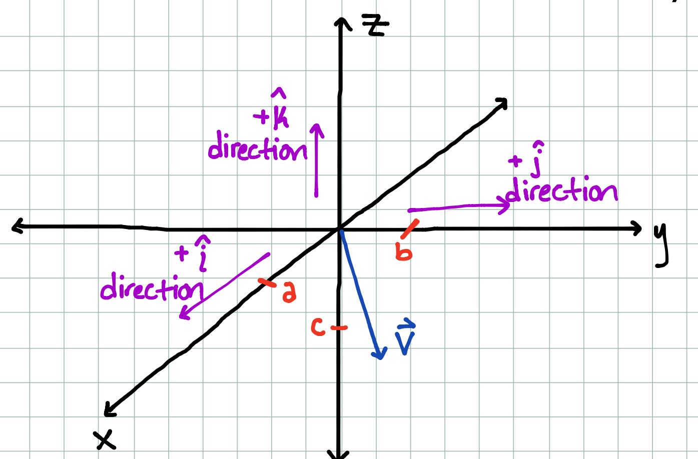
- What is a Vector?
  - A vector is a quantity that has both a **magnitude** and **direction**
    - Examples of vectors in real life:
      - Velocity
      - Force
      - Position in 2D and 3D space
      - 1000x1000 pixel grayscale image can be represented by a vector with 1,000,000 dimensions
  - Notation: How to represent a vector

$$\vec{V} = \langle a, b, c \rangle = a\hat{i} + b\hat{j} + c\hat{k}$$

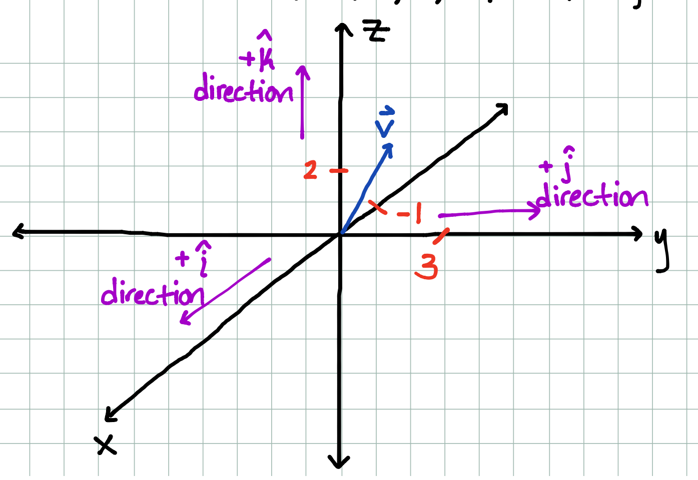
      - Index Notation: a way of labeling a vector's components

$$\vec{V} = \langle V_1, V_2, V_3 \rangle$$

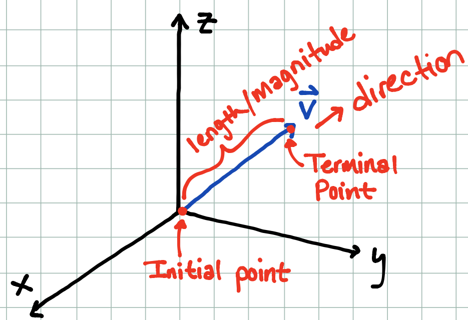
      - Vector $\vec{v}$ is composed of three components a, b, and c, which tell you how to draw $\vec{v}$ on a 3D Cartesian coordinate system



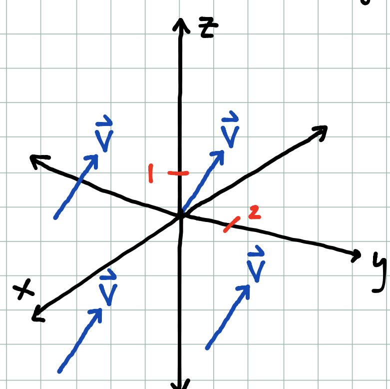
        - Note: $\hat{i}$, $\hat{j}$, and $\hat{k}$ just refer to the x, y, and z directions respectively.
  - Example: Draw the vector $\vec{v} = \langle -1, 3, 2 \rangle = -\hat{i} + 3\hat{j} + 2\hat{k}$



