K} \to 1} \mathrm{H}_{(2)}^i(\mathrm{Sh}_{\mathscr{K}}(\mathbf{G}, \mathbf{X}), \mathscr{L}_\xi)$$

be the L^2-cohomology of $\mathrm{Sh}(\mathbf{G}, \mathbf{X}) \times_{F_1, \tau_0} \mathbb{C}$ as defined in [Fal83, §6], and let

$$\mathrm{IH}^*(\mathrm{Sh}, \mathscr{L}_{\iota_\ell \xi}) := \varprojlim_{\mathscr{K} \to 1} \mathrm{IH}^*(\mathrm{Sh}_{\mathscr{K}}(\mathbf{G}, \mathbf{X}), \mathscr{L}_{\iota_\ell \xi})$$

be the ℓ-adic intersection cohomology of $\mathrm{Sh}(\mathbf{G}, \mathbf{X})$. These two cohomologies are equipped with admissible $\mathbf{G}(\mathbf{A}_{F,f})$-actions defined by Hecke correspondences. There are natural $\mathbf{G}(\mathbf{A}_{F,f})$-equivariant maps

$$(4.5) \qquad \mathrm{H}_{\mathrm{\acute{e}t},c}^i(\mathrm{Sh}, \mathscr{L}_{\iota_\ell \xi}) \to \iota_\ell \mathrm{H}_{(2)}^i(\mathrm{Sh}, \mathscr{L}_\xi) \to \mathrm{H}_{\mathrm{\acute{e}t}}^i(\mathrm{Sh}, \mathscr{L}_{\iota_\ell \xi}),$$

$$\mathrm{H}_{\mathrm{\acute{e}t},c}^i(\mathrm{Sh}, \mathscr{L}_{\iota_\ell \xi}) \to \mathrm{IH}^*(\mathrm{Sh}, \mathscr{L}_{\iota_\ell \xi}),$$

and it follows from Zucker's conjecture [Loo88, LR91, SS90] that there is a $\mathbf{G}(\mathbf{A}_{F,f})$-equivariant commutative diagram

$$(4.6) \qquad \begin{array}{ccc} \mathrm{H}_{\mathrm{\acute{e}t},c}^i(\mathrm{Sh}, \mathscr{L}_{\iota_\ell \xi}) & \longrightarrow & \mathrm{IH}^*(\mathrm{Sh}, \mathscr{L}_{\iota_\ell \xi}) \\ & \searrow & \downarrow{\scriptstyle \cong} \\ & & \iota_\ell \mathrm{H}_{(2)}^i(\mathrm{Sh}, \mathscr{L}_\xi) \end{array} \quad .$$

Lemma 4.5.5. *The maps in* (4.5) *induce isomorphisms*

$$\mathrm{H}_{\mathrm{\acute{e}t},c}^i(\mathrm{Sh}, \mathscr{L}_{\iota_\ell \xi})[\pi^\infty] \cong \iota_\ell \mathrm{H}_{(2)}^i(\mathrm{Sh}, \mathscr{L}_\xi)[\pi^\infty] \cong \mathrm{H}_{\mathrm{\acute{e}t}}^i(\mathrm{Sh}, \mathscr{L}_{\iota_\ell \xi})[\pi^\infty].$$

Moreover, $\dim \mathrm{H}_{\mathrm{\acute{e}t},c}^i(\mathrm{Sh}, \mathscr{L}_{\iota_\ell \xi})^{\mathrm{ss}}[\pi^\infty] = \dim \mathrm{H}_{\mathrm{\acute{e}t},c}^i(\mathrm{Sh}, \mathscr{L}_{\iota_\ell \xi})[\pi^\infty].$

Proof. The isomorphisms follow from Franke's spectral sequence [Fra98, Theorem 19] and the last assertion follows Borel–Casselman's decomposition of $\mathrm{H}_{(2)}^i(\mathrm{Sh}, \mathscr{L}_\xi)$ as direct sums of certain multiplies of π_f for each $\pi \in L_{\mathrm{disc}}^2(\mathbf{G}(\mathbf{A}_F))^{\mathrm{sm}}$ (thus $\mathrm{H}_{(2)}^i(\mathrm{Sh}, \mathscr{L}_\xi)$ is semisimple as a $\mathbf{G}(\mathbf{A}_{F,f})$-module); see [BC83]. The proofs are the same as that of [KS23, Lemma 8.1(1)], thus omitted here. $\square$

Theorem 4.5.6. *For all but finitely many finite places $\mathfrak{p} \in \Sigma_{F_1}^{\mathrm{fin}} \smallsetminus \Sigma_{F_1}(\Sigma \cup \{\ell\})$ and all sufficiently large positive integer j (depending on $\mathfrak{p}$),*

$$\mathrm{tr}\left(\sigma_{\mathfrak{p}}^j | \rho_{\mathrm{Sh}}^\pi\right) = m(\pi) \cdot \|\mathfrak{p}\|^{\frac{\dim_{\mathbb{C}}(\mathbf{X})}{2} \cdot j} \cdot \mathrm{tr}\left(\iota_\ell \tilde{\phi}_{\pi_{\mathfrak{p}_\flat}}^{\mathrm{GL}}(\sigma_{\mathfrak{p}}^j)\right),$$

where $\tilde{\phi}_{\pi_{\mathfrak{p}_\flat}}$ is the classical L-parameter of $\pi_{\mathfrak{p}_\flat}$. Moreover, the only non-zero term in the definition of ρ_{Sh}^π (see Definition 4.5.4) appears in the middle degree $\dim_{\mathbb{C}}(\mathbf{X})$. In particular, ρ_{Sh}^π is a genuine representation of Gal_{F_1}.

Proof. We imitate the argument of [Kot92] and [KS23, Proposition 8.2]. Consider the test function $f = f_\infty \otimes f_{\Sigma^{\mathrm{St}}} \otimes f^{\Sigma^{\mathrm{St}} \cup \Sigma_F^\infty}$ on $\mathbf{G}(\mathbf{A}_F)$ such that

- $f_\infty := (\#\Pi_\xi(\mathbf{G}(F \otimes \mathbb{R})))^{-1} \sum_{\tau_\infty \in \Pi_\xi(\mathbf{G}(F \otimes \mathbb{R}))} f_{\tau_\infty}$, i.e., the average of the pseudo-coefficients for the discrete series L-packet of $\mathbf{G}(F \otimes \mathbb{R})$ associated to ξ.

- $f^{\Sigma^{\mathrm{St}} \cup \Sigma_F^\infty} = f_{\Sigma \smallsetminus \Sigma^{\mathrm{St}}} \otimes f^{\Sigma \cup \Sigma_F^\infty}$ such that $f^{\Sigma \cup \Sigma_F^\infty} = \mathbf{1}_{\mathscr{K}^\Sigma}$, and

$$\mathrm{Tr}(f^{\Sigma^{\mathrm{St}} \cup \Sigma_F^\infty} | \Pi^{\Sigma^{\mathrm{St}} \cup \Sigma_F^\infty}) = \begin{cases} 1 & \text{if } \Pi^{\Sigma^{\mathrm{St}} \cup \Sigma_F^\infty} \cong \pi^{\Sigma^{\mathrm{St}} \cup \Sigma_F^\infty} \\ 0 & \text{otherwise} \end{cases}$$

  for all automorphic representations Π of $\mathbf{G}(\mathbf{A}_F)$ satisfying $(\Pi_f)^{\mathscr{K}} \neq 0$ and $\mathrm{tr}(f_\infty | \Pi_\infty) \neq 0$. This is possible because there are only finitely many such Π with $\Pi^{\Sigma^{\mathrm{St}} \cup \Sigma_F^\infty} \cong \pi^{\Sigma^{\mathrm{St}} \cup \Sigma_F^\infty}$, by Corollary 4.2.2.

- $f_{\Sigma^{\mathrm{St}}} = \otimes_{v \in \Sigma^{\mathrm{St}}} f_{\mathrm{Lef},v}^{\mathbf{G}}$, where each $f_{\mathrm{Lef},v}^{\mathbf{G}}$ is a Lefschetz function (see Definition A.4.7).

Consider $\mathfrak{p} \in \Sigma_{F_1}^{\mathrm{fin}} \smallsetminus \Sigma_{F_1}(\Sigma)$ with $p = \mathrm{char}\,\kappa(\mathfrak{p})$. We choose an isomorphism $\iota_p : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_p}$ such that $\iota_p \circ \tau_0 : F_1 \to \overline{\mathbb{Q}_p}$ induces the place $\mathfrak{p}$, and we denote by $\mathfrak{p}_\flat$ the underlying prime $\mathfrak{p}$ in F. Then the stabilized Langlands–Kottwitz formula Theorem 4.5.3 simplifies to

$$\sum_{i=0}^{2\dim_{\mathbb{C}}(\mathbf{X})} (-1)^i \iota_\ell^{-1} \mathrm{Tr}\left(\iota_\ell f^\infty \sigma_{\mathfrak{p}}^j | \mathrm{H}_{\mathrm{\acute{e}t},c}^i(\mathrm{Sh}, \mathscr{L}_{\iota_\ell \xi})\right) = \mathrm{ST}_{\mathrm{ell},\chi}^{\mathbf{G}^*}(h_{\xi,j}^{\mathbf{G}^*}),$$

because the stable orbital integrals of $h_v^{\mathbf{G}^\mathfrak{e}}$ vanish for $\mathbf{G}^\mathfrak{e} \neq \mathbf{G}^*$ as they equal κ-orbital integrals of $f_{\mathrm{Lef},v}^{\mathbf{G}}$ with $\kappa \neq \mathbf{1}$ up to a non-zero constant, and $f_{\mathrm{Lef},v}^{\mathbf{G}}$ is stabilizing (see Definition A.4.7); see [Lab99, Theorem 4.3.4]. Note that the left-hand side equals

$$(-1)^{\dim_{\mathbb{C}}(\mathbf{X}) + \sum_{v \in \Sigma^{\mathrm{St}}} q(\mathbf{G}_v)} \mathrm{tr}\left(\sigma_v^j | \rho_{\mathrm{Sh}}^\pi\right)$$

by definition of $f^\infty = f_{\Sigma^{\mathrm{St}}} \otimes f^{\Sigma^{\mathrm{St}} \cup \Sigma_F^\infty}$, where $q(\mathbf{G}_v)$ is the F_v-rank of $\mathbf{G}_v$.

Then it follows from the simple stable trace formula Theorem A.4.10 and Definition A.4.8 that

$$(4.7) \qquad \mathrm{ST}_{\mathrm{ell},\chi}^{\mathbf{G}^*}(h_{\xi,j}^{\mathbf{G}^*}) = \mathrm{T}_{\mathrm{cusp},\chi}^{\mathbf{G}}(f^{p\infty} f'_{p,j} f_\infty) = \sum_{\Pi \in \mathrm{Irr}_\chi^{\mathrm{cusp}}(\mathbf{G})} m(\Pi) \mathrm{tr}(f^{p\infty} | \Pi^{p\infty}) \mathrm{tr}(f'_{p,j} | \Pi_p) \mathrm{tr}(f_\infty | \Pi_\infty)$$

where $f'_{p,j} = h_{p,j}^{\mathbf{G}^*}$ via the identification $\varrho_p : \mathbf{G}^*(F \otimes \mathbb{Q}_p) \xrightarrow{\sim} \mathbf{G}(F \otimes \mathbb{Q}_p)$. The term on the right-hand side vanishes unless Π_p is unramified and both traces $\mathrm{tr}(f^{p\infty} | \Pi^{p\infty})$ and $\mathrm{tr}(f_\infty | \Pi_\infty)$ are nonzero. Hence

$$\mathrm{tr}(f^\infty | \Pi^\infty) \neq 0,$$

which yields $\Pi^{\Sigma^{\mathrm{St}} \cup \Sigma_F^\infty} \cong \pi^{\Sigma^{\mathrm{St}} \cup \Sigma^\infty}$; by strong multiplicity one (Corollary 4.2.2), it follows that $\Pi^\infty \cong \pi^\infty$. By Definition A.4.8, the right-hand side of Equation (4.7) equals

$$\frac{1}{\#\Pi_\xi(\mathbf{G}(F \otimes \mathbb{R}))} \sum_{\Pi \in A(\pi)} m(\Pi)(-1)^{\sum_{v \in \Sigma^{\mathrm{St}}} q(\mathbf{G}_v)} \mathrm{ep}(\Pi_\infty \otimes \xi) \mathrm{tr}\left(h_{p,j}^{\mathbf{G}^*} | \prod_{v \in \Sigma_F(\{p\})} \pi_v\right)$$

$$= (-1)^{\dim_{\mathbb{C}}(\mathbf{X}) + \sum_{v \in \Sigma^{\mathrm{St}}} q(\mathbf{G}_v)} m(\pi) \mathrm{tr}\left(h_{p,j}^{\mathbf{G}^*} | \prod_{v \in \Sigma_F(\{p\})} \pi_v\right).$$

Here we use that $m(\Pi) = m(\pi)$ for each $\Pi \in A(\pi)$ by Arthur's multiplicity formula. Thus

$$\mathrm{tr}\left(\sigma_v^j | \rho_{\mathrm{Sh}}^\pi\right) = m(\pi) \mathrm{tr}\left(h_{p,j}^{\mathbf{G}^*} | \prod_{v \in \Sigma_F(\{p\})} \pi_v\right).$$

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To analyze the right-hand side, consider the conjugacy class of Hodge cocharacters $\{\mu\} : \mathrm{GL}(1)_{\mathbb{C}} \to \left(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}\right)_{\mathbb{C}}$ defined in §4.4. The highest weight tilting module of $\left(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}\right)^{\wedge} \cong \prod_{\tau \in \Sigma_F^{\infty}} \widehat{\mathbf{G}_{\tau}}$ associated to $\{\mu\}$ is the standard representation on the τ_0 component and trivial on other components. Furthermore, the Satake parameter of π_p belongs to the σ_v-coset of $\left((\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}) \otimes \mathbb{Q}_p\right)^{\wedge}$, identified with $\left(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}\right)^{\wedge}$ via $\widehat{\iota_p}$, in the L-group. Thus the τ_0-component of the Satake parameter of the representation $\prod_{v \in \Sigma_F(\{p\})} \pi_v$ of $\left(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}\right) \otimes \mathbb{Q}_p$ is identified with the Satake parameter of $\pi_{\mathfrak{p}_\flat}$, since $F \xrightarrow{\tau_0} \mathbb{C} \xrightarrow{\iota_p} \overline{\mathbb{Q}_p}$ induces the place $\mathfrak{p}_\flat$. Then it follows from [Kot84, (2.2.1)] that

$$\mathrm{tr}\left(h_{p,j}^{\mathbf{G}^*}|\prod_{v \in \Sigma_F(\{p\})} \pi_v\right) = \|\mathfrak{p}\|^{\frac{\dim_{\mathbb{C}}(\mathbf{X})}{2}j}\,\mathrm{tr}\left(\iota_\ell \tilde{\phi}_{\pi_{\mathfrak{p}_\flat}}^{\mathrm{GL},\vee}(\sigma_{\mathfrak{p}}^j)\right),$$

and the first assertion follows.

To show that the only non-zero term in the definition of ρ_{Sh}^π is concentrated on the middle degree $\dim_{\mathbb{C}}(\mathbf{X})$, note that there exists a Gal_{F_1}-equivariant isomorphism

$$\mathrm{H}_{\text{ét,c}}^i(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi})[\pi^\infty] \cong \mathrm{IH}^i(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi})[\pi^\infty]$$

for each $i \in \mathbb{N}$, by Lemma 4.5.5 and (4.6). The intersection complex defined by $\iota_\ell\xi$ is pure of weight 0, so for each $\mathfrak{p}$ as above, the action of $\sigma_{\mathfrak{p}}$ on $\mathrm{IH}^i(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi})_{\mathrm{ss}}[\pi^\infty]$ is pure of weight i by a purity result of Pink [Pin92, Proposition 5.6.2] and the purity result for intersection cohomologies. (Note that the weight cocharacter of the Shimura datum which appears in [Pin92, §5.4] is trivial because $Z(\mathbf{G})$ is anisotropic). In particular, there are no cancellations between cohomologies in different degree. It follows from Theorem 4.3.6 that $\Pi_{\mathfrak{p}}$ is tempered for each $\Pi \in A(\pi)$, so the first part of the theorem and Theorem 2.3.1 imply that all eigenvalues of $\iota_\ell^{-1}\rho_2^{\mathrm{Sh}}(\sigma_{\mathfrak{p}})_{\mathrm{ss}}$ have absolute values $\|\mathfrak{p}\|^{\frac{\dim_{\mathbb{C}}(\mathbf{X})}{2}}$. Thus $\mathrm{IH}^i(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi})[\Pi^\infty] = 0$ unless $i = \dim_{\mathbb{C}}(\mathbf{X})$. $\qquad\square$

Corollary 4.5.7. $\mathrm{R}\Gamma_c(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi})[\pi^\infty]$ *is concentrated in the middle degree* $\dim_{\mathbb{C}}(\mathbf{X})$, *and for each* $w \in \Sigma_{F_1}^{\mathrm{fin}}$ *with underlying place* $v \in \Sigma_F^{\mathrm{fin}}$ *it has a subquotient isomorphic to*

$$\rho_{\pi,\ell} \otimes \iota_\ell |-|_{F_{1,w}}^{\frac{N(\mathbf{G})-1-\dim_{\mathbb{C}}(\mathbf{X})}{2}} \cong \iota_\ell\left(\tilde{\phi}_{\pi_v}^{\mathrm{GL}} \otimes |-|_{F_{1,w}}^{-\frac{\dim_{\mathbb{C}}(\mathbf{X})}{2}}\right)$$

as a $W_{F_{1,w}}$*-module, where* $\rho_{\pi,\ell}$ *is defined in* Theorem 4.3.6.

Proof. This follows from Theorem 4.5.6 and the definition of $\rho_{\pi,\ell}$ (Theorem 4.3.6), using Brauer–Nesbitt theorem and Cheboratev density theorem. $\qquad\square$

5. Local and global Shimura varieties

In this section, we connect local shtuka spaces with minuscule μ (or local Shimura varieties) to global Shimura varieties, in order to prove a key result on the cohomology of local shtuka spaces Corollary 5.2.3 using global methods.

Let p be a rational prime with a fixed isomorphism $\iota_p : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_p}$ and K be a finite extension of $\mathbb{Q}_p$. Let ℓ be a rational primes different from p with a fixed isomorphism $\iota_\ell : \mathbb{C} \to \overline{\mathbb{Q}_\ell}$, which fixes a square root $\sqrt{p}$ of p in $\overline{\mathbb{Z}_\ell}$ thus also a square root $\sqrt{p}$ of p in $\overline{\mathbb{F}_\ell}$.

5.1. **Basic uniformizations of Shimura varieties.** In this subsection we briefly review basic uniformization of the generic fiber of a Shimura variety following [Han20, §3.1] and [Ham22, §4]. We adopt the notation on local shtuka spaces from §3.2.

Let $\mathbb{G}$ be a connected reductive group over $\mathbb{Q}$ and $(\mathbb{G}, \mathbb{X})$ a Shimura datum of Abelian type with associated conjugacy class of Hodge cocharacters $\{\mu\} : \mathrm{GL}(1)_{\mathbb{C}} \to \mathbb{G}_{\mathbb{C}}$. Suppose $\mathsf{G} = \mathbb{G} \otimes \mathbb{Q}_p$ is unramified, and we fix a Borel pair (B, T) for G, then we get from $\{\mu\}$ and ι_p a dominant cocharacters μ for $\mathsf{G}_{\overline{\mathbb{Q}_p}}$, with reflex field $E_\mu/\mathbb{Q}_p$.

For each neat compact open subgroups $\mathcal{K} = \mathcal{K}_p\mathcal{K}^p \leq \mathbb{G}(\mathbf{A}_f)$, we have the adic space $\mathcal{S}_{\mathcal{K}_p\mathcal{K}^p}(\mathbb{G}, \mathbb{X})$ over $\mathrm{Spa}(E_\mu)$ associated to the Shimura variety $\mathrm{Sh}_{\mathcal{K}_p\mathcal{K}^p}(\mathbb{G}, \mathbb{X})$, and we define

$$\mathcal{S}_{\mathcal{K}^p}(\mathbb{G}, \mathbb{X}) := \varprojlim_{\mathcal{K}_p \to \mathbf{1}} \mathcal{S}_{\mathcal{K}_p\mathcal{K}^p}(\mathbb{G}, \mathbb{X}),$$

which is representable by a perfectoid space because $(\mathbb{G}, \mathbb{X})$ is of Abelian type. By the result of [Han20], there exists a canonical $\mathsf{G}(\mathbb{Q}_p)$-equivariant Hodge–Tate period map defined in [Sch15, CS17]

$$\pi_{\mathrm{HT}} : \mathcal{S}_{\mathcal{K}^p}(\mathbb{G}, \mathbb{X}) \to \mathrm{Gr}_{\mathsf{G}, \mu},$$

where $\mathrm{Gr}_{\mathsf{G}, \mu}$ is the Schubert cell of the $\mathbf{B}_{\mathrm{dR}}^+$-affine Grassmannian of [SW20] indexed by μ, defined over $\mathrm{Spa}(E_\mu)$.

We write $\mu^\bullet = -w_0(\mu) \in X_\bullet(\mathsf{G})$, where w_0 for the longest-length Weyl group element. Let $\mathsf{b} \in B(\mathsf{G}, \mu^\bullet)_{\mathrm{bas}}$ be the unique basic element and $\mathbf{1} \in B(\mathsf{G})_{\mathrm{bas}}$ be the trivial element, then we obtain the open basic Newton stratum $\mathrm{Gr}_{\mathsf{G}, \mu}^{\mathsf{b}} \subset \mathrm{Gr}_{\mathsf{G}, \mu}$ as defined in [CS17, §3.5], and let $\mathcal{S}_{\mathcal{K}^p}(\mathbb{G}, \mathbb{X})^{\mathsf{b}} \subset \mathcal{S}_{\mathcal{K}^p}(\mathbb{G}, \mathbb{X})$ be the preimage of $\mathrm{Gr}_{\mathsf{G}, \mu}^{\mathsf{b}}$ under π_{HT}, called the basic Newton stratum.

