Homotopy of paths

Let \( f, f' \) be cont. maps \( Y \to X \). We say that \( f, f' \) are homotopic if there exists \( F : Y \times [0, 1] \to X \) cont. s.t. \( F(y, 0) = f(y) \), \( F(y, 1) = f'(y) \) for all \( y \in Y \). For \( x_0, x_1 \) in \( X \) let \( \mathcal{P}_{x_0, x_1} \) be the set of \( f : [0, 1] \to X \) cont. s.t. \( f(0) = x_0, f(1) = x_1 \). We say that \( f, f' \in \mathcal{P}_{x_0, x_1} \) are path homotopic if there exists \( F : [0, 1]^2 \to X \) cont. s.t. \( F(s, 0) = f(s), F(s, 1) = f'(s), F(0, t) = x_0, F(1, t) = x_1 \).
for \( t, s \in [0, 1] \).

This is an equivalence rel. on \( \mathcal{P}_{x_0, x_1} \)

\( f, f \) in \( \mathcal{P}_{x_0, x_1} \) are path-homot.: take \( F(s, t) = f(s) \). If \( f, f' \)

in \( \mathcal{P}_{x_0, x_1} \) are path-homot. through \( F \) then \( f', f \) are path-homot.

through \( F'(s, t) = F(s, 1 - t) \). If \( f, f' \) in \( \mathcal{P}_{x_0, x_1} \) are path-homot.

through \( F \) and \( f', f'' \) in \( \mathcal{P}_{x_0, x_1} \) are path-homot. through \( F' \),

then \( f, f'' \) are path-homot. through \( F'' \) where
\[ F''(s, t) = F(s, 2t) \text{ for } t \in [0, 1/2], \quad F''(s, t) = F(s, 2t - 1) \]

for \( t \in [1/2, 1] \). Let \( \mathcal{P}_{x_0, x_1} \) be the set of equivalence classes.

Now let \( x_0, x_1, x_2 \) in \( X \). We define “composition”

\[ \mathcal{P}_{x_0, x_1} \times \mathcal{P}_{x_1, x_2} \to \mathcal{P}_{x_0, x_2} \text{ by } f, g \mapsto f \ast g \text{ where } \]

\[ f \ast g(s) = f(2s) \text{ for } s \in [0, 1/2], \]

\[ f \ast g(s, t) = f'(2s - 1) \text{ for } s \in [1/2, 1]. \]
If $f, f'$ in $\mathcal{P}_{x_0, x_1}$ are path homot. through $F$ then $f \ast g, f' \ast g$ are path-homot. through $\tilde{F} : [0, 1] \times [0, 1] \to X, \tilde{F}(s, t) = F(2s, t)$ for $s \in [0, 1/2], \tilde{F}(s, t) = F(2s - 1, t)$ for $s \in [1/2, 1]$.

Similarly if $g, g'$ in $\mathcal{P}_{x_1, x_2}$ are path homot. then

$f \ast g, f \ast g'$ are path-homot. Hence composition induces a map

$\bar{\mathcal{P}}_{x_0, x_1} \times \bar{\mathcal{P}}_{x_1, x_2} \to \bar{\mathcal{P}}_{x_0, x_2}$

also called composition.
Composition of paths is not associative: if $f \in \mathcal{P}_{x_0,x_1}$, $g \in \mathcal{P}_{x_1,x_2}$, $h \in \mathcal{P}_{x_2,x_3}$ then $((f * g) * h)(s)$ is $f(4s)$ for $s \in [0, 1/4]$, is $g(4s - 1)$ for $s \in [1/4, 1/2]$, is $h(2s - 1)$ for $s \in [1/2, 1]$ while $(f * (g * h))(s)$ is $f(2s)$ for $s \in [0, 1/2]$, is $g(4s - 2)$ for $s \in [1/2, 3/4]$, is $h(4s - 3)$ for $s \in [3/4, 1]$. Although $(f * g) * h, f * (g * h)$ are not necessarily equal, they are path-homotopic.
Define $F : [0, 1]^2 \to X$ where

$F(s, t)$ is $f(4s/(t + 1))$ if $0 \leq s \leq (t + 1)/4$

is $g(4s - t - 1)$ if $(t + 1)/4 \leq s \leq (t + 2)/4$,

is $h((4s - t - 2)/(2 - t))$ if $(t + 2)/4 \leq s \leq 1$.

This is continuous: if $s = (t + 1)/4$ then

$f(4s/(t + 1)) = f(1) = g(0) = g(4s - t - 1)$; if $s = (t + 2)/4$ then

$g(4s - t - 1) = g(1) = h(0) = h((4s - t - 2)/(2 - t))$. 
We have $F(s, 0) = ((f \ast g) \ast h)(s)$, $F(s, 1) = (f \ast (g \ast h))(s)$.

We see that composition

$$\bar{P}_{x_0, x_1} \times \bar{P}_{x_1, x_2} \rightarrow \bar{P}_{x_0, x_2}$$

is associative. For $x \in X$ let $\epsilon_x \in \mathcal{P}_{x, x}$ be the path $t \mapsto x$.