- Dissecting a Vector: Initial point, Terminal Point, Length/Magnitude, and Direction
  - Initial point: the point where the vector starts
  - Terminal point: the tip (or tail) of the vector
  - Length/Magnitude: how long the vector is; the scalar quantity attached to the vector
  - Direction: where the arrow points



- Does a vector always start at the origin?
  - A vector can start anywhere in space. The placement of the initial point is arbitrary. As long as the vector keeps the same direction and length, you can have the vector start anywhere
    - Example: $\vec{v} = \langle 0, 2, 1 \rangle = 2\hat{j} + \hat{k}$



* All of these representations of $\vec{v}$ are correct

- Length/Magnitude
  - How to Calculate the Length/Magnitude of a Vector Given its Components:

for the vector $\vec{v} = \langle a, b, c \rangle$:

$$|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$$

where $|\vec{v}|$ is the magnitude of vector $\vec{v}$

- Direction
  - Every vector is a product of its length/magnitude and direction

$$\vec{v} = \underbrace{|\vec{v}|}_{\text{length/magnitude of } \vec{v}} \; \underbrace{\hat{v}}_{\text{direction of } \vec{v}}$$

- $\hat{v}$ is called the unit vector of $\vec{v}$. It specifies the direction of $\vec{v}$ and is found by dividing $\vec{v}$ by $|\vec{v}|$ so you're left with only the direction of $\vec{v}$

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

  - Example: Find $\hat{u}$ for $\vec{u} = \langle 1, -2, 2 \rangle$

1 Find $|\vec{u}|$

$$|\vec{u}| = \sqrt{(1)^2 + (-2)^2 + (2)^2} = 3$$

2 Find $\hat{u}$ using $\hat{u} = \frac{\vec{u}}{|\vec{u}|}$

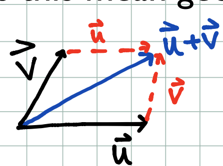
$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{\langle 1, -2, 2 \rangle}{3} = \left\langle \frac{1}{3}, \frac{-2}{3}, \frac{2}{3} \right\rangle$$

- Adding and Subtracting Vectors
  - Vector Addition
    - To add two vectors, add each respective component together

$$\vec{v} = \langle a, b, c \rangle$$
$$\vec{u} = \langle d, e, f \rangle$$

$$\Rightarrow \vec{v} + \vec{u} = \langle a, b, c \rangle + \langle d, e, f \rangle = \langle a+d, b+e, c+f \rangle$$

    - What does this mean geometrically?



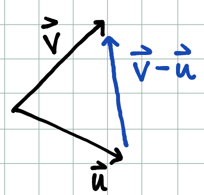
    - Example: Find $\vec{v} + \vec{u}$, where $\vec{v} = 3\hat{i} - 2\hat{j} + 5\hat{k}$ and $\vec{u} = -4\hat{i} + 4\hat{j} + 3\hat{k}$

$$\vec{v} + \vec{u} = [3\hat{i} - 2\hat{j} + 5\hat{k}] + [-4\hat{i} + 4\hat{j} + 3\hat{k}] = (3 + -4)\hat{i} + (-2+4)\hat{j} + (5+3)\hat{k}$$
$$= -\hat{i} + 2\hat{j} + 8\hat{k}$$

- Vector Subtraction
  - To subtract two vectors, subtract component by component in the correct order

$$\vec{v} = \langle a, b, c \rangle$$
$$\vec{u} = \langle d, e, f \rangle$$
$$\Rightarrow \vec{v} - \vec{u} = \langle a, b, c \rangle - \langle d, e, f \rangle = \langle a-d, b-e, c-f \rangle$$

  - What does this mean geometrically?



