# 1 Goldbach's Problems

The ternary Goldbach's conjecture is not hard to prove if we assume prime number theorem and Davenport's theorem:

**Thm. (0.0.1.1) [Davenport(37)].** For any $\theta \in \mathbb{S}^1, X \in \mathbb{R}_{\geq 2}$ and any $A \in \mathbb{R}_+$, we have

$$\mathcal{E}_\mu(\theta; X) = \sum_{n \in [X]_+} \mu(n) e^{2\pi \mathrm{i} \theta n} \ll_A \frac{X}{\log^A X}.$$

⌟

**Def. (0.0.1.2) [Goldbach's Functions].** For $k \in \mathbb{Z}_{\geq 2}$, define $G_k : \mathbb{Z}_+ \to \mathbb{R}$:

$$G_k(n) = \sum_{n_i \in \mathbb{Z}_+, \sum_{i=1}^k n_i = n} \Lambda(n_1) \cdots \Lambda(n_k).$$

⌟

**Def. (0.0.1.3) [Singular Series for Goldbach's Functions].** For $N \in 2\mathbb{Z}_+$, denote

$$\mathfrak{G}_2(N) \triangleq \sum_{d \in \mathbb{Z}_+, d | N} \frac{\mu^2(d) d}{\phi^2(d)} \sum_{c \in \mathbb{Z}_+, (c,d)=1} \frac{\mu(c)}{\phi^2(c)},$$

then

$$\mathfrak{G}_2(N) = 2 \prod_{p \in \texttt{Prime}_{\geq 3}} \left(1 - \frac{1}{(p-1)^2}\right) \cdot \prod_{p \in \texttt{Prime}_{\geq 3}, p | N} \left(\frac{p-1}{p-2}\right)$$

and is a non-zero unbounded function.

And for $N \in 2\mathbb{Z}_+ + 1$, denote

$$\mathfrak{G}_3(N) \triangleq \frac{1}{2} \sum_{c,d \in \mathbb{Z}_+, (d, cN)=1} \frac{\mu(c)\mu^2(d) d}{\phi^2(c)\phi^3(d)},$$

then

$$\mathfrak{G}_3(N) = \frac{1}{2} \prod_{p \in \texttt{Prime}} \left(1 + \frac{1}{(p-1)^3}\right) \prod_{p \in \texttt{Prime}, p | N} \left(1 - \frac{1}{p^2 - 3p + 3}\right)$$

and is a non-zero bounded function. ⌟

*Proof:*

$$\begin{aligned}
\mathfrak{G}_2(N) \prod_{p \in \texttt{Prime}_{\geq 3}} \left(1 - \frac{1}{(p-1)^2}\right)^{-1} &= \sum_{2 | d | N} \frac{\mu^2(d) d}{\phi^2(d)} \prod_{p | d, p \geq 3} \left(1 - \frac{1}{(p-1)^2}\right)^{-1} \\
&= 2 \prod_{p \in S(N) \setminus \{2\}} \left(1 + \frac{p}{(p-1)^2} \left(1 - \frac{1}{(p-1)^2}\right)^{-1}\right) \\
&= 2 \prod_{p \in \texttt{Prime}_{\geq 3}, p | N} \left(\frac{p-1}{p-2}\right).
\end{aligned}$$

$$\mathfrak{G}_3(N)\prod_{p\in\texttt{Prime}_{\geq 3}}\left(1-\frac{1}{(p-1)^2}\right)^{-1}=\sum_{d\in 2\mathbb{Z}_++1,(d,N)=1}\frac{\mu^2(d)d}{\phi^3(d)}\prod_{p|d,p\geq 3}\left(1-\frac{1}{(p-1)^2}\right)^{-1}$$
$$=\prod_{p\in\texttt{Prime}_{\geq 3},p\nmid N}\left(1+\frac{p}{(p-1)^3}\left(1-\frac{1}{(p-1)^2}\right)^{-1}\right)$$
$$=\prod_{p\in\texttt{Prime}_{\geq 3},p\nmid N}\frac{p^2-3p+3}{(p-1)(p-2)}$$

