THE ENDOSCOPIC CHARACTER IDENTITY FOR EVEN SPECIAL ORTHOGONAL GROUPS

HAO PENG

Abstract. We establish the endoscopic character identity for certain bounded A-packets of non-quasi-split even special orthogonal groups, with respect to elliptic endoscopic triples. The proof reduces the non-quasi-split case to the quasi-split case and the real Adams–Johnson case by combining the local-global compatibility principle with Arthur's multiplicity formula for non-quasi-split global even special orthogonal groups established by Chen and Zou [CZ24]. This result plays a key role in the author's work [Pen25] on the compatibility between the Fargues–Scholze local Langlands correspondence and the classical local Langlands correspondence for even special orthogonal groups.

Contents

1. Introduction

In his monumental book [Art13], Arthur established the local Langlands correspondence for symplectic and quasi-split special orthogonal groups over local fields K of characteristic zero (the endoscopic classification), along with a description of the automorphic discrete spectrum for these groups over number fields (Arthur's multiplicity formula), via the stable trace formula and the theory of endoscopy. However, for quasi-split even special orthogonal groups, the classification is given only for irreducible representations of $\mathrm{SO}(V)$ up to conjugation by $\mathrm{O}(V)$, rather than by $\mathrm{SO}(V)$ itself. Similarly, over a number field F, Arthur's multiplicity formula does not distinguish between a square-integrable automorphic representation π and its twist by the outer automorphism associated with an element of $\mathrm{O}(\mathbf{V}) \setminus \mathrm{SO}(\mathbf{V})$.

In [CZ21a] and [CZ21b], Chen–Zou extended Arthur's endoscopic classification to non-quasi-split even special orthogonal groups. In [CZ24], they further extended Arthur's multiplicity formula to non-quasi-split even special orthogonal groups. (Strictly speaking, their results are stated for non-quasi-split even orthogonal groups, but this implies the corresponding result for special orthogonal groups via the argument of Atobe–Gan [AG17]). Their approach relies on local (respectively, global) theta correspondences between even orthogonal groups and symplectic groups. As in other cases where the local Langlands correspondence was established via theta correspondence (e.g. [GT11], [GS12]), they didn't the conjectural endoscopic character identity for the L-packets (respectively, "theta packets") as formulated in [Kal16].

The main goal of this short paper is to verify the conjectural endoscopic character identity for certain local A-parameters that are bounded on the Weil group. As in the quasi-split case [Art13, Theorem 2.2.1], the character identity holds only up to outer automorphism. Let V be a quadratic space of dimension $2n$ over a non-Archimedean local field K of characteristic zero, and let $G = \mathrm{SO}(V)$ be the associated special orthogonal group. Suppose we are given a (bounded) A-parameter ψ of G and an elliptic endoscopic triple $\mathfrak{e} = (G^{\mathfrak{e}}, s^{\mathfrak{e}}, \xi^{\mathfrak{e}})$ for G (see Section 3), together with an A-parameter $\psi^{\mathfrak{e}}$ of $G^{\mathfrak{e}}$ such that $\xi^{\mathfrak{e}} \circ \psi^{\mathfrak{e}} = \psi$. Then $G^{\mathfrak{e}}$ is a product of two (possibly trivial) quasi-split special orthogonal groups $G_1 = \mathrm{SO}(V_1)$ and $G_2 = \mathrm{SO}(V_2)$ over K. Arthur [Art13] assigns to $\psi^{\mathfrak{e}}$ an A-packet $\tilde{\Pi}_{\psi^{\mathfrak{e}}}(G^{\mathfrak{e}})$ consisting of $\mathrm{O}(V_1) \times \mathrm{O}(V_2)$-conjugacy classes of irreducible unitarizable representations of $G^{\mathfrak{e}}(K)$. Similarly, in [CZ21b], a "theta packet" $\tilde{\Pi}_{\psi}(G)$ is assigned to ψ via theta correspondence, consisting of $\mathrm{O}(V)$-conjugacy classes of irreducible unitarizable representations of $G(K)$. For notational uniformity, we will also refer to them as A-packets throughout this paper.

Let G^* be the unique quasi-split inner form of G. Then G can be realized as a pure inner twist (ϱ, z) of G^*. Fix an additive character ψ_K of K and a pinning $(B^*, T^*, \{X_\alpha^*\}_{\alpha\in\Delta})$ of G^*, which together determine a Whittaker datum $\mathfrak{m}$ for G^*, cf. [KS99, §5.3]. The Whittaker datum $\mathfrak{m}$ and the pure inner twist (ϱ, z) determine a canonical map

$$\iota_{\mathfrak{m},z} : \tilde{\Pi}_{\psi}(G) \to \mathrm{Irr}(\mathfrak{S}_{\psi}), \quad \text{where } \mathfrak{S}_{\psi} := \pi_0(Z_{\widehat{G}}(\mathrm{Im}(\psi))),$$

as defined in Theorem 2.2.1.

There exists an outer automorphism ς of G^* that preserves $\mathfrak{m}$. In fact, ς can be realized as an element of the corresponding orthogonal group with determinant -1, cf. [Taï19, p. 847]. Via the isomorphism ϱ, the element ς induces a rational outer automorphism of G, cf. [Art13, Lemma 9.1.1]. Let $\mathcal{H}(G)$ denote the space of smooth compactly supported complex-valued functions on $G(K)$. Following Arthur, let $\tilde{\mathcal{H}}(G)$ denote the subspace of $\mathcal{H}(G)$ consisting of ς-invariant functions on $G(K)$. Irreducible smooth representations of $\tilde{\mathcal{H}}(G)$ then correspond to $\mathrm{O}(V)$-conjugacy classes of irreducible admissible representations of $G(K)$. Similarly, define $\tilde{\mathcal{H}}(G^{\mathfrak{e}}) := \tilde{\mathcal{H}}(G_1) \times \tilde{\mathcal{H}}(G_2)$ as a subspace of $\mathcal{H}(G^{\mathfrak{e}})$.

Fix Haar measures on $G(K)$ and $G^{\mathfrak{e}}(K)$. For each $\mathrm{O}(V)$-conjugacy class $\tilde{\pi}$ of admissible irreducible representations of $G(K)$, define the character distribution $\tilde{\Theta}_{\tilde{\pi}}$ of $\tilde{\pi}$ as follows. Choose a representative π of $\tilde{\pi}$, and define $\tilde{\Theta}_{\tilde{\pi}}$ as the restriction of the distribution Θ_π to the subspace $\tilde{\mathcal{H}}(G)$. This is independent of the representative π chosen. Arthur [Art13] defined a distribution

$$\tilde{\Theta}_{\psi^{\mathfrak{e}}} = \sum_{\tilde{\pi}\in\tilde{\Pi}_{\psi^{\mathfrak{e}}}(G^{\mathfrak{e}})} \langle \tilde{\pi}, s_{\psi^{\mathfrak{e}}} \rangle \, \tilde{\Theta}_{\tilde{\pi}},$$

which is well-defined for $f^{G^{\mathfrak{e}}} \in \tilde{\mathcal{H}}(G^{\mathfrak{e}})$. Here $\langle \tilde{\pi}, - \rangle$ denotes the character of $\mathfrak{S}_{\psi}$ defined by Arthur [Art13, Theorem 2.2.1], and $s_{\psi^{\mathfrak{e}}}$ is the image of $-1 \in \mathrm{SL}_2(\mathbb{C})$ under $\psi^{\mathfrak{e}}$. The distribution $\tilde{\Theta}_{\psi^{\mathfrak{e}}}$ is stable; that is, $\tilde{\Theta}_{\psi^{\mathfrak{e}}}(f^{G^{\mathfrak{e}}}) = 0$ whenever all stable orbital integrals of $f^{G^{\mathfrak{e}}}$ vanish.

We prove the endoscopic character identity for a certain class of A-parameters. Let ψ be an A-parameter of G such that

$$\psi = \sum_i \psi_i \boxtimes \mathrm{sp}_{b_i} : W_K \times \mathrm{SU}(2) \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$$

as a representation of $\mathrm{WD}_K \times \mathrm{SL}_2(\mathbb{C})$, where $b_i \neq b_j$ for $i \neq j$. We say that ψ is *admissible* if each b_i is odd and each $\dim(\psi_i)$ is even, see Section 2.2. In particular, every tempered A-parameter (i.e. a tempered L-parameter) is admissible. The following is the desired endoscopic character identity for G.

Theorem A. *Suppose ψ is an admissible A-parameter. Then*

$$\tilde{\Theta}_{\psi^{\mathfrak{e}}}(f^{G^{\mathfrak{e}}}) = \sum_{\tilde{\pi} \in \tilde{\Pi}_\psi(G)} \iota_{\mathfrak{m},z}(\tilde{\pi})(s^{\mathfrak{e}} s_\psi) \cdot \tilde{\Theta}_{\tilde{\pi}}(f)$$

for any pair of $\Delta[\mathfrak{m}, \mathfrak{e}, z]$-matching functions $f \in \tilde{\mathcal{H}}(G)$ and $f^{G^{\mathfrak{e}}} \in \tilde{\mathcal{H}}(G^{\mathfrak{e}})$ (see Section 3.3). Here, $\Delta[\mathfrak{m}, \mathfrak{e}, z]$ denotes the transfer factor between G and $G^{\mathfrak{e}}$ defined by Kaletha [Kal16].

Remark 1.0.1. There is no appearance of the Kottwitz sign $e(G)$ [Kot83], since it is always equal to 1 for even special orthogonal groups over K.

The main theorem was established by Arthur [Art13, Theorem 2.2.1] in the quasi-split case. This paper deals with the non-quasi-split case. Our strategy is to reduce to the quasi-split case via the localglobal compatibility principle. Suppose $\psi^{\mathfrak{e}}$ is discrete and admissible (see Section 2.2). We globalize the pair $(G, G^{\mathfrak{e}})$ over a totally real number field F, with $\mathbf{G}$ quasi-split at all finite places except one (where its localization is G); such that both $\mathbf{G}(F \otimes \mathbb{R})$ and $\mathbf{G}^{\mathfrak{e}}(F \otimes \mathbb{R})$ admit discrete series.

We globalize the local parameters $\psi^{\mathfrak{e}}$ and ψ to suitable elliptic global A-parameters of Adams–Johnson type at Archimedean places. We then apply the multiplicity formula established by Chen–Zou [CZ24]. Combined with Arthur's stable multiplicity formula [Art13], this implies that the endoscopic character identity for G follows if it holds for all other localizations of $\mathbf{G}$.

In the general case where $\psi^{\mathfrak{e}}$ is not discrete, we apply parabolic descent to reduce to the discrete case. Similar ideas were employed in the study of endoscopic character identities for inner forms of GSp(4) [CG15] and metaplectic groups [Luo20] over non-Archimedean local fields. We restrict our attention to admissible A-parameters, since for more general parameters, the globalized parameter may factor through the endoscopic group $\mathbf{G}^{\mathfrak{e}}$ in multiple ways, making it difficult to distinguish the contribution from the specified local parameter $\psi^{\mathfrak{e}}$.

The main result of this paper has been applied in the author's earlier work [Pen25], which establishes the compatibility between the Fargues–Scholze and classical local Langlands correspondences for even special orthogonal groups. Furthermore, using that compatibility, the author upgraded the ambiguous local Langlands correspondence to an "unambiguous" version—that is, not merely defined up to outer automorphism, cf. [Pen25, § 7.1].

Acknowledgements. The author would like to thank Rui Chen and Jialiang Zou for introducing him to the local Langlands correspondence for even orthogonal groups. He is also grateful to his advisor, Prof. Wei Zhang, for his detailed feedback and valuable suggestions. He further thanks Prof. Hiraku Atobe for encouraging him to establish the main result for bounded A-parameters rather than merely for tempered L-parameters.

1.1. **Notation and conventions.**

Notation 1.1.1.

- Let $\mathbb{N}$ denote the set of non-negative integers and $\mathbb{Z}_+$ the set of positive integers. As usual, we write $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ for the integers, rational numbers, real numbers, and complex numbers, respectively.

- Suppose X is a set.
  - Let $\mathbf{1} \in X$ denote the distinguished trivial element (this notation is only used when the notion of triviality is clear from context).
  - Let $\#X$ be the cardinality of X.
  - For $a, b \in X$, we define the Kronecker symbol

$$\delta_{a,b} := \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}.$$

- For a finite group A, write $\mathrm{Irr}(A)$ for the set of isomorphism classes of its irreducible complex representations.
- For a perfect field K, write Gal_K for its absolute Galois group $\mathrm{Gal}(\overline{K}/K)$.
- If K is a local field of characteristic zero, write W_K for its Weil group and WD_K for its Weil–Deligne group:

$$\mathrm{WD}_K = \begin{cases} W_K & \text{if } K = \mathbb{R} \text{ or } \mathbb{C}, \\ W_K \times \mathrm{SU}(2) & \text{otherwise.} \end{cases}$$

- If K is non-Archimedean, let I_K denote its inertia group, and $|-|_K$ the normalized absolute value on $K^\times$. Via the Artin map $\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$, we extend $|-|_K$ to a character $W_K \to \mathbb{R}_+$.
- Suppose F is a number field.
  - Let Σ_F^{fin} denote the set of finite places of F, and $\Sigma_F^\infty := \mathrm{Hom}(F, \mathbb{C})$ the set of infinite places; set $\Sigma_F = \Sigma_F^{\mathrm{fin}} \cup \Sigma_F^\infty$.
  - Write $\mathbf{A}_F$ for the ring of adeles of F, $\mathbf{A}_{F,f}$ for the finite adeles, and $\mathbf{C}_F$ for the idele class group.
- Suppose K is a local field and G is a connected reductive group over K. Let $\mathcal{H}(\mathsf{G})$ denote the test function space on $\mathsf{G}(K)$:

$$\mathcal{H}(\mathsf{G}(K)) = \begin{cases} C_c^\infty(\mathsf{G}(K)) & \text{if } K \text{ is non-Archimedean,} \\ \{\, f \in C_c^\infty(\mathsf{G}(\mathbb{R})) : \ f \text{ is bi-}\mathcal{K}\text{-finite} \,\} & \text{if } K = \mathbb{R}, \end{cases}$$

  where $\mathcal{K} \subset \mathsf{G}(K)$ is a fixed maximal compact subgroup.