Suppose $p > 2$ and G is unramified, then a result of Shen [She20, Theorem 1.2] says $(\mathbb{G}, \mathbb{X})$ has a basic uniformization at p in the following sense:

Theorem 5.1.1. *There exists a $\mathbb{Q}$-inner form $\mathbb{G}'$ of $\mathbb{G}$ such that*
- $\mathbb{G}' \otimes \mathbf{A}_f^p \cong \mathbb{G} \otimes \mathbf{A}_f^p$ *as algebraic groups over $\mathbf{A}_f^p$,*
- $\mathbb{G}' \otimes \mathbb{Q}_p \cong \mathsf{G}_{\mathsf{b}}$,
- $\mathbb{G}'(\mathbb{R})$ *is compact modulo center,*

together with a $\mathbb{G}(\mathbf{A}_f)$-equivariant isomorphism of diamonds over $\breve{E} := \breve{\mathbb{Q}}_p E_\mu$:

$$\varprojlim_{\mathcal{K}^p \to \mathbf{1}} \mathcal{S}_{\mathcal{K}^p}(\mathbb{G}, \mathbb{X})^{\mathsf{b}} \cong \left(\underline{\mathbb{G}'(\mathbb{Q}) \backslash \mathbb{G}'(\mathbf{A}_f)} \times_{\mathrm{Spd}(\breve{E})} \mathrm{Sht}(\mathsf{G}, \mathsf{b}, \mathbf{1}, \mu^\bullet) \right) / \underline{\mathsf{G}_{\mathsf{b}}(\mathbb{Q}_p)},$$

where $\mathsf{G}_{\mathsf{b}}(\mathbb{Q}_p)$ acts diagonally and $\mathbb{G}(\mathbf{A}_f) \cong \mathbb{G}'(\mathbf{A}_f^p) \times \mathsf{G}(\mathbb{Q}_p)$ acts on the right-hand side via the natural action of $\mathbb{G}'(\mathbf{A}_f^p)$ on $\mathbb{G}'(\mathbb{Q}) \backslash \mathbb{G}'(\mathbf{A}_f)$ and the action of $\mathsf{G}(\mathbb{Q}_p)$ on $\mathrm{Sht}(\mathsf{G}, \mathsf{b}, \mathbf{1}, \mu^\bullet)$. Moreover, under the identification $\mathrm{Gr}_{\mathsf{G}, \mu}^{\mathsf{b}} \cong \mathrm{Sht}(\mathsf{G}, \mathsf{b}, \mathbf{1}, \mu^\bullet) / \underline{\mathsf{G}_{\mathsf{b}}(\mathbb{Q}_p)}$, the Hodge–Tate period map:

$$\pi_{\mathrm{HT}} : \varprojlim_{\mathcal{K}^p \to \mathbf{1}} \mathcal{S}_{\mathcal{K}^p}(\mathbb{G}, \mathbb{X})^{\mathsf{b}} \to \mathrm{Gr}_{\mathsf{G}, \mu}^{\mathsf{b}}$$

identifies with the natural projection

$$\left(\underline{\mathbb{G}'(\mathbb{Q}) \backslash \mathbb{G}'(\mathbf{A}_f)} \times_{\mathrm{Spd}(\breve{E})} \mathrm{Sht}(\mathsf{G}, \mathsf{b}, \mathbf{1}, \mu^\bullet) \right) / \underline{\mathsf{G}_{\mathsf{b}}(\mathbb{Q}_p)} \to \mathrm{Sht}(\mathsf{G}, \mathsf{b}, \mathbf{1}, \mu^\bullet) / \underline{\mathsf{G}_{\mathsf{b}}(\mathbb{Q}_p)}.$$

This basic uniformization at p will allow us to deduce an isomorphism

$$\mathrm{R}\Gamma_c(\mathsf{G}, \mathsf{b}, \mathbf{1}, \mu^\bullet) \otimes \iota_\ell \left| - \right|_{E_\mu}^{\frac{-\dim(\mathbb{X})}{2}} [\dim(\mathbb{X})] \otimes_{\mathsf{G}_{\mathsf{b}}(\mathbb{Q}_p)}^{\mathsf{L}} \mathcal{A}(\mathbb{G}'(\mathbb{Q}) \backslash \mathbb{G}'(\mathbf{A}_f) / \mathcal{K}^p, \mathscr{L}_{\iota_\ell \xi})$$
$$\cong \mathrm{R}\Gamma_c \left(\mathcal{S}_{\mathcal{K}^p}(\mathbb{G}, \mathbb{X})^{\mathsf{b}}, \mathcal{L}_{\iota_\ell \xi} \right)$$

of $\mathsf{G}(\mathbb{Q}_p) \times W_{E_\mu}$-modules, for each algebraic representation ξ of $\mathbb{G}_{\mathbb{C}}$ that is trivial on Z_{ac} (see Definition 4.5.2), where $\mathcal{L}_{\iota_\ell \xi}$ is the rigid analytification of the lisse $\overline{\mathbb{Q}_\ell}$-sheaf $\mathscr{L}_{\iota_\ell \xi}$ associated to ξ (see Definition 4.5.2). When composed with the morphism

$$\mathrm{R}\Gamma_c \left(\mathcal{S}_{\mathcal{K}^p}(\mathbb{G}, \mathbb{X})^{\mathsf{b}}, \mathcal{L}_{\iota_\ell \xi} \right) \to \mathrm{R}\Gamma_c \left(\mathcal{S}_{\mathcal{K}^p}(\mathbb{G}, \mathbb{X}), \mathcal{L}_{\iota_\ell \xi} \right)$$

coming from excision with respect to the open basic stratum $\mathcal{S}(\mathbb{G}, \mathbb{X})_{\mathcal{K}^p}^{\mathsf{b}} \subset \mathcal{S}(\mathbb{G}, \mathbb{X})_{\mathcal{K}^p}$, this isomorphism gives us a uniformization map between cohomologies. The actual argument is more subtle than this because we need to use excision with integer coefficient (cf. [Ham22, Proposition 4.1]), but in the end we get:

Proposition 5.1.2. *Suppose $p > 2$, then there exists a $\mathsf{G}(\mathbb{Q}_p) \times W_{E_\mu}$-equivariant map*

$$\Theta : \mathrm{R}\Gamma_c(\mathsf{G}, \mathsf{b}, \mathbf{1}, \mu^\bullet) \otimes \iota_\ell \left| - \right|_{E_\mu}^{\frac{-\dim(\mathbb{X})}{2}} [\dim(\mathbb{X})] \otimes_{\mathsf{G}_{\mathsf{b}}(\mathbb{Q}_p)}^{\mathsf{L}} \mathcal{A}(\mathbb{G}'(\mathbb{Q}) \backslash \mathbb{G}'(\mathbf{A}_f) / \mathcal{K}^p, \mathscr{L}_{\iota_\ell \xi})$$
$$\to \mathrm{R}\Gamma_c \left(\mathcal{S}(\mathbb{G}, \mathbb{X})_{\mathcal{K}^p}, \mathcal{L}_{\iota_\ell \xi} \right)$$

functorial with respect to $\mathcal{K}^p$.

Next, we apply the "Boyer's trick", which is an analogue of results of [Boy99] that relates the supersingular part of cohomology of the local Shimura variety to cohomology of the global Shimura variety. For this, we recall the definition of being totally Hodge–Newton decomposable in the sense of [RV14, Definition 4.28] and [GHN19, Definition 3.1]:

Definition 5.1.3. Suppose G is a quasi-split reductive group over $\mathbb{Q}_p$ with a Borel pair (B, T), and μ is a dominant cocharacters for $\mathsf{G}_{\overline{\mathbb{Q}_p}}$. Then (G, μ) is called *totally Hodge–Newton decomposable* if for every non-basic φ_K-conjugacy class $\mathsf{b} \in B(\mathsf{G}, \mathbf{1}, \mu)$ (see (3.3)), there exists a properly contained standard Levi subgroup M of G, a dominant cocharacter μ_{M} of $\mathsf{M}_{\overline{\mathbb{Q}_p}}$ and an element $\mathsf{b}_{\mathsf{M}} \in B(\mathsf{M}, \mathbf{1}, \mu_{\mathsf{M}})$ such that μ_{M} is conjugate to μ under $\mathsf{G}(\overline{\mathbb{Q}_p})$-action, and b_{M} is mapped to b under the natural map $B(\mathsf{M}) \to B(\mathsf{G})$.

Example 5.1.4. Suppose
- $\mathsf{G} = \mathrm{Res}_{K/\mathbb{Q}_p} G^*$ where $K/\mathbb{Q}_p$ is unramified and G^* is a quasi-split reductive group over K defined in §2.1, moreover we assume that G^*_{ad} is geometrically simple,
- μ is the dominant cocharacters of
$$\left(\mathrm{Res}_{K/\mathbb{Q}_p} G^*\right)^*_{\overline{\mathbb{Q}_p}} \cong \prod_{v \in \mathrm{Hom}(K, \overline{\mathbb{Q}_p})} G^*_{\overline{K}}$$
that equals μ_1 defined in (2.5) on one factor and trivial on the other factors,

then $(\mathrm{Res}_{K/\mathbb{Q}_p} G^*, \mu)$ is totally Hodge–Newton decomposable: By [GHN19, Theorem 3.3, Proposition 3.4], it suffices to show that $\{\mu\}$ is minute for $\mathrm{Res}_{K/\mathbb{Q}_p} G^*_{\mathrm{ad}}$ as defined in [GHN19, Definition 3.2]. Then it suffices to show that $\{\mu_1\}$ is minute for G^* by [GHN19, §3.4], and this follows from the classification in [GHN19, Theorem 3.5]; cf. [GHN19, §3.7].

Finally, we recall the following result of Hamann [Ham22, Proposition 4.4].

Proposition 5.1.5. *If $p > 2$, G is unramified, and $(\mathsf{G}, \mu^\bullet)$ is totally Hodge–Newton decomposable, then the uniformization map Θ in Proposition 5.1.2 induces an isomorphism of W_{E_μ}-modules*

$$\Theta_{\mathrm{sc}} : \mathrm{R}\Gamma_c(\mathsf{G}, \mathsf{b}, \mathbf{1}, \mu^\bullet)_{\mathrm{sc}} \otimes \iota_\ell \left|-\right|_{E_\mu}^{\frac{-\dim(\mathbb{X})}{2}} [\dim(\mathbb{X})] \otimes^{\mathsf{L}}_{\mathsf{G}_{\mathsf{b}}(\mathbb{Q}_p)} \mathcal{A}\left(\mathbb{G}'(\mathbb{Q}) \backslash \mathbb{G}'(\mathbf{A}_f^p)/\mathcal{K}^p, \mathscr{L}_{\iota_\ell \xi}\right)$$
$$\xrightarrow{\sim} \mathrm{R}\Gamma_c\left(\mathcal{S}_{\mathcal{K}^p}(\mathbb{G}, \mathbb{X}), \mathcal{L}_{\iota_\ell \xi}\right)_{\mathrm{sc}}$$

on the summands where $\mathsf{G}(\mathbb{Q}_p)$ acts via a supercuspidal representation.

Proof. The proofs of [Ham22, Lemma 4.3, Proposition 4.4] go through verbatim. The key point is that the non-basic Newton strata of the flag varieties $\mathrm{Gr}^{\mathsf{b}}_{\mathsf{G}, \mu}$ are all parabolically induced from Newton strata on flag varieties associated to properly contained Levi subgroups of G, thus don't contribute to the supercuspidal part of cohomology. $\qquad\square$

5.2. **Globalization.** In this subsection assume $K/\mathbb{Q}_p$ is unramified. By a simple application of Krasner's lemma and weak approximation theorem (see, for example, [Art13, Lemma 6.2.1], [CZ24, Theorem F.1]), we may choose a CM or total real number field $F_1 \subset \mathbb{C}$ with maximal totally real subfield $F \subset F_1$, together with distinct rational primes p, q inert in F_1, such that

$$F_1 \otimes \mathbb{Q}_p / F \otimes \mathbb{Q}_p \cong K_1/K$$

as extensions. Write $\mathfrak{p}_1 = (p), \mathfrak{q}_1 = (q) \in \Sigma^{\mathrm{fin}}_{F_1}$ with underlying places $\mathfrak{p}, \mathfrak{q} \in \Sigma^{\mathrm{fin}}_F$, respectively, and let τ_0 denote the natural embedding $F \hookrightarrow \mathbb{C}$. Adopt the notation from §4.1 over F. In particular, we have reductive groups $\mathbf{G}, \mathbf{G}^*, \mathbf{G}^\sharp$ over F with $\mathbf{G}_{\mathrm{ad}}$ geometrically simple; if $F_1 \neq F$, we fixed a totally imaginary element $\daleth \in F_1^\times$, so each embedding $\tau : F \to \mathbb{C}$ extends to an embedding $\tau : F_1 \to \mathbb{C}$ sending $\daleth$ to $\mathbb{R}_+ i$.

Suppose $(\mathbf{V}, \mathbf{G} = \mathrm{U}(\mathbf{V})^\circ)$ is a standard indefinite pure inner form of $\mathbf{G}^*$ (see Definition 4.4.1) such that the Hasse–Witt invariant of $\mathbf{V}$ is trivial outside $\{\mathfrak{q}\} \cup \Sigma^\infty_F$ (such a pure inner twist exists uniquely). Set $G := \mathbf{G}_{\mathfrak{p}}$. Then G is one of the quasi-split reductive groups over K defined in §2.1, and we adopt the notation there.

Let $(\mathrm{Res}_{F/\mathbb{Q}} \mathbf{G}, \mathbf{X})$ be the Shimura datum defined in §4.4 with conjugacy class of Hodge cocharacters $\{\mu\}$. Via ι_p, we may regard $\{\mu\}$ as a conjugacy class of cocharacter

$$\mathrm{GL}(1)_{\overline{\mathbb{Q}_p}} \to \left(\mathrm{Res}_{F/\mathbb{Q}} \mathbf{G}\right)_{\overline{\mathbb{Q}_p}} = \prod_{v \in \mathrm{Hom}(K, \overline{\mathbb{Q}_p})} G_{\overline{K}}.$$

Then it contains a dominant cocharacter μ that equals μ_1 defined in (2.5) on one factor and trivial on the other factors. In particular, $\left(\left(\mathrm{Res}_{F/\mathbb{Q}} \mathbf{G}\right)_{\overline{\mathbb{Q}_p}}, \mu\right)$ is totally Hodge–Newton decomposable by Definition 5.1.4.

Let $(\mathbf{V}', \mathbf{G}' = \mathrm{U}(\mathbf{V}')^{\circ})$ be a standard definite pure inner twist of $\mathbf{G}^*$ (see Definition 4.4.1) such that the Hasse–Witt invariant of $\mathbf{V}'$ is nontrivial at $\mathfrak{p}$ and trivial outside $\{\mathfrak{p}, \mathfrak{q}\} \cup \Sigma_F^{\infty}$ (such a pure inner twist exists uniquely), and define $J := \mathbf{G}'_{\mathfrak{p}}$, which is isomorphic to G_{b_1}, where b_1 is the unique nontrivial basis element in $B(G)$. We also regard b_1 as the unique nontrivial basic element in $B(\mathrm{Res}_{K/\mathbb{Q}_p} G)_{\mathrm{bas}}$ via the isomorphism $B(\mathrm{Res}_{K/\mathbb{Q}_p} G) \cong B(G)$.

By [Shi12, Theorem 1.1] applied to $\mathbf{G}^*$ and Theorem 4.2.1, we obtain the following globalization result:

Proposition 5.2.1. *Suppose $\rho \in \Pi_{\mathrm{sc}}(J)$ is supercuspidal, then there exists:*

- *a control tuple $\bigstar$ for $\mathbf{G}^*$ such that $\mathfrak{p} \in \Sigma^{\mathrm{sc}}, \Sigma^{\circ} = \{\mathfrak{p}, \mathfrak{q}\}$ and $\Sigma^{\mathrm{St}} = \Sigma_F(S^{\mathrm{St}})$ where S^{St} is a non-empty finite set of rational primes.*

- *a $\bigstar$-split compact open subgroup $\mathcal{K}^{\mathfrak{p}} \leq \mathbf{G}'(\mathbf{A}_{F,f}^{\mathfrak{p}})$ (see Definition 4.3.4); and*

- *a $\bigstar$-good automorphic representation $\Pi' = \otimes'_v \Pi'_v$ of $\mathbf{G}'(\mathbf{A}_F)$ such that $\Pi'_{\mathfrak{p}} \cong \rho$ and $(\Pi'_f)^{\mathcal{K}^p} \neq 0$.*

Such a tuple $(\bigstar, \mathcal{K}^{\mathfrak{p}}, \Pi')$ is called a good globalization of ρ.

Suppose $\rho \in \Pi_{\mathrm{sc}}(J)$ is supercuspidal with supercuspidal classical L-parameter $\tilde{\phi}$, and $(\bigstar, \mathcal{K}^{\mathfrak{p}}, \Pi')$ be a good globalization of ρ, with associated Hecke-character

$$\phi_{\Pi'}^{\Sigma} : \mathbb{T}^{\Sigma} \to \mathbb{C},$$

and define $\mathfrak{m} := \iota_{\ell} \ker(\phi_{\Pi}^{\Sigma})$, which is a maximal ideal of $\iota_{\ell} \mathbb{T}^{\Sigma}$. Consider the uniformization map

$$\Theta : \mathrm{R}\Gamma_c(G, b_1, \mathbf{1}, \mu_1) \otimes \iota_{\ell} \left. |-| \right._{K_1}^{\frac{-\dim_{\mathbb{C}}(\mathbf{X})}{2}} [\dim_{\mathbb{C}}(\mathbf{X})] \otimes_{J(K)}^{\mathrm{L}} \mathcal{A}(\mathbf{G}'(F) \backslash \mathbf{G}'(\mathbf{A}_{F,f}) / \mathcal{K}^{\mathfrak{p}}; \mathscr{L}_{\iota_{\ell}\xi})$$
$$\to \mathrm{R}\Gamma_c(\mathcal{S}_{\mathcal{K}^{\mathfrak{p}}}(\mathrm{Res}_{F/\mathbb{Q}} \mathbf{G}, \mathbf{X}), \mathcal{L}_{\iota_{\ell}\xi})$$

which is $G(K) \times W_{K_1}$-equivariant and functorial with respect to $\mathcal{K}^{\mathfrak{p}}$ by Proposition 5.1.2. Here we use that $\mathrm{Sht}(\mathrm{Res}_{K/\mathbb{Q}_p} G, b_1, \mathbf{1}, \mu^{\bullet})$ is naturally isomorphic to $\mathrm{Sht}(G, b_1, \mathbf{1}, \mu_1)$; see §3.2. We localize both sides at $\mathfrak{m}$ and restrict to the parts on both sides where $G(K)$ acts via supercuspidal representations to get an isomorphism

$$\Theta_{\mathfrak{m},\mathrm{sc}} : \mathrm{R}\Gamma_c(G, b_1, \mathbf{1}, \mu_1)_{\mathrm{sc}} \otimes \iota_{\ell} \left. |-| \right._{K_1}^{\frac{-\dim_{\mathbb{C}}(\mathbf{X})}{2}} [\dim_{\mathbb{C}}(\mathbf{X})] \otimes_{J(K)}^{\mathrm{L}} \mathcal{A}(\mathbf{G}'(F) \backslash \mathbf{G}'(\mathbf{A}_{F,f}) / \mathcal{K}^{\mathfrak{p}}, \mathscr{L}_{\iota_{\ell}\xi})_{\mathfrak{m}}$$
$$\xrightarrow{\sim} \mathrm{R}\Gamma_c(\mathcal{S}_{\mathcal{K}^{\mathfrak{p}}}(\mathrm{Res}_{F/\mathbb{Q}} \mathbf{G}, \mathbf{X}), \mathcal{L}_{\iota_{\ell}\xi})_{\mathfrak{m}}$$

by the strong multiplicity one result Corollary 4.2.2 and the basic uniformization result Proposition 5.1.5 and Definition 5.1.4. Note that, by the strong multiplicity one result Corollary 4.2.2 again, the left-hand side decomposes as a direct sum

$$(5.1) \qquad \bigoplus_{\dot{\Pi}'} \mathrm{R}\Gamma_c(G, b_1, \mathbf{1}, \mu_1)_{\mathrm{sc}} \otimes \iota_{\ell} \left. |-| \right._{K_1}^{\frac{-\dim_{\mathbb{C}}(\mathbf{X})}{2}} [\dim_{\mathbb{C}}(\mathbf{X})] \otimes_{J(K)}^{\mathrm{L}} \iota_{\ell}(\dot{\Pi}'_f)^{\mathcal{K}_{\mathfrak{p}}},$$

where $\dot{\Pi}'$ runs through $\bigstar$-good automorphic representations of $\mathbf{G}'(\mathbf{A}_F)$ such that $(\dot{\Pi}')^{\Sigma} \cong (\Pi')^{\Sigma}$ and $\dot{\Pi}'_{\mathfrak{p}}$ has classical L-parameter $\tilde{\phi}$.

Corollary 5.2.2. *The map $\Theta_{\mathfrak{m},\mathrm{sc}}$ is an isomorphism of complexes of $G(K) \times W_{K_1}$-modules concentrated in the central degree $\dim_{\mathbb{C}}(\mathbf{X})$, and as a W_{K_1}-module it has a subquotient isomorphic to $\iota_{\ell} \left(\tilde{\phi}^{\mathrm{GL}} \otimes \left. |-| \right._{K_1}^{-\frac{\dim_{\mathbb{C}}(\mathbf{X})}{2}} \right)$.*

Proof. It suffices to prove the claim for each summand

$$\mathrm{R}\Gamma_c(G, b_1, \mathbf{1}, \mu_1)_{\mathrm{sc}} \otimes \iota_{\ell} \left. |-| \right._{K_1}^{\frac{-\dim_{\mathbb{C}}(\mathbf{X})}{2}} [\dim_{\mathbb{C}}(\mathbf{X})] \otimes_{J(K)}^{\mathrm{L}} \iota_{\ell}(\dot{\Pi}')_f^{\mathfrak{p}},$$

where $\dot{\Pi}'$ runs through $\bigstar$-good automorphic representations of $\mathbf{G}'(\mathbf{A})$ such that $(\dot{\Pi}')^{\Sigma} \cong (\Pi')^{\Sigma}$ and $\dot{\Pi}'_{\mathfrak{p}}$ has classical L-parameter $\tilde{\phi}$. Let $\dot{\Pi}$ be a $\bigstar$-good transfers of $\dot{\Pi}'$ to $\mathbf{G}$, and let $\rho_{\Pi',\ell} : \mathrm{Gal}_{F_1} \to \mathrm{GL}\left(N(\mathbf{G}); \overline{\mathbb{Q}_{\ell}}\right)$ be the Galois representation associated to Π' (Theorem 4.3.6). Then $\rho_{\Pi',\ell}$ is also the Galois representation associated to $\dot{\Pi}'$ and $\dot{\Pi}$ by Theorem 4.3.6, and it follows from Corollary 4.5.7 that the $\dot{\Pi}^{\Sigma} \cong (\dot{\Pi}')^{\Sigma}$-isotypic part of the target of $\Theta_{\mathfrak{m},\mathrm{sc}}$ only appears in the middle degree $\dim_{\mathbb{C}}(\mathbf{X})$, and as a W_{K_1}-module it contains a subquotient isomorphic to $\iota_{\ell} \left(\tilde{\phi}^{\mathrm{GL}} \otimes \left. |-| \right._{K_1}^{\frac{\dim_{\mathbb{C}}(\mathbf{X})}{2}} \right)$. $\square$

In particular, by the direct sum decomposition (5.1), we get the following key corollary:

Corollary 5.2.3. *If $\tilde\phi \in \tilde\Phi(J)$ is a supercuspidal L-parameter with associated packet $\tilde\Pi_{\tilde\phi}(J)$, then the direct summand of*

$$\bigoplus_{\tilde\rho' \in \tilde\Pi_{\tilde\phi}(J)} \mathrm{R}\Gamma_c(G, b_1, \mathbf{1}, \mu_1)[\iota_\ell \tilde\rho']$$

where $G(K)$ acts by supercuspidal representations, denoted by

$$\bigoplus_{\tilde\rho' \in \tilde\Pi_{\tilde\phi}(J)} \mathrm{R}\Gamma_c(G, b_1, \mathbf{1}, \mu_1)[\iota_\ell \tilde\rho']_{\mathrm{sc}},$$

is concentrated in middle degree $\dim_{\mathbb{C}}(\mathbf{X})$, and it has a subquotient isomorphic to $\iota_\ell \tilde\phi^{\mathrm{GL}}$ as a W_{K_1}-module.

6. Proof of the compatibility property

In this section we prove the main theorem Theorem A. So p is a rational prime, $K/\mathbb{Q}_p$ is an unramified finite extension, ℓ is a rational prime different from p with a fixed isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$, and $(G, \varrho_{b_0}, z_{b_0})$ is an extended pure inner form of G^* of Case O or Case U as in §2, such that G splits over an unramified finite extension of K. Let $\pi \in \Pi(G)$ be an irreducible smooth representation with classical L-parameter $\tilde\phi \in \tilde\Phi(G^*)$, we will show $\tilde\phi^{\mathrm{ss}} = \tilde\phi_\pi^{\mathrm{FS}}$ using induction on $n(G)$.

In Case O2 or Case U, when $n(G) = 1$, G is a torus, thus the assertion is known by compatibility of Fargues–Scholze LLC with local class field theory Theorem 3.1.1.

In Case U, when $n(G) = 2$, G is contained in $G^\sharp = (\mathrm{GL}_1(D) \times \mathrm{Res}_{K_1/K}\mathrm{GL}(1))/\mathrm{GL}(1)$ where D is a quaternion algebra over K and $\mathrm{GL}(1)$ acts anti-diagonally. By [Tad92, Proposition 2.2], for any $\pi \in \Pi(G)$, there exists $\pi^\sharp \in \Pi(G^\sharp)$ such that π is a subrepresentation of $\pi^\sharp|_{G(K)}$. So the assertion follows from compatibility for $G^\sharp$ [HL24, Lemma 4.7] and and compatibility of Fargues–Scholze LLC with central extensions Theorem 3.1.1.

In Case O1, when $n(G) = 1$, G is of the form $\mathrm{PGL}_1(D)$ for some quaternion algebra D over K, and the LLC for G defined in Theorem 2.3.1 equals the LLC for G via the LLC for. $\mathrm{GL}_1(D)$ constructed in [DKV84, Rog83] and the projection $\mathrm{GL}_1(D) \to \mathrm{PGL}_1(D)$; see [AG17, pp. 385–386]. Thus the main theorem follows from compatibility for inner forms of general linear groups [HKW22, Theorem 6.6.1] and compatibility of Fargues–Scholze LLC with central extensions; see Theorem 3.1.1.

In Case O2, when $n(G) = 2$, $G \cong \left(\mathrm{Res}_{K'/K}\mathrm{SL}_1(D)_{K'}\right)/\mu_2$ where K' is either $K \times K$ or the unique unramified quadratic extension of K, and D is a quaternion algebra over K; cf. [KR99, §0]. In fact we can prove the main theorem whenever K' is an étale extension of K of rank at most two. The L-parameter is constructed for G as follows (not just up to outer automorphism): For any $\pi \in \Pi(G)$, by [Tad92, Proposition 2.2] there exists $\pi^\sharp \in \Pi(G^\sharp)$, where $G^\sharp = \left(\mathrm{Res}_{K'/K}\mathrm{GL}_1(D)_{K'}\right)/\mathrm{GL}(1)$ containing G, such that π is a subrepresentation of the restriction of $\pi^\sharp$ to $G(K)$. Then the L-parameter ϕ_π is given by $\phi_{\pi^\sharp}$ composed with the natural map ${}^L G^\sharp \to {}^L G$; see [AG17, pp. 385–386]. Thus, as before, the main theorem follows from compatibility for inner forms of general linear groups [HKW22, Theorem 6.6.1] and compatibility of Fargues–Scholze LLC with central extensions Theorem 3.1.1.

Then, in all remaining cases G_{ad} is geometrically simple. We suppose throughout this section that the assertion is known for $G(n_0)$ for each $n_0 < n(G)$.