And the assertion follows. □

**Def. (0.0.1.4) [Truncated Von.Mangoldt-Function].** For $R, X\in\mathbb{R}_{\geq 2}$, we denote

$$\Lambda_R^\sharp(n)=-\sum_{d\in[R]_+,d|n}\mu(d)\log(d),\quad \Lambda_R^\flat=\Lambda-\Lambda_R^\sharp,$$

⌟

**Lemma (0.0.1.5) [The Main Contributions].** Fix some $z\in\mathbb{R}_{\geq 2}$ and $N\in 2\mathbb{Z}_++2$, if we denote

$$G_2^{\sharp\sharp}(N)=\sum_{n_1+n_2=N}\Lambda_z^\sharp(n_1)\Lambda_z^\sharp(n_2)(0.0.1.4),$$

then

$$G_2^{\sharp\sharp}(N)=\mathfrak{G}_2(N)N+O_A\left(N\log^{-A}z+\tau(N)Nz^{-1/3}+z^3\right)(0.0.1.3).$$

Heuristically, this implies that the main contribution to $G_2(N)$ comes from $\Lambda_z^\sharp$. ⌟

*Proof:*

$$G_2^{\sharp\sharp}(N)=\sum_{m_1,m_2\in[z]_+}\mu(m_1)\mu(m_2)\log(m_1)\log(m_2)\sum_{l_1,l_2\in\mathbb{Z}_+,l_1m_1+l_2m_2=N}1$$
$$=\sum_{m_1,m_2\in[z]_+}\mu(m_1)\mu(m_2)\log(m_1)\log(m_2)\left(\frac{N}{[m_1,m_2]}+O(1)\right)$$
$$=N\sum_{m_1,m_2\in[z]_+}\frac{\mu(m_1)\mu(m_2)}{[m_1,m_2]}\log(m_1)\log(m_2)+O(z^2\log^2 z)$$

And we have

$$\sum_{m_1,m_2\in[z]_+}\frac{\mu(m_1)\mu(m_2)}{[m_1,m_2]}\log(m_1)\log(m_2)=\sum_{d|N}\frac{\mu^2(d)}{d}\sum_{\substack{m_1,m_2\in[z/d]_+\\(m_1,m_2)=1\\(m_1m_2,d)=1}}\frac{\mu(m_1m_2)}{m_1m_2}\log(dm_1)\log(dm_2)$$
$$=\sum_{d|N}\frac{\mu^2(d)}{d}\sum_{\substack{m_1,m_2\in[z/d]_+\\(m_1m_2,d)=1}}\sum_{c|(m_1,m_2)}\mu(c)\frac{\mu(m_1m_2)}{m_1m_2}\log(dm_1)\log(dm_2)$$
$$=\sum_{d|N}\frac{\mu(d)}{d}\sum_{c\in[z/d]_+}\frac{\mu(cd)}{c^2}\left(\sum_{m\in[z/cd]_+,(m,cd)=1}\frac{\mu(m)}{m}\log(cdm)\right)^2$$

For $c > \sqrt{z}/d$, this is bounded by

$$\sum_{d|N} \sum_{c > \sqrt{z}/d} c^{-2} d^{-1} \log^4 z \leq 2\tau(N) z^{-1/2} \log^4 z,$$

and for $c \leq \sqrt{z}/d$, it follows from prime number theorem that

$$-\sum_{m \in [z/cd]_+, (m,cd)=1} \frac{\mu(m)}{m} \log(cdm) = \frac{cd}{\phi(cd)} \left(1 + O_A\left(\tau(cd) \log^{-A} z\right)\right)$$

