Urysohn Lemma.

Let $X$ be a normal space, let $A, B$ be two disjoint closed subsets of $X$. There exists a continuous function $f : X \to \mathbb{R}$ such that $f(x) = 0$ for $x \in A$, $f(x) = 1$ for $x \in B$, $f(x) \in [0, 1]$ for $x \in X$.

We have constructed a family of open sets $U_p$ of $X$ ($p \in \mathbb{Q}$ (the rational nrs.) such that
\[ p < q \implies \bar{U}_p \subset U_q, \]

\[ U_1 = \mathcal{C}(B), A \subset U_0, \]

\[ U_p = \emptyset \text{ if } p < 0, \quad U_p = X \text{ if } p > 1. \]

For \( x \in X \) we define a subset \( Q(x) \) of \( Q \) by

\[ Q(x) = \{ p \in Q; x \in U_p \}. \]

We have \((1, \infty) \cap Q \subset Q(x) \subset [0, \infty)\).
Recall: if $Z \subset \mathbb{R}$ is nonempty and $Z \subset [a, \infty)$ for some $a$ then there is a unique $a_0 \in \mathbb{R}$ (denoted $a_0 = glb(Z)$) s.t. $Z \subset [a_0, \infty)$ but $Z \not\subset [a_0 + \epsilon, \infty)$ for any $\epsilon > 0$. Example:

$glb[a, \infty) = a$, $glb((a, \infty) \cap \mathbb{Q}) = a$. If $Z'$ is a set like $Z$, $Z \subset Z'$, then $glb(Z') \leq glb(Z)$: Assume $a'_0 = glb(Z') > glb(Z) = a_0$.

We have $Z \not\subset [a_0 + \epsilon, \infty) = [a'_0, \infty)$ for $\epsilon = a'_0 - a_0 > 0$,

$Z \subset Z' \subset [a'_0, \infty)$ contradiction.
For $x \in X$ set $f(x) = \text{glb}(Q(x)) \in \mathbb{R}$. (Use $Q(x) \subset [0, \infty)$ and $Q(x) \neq \emptyset$ since $(1, \infty) \cap Q \subset Q(x)$.) We have $f(x) \geq 0$ since $Q(x) \subset [0, \infty)$ and $\text{glb}[0, \infty) = 0$. We have $f(x) \leq 1$ since $(1, \infty) \cap Q \subset Q(x)$ and $\text{glb}((1, \infty) \cap Q) = 1$. Thus $f(x) \in [0, 1]$.

We show: $x \in B \implies f(x) = 1$. We have $x \notin U_1$. If $p \in Q(x)$, $p \leq 1$ then $x \in U_p \subset U_1$ absurd. Thus $Q(x) \subset (1, \infty) \cap Q$ and $Q(x) = (1, \infty) \cap Q$, $f(x) = \text{glb}((1, \infty) \cap Q) = 1$. 
We show: \( x \in A \implies f(x) = 0 \). We have \( x \in U_0 \). If \( p \in Q \),

\[
p \geq 0 \text{ we have } U_0 \subset U_p \text{ hence } p \in Q(x).
\]

Thus \( [0, \infty) \cap Q \subset Q(x) \).

Now \( (-\infty, 0) \cap Q(x) = \emptyset \) hence \( Q(x) = [0, \infty) \) and

\[
f(x) = \operatorname{glb}([0, \infty) \cap Q) = 0.
\]
We show: If \( r \in \mathbb{Q}, x \in \tilde{U}_r \) then \( f(x) \leq r \).

If \( s \in \mathbb{Q}, s > r \) then \( \tilde{U}_r \subset U_s \) hence \( x \in U_s \). Thus

\[
(r, \infty) \cap \mathbb{Q} \subset Q(x) \text{ so that } f(x) \leq \text{glb}((r, \infty) \cap \mathbb{Q}) = r.
\]

We show: If \( r \in \mathbb{Q}, x \notin U_r \) then \( f(x) \geq r \).

If \( s \in \mathbb{Q}, s < r \) then \( \tilde{U}_s \subset U_r \) hence \( x \notin U_s \). Thus

\[
(-\infty, r) \cap \mathbb{Q} \cap Q(x) = \emptyset \text{ so that } Q(x) \subset [r, \infty) \cap \mathbb{Q} \text{ so that }
\]

\[
f(x) \geq \text{glb}([r, \infty) \cap \mathbb{Q}) = r.
\]
We show that $f$ is continuous. It is enough to prove:

If $c < d$ in $\mathbb{R}$ then $f^{-1}(c, d)$ is open in $X$ that is:

for any $x \in f^{-1}(c, d)$ there exists $U$ open in $X$ such that

$x \in U \subset f^{-1}(c, d)$. We have $c < f(x) < d$. Can choose $p, q$ in $\mathbb{Q}$

with $c < p < f(x) < q < d$. Now $\bar{U}_p \subset U_q$. Let $U = U_q \cap C(\bar{U}_p)$

(open in $X$). We have $x \in U_q$. (If $x \notin U_q$ then $f(x) \geq q$ absurd.)