  - Example: Find $\vec{v}-\vec{u}$, where $\vec{v} = 3\hat{i} - 2\hat{j} + 5\hat{k}$ and $\vec{u} = -4\hat{i} + 4\hat{j} + 3\hat{k}$

$$\vec{v}-\vec{u} = \left[3\hat{i} - 2\hat{j} + 5\hat{k}\right] - \left[-4\hat{i} + 4\hat{j} + 3\hat{k}\right] = (3 - -4)\hat{i} + (-2-4)\hat{j} + (5-3)\hat{k}$$
$$= 7\hat{i} - 6\hat{j} + 2\hat{k}$$

- Multiplying a Vector by a Scalar
  - To multiply a vector by a scalar, multiply each component of the vector by the scalar value

$$\vec{v} = \langle a, b, c \rangle$$
$$\lambda = \text{scalar value}$$
$$\Rightarrow \lambda\vec{v} = \lambda\langle a, b, c \rangle = \langle \lambda a, \lambda b, \lambda c \rangle$$

  - How to Interpret Scalar Multiplication
    - $\lambda > 0$: $\vec{v}$ and $\lambda\vec{v}$ are in the same direction
    - $\lambda < 0$: $\vec{v}$ and $\lambda\vec{v}$ are in opposite directions
    - $\vec{v}$ and $\lambda\vec{v}$ are parallel
  - Example: Find $\lambda\vec{v}$, where $\lambda = 3$ and $\vec{v} = \langle 1, 2, -3 \rangle$

$$\lambda\vec{v} = 3\langle 1, 2, -3 \rangle = \langle 3\cdot 1, 3\cdot 2, 3\cdot -3 \rangle = \langle 3, 6, -9 \rangle$$

- Dot Products
  - How do we multiply two vectors?
    - Answer: dot product and cross product
  - The dot product of two vectors is found by summing the products of each respective components of the vectors

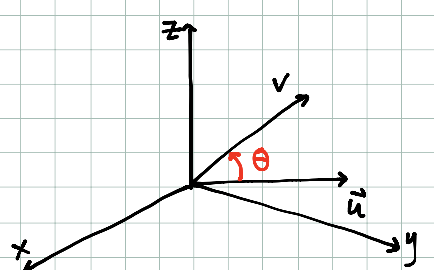
$$\vec{v} = \langle a, b, c \rangle$$
$$\vec{u} = \langle d, e, f \rangle$$
$$\Rightarrow \vec{v} \cdot \vec{u} = \langle a, b, c \rangle \cdot \langle d, e, f \rangle = (a \cdot d) + (b \cdot e) + (c \cdot f)$$

    - Note: the dot product takes two vectors and spits out a scalar
  - Example: Find $\vec{v} \cdot \vec{u}$, where $\vec{v} = \langle 1, -2, 4 \rangle$ and $\vec{u} = \langle 3, 1, -2 \rangle$

$$\vec{v} \cdot \vec{u} = \langle 1, -2, 4 \rangle \cdot \langle 3, 1, -2 \rangle = (1 \cdot 3) + (-2 \cdot 1) + (4 \cdot -2) = 3 + -2 + -8 = -7$$

  - Properties of the Dot Product
    - $\vec{v} \cdot \vec{v} = \langle a, b, c \rangle \cdot \langle a, b, c \rangle = a^2 + b^2 + c^2 = |\vec{v}|^2$
    - Obeys the usual rules of algebra:
      - $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
      - $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
      - $(\lambda \vec{u}) \cdot \vec{v} = \lambda(\vec{u} \cdot \vec{w})$
  - What does the Dot Product Tell us Geometrically?
    - The dot product can tell us the angle between the two vectors being dotted together

$$\vec{v} \cdot \vec{u} = |\vec{v}||\vec{u}|\cos\theta, \text{ where } \theta \text{ is the angle between } \vec{v} \text{ and } \vec{v}$$