So we can estimate by bounding $\phi(n)$ and $\tau(n)$ that

$$\begin{aligned} \sum_{d|N} \frac{\mu(d)}{d} \sum_{c \in [z^{1/2}/d]_+} \frac{\mu(cd)}{c^2} \left( \sum_{m \in [z/cd]_+, (m,cd)=1} \frac{\mu(m)}{m} \log(cdm) \right)^2 &= \sum_{d|N} \sum_{c \in [z^{1/2}/d]_+} \mu(d)\mu(cd) d \phi^{-2}(cd) \\ &+ O_A\left(\log^{4-A} z + \tau(N) z^{-1/3}\right) \\ &= \mathfrak{S}_2(N) + O_A\left(\log^{-A} z + \tau(N) z^{-1/3}\right) \text{ (0.0.1.3)}, \end{aligned}$$

and the assertion follows. □

**Lemma (0.0.1.6) [The Minor Contributions].** For any $(u_k)_{k \in \mathbb{Z}_+}, (v_m)_{m \in \mathbb{Z}_+} \in \mathbb{C}$ and $z \in \mathbb{R}_{\geq 2}, A \in \mathbb{R}_+$, we have

$$\sum_{l+m+n=N} u_l v_m \Lambda_z^\flat(n) \ll_A \log^{-A} z \, \|u\| \, \|v\| \, N \log N, \text{ where } N \in \mathbb{Z}_+.$$

⌟

*Proof:* For any $\alpha \in \mathbb{R}$, it follows from Davenport's inequality(0.0.1.1) and partial summation that

$$\sum_{n \in [N]_+} \mu(n) \log(n) e^{2\pi i \alpha n} = O_A\left(\frac{X}{\log^A X}\right).$$

If we write $S^\flat(\alpha; N) = \sum_{n \in [N]_+} \Lambda_z^\flat(n) e^{2\pi i \alpha n}$, then we have

$$\begin{aligned} |S^\flat(\alpha; N)| &\leq \sum_{l \in [N/z]_+} \left| \sum_{m \in \mathbb{Z}_+, z < m \leq N/l} \mu(m) \log(m) e^{2\pi i \alpha l m} \right| \\ &\ll_A \sum_{l \in [N/z]_+} \frac{N}{l} \log^{-A}(N/l) \\ &\leq_A N \log N \log^{-A} z. \end{aligned}$$

And we have

$$\sum_{l+m+n=N} u_l v_m \Lambda_z^\flat(n) = \int_0^1 \left(\sum_{l \in [N]_+} u_l e^{2\pi i \alpha l}\right) \left(\sum_{m \in [N]_+} v_m e^{2\pi i \alpha m}\right) S^\flat(\alpha; N) e^{-2\pi i \alpha N} d\alpha.$$

Then the assertion follows from Schwarz's inequality and Parseval's formula. □

**Thm& Conj.Cor. (0.0.1.7) [Goldbach's Conjecture for almost all Even Numbers].** For any $A \in \mathbb{R}_+$, there exists $C_A \in \mathbb{R}_+$ s.t. for any $X \in \mathbb{R}_{\geq 2}$, we have

$$\sum_{N \in 2\mathbb{Z}_+ \cap [X]_+} (G_2(N) - \mathfrak{G}_2(N)N)^2 \leq C_A \frac{X^3}{\log^{2A} X}.$$

In particular, for $A, B \in \mathbb{R}_{\geq 4}$, there exists $C_{A,B} \in \mathbb{R}_+$ s.t. for any $X \in \mathbb{R}_{\geq 2}$, there are at most $C_{A,B} X \log^{-A}(X)$-many $m \in [X]_+$ s.t.

$$|G_2(2m) - \mathfrak{G}_2(2m)2m| \geq B \frac{X}{\log^A X} (0.0.1.3).$$

In particular, all but at most $O_A(X \log^{-A} X)$-many even numbers in $[X]_+$ are sums of two primes. ⌟

*Proof:* For any sequence $(c_N)_{N \in \mathbb{Z}_+} \in \mathbb{C}$ supported on odd integers, we have

$$\sum_{N \in [X]_+} c_N G_2(N) = \sum_{N \in (2\mathbb{Z}_+ + 1) \cap [X]_+} c_N \left( G_2^{\sharp\sharp}(N) + 2G_2^{\sharp\flat}(N) + G_2^{\flat\flat}(N) \right).$$

And by(0.0.1.5) applied to $z = X^{1/4}$, we see

$$\sum_{N \in [X]_+} c_N G_2^{\sharp\sharp}(N) = \sum_{N \in (2\mathbb{Z}_+ + 1) \cap [X]_+} c_N G_2(N) + O_A \left( \|c\| X^{3/2} \log^{-A} X \right).$$