2. Local A-packets

In this section, we recall the properties of (ambiguous) A-packets following [AG17, CZ21a].

2.1. **The groups.** Suppose K is a local or global field of characteristic zero. Let V be a $2n$-dimensional K-vector space (with $n \in \mathbb{Z}_+$), equipped with a non-degenerate symmetric bilinear form $\langle -, - \rangle$, that is,

$$\langle au + bv, w \rangle = a \langle u, w \rangle + b \langle v, w \rangle, \quad \text{and} \quad \langle v, w \rangle = \langle w, v \rangle$$

for all $a, b \in K$ and $u, v \in V$.

Fix an orthogonal basis $\{v_1, \dots, v_{2n}\}$ of V over K such that $\langle v_i, v_i \rangle = a_i \in K^\times$. Define the discriminant of V as

$$\mathrm{disc}(V) = (-1)^n \prod_{i=1}^{2n} a_i$$

whose class in $K^\times/(K^\times)^2$ is independent of the choice of orthogonal basis.

The (normalized) Hasse–Witt invariant

$$\epsilon(V) = \left(-1, (-1)^{\binom{n}{2}} \cdot \mathrm{disc}(V)^{n-1}\right)_K \cdot \prod_{1 \le i < j \le 2n} (a_i, a_j)_K$$

where

$$(-,-)_K : \Big(K^\times/(K^\times)^2\Big) \times \Big(K^\times/(K^\times)^2\Big) \to \mathrm{Br}(K)[2] \cong \{\pm 1\}$$

denotes the Hilbert symbol.

We define $G := \mathrm{SO}(V)$. Then

- G is split if $\mathrm{disc}(V) = 1, \epsilon(V) = 1$;
- G is non-quasi-split if $\mathrm{disc}(V) = 1$ and $\epsilon(V) = -1$;
- G is quasi-split but non-split if $\mathrm{disc}(V) \neq 1$.

For each $\alpha \in K^\times/(K^\times)^2$, define the quasi-split special orthogonal group $G^* = \mathrm{SO}_{2n}^\alpha$ to be the special orthogonal group associated with a quadratic space over K with discriminant α and Hasse–Witt invariant 1. As in [Bor79, Remark 2.4.(2)], we denote by ${}^L G$ the L-group of G with component group $\mathrm{Gal}(K(\sqrt{\alpha})/K)$, and let $\widehat{G}$ denote its identity component. Then $\widehat{G} = \mathrm{SO}(2n, \mathbb{C})$, and there is a canonical identification

$$(2.1) \qquad {}^L G = \begin{cases} \mathrm{SO}(2n,\mathbb{C}) & \text{if } \alpha = 1, \\ \mathrm{O}(2n,\mathbb{C}) & \text{if } \alpha \neq 1. \end{cases}$$

Let $\widehat{\mathrm{Std}} : {}^L G \to \mathrm{GL}_{2n}(\mathbb{C})$ be the standard representation.

We fix a pinning of G^* by identifying it with $\mathrm{SO}(V^*)$ for a suitable quadratic space V^* over K, and choosing a complete flag of totally isotropic subspaces in V^*. Recall that a Whittaker datum for G^* is a $T^*(K)$-conjugacy class of generic characters of $N^*(K)$, where N^* is the unipotent radical of B^*. The set of Whittaker data for G^* forms a principal homogeneous space over the finite Abelian group

$$E = \mathrm{Coker}\,(G^*(K) \to G^*_{\mathrm{ad}}(K)) = \ker(H^1(K, Z(G^*)) \to H^1(K, G^*)),$$

cf. [GGP12, §9]. The fixed pinning $(B^*, T^*, \{X^*_\alpha\}_{\alpha\in\Delta})$ of G^*, together with the additive character ψ_K of K, determines a Whittaker datum $\mathfrak{m}$ for G^*, cf. [KS99, §5.3]. When G is unramified, there exists a unique $G(K)$-conjugacy class of hyperspecial maximal compact open subgroups compatible with $\mathfrak{m}$, in the sense of [CS80]. In this case, "unramified representations of $G(K)$" refers to those unramified with respect to such a hyperspecial subgroup.

We may realize G as a pure inner twist of G^*; that is, there exists an isomorphism $\varrho : G^*_{\overline{K}} \to G_{\overline{K}}$ and a 1-cocycle $z \in Z^1(\mathrm{Gal}_K, G^*)$ such that

$$\varrho^{-1}\sigma(\varrho) = \mathrm{Ad}(z(\sigma)) \quad \text{for all } \sigma \in \mathrm{Gal}_K\,.$$

For each parabolic pair (M, P) of G, there exists a unique standard parabolic pair (M^*, P^*) of G^* corresponding to (M, P) under ϱ, and this determines an equivalence class of pure inner twist of M^*, which we also denote by (ϱ, z) by abuse of notation.

We define $\mathrm{OAut}_N(G^*) := \mathrm{O}(2n,\mathbb{C})/\,\mathrm{SO}(2n,\mathbb{C})$, which is a subgroup of the outer automorphism group of $\widehat{G}$ of order two, and can be identified with the group of automorphisms of G^* preserving the fixed pinning. Let ς denote the nontrivial element in $\mathrm{OAut}_N(G^*)$. There is a rational action of ς on G via the inner twist ϱ, cf. [Art13, Lemma 9.1.1].

2.2. **Local A-parameters and A-packets.** In this subsection, we assume that K is a non-Archimedean local field. For any connected reductive group G over K, we let $\Psi(\mathsf{G})$ denote the set of conjugacy classes of A-parameters

$$\psi : \mathrm{WD}_K \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathsf{G}$$

such that $\psi(W_K)$ is a precompact subset of the target.[1]

Let G^* be the unique quasi-split inner form of G. Then the L-groups of G and G^* are canonically identified, so we identify $\Psi(\mathsf{G})$ with $\Psi(\mathsf{G}^*)$.

[1] Here we take the Langlands L-group ${}^L G$ in the reduced Weil form, as in [Bor79, Remark 2.4.(2)].

Let $m \in \mathbb{Z}_+$. An A-parameter for $\mathrm{GL}(m)_K$ can be regarded as an isomorphism class of m-dimensional representations

$$\psi : \mathrm{WD}_K \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(m, \mathbb{C}).$$

Every such representation is isomorphic to a finite direct sum of representations of the form $\rho \boxtimes S_a \boxtimes \mathrm{sp}_b$, where

- ρ is a smooth representation of W_K,
- S_a is the unique irreducible representation of $\mathrm{SU}(2)$ of dimension a; and
- sp_b is the unique irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension b.

An A-parameter ψ for $\mathrm{GL}(m)$ over K is said to be self-dual and irreducible if $\psi \cong \psi^\vee$ as irreducible representations of $\mathrm{WD}_K \times \mathrm{SL}_2(\mathbb{C})$. Following [GGP12, §3], we define the sign of a self-dual irreducible A-parameter ψ: There exists an isomorphism $f : \psi \cong \psi^\vee$ such that $f^\vee = b(\psi) f$ for some $b(\psi) \in \{\pm 1\}$. The value $b(\psi)$ is independent of the choice of f, and is called the sign of ψ. If $\psi = \rho \boxtimes S_a \boxtimes \mathrm{sp}_b$, where ρ is an irreducible representation of W_K, then

$$b(\psi) = b(\rho)(-1)^{a+b},$$

cf. [GGP12, Lemma 3.2], [KMSW14, §1.2.4].

We now restrict to the case $G = \mathrm{SO}(V)$. By [GGP12, Theorem 8.1] and [AG17, p. 365], there is a natural identification

$$\Psi(G^*) = \left\{ \begin{matrix} \text{admissible } \psi : \mathrm{WD}_K \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{O}(2n, \mathbb{C}) \\ \text{such that } \det(\psi) = (\mathrm{Art}_K^{-1}(-), \mathrm{disc}(V))_K \end{matrix} \right\} / \, \mathrm{SO}(2n, \mathbb{C})$$

Here ψ is called admissible if

(1) $\psi(\sigma_K)$ is semisimple;
(2) $\psi|_{I_K}$ is smooth;
(3) $\psi(W_K)$ is a precompact subset of the target;
(4) $\psi|_{\mathrm{SU}(2) \times \mathrm{SL}_2(\mathbb{C})}$ is algebraic.

The group $\mathrm{O}(2n, \mathbb{C})$ acts on $\Psi(G^*)$ by conjugation. Let $\tilde{\Psi}(G^*)$ denote the set of orbits, and write $\tilde{\psi} \in \Psi(G^*)$ for the image of $\psi \in \Psi(G^*)$ in $\tilde{\Psi}(G^*)$.

Any A-parameter $\psi \in \Psi(G^*)$ gives rise, via the standard representation $\widehat{\mathrm{Std}}$, to a self-dual representation ψ^{GL} of $\mathrm{WD}_K \times \mathrm{SL}_2(\mathbb{C})$ of dimension $2n$. The parameter ψ is determined by ψ^{GL} up to $\mathrm{O}(2n, \mathbb{C})$-conjugacy, cf. [GGP12, Theorem 8.1]. Writing

$$\psi^{\mathrm{GL}} = \sum_i m_i \rho_i \boxtimes S_{a_i} \boxtimes \mathrm{sp}_{b_i},$$

as representations of $\mathrm{WD}_K \times \mathrm{SL}_2(\mathbb{C})$, we define the following conditions:

- ψ is called *tempered* (or an L-parameter) if $\psi|_{\mathrm{SL}_2(\mathbb{C})} = \mathbf{1}$;
- ψ is called *discrete* if each ρ_i is self-dual and $m_i = 1$;
- ψ is called a *simple supercuspidal L-parameter* if it is tempered, discrete, trivial on the $\mathrm{SU}(2)$-component, and irreducible as a W_K-representation;
- ψ is called *elementary* if $m_i = 1$ and $1 \in \{a_i, b_i\}$ for all i;
- ψ is called *admissible* if all b_i are odd and ψ can be written as

$$\psi^{\mathrm{GL}} = \sum_j \phi_j \boxtimes \mathrm{sp}_{b_j},$$

where each ϕ_j is a representation of WD_K of even dimension. In particular, any tempered L-parameter is admissible.

We write ψ^{GL} as

$$\psi^{\mathrm{GL}}=\bigoplus_{i\in I_\psi^+} m_i\psi_i\oplus\bigoplus_{i\in I_\psi^-}2m_i\psi_i\oplus\bigoplus_{i\in J_\psi}m_i(\psi_i\oplus\psi_i^\vee),$$

where $m_i\in\mathbb{Z}_+$, and I_ψ^+,I_ψ^-,J_ψ index mutually inequivalent irreducible representations of $\mathrm{WD}_K\times\mathrm{SL}_2(\mathbb{C})$ such that

(1) For $i\in I_\psi^+$, ψ_i is self-dual with sign $+1$.

(2) For $i\in I_\psi^-$, ψ_i is self-dual with sign -1.

(3) For $i\in J_\psi$, ψ_i is not self-dual.

Then ψ is discrete if and only if $m_i=1$ for all i, and $I_\psi^-=J_\psi=\varnothing$.

For any $\psi\in\Psi(G^*)$, we define

$$S_\psi^\sharp:=\prod_{i\in I_\psi^+}\mathrm{O}(m_i,\mathbb{C})\times\prod_{i\in I_\psi^-}\mathrm{Sp}(2m_i,\mathbb{C})\times\prod_{i\in J_\psi}\mathrm{GL}(m_i,\mathbb{C}),$$

which formally represents the centralizer of ψ in ${}^LG$, and its formal component group

$$\mathfrak{S}_\psi^\sharp:=\pi_0(S_\psi^\sharp)\cong\bigoplus_{i\in I_\psi^+}(\mathbb{Z}/2)e_i,$$

where each e_i is a formal place-holder. There is a canonical identification

$$\mathrm{Irr}(\mathfrak{S}_\psi^\sharp)=\oplus_{i\in I_\psi^+}(\mathbb{Z}/2)e_i^\vee,\quad\text{where}\quad e_i^\vee(e_j)=\delta_{i,j}.$$

We also define

$$S_\psi:=Z_{\widehat{G}}(\psi)$$

which is naturally a subgroup of $S_\psi^\sharp$ of index at most two. There is a homomorphism

$$\det{}_\psi:\mathfrak{S}_\psi^\sharp\to\mathbb{Z}/2,\quad\sum_{i\in I_\psi^+}x_ie_i\mapsto\sum_{i\in I_\psi^+}x_i\dim(\psi_i),$$

and we define the formal component group $\mathfrak{S}_\psi:=\ker(\det_\psi)=\pi_0(S_\psi)$. We define the central element

$$z_\psi:=\sum_{i\in I_\psi^+}m_ie_i\in\mathfrak{S}_\psi,$$

and define the reduced component group

$$\overline{\mathfrak{S}}_\psi:=\mathfrak{S}_\psi/\langle z_\psi\rangle.$$

We denote by s_ψ the image of $-1\in\mathrm{SL}_2(\mathbb{C})$ under ψ, which lies in S_ψ.