*Note: $0 \leq \theta \leq \pi$

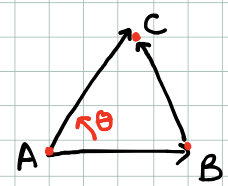
    - Since the dot product gives us information about the angle between the two vectors being dotted together, we can find out if the two vectors are perpendicular to each other very easily
      - If the two vectors are perpendicular to each other, $\theta = \frac{\pi}{2}$, so $\cos\theta = 0$

$$\vec{v} \cdot \vec{u} = |\vec{v}||\vec{u}| \cos(\tfrac{\pi}{2}) = 0$$

        - Thus, the dot product of two vectors is equal to zero if and only of the two vectors are perpendicular

- Example: For a triangle in space, with vertices A=(1,0,0), B=(1,1,-1), and C=(-1,1,0), what is the angle at A?

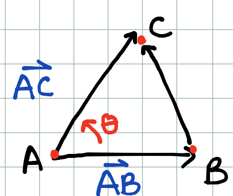
① Draw a picture



② Find the vectors from each initial and terminal point

$\vec{AB} = \langle 1-1, 1-0, -1-0 \rangle = \langle 0,1,-1 \rangle$

$\vec{AC} = \langle -1-1, 1-0, 0-0 \rangle = \langle -2,1,0 \rangle$



③ Recall relationship between the dot product & angle between the vectors

$$\vec{AB} \cdot \vec{AC} = |\vec{AB}||\vec{AC}| \cos\theta$$

$$\Rightarrow \cos\theta = \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}||\vec{AC}|}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}||\vec{AC}|}\right)$$

④ Find $|\vec{AB}|$ and $|\vec{AC}|$

$$|\vec{AB}| = \sqrt{(0)^2+(1)^2+(-1)^2} = \sqrt{2}$$

$$|\vec{AC}| = \sqrt{(-2)^2+(1)^2+(0)^2} = \sqrt{5}$$

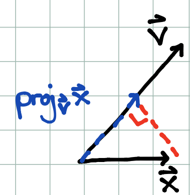
⑤ Compute $\vec{AB} \cdot \vec{AC}$

$$\vec{AB} \cdot \vec{AC} = \langle 0,1,-1 \rangle \cdot \langle -2,1,0 \rangle = (0 \cdot -2) + (1 \cdot 1) + (-1 \cdot 0) = 1$$

⑥ Complete by algebra

$$\theta = \cos^{-1}\left(\frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}||\vec{AC}|}\right) = \cos^{-1}\left(\frac{1}{(\sqrt{2})(\sqrt{5})}\right) = \cos^{-1}\left(\frac{1}{\sqrt{10}}\right)$$

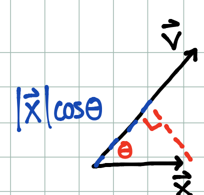
- Vector Projections
  - We can use our knowledge of dot products to project one vector on another vector
    - This comes in handy in Physics, when it is helpful to manipulate coordinate systems to make a problem simpler
  - The vector projection of $\vec{x}$ onto $\vec{v}$ is the vector given by the multiple of $\vec{v}$ obtained by dropping down a perpendicular line from $\vec{x}$



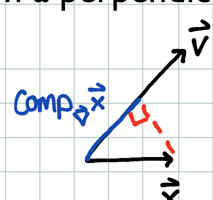
    - The vector projection of $\vec{x}$ onto $\vec{v}$ is the **closest point** to $\vec{x}$ on the line given by all multiples of $\vec{v}$
    - Algebraically: The Vector Projection of $\vec{x}$ onto $\vec{v}$ is:

$$proj_{\vec{v}}\vec{x} = (\vec{x}\cdot\vec{v})\frac{\vec{v}}{|\vec{v}|^2}$$

    - If θ is the angle between $\vec{x}$ and $\vec{v}$, the vector projection of $\vec{x}$ onto $\vec{v}$ is the vector of length/magnitude $|\vec{x}|\cos\theta$ that is in the **direction** of $\vec{v}$