Firstly, if π is non-supercuspidal, then the assertion is true:

Proposition 6.0.1. *If $\pi \in \Pi(G)$ is a subquotient of a parabolic induction, then $\tilde\phi_\pi^{\mathrm{ss}} = \tilde\phi_\pi^{\mathrm{FS}}$.*

Proof. Suppose π is a subquotient of $\mathrm{I}_P^G(\sigma)$ where $P \leq G$ is a properly contained parabolic subgroup with Levi subgroup M, and $\sigma \in \Pi(M)$. By compatibility of Fargues–Scholze LLC with parabolic inductions Theorem 3.1.1 and compatibility of classical LLC with parabolic inductions Proposition 2.4.3, the assertion for π follows from the assertion for σ. The Levi subgroup M of G is of the form $G(n_0) \times \mathrm{Res}_{K_1/K}(H)$ for some integer $0 < n_0 < n(G)$, and H is a product of general linear groups. So the assertion follows from the induction hypothesis and compatibility of Fargues–Scholze LLC with products Theorem 3.1.1 and compatibility of Fargues-Scholze LLC with classical LLC for general linear groups Theorem 3.1.1. □

Assume now that π is supercuspidal, so $\tilde\phi$ is a discrete L-parameter by Theorem 2.3.1. There are two cases that can happen to the L-packet $\tilde\Pi_{\tilde\phi}(G^*)$:

(1) Case (1): $\tilde\Pi_{\tilde\phi}(G^*)$ consists entirely of supercuspidal representations,

(2) Case (2): $\tilde{\Pi}_{\tilde{\phi}}(G^*)$ contains a non-supercuspidal representation.

In the following subsections, we prove the main theorem in each case.

6.1. The first case.

In Case (1), $\Pi_{\tilde{\phi}}(G^*)$ consists entirely of supercuspidal representations, so it follows from Corollary 2.5.2 that $\tilde{\phi}$ is a supercuspidal L-parameter.

We first prove the compatibility for representations of G_{b_1}, where b_1 is the unique nontrivial basic element in $B(G)_{\mathrm{bas}}$.

Proposition 6.1.1. *If $\tilde{\phi} \in \tilde{\Pi}_{\mathrm{sc}}(G^*)$ is a supercuspidal L-parameter, then $(\tilde{\phi}_{\tilde{\rho}})^{\mathrm{FS}} = \tilde{\phi}$ for any $\tilde{\rho} \in \tilde{\Pi}_{\tilde{\phi}}(G_{b_1})$.*

Proof. We first consider the case when $\kappa_{b_0}(-1) = 1$, in particular $G \cong G^*$. We write $\tilde{\phi}^{\mathrm{GL}} = \phi_1 + \ldots + \phi_r$, where each ϕ_i is an irreducible representation of W_{K_1}. By Corollary 5.2.3, (3.5) and the compatibility of Fargues–Scholze LLC with contragredients (see Theorem 3.1.1), we see that ϕ_i appears in

$$\bigoplus_{\tilde{\rho} \in \tilde{\Pi}_{\tilde{\phi}}(G_{b_1})} \mathrm{Mant}_{G, b_1, \mu_1}(\iota_\ell \tilde{\rho})$$

as a representation of W_{K_1}, for each i. It follows that ϕ_i is an irreducible subquotient of $(\tilde{\phi}_{\tilde{\rho}}^{\mathrm{FS}})^{\mathrm{GL}}$ for each i, by Corollary 3.4.3 and [Kos21, Theorem 1.3].[9] Since $\tilde{\phi}^{\mathrm{GL}}$ and $(\tilde{\phi}_{\tilde{\rho}}^{\mathrm{FS}})^{\mathrm{GL}}$ are both semisimple, we see that they are equal. Thus it follows from [GGP12, Theorem 8.1.(ii)] that $\tilde{\phi} = \tilde{\phi}_{\tilde{\rho}}^{\mathrm{FS}}$.

Next we consider the case when $\kappa_{b_0}(-1) = -1$, in particular $G_{b_1} \cong G^*$. It follows from the weak Kottwitz conjecture Corollary 3.4.2 that some $\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G^*)$ appears in $\mathrm{Mant}_{G, b_1, \mu_1}(\tilde{\rho})$. We then have $\tilde{\phi}_{\tilde{\pi}}^{\mathrm{FS}} = \tilde{\phi}_{\tilde{\rho}}^{\mathrm{FS}}$ by Proposition 3.2.3. Thus

$$\tilde{\phi}_{\tilde{\rho}}^{\mathrm{FS}} = \tilde{\phi}_{\tilde{\pi}}^{\mathrm{FS}} = \tilde{\phi},$$

where the second equality follows from the first Case Upon replacing $(G, \varrho_{b_0}, z_{b_0})$ with $(G_{b_1}, \varrho_{b_0 + b_1}, z_{b_0 + b_1})$. $\square$

Finally, to deduce the compatibility for $\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)$, we apply this result with $(G, \varrho_{b_0}, z_{b_0})$ replaced by $(G_{b_1}, \varrho_{b_0 + b_1}, z_{b_0 + b_1})$, because then the unique nontrivial basic element in $B(G_{b_1})_{\mathrm{bas}}$ induces an inner form of G_{b_1} that is isomorphic to G.

6.2. The second case.

In the second case where $\tilde{\Pi}_{\tilde{\phi}}(G^*)$ contains a non-supercuspidal representation, we write $\tilde{\phi}^{\mathrm{GL}} = \phi_1 + \ldots + \phi_k + \phi_{k+1} + \ldots + \phi_r$ where ϕ_i is an irreducible representations of $W_{K_1} \times \mathrm{SL}(2, \mathbb{C})$ of dimension d_i for each i such that d_i is odd if and only if $i \leq k$.

We now prove the main theorem in this case:

Proposition 6.2.1. *If $\tilde{\Pi}_{\tilde{\phi}}(G^*)$ contains a non-supercuspidal representation $\tilde{\rho}_{\mathrm{nsc}}^*$, then $\tilde{\phi}_{\tilde{\pi}}^{\mathrm{FS}} = \tilde{\phi}^{\mathrm{ss}}$ for every $\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)$.*

Proof. Write $\tilde{\rho}_{\mathrm{nsc}} = \tilde{\pi}_{[I]}$ for some $I \in \mathscr{P}([r]_+)/\sim_k$ with $\#[I] \equiv 0 \pmod 2$. We will use induction on $d \in \mathbb{N}$ to show that $\tilde{\phi}_{\tilde{\pi}}^{\mathrm{FS}} = \tilde{\phi}^{\mathrm{ss}}$ whenever $\tilde{\pi} = \tilde{\pi}_{[I \oplus J]} \in \tilde{\Pi}_{\tilde{\phi}}(G) \cup \tilde{\Pi}_{\tilde{\phi}}(G_{b_1})$ for any $J \subset [r]_+$ with $\#J = d$.

If $d = 0$, then $\tilde{\pi} = \tilde{\rho}_{\mathrm{nsc}}$ and the assertion is true by Proposition 6.0.1. Suppose $d > 0$ and suppose by induction hypothesis that the assertion is proven for any smaller d. Given any $\tilde{\pi} = \tilde{\pi}_{[I \oplus J]}$ with $\#J = d$, if $\tilde{\pi}$ is not supercuspidal, then $\tilde{\phi}_{\tilde{\pi}}^{\mathrm{FS}} = \tilde{\phi}^{\mathrm{ss}}$ by Proposition 6.0.1. On the other hand, if $\tilde{\pi}$ is supercuspidal, we take an arbitrary subset $J' \subset J$ with $\#J' = d - 1$, then $\tilde{\pi}' = \tilde{\pi}_{[I \oplus J']} \in \tilde{\Pi}_{\tilde{\phi}}(G_{b'}^*)$, where $b' \in B(G^*)_{\mathrm{bas}}$ is the unique basic element with $\kappa_{G^*}(b')(-1) = (-1)^{d-1}$, and $\tilde{\phi}_{\tilde{\pi}'}^{\mathrm{FS}} = \tilde{\phi}^{\mathrm{ss}}$. Then it follows from Corollary 3.4.2 that $\tilde{\pi}$ occurs in $\mathrm{Mant}_{G_{b'}^*, b_1, \{\mu_1\}}[\tilde{\pi}_{[I \oplus J']}]$, where $b_1 \in B(G_{b'}^*)_{\mathrm{bas}}$ is the unique nontrivial basic element (here we use that $\tilde{\pi}$ is supercuspidal). So it follows from Proposition 3.2.3 that $\tilde{\phi}_{\tilde{\pi}}^{\mathrm{FS}} = \tilde{\phi}_{\tilde{\pi}_{[I \oplus J']}}^{\mathrm{FS}} = \tilde{\phi}^{\mathrm{ss}}$. $\square$

---

[9] Note that the notation $\mathcal{M}_{(G, b, \mu), K}$ in [Kos21] is just our $\mathrm{Sht}_{\mathscr{K}}(G, b, \{\mu^{-1}\})$, so there exists no dual appearing; cf. [Ham22, Remark 3.8].

7. Applications

7.1. **Unambiguous local Langlands correspondence for even orthogonal groups.** Combining the Fargues–Scholze LLC and the classical LLC, we can get an unambiguous local Langlands correspondence for even orthogonal groups, i.e., the L-parameter is constructed up to conjugation by $\mathrm{SO}(2n(G), \mathbb{C})$ instead of $\mathrm{O}(2n(G), \mathbb{C})$: Notation as in §2. In particular, p is a rational prime, $K/\mathbb{Q}_p$ is an unramified finite extension, and $(G, \varrho_{b_0}, z_{b_0})$ is a pure inner twist of G^* of type O2.

Theorem 7.1.1. *In case O2, suppose $(G, \varrho_{b_0}, z_{b_0})$ is a pure inner twist of G^* and $\mathrm{ord}_K(\mathrm{disc}(G)) \equiv 0 (\mathrm{mod}\ 2)$, then there exists a map $\mathrm{rec}_G^\flat : \Pi(G) \to \Phi(G^*)$ fitting into a commutative diagram*

$$
\begin{array}{ccc}
\Pi(G) & \xrightarrow{\ \mathrm{rec}_G^\flat\ } & \Phi(G^*) \\
\downarrow & & \downarrow \\
\tilde{\Pi}(G) & \xrightarrow{\ \mathrm{rec}_G\ } & \tilde{\Phi}(G^*)
\end{array}
$$

For any $\phi \in \Phi(G^*)$, we write $\Pi_\phi(G) := (\mathrm{rec}_G^\flat)^{-1}(\phi)$, called the L-packet for ϕ. This correspondence satisfies the following properties:

(1) If $\phi \in \Phi(G^*)$ is not relevant for G in the sense of [KMSW14, Definition 0.4.14], then $\Pi_\phi(G) = \varnothing$.

(2) For each $\phi \in \Phi(G^*)$ and $\pi \in \Pi_{\tilde{\phi}}(G)$, π is tempered if and only if ϕ is tempered, and π is a discrete series representation if and only if ϕ is discrete.

(3) $\mathrm{rec}_G^\flat$ only depends on G but not in ϱ_{b_0} and z_{b_0}. For the fixed Whittaker datum $\mathfrak{m}$ of G^*, there exists a canonical bijection

$$
\iota_{\mathfrak{m}, b_0} : \Pi_\phi(G) \xrightarrow{\sim} \mathrm{Irr}(\mathfrak{S}_\phi; \kappa_{b_0})
$$

for each $\phi \in \Phi(G^*)$, where $\mathrm{Irr}(\mathfrak{S}_\phi; \kappa_{b_0})$ is the set of characters η of $\mathfrak{S}_\phi$ such that $\eta(z_\phi) = \kappa_{b_0}(-1)$. We write $\pi = \pi_{\mathfrak{m}, b_0}(\phi, \eta)$ if $\pi \in \Pi_\phi(G)$ corresponds to $\eta \in \mathrm{Irr}(\mathfrak{S}_\phi)$ via $\iota_{\mathfrak{m}, b_0}$.

(4) (Compatibility with Langlands quotient) If $P \le G$ is a parabolic subgroup of G with a Levi factor

$$
M \cong \mathrm{GL}(d_1) \times \cdots \times \mathrm{GL}(d_r) \times G(n_0),
$$

such that $M = \varrho_{b_0}(M^*)$ where M^* is a standard Levi subgroup of G^*, and $\pi \in \Pi(G)$ is the unique irreducible quotient of

$$
\mathrm{I}_P^G \left(\tau_1 \otimes \nu^{s_1} \boxtimes \cdots \boxtimes \tau_r \otimes \nu^{s_r} \boxtimes \pi_0 \right),
$$

where $d_1 + \ldots + d_r + n_0 = n$, $s_1 \ge s_2 \ge \ldots \ge s_r > 0$, $\pi_0 \in \Pi_{\mathrm{temp}}(G(n_0))$ is a tempered representation with $\phi_0 := \mathrm{rec}_G^\flat(\pi_0)$ and each $\tau_i \in \Pi_{2,\mathrm{temp}}(\mathrm{GL}(d_i))$ is a discrete series representations with classical parameter ϕ_i, then $\mathrm{rec}_G^\flat(\pi)$ equals the image of

$$
\phi_1 \left| - \right|_{K_1}^{s_1} \times \cdots \times \phi_r \left| - \right|_{K_1}^{s_r} \times \phi_0 \in \Phi(M)
$$

composed the canonical embedding $^L M \to {}^L G$. Moreover, there exists a natural identification $\mathfrak{S}_{\phi_0} \cong \mathfrak{S}_\phi$ under which $\iota_{\mathfrak{m}, b_0}(\pi) = \iota_{\mathfrak{m}_0, b_0}(\pi_0)$, where $\mathfrak{m}_0$ is the induced Whittaker datum on M^*.

(5) (Compatibility with standard γ-factors) Suppose $\pi \in \Pi(G)$ with $\phi := \mathrm{rec}_G^\flat(\pi)$, then for any character χ of $K^\times$,

$$
\gamma(\pi, \chi, \psi_K; s) = \gamma(\phi^{\mathrm{GL}} \otimes \chi, \psi_K; s),
$$

where the left-hand side is the standard γ-factor defined by Lapid–Rallis using the doubling zeta integral [LR05] but modified in [GI14], and the right-hand side is the γ-factor defined in [Tat79].

(6) (Compatibility with Plancherel measures) Suppose $\pi \in \Pi(G)$ with $\phi := \mathrm{rec}_G^\flat(\pi)$, then for any $\tau \in \Pi(\mathrm{GL}(n))$ with L-parameter ϕ_τ,

$$
\begin{aligned}
\mu_{\psi_K}(\tau \otimes \nu^s \boxtimes \pi) = {} & \gamma(\phi_\tau \otimes (\phi^{\mathrm{GL}})^\vee, \psi_{K_1}; s) \cdot \gamma(\psi_\tau^\vee \otimes \phi^{\mathrm{GL}}, \psi_K^{-1}, -s) \\
& \times \gamma(\wedge^2(\phi_\tau), \psi_K; 2s) \cdot \gamma(\wedge^2(\phi_\tau^\vee), \psi_K^{-1}; -2s),
\end{aligned}
$$

where the left-hand side is the Plancherel measure defined in [GI14, §12]; cf.[GI16, §A.7].

(7) (Local intertwining relations) Suppose $P \leq G$ is a maximal parabolic subgroup with a Levi factor

$$M \cong \mathrm{GL}(d) \times G(n-d),$$

and $\pi \in \Pi_{\mathrm{temp}}(G)$ is a subrepresentation of $\mathrm{I}_P^G(\tau \boxtimes \pi_0)$. Assume that $M = \varrho_{b_0}(M^*)$ where M^* is a standard Levi subgroup of G^*, and $\mathfrak{m}_0$ is the induced Whittaker datum on M^*, then $\phi := \mathrm{rec}_G^\natural(\pi)$ equals the image of

$$\phi_\tau \times \phi_0 \in \Phi(M)$$

composed with the canonical embedding $^L M \to {}^L G$. Furthermore, if ϕ_τ is self-dual of sign 1 and the normalized intertwining operator

$$\mathrm{R}_{\mathfrak{m}}(w, \tau \boxtimes \pi_0) \in \mathrm{End}_{G(K)} \left(\mathrm{I}_P^G(\tau \boxtimes \pi_0) \right)$$

defined in [CZ21, §7.1] acts on π by an element $\epsilon \in \{\pm 1\}$, where w is the unique nontrivial element in the relative Weyl group for M, then

$$\eta := \iota_{\mathfrak{m}, b_0}(\pi) \in \mathrm{Irr}(\mathfrak{S}_\phi)$$

restricts to $\iota_{\mathfrak{m}_0, b_0}(\pi_0) \in \mathfrak{S}_{\phi_0}$ under the natural embedding $\mathfrak{S}_{\phi_0} \hookrightarrow \mathfrak{S}_\phi$, and satisfies $\eta(e_\tau) = \epsilon$, where e_τ is the element of $\mathfrak{S}_\phi$ corresponding to ϕ_τ.

(8) (Compatibility with Fargues–Scholze LLC) Suppose ℓ is a rational prime distinct from p with a fixed isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$. For any $\pi \in \Pi(G)$ with L-parameter $\phi = \mathrm{rec}_G^\natural(\pi)$, the semisimplification satisfies $\phi^{\mathrm{ss}} = \iota_\ell^{-1} \phi_{\iota_\ell \pi}^{\mathrm{FS}} \in \Phi^{\mathrm{ss}}(G)$.

Proof. Note that the fiber of $\Phi_2(G^*) \to \tilde{\Phi}_2(G^*)$ over $\tilde{\phi}$ has cardinality 1 if and only if each subrepresentation $\rho \boxtimes \mathrm{sp}_a$ of $W_K \times \mathrm{SL}(2, \mathbb{C})$ contained $\tilde{\phi}^{\mathrm{GL}}$ has even dimension [GGP12, Theorem 8.1(ii)]. But note that each irreducible subrepresentation of the semi-simplification

$$(\rho \boxtimes \mathrm{sp}_a)^{\mathrm{ss}} = \rho \otimes |-|_{K_1}^{\frac{a-1}{2}} + \rho \otimes |-|_{K_1}^{\frac{a-3}{2}} + \ldots + \rho \otimes |-|_{K_1}^{\frac{1-a}{2}}$$

also has even dimension. Indeed, if $\dim(\rho)$ is odd, then a is even and $b(\rho) = 1$, but then $b(\rho \boxtimes \mathrm{sp}_a) = -1 \neq b(G)$, which is a contradiction.

By Theorem 2.3.1 and the Theorem A, for each $\pi \in \Pi_{2, \mathrm{temp}}(G)$ with classical L-parameter $\tilde{\phi} \in \tilde{\Phi}_2(G)$, we may define $\phi = \mathrm{rec}_G^\natural(\pi)$ to be the unique element in the fiber of $\Phi_2(G^*) \to \tilde{\Phi}_2(G^*)$ over $\tilde{\phi}$ such that $\phi^{\mathrm{ss}} = \iota_\ell^{-1} \phi_{\iota_\ell \pi}^{\mathrm{FS}}$, where ℓ is an arbitrary rational prime different from p with a fixed isomorphism $\iota_\ell : \mathbb{C} \cong \overline{\mathbb{Q}_\ell}$. We then extend the definition of $\mathrm{rec}_G^\natural$ to tempered representations $\pi \in \Pi_{\mathrm{temp}}(G)$ inductively using (7) as in the proof of Proposition 2.4.3, and extend the definition of $\mathrm{rec}_G^\natural$ to any irreducible smooth representations $\pi \in \Pi(G)$ using (4). Then the theorem follows from Theorem 2.3.1 and the compatibility properties of Fargues–Scholze LLC Theorem 3.1.1. $\qquad\square$

And the weak version of the Kottwitz conjecture Theorem 3.4.1 can be strengthened as follows:

Theorem 7.1.2. *Suppose $(G, \varrho_{b_0}, z_{b_0})$ is a pure inner twist of G^* associated to $b_0 \in B(G^*)_{\mathrm{bas}}$, μ be a dominant cocharacter of $\mathsf{G}_{\overline{K}}^*$ and $b \in B(G^*, b_0, \mu)_{\mathrm{bas}}$ be the unique basic element. If $\phi \in \Phi_2(G^*)$ is a discrete L-parameter and $\rho \in \Pi_\phi(G_b)$, then*

$$\mathrm{Mant}_{G, b, \mu}(\iota_\ell \rho) = \sum_{\pi \in \Pi_\phi(G)} \dim \mathrm{Hom}_{\mathfrak{S}_\phi}(\delta[\pi, \rho], \mathcal{T}_\mu)[\iota_\ell \pi] + \mathrm{Err}$$

in $\mathrm{K}_0(G, \overline{\mathbb{Q}_\ell})$, where $\mathrm{Err} \in \mathrm{K}_0(G, \overline{\mathbb{Q}_\ell})$ is a virtual representation whose character vanishes on $G(K)_{\mathrm{s.reg,ell}}$. Moreover, if ϕ is supercuspidal, then $\mathrm{Err} = 0$.

Proof. This follows from Theorem 3.4.1 and Proposition 3.2.3, by extracting the terms whose Fargues–Scholze parameters equal ϕ^{ss}, noticing that $\mathrm{Mant}_{G, b, \{\mu\}}$ commutes with ς. $\qquad\square$

We conjecture that LLC defined in Theorem 7.1.1 satisfies the "unambiguous" endoscopic character identities defined in [Kal16], which is related to the fact that Fargues–Scholze LLC is compatible with endoscopic transfers. More precisely, we expect that the endoscopic character identities follows from an ambiguous version of the conjectural Shin's averaging formula stated in [Ham24, Conjecture C.2], where ambiguity means we conflate representation conjugated by some outer automorphism. For example, in the

trivial endoscopy case, using the Kottwitz conjecture Theorem 7.1.2, we prove the following endoscopic character identity between G and G^*:

Theorem 7.1.3. *Suppose* $\phi \in \Phi_2(G^*)$, *and* $g \in G(K)_{\mathrm{s.reg,ell}}, h \in G^*(K)_{\mathrm{s.reg,ell}}$ *are stably conjugate, then*

$$e(G) \sum_{\rho \in \Pi_\phi(G)} \Theta_\rho(g) = \mathrm{S}\Theta_\phi(h) = \sum_{\pi \in \Pi_\phi(G^*)} \Theta_\pi(h),$$

Proof. This is established by reversing the argument in the proof of Theorem 3.4.1. Given g, h, we may choose a dominant cocharacter μ of $G^*_{\overline{K}}$ such that $(g, h, \mu) \in \mathrm{Rel}_{b_0}$. Here we identify G_{b_0} with G^*. We apply Theorem 7.1.2, and it follows from Proposition 3.4.6 and the proof of Proposition 3.4.7 that

$$(7.1) \qquad e(G) \sum_{(h', \lambda)} \Theta_\pi(h') \dim \mathcal{T}_\mu[\lambda] = \sum_{\rho \in \Pi_\phi(G)} \dim \mathrm{Hom}_{\mathfrak{S}_\phi}(\delta[\rho, \pi], \mathcal{T}_\mu) \Theta_\rho(g).$$

for any $\pi \in \Pi_\phi(G^*)$. For each $\rho \in \Pi_\phi(G)$, when π runs through all elements of $\Pi_\phi(G^*)$, $\delta[\rho, \pi]$ runs through every element of $\mathrm{Irr}(\mathfrak{S}_\phi; \kappa_{b_0})$ exactly once. Note that for the fixed g, an character $\lambda \in X_\bullet(Z_G(g)) = X^\bullet(\widehat{T})$ can be extended uniquely to a triple $(g, h', \lambda) \in \mathrm{Rel}_{b_0}$ if and only if the restriction of λ to $Z(\widehat{G})^{\mathrm{Gal}_K}$ equals κ_{b_0}; see [HKW22, p. 17]. So we sum Equation (7.1) over $\pi \in \Pi_\phi(G^*)$ to get

$$(7.2) \qquad e(G) \sum_{(h', \lambda)} \dim \mathcal{T}_\mu[\lambda] \mathrm{S}\Theta_\phi(h')$$

$$(7.3) \qquad = \dim \mathrm{Hom}_{Z(\widehat{G})^{\mathrm{Gal}_K}}(\kappa_{b_0}, \mathcal{T}_\mu) \sum_{\rho \in \Pi_\phi(G)} \Theta_\rho(g)$$

$$(7.4) \qquad = \left(\sum_{(h', \lambda)} \dim \mathcal{T}_\mu[\lambda] \right) \cdot \sum_{\rho \in \Pi_\phi(G)} \Theta_\rho(g).$$

If there exists a pair (h', λ) such that $\mathcal{T}_\mu[\lambda] \neq 0$, $(g, h', \lambda) \in \mathrm{Rel}_{b_0}$ and $\lambda \neq \mu$, then we may replace μ by λ to get a new equation. By the highest weight theory of representations, after finitely many steps we may replace the original μ by another μ' such that the only term in the summation of (7.2) is $(h', \lambda) = (h, \mu')$. Thus we get:

$$e(G) \sum_{\rho \in \Pi_\phi(G)} \Theta_\rho(g) = \mathrm{S}\Theta_\phi(h).$$

$\square$

The arguments above show a general strategy to eliminate ambiguity in local Langlands correspondence caused by outer automorphisms: If we can construct a coarse local Langlands correspondence for all extended pure inner twists of a quasi-split reductive group G^* over K up to action of a finite group $\mathfrak{A}$ acting by outer automorphisms, character twists or taking contragredients, and verify endoscopic character identities in the sense of [Kal16] but up to action by $\mathfrak{A}$, then we may use it to deduce a weak version of Kottwitz conjecture up to action by $\mathfrak{A}$. If we can also show the local Langlands correspondence constructed is compatible with Fargues–Scholze parameters up to action by $\mathfrak{A}$ in the sense of Theorem A, and the semi-simplification map $\Phi(G) \to \Phi^{\mathrm{ss}}(G)$ is injective on each orbit of the $\mathfrak{A}$-action, then we may use the action of $\mathfrak{A}$ on the local shtuka space and [Kos21, Theorem 1.3] to extract a local Langlands correspondence not up to action by $\mathfrak{A}$. For example, we expect the strategy to hold for constructing local Langlands correspondence for pure inner forms of the the even rank unitary similitude group $\mathrm{GU}(2n)$ with respect to unramified quadratic extensions or general even special orthogonal groups $\mathrm{GSO}(2n)$, following work of Xu [Xu16].

