27 April 2012
18.085
Computational Science and Engineering I
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Problem Set 6

Due: **3 May 2012** in class.

Print or write out any Matlab input and output.

1. (10 points) Using the Taylor series of $u(x)$ around $x = 0$:
   (a) Show that $\frac{u(\Delta x)-u(0)}{\Delta x}$ is a first-order approximation of $\frac{du}{dx}(0)$.
   (b) Find constants c_0, c_1, c_2 such that $\frac{c_2 u(2\Delta x)+c_1 u(\Delta x)+c_0 u(0)}{\Delta x}$ is a second-order approximation of $\frac{du}{dx}(0)$.

2. (10 points) By hand, find the solution to

$$-\frac{d^2u}{dx^2} = x$$
$$\frac{du}{dx}(0) = 0 \qquad (*)$$
$$u(1) = 0$$

3. (20 points) In this problem, you will solve (*) by a finite difference method. Use a grid of N equally spaced points from $x = 0$ to $x = 1$, inclusive. Thus, $\Delta x = \frac{1}{N-1}$.
   (a) Write out by hand the finite difference linear system to be solved in the $N = 5$ case. Approximate $\frac{d^2u}{dx^2}$ by the standard 3-point discretization. Approximate $\frac{du}{dx}(0)$ by $\frac{u(\Delta x)-u(0)}{\Delta x}$.
   (b) Use Matlab to construct and solve the corresponding linear system for $N = 100$ and $N = 200$.
   (c) What is the maximal error in each of these solutions? Is the method first or second order accurate in Δx.

4. (20 points)
   (a) Show that any solution to (*) satisfies the following weak form

$$\int_0^1 \frac{du}{dx}\frac{d\phi}{dx}dx = \int_0^1 x\phi(x)dx \text{ for all } \phi \text{ such that } \phi(1) = 0.$$

   Note that $\phi(0)$ is not assumed to be zero.
   (b) Consider a grid $x_i = i\Delta x$ with $\Delta x = \frac{1}{N}$. For $i = 0, \cdots, N-1$, let ϕ_i be the piecewise linear function such that $\phi_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. In the case $N = 5$, draw $\phi_0(x), \cdots, \phi_4(x)$ and write out the linear system arising from a finite element method with these basis functions.
   (c) Write a Matlab program to generate and solve the finite element linear system for the cases $N = 100$ and $N = 200$.

(d) What is the maximal error in each of these two solutions. Is the method first or second order accurate in Δx.

5. (20 Points) Consider the 2π periodic boundary value problem

$$
\begin{aligned}
-\frac{d^2u}{dx^2}(x) &= f(x) \\
u(-\pi) &= u(\pi) \\
\frac{du}{dx}(-\pi) &= \frac{du}{dx}(\pi)
\end{aligned}
$$

where $f(x) = \delta(x - \pi/2) - \delta(x + \pi/2)$.

(a) Sketch a solution to the boundary value problem.

(b) Expand f as a Fourier series: $f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$. Find $\hat{f}(n)$.

(c) Expand u as a Fourier series: $u(x) = \sum_{n=-\infty}^{\infty} \hat{u}(n)e^{inx}$. Compute $\frac{d^2u}{dx^2}$ and use the differential equation to solve for $\hat{u}(n)$. Argue that $\hat{u}(0)$ can be anything.

(d) Set $\hat{u}(0) = 0$. Use Matlab to plot $\sum_{n=-20}^{20} \hat{u}(n)e^{inx}$ for $-\pi \le x \le \pi$.