If ψ_1 and ψ_2 are $\mathrm{O}(2n,\mathbb{C})$-conjugate, then $\psi_1^{\mathrm{GL}}=\psi_2^{\mathrm{GL}}$. In particular, the definitions of $\psi^{\mathrm{GL}},I_\psi^+,I_\psi^-,J_\psi,S_\psi^\sharp,S_\psi,\mathfrak{S}_\psi,\overline{\mathfrak{S}}_\psi,z_\psi$, and s_ψ are well-defined for $\tilde\psi\in\tilde\Psi(G^*)$.

There exists an outer automorphism ς of G^* preserving the Whittaker datum $\mathfrak{m}$. It can be realized as an element of the corresponding orthogonal group with determinant -1, cf. [Taï19, p. 847]. Via the inner twisting ϱ, the element ς induces a rational outer automorphism of G, cf. [Art13, Lemma 9.1.1]. Following Arthur, we define $\tilde{\mathcal{H}}(G)$ as the subspace of $\mathcal{H}(G)$ consisting of ς-invariant distributions on $G(K)$. The irreducible smooth representations of $\tilde{\mathcal{H}}(G)$ then correspond to $\mathrm{O}(V)$-conjugacy classes of irreducible admissible representations of $G(K)$.

We now assume that K is non-Archimedean. Let $\psi\in\Psi(G^*)$ be an A-parameter. Using theta correspondence, Chen–Zou [CZ21b] constructed a finite set $\tilde\Pi_\psi(G)$ of $\mathrm{O}(V)$-conjugacy classes of irreducible unitarizable representations of G, and a map

$$\iota_{\mathfrak{m},z}:\tilde\Pi_\psi(G)\to\mathrm{Irr}(\mathfrak{S}_\psi).$$

We refer to $\tilde{\Pi}_{\psi}(G)$ as the (ambiguous) A-packet associated with ψ, since both their packet and the map $\iota_{\mathfrak{m},z}$ agree with Arthur's packet [Art13] when G is quasi-split, see [CZ21b, Theorem 8.7]. The A-packet $\tilde{\Pi}_{\psi}(G)$ depends only on the $\mathrm{O}(2n,\mathbb{C})$-conjugacy class $\tilde{\psi}$ of ψ, and we may thus write it as $\tilde{\Pi}_{\tilde{\psi}}(G)$.

We restate their result in the following form.

Theorem 2.2.1. *For each $\tilde{\psi} \in \tilde{\Psi}(G^*)$, there exists a finite set $\tilde{\Pi}_{\tilde{\psi}}(G)$ of ς-conjugacy classes of irreducible unitarizable representations of $G(K)$, called the (ambiguous) A-packet associated with $\tilde{\psi}$. These A-packets satisfy the following properties:*

(1) If $\tilde{\psi} \in \tilde{\Psi}(G^*)$ is tempered, then every $\tilde{\pi} \in \tilde{\Pi}_{\tilde{\psi}}(G)$ is tempered; if $\tilde{\psi}$ is moreover discrete, then every $\tilde{\pi}$ lies in the discrete series.

(2) The packet $\tilde{\Pi}_{\tilde{\psi}}(G)$ depends only on G, and not on the choice of inner twist (ϱ, z). Given the fixed Whittaker datum $\mathfrak{m}$ of G^*, which induces a Whittaker datum on each standard Levi subgroup of G^*, there exists a canonical map

$$\iota_{\mathfrak{m},z} : \tilde{\Pi}_{\tilde{\psi}}(G) \to \mathrm{Irr}(\mathfrak{S}_{\tilde{\psi}}).$$

For each $\eta \in \mathrm{Irr}(\mathfrak{S}_{\tilde{\psi}})$, we write $\tilde{\pi}_{\mathfrak{m},z}(\tilde{\psi},\eta)$ for the direct sum (possibly zero) of elements in the fiber of $\iota_{\mathfrak{m},z}$ over $\eta \in \mathrm{Irr}(\mathfrak{S}_{\tilde{\psi}})$, viewed as a $\tilde{\mathcal{H}}(G)$-module.

(3) (Local intertwining relations) Suppose $\tilde{\psi} \in \tilde{\Psi}(G^*)$ satisfies $\tilde{\psi}^{\mathrm{GL}} = \psi_\tau + \tilde{\psi}_0^{\mathrm{GL}} + \psi_\tau^\vee$, where $\psi_\tau \in \Psi(\mathrm{GL}(d))$ corresponds to an irreducible unitarizable representation τ of $(\mathrm{GL}(d))$ of Arthur type, and $\tilde{\psi}_0 \in \tilde{\Psi}\left(\mathrm{SO}(2n-2d)^{\mathrm{disc}(V)}\right)$. Let $P^* \leq G^*$ be a standard maximal parabolic subgroup with a Levi factor

$$M^* \cong \mathrm{GL}(d) \times \mathrm{SO}(2n-2d)^{\mathrm{disc}(V)}.$$

Assume that $P = \varrho(P^*)$ is a parabolic subgroup of G, and let $\mathfrak{m}_0$ denote the induced Whittaker datum of $\mathfrak{m}$ on M^*. Then for each $\eta_0 \in \mathrm{Irr}(\mathfrak{S}_{\tilde{\psi}_0})$,

$$\mathrm{I}_P^G(\tau \boxtimes \tilde{\pi}_{\mathfrak{m}_0,z}(\eta_0)) = \bigoplus_{\eta} \tilde{\pi}_{\mathfrak{m},z}(\tilde{\psi},\eta),$$

as a $\tilde{\mathcal{H}}(G)$-module, where η runs over all characters of $\mathfrak{S}_{\tilde{\psi}}$ that restricts to η_0 under the natural embedding $\mathfrak{S}_{\tilde{\psi}_0} \hookrightarrow \mathfrak{S}_{\tilde{\psi}}$.

Proof. This theorem follows from [CZ21a, Theorem A.1] and [CZ21b, Corollary 5.4, Theorem 7.3]. □

Moreover, by results of Moeglin [Mg06] and Xu [Xu17], the structure of the A-packet is better understood when ψ is elementary.

Lemma 2.2.2. *Suppose (ϱ, z) is the trivial pure inner twist (i.e., $\varrho = \mathrm{id} : G^* \to G$ and $z = 1$), so that $G = G^*$. If ψ is an elementary A-parameter for G, then $\iota_{\mathfrak{m},z}$ induces a canonical bijection*

$$\tilde{\Pi}_{\tilde{\psi}}(G) \cong \mathrm{Irr}(\overline{\mathfrak{S}}_{\tilde{\psi}}).$$

Proof. This result follows from [Xu17, Theorem 6.1]. □

3. Endoscopy theory

Let F be a local or global field of characteristic zero. In this section, we recall the theory of endoscopy for even special orthogonal groups over F.

3.1. **Endoscopic triples.** We begin by recalling the definition of elliptic endoscopic triples, following [KS99, Wal10]. For any connected reductive group G over F, an *extended endoscopic triple* for G is a triple $\mathfrak{e} = (\mathsf{G}^{\mathfrak{e}}, \mathsf{s}^{\mathfrak{e}}, {}^L\xi^{\mathfrak{e}})$, where $\mathsf{G}^{\mathfrak{e}}$ is a connected reductive group over F, $\mathsf{s}^{\mathfrak{e}} \in \widehat{\mathsf{G}}$ is a semisimple element, and ${}^L\xi^{\mathfrak{e}} : {}^L\mathsf{G}^{\mathfrak{e}} \to {}^L\mathsf{G}$ is an L-embedding such that

$$\mathrm{Ad}(\mathsf{s}^{\mathfrak{e}}) \circ {}^L\xi^{\mathfrak{e}} = {}^L\xi^{\mathfrak{e}}$$

and ${}^L\xi^{\mathfrak{e}}(\widehat{\mathsf{G}^{\mathfrak{e}}})$ is a connected component of the subgroup of $\mathrm{Ad}(\mathsf{s}^{\mathfrak{e}})$-fixed elements of $\widehat{\mathsf{G}}$.

The triple $\mathfrak{e}$ is called *elliptic* if

$${}^L\xi^{\mathfrak{e}}\left(Z(\widehat{\mathsf{G}^{\mathfrak{e}}})^{\mathrm{Gal}_F}\right)^{\circ} \subset Z(\widehat{\mathsf{G}}),$$

i.e., the identity component of the Galois-invariant center of $\widehat{\mathsf{G}^{\mathfrak{e}}}$ maps into the center of $\widehat{\mathsf{G}}$ under ${}^L\xi^{\mathfrak{e}}$. An isomorphism between two endoscopic triples $\mathfrak{e}, \mathfrak{e}'$ is an element $\mathsf{g} \in \widehat{\mathsf{G}}$ such that

(1) $\mathsf{g}{}^L\xi^{\mathfrak{e}}({}^L\mathsf{G}^{\mathfrak{e}})\mathsf{g}^{-1} = {}^L\xi^{\mathfrak{e}'}({}^L\mathsf{G}^{\mathfrak{e}'})$, and

(2) $\mathsf{g}\mathsf{s}^{\mathfrak{e}}\mathsf{g}^{-1} = \mathsf{s}^{\mathfrak{e}'}$ modulo $Z(\widehat{\mathsf{G}})$.

We denote by $\mathcal{E}(\mathsf{G}^*)$ the set of isomorphism classes of extended endoscopic triples for G^*, and by $\mathcal{E}_{\mathrm{ell}}(\mathsf{G}^*)$ the subset of elliptic classes.

Suppose $\mathfrak{e} \in \mathcal{E}(\mathsf{G}^*)$. Then for each $g^{\mathfrak{e}} \in \widehat{\mathsf{G}}^{\mathfrak{e}}$, the image ${}^L\xi^{\mathfrak{e}}(g^{\mathfrak{e}}) \in {}^LG$ defines an automorphism of $\mathfrak{e}$. This allows us to define the outer automorphism group of $\mathfrak{e}$ as

$$\mathrm{OAut}(\mathfrak{e}) := \mathrm{Aut}(\mathfrak{e})/{}^L\xi^{\mathfrak{e}}(\widehat{\mathsf{G}^{\mathfrak{e}}}). \tag{3.1}$$

We now specialize to the case $G = \mathrm{SO}(V)$. According to [Wal10, §1.8], the isomorphism classes of elliptic extended endoscopic triples for G are determined by the endoscopic group

$$G^{\mathfrak{e}} = \mathrm{SO}(2n_1)^{\mathfrak{d}_1} \times \mathrm{SO}(2n_2)^{\mathfrak{d}_2},$$

where $n_1 + n_2 = n$ and $\mathfrak{d}_1\mathfrak{d}_2 = \mathfrak{d}$. Here it is required that $\mathfrak{d}_i = 1$ when $n_i = 0$, and $(n_i, \mathfrak{d}_i) \neq (1, 1)$ for each $i \in \{1, 2\}$.

The outer automorphism group $\mathrm{OAut}(\mathfrak{e})$ is trivial unless $n_1 n_2 \neq 0$, in which case there exists a nontrivial outer automorphism given by simultaneous outer conjugation on both factors of $\widehat{G^{\mathfrak{e}}}$. Furthermore, if $n_1 = n_2$ and $\mathfrak{d} = 1$, there exists an additional nontrivial automorphism swapping the two factors. In this case, the full outer automorphism group satisfies

$$\mathrm{OAut}(\mathfrak{e}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2.$$

3.2. **Orbital integrals.** We briefly recall some of the discussion from [Art13, §2.1]. Suppose $F = K$ is a local field and let G be a connected reductive group over K. We denote by $\mathsf{G}(K)_{\mathrm{s.reg}} \subset \mathsf{G}(K)$ the open subset of strongly regular semisimple elements, i.e., regular semisimple elements whose centralizer is connected (in fact, a maximal torus).

Fix a Haar measure on $\mathsf{G}(K)$. For $\delta \in \mathsf{G}(K)_{\mathrm{s.reg}}$, the *Weyl discriminant* of δ is defined as

$$D^{\mathsf{G}}(\delta) := \det(1 - \mathrm{Ad}(\delta)|\mathfrak{g}/\mathfrak{g}_\delta) \in K^{\times},$$

where $\mathfrak{g}$ and $\mathfrak{g}_\delta$ are the Lie algebras of G and $Z_{\mathsf{G}}(\delta)$, respectively. Fix a Haar measure on the torus $Z_{\mathsf{G}}(\delta)$, which induces a quotient measure on $Z_{\mathsf{G}}(\delta)(K)\backslash\mathsf{G}(K)$.