And(0.0.1.6) applied to $z = X^{1/4}$ implies that

$$\sum_{N \in [X]_+} c_N (2G_2^{\sharp\flat}(N) + G_2^{\flat\flat}(N)) = O_A \left( \|c\| \left( 2 \left\| \Lambda^{\sharp}_{X^{1/4}} \right\| + \left\| \Lambda^{\flat}_{X^{1/4}} \right\| \right) X \log^{-A} X \right) = O_A \left( \|c\| X^{3/2} \log^{-A} X \right).$$

So we get

$$\sum_{N \in [X]_+} c_N G_2(N) = \sum_{N \in [X]_+} c_N \mathfrak{G}_2(N) N + O_A \left( \|c\| X^{3/2} \log^{-A} X \right).$$

The assertion follows by applying this to $c_N = G_2(N) - \mathfrak{G}_2(N)N$. □

**Cor. (0.0.1.8) [Ternary Goldbach's Function].** For any $A \in \mathbb{R}_+$, there exists $C_A \in \mathbb{R}_+$ s.t. for any $N \in 2\mathbb{Z}_+ + 5$, we have

$$\left| G_3(N) - \mathfrak{G}_3(N)N^2 \right| \leq C_A \frac{N^2}{\log^A N}, \quad \left| R_3(N) - \mathfrak{G}_3(N) \frac{N^2}{\log^3 N} \right| \leq C_A \frac{N^2}{\log^A N} (0.0.1.3),$$

where $R_3(N)$ is the number of ternary Goldbach-representations of $N$. In particular, any sufficiently large odd number $N$ is a sum of three primes. ⌟

*Proof:* By the proof of(0.0.1.7), we see that for any sequence $(c_N)_{N \in \mathbb{Z}_+} \in \mathbb{C}$ supported on odd integers, we have

$$\sum_{N \in [X]_+} c_N G_2(N) = \sum_{N \in [X]_+} c_N \mathfrak{G}_2(N) N + O_A \left( \|c\| X^{3/2} \log^{-A} X \right).$$

Apply this to

$$\begin{aligned} G_3(N) &= \sum_{n\in[N-1]_+} \Lambda(n) G_2(N-n) \\ &= \sum_{n\in[N-1]_+} \Lambda(n)\mathfrak{G}_2(N-n)(N-n) + O_A(N^2\log^{-A} N) \\ \text{(0.0.1.3)} &= \sum_{(c,d)=1} \frac{\mu(c)\mu^2(d)d}{\phi^2(c)\phi^2(d)} \sum_{n<N, n\equiv N(\bmod\ d)} \Lambda(n)(N-n) + O_A(N^2\log^{-A} N). \end{aligned}$$

If $d \nmid N$, then

$$\sum_{n<N, n\equiv N(\bmod\ d)} \Lambda(n)(N-n) = \frac{N^2}{2\phi(d)} + O_A(N^2\log^{-A} N)$$

by prime number theorem and partial summation. And if $d = p \in S(N)$, then this summation is bounded by $N\log N$, and the contribution is smaller than $N\log^2 N$. Hence we get

$$G_3(N) = \frac{N^2}{2} \sum_{c,d\in\mathbb{Z}_+,(c,dN)=1} \frac{\mu(c)\mu^2(d)d}{\phi^2(c)\phi^3(d)} + O_A(N^2\log^{-A} N) = \mathfrak{G}_3(N)N^2 + O_A\left(\frac{N^2}{\log^A N}\right).$$

The proof for $R_3(N)$ is similar. □

## 2 Average of Multiplicative Functions

**Prop. (0.0.2.1).** If $f:\mathbb{Z}_+\to\mathbb{R}_{\geq 0}$ is multiplicative, then for each $X\in\mathbb{R}_+$, we have

$$\mathcal{M}_f(X) \leq X \sum_{n\in[X]_+} \frac{f(n)}{n} \leq X \prod_{p\in\texttt{Prime}_{\leq X}} \left(\sum_{k\in\mathbb{N}} f(p^k)p^{-k}\right).$$