If $f \in \mathcal{P}_{x_0, x_1}$ then $(f \ast \epsilon_{x_1})(s)$ is $f(2s)$ if $s \in [0, 1/2]$

and is $x_1$ if $s \in [1/2, 1]$. Although $f \ast \epsilon_{x_1}, f$ are not necessarily equal they are path-homot.
Define $F : [0, 1]^2 \to X$ where $F(s, t) = f(2s/(2 - t))$ if

$0 \leq s \leq (2 - t)/2$ and is $x_1$ if $(2 - t)/2 \leq s \leq 1$. This

is continuous: if $s = (2 - t)/2$ then $f(2s/(2 - t)) = f(1) = x_1$.

We have $F(s, 1) = (f * \epsilon_{x_1})(s)$, $F(s, 0) = f(s)$.

Now $(\epsilon_{x_0} * f)(s)$ is $x_0$ if $s \in [0, 1/2]$ and is $f(2s - 1)$

if $s \in [1/2, 1]$. Now, although $\epsilon_{x_0} * f$, $f$ are not necessarily

equal they are path-homot.
Define $F : [0, 1]^2 \to X$ where $F(s, t)$ is $x_0$ if $0 \leq s \leq t/2$

and is $f((2s - t)/(2 - t))$ if $t/2 \leq s \leq 1$. This is continuous.

We have $F(s, 1) = (\epsilon_{x_0} * f)(s)$, $F(s, 0) = f(s)$. 
For $f \in \mathcal{P}_{x_0,x_1}$ we define $\bar{f} \in \mathcal{P}_{x_1,x_0}$ by $\bar{f}(s) = f(1 - s)$. Then $f \ast \bar{f} \in \mathcal{P}_{x_0,x_0}$ is

$s \mapsto f(2s)$ if $s \in [0, 1/2]$ and is $s \mapsto f(2 - 2s)$ if $s \in [1/2, 1]$;

this is not necessarily equal to $\epsilon_{x_0}$ but is path homot. to it.

Define $F : [0, 1]^2 \to X$ where $F(s, t)$ is $f(2ts)$ if $0 \leq s \leq 1/2$

and is $f(2t - 2ts)$ if $1/2 \leq s \leq 1$. 
This is continuous: if $s = 1/2$ we have

$$f(2ts) = f(2t - 2ts) = f(t).$$

We have $F(s, 1) = (f * \bar{f})(s)$, $F(s, 0) = f(0) = \epsilon_{x_0}(s)$.

A group is a set $G$ together with a map $G \times G \to G$,

$$(g, g') \mapsto gg'$$

such that (1) $(gg')g'' = g(g'g'')$ for any $g, g', g''$ in $G$ (associativity); (2) there exists $1 \in G$ such that

$$1g = g, g1 = g$$

for any $g$ (it is necessarily unique);
(3) for any \( g \in G \) there exists \( g' \in G \) such that \( gg' = g'g = 1 \) (it is necessarily unique). We denote \( g' = g^{-1} \). For example

the integers with \(+\) is a group; the nonzero real numbers with \( \times \) is a group. The set with one element is a group.

For any \( x_0 \in X \) the set \( \bar{P}_{x_0,x_1} \) is a

group with operation \( \ast \) denoted \( \pi_1(X,x_0) \).
\[ \pi_1(X, x_0) \] is called the fundamental group of \( X \) at \( x_0 \).

If \( X \) is path connected and \( \pi_1(X, x_0) = \{1\} \) for \( x_0 \in X \),

we say that \( X \) is simply connected.
If $G, G'$ are groups, a map $f : G \to G'$ is said to be a homomorphism if (1) $f(gg') = f(g)f(g')$ for any $g, g'$ in $G$;

(2) $f(1) = 1$; (3) $f(g^{-1}) = f(g)^{-1}$ for any $g \in G$.

Actually (2),(3) are automatically true if (1) holds.

Indeed if (1) holds, then $f(1) = f(11) = f(1)f(1)$. Hence

$1 = f(1)^{-1}f(1) = f(1)$ and (2) holds. Also,

$1 = f(g^{-1}g) = f(g^{-1})f(g)$ hence $f(g^{-1}) = f(g)^{-1}$ and (3) holds.
If $G, G'$ are groups, a map $f : G \rightarrow G'$ is said to be an isomorphism if it is a homomorphism and a bijection. Then its inverse is also an isomorphism. We show:

**If** $x_0 \in X, \ x_1 \in X$ **are in the same path-component**

then $\pi_1(X, x_0), \pi_1(X, x_1)$ **are isomorphic groups.**

Choose a path $p \in \mathcal{P}(x_1, x_0)$. Define

$h : \bar{\mathcal{P}}(x_0, x_0) \rightarrow \bar{\mathcal{P}}(x_1, x_1)$
by $f \mapsto h(f) = p \ast f \ast \bar{p}$. Recall that $\bar{p} \in \mathcal{P}_{x_0,x_1}$.