For $f \in \mathcal{H}(\mathsf{G})$ and $\delta \in \mathsf{G}(K)_{\mathrm{s.reg}}$, the *normalized orbital integral* of f at δ is defined as

$$\mathrm{Orb}_\delta(f) := \left|D^{\mathsf{G}}(\delta)\right|^{\frac{1}{2}} \int_{Z_{\mathsf{G}}(\delta)(K)\backslash\mathsf{G}(K)} f(x^{-1}\delta x)\mathrm{d}x.$$

If G is quasi-split, the normalized stable orbital integral of f at δ is defined by summing over its rational conjugacy class:

$$\mathrm{SOrb}_\delta(f) := \sum_{\delta'} \mathrm{Orb}_{\delta'}(f),$$

where δ' runs over a set of representatives for the $\mathsf{G}(K)$-conjugacy classes of those elements that are $\mathsf{G}(\overline{K})$-conjugate to δ.

3.3. **Local transfer.** In this subsection, we suppose $F = K$ is a local field and continue with the notations established in Section 2.1. In particular, $G = \mathrm{SO}(V)$ is an even special orthogonal group, and (G, ϱ, z) is a pure inner twist of the quasi-split inner form G^*. We have fixed a Whittaker datum $\mathfrak{m}$ for G^*.

Recall the subspace $\tilde{\mathcal{H}}(G)$ of $\mathcal{H}(G)$ consisting of ς-invariant test functions. Similarly, for each $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(G)$, we can define the analogous subspace $\tilde{\mathcal{H}}(G^{\mathfrak{e}}) = \tilde{\mathcal{H}}(G_1) \times \tilde{\mathcal{H}}(G_2)$ of invariant test functions on $G^{\mathfrak{e}}$, because $G^{\mathfrak{e}}$ is a product of two (possibly trivial) even special orthogonal groups over K.

For each extended endoscopic triple $\mathfrak{e} \in \mathcal{E}(G^*)$, a *transfer factor*

$$\Delta[\mathfrak{m}, \mathfrak{e}, z] : G^{\mathfrak{e}}(K)_{\mathrm{s.reg}} \times G(K)_{\mathrm{s.reg}} \to \mathbb{C}$$

is defined in [Kal16, p. 233, Equation (6)], such that $\Delta[\mathfrak{m}, \mathfrak{e}, z]$ is a function on stable conjugacy classes of $G^{\mathfrak{e}}(K)_{\mathrm{s.reg}}$ and conjugacy classes of $G(K)_{\mathrm{s.reg}}$. With the transfer factor in hand, we can now recall the notion of matching test functions from [KS99, §5.5].

Definition 3.3.1. Let $f^{G^{\mathfrak{e}}} \in \mathcal{H}(G^{\mathfrak{e}})$ and $f \in \mathcal{H}(G)$. We say that $f^{G^{\mathfrak{e}}}$ and f are ($\Delta[\mathfrak{m}, \mathfrak{e}, z]$-) *matching test functions* if

$$\mathrm{SOrb}_{\gamma}(f^{G^{\mathfrak{e}}}) = \sum_{\delta \in G(K)_{\mathrm{s.reg}}/\mathrm{conj}} \Delta[\mathfrak{m}, \mathfrak{e}, z](\gamma, \delta)\, \mathrm{Orb}_{\delta}(f)$$

for every $\gamma \in G^{\mathfrak{e}}(K)_{\mathrm{s.reg}}$. In this case, we will also say that $f^{G^{\mathfrak{e}}}$ is a transfer of f to $G^{\mathfrak{e}}$.

Remark 3.3.2. Since the orbital integrals $\mathrm{Orb}_{\delta}(f)$ depend on the choices of measures on $G(K)$ and $Z_G(\delta)(K)$, the notion of matching test functions also depends on the choice of Haar measures on $G(K), G^{\mathfrak{e}}(K)$, and all tori in G and $G^{\mathfrak{e}}$. There exists a way to synchronize the various tori; see [AK24, Remark 5.1.2].

We now state a theorem asserting the existence of transfers of test functions. When $K = \mathbb{R}$, this is a fundamental result of Shelstad [She82, She08]. When K is non-Archimedean, it is the culmination of extensive work of many people, including Langlands and Shelstad [LS87, LS90], Waldspurger [Wal97, Wal06], and Ngô [Ngô10].

Theorem 3.3.3. *Let $f \in \mathcal{H}(G)$ and let $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(G)$. Then there exists a transfer $f^{G^{\mathfrak{e}}}$ of f to $G^{\mathfrak{e}}$. Moreover, suppose $\mathcal{K}$ is a ς-stable hyperspecial maximal compact open subgroup of $\mathsf{G}(K)$. Then there exists a hyperspecial maximal compact open subgroup $\mathcal{K}^{\mathfrak{e}}$ of $\mathsf{G}^{\mathfrak{e}}(K)$ such that the characteristic function $\mathbf{1}_{\mathcal{K}^{\mathfrak{e}}}$ is a transfer of the characteristic function $\mathbf{1}_{\mathcal{K}}$ to $\mathsf{G}^{\mathfrak{e}}$, provided that the Haar measures are chosen so that* $\mathrm{Vol}(\mathcal{K}) = \mathrm{Vol}(\mathcal{K}^{\mathfrak{e}}) = 1$.

Moreover, if f lies in $\tilde{\mathcal{H}}(G)$, then there exists a transfer $f^{G^{\mathfrak{e}}}$ of f to $G^{\mathfrak{e}}$ that is contained in $\tilde{\mathcal{H}}(G^{\mathfrak{e}})$.

Proof. The final assertion follows from the invariant properties of the transfer factor; see [Taï19, p. 860]. ☐

Similarly, when F is global and $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(G)$, a function $f^{G^{\mathfrak{e}}} = \otimes_v f_v^{G^{\mathfrak{e}}} \in \tilde{\mathcal{H}}(\mathbf{G}^{\mathfrak{e}}(\mathbf{A}_F)) := \otimes'_v \mathcal{H}(\mathbf{G}^{\mathfrak{e}} \otimes_F F_v)$ is called an adelic transfer of $f = \otimes_v f_v \in \tilde{\mathcal{H}}(\mathbf{G}(\mathbf{A}_F)) := \otimes'_v \mathcal{H}(\mathbf{G} \otimes_F F_v)$ if for each place $v \in \Sigma_F$, $f_v^{G^{\mathfrak{e}}}$ is a transfer of f_v. The existence of adelic transfer then follows from Theorem 3.3.3.

4. Adams–Johnson packets

In this section, we recall Adams–Johnson parameters and the associated packets studied in [AJ87], which play a key role in the globalization process. We continue with the notation introduced in Section 2, and assume throughout this section that $K = \mathbb{R}$ and $G = \mathrm{SO}(V)$.

An A-parameter $\psi : W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ is called an *Adams–Johnson parameter* if its infinitesimal character is C-algebraic regular in the sense of [BG14]. This is always true if ψ is tempered and discrete, though there are other non-tempered examples as well. An Adams–Johnson parameter is always discrete, but not every discrete A-parameter is of Adams–Johnson type. For further details, see [Taï17, §4.2.2] and [AMR18, §8.1].

For each Adams–Johnson parameter ψ of G, Adams–Johnson [AJ87] constructed a packet $\Pi_\psi^{\mathrm{AJ}}(G)$ of irreducible admissible representations of $G(\mathbb{R})$. Using the Whittaker datum $\mathfrak{m}$ for G^* and the pure inner twist (ϱ, z), Taïbi [Taï19, §3.2.2-3.2.3] attached to each element of $\Pi_\psi^{\mathrm{AJ}}(G)$ a character of $S_\psi = \mathfrak{S}_\psi$. We will omit the superscript "AJ", since for quasi-split G, the works of Arancibia, Moeglin, and Renard [AMR18] show that the Adams–Johnson packet coincides with Arthur's packet, and the two constructions of the map to $\mathrm{Irr}(\mathfrak{S}_\psi)$ coincide.

As in the non-Archimedean case, we may define ς-equivalence classes of Adams–Johnson parameters, denoted $\tilde{\psi}$, and for each such $\tilde{\psi}$, there is a corresponding packet $\tilde{\Pi}_{\tilde{\psi}}(G)$ consisting of $\mathrm{O}(V)$-conjugacy classes of irreducible admissible representations of $G(\mathbb{R})$. The following endoscopic character identity, proved by Taïbi [Taï19, Proposition 3.2.6], plays a central role.

Theorem 4.0.1. *Assume that G^* admits discrete series. Let ψ be an Adams–Johnson parameter of G, and let $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(G)$ be an elliptic extended endoscopic triple. If $\psi^{\mathfrak{e}}$ is an A-parameter for $G^{\mathfrak{e}}$ such that ${}^L\xi^{\mathfrak{e}} \circ \psi^{\mathfrak{e}} = \psi$, then $\psi^{\mathfrak{e}}$ is also an Adams–Johnson parameter. Moreover, for any $f \in \widetilde{\mathcal{H}}(G)$, a transfer $f^{G^{\mathfrak{e}}}$ of f to $G^{\mathfrak{e}}$ may be chosen in $\widetilde{\mathcal{H}}(G^{\mathfrak{e}})$, and the following endoscopic character identity holds:*

$$\sum_{\tilde{\pi} \in \tilde{\Pi}_{\tilde{\psi}}(G)} \iota_{\mathfrak{m},z}(\pi)(s^{\mathfrak{e}} s_\psi) \tilde{\Theta}_{\tilde{\pi}}(f) = \tilde{\Theta}_{\tilde{\psi}^{\mathfrak{e}}}(f^{G^{\mathfrak{e}}}).$$

5. Arthur's multiplicity formula

Let F be a totally real number field. Let $\mathbf{V}$ be a quadratic space of dimension $2n$ over F, and let $\mathbf{G} = \mathrm{SO}(\mathbf{V})$ be the associated even special orthogonal group over F. Denote by $\mathbf{G}^*$ the unique quasi-split inner form of $\mathbf{G}$. We may fix a pure inner twisting (ϱ, z) from $\mathbf{G}^*$ to $\mathbf{G}$, which localizes at each place v of F to a pure inner twist (ϱ_v, z_v) from $\mathbf{G}_v^*$ to $\mathbf{G}_v$.

For any Archimedean place $\tau \in \Sigma_F^\infty$, we fix a maximal compact subgroup $\mathscr{K}_\tau$ of $\mathbf{G}(F_\tau)$. Let $\mathcal{A}(\mathbf{G})$ denote the space of automorphic forms for $\mathbf{G}$ in the sense of [BJ79], with respect to the fixed subgroup $\prod_{\tau \in \Sigma_F^\infty} \mathscr{K}_\tau$. We write $\mathcal{A}_2(\mathbf{G})$ for the space of square-integrable automorphic forms for $\mathbf{G}$.

There is a decomposition of $\mathcal{A}_2(\mathbf{G})$ as an orthogonal sum of discrete series representations $\pi = \otimes_v' \pi_v$ of $\mathbf{G}(\mathbf{A}_F)$:

$$\mathcal{A}_2(\mathbf{G}) \cong \bigoplus_\pi \pi^{m(\pi)}.$$

Here the positive integer $m(\pi)$ is the *automorphic multiplicity* of π.

Let $\mathrm{Irr}_{\mathrm{disc}}(\mathbf{G}(\mathbf{A}_F))$ denote the set of irreducible discrete automorphic representations of $\mathbf{G}(\mathbf{A}_F)$, i.e. those with $m(\pi) > 0$. We also define $\widetilde{\mathrm{Irr}}_{\mathrm{disc}}(\mathbf{G}(\mathbf{A}_F))$ to be the set of irreducible $\tilde{\mathcal{H}}(\mathbf{G}(\mathbf{A}_F))$-modules appearing in $\mathcal{A}_2(\mathbf{G})$, where

$$\tilde{\mathcal{H}}(\mathbf{G}(\mathbf{A}_F)) := \bigotimes_v{}' \tilde{\mathcal{H}}(\mathbf{G}(F_v)).$$

In order to formulate his multiplicity formula, Arthur circumvented the absence of the hypothetical Langlands group by a surrogate notion of global Arthur–Langlands parameters.

Recall that a global A-parameter for $\mathbf{G}$ is a formal finite sum

$$\boldsymbol{\psi} = \sum_i \boldsymbol{\psi}_i \boxtimes \mathrm{sp}_{d_i}$$

where

- each $\boldsymbol{\psi}_i$ is an irreducible self-dual cuspidal automorphic representation of $\mathrm{GL}(n_i; \mathbf{A}_F)$ with sign $(-1)^{d_i-1}$, in the sense of [GGP12, p. 94];
- sp_{d_i} denotes the d_i-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$;
- the total dimension satisfies $\sum_i n_i d_i = 2n$; and
- if ω_i is the central character of $\boldsymbol{\psi}_i$, then $\prod_i \omega_i^{d_i} = \chi_{\mathbf{V}}$, where $\chi_{\mathbf{V}}$ is the quadratic character of $\mathbf{C}_F$ associated with $\mathrm{disc}(\mathbf{V})$.

The parameter $\boldsymbol{\psi}$ is called *elliptic* (or *discrete*) if the pairs $(\boldsymbol{\psi}_i, d_i)$ are pairwise distinct. It is called *generic* (or *tempered*) if $d_i = 1$ for all i.