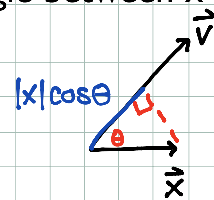
  - Scalar Vector Projections
    - A scalar vector projection is when you project a component of a vector onto another vector
      - Since the components of vectors are scalars, this is referred to as a scalar projection
    - The component of $\vec{x}$ along $\vec{v}$ is the distance along $\vec{v}$ obtained by dropping down a perpendicular line from $\vec{x}$



- Algebraically: The Scalar Projection of the component of $\vec{x}$ along $\vec{v}$ is:

$$\text{comp}_{\vec{v}}\vec{x} = \frac{\vec{x}\cdot\vec{v}}{|\vec{v}|}$$

- If θ is the angle between $\vec{x}$ and $\vec{v}$, the component of $\vec{x}$ along $\vec{v}$ is $|\vec{x}|\cos\theta$

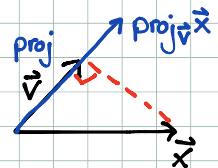


- Note: a vector component is negative if the two vectors are more than $\frac{\pi}{2}$ apart in angle
    - Example: Find the projection of $\vec{x}$ onto $\vec{v}$.

$\vec{x} = \langle x_1, x_2, x_3 \rangle$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

1) Draw a Picture



2) Find the direction of the desired vector

$\text{proj}_{\vec{v}}\vec{x}$ points in the direction of $\vec{v}$

$$\Rightarrow \text{dir}\left(\text{proj}_{\vec{v}}\vec{x}\right) = \text{dir}(\vec{v}) = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle v_1, v_2, v_3\rangle}{\sqrt{v_1^2+v_2^2+v_3^2}}$$

3) Find the length/magnitude of the desired vector

$|\vec{x}|\cos\theta$ is the length/magnitude of $\text{proj}_{\vec{v}}\vec{x}$

$$\vec{x}\cdot\vec{v} = |\vec{x}||\vec{v}|\cos\theta$$

$$\Rightarrow |\vec{x}|\cos\theta = \frac{\vec{x}\cdot\vec{v}}{|\vec{v}|}$$

4) Construct the desired Vector

vector = length/magnitude $\cdot$ direction

$$\Rightarrow \text{proj}_{\vec{v}}\vec{x} = |\vec{x}|\cos\theta \cdot \text{dir}\left(\text{proj}_{\vec{v}}\vec{x}\right) = \frac{\vec{x}\cdot\vec{v}}{|\vec{v}|}\cdot\frac{\langle v_1, v_2, v_3\rangle}{\sqrt{v_1^2+v_2^2+v_3^2}}$$

$$= \frac{x_1v_1 + x_2v_2 + x_3v_3}{\sqrt{v_1^2+v_2^2+v_3^2}}\langle v_1, v_2, v_3\rangle$$

- Matrices
  - What is a matrix?
    - The basic idea of a matrix is to organize an array of many numbers to make mass computations much easier
    - Some uses of matrices that you will see in 18.02:
        - Compact representation of linear equations
        - Performing operations on vectors
    - What does a matrix look like?
      - A matrix can have any dimensions you would like it to have, but for this class we will only work with square matrices (nxn collection of numbers)

<u>2×2 Matrix</u>

$$\begin{array}{cc} & n=2 \text{ columns} \\ \begin{array}{c} n=2 \\ \text{rows} \end{array} & \begin{pmatrix} a & c \\ b & d \end{pmatrix} \end{array}$$

<u>3×3 Matrix</u>

$$\begin{array}{cc} & n=3 \text{ columns} \\ \begin{array}{c} n=3 \\ \text{rows} \end{array} & \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \end{array}$$

    - Example: What are the dimensions of matrix A?