7.2. **Naturality of Fargues–Scholze LLC.** In this subsection, we will prove the following "naturally" property of Fargues–Scholze LLC for G, which is just [Ham24, Assumption 7.5], and a weaker result for a central extension of $\mathrm{Res}_{K/\mathbb{Q}_p} G$, which will be used to prove a vanishing result for relevant Shimura varieties in §8. We first recall some notation from [HL24]: For a quasi-split reductive group G over a non-Archimedean local field K of characteristic zero with a Borel pair (B, T), if $\mathsf{b} \in B(\mathsf{G})_{\mathrm{un}}$, there exists a standard Levi subgroup M_b of a parabolic subgroup P_b of G containing B such that $\mathsf{G}_\mathsf{b} \cong \mathsf{M}_\mathsf{b}$ under the inner twisting by

b, and $\mathsf{B} \le M_{\mathsf{b}}$ transfers to a Borel subgroup $\mathsf{B}_{\mathsf{b}} \le \mathsf{G}_{\mathsf{b}}$. Let $W_{\mathsf{b}} := W_{\mathsf{G}}/W_{M_{\mathsf{b}}}$, and we identify an element of W_{b} with a representative in W_{G} of minimal length. For a character χ of $\mathsf{T}(K)$ and $\mathsf{w} \in W_{\mathsf{b}}$, we set

$$(7.5) \qquad \rho_{\mathsf{b},\mathsf{w}}^{\chi} := \mathrm{I}_{\mathsf{B}_{\mathsf{b}}}^{\mathsf{G}_{\mathsf{b}}}(\chi^{\mathsf{w}}) \otimes \delta_{\mathsf{P}_{\mathsf{b}}}^{-1/2}.$$

We now verify a property of the Fargues–Scholze LLC, stated as an assumption in [Ham24, Assumption 7.5].

Proposition 7.2.1.

(1) *For each $b \in B(G^*)$, the classical LLC $\mathrm{rec}_{G_b^*} : \Pi(G_b^*) \to \Phi(G_b)$ is compatible with the Fargues–Scholze LLC, i.e., $\iota_\ell^{-1}\phi_{\iota_\ell\pi}^{\mathrm{FS}} = \phi^{\mathrm{ss}}$ for each $\pi \in \Pi(G_b^*)$.*

(2) *For $b \in B(G^*)$ and $\rho \in \Pi(G_b^*)$, let $\phi \in \Phi(G)$ be the composition of $\phi_\rho \in \Phi(G_b^*)$ with the twisted embedding ${}^L G_b \cong {}^L M_b \to {}^L G$ (as defined in [FS24, §IX.7.1]), then ϕ factors through the canonical embedding ${}^L T^* \to {}^L G^*$ only if $b \in B(G^*)_{\mathrm{un}}$.*

(3) *Situation as in (2), if b is unramified and ϕ factors through $\phi_{T^*} \in \Phi(T^*)$, then ρ is isomorphic to an irreducible constituent of $\rho_{b,w}^{\chi}$, where $w \in W_b$ and χ is the character of T^* attached to ϕ_{T^*} via local Langlands correspondence for tori.*

Proof. For (1): each G_b is an inner form of a Levi subgroup M_b of a parabolic subgroup of G^*, thus is of the form $\mathrm{Res}_{K_1/K}H \times G'$, where G' is a special orthogonal or unitary group that splits over an unramified quadratic extension of K, and H is a product of general linear groups. So the assertion follows from Theorem 7.1.1.

For (2): suppose $b \notin B(G^*)_{\mathrm{un}}$. By [Ham24, Lemma 2.12], G_b is not quasi-split. Consequently, T^* is not relevant for G_b in the sense of [KMSW14, Definition 0.4.14], and hence $\Pi_{\phi_\rho}(G_b)$ is empty. Indeed, by Jacquet–Langlands correspondence [DKV84], this follows from Theorem 7.1.1 together with the corresponding results for inner forms of general linear groups. This yields a contradiction.

For (3): first, we note ϕ is semisimple, since ${}^L T$ consists of semisimple elements. Since $G_b \cong M_b$, the preimage of ϕ under the natural embedding $\Phi(M_b^*) \to \Phi(G^*)$ is parametrized by a set of minimal length representatives of W_b. So the assertion follows from the compatibility of classical LLC with parabolic inductions Theorem 7.1.1, noticing that the twisting by $\delta_{P_b^*}^{-1/2}$ appears to neutralize the twisting in the definition of twisted embedding ${}^L G_b \cong {}^L M_b \to {}^L G$ as defined in [FS24, §IX.7.1]. $\qquad\square$

We also need to prove a weaker result for a central extension of $\mathrm{Res}_{K/\mathbb{Q}_p}G^*$. More generally, we impose the following global setup for future use:

Setup 7.2.2.

- Let F be a totally real number field and F_1 be either F or a CM field containing F, and let $\mathsf{c} \in \mathrm{Gal}(F_1/F)$ be the element with fixed field F,

- Let p be a rational prime that is unramified in F, with a fixed isomorphism $\iota_p : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_p}$, and we write $K = F \otimes \mathbb{Q}_p$, which is a finite product of unramified finite extensions of $\mathbb{Q}_p$,

- Let $\mathbf{G}$ be a standard indefinite special orthogonal or unitary group over F defined by a c-Hermitian space $\mathbf{V}$ as in Definition 4.4.1, such that $G^* := \mathbf{G} \otimes_F K$ is quasi-split and splits over an unramified finite extension of $\mathbb{Q}_p$.

- for a each quadratic imaginary element $\daleth \in \mathbb{R}_+\mathrm{i}$ (in particular $\daleth^2 \in \mathbb{Q}_-$), we defined in §4.1 a central extension

$$1 \to \mathbf{Z}^{\mathbb{Q}} \to \mathbf{G}^{\sharp} \to \mathbf{G} \to 1$$

where

$$\mathbf{Z}^{\mathbb{Q}} = \begin{cases} \{z \in \mathrm{Res}_{F(\daleth)/\mathbb{Q}}\mathrm{GL}(1) : \mathrm{Nm}_{F(\daleth)/F}(z) \in \mathbb{Q}^{\times}\} & \text{in Case O} \\ \{z \in \mathrm{Res}_{F_1/\mathbb{Q}}\mathrm{GL}(1) : \mathrm{Nm}_{F_1/F}(z) \in \mathbb{Q}^{\times}\} & \text{in Case U.} \end{cases}$$

Moreover, this central extension splits in Case U. In Case O, we assume that $\mathbb{Q}(\daleth)/\mathbb{Q}$ is split at p, so

$$\mathbf{Z}^{\mathbb{Q}} \otimes \mathbb{Q}_p \cong \mathrm{GL}(1) \times \mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}(1), \quad \mathbf{G}^{\sharp} \otimes \mathbb{Q}_p \cong \mathrm{GL}(1) \times \mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GSpin}(V^*).$$

Set

$$(7.6) \qquad G^{\sharp} := \mathbf{G}^{\sharp} \otimes \mathbb{Q}_p,$$

which is a central extension of $\mathrm{Res}_{K/\mathbb{Q}}G^*$. We fix a Borel pair $(B^\sharp, T^\sharp)$ for $G^\sharp$ with image (B, T) in $\mathrm{Res}_{K/\mathbb{Q}}G^*$.

We now prove a version of Proposition 7.2.1 for those L-parameters of $G^\sharp$ coming from L-parameters of $\mathrm{Res}_{K/\mathbb{Q}_p}G^*$:

Theorem 7.2.3. *Let $\phi \in \Phi^{\mathrm{ss}}(\mathrm{Res}_{K/\mathbb{Q}_p}G^*)$ and let $\phi^\sharp$ be the image of ϕ under the natural L-homomorphism $^L(\mathrm{Res}_{K/\mathbb{Q}_p}G^*) \to {}^LG^\sharp$.*

(1) Suppose $b^\sharp \in B(G^\sharp)$ and $\rho^\sharp \in \Pi(G^\sharp_{b^\sharp})$ such that $\phi^{\mathrm{FS}}_{\iota_\ell \rho^\sharp}$ under the twisted embedding $^LG^\sharp_{b^\sharp} \cong {}^LM_{b^\sharp} \to {}^LG^\sharp$ (as defined in [FS24, §IX.7.1]) equals $\phi^\sharp$, then $\rho^\sharp$ factors through a representation $\rho \in \Pi((\mathrm{Res}_{K/\mathbb{Q}_p}G^*)_b)$, where b is the image of $b^\sharp$ under the map $B(G^\sharp) \to B(\mathrm{Res}_{K/\mathbb{Q}_p}G^*)$, and the classical LLC $\phi_\rho \in \Phi((\mathrm{Res}_{K/\mathbb{Q}_p}G^*)_b)$ is defined, and the image of ϕ^{ss}_ρ under the natural L-homomorphism $^L(\mathrm{Res}_{K/\mathbb{Q}_p}G^*)_b \to {}^LG^\sharp_{b^\sharp}$ equals $\iota_\ell^{-1}\phi^{\mathrm{FS}}_{\iota_\ell \rho^\sharp}$.

(2) Situation as in (1), if $\phi^\sharp$ factors through the canonical embedding $^LT^\sharp \to {}^LG^\sharp$, then $b^\sharp \in B(G^\sharp)_{\mathrm{un}}$.

(3) Situation as in (1), if $b^\sharp$ is unramified and $\phi^\sharp$ factors through $\phi_{T^\sharp} \in \Phi^{\mathrm{ss}}(T^\sharp)$, then $\rho^\sharp$ is isomorphic to an irreducible constituent of $\rho^{\chi^\sharp}_{b^\sharp, w^\sharp}$, where $w^\sharp \in W_{b^\sharp}$ and $\chi^\sharp$ is the character of $T^\sharp$ attached to $\phi_{T^\sharp}$ via local Langlands correspondence for tori.

Proof. Note that $G^\sharp_{b^\sharp}$ is a product of Weil restrictions of general linear groups, unitary similitude maps or general spinor groups, so for any $b^\sharp \in B(G^\sharp)$ and $\rho^\sharp \in \Pi(G^\sharp_{b^\sharp})$ as in (1), $\rho^\sharp$ is trivial on the kernel Z of the map $G^\sharp_{b^\sharp}(\mathbb{Q}_p) \to (\mathrm{Res}_{K/\mathbb{Q}_p}G^*)_b(\mathbb{Q}_p)$, by compatibility of Fargues–Scholze LLC with central characters Theorem 3.1.1. Thus $\rho^\sharp$ factors through a representation of $(\mathrm{Res}_{K/\mathbb{Q}_p}G^*)_b(\mathbb{Q}_p)$ because either the central extension

$$1 \to Z \to G^\sharp \to \mathrm{Res}_{K/\mathbb{Q}_p}G^* \to 1$$

is split or the kernel Z is an induced torus, i.e., a product of tori of the form $\mathrm{Res}_{L_i/\mathbb{Q}_p}\mathrm{GL}(1)$ for finite extensions $L_i/\mathbb{Q}_p$. Now (1) follows from compatibility of Fargues–Scholze LLC with central extensions Theorem 3.1.1, and (2)(3) follows from Proposition 7.2.1 and the following lemma Lemma 7.2.4 showing that $\Phi(\mathrm{Res}_{K/\mathbb{Q}_p}G^*) \to \Phi(G^\sharp)$ is injective. $\square$

Lemma 7.2.4. *Suppose K is a non-Archimedean local field of characteristic zero and $1 \to \mathsf{Z} \to \mathsf{G}' \to \mathsf{G} \to 1$ is a central extension of reductive groups over K such that Z is a torus and $\mathsf{G}'(K) \to \mathsf{G}(K)$ is surjective, then the natural homomorphism $^L\mathsf{G} \to {}^L\mathsf{G}'$ induces an injection $\Phi(\mathsf{G}) \hookrightarrow \Phi(\mathsf{G}')$.*

Proof. It follows from [Kot85, (1.9.1)] that there is a long exact sequence

$$1 \to \mathsf{Z}(K) \to \mathsf{G}'(K) \to \mathsf{G}(K) \to B(\mathsf{Z}) \to B(\mathsf{G}') \to B(\mathsf{G}).$$

So the hypothesis implies that $B(\mathsf{Z}) \to B(\mathsf{G}')$ is injective. By the Kottwitz isomorphism this implies that

$$Z(\widehat{\mathsf{G}'})^{\mathrm{Gal}_K} \to \widehat{\mathsf{Z}}^{\mathrm{Gal}_K}$$

is surjective. In particular

$$\widehat{\mathsf{G}'}^{\mathrm{Gal}_K} \to \widehat{\mathsf{Z}}^{\mathrm{Gal}_K}$$

is surjective. Now the short exact sequence $1 \to \widehat{\mathsf{G}} \to \widehat{\mathsf{G}'} \to \widehat{\mathsf{Z}} \to 1$ induces another long exact sequence

$$1 \to \widehat{\mathsf{G}}^{\mathrm{Gal}_K} \to \widehat{\mathsf{G}'}^{\mathrm{Gal}_K} \to \widehat{\mathsf{Z}}^{\mathrm{Gal}_K} \to \Phi(\mathsf{G}) \to \Phi(\mathsf{G}') \to \Phi(\mathsf{Z}).$$

which implies that $\Phi(\mathsf{G}) \to \Phi(\mathsf{G}')$ is injective. $\square$

7.3. **Strong Kottwitz conjecture.** We now revisit the Kottwitz conjecture discussed in §3.2 and §3.4. Following [MHN24], we combine the compatibility result and the Act-functors defined in §3.3 to describe the complexes of $G(K) \times W_{E_\mu}$-module $\mathrm{R}\Gamma(G, b, \mu)[\rho]$ without passing to the Grothendieck group and without the condition that μ is minuscule. We adopt the notation from §3, but without modulo outer automorphisms in Case O2, because we now have the unambiguous LLC Theorem 7.1.1. In particular, K is a non-Archimedean local field with residue characteristic p, and ℓ is a rational prime different from p with a fixed isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$.

We first recall the following general result of Hansen [Han20, Theorem 1.1].

Theorem 7.3.1. *Suppose* G *is a quasi-split reductive group over* $\mathbb{Q}_p$ *with a Borel pair* (B, T), $\mathsf{b}_0 \in B(\mathsf{G})_{\mathrm{bas}}$ *is a basic element,* μ *is a minuscule dominant cocharacter of* $\mathsf{G}_{\overline{\mathbb{Q}_p}}$, *and* $\mathsf{b} \in B(\mathsf{G}, \mathsf{b}_0, \mu^\bullet)$ *(see* (3.3)*), so* $(\mathsf{G}_{\mathsf{b}_0}, \mathsf{b}, \mu^\bullet)$ *is a local Shimura datum in the sense of* [RV14, Definition 5.1]*, and* $\rho \in \Pi(\mathsf{G}_\mathsf{b}, \overline{\mathbb{Q}_\ell})$. *Suppose the following conditions hold:*

- $\mathrm{Sht}(\mathsf{G}, \mathsf{b}, \mathsf{b}_0, \mu)$ *appears in the basic uniformization at* p *of a global Shimura variety in the sense of Theorem 5.1.1.*
- *The Fargues–Scholze L-parameter* ϕ_ρ^{FS} *is supercuspidal.*

Then the complex $\mathrm{R}\Gamma_c(\mathsf{G}, \mathsf{b}, \mathsf{b}_0, \mu^\bullet)[\rho]$ *is concentrated in middle degree, which is 0 under our normalization.*

Note that $\mathrm{Sht}(G^*, b_1, \mathbf{1}, \mu_1)$ appears in the basic uniformization of a global Shimura variety of Abelian type defined in §4.4, where $b_1 \in B(G^*)_{\mathrm{bas}}$ is the unique nontrivial basic element and μ_1 is defined in (2.5).

For the remainder of this subsection, fix a supercuspidal L-parameter $\phi \in \Phi_{\mathrm{sc}}(G^*)$ such that

$$\tilde{\phi}^{\mathrm{GL}} = \phi_1 + \ldots + \phi_k + \phi_{k+1} + \ldots + \phi_r,$$

where the ϕ_i are distinct irreducible representations of W_{K_1} of dimension d_i, with d_i odd if and only if $i \leq k$. Adopt the combinatorial notation on L-parameters introduced in §2.6. By the weak version of Kottwitz conjecture together with the preceding theorem, the following holds.

Corollary 7.3.2. *For each* $I \subset [r]_+$ *with* $\#I \equiv 1 (\mathrm{mod}\ 2)$, *there is an isomorphism*

$$\mathrm{R}\Gamma_c(G^*, b_1, \mathbf{1}, \mu_1)[\iota_\ell \pi_{[I]}] \cong \bigoplus_{i=1}^r d_i \iota_\ell \pi_{[I \oplus \{i\}]}$$

of complexes of representations of $G(K)$.

Proof. It follows from the main theorem Theorem A that $\iota_\ell^{-1} \phi_{\iota_\ell \pi_{[I]}}^{\mathrm{FS}} = \phi$ is supercuspidal, so Theorem 7.1.2 and Proposition 3.2.2 imply that

$$\left[\mathrm{R}\Gamma_c(G^*, b_1, \mathbf{1}, \mu_1)[\iota_\ell \pi_{[I]}] \right] = \sum_{i=1}^r d_i [\iota_\ell \pi_{[I \oplus \{i\}]}] \in \mathrm{K}_0(G, \overline{\mathbb{Q}_\ell}).$$

Moreover, Theorem 7.3.1 implies that $\mathrm{R}\Gamma_c(G^*, b_1, \mathbf{1}, \mu_1)[\iota_\ell \pi_{[I]}]$ is concentrated in degree 0, so it has a filtration with subquotients given by $\iota_\ell \pi_{[I \oplus \{i\}]}$, where $i \in [r]_+$. By Corollary 2.5.2, $\pi_{[I \oplus \{i\}]} \in \Pi(G)$ is supercuspidal for each $i \in [r]_+$. Since supercuspidal representations are injective and projective in the category of smooth representations with a fixed central character, this filtration must split, and the assertion follows. $\square$

These Act-functors defined in §3.3 are related to the cohomology of local shtuka spaces via the following result of Fargues–Scholze [FS24, §X.2] and Hamann [Ham22, Corollary 3.11]. Recall that we write $\phi^\natural : W_K \to {}^L G$ for the L-parameter corresponding to the supercuspidal L-parameter $\phi \in \Phi^{\mathrm{sc}}(G)$; see §2.2.

Theorem 7.3.3. *Suppose* $b_0 \in B(G^*)_{\mathrm{bas}}$ *is a basic element,* μ *is a dominant cocharacter of* $G^*_{\overline{\mathbb{Q}_p}}$, *and* $b \in B(G^*, b_0, \mu)$, *the highest weight tilting module* $\mathcal{T}_\mu$ *of* $\widehat{G}$ *naturally extends to a representation* $\mathcal{T}_\mu$ *of* $\widehat{G} \rtimes W_{E_\mu}$ *as defined in* [Kot84, Lemma 2.1.2]*, then for each* $\rho \in \Pi_\phi(G^*_b)$ *there exists an isomorphism*

$$\mathrm{R}\Gamma_c(G^*, b, b_0, \mu)[\iota_\ell \rho] \cong \bigoplus_{\eta \in \mathrm{Irr}(\mathfrak{S}_\phi)} \mathrm{Act}_\eta(\iota_\ell \rho) \boxtimes \iota_\ell \mathrm{Hom}_{\mathfrak{S}_\phi}\left(\eta, \mathcal{T}_\mu \circ \left(\phi^\natural|_{W_{E_\mu}} \right) \right)$$

as $G^*_{b_0}(K) \times W_{E_\mu}$*-modules.*

Combining this strong Kottwitz conjecture with the monoidal property of Act-functors, we deduce the analogous results for non-minuscule μ, extending [Ham22, Theorem 8.2] and [MHN24, Theorem 4.6, Theorem 4.22] to special orthogonal groups and unitary groups.

Theorem 7.3.4. *For each* $I \subset [r]_+$, *there exists a bijection of multisets*

(7.7) $$\{\mathrm{Act}_{[\{x\}]}(\pi_{[I]})\}_{x \in [r]_+} \cong \{\pi_{[I \oplus \{x\}]}\}_{x \in [r]_+}.$$

We choose a permutation $\sigma_\varnothing$ *of* $[r]_+$ *such that*

$$\mathrm{Act}_{[\{x\}]}(\pi_{[\varnothing]}) \cong \pi_{[\{\sigma_\varnothing(x)\}]}$$

for each $x \in [r]_+$. For each $b_0 \in B(G^*)_{\text{bas}}$ and dominant cocharacter μ of $G^*_{\overline{K}}$ with reflex field E_μ, if $b \in B(G^*, b_0, \mu)_{\text{bas}}$, we write

$$\mathcal{T}_\mu \circ \left(\phi^\natural|_{W_{E_\mu}} \right) = \bigoplus_{j=1}^m \eta_{[I_j^\mu]} \boxtimes \sigma_j^\mu$$

as a sum of irreducible representations of $\mathfrak{S}_\phi \times W_{E_\mu}$. Then for any $I \subset [r]_+$ with $\#I \equiv \frac{\kappa_b(-1)-1}{2} (\text{mod } 2)$, there is an isomorphism of $G^*_{b_0}(K) \times W_{E_\mu}$-modules

$$(7.8) \qquad \iota_\ell^{-1} \mathrm{R}\Gamma_c(G^*, b, b_0, \mu)[\iota_\ell \pi_{[I]}] \cong \bigoplus_{j=1}^m \pi_{[I \oplus \sigma_\varnothing(I_j^\mu)]} \boxtimes \sigma_j^\mu.$$

Remark 7.3.5. The theorem implies that, if Equation (7.8) is known for any (b, b_0, μ) with $\mu = \mu_1$ and any $I \subset [r]_+$ with $\#I \equiv \frac{\kappa_b(-1)-1}{2} (\text{mod } 2)$, then it is known for any (b, b_0, μ) and any $I \subset [r]_+$ with $\#I \equiv \frac{\kappa_b(-1)-1}{2} (\text{mod } 2)$. We expect that this can be shown using the basic uniformization and more careful analysis of cohomology of global Shimura varieties.

Proof. If we apply Theorem 7.3.3 to $b = b_1, b_0 = \mathbf{1}$ and $\mu = \mu_1$, we get an isomorphism

$$\mathrm{R}\Gamma_c(G^*, b_1, \mathbf{1}, \mu_1)[\pi_{[I]}] \cong \bigoplus_{j=1}^r \iota_\ell \mathrm{Act}_{[j]}(\pi_{[I]}) \boxtimes \phi_i$$

of representations of $G^*_{b_1}(K) \times W_{K_1}$ for each $[I] \subset \mathscr{P}([r]_+)/\sim_k$ with $\#[I] \equiv 1 (\text{mod } 2)$. So it follows from Corollary 7.3.2 and Schur's lemma (noticing that each $\mathrm{Act}_{[j]}(\pi_{[I]})$ is irreducible) that there exists a bijection of multisets

$$(7.9) \qquad \{\mathrm{Act}_{[\{x\}]}(\pi_{[I]})\}_{x \in [r]_+} \cong \{\pi_{[I] \oplus [\{x\}]}\}_{x \in [r]_+}$$

(The only subtlety comes from the case when $k = 2$, but it is easy to verify that this bijection holds in this case). Moreover, it follows from the the monoidal property of Act (see §3.3) that

$$\mathrm{Act}_{[\{j\}]}(\pi_{[I] \oplus \sigma_{[I]}([\{j\}])}) \cong \mathrm{Act}_{[\{j\}]} \circ \mathrm{Act}_{[\{j\}]}(\pi_{[I]}) \cong \mathrm{Act}_{[\varnothing]}(\pi_{[I]}) \cong \pi_{[I]}.$$

So for each $[I'] \subset \mathscr{P}([r]_+)/\sim_k$ with $\#[I'] \equiv 0 (\text{mod } 2)$ and $j \in [r]_+$, it follows from the bijection (7.9) for $[I] = [I'] \oplus [\{j\}]$ that there exists $x_j \in [r]_+$ such that $\mathrm{Act}_{[\{x_j\}]}(\pi_{[I]}) \cong \pi_{[I'] \oplus [\{j\}]}$. We then get a bijection of multisets

$$(7.10) \qquad \{\mathrm{Act}_{[\{x_j\}]}(\pi_{[I']})\}_{j \in [r]_+} \cong \{\pi_{[I'] \oplus [\{j\}]}\}_{j \in [r]_+}.$$

It follows from numerical counting that the left-hand side must equal the multiset $\{\mathrm{Act}_{[\{j\}]}(\pi_{[I']})\}_{j \in [r]_+}$, and there is a bijection of multisets

$$(7.11) \qquad \{\mathrm{Act}_{[\{j\}]}(\pi_{[I']})\}_{j \in [r]_+} \cong \{\pi_{[I'] \oplus [\{j\}]}\}_{j \in [r]_+}.$$

Now the first assertion follows from (7.9) or (7.11) depending on the cardinality of I.