Given an elliptic global parameter

$$\boldsymbol{\psi} = \boldsymbol{\psi}_1 \boxtimes \mathrm{sp}_{d_1} + \ldots + \boldsymbol{\psi}_k \boxtimes \mathrm{sp}_{d_k}$$

for $\mathbf{G}$, define the formal extended component group

$$\mathfrak{S}^{\sharp}_{\boldsymbol{\psi}} := \bigoplus_i (\mathbb{Z}/2) e_i,$$

where e_i is a formal coordinate associated with the summand $\boldsymbol{\psi}_i \boxtimes \mathrm{sp}_{d_i}$. Denote the determinant character

$$\det_{\boldsymbol{\psi}} : \mathfrak{S}^{\sharp}_{\boldsymbol{\psi}} \to \mathbb{Z}/2, \sum_{1 \le i \le k} x_i e_i \mapsto \sum_{1 \le i \le k} n_i x_i.$$

Then the formal global component group is defined as the kernel

$$\mathfrak{S}_{\boldsymbol{\psi}} := \ker(\det_{\boldsymbol{\psi}}).$$

Moreover, define the quotient group

$$\overline{\mathfrak{S}}_{\boldsymbol{\psi}} := \mathfrak{S}_{\boldsymbol{\psi}} / \langle e_1 + \ldots + e_k \rangle .$$

Finally, Arthur defines a canonical character $\varepsilon_{\boldsymbol{\psi}}$ of $\mathfrak{S}_{\boldsymbol{\psi}}$ as in [Art13, Equation (1.5.6)].

For each elliptic global parameter $\boldsymbol{\psi}$ of $\mathbf{G}$ and each place $v \in \Sigma_F$, there exists a localization $\tilde{\boldsymbol{\psi}}_v$—an $\mathrm{O}(2n, \mathbb{C})$-conjugacy class of local A-parameters of $\mathbf{G}_{F_v}$. Moreover, there is a natural map of component groups

$$\mathfrak{S}_{\boldsymbol{\psi}} \to \mathfrak{S}_{\tilde{\boldsymbol{\psi}}_v},$$

as described in [CZ24, p. 7]

Furthermore, associated with each local parameter $\tilde{\boldsymbol{\psi}}_v$, we define an $\mathrm{O}(2n, \mathbb{C})$-conjugacy class of L-parameters $\tilde{\phi}_{\tilde{\boldsymbol{\psi}}_v}$ as follows: choose a representative $\boldsymbol{\psi}_v$ of $\tilde{\boldsymbol{\psi}}_v$, and define

$$\phi_{\boldsymbol{\psi}_v}(w) = \boldsymbol{\psi}_v \left(w, \begin{bmatrix} |w|^{\frac{1}{2}} & \\ & |w|^{-\frac{1}{2}} \end{bmatrix} \right),$$

which is an L-parameter for $\mathbf{G}_{F_v}$. The conjugacy class $\tilde{\phi}_{\tilde{\boldsymbol{\psi}}_v}$ is independent of the choice of representative $\boldsymbol{\psi}_v$.

We now recall the "weak form" of "Arthur's multiplicity formula", as established by Chen–Zou [CZ24]. In general, this formula has been established only for elliptic generic global A-parameters. However, if $\mathbf{G}$ is either quasi-split or has F-rank at most one, then the result holds for all elliptic A-parameters.

Theorem 5.0.1. *Suppose* $\mathbf{G}$ *is either quasi-split or has* F*-rank at most one. Let* $\mathfrak{m}$ *be a Whittaker datum for* $\mathbf{G}^*$ *and* $\boldsymbol{\psi} = \boldsymbol{\psi}_1 \boxtimes \mathrm{sp}_{d_1} + \ldots + \boldsymbol{\psi}_k \boxtimes \mathrm{sp}_{d_k}$ *be an elliptic global A-parameter for* $\mathbf{G}$*. Let* $L^2_{\boldsymbol{\psi}}(\mathbf{G})$ *be the direct sum of irreducible admissible representations* π *in* $L^2_{\mathrm{disc}}(\mathbf{G})$ *such that the L-parameter of* π_v *is* $\tilde{\phi}_{\tilde{\psi}_v}$ *for almost every place of* F*. Then there exists a natural diagonal map*

$$\Delta : \mathfrak{S}_{\boldsymbol{\psi}} \to \mathfrak{S}_{\boldsymbol{\psi},\mathbf{A}_F} := \prod_{v\in\Sigma_F} \mathfrak{S}_{\tilde{\psi}_v},$$

and a decomposition of $\tilde{\mathcal{H}}(\mathbf{G}(\mathbf{A}_F))$*-modules:*

$$L^2_{\boldsymbol{\psi}}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbf{A}_F)) \cong m_{\boldsymbol{\psi}} \sum_{\substack{\eta\in\mathrm{Irr}(\mathfrak{S}_{\boldsymbol{\psi},\mathbf{A}_F})\\ \Delta^*(\eta)=\varepsilon_{\boldsymbol{\psi}}}} \tilde{\pi}_{\mathfrak{m},z}(\eta),$$

where each $\tilde{\pi}_{\mathfrak{m},z}(\eta) = \otimes'_v \tilde{\pi}_{\mathfrak{m}_v,z_v}(\eta_v)$ *is the global restricted tensor product of local representations (which is not necessarily irreducible). Moreover, the multiplicity* $m_{\boldsymbol{\psi}}$ *satisfies* $m_{\boldsymbol{\psi}} = 1$ *unless* $n_i d_i$ *is even for each* i*, in which case* $m_{\boldsymbol{\psi}} = 2$*.*

Proof. This result follows from [Art13, Theorem 1.5.2] when $\mathbf{G}$ is quasi-split, and from [CZ24, Theorem 7.7] when $\mathbf{G}$ has F-rank at most one. Although Chen–Zou [CZ24] states the theorem for the full orthogonal group $\mathrm{O}(\mathbf{V})$, the argument in [AG17, Chapter 7] implies the corresponding result for the special orthogonal group $\mathrm{SO}(\mathbf{V})$. □

6. Stable trace formula

In this section, we recall the stable trace formula for the global even orthogonal group $\mathbf{G}$.

For each extended endoscopic triple $\mathfrak{e} \in \mathcal{E}(\mathbf{G})$, denote

$$\iota(\mathfrak{e}) = \tau(\mathbf{G})\tau(\mathbf{G}^{\mathfrak{e}})^{-1}\#\mathrm{OAut}(\mathfrak{e})^{-1} \in \mathbb{Q}$$

to be the global coefficient introduced by Kottwitz and Shelstad, cf. [Art13, Equation (3.2.4)]. For instance, $\iota(\mathfrak{e}) = 1$ if $\mathbf{G}^{\mathfrak{e}} = \mathbf{G}^*$, and $\iota(\mathfrak{e}) = 1/4$ if $\mathbf{G}^{\mathfrak{e}} = \mathrm{SO}(2n_1)^{\mathfrak{d}_1} \times \mathrm{SO}(2n_2)^{\mathfrak{d}_2}$ with $(n_1, \mathfrak{d}_1) \neq (n_2, \mathfrak{d}_2)$.

Theorem 6.0.1 (Discrete part of stable trace formula, [Art13, p. 135])**.** *There is a spectral decomposition*

$$\mathrm{I}^{\mathbf{G}}_{\mathrm{disc}}(f) = \sum_{\mathfrak{e}\in\mathcal{E}_{\mathrm{ell}}(\mathbf{G})} \iota(\mathfrak{e})\mathrm{ST}^{\mathbf{G}^{\mathfrak{e}}}_{\mathrm{disc}}(f^{\mathbf{G}^{\mathfrak{e}}}) \tag{6.1}$$

for any $f^{\mathbf{G}^{\mathfrak{e}}}$ *that is matching functions with* f*. Here*

$$\mathrm{I}^{\mathbf{G}}_{\mathrm{disc}}(f) = \sum_{\mathbf{M}} \frac{1}{\#W(\mathbf{G},\mathbf{M})} \sum_{s\in W(\mathbf{G},\mathbf{M})_{\mathrm{reg}}} \frac{1}{\det(s-1)_{\mathfrak{a}^{\mathbf{G}}_{\mathbf{M}}}} \mathrm{tr}\left(M_{\mathbf{P}}(s,0)\mathcal{I}^P_{\mathrm{disc}}(0,f)\right),$$

where $\mathbf{M}$ *runs through standard Levi subgroups of* $\mathbf{G}$*, cf.* [Art13, Page 124]*, and* $\mathrm{ST}^{\mathbf{G}^{\mathfrak{e}}}(f^{\mathbf{G}^{\mathfrak{e}}})$ *is the stable linear form on* $\mathbf{G}^{\mathfrak{e}}$ *appearing in the discrete part of Arthur's stabilization of the trace formula.*

Remark 6.0.2. We refer the reader to [Art13, Chapter 3] for the precise definitions of other terms in the RHS of the formula above. For our purpose, we simply note that the term corresponding to $\mathbf{M} = \mathbf{G}$ is the sum of $m(\pi)\mathrm{tr}(\pi(f))$, where π runs over the set of discrete automorphic representations π of $\mathbf{G}$ with formal parameter ψ.

Note that there are only finitely many isomorphism classes of elliptic endoscopic data unramified outside a given finite set of places of F by [Lan83]. Combining with [Kot86, Proposition 7.5], this implies that there are only finitely many non-zero terms on the right-hand side of Equation (6.1).

7. Quasi-split case

Our main result, Theorem A, was established by Arthur [Art13, Theorem 2.2.1] when K is non-Archimedean and G is quasi-split. In fact, it was established without the hypothesis that ψ is tempered, and it was also established when K is an Archimedean local field. Moreover, we can deduce the following stable multiplicity formula for any (not necessarily quasi-split) global special orthogonal group $\mathbf{G} = \mathrm{SO}(\mathbf{V})$ over F.

For any elliptic endoscopic triple $\mathfrak{e}$ for $\mathbf{G}$, the endoscopic group $\mathbf{G}^{\mathfrak{e}} = \mathbf{G}_1 \times \mathbf{G}_2$ is a product of two (possibly trivial) even special orthogonal groups over F. For each elliptic parameter $\boldsymbol{\psi}$ for $\mathbf{G}$, we write $\Psi(\mathbf{G}^{\mathfrak{e}}, \boldsymbol{\psi})$ for the set of pairs of elliptic parameters $(\boldsymbol{\psi}_1, \boldsymbol{\psi}_2)$ for $\mathbf{G}_1$ and $\mathbf{G}_2$ such that $\boldsymbol{\psi} = \boldsymbol{\psi}_1 + \boldsymbol{\psi}_2$ as a formal sum.

Theorem 7.0.1 (Stable multiplicity formula, [Art13, Corollary 4.1.3])**.** *Let $\mathfrak{e}$ be an elliptic endoscopic triple for $\mathbf{G}$, and let $\boldsymbol{\psi}$ be an elliptic A-parameter for $\mathbf{G}$. If $f = \otimes_v f_v \in \tilde{\mathcal{H}}(\mathbf{G}(\mathbf{A}_F))$ adelically transfers to $f^{\mathbf{G}^{\mathfrak{e}}} = \otimes_v f_v^{\mathbf{G}^{\mathfrak{e}}} \in \tilde{\mathcal{H}}(\mathbf{G}^{\mathfrak{e}})$, then*

$$\mathrm{ST}^{\mathbf{G}^{\mathfrak{e}}}_{\mathrm{disc},\boldsymbol{\psi}}(f^{\mathbf{G}^{\mathfrak{e}}}) = \sum_{\boldsymbol{\psi}^{\mathfrak{e}} \in \Psi(\mathbf{G}^{\mathfrak{e}},\boldsymbol{\psi})} \frac{m_{\boldsymbol{\psi}^{\mathfrak{e}}} \varepsilon_{\boldsymbol{\psi}^{\mathfrak{e}}}(s_{\boldsymbol{\psi}^{\mathfrak{e}}})}{\#\overline{\mathfrak{S}}_{\boldsymbol{\psi}^{\mathfrak{e}}}} \prod_v \Theta_{\tilde{\psi}^{\mathfrak{e}}_v}(f_v^{\mathbf{G}^{\mathfrak{e}}}). \tag{7.1}$$

Here $m_{\boldsymbol{\psi}^{\mathfrak{e}}}, \#\overline{\mathfrak{S}}_{\boldsymbol{\psi}^{\mathfrak{e}}}$ and $\varepsilon_{\boldsymbol{\psi}^{\mathfrak{e}}}$ are defined as follows: If $\mathbf{G}^{\mathfrak{e}} = \prod_i \mathbf{G}_i$ is a product of special orthogonal groups and $\boldsymbol{\psi}^{\mathfrak{e}} = \prod_i \boldsymbol{\psi}_i$ where $\boldsymbol{\psi}_i$ is a global parameter for $\mathbf{G}_i$ for each i, then

$$m_{\boldsymbol{\psi}^{\mathfrak{e}}} := \prod_i m_{\boldsymbol{\psi}_i}, \quad \#\overline{\mathfrak{S}}_{\boldsymbol{\psi}^{\mathfrak{e}}} := \prod_i \#\overline{\mathfrak{S}}_{\boldsymbol{\psi}_i}, \quad \varepsilon_{\boldsymbol{\psi}^{\mathfrak{e}}}(s_{\boldsymbol{\psi}^{\mathfrak{e}}}) := \prod_i \varepsilon_{\boldsymbol{\psi}_i}(s_{\boldsymbol{\psi}_i}).$$

As a corollary of Theorem 7.0.1, we have the following simple stable trace formula for any even special orthogonal group over a totally real number field F.