$$A = \begin{pmatrix} 1 & 12 & -6 \\ 3 & 4 & 2 \end{pmatrix}$$

Answer: $\underset{\text{rows}}{2} \times \underset{\text{columns}}{3}$

  - Matrix Determinants
    - The determinant of a matrix provides a great deal of information about the the matrix you are working with
    - How to Find the Determinant of a 2x2 Matrix

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\Rightarrow \det(A) = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

    - How to Find the Determinant of a 3x3 Matrix

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$$\Rightarrow \det(A) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- Example: Find det(A) and det(B)

$$A = \begin{pmatrix} 1 & 6 \\ -3 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 1 & -1 \\ -6 & 0 & 0 \\ -4 & 2 & 1 \end{pmatrix}$$

det(A)

$$\det(A) = \begin{vmatrix} 1 & 6 \\ -3 & 2 \end{vmatrix} = (1\cdot 2) - (-3\cdot 6) = 2 - -18 = 20$$

det(B)

$$\det(B) = \begin{vmatrix} 3 & 1 & -1 \\ -6 & 0 & 0 \\ -4 & 2 & 1 \end{vmatrix} = 3\begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} -6 & 0 \\ -4 & 1 \end{vmatrix} + -1\begin{vmatrix} -6 & 0 \\ -4 & 2 \end{vmatrix}$$

$$= 3\big((0\cdot 1) - (2\cdot 0)\big) - 1\big((-6\cdot 1) - (-4\cdot 0)\big) + -1\big((-6\cdot 2) - (-4\cdot 0)\big)$$

$$= 3(0) - 1(-6) + -1(-12) = 6 + 12 = 18$$

- Cross Products
  - How do we multiply two vectors?
    - Answer: dot product and cross product
  - The cross product of two vectors produces a **vector** that is perpendicular to both vectors being crossed
  - How to Find the Cross Product of Two Vectors:

To find $\vec{v} \times \vec{u}$:

① rewrite the two vectors as a matrix

$$\vec{v} = \langle a, b, c \rangle = a\hat{i} + b\hat{j} + c\hat{k}$$

$$\vec{u} = \langle d, e, f \rangle = d\hat{i} + e\hat{j} + f\hat{k}$$

$$\Rightarrow \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ d & e & f \end{pmatrix}$$

② Find the determinant of the matrix

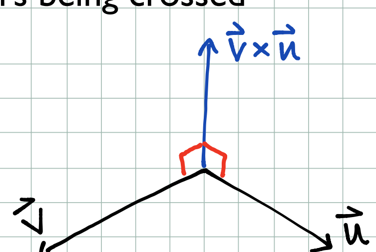
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ d & e & f \end{vmatrix} = \hat{i}\begin{vmatrix} b & c \\ e & f \end{vmatrix} - \hat{j}\begin{vmatrix} a & c \\ d & f \end{vmatrix} + \hat{k}\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$= \hat{i}(bf - ec) - \hat{j}(af - dc) + \hat{k}(ae - db)$$

③ Recognize that the determinant is the resulting vector of $\vec{v} \times \vec{u}$

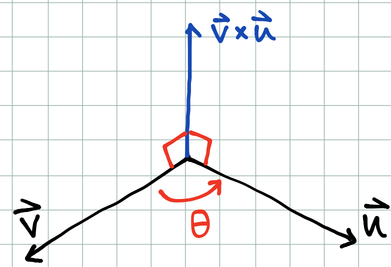
$$\hat{i}(bf - ec) - \hat{j}(af - dc) + \hat{k}(ae - db) = \langle bf - ec, af - dc, ae - db \rangle$$

  - What does the cross product tell us geometrically?
    - The cross product of two vectors gives us a vector perpendicular to the two vectors being crossed

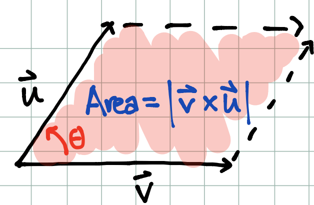


- The magnitude of the cross product is related to the angle between the two vectors being crossed