For the second assertion, by Theorem 7.3.3, it suffices to show that $\mathrm{Act}_{[I]}(\pi_{[J]}) = \pi_{[I \oplus \sigma_\varnothing(J)]}$ for any $I, J \subset [r]_+$. By the monoidal property of Act-functors §3.3, it suffices to prove for any $I \subset [r]_+$ that

$$(7.12) \qquad \mathrm{Act}_{[I]}(\pi_{[\varnothing]}) = \pi_{[\sigma_\varnothing(I)]}.$$

Now for any $I \subset [r]_+$, we use induction on $\#I$ to show (7.12). When $\#I = 0$, this follows from §3.3, and when $\#I = 1$ this follows from the hypothesis. Now suppose $\#I \geq 2$, then for each $i \in I$, it follows from the induction hypothesis and the monoidal property that

$$(7.13) \qquad \mathrm{Act}_{[I]}(\pi_{[\varnothing]}) \cong \mathrm{Act}_{[\{i\}]} \circ \mathrm{Act}_{[I \smallsetminus \{i\}]}(\pi_{[\varnothing]}) \cong \mathrm{Act}_{[\{i\}]}(\pi_{[\sigma_\varnothing(I \smallsetminus \{i\})]}).$$

Recall from (7.11) that there is a bijection of multisets

$$\{\mathrm{Act}_{[\{j\}]}(\pi_{[\sigma_\varnothing(I \smallsetminus \{i\})]})\}_{j \in [r]_+} \cong \{\pi_{[I_0] \oplus [\sigma_\varnothing(I \smallsetminus \{i\})] \oplus [\{\sigma_\varnothing(j)\}]}\}_{j \in [r]_+},$$

but for each $j \in I \smallsetminus \{i\}$, we also have

$$\mathrm{Act}_{[\{j\}]}(\pi_{[\sigma_\varnothing(I \smallsetminus \{i\})]}) \cong \pi_{[\sigma_\varnothing(I \smallsetminus \{i,j\})]} \cong \pi_{[\sigma_\varnothing(I \smallsetminus \{i\})] \oplus [\{\sigma_\varnothing(j)\}]}$$

by induction hypothesis. So we obtain a bijection of multisets

$$(7.14) \qquad \{\mathrm{Act}_{[\{j\}]}(\pi_{[\sigma_\varnothing(I \smallsetminus \{i\})]})\}_{j \in [r]_+ \smallsetminus (I \smallsetminus \{i\})} \cong \{\pi_{[\sigma_\varnothing(I \smallsetminus \{i\})] \oplus [\{\sigma_\varnothing(j)\}]}\}_{j \in [r]_+ \smallsetminus (I \smallsetminus \{i\})}.$$

When i runs through I, the intersection of the left-hand side of (7.14) contains $\mathrm{Act}_{[I]}(\pi_{[\varnothing]})$ by (7.13), and the intersection of the right-hand side of (7.14) is a multiset supported on the single element $\pi_{[\sigma_\varnothing(I)]}$. So there is an isomorphism $\mathrm{Act}_{[I]}(\pi_{[\varnothing]}) \cong \pi_{[\sigma_\varnothing(I)]}$, and the induction process is finished. $\qquad\square$

In particular, understanding the part of the cohomology of local shtuka spaces with supersingular L-parameters is reduced to understanding the decomposition of $\mathcal{T}_\mu \circ \left(\phi^\natural|_{W_{E_\mu}} \right)$ as $\mathfrak{S}_\phi \times W_{E_\mu}$-modules, up to permutation of the L-packet $\Pi_\phi(G)$.

Moreover, Theorem 7.3.4 is more naturally understood in terms of eigensheaves. We recall from [Far16, Conjecture 4.4] that a *Hecke eigensheaf* for ϕ is an object $\mathcal{G}_\phi \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{G^*}, \overline{\mathbb{Q}_\ell})$ such that for any finite index set I and $(V, r_V) \in \mathrm{Rep}_{\overline{\mathbb{Q}_\ell}}(^L G^{*I})$, there exists an isomorphism

$$(7.15) \qquad \eta_{V,I} : T_V(\mathcal{G}_\phi) \xrightarrow{\sim} \mathcal{G}_\phi \boxtimes r_V \circ \phi \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{G^*}, \overline{\mathbb{Q}_\ell})^{BW_{\mathbb{Q}_K}}$$

that is natural in I and V and compatible with compositions of Hecke operators.

Following [MHN24, §4.1.3], we construct eigensheaves attached to the supercuspidal L-parameters ϕ:

Theorem 7.3.6. *Set*

$$\mathcal{G}_\phi := \bigoplus_{\eta \in \mathrm{Irr}(\mathfrak{S}_\phi)} \mathrm{Act}_\eta(\pi_{[\{\varnothing\}]}) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{G^*}, \overline{\mathbb{Q}_\ell}),$$

and let $\mathfrak{S}_\phi$ acts on $\mathcal{G}_\phi$ by acting via $\eta \in \mathrm{Irr}(\phi)$ on $\mathrm{Act}_\eta(\pi)$.

- *$\mathcal{G}_\phi$ is supported on $B(G^*)_{\mathrm{bas}} \subset |\mathrm{Bun}_{G^*}|$, i.e., the semi-stable locus of Bun_{G^*}.*
- *For each $b \in B(G^*)_{\mathrm{bas}}$, under the natural identification $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{G^*}^b, \overline{\mathbb{Q}_\ell})$ with $\mathrm{D}_{\mathrm{lis}}(G_b^*, \overline{\mathbb{Q}_\ell})$, there exists an isomorphism*

$$i_b^* \mathcal{G}_\phi \cong \bigoplus_{\substack{[I] \in \mathscr{P}([r]_+)/\sim_k, \\ \#I \equiv \frac{\kappa_b(-1)-1}{2} (\mathrm{mod}\ 2)}} \eta_{[I]} \boxtimes \pi_{[\sigma_\varnothing(I)]}$$

  of representations of $\mathfrak{S}_\phi \times G_b(K)$.
- *$\mathcal{G}_\phi$ is a Hecke eigensheaf for ϕ, i.e., (7.15) holds.*

Proof. These assertions follow from the bijection (7.7) and the symmetric monoidal property of the Act-functors defined in §3.3 the same way as in the proof of [MHN24, Proposition 4.18, Theorem 4.19], and is omitted here. $\qquad\square$

8. A VANISHING RESULT FOR TORSION COHOMOLOGY OF SHIMURA VARIETIES

We use the compatibility result to prove a vanishing result for the generic part of the cohomology of orthogonal or unitary Shimura varieties with torsion coefficients.

We first recall the general torsion vanishing conjecture of [Han23, Ham24]. Let $(\mathbb{G}, \mathbb{X})$ be a Shimura datum with reflex field $E \subset \mathbb{C}$ (which is a number field), and p be a rational prime coprime to $2 \cdot \#\pi_1([\mathbb{G}, \mathbb{G}])$, with a fixed isomorphism $\iota_p : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_p}$. The isomorphism ι_p induces a place $\mathfrak{p}$ of E over p, and we write $\mathbb{C}_p$ for the completion of the algebraic closure of $E_\mathfrak{p} \subset \overline{\mathbb{Q}_p}$. We write $G := \mathbb{G} \otimes \mathbb{Q}_p$. Let ℓ be a rational prime that is coprime to $p \cdot \#\pi_0(Z(G))$, with a fixed isomorphism $\iota_\ell : \mathbb{C} \to \overline{\mathbb{Q}_\ell}$, which fixes a square root $\sqrt{p} \in \overline{\mathbb{Z}_\ell}$ thus also $\sqrt{p} \in \overline{\mathbb{F}_\ell}$. Let $\Lambda \in \{\overline{\mathbb{Q}_\ell}, \overline{\mathbb{F}_\ell}\}$. Whenever we consider $\overline{\mathbb{F}_\ell}$-coefficients, we assume that $\pi_0(Z(G))$ is invertible in Λ to avoid complications in this ℓ-modular setting.

For neat compact open subgroup $\mathcal{K} \le \mathbb{G}(\mathbf{A}_f)$, let $\mathcal{S}_\mathcal{K}(\mathbb{G}, \mathbb{X})$ be the adic space over $\mathrm{Spa}(E_\mathfrak{p})$ associated to the Shimura variety $\mathrm{Sh}_\mathcal{K}(\mathbb{G}, \mathbb{X})$. If $\mathcal{K}^p \le \mathbb{G}(\mathbf{A}_f^p)$ is a neat compact open subgroup, we define

$$\mathcal{S}_{\mathcal{K}^p}(\mathbb{G}, \mathbb{X}) := \varprojlim_{\mathcal{K}_p} \mathcal{S}_{\mathcal{K}_p \mathcal{K}^p}(\mathbb{G}, \mathbb{X}),$$

where $\mathcal{K}_p$ runs through all open compact subgroups of $\mathbb{G}(\mathbb{Q}_p)$.

The $G(\mathbb{Q}_p) \times W_{E_\mathfrak{p}}$-representation on

$$\mathrm{R\Gamma}_c(\mathcal{S}(\mathbb{G}, \mathbb{X})_{\mathcal{K}^p, \mathbb{C}_p}, \Lambda)$$

decomposes as

$$\mathrm{R\Gamma}_c(\mathcal{S}_{\mathcal{K}^p}(\mathbb{G}, \mathbb{X})_{\mathbb{C}_p}, \Lambda) = \bigoplus_\phi \mathrm{R\Gamma}_c(\mathcal{S}_{\mathcal{K}^p}(\mathbb{G}, \mathbb{X})_{\mathbb{C}_p}, \Lambda)_\phi$$

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according to Fargues–Scholze parameters of irreducible subquotients, where ϕ runs through semisimple L-parameters $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}; \Lambda)$; see [HL24, Corollary 4.3].

We now recall the concept of (weakly) Langlands–Shahidi type L-parameters, as defined in [Ham24, Definition 6.2].

Definition 8.0.1. Suppose G is a general quasi-split reductive group over a non-Archimedean local field K of characteristic zero with a Borel pair (B, T), and $\phi_\mathsf{T} \in \Phi^{\mathrm{ss}}(\mathsf{T}, \Lambda)$ is a semisimple L-parameter. We write $\phi_\mathsf{T}^\vee$ for the Chevalley dual of ϕ_T, then

- we say ϕ is *generic* (or of Langlands–Shahidi type) if the Galois cohomologies

$$\mathrm{R\Gamma}(W_K, {}^L\mathcal{T}_\mu \circ \phi_\mathsf{T}), \quad \mathrm{R\Gamma}(W_K, {}^L\mathcal{T}_\mu \circ \phi_\mathsf{T}^\vee)$$

are both trivial for each dominant coroot $\mu \in \Phi^\vee(\mathsf{G}, \mathsf{T})^+ \subset X_\bullet(\mathsf{T})$.[10]

- we say ϕ is of *weakly Langlands–Shahidi type* if the Galois cohomology groups

$$\mathrm{H}^2(W_K, {}^L\mathcal{T}_\mu \circ \phi_\mathsf{T}), \quad \mathrm{H}^2(W_K, {}^L\mathcal{T}_\mu \circ \phi_\mathsf{T}^\vee)$$

are both trivial for each dominant coroot $\mu \in \Phi^\vee(\mathsf{G}, \mathsf{T})^+ \subset X_\bullet(\mathsf{T})$.

By [HL24, Remark 6.3], these conditions depend only on the image ϕ of ϕ_T under the natural embedding ${}^L\mathsf{T} \to {}^L\mathsf{G}$, which we call semisimple toral L-parameters for G. For a semisimple toral L-parameter ϕ, we also write $\phi^\vee$ for the image of $\phi_\mathsf{T}^\vee$ under the natural embedding ${}^L\mathsf{T} \to {}^L\mathsf{G}$.

Moreover, it follows from [HL24, Lemma 4.21] that: For a finite splitting field extension K'/K for G, ϕ is generic (resp. of weakly Langlands–Shahidi type) if and only if $\phi|_{W_{K'}}$ is. Moreover, by the local Tate duality, genericity for ϕ is equivalent to $\alpha \circ \phi|_{W_{K'}}$ not equaling $\mathbf{1}$ or $|-|_{K'}^{\pm 1}$ for each coroot α of G.

Going back to the global situation, we recall the following conjecture by Hamann and Lee [HL24, Conj. 6.6] on vanishing of cohomology of Shimura varieties with torsion coefficients:

Conjecture 8.0.2. Let $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}; \overline{\mathbb{F}_\ell})$ be a semisimple toral L-parameter of weakly Langlands–Shahidi type, then the complex $\mathrm{R\Gamma}_c(\mathcal{S}(\mathbb{G}, \mathbb{X})_{\mathcal{K}^p, \mathbb{C}_p}, \overline{\mathbb{F}_\ell})_\phi$ (resp. $\mathrm{R\Gamma}(\mathcal{S}(\mathbb{G}, \mathbb{X})_{\mathcal{K}^p, \mathbb{C}_p}, \overline{\mathbb{F}_\ell})_\phi$) is concentrated in degrees $0 \le i \le \dim_\mathbb{C}(\mathbb{X})$ (resp. $\dim_\mathbb{C}(\mathbb{X}) \le i \le 2\dim_\mathbb{C}(\mathbb{X})$).

Remark 8.0.3. Suppose $F^+ \ne \mathbb{Q}$ is a totally real field and $\mathbb{G} = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{U}(n, n)$ is the Weil restriction of a quasi-split unitary group of even rank, and we assume that $\mathbb{G}$ splits at p, i.e.,

$$\mathbb{G} \otimes \mathbb{Q}_p \cong \prod_{i=1}^{[F^+:\mathbb{Q}]} \mathrm{GL}(2n)_{\mathbb{Q}_p}.$$

Then the conjecture is true for any semisimple toral L-parameter $\phi \in \Phi^{\mathrm{ss}}(\mathbb{G} \otimes \mathbb{Q}_p, \overline{\mathbb{F}_\ell})$ of weakly Langlands–Shahidi type by [CS24, Theorem 1.1]. Note that if ϕ is an unramified character $\phi = \mathrm{diag}(\chi_1, \ldots, \chi_{2n}) \in \Phi(\mathrm{GL}(2n)_{\mathbb{Q}_p}, \overline{\mathbb{F}_\ell})$, then ϕ is of weakly Langlands–Shahidi type if and only if $\chi_i \ne \chi_j \otimes |-|_{\mathbb{Q}_p}$ for any $i \ne j \in [2n]_+$,

8.1. **Generic semisimple L-parameters.** In this subsection, we study generic semisimple toral L-parameters. We import Setup 7.2.2. In particular, F is a totally real number field unramified at a prime p, $\mathbf{G}$ is a special orthogonal or unitary group over F with $G^* = \mathbf{G} \otimes_F (F \otimes \mathbb{Q}_p)$, and $G^\sharp$ is a central extension of $\mathbf{G}$ with $G^\sharp$. We consider the Hodge cocharacter $\mu^\sharp$ of $\mathbf{G}^\sharp$ corresponding to the Deligne homomorphism $h_0^\sharp$ (see (4.2)). When viewed as a cocharacter of

$$(8.1) \qquad G_{\overline{\mathbb{Q}_p}}^\sharp \cong \begin{cases} \mathrm{GL}(1)_{\overline{\mathbb{Q}_p}} \times \left(\mathrm{Res}_{K/\mathbb{Q}_p} \mathrm{GSpin}(V^*)\right)_{\overline{\mathbb{Q}_p}} & = \mathrm{GL}(1)_{\overline{\mathbb{Q}_p}} \times \prod\limits_{v \in \mathrm{Hom}(K, \overline{\mathbb{Q}_p})} \mathrm{GSpin}(V^* \otimes_{K,v} \overline{\mathbb{Q}_p}) \\ \text{in Case O} \\ \mathrm{GL}(1)_{\overline{\mathbb{Q}_p}} \times \left(\mathrm{Res}_{K/\mathbb{Q}_p} \mathrm{GU}(V^*)\right)_{\overline{\mathbb{Q}_p}} & = \mathrm{GL}(1)_{\overline{\mathbb{Q}_p}} \times \prod\limits_{v \in \mathrm{Hom}(K, \overline{\mathbb{Q}_p})} \mathrm{GU}(V^* \otimes_{K_1,v} \overline{\mathbb{Q}_p}) \\ \text{in Case U} \end{cases}$$

via the isomorphism ι_p, it is the inverse of the identity map on the $\mathrm{GL}(1)$-factor and only nontrivial on one another factor, where it is a lift $\mu_1^\sharp$ of the dominant inverse of the cocharacter μ_1 of $G_{\overline{K}}^* = \mathrm{U}(V^*)_{\overline{K}}^\circ$ defined in (2.5).

---

[10]Recall that ${}^L T_\mu \in \mathrm{Rep}_\Lambda({}^L\mathsf{T})$ is the extended highest weight tilting module attached to μ as defined in (3.6).

We will prove a special case of Conjecture 8.0.2 for generic toral parameters ϕ with some other conditions called "regularity"; cf. [HL24, Definition 4.12].

Definition 8.1.1. For a quasi-split reductive group G over a p-adic number field K with a Borel pair (B, T), a generic toral semisimple L-parameter $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{F}_\ell})$ is called *regular* if the character χ of $\mathsf{T}(K)$ attached to ϕ_T via LLC satisfies $\chi \not\cong \chi^w$ for any nontrivial element $w \in W_\mathsf{G}$. On the other hand, it is called *weakly normalized regular* if

$$\chi \otimes \delta_\mathsf{B}^{1/2} \not\cong (\chi \otimes \delta_\mathsf{B}^{-1/2})^w$$

and

$$\chi^{w_0} \otimes \delta_\mathsf{B}^{1/2} \not\cong (\chi^{w_0} \otimes \delta_\mathsf{B}^{-1/2})^w$$

for any nontrivial element $w \in W_\mathsf{G}$. Here we write w_0 for the longest-length Weyl group element.

We have the following lemma showing that to regularity or weakly normalized regularity can be checked after base change:

Lemma 8.1.2. *Suppose G is a quasi-split reductive group over a over a p-adic number field K with a Borel pair (B, T), and K'/K is a finite extension field. If $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{F}_\ell})$ is a semisimple toral L-parameter, then ϕ is regular (resp. weakly normalized regular) if the restriction of ϕ to $W_{K'}$ is.*

Proof. By local class field theory, if χ is the character of $\mathsf{T}(K)$ corresponding to ϕ_T, then $\chi \circ \mathrm{Nm}_{K'/K}$ is the character of $\mathsf{T}(K')$ corresponding to $\phi_\mathsf{T}|_{W_{K'}}$. By the definition of δ_B as the left multiplication transformation factor of a right Haar measure on $\mathsf{B}(K)$, we see that $\delta_{\mathsf{B}_{K'}} = \delta_\mathsf{B} \circ \mathrm{Nm}_{K'/K}$. Now it is clear that is ϕ is not regular (resp. weakly normalized regular), then neither is $\phi|_{W_{K'}}$. $\qquad\square$

And for later use, we prove the following result regarding regularity of L-parameters and central extensions:

Lemma 8.1.3. *Suppose*

$$1 \to Z \to \mathsf{G}^\sharp \to \mathsf{G} \to 1$$

is a central extension of quasi-split reductive groups over a non-Archimedean local field K of characteristic zero, and we assume either this extension splits or Z is an induced torus. Let $(\mathsf{B}^\sharp, \mathsf{T}^\sharp)$ be a Borel pair of $\mathsf{G}^\sharp$ with image (B, T) in G. If $\phi \in \Pi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{F}_\ell})$ is a semisimple toral L-parameter which may be regarded as a semisimple toral L-parameter $\phi^\sharp \in \Phi^{\mathrm{ss}}(\mathsf{G}^\sharp, \overline{\mathbb{F}_\ell})$ for $\mathsf{G}^\sharp$ via the canonical embedding ${}^L\mathsf{G}(\overline{\mathbb{F}_\ell}) \to {}^L\mathsf{G}^\sharp(\overline{\mathbb{F}_\ell})$, then $\phi^\sharp$ is regular (resp. weakly normalized regular) if and only if ϕ is.

Proof. Note that $\mathsf{T}^\sharp(K) \to \mathsf{T}(K)$ is surjective by Shapiro's lemma, and $W_{\mathsf{G}^\sharp} \cong W_\mathsf{G}$. So the assertion is clear from the definition. $\qquad\square$

We then have the following result generalizing [HL24, Lemma 4.22] to special orthogonal groups and unitary groups:

Lemma 8.1.4. *If $\phi \in \Phi^{\mathrm{ss}}(\mathrm{Res}_{K/\mathbb{Q}_p} G^*, \overline{\mathbb{F}_\ell})$ is a generic semisimple toral L-parameter, then ϕ is regular. Furthermore, if we regard ϕ as a semisimple toral L-parameter $\phi^\sharp \in \Phi^{\mathrm{ss}}(G^\sharp, \overline{\mathbb{F}_\ell})$ via the natural embedding ${}^L(\mathrm{Res}_{K/\mathbb{Q}_p} G^*)(\overline{\mathbb{F}_\ell}) \to {}^L G^\sharp(\overline{\mathbb{F}_\ell})$, then it is also regular for $G^\sharp(\overline{\mathbb{F}_\ell})$.*

Proof. By Lemma 8.1.3, the second assertion follows from the first. For the first assertion, by base change Lemma 8.1.2 and [HL24, Lemma 4.21], it suffices to prove for the split special orthogonal group $\mathrm{SO}(d(G^*))_K$ (in Case O) and the general linear group $\mathrm{GL}(d(G^*))_{K_1}$ (in Case U). Under the isomorphism

$$\mathrm{GL}(1)^{n(G^*)} \cong T^* : (t_1, \ldots, t_{n(G^*)}) \mapsto \begin{cases} \mathrm{diag}(t_1, \ldots, t_{n(G^*)}) & \text{in Case U} \\ \mathrm{diag}(t_1, \ldots, t_{n(G^*)}, 1, t_{n(G^*)}^{-1}, \ldots, t_1^{-1}) & \text{in Case O1} \\ \mathrm{diag}(t_1, \ldots, t_{n(G^*)}, t_{n(G^*)}^{-1}, \ldots, t_1^{-1}) & \text{in Case O2} \end{cases},$$

W_{G^*} is the semi-direct product of the group $\mathrm{Sym}_{n(G^*)}$ acting by permutation on the group $\{\pm 1\}^{n(G^*)}$ in Case O1, and is the semi-direct product of the group $\mathrm{Sym}_{n(G^*)}$ acting by permutation on the kernel of the determinant map $\det : \{\pm 1\}^{n(G^*)} \to \{\pm 1\} : (\epsilon_1, \ldots, \epsilon_{n(G^*)}) \mapsto \prod_i \epsilon_i$ in Case O2.

For each $i \in [n(G^*)]_+$, denote by ε_i the cocharacters $\mathrm{GL}(1) \to T^* : t \mapsto (1, \ldots, t, 1, \ldots)$ where the t is at the i-th coordinate; and set $\varepsilon_{-i} := -\varepsilon_i$.

We write $\chi = \chi_1 \boxtimes \ldots \boxtimes \chi_{n(G^*)}$, and write $\chi_{-i} := \chi_i^{-1}$ for $i \in [n(G^*)]$. Suppose for the contrary that

$$\chi = \chi^w \tag{8.2}$$

for some nontrivial element $w \in W_{G^*}$. Suppose ε_j is not fixed by W_{G^*}, and we write $\varepsilon_j^w = \varepsilon_k$, then we evaluate Equation (8.2) at ε_j to get

$$\chi_j \chi_k^{-1} = |-|_K^{k-j}.$$

And this contradicts generality because $\varepsilon_j - \varepsilon_i$ is a root of $\mathrm{SO}(d(G^*))$ or $\mathrm{GL}(d(G^*))$. $\qquad\square$

In the setting of Definition 8.1.1, the point of this weakly normalized regular condition is the following: if ϕ is weakly normalized regular, then by [Ham24, Theorem 10.10] there exists an object

$$\mathrm{nEis}(\mathcal{S}_{\phi_T}) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{F}_\ell}),$$

which is a perverse filtered Hecke eigensheaf on Bun_G with eigenvalue ϕ in the sense of [Ham24, Corollary 7.4]. In particular, if μ is a dominant cocharacter of $G_{\overline{K}}$ with extended highest weight tilting module $\mathcal{T}_\mu \in \mathrm{Rep}_{\overline{\mathbb{F}_p}}(^L G)$ as defined in (3.6), and T_μ is the Hecke operator attached to $^L\mathcal{T}_\mu$ as defined in §3.1, then $T_\mu(\mathcal{S}_{\phi_T})$ admits a W_K-equivariant filtration indexed by Gal_K-orbits $\mathrm{Gal}_K.\nu_i$ in $X_\bullet(T)$; the graded piece indexed by $\mathrm{Gal}_K.\nu_i$ is

$$\mathrm{nEis}(\mathcal{S}_{\phi_T}) \otimes (^L\mathcal{T}_{\nu_i} \circ \phi_T) \otimes {}^L\mathcal{T}_\mu[\mathrm{Gal}_K.\nu_i].$$

Moreover, when this filtration splits, there is an isomorphism

$$T_\mu(\mathcal{S}_{\phi_T}) \cong \mathcal{S}_{\phi_T} \otimes {}^L\mathcal{T}_\mu \circ \phi$$

of sheaves in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^{BW_K}$. We say that ϕ is μ-*regular* if this filtration splits. By [Ham24, Theorem 1.17], this μ-regularity condition is implied by the following strongly μ-regularity condition (cf. [HL24, Definition 4.13]).

Definition 8.1.5. For a quasi-split reductive group G over a non-Archimedean local field K of characteristic zero with a Borel pair (B, T) and a dominant cocharacter μ of $G_{\overline{K}}$, a toral semisimple toral L-parameter $\phi \in \Phi^{\mathrm{ss}}(G, \overline{\mathbb{F}_\ell})$ is called *strongly μ-regular* if the Galois cohomologies

$$\mathrm{R\Gamma}(W_{\mathbb{Q}_p}, {}^L\mathcal{T}_{\nu-\nu'} \circ \phi_T)$$

are trivial for any ν, ν' defining distinct Gal_K-orbits of weights in the extended highest weight tilting module $^L\mathcal{T}_\mu$ attached to μ.