Corollary 7.0.2. *Suppose $\mathbf{G}$ is either quasi-split or has F-rank ≤ 1. Let $\boldsymbol{\psi}$ be an elliptic A-parameter of $\mathbf{G}$ such that, at each real place of F, its localization is an Adams–Johnson parameter. Then, for any $f = \otimes_v f_v \in \tilde{\mathcal{H}}(\mathbf{G}(\mathbf{A}_F))$,*

$$m_{\boldsymbol{\psi}} \sum_{\substack{\otimes_v \eta_v \in \mathrm{Irr}(\mathfrak{S}_{\boldsymbol{\psi},\mathbf{A}_F}) \\ \Delta^*(\otimes_v \eta_v) = \varepsilon_{\boldsymbol{\psi}}}} \prod_v \Theta_{\tilde{\pi}^{\mathbf{G}}_{\eta_v}}(f_v) = \sum_{\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(\mathbf{G})} \iota(\mathfrak{e}) \sum_{\boldsymbol{\psi}^{\mathfrak{e}} \in \Psi(\mathbf{G}^{\mathfrak{e}},\boldsymbol{\psi})} \frac{m_{\boldsymbol{\psi}^{\mathfrak{e}}} \varepsilon_{\boldsymbol{\psi}^{\mathfrak{e}}}(s_{\boldsymbol{\psi}^{\mathfrak{e}}})}{\#\overline{\mathfrak{S}}_{\boldsymbol{\psi}^{\mathfrak{e}}}} \prod_v \Theta_{\tilde{\psi}^{\mathfrak{e}}_v}(f_v^{\mathbf{G}^{\mathfrak{e}}}), \tag{7.2}$$

Here $f^{\mathbf{G}^{\mathfrak{e}}} = \otimes_v f_v^{\mathbf{G}^{\mathfrak{e}}} \in \tilde{\mathcal{H}}(\mathbf{G}^{\mathfrak{e}})$ are adelic transfers of f.

Proof. It follows from Theorem 6.0.1 and Arthur's standard procedure on extracting the ψ-part of the stable trace formula (cf. [Art13, Chapter 3]) that

$$\mathrm{I}^{\mathbf{G}}_{\mathrm{disc},\boldsymbol{\psi}}(f) = \sum_{\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(\mathbf{G})} \iota(\mathfrak{e}) \mathrm{ST}^{\mathbf{G}^{\mathfrak{e}}}_{\mathrm{disc},\boldsymbol{\psi}}(f^{\mathbf{G}^{\mathfrak{e}}}). \tag{7.3}$$

Next, we claim $\mathrm{I}^{\mathbf{G}}_{\mathrm{disc},\boldsymbol{\psi}}(f)$ is the sum of $m(\pi)\mathrm{tr}(\pi(f))$ for π running over the set of discrete automorphic representations π of $\mathbf{G}$ with global A-parameter $\boldsymbol{\psi}$. This is true because the contributions from properly-contained Levi subgroups $\mathbf{M}$ of $\mathbf{G}$ only involve representations of $\mathbf{G}(\mathbf{A}_F)$ with non-regular infinitesimal character at all infinite places of F.

It then follows from Arthur's multiplicity formula Theorem 5.0.1 that

$$\mathrm{I}^{\mathbf{G}}_{\mathrm{disc},\boldsymbol{\psi}}(f) = m_{\boldsymbol{\psi}} \sum_{\substack{\otimes_v \eta_v \in \mathrm{Irr}(\mathfrak{S}_{\boldsymbol{\psi},\mathbf{A}_F}) \\ \Delta^*(\otimes_v \eta_v) = \varepsilon_{\boldsymbol{\psi}}}} \prod_v \Theta_{\tilde{\pi}^{\mathbf{G}}_{\eta_v}}(f_v). \tag{7.4}$$

On the other hand, it follows from Theorem 7.0.1 that

$$\mathrm{ST}^{\mathbf{G}^{\mathfrak{e}}}_{\mathrm{disc},\boldsymbol{\psi}}(f^{\mathbf{G}^{\mathfrak{e}}}) = \sum_{\psi^{\mathfrak{e}} \in \Psi(\mathbf{G}^{\mathfrak{e}},\boldsymbol{\psi})} \frac{m_{\psi^{\mathfrak{e}}} \varepsilon_{\psi^{\mathfrak{e}}}(s_{\psi^{\mathfrak{e}}})}{\#\overline{\mathfrak{S}}_{\psi^{\mathfrak{e}}}} \prod_v \Theta_{\tilde{\psi}^{\mathfrak{e}}_v}(f_v^{\mathbf{G}^{\mathfrak{e}}}). \tag{7.5}$$

So the assertion follows. ☐

8. Discrete case

In this section, we prove the main theorem when $\psi^{\mathfrak{e}}$ is discrete and admissible. In particular, we can write $\psi^{\mathfrak{e}} = \psi_1 + \psi_2$, and write

$$\psi_1^{\mathrm{GL}} = \phi_1 \boxtimes \mathrm{sp}_{a_1} + \ldots + \phi_r \boxtimes \mathrm{sp}_{a_r}, \quad \psi_2^{\mathrm{GL}} = \phi_{r+1} \boxtimes \mathrm{sp}_{a_{r+1}} + \ldots + \phi_k \boxtimes \mathrm{sp}_{a_k},$$

where a_i is odd for each $1 \leq i \leq k$, and $a_i \neq a_j$ when $i \neq j \leq r$ or $r < i \leq j$. We assume that G is not quasi-split. In particular, $n \geq 2$ and $\mathrm{disc}(V) = 1$. To prove the main theorem, firstly we globalize the local field K to a totally real number field:

Lemma 8.0.1. *For any non-Archimedean local field K, there exists a totally real number field F with finite places $v_0, v_1 \in \Sigma_F^{\mathrm{fin}}$ such that $F_{v_0} \cong F_{v_1} \cong K$.*

Proof. This is an easy application of Krasner's lemma, see for example the proof of [Art13, Lemma 6.2.1] and [Ish24, Lemma 6.2]. ☐

Fix such a totally real number field F as in Lemma 8.0.1, together with a fixed embedding $\tau_0 : F \to \mathbb{R}$. We now globalize the endoscopic datum $\mathfrak{e}$. Suppose $G^{\mathfrak{e}} = \mathrm{SO}(2n')^{\mathfrak{d}} \times \mathrm{SO}(2n'')^{\mathfrak{d}}$ with $n' \geq n'' \geq 0$. We note that each ϕ_i is a discrete tempered A-parameter (i.e. discrete L-parameter) for some even orthogonal group $\mathrm{SO}(2n_i)^{\mathfrak{d}_i}$, such that

$$\prod_{i=1}^{r} \mathfrak{d}_i = \prod_{i=r+1}^{k} \mathfrak{d}_i = \mathfrak{d}.$$

We choose elements $\mathfrak{D}_1, \ldots, \mathfrak{D}_r \in F^\times$ such that $\mathfrak{D}_1 = \mathfrak{d} \in F_{v_1}^\times/(F_{v_1}^\times)^2$ and $(-1)^{n_1}\mathfrak{D}_1$ is totally positive. If $n'' \neq 0$, we further choose $\mathfrak{D}_2 \in F^\times$ such that

1. $\mathfrak{D}_i = \mathfrak{d}_i \in F_{v_1}^\times/(F_{v_1}^\times)^2$ for each $1 \leq i \leq r$;
2. $(-1)^{n_i}\mathfrak{D}_i$ is totally positive for each $1 \leq i \leq r$;
3. For every two finite sets $S, T \subset \{1, 2, \ldots, k\}$, if $\prod_{i \in S} \mathfrak{D}_i = \prod_{i \in T} \mathfrak{D}_i \in F^\times/(F^\times)^2$, then $S = T$.

This is possible because of the weak approximation theorem.

We define

$$\mathfrak{D}' := \prod_{i=1}^{r} \mathfrak{D}_i, \quad \mathfrak{D}'' := \prod_{i=r+1}^{k} \mathfrak{D}_i,$$

and choose a quadratic space $\mathbf{V}$ over F of rank $2n$ with discriminant $\mathfrak{D} = \mathfrak{D}'\mathfrak{D}''$, such that

1. $\mathbf{V}$ is positive definite at each real place of F except for τ_0;
2. For each finite place $v \in \Sigma_F^{\mathrm{fin}}$ except for $v = v_1$, the Hasse–Witt invariant $\epsilon(\mathbf{V} \otimes_F F_v)$ is 1;
3. $\epsilon(\mathbf{V} \otimes_{F,\tau_0} R) = -1$.

Note that such a quadratic space exists by the Hasse–Minkowski theorem [Gro21, Theorem 2.1]. We write $\mathbf{G} = \mathrm{SO}(\mathbf{V})$. Then there exists an elliptic endoscopic datum $\mathfrak{E}$ for $\mathbf{G}$ such that $\mathbf{G}^{\mathfrak{E}} \cong \mathrm{SO}(2n')^{\mathfrak{D}'} \times \mathrm{SO}(2n'')^{\mathfrak{D}''}$. By the choices of $\mathfrak{D}_1, \ldots, \mathfrak{D}_i$, we have achieved the following:

1. $\mathbf{G}$ is anisotropic;
2. Both $\mathbf{G}^{\mathfrak{E}}(F \otimes \mathbb{R})$ and $\mathbf{G}(F \otimes \mathbb{R})$ admit discrete series representations;
3. $\mathbf{G} \otimes_F F_v$ is quasi-split for each finite place v of F except for v_1, where $\mathbf{G} \otimes_F F_{v_1} \cong G$.

Let $\mathbf{G}^*$ be the unique quasi-split inner form of $\mathbf{G}$. We can always fix a pure inner twisting (ϱ, z) between $\mathbf{G}^*$ and $\mathbf{G}$, which localizes to a pure inner twist (ϱ_v, z_v) between $\mathbf{G}_v^*$ and $\mathbf{G}_v$ for each place v of F. We further assume that (ϱ_{v_0}, z_{v_0}) is a trivial pure inner twist.

We now globalize the local A-parameter $\psi^{\mathfrak{e}}$.

Lemma 8.0.2. *There exist elliptic global A-parameters* $\boldsymbol{\psi}', \boldsymbol{\psi}''$ *for* $\mathrm{SO}(2n')^{\mathfrak{D}'}$ *and* $\mathrm{SO}(2n'')^{\mathfrak{D}''}$, *respectively, such that*

(1)

$$(\boldsymbol{\psi}')^{\mathrm{GL}} = \boldsymbol{\psi}_1 \boxtimes \mathrm{sp}_{a_1} + \ldots + \boldsymbol{\psi}_r \boxtimes \mathrm{sp}_{a_r}, \quad (\boldsymbol{\psi}'')^{\mathrm{GL}} = \boldsymbol{\psi}_{r+1} \boxtimes \mathrm{sp}_{a_{r+1}} + \ldots + \boldsymbol{\psi}_k \boxtimes \mathrm{sp}_{a_k},$$

where $\boldsymbol{\psi}_i$ *is an irreducible self-dual cuspidal automorphic representation of* $\mathrm{GL}(2n_i; \mathbf{A}_F)$ *with central character being the quadratic character of* $\mathbf{C}_F$ *associated with* $\mathfrak{D}_i$, *for each* $1 \leq i \leq k$;

(2) $\tilde{\boldsymbol{\psi}}_{i,v_0}$ *is a local simple supercuspidal L-parameter for* $\mathrm{SO}(2n_i)^{\mathfrak{D}_i} \otimes_F F_{v_0}$ *for each* $1 \leq i \leq k$, *and* $\tilde{\boldsymbol{\psi}}_{i,v_0} \neq \tilde{\boldsymbol{\psi}}_{j,v_0}$ *if* $i \neq j$;

(3) $\tilde{\boldsymbol{\psi}}_{i,v_1} = \phi_i$ *for each* $1 \leq i \leq k$.

(4) $\boldsymbol{\psi} = \boldsymbol{\psi}' + \boldsymbol{\psi}''$ *is an elliptic global A-parameter for* $\mathbf{G}$. *Moreover, for any real place* τ *of* F, *the localization* $\tilde{\boldsymbol{\psi}}_\tau$ *is a* (ς*-equivalence class of) Adams–Johnson parameter for* $\mathbf{G} \otimes_{F,\tau} \mathbb{R}$.