This is a homomorphism:

$$h(f)h(f') = p \ast f \ast \bar{p} \ast p \ast f' \ast \bar{p} = p \ast f \ast e_{x_0} \ast f' \ast \bar{p} =$$

$$= p \ast f \ast f' \ast \bar{p} = h(f \ast f').$$

Similarly $h' : \bar{\mathcal{P}}(x_1, x_1) \rightarrow \bar{\mathcal{P}}(x_0, x_0)$,

$$g \mapsto h'(g) = \bar{p} \ast g \ast p$$

is a homomorphism. We have

$$h'(h(f)) = \bar{p} \ast h(f) \ast p = \bar{p} \ast p \ast f \ast \bar{p} \ast p = f.$$
Thus $h'h = Identity$. Similarly $hh' = Identity$. Hence $h$ is an isomorphism.

**The fundamental group of the circle $S^1$.**

Define $p : \mathbb{R} \to S^1 = \{z \in \mathbb{C}; |z| = 1\}$ by

$$x \mapsto p(x) = (\cos(2\pi x), \sin(2\pi x)).$$

We have $p(x + n) = p(x)$ for any $x \in \mathbb{R}$, $n \in \mathbb{Z}$. 
We have $p^{-1}(S^1 - \{1\}) = \bigcup_{n \in \mathbb{Z}} (n, n + 1)$

and $p$ restricts to a homeomorphism $p_n : (n, n + 1) \rightarrow S^1 - \{1\}$

for any $n \in \mathbb{Z}$. We have $p^{-1}(S^1 - \{-1\}) = \bigcup_{n \in \mathbb{Z}} (n - 1/2, n + 1/2)$

and $p$ restricts to a homeomorphism $p_{n-1/2} : (n - 1/2, n + 1/2) \rightarrow S^1 - \{-1\}$

for any $n \in \mathbb{Z}$. 
Let $F : [0, 1]^2 \to S^1$ be cont.s.t. $F(0, 0) = 1$. Let $m \in \mathbb{Z}$.

There exists $\tilde{F} : [0, 1]^2 \to \mathbb{R}$ cont.s.t. $\tilde{F}(0, 0) = m$, $\rho \tilde{F} = F$.

By the existence of Lebesgue number for the covering of $[0, 1]^2$ formed by the two open sets $F^{-1}(S^1 - \{\epsilon\})$, ($\epsilon = 1, -1$), can find $k \geq 1$ s.t. for any $(t, t') \in [0, k - 1]^2$, $I_{t,t'} = [t/k, (t + 1)/k] \times [t'/k, (t' + 1)/k]$ is contained in one of these two open sets hence $F(I_{t,t'})$ is contained in $S^1 - \{\epsilon\}$, $\epsilon = 1$ or $-1$. 
We arrange the squares $I_{t,t'}$ in a sequence $I^1, I^2, ..., I^{k^2}$

given by $I_{0,0}, I_{0,1}, ..., I_{0,k-1}, I_{1,0}, I_{1,1}, ..., I_{1,k-1}, ...$

$I_{k-1,0}, I_{k-1,1}, ..., I_{k-1,k-1}$. Note that for any $u \in [2, k^2]$

$I^u \cap (I^1 \cup I^2 \cup ... \cup I^{u-1})$ is a connected subset $J^u$ of $I^u$

(homeomorphic to $[0, 1]$). We define by induction on $u$ cont.maps

$\tilde{F}_u : I^1 \cup I^2 \cup ... \cup I^u \to \mathbb{R}, u \in [1, k^2]$ such that

$\tilde{F}_u(0, 0) = m$, 

$\tilde{F}_u(0, 1) = m_0$, 

$\tilde{F}_u(1, 0) = m_1$, 

$\tilde{F}_u(1, 1) = m_2$, 

$\tilde{F}_u(0, 2) = m_3$, 

$\tilde{F}_u(2, 0) = m_4$, 

$\tilde{F}_u(2, 1) = m_5$, 

$\tilde{F}_u(1, 2) = m_6$, 

$\tilde{F}_u(2, 2) = m_7$, 

$\tilde{F}_u(0, 3) = m_8$, 

$\tilde{F}_u(3, 0) = m_9$, 

$\tilde{F}_u(3, 1) = m_{10}$, 

$\tilde{F}_u(1, 3) = m_{11}$, 

$\tilde{F}_u(3, 2) = m_{12}$, 

$\tilde{F}_u(2, 3) = m_{13}$, 

$\tilde{F}_u(3, 3) = m_{14}$.
\[ \tilde{F}_u = \tilde{F}_{u-1} \text{ on } I^1 \cup I^2 \cup \ldots \cup I^{u-1}, \ u \in [2, k^2] \]

\[ p(\tilde{F}_u(a, a')) = F(a, a') \text{ for } (a, a') \in I^1 \cup I^2 \cup \ldots \cup I^u. \]

Now \( F(I^1) \) is contained in \( S^1 - \{1\} \) or in \( S^1 - \{-1\} \).

and \( F(0, 0) \notin S^1 - \{1\} \) so that \( F(I^1) \) is contained in \( S^1 - \{-1\} \).