And for later use, we prove the following result regarding strongly μ-regularity of L-parameters and central extensions:

We then have the following result generalizing [HL24, Lemma 4.22] to special orthogonal groups and unitary groups:

Lemma 8.1.6. *Suppose $\ell \neq 2$ and $(\ell, n!) = 1$ if we are in Case O1. If $\phi \in \Phi^{\mathrm{ss}}(G^*, \overline{\mathbb{F}_\ell})$ is a generic semisimple toral L-parameter, and $\mu \in X_\bullet(G^*)$ is a dominant cocharacter, then ϕ is μ-regular.*

Similarly, if $\phi^\sharp \in \Phi^{\mathrm{ss}}(G^\sharp, \overline{\mathbb{F}_\ell})$ is a generic semisimple toral L-parameter, and $\mu^\sharp \in X_\bullet(G^\sharp)$ is a dominant cocharacter such that $(\ell, n!) = 1$ if we are in Case O1, then $\phi^\sharp$ is $\mu^\sharp$-regular.

In particular, $\mu^\sharp$ can be chosen to be not fixed by any nontrivial element of $W_{G^\sharp}$.

Proof. By base change [HL24, Lemma 4.22] and the isomorphism (8.1), it suffices to prove for the general linear group $\mathrm{GL}(n)$, the split special orthogonal group $\mathrm{SO}(d(G))$ and the split general spin group $\mathrm{GSpin}(d(G))$. For general linear groups (i.e., in Case U) this is established in [HL24, Lemma 4.22], and the argument also works for $G \in \{\mathrm{SO}(d(G)), \mathrm{GSpin}(d(G))\}$:

The standard representation $\widehat{\mathrm{Std}}$ of $\mathrm{Sp}(2n(G^*))$ or $\mathrm{SO}(2n(G^*))$ has weights given by Weyl orbits of ω_1. The standard representation extends to a standard representation of $\mathrm{GSp}(2n(G^*))$ or $\mathrm{GSO}(2n(G^*))$, and we write the highest weight of the standard representation of $\mathrm{GSp}(2n(G^*))$ or $\mathrm{GSO}(2n(G^*))$ as $\omega_1^\sharp$. Then it is clear that difference of the weights appearing in $\widehat{\mathrm{Std}}$ are coroots of G, thus ϕ is strongly μ-regular and $\phi^\sharp$ is strongly $\mu^\sharp$-regular, by definition of genericity. Thus, they are also μ-regular (resp. $\mu^\sharp$-regular) by the proof of [Ham24, Theorem 10.10].

For other cocharacters, we first recall that with $\overline{\mathbb{Q}_\ell}$-coefficients, the highest weight tilting module $\mathcal{T}_{\omega_i}$ associated to $\omega_i \in X_\bullet(\mathrm{Sp}(2n(G^*)))$ is realized on the space of harmonic elements in $\wedge^i(\widehat{\mathrm{Std}})$ as defined in

[GW09, §5.5.2], and it extends to a standard representation of $\mathrm{GSp}(2n(G^*))$ with highest weight, denoted by $\omega_i^\sharp$, and the same is true with $\overline{\mathbb{F}_\ell}$-coefficients, where each highest weight tilting module associated to $\omega_i^\sharp$ of $\mathrm{GSp}(2n(G^*))$ is a direct sum of $\wedge^i(\widehat{\mathrm{Std}})$, by our assumption on ℓ; cf. [Jan03, pp. 286–287], [Ham24, §10.1, Appendix B.2]. Similarly, with $\overline{\mathbb{Q}_\ell}$-coefficients, $\wedge^i(\widehat{\mathrm{Std}})$ is isomorphic to the highest weight tilting module $\mathcal{T}_{\omega_i}$ associated to $\omega_i \in X_\bullet(\mathrm{SO}(2n(G^*)))$ for $1 \leq i \leq n(G^*) - 2$, and $\wedge^{n(G^*)-1}(\widehat{\mathrm{Std}})$ is isomorphic to the highest weight tilting module $\mathcal{T}_{\omega_{n(G^*)-1}+\omega_{n(G^*)}}$ associated to $\omega_{n(G^*)-1} + \omega_{n(G^*)} \in X_\bullet(\mathrm{SO}(2n(G^*)))$. On the other hand, $\wedge^{n(G^*)-1}(\widehat{\mathrm{Std}})$ is isomorphic to the direct sum of highest weight tilting modules $\mathcal{T}_{2\omega_{n(G^*)-1}}$ and $\mathcal{T}_{2\omega_{n(G^*)}}$ (see [GW09, Theorem 5.5.13]), and these extends to a standard representation of $\mathrm{GSp}(2n(G^*))$ with highest weight, denoted by

$$\omega_1^\sharp, \ldots, \omega_{n-2}^\sharp, \omega_{n-1}^\sharp + \omega_{n-2}^\sharp, 2\omega_{n-1}^\sharp, 2\omega_{n-2}^\sharp,$$

respectively. By our assumption on ℓ, the same is true with $\overline{\mathbb{F}_\ell}$-coefficients, where each highest weight tilting module associated to the above cocharacters of $\mathrm{GSp}(2n(G^*))$ is a direct sum of $\wedge^i(\widehat{\mathrm{Std}})$; cf. [Jan03, pp. 286–287], [Ham24, §10.1, Appendix B.2].

Now all these highest weight tilting modules of $G^\sharp$ with fundamental weights $\mu^\sharp$ appear as direct summand of tensor products of $\widehat{\mathrm{Std}}$, so $\phi^\sharp$ is $\mu^\sharp$-regular by [Ham24, Proposition 10.12].

Finally, we may choose $\mu^\sharp$ appropriately such that under the isomorphism (8.1)

$$G^\sharp_{\overline{\mathbb{Q}_p}} \cong \begin{cases} \mathrm{GL}(1)_{\overline{\mathbb{Q}_p}} \times \prod_{v \in \mathrm{Hom}(K, \overline{\mathbb{Q}_p})} \mathrm{GSpin}(V^* \otimes_{K,v} \overline{\mathbb{Q}_p}) & \text{in Case O} \\ \mathrm{GL}(1)_{\overline{\mathbb{Q}_p}} \times \prod_{v \in \mathrm{Hom}(K, \overline{\mathbb{Q}_p})} \mathrm{GU}(V^* \otimes_{K_1,v} \overline{\mathbb{Q}_p}) & \text{in Case U} \end{cases},$$

it is of the form $(0, \mu^{\sharp\prime}, \ldots, \mu^{\sharp\prime})$ (i.e., trivial on the $\mathrm{GL}(1)$ factor and identical on the other factors), where $\mu^{\sharp\prime}$ is not fixed by any nontrivial Weyl group element. Then $\mu^\sharp$ is not fixed by any nontrivial Weyl group element. $\qquad\square$

8.2. **Perverse t-exactness and vanishing results.** In this subsection, we prove a perverse t-exactness result for Hecke operators, and deduce a vanishing result for cohomology of Shimura varieties with torsion coefficients. We adopt the notation related to Bun_G from §3.1.

For any reductive group G over a non-Archimedean local field K of characteristic zero and any open substack $U \subset \mathrm{Bun}_\mathsf{G}$, there exists a perverse t-structure on $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_\mathsf{G}, \overline{\mathbb{F}_\ell})$ defined as follows [HL24, Definition 4.11]: For each $\mathsf{b} \in B(\mathsf{G})$, we define $d_\mathsf{b} := \langle 2\rho_\mathsf{G}, \nu_\mathsf{b} \rangle$, where ν_b is the slope homomorphism of b. Then an object A is contained in ${}^p\mathrm{D}^{\leq 0}(U, \overline{\mathbb{F}_\ell})$ if $i_\mathsf{b}^* A \in \mathrm{D}^{\leq d_\mathsf{b}}(\mathsf{G}_\mathsf{b}, \Lambda)$, and A is contained in ${}^p\mathrm{D}^{\geq 0}(U, \overline{\mathbb{F}_\ell})$ if $i_\mathsf{b}^! A \in \mathrm{D}^{\geq d_\mathsf{b}}(\mathsf{G}_\mathsf{b}, \Lambda)$. Here we recall that i_b is the inclusion $\mathrm{Bun}_\mathsf{G}^\mathsf{b} \subset \mathrm{Bun}_\mathsf{G}$.

We also need the notion of universally locally acyclic (ULA) objects [FS24, Definition IV.2.31]: The full subcategory $\mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_\mathsf{G}, \overline{\mathbb{F}_\ell}) \subset \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_\mathsf{G}, \overline{\mathbb{F}_\ell})$ consists of objects A such that $i_\mathsf{b}^* A \in \mathrm{D}^{\mathrm{adm}}(\mathsf{G}_\mathsf{b}, \overline{\mathbb{F}_\ell})$ for each $\mathsf{b} \in B(\mathsf{G})$.

We import Setup 7.2.2. In particular, F is a totally real number field unramified at a prime p, $\mathbf{G}$ is a special orthogonal or unitary group over F with $G^* = \mathbf{G} \otimes_F (F \otimes \mathbb{Q}_p)$, and $G^\sharp$ is a central extension of $\mathbf{G}$ with $G^\sharp$. We then have the following local result on the perverse t-exactness of Hecke operators, which generalizes [HL24, Corollary 4.24] to special orthogonal groups and unitary groups:

Theorem 8.2.1. *Suppose $\phi \in \Phi^{\mathrm{ss}}(\mathrm{Res}_{K/\mathbb{Q}_p} G^*, \overline{\mathbb{F}_\ell})$ is generic and weakly normalized regular (see Definition 8.1.1), and we regard ϕ as a semisimple toral L-parameter $\phi^\sharp \in \Phi^{\mathrm{ss}}(G^\sharp, \overline{\mathbb{F}_\ell})$ via the natural embedding ${}^L(\mathrm{Res}_{K/\mathbb{Q}_p} G^*)(\overline{\mathbb{F}_\ell}) \to {}^L G^\sharp(\overline{\mathbb{F}_\ell})$. We assume further that $\ell \neq 2$ and moreover $(\ell, n!) = 1$ in Case O1. Then for any dominant cocharacter $\mu^\sharp$ of $G^\sharp_{\overline{\mathbb{Q}_p}}$, the Hecke operator $\mathrm{T}_{\mu^\sharp}$ attached to the extended highest weight tilting module ${}^L\mathcal{T}_{\mu^\sharp}$ as defined in (3.2) preserves ULA objects, and the induced functor*

$$i_\mathbf{1}^* \mathrm{T}_{\mu^\sharp} : \mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_{G^\sharp}, \overline{\mathbb{F}_\ell})_{\phi^\sharp} \to \mathrm{D}^{\mathrm{adm}}(G^\sharp, \overline{\mathbb{F}_\ell})_{\phi^\sharp}$$

(where $\mathbf{1} \in B(G^\sharp)$ is the trivial element) is exact with respect to the perverse t-structure on the source and the natural t-structure on the target.

Proof. For the first assertion, by [HL24, Theorem 4.20] and [Ham24, Theorem 1.17], it suffices to check the following claims all hold:

- [Ham22, Assumption 7.5] is true for $\phi^\sharp$,
- $\phi^\sharp$ is regular and weakly normalized regular,

- $\phi^\sharp$ is $\mu^\sharp$-regular, and there exists a cocharacter $\tilde{\mu}^\sharp$ of $G^\sharp_{\overline{\mathbb{Q}_p}}$ that is not fixed by any nontrivial element $w \in W_{G^\sharp}$, such that $\phi^\sharp$ is $\tilde{\mu}^\sharp$-regular.
- $\rho^{\chi^\sharp}_{b^\sharp, w^\sharp}$ (as defined in (7.5)) is irreducible for any $b^\sharp \in B(G^\sharp)_{\mathrm{un}}$ and $w^\sharp \in W_{b^\sharp}$, where the character $\chi^\sharp$ is attached to $\phi^\sharp_{\mathsf{T}^\sharp}$ via local Langlands correspondence for tori.

The first claim follows from Theorem 7.2.3, the second claim follows from Lemma 8.1.4 and the assumption, and the third claim follows from Lemma 8.1.6. Finally, we show that $\rho^{\chi^\sharp}_{b^\sharp, w^\sharp}$ is irreducible for any $b^\sharp \in B(G^\sharp)_{\mathrm{un}}$ and $w^\sharp \in W_{b^\sharp}$. Note that $G^\sharp_{b^\sharp}$ is isomorphic to a Levi factor of a parabolic subgroup of $G^\sharp$, which is of the form

$$\mathrm{GL}(1)_{\mathbb{Q}_p} \times \mathrm{Res}_{K/\mathbb{Q}_p} G' \times \mathrm{Res}_{K_1/\mathbb{Q}_p} H,$$

where H is a product of general linear groups and G' is a general spinor or general unitary group over K that splits over an unramified quadratic extension. So the first three assertions hold true with $G^\sharp$ replaced by $G^\sharp_{b^\sharp}$, and the desired claim follows from [HL24, Lemma 4.18] and [Ham24, Proposition A.2]. $\qquad\square$

Next, we recall a perversity result, which will be a crucial ingredient in the proof of torsion vanishing result later. We impose the following global assumptions for future use:

Setup 8.2.2.
- $\mathrm{Sh}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)$ is proper, and there exists a Shimura datum of Hodge type $(\mathbb{G}^\sharp, \mathbb{X}^\sharp)$ with a map of Shimura data $(\mathbb{G}^\sharp, \mathbb{X}^\sharp) \to (\mathbb{G}, \mathbb{X})$ such that $\mathbb{G}^\sharp_{\mathrm{ad}} \to \mathbb{G}_{\mathrm{ad}}$ is an isomorphism and $\#\pi_0(Z(\mathbb{G}^\sharp))$ is coprime to ℓ. Let $E^\sharp \subset \mathbb{C}$ be the common reflex field.
- $\mathbb{G}$ and $\mathbb{G}^\sharp$ are unramified at p. Set $\mathsf{G}^\sharp := \mathbb{G}^\sharp \otimes \mathbb{Q}_p, \mathsf{G} := \mathbb{G} \otimes \mathbb{Q}_p$, and fix a Borel pair $(\mathsf{B}^\sharp, \mathsf{T}^\sharp)$ of $\mathsf{G}^\sharp$ with image (B, T) in G.
- The central extension $\mathsf{G}^\sharp \to \mathsf{G}$ extends to a map of reductive integral models $\mathcal{G}^\sharp \to \mathcal{G}$ over $\mathbb{Z}_p$, and we define $\mathcal{K}^\sharp_p := \mathcal{G}^\sharp(\mathbb{Z}_p), \mathcal{K}_p := \mathcal{G}(\mathbb{Z}_p)$.
- $\mathcal{K}^{\sharp p} \le \mathbb{G}^\sharp(\mathbf{A}^p_f)$ is a neat compact open subgroup with image $\mathcal{K}^p$ in $\mathbb{G}(\mathbf{A}_f)$. Set $\mathcal{K}^\sharp := \mathcal{K}^\sharp_p \mathcal{K}^{\sharp p} \le \mathbb{G}^\sharp(\mathbf{A}_f)$ and $\mathcal{K} := \mathcal{K}_p \mathcal{K}^p \le \mathbb{G}(\mathbf{A}_f)$.
- We write $\mathcal{H}_{\mathcal{K}^\sharp_p} := \overline{\mathbb{F}_\ell}[\mathcal{K}^\sharp_p \backslash \mathsf{G}^\sharp(\mathbb{Q}_p)/\mathcal{K}^\sharp_p], \mathcal{H}_{\mathcal{K}_p} := \overline{\mathbb{F}_\ell}[\mathcal{K}_p \backslash \mathsf{G}(\mathbb{Q}_p)/\mathcal{K}_p]$ for the Hecke algebras with $\overline{\mathbb{F}_\ell}$-coefficients, respectively.
- Let $\mathfrak{m} \subset \mathcal{H}_{\mathcal{K}_p}$ be a maximal ideal with inverse image $\mathfrak{m}^\sharp \subset \mathcal{H}_{\mathcal{K}^\sharp_p}$ and corresponding semisimple toral L-parameters $\phi_\mathfrak{m} \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{F}_\ell})$ and $\phi_{\mathfrak{m}^\sharp} \in \Phi^{\mathrm{ss}}(\mathsf{G}^\sharp, \overline{\mathbb{F}_\ell})$, respectively.

We then have a finite Galois covering of Shimura varieties

$$\mathrm{Sh}_{\mathcal{K}^\sharp}(\mathbb{G}^\sharp, \mathbb{X}^\sharp) \to \mathrm{Sh}_{\mathcal{K}}(\mathbb{G}, \mathbb{X}).$$

over $E^\sharp$. Note that $\iota_p : \mathbb{C} \to \overline{\mathbb{Q}_p}$ induces an embedding $E^\sharp \to \overline{\mathbb{Q}_p}$.

We now use the Igusa varieties for Hodge type Shimura varieties defined in [DvHKZ24]. If we write

$$\mathrm{Sh}_{\mathcal{K}^{\sharp p}}(\mathbb{G}^\sharp, \mathbb{X}^\sharp) := \varprojlim_{\tilde{\mathcal{K}}_p} \mathrm{Sh}_{\tilde{\mathcal{K}}_p \mathcal{K}^{\sharp p}}(\mathbb{G}^\sharp, \mathbb{X}^\sharp),$$

where $\tilde{\mathcal{K}}_p$ runs through all compact open subgroups of $\mathsf{G}^\sharp(\mathbb{Q}_p)$, then we have the Hodge–Tate period map

$$\pi_{\mathrm{HT}} : \mathrm{Sh}_{\mathcal{K}^{\sharp p}}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)^{\mathrm{an}} \to \mathrm{Gr}_{\mathsf{G}^\sharp, \mu^\sharp}.$$

We then have the following Igusa stack $\mathrm{Ig}_{\mathcal{K}^{\sharp p}}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)$; see [DvHKZ24, Theorem I].

Theorem 8.2.3. *There is an Artin v-stack $\mathrm{Ig}_{\mathcal{K}^{\sharp p}}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)$ on $\mathfrak{Perfd}_{\overline{\kappa}}$ sitting in a Cartesian diagram*

$$
\begin{array}{ccc}
\mathrm{Sh}_{\mathcal{K}^{\sharp p}}(\mathbb{G}^\sharp, \mathbb{X}^\sharp) & \xrightarrow{\pi_{\mathrm{HT}}} & \mathrm{Gr}_{\mathsf{G}^\sharp, \mu^\sharp} \\
\downarrow{\scriptstyle \mathrm{pr}_{\mathrm{Ig}}} & & \downarrow{\scriptstyle \mathrm{BL}} \\
\mathrm{Ig}_{\mathcal{K}^{\sharp p}}(\mathbb{G}^\sharp, \mathbb{X}^\sharp) & \xrightarrow{\pi^{\mathrm{Ig}}_{\mathrm{HT}}} & \mathrm{Bun}_{\mathsf{G}^\sharp, \mu^\sharp \bullet},
\end{array}
$$

where BL *is the Beauville–Laszlo map from* [FS24, Proposition III.3.1]. *Moreover,* $\mathrm{Ig}_{\mathcal{K}^{\sharp p}}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)$ *is ℓ-cohomologically smooth of dimension 0, and its dualizing sheaf is isomorphic to $\overline{\mathbb{F}_\ell}[0]$.*

Define
$$\mathcal{F} := \mathrm{R}\pi_{\mathrm{HT},*}^{\mathrm{Ig}}(\overline{\mathbb{F}_\ell}) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}^\sharp, \mu^\sharp\bullet}, \overline{\mathbb{F}_\ell}).$$

which is universally locally acyclic by [DvHKZ24, Corollary 8.5.4]. Moreover, we recall the following perversity result from [DvHKZ24, Theorem 8.6.3]:

Theorem 8.2.4. *$\mathcal{F}$ is perverse, i.e.,*
$$\mathcal{F} \in {}^p\mathrm{D}^{\geq 0}(\mathrm{Bun}_{\mathsf{G}^\sharp, \mu^\sharp\bullet}, \overline{\mathbb{F}_\ell}) \cap {}^p\mathrm{D}^{\leq 0}(\mathrm{Bun}_{\mathsf{G}^\sharp, \mu^\sharp\bullet}, \overline{\mathbb{F}_\ell}).$$

The sheaf $\mathcal{F}$ is significant: by Theorem 8.2.3, [DvHKZ24, Theorem 8.4.10] yields the following relation between the cohomology of the relevant Shimura varieties and the value of the corresponding Hecke operator on $\mathcal{F}$.

Theorem 8.2.5. *There is an isomorphism*
$$\mathrm{R}\Gamma(\mathrm{Sh}_{\mathcal{K}^p}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)_{\mathbb{C}_p}, \overline{\mathbb{F}_\ell}) \cong i_{\mathbf{1}}^* \mathrm{T}_{\mu^\sharp\bullet}(\mathcal{F}[-\dim_{\mathbb{C}}(\mathbb{X})]) \left(-\frac{\dim_{\mathbb{C}}(\mathbb{X})}{2} \right)$$

in $\mathrm{D}(\mathsf{G}^\sharp, \overline{\mathbb{F}_\ell})$.

We now state our first main theorem on torsion vanishing result for cohomologies of orthogonal or unitary Shimura varieties away from central dimension. We impose Setup 7.2.2 with $F \neq \mathbb{Q}$, and extend the map $G^\sharp \to \mathrm{Res}_{K/\mathbb{Q}_p} G$ to a map of reductive integral models $\mathcal{G}^\sharp \to \mathcal{G}$ over $\mathbb{Z}_p$ by fixing a $\mathbb{Z}_p$-lattice of $\mathrm{Res}_{K/\mathbb{Q}_p}(\mathbf{V} \otimes_F K)$, and define $\mathcal{K}_p^\sharp := \mathcal{G}^\sharp(\mathbb{Z}_p), \mathcal{K}_p := \mathcal{G}(\mathbb{Z}_p)$. Let $\mathcal{K}^{p,\sharp} \leq \mathbf{G}^\sharp(\mathbb{A}_f^p)$ be a compact open subgroup with image $\mathcal{K}^p \leq \left(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}\right)(\mathbb{A}_f^p)$, such that $\mathcal{K} := \mathcal{K}_p \mathcal{K}^p \leq \mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}(\mathbb{A}_f)$ and $\mathcal{K}^\sharp := \mathcal{K}_p^\sharp \mathcal{K}^{\sharp p} \leq \mathbf{G}^\sharp(\mathbb{A}_f)$ are both neat. Let ℓ be a rational prime coprime to p, and we assume moreover $(\ell, n!) = 1$ in Case O1. Let
$$\mathcal{H}_{\mathcal{K}_p^\sharp} := \overline{\mathbb{F}_\ell}[\mathcal{K}_p^\sharp \backslash G^\sharp(\mathbb{Q}_p)/\mathcal{K}_p^\sharp], \mathcal{H}_{\mathcal{K}_p} := \overline{\mathbb{F}_\ell}[\mathcal{K}_p \backslash G^*(K)/\mathcal{K}_p]$$

be Hecke algebras with $\overline{\mathbb{F}_\ell}$-coefficients. Let $\mathfrak{m} \subset \mathcal{H}_{\mathcal{K}_p}$ be a maximal ideal with inverse image $\mathfrak{m}^\sharp \subset \mathcal{H}_{\mathcal{K}_p^\sharp}$ and corresponding semisimple toral L-parameters $\phi_{\mathfrak{m}} \in \Phi^{\mathrm{ss}}(\mathrm{Res}_{K/\mathbb{Q}} G^*, \overline{\mathbb{F}_\ell})$ and $\phi_{\mathfrak{m}^\sharp} \in \Phi^{\mathrm{ss}}(G^\sharp, \overline{\mathbb{F}_\ell})$, respectively. Moreover, we assume that p is coprime to $2\dim(\mathbf{V})$ in Case U, so we work in the setting of Setup 8.2.2.

Theorem 8.2.6. *Suppose $F \neq \mathbb{Q}$, and suppose $\phi_{\mathfrak{m}} \in \Phi^{\mathrm{ss}}(\mathrm{Res}_{K/\mathbb{Q}_p} G, \overline{\mathbb{F}_\ell})$ is generic and weakly normalized regular.*
   (1) $\mathrm{H}_{\text{ét}}^i(\mathrm{Sh}_{\mathcal{K}^\sharp}(\mathbf{G}^\sharp, \mathbf{X}^\sharp)_{\overline{E^\sharp}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}^\sharp}$ vanishes unless $i = \dim_{\mathbb{C}}(\mathbf{X})$.
   (2) $\mathrm{H}_{\text{ét}}^i(\mathrm{Sh}_{\mathcal{K}}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})_{\overline{E}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}^\sharp}$ vanishes unless $i = \dim_{\mathbb{C}}(\mathbf{X})$.

Proof. The second assertion follows from the first one: $\mathrm{Sh}_{\mathcal{K}^\sharp}(\mathbf{G}^\sharp, \mathbf{X})_{\overline{E^\sharp}}$ is a finite Galois covering of an open close subset M of $\mathrm{Sh}_{\mathcal{K}}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})_{\overline{E^\sharp}}$ with Galois group denoted by $\mathfrak{T}$. The cohomology of M is equipped with Hecke action by $\mathcal{H}_{\mathcal{K}_p}$. By the Hochschild–Serre spectral sequence,
$$\mathrm{R}\Gamma\left(M, \overline{\mathbb{F}_\ell}\right)_{\mathfrak{m}} = \mathrm{R}\Gamma\left(\mathfrak{T}, \mathrm{R}\Gamma(\mathrm{Sh}_{\mathcal{K}^\sharp}(\mathbf{G}^\sharp, \mathbf{X}^\sharp)_{\overline{E^\sharp}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}^\sharp}\right)_{\mathfrak{m}},$$

which is concentrated in degree at least $\dim_{\mathbb{C}}(\mathbf{X})$ by (1). Now $\mathrm{R}\Gamma\left(\mathrm{Sh}_{\mathcal{K}}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})_{\overline{E}}, \overline{\mathbb{F}_\ell}\right)_{\mathfrak{m}}$ is a finite direct sum of copies of the complex $\mathrm{R}\Gamma\left(M, \overline{\mathbb{F}_\ell}\right)_{\mathfrak{m}}$, which is also concentrated in degree at least $\dim_{\mathbb{C}}(\mathbf{X})$.