And if $f$ is non-decreasing on prime powers, then $h=\mu * f$ is non-negative, in fact $h(1)=1$ and $h(p^k)=f(p^k)-f(p^{k-1})$. Writing $f=h*\mathbf{1}$, we get

$$\begin{aligned} \mathcal{M}_f(X) &= \sum_{m\in[X]_+} h(m)\lfloor\frac{X}{m}\rfloor \leq X\sum_{m\in[X]_+}\frac{h(m)}{m} \\ &\leq X\cdot \prod_{p\in\texttt{Prime}_{\leq X}} \left(\sum_{k\in\mathbb{N}} h(p^k)p^{-k}\right) \\ &= X\cdot \prod_{p\in\texttt{Prime}_{\leq X}} \left(\sum_{k\in\mathbb{N}} f(p^k)p^{-k}\right)\left(1-\frac{1}{p}\right) \end{aligned}$$

⌟

**Cor. (0.0.2.2).** Situation as in(0.0.2.1), if $f$ is supported on square-free numbers, then we can take $h:\mathbb{Z}_+\to\mathbb{R}$ to be the multiplicative function s.t. for each $p\in\texttt{Prime}$ and $k\in\mathbb{Z}_+$, we have

$$h(p^k) = \begin{cases} \max(f(p)-1,0) & ,k=1 \\ 0 & ,k\geq 2 \end{cases}.$$

Then we have $f \leq h * 1$, and

$$\mathcal{M}_f(X) \leq X \cdot \prod_{p \in \texttt{Prime}_{\leq X}} \left(1 + \frac{\max(f(p)-1, 0)}{p}\right).$$

⌟

**Prop. (0.0.2.3).** Suppose $f : \mathbb{Z}_+ \to \mathbb{R}_{\geq 0}$ satisfies:

- $f$ is completely sub-multiplicative, i.e. $f(mn) \leq f(m)f(n)$ for $m, n \in \mathbb{Z}_+$.
- There exists $c \in \mathbb{R}_+$ s.t. $\sum_{m \in [X]_+} f(m)\Lambda(m) \leq cX$ for any $X \in \mathbb{R}_{\geq 2}$(By prime number theorem, this can be understood as $f(p)$ being bounded by $c$ on average).

Then by partial summation, we have

$$\mathcal{M}_f(X) \leq 3cd \frac{X}{\log X} \prod_{p \in \texttt{Prime}_{\leq X}} \left(1 + \frac{f(p)}{p}\right), \text{ where } d = \sum_{n \in [X]_+} \frac{f(n)^2}{n^2}.$$

⌟

*Proof:* Cf.[Iwaniek-Kowalski04]P24. □

**Thm. (0.0.2.4) [Wirsing's Formula].** If $f : \mathbb{Z}_+ \to \mathbb{R}$ is multiplicative and there exists $\kappa \in \mathbb{R}_{>-1/2}$ s.t.

- $\sum_{n \in [X]_+} \Lambda_f(n) = \kappa \log X + O(1)$ for $X \in \mathbb{R}_+$(By Merten's theorem, this can be understood as $f(p)$ being about $\kappa p^{-1}$ on average).
- $\mathcal{M}_{|f|}(X) \ll \log^{|\kappa|}(X)$ for $X \in \mathbb{R}_+$.

Then for $X \in \mathbb{R}_+$, we have

$$\mathcal{M}_f(X) = C(f) \log^{\kappa}(X) + O(\log^{|\kappa|-1} X),$$

where

$$C(f) = \frac{1}{\Gamma(\kappa+1)} \prod_{p \in \texttt{Prime}} \left(1 - \frac{1}{p}\right)^{\kappa} \sum_{k \in \mathbb{N}} f(p^k).$$

⌟

**Remark (0.0.2.5).** I cannot verify that $C(f)$ converges, but when applying this theorem, it is easy to verify that $C(f)$ converges, and under this assumption, the proof below goes through. ⌟

*Proof:* Because $f \cdot L = f * \Lambda_f$, we have

$$\sum_{n \in [X]_+} f(n) \log n = \sum_{d \in [X]_+} f(d) \sum_{m \in [X/d]_+} \Lambda_f(m) = \sum_{d \in [X]_+} f(d) \left(\kappa \log \frac{X}{d} + O(1)\right).$$