Define \( \tilde{F}_1 : I^1 \rightarrow \mathbb{R} \) by \( (a, a') \mapsto p_{m-1/2}^{-1}(F(a, a')) \). We have

\[ \tilde{F}_1(0, 0) = p_{m-1/2}^{-1}(1) = m, \ p(\tilde{F}_1(a, a')) = F(a, a') \text{ for } (a, a') \in I^1. \]
Assume that $\tilde{F}_{r-1}$ are already defined for some $r \in [2, k^2]$.

Now $F(I^r)$ is contained in $S^1 - \{\epsilon\}$ for some $\epsilon$ hence

$$F_{J^r} \in S^1 - \{\epsilon\} \text{ and } \tilde{F}_{r-1}(J^r) \in p^{-1}(S^1 - \{\epsilon\})$$

that is

$$\tilde{F}_{r-1}(J^r) \in (e, e + 1) \text{ if } \epsilon = 1, \quad \tilde{F}_{r-1}(J^r) \in (e - 1/2, e + 1/2)$$

if $\epsilon = -1$. (We use that $\tilde{F}_{r-1}(J^r)$ is connected hence is contained in

a connected component of $p^{-1}(S^1 - \{\epsilon\})$.)
We define $\tilde{F}'_r : I^r \to \mathbb{R}$ by $(a, a') \mapsto p^{-1}(F(a, a'))$

if $\epsilon = 1$ and by $(a, a') \mapsto p_{e-1/2}^{-1}(F(a, a'))$ if $\epsilon = -1$. 

We have $p(\tilde{F}'_r(a, a')) = F(a, a')$ for $(a, a') \in I^r,$

$\tilde{F}'_r(a, a') = \tilde{F}'_{r-1}(a, a')$ for $(a, a') \in J^r$

since both sides are in $(e, e + 1)$ (if $\epsilon = 1$) or in

$(e - 1/2, e + 1/2)$ (if $\epsilon = -1$) and both sides have the

same image under $p$. 
There is a unique cont. function $\tilde{F}_r : I^1 \cup ... \cup I^r \to \mathbb{R}$

whose restriction to $I^1 \cup ... \cup I^{r-1}$ is $\tilde{F}_{r-1}$ and

whose restriction to $I^r$ is $\tilde{F}'_r$.

Taking $r = k^2$ we obtain a cont. function

$\tilde{F} = \tilde{F}_{k^2} : [0, 1]^2 = I^1 \cup ... \cup I^{k^2} \to \mathbb{R}$ which

satisfies the requirements.
Let $f : [0, 1] \to S^1$ be cont.s.t. $f(0) = 1$. Let $m \in \mathbb{Z}$.

There exists $\tilde{f} : [0, 1] \to \mathbb{R}$ cont.s.t. $\tilde{f}(0) = m$, $p\tilde{f} = f$.

Define $F : [0, 1]^2 \to S^1$ by $F(a, a') = f(a)$. Then $F(0, 0) = 1$. By the previous result there exists $\tilde{F} : [0, 1]^2 \to \mathbb{R}$ cont.s.t. $\tilde{F}(0) = m$, $p\tilde{F} = F$. Define $\tilde{f} : [0, 1] \to \mathbb{R}$ by $\tilde{f}(a) = \tilde{F}(a, 0)$. We have $\tilde{f}(0) = \tilde{F}(0, 0) = m$, $p(\tilde{f}(a)) = p(\tilde{F}(a, 0)) = F(a, 0) = f(a)$.

Hence $\tilde{f}$ satisfies the requirements.
If $f, m$ are given, then $\tilde{f}$ is uniquely determined.

Assume that $\tilde{f}' : [0, 1] \to \mathbb{R}$ is another cont. function s.t. $\tilde{f}'(0) = m, p\tilde{f}' = f$. For any $a \in [0, 1]$ we have

$p(\tilde{f}'(a)) = p(\tilde{f}(a))$ hence $\tilde{f}'(a) - \tilde{f}(a)$ is an integer. Then $a \mapsto \tilde{f}'(a) - \tilde{f}(a)$ is a cont.fn. $[0, 1] \to \mathbb{Z}$ hence its image is a single integer $n$. Thus $\tilde{f}'(a) = \tilde{f}(a) + n$.

Taking $a = 0$ we get $m = m + n$ hence $n = 0$ and $\tilde{f}' = \tilde{f}$. 
Let $f : [0, 1] \to S^1$ be a path such that $f(0) = f(1) = 1$.

We attach to $f$ an integer $n = \text{deg}(f)$. Let $\tilde{f} : [0, 1] \to \mathbb{R}$ be cont.s.t. $\tilde{f}(0) = 0$, $p\tilde{f} = f$. (It is unique.)

Now $p(\tilde{f}(1)) = f(1) = 1$ hence $\tilde{f}(1) = n \in \mathbb{Z}$.

We set $\text{deg}(f) = n$. 
Assume \( f, f' : [0, 1] \to S^1 \) satisfy \( f(0) = f(1) = f'(0) = f'(1) = 1 \).