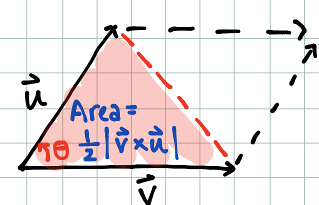
$$\vec{v} \times \vec{u} = |\vec{v}||\vec{u}|\sin\theta$$



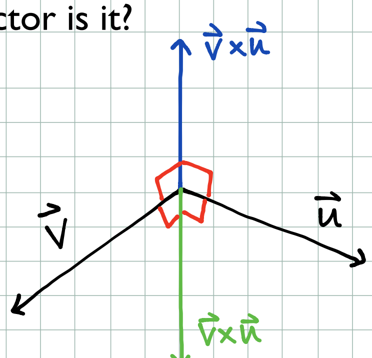
- The magnitude of the cross product tells us the area of the parallelogram made by the two vectors being crossed



- The magnitude of the cross product divided by 2 tells us the area of the triangle made by the two vectors being crossed



○ When we cross two vectors, there are two vectors that are perpendicular. Which vector is it?



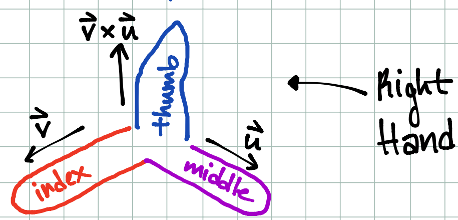
Which one???

- To figure out which direction is correct, we must use the **Right Hand Rule**

For $\vec{v} \times \vec{u}$

Using your Right Hand:

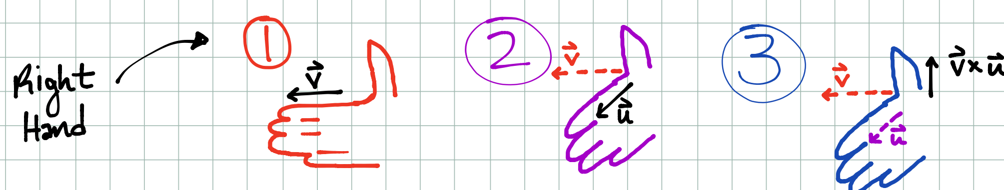
1. Point your index finger in the direction of $\vec{v}$
2. Point your middle finger in the direction of $\vec{u}$
3. Your thumb points in the direction of $\vec{v} \times \vec{u}$



Another way:

Using your Right Hand:

1. Point your four fingers in the direction of $\vec{v}$
2. Bend your fingers so they point in the direction of $\vec{u}$
3. Your thumb points in the direction of $\vec{v} \times \vec{u}$



- Cross products aren't commutative, so if you use the Right Hand Rule to find $\vec{v} \times \vec{u}$ and $\vec{u} \times \vec{v}$ the resulting vectors will point in opposite directions
- Note: There are a couple of different ways to go about the Right Hand Rule. Use whatever works for you.
- Example: Find the direction of the resulting vector when calculating $\vec{v} \times \vec{u}$

down

- Properties of the Cross Product
  - $\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$
    - Cross Products are not commutative!!!
  - $(\lambda \vec{v}) \times \vec{u} = \lambda(\vec{v} \times \vec{u}) = \vec{v} \times (\lambda \vec{u})$
  - $\vec{v} \times (\vec{u} + \vec{w}) = \vec{v} \times \vec{u} + \vec{v} \times \vec{w}$
  - $(\vec{v} + \vec{u}) \times \vec{w} = \vec{v} \times \vec{w} + \vec{u} \times \vec{w}$
  - $\vec{v} \cdot (\vec{u} \times \vec{w}) = (\vec{v} \times \vec{u}) \cdot \vec{w}$
    - Called the triple scalar product; its value gives the volume of the parallel piped made by the the three vectors
  - $\vec{v} \times (\vec{u} \times \vec{w}) = (\vec{v} \cdot \vec{w})\vec{u} - (\vec{v} \cdot \vec{u})\vec{w}$