Proof. Using Arthur's multiplicity formula Theorem 5.0.1, this follows from standard Plancherel density theorems for the quasi-split global even special orthogonal groups $\mathrm{SO}(2n_i)^{\mathfrak{D}_i}$, for example [Shi12, Theorem 1.1.(i)]. We just note that, by choosing algebraic irreducible representations ξ_i of $\left(\mathrm{Res}_{F/\mathbb{Q}} \mathrm{SO}(2n_i)^{\mathfrak{D}_i}\right) \otimes \mathbb{C}$ with highest weights sufficiently regular and sufficiently distinct from each other, we can make sure that the $\tilde{\boldsymbol{\psi}}_\tau$ has C-algebraic and regular infinitesimal character. □

We choose a global elliptic parameter $\boldsymbol{\psi}$ for $\mathbf{G}$ satisfying the requirement of Lemma 8.0.2. Then the natural map

$$\mathfrak{S}_{\boldsymbol{\psi}} \to \mathfrak{S}_{\tilde{\boldsymbol{\psi}}_{v_0}}$$

is an isomorphism. Consider $f = \otimes_v f_v \in \tilde{\mathcal{H}}(\mathbf{G}(\mathbf{A}_F))$ such that

$$\tilde{\Theta}_{\tilde{\pi}_{v_0}}(f_{v_0}) \cdot \iota_{\mathfrak{m}_{v_0}, z_{v_0}}(\tilde{\pi}_{v_0})(s^{\mathfrak{E}} s_{\tilde{\boldsymbol{\psi}}_{v_0}}) = 1$$

for each $\tilde{\pi}_{v_0} \in \tilde{\Pi}_{\tilde{\boldsymbol{\psi}}_{v_0}}(\mathbf{G}_{v_0})$, and the other components f_v for $v \in \Sigma_F \setminus \{v_0\}$ are arbitrary.

We apply Equation (7.2) to f. For each $\mathfrak{E}' \in \mathcal{E}_{\mathrm{ell}}(\mathbf{G})$, let $f^{\mathbf{G}^{\mathfrak{E}'}} \in \tilde{\mathcal{H}}(\mathbf{G}^{\mathfrak{E}}(\mathbf{A}_F))$ be an adelic transfer of f to $\mathbf{G}^{\mathfrak{E}'}$. It follows from the definition of f_{v_0} and the main result in the quasi-split case that the terms on the RHS of Equation (7.2) vanish unless $\mathbf{G}^{\mathfrak{E}'}_{v_0} \cong \mathbf{G}^{\mathfrak{E}}_{v_0}$. Furthermore, it follows from the shape of $\boldsymbol{\psi}$ and our choices of $\mathfrak{D}_i$ that the terms on the RHS of Equation (7.2) vanishes unless $\mathbf{G}^{\mathfrak{E}'} \cong \mathbf{G}^{\mathfrak{E}}$. Then it follows from the simple stable trace formula Corollary 7.0.2 and linear independence of characters at unramified places of F that

$$\sum_{\tilde{\Pi}} \prod_{v \in \Sigma} \tilde{\Theta}_{\tilde{\Pi}_v}(f_v) = \iota(\mathfrak{E}) \frac{m_{\boldsymbol{\psi}^{\mathfrak{e}}} \varepsilon_{\boldsymbol{\psi}^{\mathfrak{e}}}(s_{\boldsymbol{\psi}^{\mathfrak{e}}})}{m_{\boldsymbol{\psi}} \# \overline{\mathfrak{S}}_{\boldsymbol{\psi}^{\mathfrak{e}}}} \prod_{v \in \Sigma} \tilde{\Theta}_{\tilde{\boldsymbol{\psi}}^{\mathfrak{e}}_v}(f_v^{\mathbf{G}^{\mathfrak{E}}}) = \frac{\varepsilon_{\boldsymbol{\psi}^{\mathfrak{e}}}(s_{\boldsymbol{\psi}^{\mathfrak{e}}})}{\# \overline{\mathfrak{S}}_{\boldsymbol{\psi}}} \prod_{v \in \Sigma} \tilde{\Theta}_{\tilde{\boldsymbol{\psi}}^{\mathfrak{e}}_v}(f_v^{\mathbf{G}^{\mathfrak{E}}}). \tag{8.1}$$

Here Σ is a set of places of F containing $\Sigma_F^\infty \cup \{v_0, v_1, v_2\}$ such that $\tilde{\phi}_{\tilde{\boldsymbol{\psi}}_v}$ is unramified for each finite place $v \in \Sigma_F^{\mathrm{fin}} \setminus \Sigma$, and the sum on the LHS runs through all discrete automorphic representations $\tilde{\Pi} \in \widetilde{\mathrm{Irr}}_{\mathrm{disc}}(\mathbf{G}(\mathbf{A}_F))$ such that

(1) $\tilde{\Pi}_v$ is unramified with L-parameter $\tilde{\phi}_{\tilde{\boldsymbol{\psi}}_v}$ for any $v \in \Sigma_F^{\mathrm{fin}} \setminus \Sigma$;

(2) $\tilde{\Pi}_v \in \tilde{\Pi}_{\tilde{\boldsymbol{\psi}}_v}(\mathbf{G} \otimes_F F_v)$ for each $v \in \Sigma$.

(cf. [CG15, §11.5] for a similar argument).

We now analyze the contribution to both sides of Equation (8.1) for each $v \in \Sigma \setminus \{v_1\}$.

- Suppose $v = v_0$. Then

$$\tilde{\Theta}_{\tilde{\Pi}_{v_0}}(f_{v_0}) \cdot \iota_{\mathfrak{m}_{v_0}, z_{v_0}}(\tilde{\Pi}_{v_0})(s^{\mathfrak{E}} s_{\tilde{\boldsymbol{\psi}}_{v_0}}) = 1 \tag{8.2}$$

by the choice of f_{v_0}. On the other hand, by the main theorem in the quasi-split case,

$$\tilde{\Theta}_{\tilde{\psi}_{v_0}}(f_{v_0}^{\mathbf{G}^{\mathfrak{E}}}) = \sum_{\tilde{\pi}_{v_0}\in\tilde{\Pi}_{\tilde{\psi}_{v_0}}(\mathbf{G}\otimes_F F_{v_0})} \iota_{\mathfrak{m},z}(\tilde{\pi}_{v_0})(s^{\mathfrak{E}}s_{\tilde{\psi}_{v_0}})\tilde{\Theta}_{\tilde{\pi}_{v_0}}(f_{v_0}) = \#\tilde{\Pi}_{\tilde{\psi}_{v_0}}(\mathbf{G}\otimes_F F_{v_0}) = \#\overline{\mathfrak{S}}_{\boldsymbol{\psi}}. \tag{8.3}$$

Here for the last equality we used Lemma 2.2.2, because $\tilde{\boldsymbol{\psi}}_{v_0}$ is elementary.

- Suppose $v\in\Sigma\smallsetminus(\Sigma_F^\infty\cup\{v_0,v_1\})$. As $\mathbf{G}_v$ is quasi-split, it follows from the main theorem in the quasi-split case that

$$\sum_{\tilde{\pi}_{v}\in\tilde{\Pi}_{\tilde{\psi}_{v}}(\mathbf{G}\otimes_F F_{v})} \iota_{\mathfrak{m},z}(\tilde{\pi}_{v})(s^{\mathfrak{E}}s_{\tilde{\psi}_{v}})\tilde{\Theta}_{\tilde{\pi}_{v}}(f_{v}) = \tilde{\Theta}_{\tilde{\psi}^{\mathfrak{e}}_{v_1}}(f_v^{\mathbf{G}^{\mathfrak{E}}}). \tag{8.4}$$

- Suppose $\tau\in\Sigma_F^\infty$. As $\tilde{\boldsymbol{\psi}}_\tau$ is an Adams–Johnson parameter, by Theorem 4.0.1, we also have

$$\sum_{\tilde{\pi}_{\tau}\in\tilde{\Pi}_{\tilde{\psi}_{\tau}}(G\otimes_{F,\tau}\mathbb{R})} \iota_{\mathfrak{m},z}(\tilde{\pi}_{\tau})(s^{\mathfrak{E}}s_{\tilde{\psi}_{\tau}})\tilde{\Theta}_{\tilde{\pi}_{\tau}}(f_{\tau}) = \tilde{\Theta}_{\tilde{\psi}^{\mathfrak{e}}_{\tau}}(f_\tau^{\mathbf{G}^{\mathfrak{E}}}). \tag{8.5}$$

For each $\tilde{\Pi}$ appearing in the LHS of Equation (8.1), it follows from Arthur's multiplicity formula Theorem 5.0.1 that

$$\prod_{v\in\Sigma}\eta_{\tilde{\Pi}_v}(s^{\mathfrak{E}}s_{\boldsymbol{\psi}}) = \varepsilon_{\boldsymbol{\psi}}(s^{\mathfrak{E}}s_{\boldsymbol{\psi}}). \tag{8.6}$$

Conversely, it follows from Lemma 2.2.2 that for any

$$\bigotimes_{v\in\Sigma\smallsetminus\{v_0\}}\tilde{\Pi}_v^\dagger\in\prod_{v\in\Sigma\smallsetminus\{v_0\}}\tilde{\Pi}_{\tilde{\psi}_v}(\mathbf{G}\otimes_F F_v),$$

there exists a unique $\tilde{\Pi}\in\widetilde{\mathrm{Irr}}_{\mathrm{disc}}(\mathbf{G}(\mathbf{A}_F))$ appearing in the LHS of Equation (8.1) with $\prod_{v\in\Sigma\smallsetminus\{v_0\}}\tilde{\Pi}_v\cong\prod_{v\in\Sigma\smallsetminus\{v_0\}}\tilde{\Pi}_v^\dagger$.

Finally, it follows from [Art13, Lemma 4.4.1] that

$$\varepsilon_{\boldsymbol{\psi}^{\mathfrak{e}}}(s_{\boldsymbol{\psi}^{\mathfrak{e}}}) = \varepsilon_{\boldsymbol{\psi}}(s^{\mathfrak{E}}s_{\boldsymbol{\psi}}). \tag{8.7}$$

So we conclude from Equations (8.1) to (8.7) that

$$\sum_{\tilde{\pi}_{v_1}\in\tilde{\Pi}_{\psi_{v_1}}(\mathbf{G}\otimes_F F_{v_1})} \iota_{\mathfrak{m},z}(\tilde{\pi}_{v_1})(s^{\mathfrak{e}}s_{\psi})\tilde{\Theta}_{\tilde{\pi}_{v_1}}(f_{v_1}) = \tilde{\Theta}_{\psi_{v_1}}(f_{v_1}^{\mathbf{G}^{\mathfrak{e}}}),$$

which proves the main theorem for $\tilde{\psi}=\tilde{\boldsymbol{\psi}}_{v_1}$.

9. Non-discrete case

In this section, suppose K is non-Archimedean. We prove the main theorem for local A-parameters $\psi\in\Psi(G)$ that factors through an admissible local A-parameter $\psi^{\mathfrak{e}}$ of $G^{\mathfrak{e}}$ that is not discrete. We use induction on the geometric rank of G, so we assume that the main theorem is proved when G is replaced by an even special orthogonal group of smaller geometric rank. We write $\mathfrak{d}=\mathrm{disc}(V)$ for the discriminant of V.

In this case there is a properly contained Levi subgroup $M_{\mathfrak{e}}$ of $G^{\mathfrak{e}}$ such that $\psi^{\mathfrak{e}}$ factors through an admissible discrete A-parameter for $M_{\mathfrak{e}}$. We know $M_{\mathfrak{e}}$ is of the form

$$M_{\mathfrak{e}} = \left(\mathrm{SO}(2m_1)^{\mathfrak{d}_1}\times\prod_{i\in I_1}\mathrm{GL}(n_i)\right)\times\left(\mathrm{SO}(2m_2)^{\mathfrak{d}_2}\times\prod_{i\in I_2}\mathrm{GL}(n_i)\right),$$

where $\mathfrak{d}_1\mathfrak{d}_2 = \mathfrak{d} \in K^\times/(K^\times)^2$. So the A-parameter ψ factors through an A-parameter ψ_M of a Levi subgroup $\widehat{M^*}$ of $\widehat{G}$, where M^* is a standard Levi subgroup of G^* such that

$$M^* \cong \mathrm{SO}(2n_0)^{\mathfrak{d}} \times \prod_{i\in I} \mathrm{GL}(n_i)$$

with $I = I_1 \coprod I_2$ and $n_0 = m_1 + m_2$. We may write

$$\psi^{\mathrm{GL}} = \psi_\tau + \psi_0^{\mathrm{GL}} + \psi_\tau^\vee,$$

where ψ_τ is a representation of $\mathrm{WD}_K \times \mathrm{SL}_2(\mathbb{C})$ corresponding to an admissible irreducible unitarizable representation $\tau = \prod_{i\in I} \tau_i$ of $\prod_{i\in I} \mathrm{GL}(n_i)$ of Arthur type, and ψ_0 is a discrete parameter for $\mathrm{SO}(2n_0)^{\mathfrak{d}}$. We write $M = \varrho(M^*)$, which is a Levi subgroup of G. Then $M_{\mathfrak{e}}$ is an endoscopic group for M, i.e. there exists an endoscopic triple $\mathfrak{e}_0$ for M such that $M^{\mathfrak{e}_0} = M_{\mathfrak{e}}$. We can choose $\mathfrak{e}_0$ such that under the commutative diagram

$$\begin{array}{ccc} S_{\psi_M} & \overset{\iota}{\hookrightarrow} & S_\psi \\ \downarrow & & \downarrow \\ \mathfrak{S}_{\psi_M} & \overset{\bar{\iota}}{\hookrightarrow} & \mathfrak{S}_\psi \end{array},$$

the image of $s^{\mathfrak{e}_0} \in Z_{\widehat{M}}(\psi_M)$ in $\mathfrak{S}_{\psi_M} = \pi_0(Z_{\widehat{M}}(\psi_M))$ is mapped by $\bar{\iota}$ to the image of $s^{\mathfrak{e}} \in S_\psi$ in $\mathfrak{S}_\psi$.