Assume \( f, f' \) are path-homotopic: there exists \( F : [0, 1]^2 \to S^1 \) cont. s.t. \( F(s, 0) = f(s), F(s, 1) = f'(s), F(0, t) = 1, F(1, t) = 1 \).

We show: \( \text{deg}(f) = \text{deg}(f') \).

Let \( \tilde{f} : [0, 1] \to \mathbb{R} \) be cont.s.t. \( \tilde{f}(0) = 0, p\tilde{f} = f \).

Let \( \tilde{f}' : [0, 1] \to \mathbb{R} \) be cont.s.t. \( \tilde{f}'(0) = 0, p\tilde{f}' = f' \).

Must show \( \tilde{f}(1) = \tilde{f}'(1) \).
Can find $\tilde{F} : [0, 1]^2 \to \mathbb{R}$ cont.s.t. $p\tilde{F} = F$, $\tilde{F}(0, 0) = 0$. Now

$$\tilde{F}(1, t) \in p^{-1}(F(1, t)) = p^{-1}(1) = \mathbb{Z}, \quad \tilde{F}(0, t) \in p^{-1}(F(0, t))$$

$$= p^{-1}(1) = \mathbb{Z}.$$  Hence $t \mapsto \tilde{F}(1, t)$ (resp. $t \mapsto \tilde{F}(0, t)$) is a cont. function $[0, 1] \to \mathbb{Z}$; hence its image is a single integer.

In particular $\tilde{F}(1, 0) = \tilde{F}(1, 1)$ (\ast), $\tilde{F}(0, 0) = \tilde{F}(0, 1) = 0$ (\ast\ast)
Now $s \mapsto \tilde{F}(s, 0)$ (resp. $s \mapsto \tilde{F}(s, 1)$) satisfies the definition of $\tilde{f}$ (resp. $\tilde{f}'$) (use (**) ) ; hence

$$\tilde{F}(s, 0) = \tilde{f}(s), \quad \tilde{F}(s, 1) = \tilde{f}'(s).$$

The equation $\deg(f) = \deg(f')$ is the same as $\tilde{f}(1) = \tilde{f}'(1)$.

and the same as $\tilde{F}(1, 0) = \tilde{F}(1, 1)$ (use (*)).
We see that $f \mapsto \text{deg}(f)$ defines a map

$$\pi_1(S^1, 1) = \tilde{\mathcal{P}}_{1, 1} \to \mathbb{Z}.$$ 

We show that this map is a group homomorphism.

Let $f : [0, 1] \to S^1$, $f' : [0, 1] \to S^1$, $f'' : [0, 1] \to S^1$

be cont. with $f(0) = f(1) = f'(0) = f'(1) = f''(0) = f''(1) = 1$

and $f''(t) = f(2t)$ if $t \in [0, 1/2]$, $f''(t) = f'(2t - 1)$

if $t \in [1/2, 1]$. We show: $\text{deg}(f'') = \text{deg}(f) + \text{deg}(f').$
Let \( \tilde{f} : [0, 1] \to \mathbb{R}, \tilde{f}' : [0, 1] \to \mathbb{R}, \tilde{f}'' : [0, 1] \to \mathbb{R}, \)

be cont. with \( p \tilde{f} = f, p \tilde{f}' = f', p \tilde{f}'' = f'', \)

\( \tilde{f}(0) = 0, \tilde{f}'(0) = 0, \tilde{f}''(0) = 0. \) Recall

\( \text{deg } f = \tilde{f}(1), \text{deg } (f') = \tilde{f}'(1). \) Define \( g : [0, 1] \to \mathbb{R} \) by

\( g(t) = \tilde{f}(2t) \) for \( t \in [0, 1/2], \)

\( g(t) = \tilde{f}'(2t - 1) + \text{deg } (f) \) for \( t \in [1/2, 1]. \) This is cont. since \( \tilde{f}(1) = \tilde{f}'(0) + \text{deg } (f). \)

We have \( g(0) = 0, pg = f'' \)
hence $g = \tilde{f}''$. Now $\text{deg}(f'') = g(1) = \text{deg}(f') + \text{deg}(f)$.

We show $\text{deg} : \pi_1(S^1, 1) \to \mathbb{Z}$ is surjective.

Let $n \in \mathbb{Z}$. Define $\tilde{f} : [0, 1] \to \mathbb{R}$ by $a \mapsto na$.

This is cont., $\tilde{f}(0) = 0$ and if $f = p\tilde{f}$ then

$f : [0, 1] \to S^1, f(0) = p(0) = 1, f(1) = p(n) = 1,$

$\text{deg}(f) = n.$
We show \( \text{deg} : \pi_1(S^1, 1) \to \mathbb{Z} \) is injective.

Let \( f, f' : [0, 1] \to S^1 \) be cont.s.t. \( f(0) = f(1) = f'(0) = f'(1) = 1 \).

Assume \( \text{deg}(f) = \text{deg}(f') \). We show: \( f, f' \) are path-homotopic.