Then we get

$$(\kappa+1) \sum_{n \in [X]_+} f(n) \log n = \kappa \mathcal{M}_f(X) \log X + O(\log^{|\kappa|} X).$$

Using the equation

$$\sum_{n \in [X]_+} f(n) \log \frac{X}{n} = \int_1^X \mathcal{M}_f(y) y^{-1} \mathrm{d}y,$$

we see that

$$\Delta(X) \stackrel{\triangle}{=} \mathcal{M}_f(X) \log X - (\kappa+1) \int_2^X \mathcal{M}_f(y) y^{-1} \mathrm{d}y = O(\log^{|\kappa|} X).$$

So

$$\int_2^X \Delta(y)y^{-1}\log^{-\kappa-2}(y)\mathrm{d}y = \int_2^X \mathcal{M}_f(y)y^{-1}\log^{-\kappa-1}(y)\mathrm{d}y - (\kappa+1)\int_2^X y^{-1}\log^{-\kappa-2}(y)\left(\int_2^y \mathcal{M}_f(u)u^{-1}\mathrm{d}u\right)\mathrm{d}y$$

(change order of integration) $= \log^{-\kappa-1}(X)\int_2^X \mathcal{M}_f(u)u^{-1}\mathrm{d}u.$

Plug in this back to the definition of $\Delta(X)$ we see that

$$\mathcal{M}_f(X) = \log^{\kappa}(X)\int_2^X(-\Delta(y))\mathrm{d}\left(\log^{-\kappa-1}(y)\right) + \frac{\Delta(X)}{\log X}.$$

It can be checked that this integral is absolutely convergent, so we have

$$\mathcal{M}_f(X) = (c(f) + r_f(X))\log^{\kappa}(X),$$

where

$$c(f) = -\int_2^\infty \Delta(y)\mathrm{d}(\log^{-\kappa-1}(y)), \quad r_f(X) = \int_X^\infty (\Delta(y)-\Delta(X))\mathrm{d}(\log^{-\kappa-1}(y)).$$

It follows from the fact that $\Delta(X) = O(\log^{|\kappa|}(X))$ that we see

$$r_f(X) = O(\log^{|\kappa|-\kappa-1}(X)).$$

To conclude, it suffices to show that

$$c(f) = \frac{1}{\Gamma(\kappa+1)}\prod_{p\in\mathtt{Prime}}\left(1-\frac{1}{p}\right)^{\kappa}\sum_{k\in\mathbb{N}} f(p^k).$$

For this, notice that $L(f;s)$ is absolutely convergent for $\mathrm{Re}(s) > 0$, and for these $s$, we have

$$\begin{aligned} L(f;s) &= \int_1^\infty y^{-s}\mathrm{d}\mathcal{M}_f(y) \\ &= -\int_1^\infty \mathcal{M}_f(y)\mathrm{d}(y^{-s}) \\ &= -\int_0^\infty \mathcal{M}_f(e^t)\mathrm{d}(e^{-st}) \\ &= -\int_0^\infty \left(c(f) + O(t^{|\kappa|-\kappa-1})\right)t^{\kappa}\mathrm{d}(e^{-st}) \end{aligned}$$

which equals $\left(c(f) + O(s^{\kappa+1-|\kappa|})\right)s^{-\kappa}\Gamma(\kappa+1)$ when $s \to 0_+$. So

$$c(f)\Gamma(\kappa+1) = \lim_{s\to 0_+}\left(\zeta(s+1)^{-\kappa}L(f;s)\right) = \prod_{p\in\mathtt{Prime}}\left(1-\frac{1}{p}\right)^{\kappa}\left(1+f(p)+f(p^2)+\dots\right),$$

and the assertion follows. □

**Cor. (0.0.2.6) [Forbidden Prime Divisors].** If $\mathcal{P}\subset\mathtt{Prime}$ is a set of primes and $\kappa\in[0,1/2)$ s.t. for each $X\in\mathbb{R}_+$ we have

$$\sum_{p\in\mathtt{Prime}_{\le X}}\frac{\log p}{p} = \kappa\log X + O(1),$$

then the number $S(\mathcal{P},X)$ of integers in $[X]_+$ having no prime divisors in $\mathcal{P}$ satisfies

$$S(\mathcal{P},X) = c(\mathcal{P})\frac{X}{\log^{\kappa} X}\left(1 + O\left(\frac{1}{\log^{1-2\kappa} X}\right)\right),$$

where

$$c(\mathcal{P}) = \frac{1}{\Gamma(1-\kappa)}\prod_{p\in\texttt{Prime}}\left(1-\frac{f(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-\kappa}.$$