We recall the following lemma on descent property of transfer pairings.

Lemma 9.0.1.

(1) For each $\gamma \in M^{\mathfrak{e}_0}(K)$ and $\delta \in M(K)$, the following identity holds:

$$\Delta[\mathfrak{m}_0, \mathfrak{e}_0, z](\gamma, \delta) = \Delta[\mathfrak{m}, \mathfrak{e}, z](\gamma, \delta) \cdot \left| \frac{D^{G^{\mathfrak{e}}}_{M^{\mathfrak{e}_0}}(\gamma)}{D^G_M(\delta)} \right|^{\frac{1}{2}}.$$

Here $\mathfrak{m}_0$ is the induced Whittaker datum on M^*, and $D^{G^{\mathfrak{e}}}_{M^{\mathfrak{e}_0}}(\gamma)$ and $D^G_M(\delta)$ are the relative Weyl discriminants.

(2) If $f \in \mathcal{H}(G)$ and $f^{\mathfrak{e}} \in \mathcal{H}(G^{\mathfrak{e}})$ are $\Delta[\mathfrak{m}, \mathfrak{e}, z]$-matching, then their constant terms $f_M \in \mathcal{H}(M)$ and $f^{\mathfrak{e}}_{M^{\mathfrak{e}_0}} \in \mathcal{H}(M^{\mathfrak{e}_0})$ is $\Delta[\mathfrak{m}_0, \mathfrak{e}_0, z]$-matching.

Proof. The proof is similar to that of [KMSW14, Lemma 1.1.4]. □

We now consider the L-packet corresponding to $\psi^{\mathfrak{e}}$ and ψ. It follows from Theorem 2.2.1 that

$$\tilde{\Pi}_{\psi^{\mathfrak{e}}}(G^{\mathfrak{e}}) = \coprod_{\tilde{\sigma} \in \tilde{\Pi}_\psi(M^{\mathfrak{e}_0})} \{\text{irreducible components of } \mathrm{I}^{G^{\mathfrak{e}}}_{M^{\mathfrak{e}_0}}(\tilde{\sigma})\}$$

and

$$\tilde{\Pi}_\psi(G) = \coprod_{\tilde{\pi} \in \tilde{\Pi}_{\psi_M}(M)} \{\text{irreducible components of } \mathrm{I}^G_M(\tilde{\pi})\}.^{2}$$

Here the induced representations are multiplicity-free, and its ς-equivalence class is independent of the representative chosen. Hence

(9.1)
$$\sum_{\tilde{\pi} \in \tilde{\Pi}_\psi(G)} \iota_{\mathfrak{m},z}(\tilde{\pi})(s^{\mathfrak{e}} s_\psi)\tilde{\Theta}_{\tilde{\pi}} = \sum_{\tilde{\pi} \in \tilde{\Pi}_\psi(G)} \iota_{\mathfrak{m},z}(\tilde{\pi})(\iota(s^{\mathfrak{e}_0} s_{\psi_M}))\tilde{\Theta}_{\tilde{\pi}} = \sum_{\tilde{\pi} \in \tilde{\Pi}_{\psi_M}(M)} \iota_{\mathfrak{m}_0,z}(\tilde{\pi})(s^{\mathfrak{e}_0} s_{\psi_M})\tilde{\Theta}_{\mathrm{I}^G_M(\tilde{\pi})}.$$

[2]Note that M is a product of a special orthogonal group and general linear groups, so the L-packet and also the pairing $\iota_{\mathfrak{m}_0,z}$ is obviously defined.

It follows from Equation (9.1) and the parabolic descent formula that

$$\sum_{\tilde{\pi}\in\tilde{\Pi}_{\psi}(G)} \iota_{\mathfrak{m},z}(\tilde{\pi})(s^{\mathfrak{e}}s_{\psi})\tilde{\Theta}_{\tilde{\pi}}(f) = \sum_{\tilde{\pi}\in\tilde{\Pi}_{\psi_M}(M)} \iota_{\mathfrak{m}_0,z}(\tilde{\pi})(s^{\mathfrak{e}_0}s_{\psi_M})\tilde{\Theta}_{\tilde{\pi}}(f_M).$$

Similarly, it follows from the parabolic descent formula that

$$\tilde{\Theta}_{\psi^{\mathfrak{e}}}(f^{\mathfrak{e}}) = \tilde{\Theta}_{\psi^{\mathfrak{e}}}(f^{\mathfrak{e}}_{M^{\mathfrak{e}_0}}).$$

Here we note that $f^{\mathfrak{e}}_{M^{\mathfrak{e}_0}}$ is ς-invariant.

We now prove the main theorem Theorem A. By Lemma 9.0.1, $f^{\mathfrak{e}}_{M^{\mathfrak{e}_0}}$ and f_M are $\Delta[\mathfrak{m}_0,\mathfrak{e}_0,z]$-matching test functions. Thus, it follows from the induction hypothesis that

$$\tilde{\Theta}_{\psi^{\mathfrak{e}}}(f^{\mathfrak{e}}) = \tilde{\Theta}_{\psi^{\mathfrak{e}}}(f^{\mathfrak{e}}_{M^{\mathfrak{e}_0}}) = \sum_{\tilde{\pi}\in\tilde{\Pi}_{\psi_M}(M)} \iota_{\mathfrak{m}_0,z}(\tilde{\pi})(s^{\mathfrak{e}_0}s_{\psi_M})\tilde{\Theta}_{\tilde{\pi}}(f_M) = \sum_{\tilde{\pi}\in\tilde{\Pi}_{\psi}(G)} \iota_{\mathfrak{m},z}(\tilde{\pi})(s^{\mathfrak{e}}s_{\psi})\tilde{\Theta}_{\tilde{\pi}}(f).$$

So the main theorem is proved when the admissible A-parameter $\psi^{\mathfrak{e}}$ is not discrete.

References

[AG17] H. Atobe and W. T. Gan, *On the local Langlands correspondence and Arthur conjecture for even orthogonal groups*, Represent. Theory **21** (2017), 354–415. MR3708200

[AJ87] J. Adams and J. F. Johnson, *Endoscopic groups and packets of nontempered representations*, Compositio Math. **64** (1987), no. 3, 271–309. MR918414

[AK24] J. Adams and T. Kaletha, *Discrete series L-packets for real reductive groups*, Preprint (2024), available at arXiv:2409.13375.

[AMR18] N. Arancibia, C. Moeglin, and D. Renard, *Paquets d'Arthur des groupes classiques et unitaires*, Ann. Fac. Sci. Toulouse Math. (6) **27** (2018), no. 5, 1023–1105. MR3919547

[Art13] J. Arthur, *The endoscopic classification of representations*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013. Orthogonal and symplectic groups. MR3135650

[BG14] K. Buzzard and T. Gee, *The conjectural connections between automorphic representations and Galois representations*, Automorphic forms and Galois representations. Vol. 1, 2014, pp. 135–187. MR3444225

[BJ79] A. Borel and H. Jacquet, *Automorphic forms and automorphic representations*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, 1979, pp. 189–207. With a supplement "On the notion of an automorphic representation" by R. P. Langlands. MR546598

[Bor79] A. Borel, *Automorphic L-functions*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, 1979, pp. 27–61. MR546608

[CG15] P.-S. Chan and W. T. Gan, *The local Langlands conjecture for* GSp(4) *III: Stability and twisted endoscopy*, J. Number Theory **146** (2015), 69–133. MR3267112

[CS80] W. Casselman and J. Shalika, *The unramified principal series of p-adic groups. II. The Whittaker function*, Compositio Math. **41** (1980), no. 2, 207–231. MR581582

[CZ21a] R. Chen and J. Zou, *Local Langlands correspondence for even orthogonal groups via theta lifts*, Selecta Math. (N.S.) **27** (2021), no. 5, Paper No. 88, 71. MR4308934

[CZ21b] ———, *Theta correspondence and Arthur packets*, Preprint (2021), available at arXiv:2104.12354.

[CZ24] ———, *Arthur's multiplicity formula for even orthogonal and unitary groups*, Preprint (2024), available at arXiv:2103.07956.

[GGP12] W. T. Gan, B. H. Gross, and D. Prasad, *Symplectic local root numbers, central critical l-values, and restriction problems in the representation theory of classical groups*, Sur les conjectures de gross et prasad. i, 2012, pp. 1–109 (en). MR3202556

[Gro21] B. H. Gross, *Incoherent definite spaces and Shimura varieties*, Relative trace formulas, 2021 ©2021, pp. 187–215. MR4611946

[GS12] W. T. Gan and G. Savin, *Representations of metaplectic groups I: epsilon dichotomy and local Langlands correspondence*, Compos. Math. **148** (2012), no. 6, 1655–1694. MR2999299

[GT11] W. T. Gan and S. Takeda, *The local Langlands conjecture for* GSp(4), Ann. of Math. (2) **173** (2011), no. 3, 1841–1882. MR2800725

[Ish24] H. Ishimoto, *The endoscopic classification of representations of non-quasi-split odd special orthogonal groups*, Int. Math. Res. Not. IMRN **2024** (2024), no. 14, 10939–11012. MR4776199

[Kal16] T. Kaletha, *The local Langlands conjectures for non-quasi-split groups*, Families of automorphic forms and the trace formula, 2016, pp. 217–257. MR3675168

[KMSW14] T. Kaletha, A. Minguez, S. W. Shin, and P.-J. White, *Endoscopic classification of representations: Inner forms of unitary groups*, Preprint (2014), available at arXiv:1409.3731.

[Kot83] R. E. Kottwitz, *Sign changes in harmonic analysis on reductive groups*, Trans. Amer. Math. Soc. **278** (1983), no. 1, 289–297. MR697075

[Kot86] ———, *Stable trace formula: elliptic singular terms*, Math. Ann. **275** (1986), no. 3, 365–399. MR858284

[KS99] R. E. Kottwitz and D. Shelstad, *Foundations of twisted endoscopy*, Astérisque **255** (1999), vi+190. MR1687096

[Lan83] R. P. Langlands, *Les débuts d'une formule des traces stable*, Publications Mathématiques de l'Université Paris VII [Mathematical Publications of the University of Paris VII], vol. 13, Université de Paris VII, U.E.R. de Mathématiques, Paris, 1983. MR697567

[LS87] R. P. Langlands and D. Shelstad, *On the definition of transfer factors*, Math. Ann. **278** (1987), no. 1-4, 219–271. MR909227

[LS90] R. Langlands and D. Shelstad, *Descent for transfer factors*, The Grothendieck Festschrift, Vol. II, 1990, pp. 485–563. MR1106907

[Luo20] C. Luo, *Endoscopic character identities for metaplectic groups*, J. Reine Angew. Math. **768** (2020), 1–37. MR4168685

[Mg06] C. Mœ glin, *Sur certains paquets d'Arthur et involution d'Aubert-Schneider-Stuhler généralisée*, Represent. Theory **10** (2006), 86–129. MR2209850

[Ngô10] B. C. Ngô, *Le lemme fondamental pour les algèbres de Lie*, Publ. Math. Inst. Hautes Études Sci. **111** (2010), 1–169. MR2653248

[Pen25] H. Peng, *Fargues-Scholze parameters and torsion vanishing for special orthogonal and unitary groups*, Preprint (2025), available at arXiv:2503.04623.

[She08] D. Shelstad, *Tempered endoscopy for real groups. III. Inversion of transfer and L-packet structure*, Represent. Theory **12** (2008), 369–402. MR2448289

[She82] ———, *L-inadistinguishability for real groups*, Math. Ann. **259** (1982), no. 3, 385–430. MR661206

[Shi12] S. W. Shin, *Automorphic Plancherel density theorem*, Israel J. Math. **192** (2012), no. 1, 83–120. MR3004076

[Taï17] O. Taïbi, *Dimensions of spaces of level one automorphic forms for split classical groups using the trace formula*, Ann. Sci. Éc. Norm. Supér. (4) **50** (2017), no. 2, 269–344. MR3621432

[Taï19] ———, *Arthur's multiplicity formula for certain inner forms of special orthogonal and symplectic groups*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 3, 839–871. MR3908767

[Wal06] J.-L. Waldspurger, *Endoscopie et changement de caractéristique*, J. Inst. Math. Jussieu **5** (2006), no. 3, 423–525. MR2241929

[Wal10] ———, *Les facteurs de transfert pour les groupes classiques: un formulaire*, Manuscripta Math. **133** (2010), no. 1-2, 41–82. MR2672539

[Wal97] ———, *Le lemme fondamental implique le transfert*, Compositio Math. **105** (1997), no. 2, 153–236. MR1440722

[Xu17] B. Xu, *On the cuspidal support of discrete series for p-adic quasisplit $Sp(N)$ and $SO(N)$*, Manuscripta Math. **154** (2017), no. 3-4, 441–502. MR3713922

Hao Peng
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
Email: hao_peng@mit.edu