Let \( \tilde{f} : [0, 1] \to \mathbb{R}, \tilde{f}' : [0, 1] \to \mathbb{R} \) be cont. with

\[
p\tilde{f} = f, \quad p\tilde{f}' = f', \quad \tilde{f}(0) = 0, \tilde{f}'(0) = 0. \quad \text{We have}
\]

\[
\tilde{f}(1) = \tilde{f}'(1). \quad \text{Define } \tilde{F} : [0, 1]^2 \to \mathbb{R} \text{ by}
\]

\[
\tilde{F}(s, t) = (1 - t)\tilde{f}(s) + t\tilde{f}'(s).
\]
Define $F : [0, 1]^2 \to S^1$ by $F(s, t) = p\tilde{F}(s, t)$.

Then $F(s, 0) = p\tilde{f}(s) = f(s)$, $F(s, 1) = p\tilde{f}'(s) = f'(s)$,

$F(0, t) = p((1 - t)\tilde{f}(0) + t\tilde{f}'(0)) = p(0) = 1$

$F(1, t) = p((1 - t)\tilde{f}(1) + t\tilde{f}'(1)) = p(\deg f) = 1$

hence $f, f'$ are path-homotopic. We see:

$\deg : \pi_1(S^1, 1) \to \mathbb{Z}$ is a group isomorphism.
Let $g : Y \to X$ be a cont.map of top.spaces. Let $y_0 \in Y,$

$x_0 = g(y_0) \in X$. Define $g_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$

by $f \mapsto gf$. (Here $f \in \mathcal{P}_{y_0}(Y)$ so that $gf \in \mathcal{P}_{x_0}(X).$)

If $f, f'$ are equivalent then $gf, gf'$ are equivalent so $g_*$ is well defined. Note: $g_*$ is a group homomorphism. Let $f, f'$ be in \mathcal{P}_{y_0}(Y).$ Enough to show: $g(f * f') = (gf) * (gf').$ Both sides take $t$ to $g(f(2t))$ if $t \in [0, 1/2]$, to $g(f'(2t - 1))$ if $t \in [1/2, 1]$. 
Assume that $g, g'$ are cont. maps $Y \to X$ s.t. $g(y_0) = g'(y_0) = x_0$.

Assume that $g, g'$ are homotopic through $F : Y \times [0, 1] \to X$ cont.

$F(y, 1) = g'(y)$ for all $y \in Y$. Assume also that $F(y_0, t) = x_0$ for all $t$. We show: $g_* = g'_*$ as homom. $\pi_1(Y, y_0) \to \pi_1(X, x_0)$.

Let $f \in \mathcal{P}_{y_0}(Y)$. We must show $gf, g'f \in \mathcal{P}_{x_0}(X)$ are equivalent.

Define $\phi : [0, 1] \times [0, 1] \to X$ by $\phi(s, t) = F(f(s), t))$. Then

$\phi(s, 0) = F(f(s), 0) = gf(s), \ \phi(s, 1) = F(f(s), 1) = g'f(s)$,
\[ \phi(0, t) = \phi(1, t) = F(y_0, t) = x_0, \]

as required.
If $X = Y = S^1$ and $g : S^1 \to S^1$ is cont. with $g(1) = 1$ then

$g_* : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ is defined. We can identify $\pi_1(S^1, 1) = \mathbb{Z}$ using $\text{deg}$. Then $g_*$ becomes a homomorphism $g_* : \mathbb{Z} \to \mathbb{Z}$. This takes 1 to an integer $n$ and then it takes any integer $m$ to $mn$. We say that $n = \text{deg}(g)$.

Now let $g : S^1 \to S^1$ be cont. but we don’t assume $g(1) = 1$.

Set $z_0 = g(1)$. We define $g' : S^1 \to S^1$ by $g'(z) = g(z)z_0^{-1}$. 
We use that $S^1$ is a group under multiplication of complex numbers $(x + iy)(x' + iy') = xx' - yy' + i(xy' - x'y)$.

Now $g'$ is cont. and $g'(1) = 1$ hence $deg(g') \in \mathbb{Z}$ is defined.

We set $deg(g) = deg(g')$. We show:

Assume that $g_0, g_1$ are cont. maps $S^1 \to S^1$ and are homotopic. Then $deg(g_0) = deg(g_1)$. 
Let $F : S^1 \times [0, 1] \to S^1$ be a homotopy between $g_0, g_1$. Define $g'_0, g'_1$ by $g'_0(z) = g_0(z)g_0(1)^{-1}$, $g'_1(z) = g_1(z)g_1(1)^{-1}$. Define a homotopy $F' : S^1 \times [0, 1] \to S^1$ by $F'(z, t) = F(z, t)F(z, 1)^{-1}$. It respects the base point 1. Hence $g'_0* = g'_1*$ as maps $\pi_1(S^1, 1) \to \pi_1(S^1, 1)$ and $\deg(g'_0) = \deg(g'_1)$.

Hence $\deg(g_0) = \deg(g_1)$. We show:
If $g : S^1 \to S^1$ has image a single point then $\text{deg}(g) = 0$.

Set $g'(z) = g(z)g(1)^{-1}$. Then $g' : S^1 \to S^1$ has image 1.
For any $f \in \mathcal{P}_1(S^1)$ we have $g'f = e_1$ (constant path at 1).