⌟

*Proof:* Let $f : \mathbb{Z}_+ \to \mathbb{R}$ be the completely multiplicative function s.t. $f(p) = \mathbf{1}_{\mathcal{P}}(p)$ for $p \in \texttt{Prime}$, and let $g = (\mu \cdot f) * \mathbf{1}$, then

$$g(n) = \prod_{p\in S(n)}(1 - f(p)),$$

and

$$S(\mathcal{P},X) = \mathcal{M}_g(X) = \sum_{d\in[X]_+}\mu(d)f(d)\lfloor\frac{X}{d}\rfloor = X\sum_{d\in[X]_+}\mu(d)\frac{f(d)}{d} + O(\mathcal{M}_f(X))$$

Wirsing's formula(0.0.2.4) applied to $\mu(n)f(n)/n$ implies that

$$\sum_{d\in[X]_+}\mu(d)\frac{f(d)}{d} = c(\mathcal{P})\log^{-\kappa} X + O(\log^{\kappa-1} X)$$

where

$$c(\mathcal{P}) = \frac{1}{\Gamma(1-\kappa)}\prod_{p\in\texttt{Prime}}\left(1-\frac{f(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-\kappa}.$$

And also

$$\mathcal{M}_f(X) = O\left(X\log^{\kappa-1}(X)\right)$$

by(0.0.2.3), noticing that

$$\sum_{p\in\texttt{Prime}_{\le X}}\frac{f(p)}{p} \le \kappa\log_{(2)} X$$

by partial summation. Thus the assertion follows. □

**Prop. (0.0.2.7) [Landau1908].** For $X \in \mathbb{R}_+$, we have

$$B(X) = \#\{n\in[X]_+ : n = a^2 + b^2 \text{ for some } a,b\in\mathbb{Z}_+\} = C\frac{X}{\sqrt{\log X}} + O\left(\frac{X}{\log^{3/4} X}\right),$$

where

$$C = \frac{1}{\sqrt{2}}\prod_{p\equiv 3 \pmod 4}\left(1-\frac{1}{p^2}\right)^{-1/2}.$$

⌟

*Proof:* Cf.[Topics in Number Theory, 2, LeVeque]P261. We only prove the asymptotic.

For $n \in \mathbb{Z}_+$, let $b(n) = \begin{cases} 1 & , n = a^2 + b^2 \text{ for some } a, b \in \mathbb{Z}_+ \\ 0 & , \text{otherwise} \end{cases}$, then $B(X) = \mathcal{M}_b(X)$, and we have

$$L(b; s) = \frac{1}{1 - 2^{-s}} \prod_{p \equiv 1 \pmod 4} \frac{1}{1 - p^{-s}} \prod_{p \equiv 3 \pmod 4} \frac{1}{1 - p^{-2s}}.$$

So

$$L(b; s)^2 = \zeta(s) L(\chi_{-4}; s) \frac{1}{1 - 2^{-s}} \prod_{p \equiv 3 \pmod 4} \frac{1}{1 - p^{-2s}}.$$

So $g(s) = (s - 1)L(b; s)^2$ is holomorphic and non-vanishing on a nbhd of the half plane $\{s \in \mathbb{C} : \mathrm{Re}(s) \geq 1\}$, and

$$g(1) = \frac{\pi}{2} \prod_{p \equiv 3 \pmod 4} \left(1 - \frac{1}{p^2}\right)^{-1}.$$

So we get

$$B(X) \sim C \frac{X}{\sqrt{\log X}}$$

by the Tauberian theorem. □