But by Poincaré duality and [HL24, Corollary A.7], it follows that
$$\mathrm{R}\Gamma(\mathrm{Sh}_{\mathcal{K}}(\mathbf{G}, \mathbf{X})_{\overline{E}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}} \cong \mathrm{R}\Gamma(\mathrm{Sh}_{\mathcal{K}}(\mathbf{G}, \mathbf{X})_{\overline{E}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}^\vee}$$

is concentrated in degree at most $\dim_{\mathbb{C}}(\mathbf{X})$, where $\mathfrak{m}^\vee$ is the maximal ideal corresponding to $\phi_{\mathfrak{m}}^\vee$.

For the first assertion, note that $\mathrm{Sh}_{\mathcal{K}^\sharp}(\mathbf{G}^\sharp, \mathbf{X}^\sharp)$ is proper because $[\mathbf{G}^\sharp, \mathbf{G}^\sharp]$ is anisotropic, so the assertion follows Theorem 8.2.4, Theorem 8.2.5, Theorem 8.2.1 and Poincaré duality, as $\phi_{\mathfrak{m}}$ and $\phi_{\mathfrak{m}}^\vee$ are both generic and weakly normalized regular. $\square$

If we look at the cohomology of an Abelian type Shimura variety with torsion coefficients localized at a split place, then the above theorem can be generalized. In fact, we do not need the full strength of the compatibility of Fargues–Scholze LLC with "classical local Langlands correspondence" in the sense of [Ham24, Assumption 7.5], but only one property of the Fargues–Scholze LLC:

Axiom 8.2.7. Suppose G/K is a quasi-split reductive group with a Borel pair (B, T) and $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{Q}_\ell})$ is a semisimple generic toral L-parameter. Then for any $\mathsf{b} \in B(\mathsf{G})$ and any $\rho \in \Pi(\mathsf{G}_{\mathsf{b}}, \overline{\mathbb{Q}_\ell})$, if the composition of $\phi_\rho^{\mathrm{FS}} : W_K \to {}^L\mathsf{G}_{\mathsf{b}}(\overline{\mathbb{Q}_\ell})$ with the twisted embedding ${}^L\mathsf{G}_{\mathsf{b}}(\overline{\mathbb{Q}_\ell}) \to {}^L\mathsf{G}(\overline{\mathbb{Q}_\ell})$ (as defined in [FS24, §IX.7.1]) equals ϕ, then b is unramified.

Remark 8.2.8. Note that, by [Ham24, Lemma 3.7], this axiom is a consequence of [Ham24, Assumption 7.5]: The hypothesis that $\phi_{\iota_\ell^{-1}\rho}^{\mathrm{ss}} = \iota_\ell^{-1}\phi_\rho^{\mathrm{FS}}$ factors through a generic parameter ϕ_{T} of T implies that $\phi_{\iota_\ell^{-1}\rho} = \phi_{\iota_\ell^{-1}\rho}^{\mathrm{ss}}$ by [Ham24, Lemma 3.17], which implies T is relevant for G_{b} by Theorem 2.3.1. So G_{b} is quasi-split, or equivalently, b is unramified.

Observe that this axiom is stable under duality: if it holds for ϕ, then it holds for $\phi^\vee$. The following invariance property may be of independent interest.

Proposition 8.2.9. *Suppose $\mathsf{G}' \to \mathsf{G}$ is a map of quasi-split reductive groups over a non-Archimedean local field K of characteristic zero that induces an isomorphism on adjoint groups. Let (B, T) and $(\mathsf{B}', \mathsf{T}')$ be compatible Borel pairs of G and G', respectively. If $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{Q}_\ell})$ is a semisimple generic toral L-parameter, and we write $\phi' \in \Phi^{\mathrm{ss}}(\mathsf{G}', \overline{\mathbb{Q}_\ell})$ for the image of ϕ under the natural map $\Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{Q}_\ell}) \to \Phi^{\mathrm{ss}}(\mathsf{G}', \overline{\mathbb{Q}_\ell})$. then Axiom 8.2.7 for ϕ implies that it holds for ϕ'.*

In particular, if $\mathsf{G}' \to \mathsf{G}$ is an injection, then Axiom 8.2.7 for G implies that it holds for G'.

Proof. Suppose $\mathsf{b}' \in B(\mathsf{G}')$ maps to $\mathsf{b} \in B(\mathsf{G})$, then there is a map $\mathsf{G}_{\mathsf{b}'} \to \mathsf{G}_{\mathsf{b}}$ that induces an isomorphism on adjoint groups. Suppose $\rho' \in \Pi(\mathsf{G}'_{\mathsf{b}'}, \overline{\mathbb{Q}_\ell})$ such that the composition of $\phi_{\rho'}^{\mathrm{FS}} : W_K \to {}^L\mathsf{G}'_{\mathsf{b}'}(\overline{\mathbb{Q}_\ell})$ with the twisted embedding ${}^L\mathsf{G}'_{\mathsf{b}'}(\overline{\mathbb{Q}_\ell}) \to {}^L\mathsf{G}'(\overline{\mathbb{Q}_\ell})$ equals $\phi' \in \Phi^{\mathrm{ss}}(\mathsf{G}'_{\mathsf{b}'}, \overline{\mathbb{Q}_\ell})$. Then it follows from the compatibility of Fargues–Scholze parameters with central characters Theorem 3.1.1 that ρ' factors through $\mathrm{Im}(\mathsf{G}'_{\mathsf{b}'}(K) \to \mathsf{G}_{\mathsf{b}}(K))$. Then it follows from [GK82, Lemma 2.3] that there exists an irreducible smooth representation $\rho \in \Pi(\mathsf{G}_{\mathsf{b}}, \overline{\mathbb{Q}_\ell})$ such that ρ' is a subquotient of $\rho|_{\mathsf{G}'_{\mathsf{b}'}(K)}$. It follows from compatibility of Fargues–Scholze correspondence with central extensions Theorem 3.1.1 that $\phi_{\rho'}^{\mathrm{FS}}$ is the image of ϕ_ρ^{FS} under the natural map $\Phi^{\mathrm{ss}}(\mathsf{G}_{\mathsf{b}}, \overline{\mathbb{Q}_\ell}) \to \Phi^{\mathrm{ss}}(\mathsf{G}'_{\mathsf{b}'}, \overline{\mathbb{Q}_\ell})$. So the composition of $\phi_\rho^{\mathrm{FS}} : W_K \to {}^L\mathsf{G}_{\mathsf{b}}(\overline{\mathbb{Q}_\ell})$ with the twisted embedding ${}^L\mathsf{G}_{\mathsf{b}}(\overline{\mathbb{Q}_\ell}) \to {}^L\mathsf{G}(\overline{\mathbb{Q}_\ell})$ is a parameter $\tilde{\phi}$ whose image under the natural map $\Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{Q}_\ell}) \to \Phi^{\mathrm{ss}}(\mathsf{G}', \overline{\mathbb{Q}_\ell})$ equals ϕ'. So it follows from the compatibility of Fargues–Scholze correspondence with character twists and [Xu17, Appendix A] that we may twist ρ by a character of $\mathrm{Coker}(\mathsf{G}'_{\mathsf{b}}(K) \to \mathsf{G}_{\mathsf{b}}(K))$ to make sure that $\tilde{\phi} = \phi$. Now the axiom for ϕ implies that b is unramified. In particular, G_{b} is quasi-split. Suppose $\mathsf{B}_{\mathsf{b}} \leq \mathsf{G}_{\mathsf{b}}$ is a Borel subgroup, then $\mathsf{B}_{\mathsf{b}'} = \mathsf{B}_{\mathsf{b}} \cap \mathsf{G}'_{\mathsf{b}'}$ is a Borel subgroup of $\mathsf{G}'_{\mathsf{b}'}$. Thus b' is unramified.

For the last assertion, it suffices to note that if $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{Q}_\ell})$ has toral generic image $\phi' \in \Phi^{\mathrm{ss}}(\mathsf{G}', \overline{\mathbb{Q}_\ell})$, then ϕ is itself toral and generic. $\square$

We now prove the second main theorem on vanishing result for torsion cohomology of Shimura varieties of Abelian type under Axiom 8.2.7:

Theorem 8.2.10. *We work in the setting of* Setup 8.2.2, *and furthermore we assume that the set of unramified $\mu_{\mathrm{ad}}^\bullet$-acceptable elements $B(\mathsf{G}_{\mathrm{ad}}, \mu_{\mathrm{ad}}^\bullet)_{\mathrm{un}}$ is a singleton. Suppose $\phi_{\mathfrak{m}}$ is generic and Axiom 8.2.7 holds for $\phi_{\mathfrak{m}}$.*

(1) $\mathrm{H}^i_{\text{ét}}(\mathrm{Sh}_{\mathcal{K}^\sharp}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)_{\overline{E^\sharp}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}^\sharp}$ vanishes unless $i = \dim_{\mathbb{C}}(\mathbb{X})$.

(2) $\mathrm{H}^i_{\text{ét}}(\mathrm{Sh}_{\mathcal{K}}(\mathbb{G}, \mathbb{X})_{\overline{E^\sharp}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}}$ vanishes unless $i = \dim_{\mathbb{C}}(\mathbb{X})$.

Remark 8.2.11. By [XZ17, Corollary 4.2.4], the conditions that is $B(\mathsf{G}_{\mathrm{ad}}, \mu_{\mathrm{ad}}^\bullet)_{\mathrm{un}}$ is a singleton (which is the μ_{ad}-ordinary element) is guaranteed when G_{ad} is a product of unramified Weil restrictions of split simple groups $\prod_{i=1}^k \mathrm{Res}_{L_i/\mathbb{Q}_p} \mathsf{H}_i$, and the conjugacy class of Hodge cocharacters $\{\mu\}$ associated to $\mathbb{X}^\sharp$ induces a dominant cocharacter $\mu_{\mathrm{ad}} = (\mu_1, \dots, \mu_k)$ of $\mathsf{G}_{\overline{\mathbb{Q}_p}}$ via ι_p, such that each μ_i is trivial on all but except possibly one simple factor of $(\mathsf{H}_i)_{\overline{\mathbb{Q}_p}} \cong \prod_{\mathrm{Hom}(L_i, \overline{\mathbb{Q}_p})} (\mathsf{H}_i)_{\overline{L_i}}$.

Remark 8.2.12. The first assertion is established in [DvHKZ24, Theorem 10.1.6] under the assumption that the Fargues–Scholze LLC for $\mathsf{G}^\sharp$ is "natural" in the sense of [Ham24, Assumption 7.5]. However, this naturality is established in limited cases. For example, "classical local Langlands correspondence" for pure inner forms of $\mathrm{GSpin}(N)$ when $N \geq 8$ haven't been constructed, except for those irreducible representations with central character being a square of another character; see [GT19]. Nonetheless, we can still establish the theorem under this weaker axiom by modifying the argument in [DvHKZ24].

Proof. Firstly, The second assertion follows from the first one: We write $\mathcal{H}' = \mathrm{Im}(\mathcal{H}_{\mathcal{K}_p^\sharp} \to \mathcal{H}_{\mathcal{K}_p})$, and $\mathfrak{m}'$ the inverse image of $\mathfrak{m}$ in $\mathcal{H}'$. By functoriality of Shimura varieties, $\mathrm{Sh}_{\mathcal{K}^\sharp}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)$ is a finite Galois covering of an open closed subset M of $\mathrm{Sh}_{\mathcal{K}}(\mathbb{G}, \mathbb{X})_{\overline{E^\sharp}}$, with Galois group denoted by $\mathfrak{T}$. The cohomology of M is equipped with Hecke action by $\mathcal{H}'$. By the Hochschild–Serre spectral sequence,

$$\mathrm{R}\Gamma\left(M, \overline{\mathbb{F}_\ell}\right)_{\mathfrak{m}'} = \mathrm{R}\Gamma\left(\mathfrak{T}, \mathrm{R}\Gamma(\mathrm{Sh}_{\mathcal{K}^\sharp}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)_{\overline{E^\sharp}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}^\sharp}\right)_{\mathfrak{m}'},$$

which is concentrated in degree$\geq \dim_{\mathbb{C}}(\mathbb{X})$ by (1). Now $\mathrm{R}\Gamma\left(\mathrm{Sh}_{\mathcal{K}}(\mathbb{G}, \mathbb{X})_{\overline{E^\sharp}}, \overline{\mathbb{F}_\ell}\right)_{\mathfrak{m}}$ is a finite direct sum of the copies of the complex

$$\mathrm{RHom}_{\mathcal{H}'_{\mathfrak{m}'}}\left((\mathcal{H}_{\mathcal{K}_p})_{\mathfrak{m}}, \mathrm{R}\Gamma\left(M, \overline{\mathbb{F}_\ell}\right)_{\mathfrak{m}'}\right),$$

which is also concentrated in degree$\geq \dim_{\mathbb{C}}(\mathbb{X})$.

But by Poincaré duality and [HL24, Corollary A.7], it follows that

$$\mathrm{R}\Gamma(\mathrm{Sh}_{\mathcal{K}}(\mathbb{G}, \mathbb{X})_{\overline{E^\sharp}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}} \cong \mathrm{R}\Gamma(\mathrm{Sh}_{\mathcal{K}^\sharp}(\mathbb{G}, \mathbb{X})_{\overline{E^\sharp}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}^\vee}$$

is concentrated in degree$\leq \dim_{\mathbb{C}}(\mathbb{X})$, where $\mathfrak{m}^\vee$ is the maximal ideal corresponding to $\phi_{\mathfrak{m}}^\vee$.

For the first assertion, note that $\phi_{\mathfrak{m}^\sharp}$ and $\phi_{\mathfrak{m}^\sharp}^\vee$ are both generic. So it follows Theorem 8.2.4, Theorem 8.2.5 and Poincaré duality that it suffices to show the t-exactness of the functor

$$i_{\mathbf{1}}^* \mathrm{T}_{\mu^{\sharp\bullet}} : \mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_{\mathsf{G}^\sharp}, \overline{\mathbb{F}_\ell})_{\overline{\phi}} \to \mathrm{D}^{\mathrm{adm}}(\mathsf{G}^\sharp, \overline{\mathbb{F}_\ell})_{\overline{\phi}}$$

(as defined in Theorem 8.2.1) for any semisimple toral generic L-parameter $\overline{\phi} \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{F}_\ell})$ satisfying Axiom 8.2.7, where we regard $\overline{\phi}$ as an semisimple toral generic L-parameter for $\mathsf{G}^\sharp$ under the natural L-homomorphism ${}^L\mathsf{G}(\overline{\mathbb{F}_\ell}) \to {}^L\mathsf{G}^\sharp(\overline{\mathbb{F}_\ell})$. By the description of the set $B(\mathsf{G}^\sharp, \mu^{\sharp\bullet})_{\mathrm{un}}$ in [XZ17, Corollary 4.2.4], we see that

$$B(\mathsf{G}^\sharp, \mu^{\sharp\bullet})_{\mathrm{un}} \subset B(\mathsf{G}_{\mathrm{ad}}, \mu_{\mathrm{ad}}^\bullet)_{\mathrm{un}}$$

is a singleton, by our assumption. So it follows from [DvHKZ24, Proposition 10.2.5] that it suffices to show that $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}^\sharp}, \overline{\mathbb{F}_\ell})_{\overline{\phi}}$ is supported on the unramified strata.

By [HL24, Lemma 4.2(1)] and [FS24, §IX.7.1], if

$$\overline{\rho}^\sharp \in \mathrm{D}(\mathrm{Bun}_{\mathsf{G}^\sharp}^{\mathsf{b}^\sharp}, \overline{\mathbb{F}_\ell})_{\overline{\phi}} \subset \mathrm{D}(\mathrm{Bun}_{\mathsf{G}^\sharp}^{\mathsf{b}^\sharp}, \overline{\mathbb{F}_\ell}) \cong \mathrm{D}(\mathsf{G}_{\mathsf{b}^\sharp}^\sharp, \overline{\mathbb{F}_\ell})$$

is an irreducible admissible representation of $\mathsf{G}_{\mathsf{b}^\sharp}^\sharp(\mathbb{Q}_p)$, then the Fargues–Scholze parameter $\phi_{\overline{\rho}^\sharp}^{\mathrm{FS}} \in \Phi^{\mathrm{ss}}(\mathsf{G}_{\mathsf{b}^\sharp}^\sharp, \overline{\mathbb{F}_\ell})$ composed with the twisted embedding ${}^L\mathsf{G}_{\mathsf{b}^\sharp}^\sharp(\overline{\mathbb{F}_\ell}) \to {}^L\mathsf{G}^\sharp(\overline{\mathbb{F}_\ell})$ (as defined in [FS24, §IX.7.1]) agrees with $\overline{\phi}$. By [Dat05, Lemma 6.8], we may lift $\overline{\rho}^\sharp$ to an irreducible admissible $\overline{\mathbb{Q}_\ell}$-representation $\rho^\sharp$ of $\mathsf{G}_{\mathsf{b}^\sharp}^\sharp(\mathbb{Q}_p)$ admitting a $\mathsf{G}_{\mathsf{b}^\sharp}^\sharp(\mathbb{Q}_p)$-stable $\overline{\mathbb{Z}_\ell}$-lattice such that $\overline{\rho}^\sharp$ occurs as a subquotient of $\rho^\sharp$ modulo ℓ. Since the Fargues–Scholze parameter is compatible with reduction modulo ℓ Theorem 3.1.1, it follows that the Fargues–Scholze parameter $\phi_{\rho^\sharp}^{\mathrm{FS}}$ factors through ${}^L\mathsf{G}_{\mathsf{b}^\sharp}^\sharp(\overline{\mathbb{Z}_\ell})$, and its reduction modulo ℓ equals $\overline{\phi}$. So the composition of $\phi_{\rho^\sharp}^{\mathrm{FS}}$ with the twisted embedding ${}^L\mathsf{G}_{\mathsf{b}^\sharp}^\sharp(\overline{\mathbb{Q}_\ell}) \to {}^L\mathsf{G}^\sharp(\overline{\mathbb{Q}_\ell})$ also factors through a parameter $\phi_{\mathsf{T}^\sharp} \in \Phi^{\mathrm{ss}}(\mathsf{T}^\sharp, \overline{\mathbb{Q}_\ell})$: This follows from standard deformation theory, because

$$\mathrm{H}^1(W_K, \alpha \circ \overline{\phi}_{\mathsf{T}^\sharp}) = 0$$

for each dominant coroots $\alpha \in \Phi^\vee(\mathsf{G}^\sharp, \mathsf{T}^\sharp)^+$ (cf. [CS17, Lemma 6.2.2]). It is also easy to see that $\phi_{\mathsf{T}^\sharp}$ is generic since its reduction modulo ℓ is. Now the assertion follows from Axiom 8.2.7 and Proposition 8.2.9. $\qquad\square$

Appendix A. (Twisted) Endoscopy theory

In this appendix we review some standard definitions related to the trace formulas used in the main text and fix notation. §A.1 recalls (twisted) endoscopic triples for reductive groups over both local and global fields. §A.2 reviews local transfers and pseudo-coefficients for square-integrable irreducible admissible representations. §A.4 records Lefschetz functions and also a simple stable trace formula that will be used during the Langlands–Kottwitz method §4.5.

A.1. **Endoscopic triples.** We recall here some general definitions of extended endoscopic triples from [LS87, §1.2] and [KS99, §2.1]. Note that the more general notion of endoscopic data are not needed in the cases we consider in this paper.

Let F be a local or global field of characteristic zero, and consider a pair (G^*, θ^*) where

(1) G^* is a quasi-split reductive group over F with a fixed Gal_F-invariant pinning on $\widehat{\mathsf{G}^*}$,

(2) θ^* is a pinned automorphism G^*.

Then θ^* induces an automorphism $\widehat{\theta}$ of $\widehat{\mathsf{G}^*}$ preserving the fixed Gal_F-invariant pinning of $\widehat{\mathsf{G}}$ [KS99, §1.2]. Set $^L\theta = \widehat{\theta} \times \mathrm{id}_{W_F}$, which is an automorphism of $^L\mathsf{G}$. Then an *extended endoscopic triple* for (G^*, θ^*) is a triple $\mathfrak{e} = (\mathsf{G}^{\mathfrak{e}}, \mathsf{s}^{\mathfrak{e}}, {}^L\xi^{\mathfrak{e}})$, where $\mathsf{G}^{\mathfrak{e}}$ is a quasi-split reductive group over F, $\mathsf{s}^{\mathfrak{e}} \in \widehat{\mathsf{G}^*}$, and $^L\xi^{\mathfrak{e}} : {}^L\mathsf{G}^{\mathfrak{e}} \to {}^L\mathsf{G}^*$ is an L-homomorphism such that

(1) $\mathrm{Ad}(\mathsf{s}) \circ \widehat{\theta}$ preserves a pair of a Borel subgroup and a maximal torus in $\widehat{\mathsf{G}}$, and $\mathrm{Ad}(\mathsf{s}) \circ \widehat{\theta} \circ {}^L\xi^{\mathfrak{e}} = {}^L\xi^{\mathfrak{e}}$,

(2) $^L\xi^{\mathfrak{e}}(\widehat{\mathsf{G}^{\mathfrak{e}}})$ is the connected component of the subgroup of $\mathrm{Ad}(s) \circ \widehat{\theta}$-fixed elements in $\widehat{\mathsf{G}^*}$.

$\mathfrak{e}$ is called *elliptic* if $^L\xi^{\mathfrak{e}}\big(Z(\widehat{\mathsf{G}^{\mathfrak{e}}})^{\mathrm{Gal}_K}\big)^{\circ} \subset Z(\widehat{\mathsf{G}^*})$.

Following [KMSW14], we define an isomorphism between two extended endoscopic triples $\mathfrak{e}, \mathfrak{e}'$ to be an element $\mathsf{g} \in \widehat{\mathsf{G}^*}$ such that

$$\mathsf{g}^L\xi^{\mathfrak{e}}(^L\mathsf{G}^{\mathfrak{e}})\mathsf{g}^{-1} = {}^L\xi^{\mathfrak{e}'}(^L\mathsf{G}^{\mathfrak{e}'}), \quad g\mathsf{s}^{\mathfrak{e}}\widehat{\theta}(g)^{-1} = \mathsf{s}^{\mathfrak{e}'}(\mathrm{mod}\ Z(\widehat{\mathsf{G}})).$$

Denote by $\mathcal{E}(\mathsf{G}^* \rtimes \theta^*)$ the set of isomorphism classes of extended endoscopic triples for (G^*, θ^*). When $\theta^* = \mathrm{id}_{\mathsf{G}^*}$, we also write $\mathcal{E}(\mathsf{G}^*)$ for $\mathcal{E}(\mathsf{G}^* \rtimes \theta^*)$. Also we write $\mathcal{E}_{\mathrm{ell}}(\mathsf{G}^* \rtimes \theta^*)$ or $\mathcal{E}_{\mathrm{ell}}(\mathsf{G}^*)$ for the subset of elliptic extended endoscopic triples.

Suppose $\mathfrak{e} \in \mathcal{E}(\mathsf{G}^* \rtimes \theta^*)$. For each $g^{\mathfrak{e}} \in \widehat{\mathsf{G}^{\mathfrak{e}}}$, the $^L\xi^{\mathfrak{e}}(g^{\mathfrak{e}}) \in {}^L G$ induces an automorphism of $\mathfrak{e}$. Define the outer automorphism group of $\mathfrak{e}$ by

(A.1) $$\mathrm{OAut}(\mathfrak{e}) := \mathrm{Aut}(\mathfrak{e})/{}^L\xi^{\mathfrak{e}}(\widehat{\mathsf{G}^{\mathfrak{e}}}).$$

A.2. **Transfer of orbital integrals.** Here we recall some notions on the theory of transfer, following Arthur [Art13, §2.1] and Mok [Mok15, §3.1].

Let K be a local field of characteristic zero, (G^*, θ^*) be a pair as in Appendix §A.1, and (G, ϱ, z) be a pure inner twist of G^*. We get an automorphism $\theta := \varrho \circ \theta^* \circ \varrho^{-1}$, which we assume to be a rational automorphism of G. Moreover, we fix a Whittaker datum $\mathfrak{m}$ for G^*. We write $\mathsf{G} \times \theta$ for the twisted group (or bitorsor) over G. If θ is the identity, then of course $\mathsf{G} \times \theta = \mathsf{G}$ is just the trivial bitorsor. Given $\delta \in \mathsf{G} \times \theta$, we write $Z_{\mathsf{G}}(\delta)$ for the centralizer of δ in G. We write $(\mathsf{G} \times \theta)(K)_{\mathrm{s.reg}} \subset (\mathsf{G} \times \theta)(K)$ for the open subset of strongly regular semisimple elements, meaning those regular semisimple elements whose centralizer is connected, i.e., a maximal torus. We fix a Haar measure on $\mathsf{G}(K)$. For $\delta \in \mathsf{G}(K)_{\mathrm{s.reg}}$, the Weyl discriminant of δ is defined as

$$D^{\mathsf{G}}(\delta) := \det(1 - \mathrm{Ad}(\delta)|\mathfrak{g}/\mathfrak{g}_{\delta}) \in K^{\times},$$

where $\mathfrak{g}$ and $\mathfrak{g}_{\delta}$ are the Lie algebras of G and $Z_{\mathsf{G}}(\delta)$, respectively. We fix a Haar measure on the torus $Z_{\mathsf{G}}(\delta)$, which induces a quotient measure on $Z_{\mathsf{G}}(\delta)(K)\backslash\mathsf{G}(K)$.