Hence $g'(f)$ is the unit element of $\mathcal{P}_1(S^1)$ hence $g' : \mathbb{Z} \to \mathbb{Z}$ has image 0. Thus $\deg(g') = 0$ and $\deg(g) = 0$. We have $g'f = e_1$ (constant path at 1). We deduce:

If $g : S^1 \to S^1$ cont. is homotopic to a constant map then $\deg(g) = 0$.

We show:
Let $g : S^1 \to S^1$ be cont. Assume $g$ extends to a cont. map $	ilde{g} : \{z \in \mathbb{C}; |z| \leq 1\} \to S^1$. Then $\text{deg}(g) = 0$.

Define $F : S^1 \times [0, 1] \to S^1$ by $F(z, t) = \tilde{g}((1 - t)z)$. Note $|(1 - t)z| = 1 - t \leq 1$. We have $F(z, 0) = \tilde{g}(z) = g(z)$, $F(z, 1) = \tilde{g}(0)$, a constant map. We show:
Define $g : S^1 \to S^1$ by $g(z) = z^n$, $n \in \mathbb{Z}$. Then $\text{deg}(g) = n$.

Recall that if $f_m \in \mathcal{P}_1(S^1)$ is $t \mapsto \exp(2\pi itn)$ then $\text{deg}(f_m) = m$.

(See proof of surjectivity of $\text{deg}$.) We have $gf_m : t \mapsto \exp(2\pi itnm)$

hence $gf_m = f_{mn}$. Thus $g_\ast : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ can be

identified with $\mathbb{Z} \to \mathbb{Z}$, $m \mapsto mn$. Hence $\text{deg}(g) = n$. 
Let $a_0, a_1, ..., a_{n-1}$ be complex numbers ($n \geq 1$) such that

$$|a_0| + |a_1| + ... + |a_{n-1}| < 1.$$ There exists $z \in \mathbb{C}$ with $|z| \leq 1$,

$$z^n + a_{n-1}z^{n-1} + ... + a_0 = 0.$$

Assume it is not true. Then $z \mapsto g(z) = z^n + a_{n-1}z^{n-1} + ... + a_0$

is a cont. function $\{z \in \mathbb{C}; |z| \leq 1\} \to \mathbb{C} - \{0\}$. This restricts

to a cont. function $f : S^1 \to \mathbb{C} - \{0\}$. Now $f$ is homotopic to a

constant map through $F(z, t) = g((1 - t)z)$, $z \in S^1$;
\[ F(z, 0) = f(z), \; F(z, 1) = g(0). \]

Now \( f \) is also homotopic to \( h : S^1 \to \mathbb{C} - \{0\}, \; h(z) = z^n \). Indeed define \( F(z, t) = z^n + t(a_{n-1}z^{n-1} + \ldots + a_0), \; z \in S^1, \; t \in [0, 1]. \)

We must show \( F(z, t) \neq 0 \). Now

\[ |F(z, t)| > |z^n| - t|a_{n-1}z^{n-1} + \ldots + a_0| \geq 1 - t(|a_{n-1}| + \ldots + |a_0|). \]

If \( t < 1 \) this is \( \geq 1 - t > 0; \) if \( t = 1 \) this is

\[ \geq 1 - (|a_{n-1}| + \ldots + |a_0|) > 0. \]
In any case $F(z, t) \neq 0$. We have $F(z, 0) = z^n = h(z)$,

$$F(z, 1) = f(z).$$

Thus

$$h : S^1 \to \mathbb{C} - \{0\}$$

is homotopic to a constant map $S^1 \to \mathbb{C} - \{0\}$. 
Composing these maps with $\mathbb{C} - \{0\} \to S^1$, $z \mapsto z/|z|$ we see that the map $S^1 \to S^1$, $z \mapsto z^n$ is homotopic to a constant map $S^1 \to S^1$. Hence the last two maps have the same degree.

Hence $n = 0$, contradiction. QED.

**Fundamental theorem of algebra.** Let $a_0, a_1, \ldots, a_{n-1}$ be complex numbers $(n \geq 1)$ There exists $z \in \mathbb{C}$ with

$$z^n + a_{n-1}z^{n-1} + \ldots + a_0 = 0.$$
Write $z = cy$ where $c \in \mathbb{R}, c > 0$. In terms of $y$ the equation is

$$(cy)^n + a_{n-1}(cy)^{n-1} + \ldots + a_0 = 0$$

that is

$$y^n + \frac{a_{n-1}}{c}y^{n-1} + \frac{a_{n-2}}{c^2} + \ldots + \frac{a_0}{c^n} = 0.\]$$

We can choose $c > 0$ so that $|\frac{a_{n-1}}{c}| + |\frac{a_{n-2}}{c^2}| + \ldots + |\frac{a_0}{c^n}| < 1.$

Then the previous result is applicable. QED.
Let $X$ be a top.space. Assume $U, V$ are open in $X$ with $X = U \cup V$, $U \cap V$ is path-connected. Let $x_0 \in U \cap V$.

Let $i : U \to X, j : V \to X$ be the inclusions. Then

any element in $\pi_1(X, x_0)$ is of the form

$f_1 f_2 \ldots f_n$ where each $f_k$ belongs to the image of

$i_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$ or to the image of

$j_* : \pi_1(VU, x_0) \to \pi_1(X, x_0)$. 
In particular if $U, V$ are simply connected then $X$ is simply connected.

Let $f : [0, 1] \to X$ be cont.s.t. $f(0) = f(1) = x_0$. Using the Lebesgue number for the open covering of $[0, 1]$ by $f^{-1}(U), f^{-1}(V)$, we can find $0 = b_0 < b_1 < \ldots < b_k = 1$ such that for $r = 0, 1, \ldots, k - 1$, $[b_{r-1}, b_r]$ is contained in $f^{-1}(U)$ or in $f^{-1}(V)$. 
We can assume that $k$ is minimum possible. We show:

$$f(b_i) \in U \cap V \text{ for } i = 0, 1, \ldots, k.$$ Assume $f(b_i) \notin U \cap V$ for some $i$. We must have $i \neq 0, i \neq k$ since $x_0 \in U \cap V$.

Now each of $f([b_{i-1}, b_i]), f([b_i, b_{i+1}])$ is contained in $U$ or in $V$. 
If \( f(b_i) \in U \cap C(V) \) then \( f([b_{i-1}, b_i]) \not\subset V \) hence

\[ f([b_{i-1}, b_i]) \subset U. \] Also \( f([b_i, b_{i+1}]) \not\subset V \) hence

\[ f([b_i, b_{i+1}]) \subset U. \] Thus

\[ f([b_{i-1}, b_{i+1}]) = f([b_{i-1}, b_i]) \cup f([b_i, b_{i+1}]) \subset U. \]

Then we can replace \( 0 = b_0 < b_1 < ... < b_k = 1 \) by

\[ 0 = b_0 < b_1 < ... < b_{i-1} < b_{i+1} < ... < b_k = 1 \]

and we get a contradiction with the minimality of \( k \).
Similarly, if $f(b_i) \in V \cap \mathcal{C}(U)$ we get a contradiction (use symmetry of $U, V$). Thus neither $f(b_i) \in U \cap \mathcal{C}(V)$ or $f(b_i) \in V \cap \mathcal{C}(U)$ can hold. Since $f(b_i) \in U \cup V$ this forces $f(b_i) \in U \cap V$. For $i = 1, ..., k$ we define

$$f_i : [0, 1] \rightarrow X \text{ by } f_i(t) = f((1 - t)b_{i-1} + tb_i).$$

This is cont. We have $f_i([0, 1]) = f([b_{i-1}, b_i])$ so it is contained in $U$ or in $V$. 
One can verify that $f$ is homotopic to $f_1 * f_2 * \cdots * f_k$.

For each $i = 0, 1, \ldots, k$ we choose a path $\alpha_i : [0, 1] \to U \cap V$ s.t. $\alpha_i(0) = x_0$, $\alpha_i(1) = f(b_i)$. We can assume that $\alpha_0$ is the constant map at $x_0 = f(b_0) = f(b_k)$. Let $g_i = \alpha_{i-1} * f_i * \bar{\alpha}_i$ a path from $x_0$ to $x_0$ with image contained either in $U$

or in $V$. We have $f_1 * f_2 * \cdots * f_k$ equiv.to $g_1 * g_2 * \cdots * g_k$. Hence

$f$ equiv.to $g_1 * g_2 * \cdots * g_k$. 

Example. Take $X$ to be the unit sphere in $\mathbb{R}^n$, $n \geq 3$. It is given by $\{(x_1, ..., x_n) \in \mathbb{R}^n; x_1^2 + ... x_n^2 = 1\}$.

Define $U = X - \{0, 0, ..., 0, 1\}$, $V = X - \{0, 0, ..., 0, -1\}$. Then $U, V$ are open in $X$, $U \cup V = X$. We have a bijection

$$U \rightarrow \mathbb{R}^{n-1}, (x_1, ..., x_n) \mapsto \left(\frac{x_1}{1-x_n}, ..., \frac{x_{n-1}}{1-x_n}\right)$$

with inverse $y = (y_1, ..., y_{n-1}) \mapsto (t(y)y_1, ..., t(y)y_{n-1}, 1 - t(y))$

where $t(y) = 2/(1 + |y|^2)$. This is homeomorphism.
Under this homeomorphism $U \cap V$ corresponds to

$\mathbb{R}^{n-1} - \{(0, 0, ..., 0)\}$

hence is path-connected (earlier result; here we use $n - 1 \geq 2$).

We have a homeomorphism $U \rightarrow V$,

$$(x_1, ..., x_n) \mapsto (x_1, ..., x_n - 1, -x_n).$$

Hence $V$ is homeomorphic to $\mathbb{R}^{n-1}$. Since $\mathbb{R}^{n-1}$ is simply

connected it follows that $X$ is simply connected.

In particular $X$ is not homeomorphic to the circle.