If K is non-Archimedean, we let $\mathcal{H}(\mathsf{G} \times \theta)$ be the space of smooth compactly supported functions on $(\mathsf{G} \times \theta)(K)$ with complex coefficients. If $K = \mathbb{R}$, we fix a maximal compact subgroup $\mathcal{K}$ of $\mathsf{G}(\mathbb{R})$ and let $\mathcal{H}(\mathsf{G} \times \theta)$ be the space of bi-$\mathcal{K}$-finite smooth compactly supported functions on $(\mathsf{G} \times \theta)(\mathbb{R})$ with complex coefficients.

For $f \in \mathcal{H}(\mathsf{G} \times \theta)$ and $\delta \in (\mathsf{G} \times \theta)(K)_{\mathrm{s.reg}}$, the normalized orbital integral of f along the conjugacy class of δ is defined as

$$\mathrm{Orb}_{\delta}(f) := \left|D^{\mathsf{G}}(\delta)\right|^{\frac{1}{2}} \int_{Z_{\mathsf{G}}(\delta)(K)\backslash\mathsf{G}(K)} f(x^{-1}\delta x)\mathrm{d}x,$$

where Moreover, when G is quasi-split and $\theta^* = \mathrm{id}$, the normalized stable orbital integral of f along δ is defined as

$$\mathrm{SOrb}_{\delta}(f) := \sum_{\delta'} \mathrm{Orb}_{\delta}(f),$$

where δ' runs over a set of representatives for the $\mathsf{G}(K)$-conjugacy classes of those elements that are $\mathsf{G}(\overline{K})$-conjugate to δ.

Assume that either $(\mathsf{G}, \varrho, z) = (\mathsf{G}^*, \mathrm{id}, \mathbf{1})$ or $\theta = \mathrm{id}$. For $\mathfrak{e} \in \mathcal{E}(\mathsf{G}^* \rtimes \theta^*)$, a *transfer factor*

$$\Delta[\mathfrak{m}, \mathfrak{e}, z] : \mathsf{G}^{\mathfrak{e}}(K)_{\mathrm{s.reg}} \times (\mathsf{G} \times \theta)(K)_{\mathrm{s.reg}} \to \mathbb{C}$$

is defined in [KMSW14, §1.1.2], such that $\Delta[\mathfrak{m}, \mathfrak{e}, z]$ is a function on stable conjugacy classes of $\mathsf{G}^{\mathfrak{e}}(K)_{\mathrm{s.reg}}$ and $\mathsf{G}(K)$-conjugacy classes of $(\mathsf{G} \times \theta)(K)_{\mathrm{s.reg}}$. With the transfer factor in hand, we now recall the notion of matching test functions from [KS99, §5.5].

Definition A.2.1. Two functions $f^{\mathsf{G}^{\mathfrak{e}}} \in \mathcal{H}(\mathsf{G}^{\mathfrak{e}})$ and $f \in \mathcal{H}(\mathsf{G} \times \theta)$ are called $(\Delta[\mathfrak{m}, \mathfrak{e}, z]\text{-})$ *matching test functions* if, for every $\gamma \in \mathsf{G}^{\mathfrak{e}}(K)_{\mathrm{s.reg}}$,

$$\mathrm{SOrb}_{\gamma}(f^{\mathsf{G}^{\mathfrak{e}}}) = \sum_{\delta \in (\mathsf{G} \times \theta)(K)_{\mathrm{s.reg}}/\mathsf{G}(K)\text{-conj}} \Delta[\mathfrak{m}, \mathfrak{e}, z](\gamma, \delta) \, \mathrm{Orb}_{\delta}(f).$$

For brevity, say that $f^{\mathsf{G}^{\mathfrak{e}}}$ is a transfer of f to $\mathsf{G}^{\mathfrak{e}}$.

Remark A.2.2. Since the orbital integrals $\mathrm{Orb}_{\delta}(f)$ depend on the choices of measures on $\mathsf{G}(K)$ and $Z_{\mathsf{G}}(\delta)(K)$, the concept of matching functions also depends on the choice of Haar measures on $\mathsf{G}(K)$ and $\mathsf{G}^{\mathfrak{e}}(K)$, and all tori in G and $\mathsf{G}^{\mathfrak{e}}$. There is a way to synchronize the various tori; cf. [AK24, Remark 5.1.2].

We now state a theorem asserting existence of transfer of orbital integrals. When $K = \mathbb{R}$, it is a fundamental result of Shelstad [She82, She08]. When K is non-Archimedean, it is a culmination of the work of many people, including Langlands and Shelstad [LS87, LS90], Waldspurger [Wal97, Wal06], and Ngô [Ngô10].

Theorem A.2.3. *Let $f \in \mathcal{H}(\mathsf{G} \times \theta)$ and $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(\mathsf{G})$. Then there exists a transfer $f^{\mathsf{G}^{\mathfrak{e}}}$ of f to $\mathsf{G}^{\mathfrak{e}}$.*

Moreover, suppose $\mathcal{K}$ is a θ-stable hyperspecial maximal compact open subgroup of $\mathsf{G}^{\mathfrak{e}}(K)$. Then there exists a hyperspecial maximal compact open subgroup $\mathcal{K}^{\mathfrak{e}} \subset \mathsf{G}^{\mathfrak{e}}(K)$ such that the characteristic function $\mathbf{1}_{\mathcal{K}^{\mathfrak{e}}}$ is a transfer of the characteristic function $\mathbf{1}_{\mathcal{K} \times \theta}$ to $\mathsf{G}^{\mathfrak{e}}$, provided the Haar measure is chosen such that $\mathrm{Vol}(\mathcal{K}) = \mathrm{Vol}(\mathcal{K}^{\mathfrak{e}}) = 1$.

A.3. **Cuspidal functions.** In this subsection, we recall the definition of cuspidal and stabilizing functions, following Labesse [Lab99, Definition 3.8.1, 3.8.2]. Recall that $\gamma \in \mathsf{G}(K)$ is called an elliptic element if the maximal split sub-torus of the center of $Z_{\mathsf{G}}(\gamma)$ is equal to the maximal split sub-torus of $Z(\mathsf{G})$.

Definition A.3.1. Suppose K is non-Archimedean of characteristic zero. A function $\phi \in \mathcal{H}(\mathsf{G})$ is called
- *cuspidal* if the orbital integral of ϕ vanishes on all regular semisimple non-elliptic elements.
- *strongly cuspidal* if the orbital integral of ϕ vanishes outside regular semisimple elliptic elements.
- *stabilizing* if it is cuspidal and the κ-orbital integral of ϕ (as defined in [Lab99, p. 68]) vanishes on all semisimple elements γ and all nontrivial κ.

We recall the notion of pseudo-coefficients of [Kaz86, Clo86]:

Definition A.3.2. Fix a Haar measure on $\mathsf{G}(K)$. For a square-integrable irreducible admissible representation π of $\mathsf{G}(K)$, a function $f_{\pi} \in \mathcal{H}(\mathsf{G}(K))$ is called a *pseudo-coefficient* for π if $\mathrm{tr}(f_{\pi}|\pi') = \delta_{\pi,\pi'}$ for each tempered representation π' of $\mathsf{G}(K)$.

Proposition A.3.3 ([Kaz86, Clo86])**.** *Suppose K is non-Archimedean of characteristic zero, and fix a Haar measure on $\mathsf{G}(K)$. For any square-integrable irreducible admissible representation π of $\mathsf{G}(K)$, there exists a pseudo-coefficient f_{π}. Moreover, for any such f_{π}:*
(1) *$\mathrm{tr}(f_{\pi}|\pi') = 0$ for any finite-length admissible smooth representation π' of $\mathsf{G}(K)$ that is parabolically induced from a properly contained parabolic subgroup of G.*
(2) *For every regular elliptic element $\gamma \in \mathsf{G}(K)$,*

$$\int_{\mathsf{G}(K)} f_{\pi}(g^{-1}xg)\mathrm{d}g = \Theta_{\pi}(\gamma),$$

where Θ_{π} denotes the Harish-Chandra character of π.
(3) *f_{π} is cuspidal.*
(4) *If π is supercuspidal, then $\mathrm{tr}(f_{\pi}|\pi') = \delta_{\pi,\pi'}$ for every irreducible admissible representation π' of $\mathsf{G}(K)$.*

Proof. The existence of f_π is established by Kazhdan [Kaz86, Theorem K] and Clozel [Clo86, Proposition 1]. The trace values $\Theta_{\pi'}(f_\pi)$ for admissible representations π' of $\mathsf{G}(K)$ and the orbital integrals $\int_{\mathsf{G}(K)} f_\pi(g^{-1}xg)\mathrm{d}g$ for any regular elliptic element $\gamma \in \mathsf{G}(K)$ are independent of f_π chosen by [Kaz86, Theorem 0]. It is built in from Kazhdan's construction [Kaz86, Theorem K] and also [Kaz86, Theorem A.(b)] that (i) and (iii) is true. (ii) follows from [Kaz86, Theorem K].

(iv) follows from (i) because any non-supercuspidal admissible irreducible representation π' of $\mathsf{G}(K)$ has image in the Grothendieck group of finite-length admissible representations of $\mathsf{G}(K)$ given by a linear combinations of representations induced from proper parabolic subgroups of G (see [Clo86, Proposition 2]), so $\mathrm{tr}(f_\pi|\pi') = 0$. $\qquad\square$

A.4. **Simple stable trace formulas.** Here we recall some results on simple stable trace formulas from [KS23, Ham22]. We work in the following setting.

Notation A.4.1.
- F is a totally real number field.
- $\mathbb{G}^*$ is a quasi-split reductive group over F that is simple over $\overline{F}$; assume $\mathbb{G}(F \otimes \mathbb{R})$ admits discrete series.
- $(\mathbb{G}, \varrho, z)$ is a pure inner twist of $\mathbb{G}^*$.
- Let Z denote the center of $\mathbb{G}^*$, and let A_Z denote the maximal split torus of $\mathrm{Res}_{F/\mathbb{Q}}Z$; write $A_{Z,\infty} := A_Z(\mathbb{R})^\circ$.
- Let $\mathscr{K}_\infty = \prod_{v \in \Sigma_F^\infty} \mathscr{K}_v \leq \mathbb{G}(F \otimes \mathbb{R})$ be the product of a maximal compact subgroup with $Z_\mathbb{G}(F \otimes \mathbb{R})$.
- For each finite place v of F, set $q(\mathbb{G}_v)$ to be the F_v-rank of $\mathbb{G}_{v,\mathrm{ad}}$.
- For each infinite place v of F, set $q(\mathbb{G}_v)$ to be the real dimension of the locally symmetric space $\mathbb{G}(F_v)/\mathscr{K}_v$.
- Set
$$\mathbb{G}(\mathbf{A}_F)^1 := \bigcap_{\chi \in X^*(\mathbb{G})} \ker(\|-\| \circ \chi : \mathbb{G}(\mathbf{A}_F) \to \mathbb{R}_+).$$
In particular, $\mathbb{G}(\mathbf{A}_F) = \mathbb{G}(\mathbf{A}_F)^1 \times A_{Z,\infty}$.

Definition A.4.2. A *central character datum* for $\mathbb{G}$ is a pair $(\mathfrak{X}, \chi)$ where $\mathfrak{X}$ is closed subgroup of $Z(\mathbf{A}_F)$ containing $A_{Z,\infty}$ such that $Z(F)\mathfrak{X}$ is closed in $Z(\mathbf{A}_F)$, and $\chi : (\mathfrak{X} \cap Z(F))\backslash\mathfrak{X} \to \mathbb{C}^\times$ is a continuous character. In particular, $Z(F)\mathfrak{X}$ is cocompact in $Z(\mathbf{A}_F)$ because $Z(F)\backslash Z(\mathbf{A}_F)/A_{Z,\infty}$ is compact.

For our purposes, it suffices to consider the cases when $\mathfrak{X} = \prod_{v \in \Sigma_F} \mathfrak{X}_v$ where $\mathfrak{X}_\tau = Z(F_\tau)$ for each $\tau \in \Sigma_F^\infty$.

Note that the center of $\mathbb{G}$ is are isomorphic to Z via ϱ, so any central character datum for $\mathbb{G}^*$ may be regarded as a central character datum for $\mathbb{G}$.

Remark A.4.3. We suppress the choice of Haar measures for various groups below as they are standard.

Definition A.4.4. Given a central character datum $(\mathfrak{X}, \chi)$ for $\mathbb{G}^*$ of the form $\mathfrak{X} = \prod_{v \in \Sigma_F} \mathfrak{X}_v$, for each $v \in \Sigma_F$, let $\mathcal{H}(\mathbb{G}(F_v), \chi_v^{-1})$ be the space of smooth functions on $\mathbb{G}(F_v)$ that is compactly supported modulo center and transforms under $\mathfrak{X}_v$ by the character χ_v^{-1} and moreover $\mathscr{K}_v$-finite when $v \in \Sigma_F^\infty$.

Given a semisimple element $\gamma_v \in \mathbb{G}(F_v)$ with $I_{\gamma_v} = Z_{\mathbb{G}_{F_v}}(\gamma_v)^\circ$, we define the orbital integral on $\mathcal{H}(\mathbb{G}(F_v), \chi_v^{-1})$ to be
$$\mathrm{Orb}_{\gamma_v}(f_v) := \int_{I_{\gamma_v}\backslash\mathbb{G}(F_v)} f_v(x_v^{-1}\gamma_v x_v)\mathrm{d}x_v,$$
where $I_{\gamma_v}\backslash\mathbb{G}(F_v)$ is given the Euler-Poincaré measure as defined in [Kot88, §1]. If π_v is an admissible representation of $\mathbb{G}(F_v)$ with central character χ_v on $\mathfrak{X}_v$, we define the trace character on $\mathcal{H}(\mathbb{G}(F_v), \chi_v^{-1})$ to be
$$\Theta_{\pi_v}(f_v) := \mathrm{tr}\left(\int_{\mathbb{G}(F_v)/\mathfrak{X}_v} f_v(g)\pi_v(g)\mathrm{d}g\right).$$

We also define the adelic Hecke algebra $\mathcal{H}(\mathbb{G}(\mathbf{A}_F), \chi^{-1})$ as well as adelic orbital integrals and adelic trace characters by taking restricted tensor product over the local cases considered above.

Definition A.4.5. Given a central character datum $(\mathfrak{X}, \chi)$ for $\mathbb{G}^*$, we write

- $\Gamma_{\mathrm{ell},\mathfrak{X}}(\mathbb{G})$ for the set of $\mathfrak{X}$-orbits of elliptic conjugacy classes in $\mathbb{G}(F)$,
- $\Sigma_{\mathrm{ell},\mathfrak{X}}(\mathbb{G})$ for the set of $\mathfrak{X}$-orbits of elliptic stable conjugacy classes in $\mathbb{G}(F)$.
- $L^2_{\mathrm{disc},\chi}(\mathbb{G}(F)\backslash\mathbb{G}(\mathbf{A}_F))$ for the space of measurable functions on $\mathbb{G}(F)\backslash\mathbb{G}(\mathbf{A}_F)$ transforming under $\mathfrak{X}$ by χ and square-integrable on $\mathbb{G}(F)\backslash\mathbb{G}(\mathbf{A}_F)^1 / \left(\mathfrak{X} \cap \mathbb{G}(\mathbf{A}_F)^1\right)$,
- $L^2_{\mathrm{cusp},\chi}(\mathbb{G}(F)\backslash\mathbb{G}(\mathbf{A}_F))$ for the space of cuspidal measurable functions on $\mathbb{G}(F)\backslash\mathbb{G}(\mathbf{A}_F)$ transforming under $\mathfrak{X}$ by χ and square-integrable on $\mathbb{G}(F)\backslash\mathbb{G}(\mathbf{A}_F)^1 / \left(\mathfrak{X} \cap \mathbb{G}(\mathbf{A}_F)^1\right)$,
- $\mathrm{Irr}^{\mathrm{cusp}}_\chi(\mathbb{G})$ the set of isomorphism classes of cuspidal automorphic representations of $\mathbb{G}(\mathbf{A}_F)$ whose central characters under $\mathfrak{X}$ are χ.

We define the following invariant distributions on $\mathcal{H}(\mathbb{G}(\mathbf{A}_F), \chi^{-1})$:

$$\mathrm{T}^{\mathbb{G}}_{\mathrm{ell},\chi}(f) := \sum_{\gamma \in \Gamma_{\mathrm{ell},\mathfrak{X}}(\mathbb{G})} \frac{1}{\#\pi_0(Z_{\mathbb{G}}(\gamma))} \mathrm{Vol}\left(I_\gamma(F)\backslash I_\gamma(\mathbf{A}_F)/\mathfrak{X}\right) \mathrm{Orb}_\gamma(f),$$

$$\mathrm{T}^{\mathbb{G}}_{\mathrm{disc},\chi}(f) := \mathrm{tr}\left(f|L^2_{\mathrm{disc},\chi}(\mathbb{G}(F)\backslash\mathbb{G}(\mathbf{A}_F))\right),$$

$$\mathrm{T}^{\mathbb{G}}_{\mathrm{cusp},\chi}(f) := \mathrm{tr}\left(f|L^2_{\mathrm{cusp},\chi}(\mathbb{G}(F)\backslash\mathbb{G}(\mathbf{A}_F))\right),$$

Next, we recall the definition of unramified twists of Steinberg representations and Lefschetz functions.

Definition A.4.6. The Steinberg representation $\mathrm{St}_{\mathbb{G}_v}$ is the discrete series representation defined in [BW00, 10.4.6]. An unramified twist of $\mathrm{St}_{\mathbb{G}_v}$ is just the twist of the Steinberg representation by an unramified character of $\mathbb{G}(F_v)$, where a character of $\mathbb{G}(F_v)$ is unramified if it is trivial on all compact subgroups of $\mathbb{G}(F_v)$; see [Cas95, p. 17].

By [Kot88, Theorem 2 and Theorem 2'] and [KS23, Proposition A.1, A.4 and Lemma A.7], we introduce the following non-Archimedean Lefschetz function:

Definition A.4.7. Let v be a finite place of F. If Z_{F_v} is anisotropic, there exists a *Lefschetz function* $f^{\mathbb{G}}_{\mathrm{Lef},v} \in \mathcal{H}(\mathbb{G}(F_v))$ such that

- If $\mathbb{G}_{\mathrm{ad}}$ is simple, then for each irreducible admissible representation π of $\mathbb{G}(F)$,

$$\mathrm{tr}\left(f^{\mathbb{G}}_{\mathrm{Lef},v}|\pi\right) = \begin{cases} 1 & \pi = \mathbf{1} \\ (-1)^{q(\mathbb{G}_v)} & \pi = \mathrm{St}_{\mathbb{G}_v} \\ 0 & \text{otherwise} \end{cases}.$$

- If $\gamma_v \in \mathbb{G}(F_v)$ is semisimple with $I_{\gamma_v} = Z_{\mathbb{G}_v}(\gamma_v)^\circ$, then the orbital integral

$$\mathrm{Orb}_{\gamma_v}(f^{\mathbb{G}}_{\mathrm{Lef},v}) = \int_{I_{\gamma_v}(F_v)\backslash\mathbb{G}(F_v)} f^{\mathbb{G}}_{\mathrm{Lef},v}(g_v^{-1}\gamma_v g_v)\mathrm{d}g_v$$

  vanishes unless $Z(I_{\gamma_v}(F))$ is compact, in which case $\mathrm{Orb}_{\gamma_v}(f^{\mathbb{G}}_{\mathrm{Lef},v}) = 1$. Here $I_{\gamma_v}\backslash\mathbf{G}(F_v)$ is endowed with the Euler–Poincaré measure.

In general, set A_v to be the maximal split torus of $\mathbb{G}_v$, and set $\mathbb{G}'_v = \mathbb{G}_v/A_v$. Let $\nu : \mathbb{G}(F) \to X^\bullet(A_v)\otimes\mathbb{R}$ denote the valuation map as in [Lab99, §3.9] with kernel $\mathbb{G}(F)^1$, and set

$$f^{\mathbb{G}_v}_{\mathrm{Lef},v} = \mathbf{1}_{\mathbb{G}(F)^1} \cdot f^{\mathbb{G}'_v}_{\mathrm{Lef},v}.$$

- $f^{\mathbb{G}}_{\mathrm{Lef},v}$ is strongly cuspidal and stabilizing (see Definition A.3.1);
- If $\mathrm{tr}(f^{\mathbb{G}}_{\mathrm{Lef},v}|\pi_v) \neq 0$ for some irreducible unitary representation π_v of $\mathbb{G}_v$, then π_v is an unramified character twist of either the trivial representation or the Steinberg representation (see Definition A.4.6);
- $(-1)^{q(\mathbb{G}_v)} f^{\mathbb{G}}_{\mathrm{Lef},v}$ and $(-1)^{q(\mathbb{G}^*_v)} f^{\mathbb{G}^*}_{\mathrm{Lef},v}$ are associated.

If $\mathbf{G}(F \otimes \mathbb{R})$ admits discrete series, we introduce the following Archimedean Lefschetz function:

Definition A.4.8. Suppose $\tau \in \Sigma_F^\infty$ and ξ_τ is an irreducible algebraic representation of $\mathbb{G}_\tau$ with regular highest weight. Denote by $\chi_{\xi_\tau} : Z(F_\tau) \to \mathbb{C}^\times$ the inverse of the central character of ξ_τ. Then there exists a *Lefschetz function* $f_{\xi_\tau}^\mathbb{G} \in \mathcal{H}(\mathbb{G}(F \otimes \mathbb{R}), \chi_{\xi_\tau}^{-1})$ associated to ξ_τ such that

$$\mathrm{tr}(f_{\xi_\tau}|\pi_\tau) = \mathrm{ep}_{\mathscr{K}_\tau}(\pi_\tau \otimes \xi_\tau) := \sum_{i \in \mathbb{N}} (-1)^i \dim \mathrm{H}^i(\mathrm{Lie}(\mathbb{G}_\tau(\mathbb{R})), \mathscr{K}_\tau; \pi_\tau \otimes \xi_\tau)$$

for each irreducible admissible representation π_τ of $\mathbb{G}_\tau(\mathbb{R})$ whose central character equals the inverse of the central character of ξ.

For any irreducible admissible $(\mathfrak{g}_\tau, \mathscr{K}_\tau)$-module π_v such that $\mathrm{tr}\big(f_{\xi_\tau}^\mathbb{G}|\pi_\tau\big) \neq 0$, we know π_τ is a discrete series representation cohomological for ξ (i.e., π_τ has the same central character and infinitesimal character as $\xi^\vee$), and $\mathrm{tr}\big(f_{\xi_\tau}^\mathbb{G}|\pi_\tau\big) = (-1)^{q(\mathbb{G}_\tau)}$, by Vogan–Zuckerman's classification of unitary cohomological representations; cf. [Shi12, Lemma 2.7].

Definition A.4.9. Given a central character datum $(\mathfrak{X}, \chi)$ for $\mathbb{G}^*$, we define the stably invariant distributions on $\mathcal{H}(\mathbb{G}^*(\mathbf{A}_F), \chi^{-1})$:

$$\mathrm{ST}_{\mathrm{ell}, \chi}^{\mathbb{G}^*}(f^*) := \tau(\mathbb{G}^*) \sum_{\gamma \in \Sigma_{\mathrm{ell}, \mathfrak{X}}(\mathbb{G}^*)} \frac{1}{\#\pi_0(Z_\mathbb{G}(\gamma))^{\mathrm{Gal}_F}} \mathrm{SOrb}_{\gamma, \chi}^{\mathbb{G}^*}(f^*),$$

where $\tau(\mathbb{G}^*)$ is the Tamagawa number of $\mathbb{G}^*$ and

$$\mathrm{SOrb}_{\gamma, \chi}^{\mathbb{G}^*} = \sum_{\gamma'} \mathrm{Orb}_{\gamma, \chi}^{\mathbb{G}^*}(f^*),$$

is the stable orbital integral at γ, where γ' runs through (a set of representatives for) the set of F-conjugacy classes inside the stable conjugacy class of γ.

Theorem A.4.10. *Fix a central character datum $(Z(F \otimes \mathbb{R}), \chi)$ where χ equal to the inverse of central character of some irreducible representation ξ of $\mathbb{G}_\mathbb{C}$ with regular weight, and suppose $f \in \mathcal{H}(\mathbb{G}(\mathbf{A}_F), \chi^{-1})$ such that*

- *$f_\infty = f_\xi^\mathbb{G}$ is a Lefschetz function,*

- *there exists a finite place v of F where $f_v = f_{\mathrm{Lef}, v}^\mathbb{G}$ is a Lefschetz function,*

then

$$\mathrm{ST}_{\mathrm{ell}, \chi}^{\mathbb{G}^*}(f^*) = \mathrm{T}_{\mathrm{ell}, \chi}^\mathbb{G}(f) = \mathrm{T}_{\mathrm{disc}, \chi}^\mathbb{G}(f) = \mathrm{T}_{\mathrm{cusp}, \chi}^\mathbb{G}(f),$$

where $f^* \in \mathcal{H}(\mathbb{G}^*(\mathbf{A}_F, \chi^{-1})$ is a transfer of f to $\mathbb{G}^*$.

Proof. This follows from [KS23, Lemma 6.1, 6.2]. $\square$

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