We have $x \notin \bar{U}_p$. (If $x \in \bar{U}_p$ then $f(x) \leq p$ absurd.)
From $x \in U_q$, $x \notin \bar{U}_p$ we deduce $x \in U$.

Let $x' \in U$. Then $x' \in U_q$ so that $f(x') \leq q$.

Since $x' \in \mathcal{C}(\bar{U}_p)$ we have $x' \in \mathcal{C}(U_p)$ so

that $f(x') \geq p$. Thus $f(x') \subset [p, q] \subset (c, d)$.

Thus $f(U) \subset (c, d)$ that is $U \subset f^{-1}(c, d)$.

This completes the proof of continuity of $f$. Urysohn’s lemma is proved.
Tietze extension theorem.

Let $a < b$ in $\mathbb{R}$. Let $X$ be a normal space, let $A \subset X$ be closed and let $f : A \to [a, b]$ be a continuous function. There exists $g : X \to [a, b]$ cont. such that $g(x) = f(x)$ for all $x \in A$.

Proof. If we know this for some $a < b$ then we know it for any $a' < b'$ (use a homeomorphism $[a, b] \to [a', b']$).

STEP 1. Assume $[a, b] = [-r, r]$ where $r > 0$. Write
\[ [-r, r] = I_1 \cup I_2 \cup I_3 \] where

\[ I_1 = [-r, -r/3], I_2 = [-r/3, r/3], I_3 = [r/3, r]. \]

Let \( B = f^{-1}(I_1), C = f^{-1}(I_3) \). Then \( B, C \) are closed disjoint subsets of \( A \) hence are closed in \( X \). By Urysohn, can find

\[ g : X \to [-r/3, r/3] \] cont. s.t. \( g(x) = -r/3 \) for \( x \in B \),

\[ g(x) = r/3 \] for \( x \in C \), \(-r/3 \leq g(x) \leq r/3 \) for \( x \in X \).
We show: for \( x \in A \) we have \(|f(x) - g(x)| \leq 2r/3 \) (*).

1) If \( x \in B \) then \( f(x) \in I_1, g(x) \in I_1 \) hence (*) holds.

2) If \( x \in C \) then \( f(x) \in I_3, g(x) \in I_3 \) hence (*) holds.

3) If \( x \notin B \cup C \) then \( f(x) \in I_2, g(x) \in I_2 \) hence (*) holds.

STEP 2. We take \([a, b] = [-1, 1]\).
By Step 1 can find $g_1 : X \to [-1/3, 1/3]$ cont. with

$$|f(x) - g_1(x)| \leq 2/3 \text{ for } x \in A.$$  Consider

$f - g_1 : A \to [-2/3, 2/3]$. By Step 1 with $r = 2/3$ can find

$g_2 : X \to [-2/3^2, 2/3^2]$ (cont.) with

$$|f(x) - g_1(x) - g_2(x)| \leq 2^2/3^2 \text{ for } x \in A.$$  Continuing we find

$g_1, g_2, g_3, \ldots$ where $g_n : X \to [-2^{n-1}/3^n, 2^{n-1}/3^n]$,

$$|f(x) - g_1(x) - g_2(x) - \ldots - g_n(x)| \leq 2^n/3^n \text{ for } x \in A.$$. 
Recall (calculus): (1) If $c_1 \leq c_2 \leq c_3 \leq ...$ is a sequence in $\mathbb{R}$

s.t. for some $c \in \mathbb{R}$ we have $c_i \leq c$ for all $i$, then $c_i$ is convergent.

(2) Let $a_1, a_2, ...$ and $b_1, b_2, ...$ be two sequences in $\mathbb{R}$ such that

$|a_i| \leq b_i$ for all $i$. If $b_1, b_1 + b_2, b_1 + b_2 + b_3, ...$ is convergent

then $a_1, a_1 + a_2, a_1 + a_2 + a_3, ...$ is convergent.

(3) Let $d_i$ be a seq. in $\mathbb{R}$ s.t. for any $\epsilon > 0$ there exists $N \geq 1$

s.t. for $n \geq N, m \geq N$ we have $|d_n - d_m| < \epsilon$. Then $d_i$ converges.
A sequence as in (3) is said to be Cauchy.

We show: (1) follows from (3).

It is enough to show that $c_i$ is Cauchy. Assume it is not. Can find $\epsilon > 0$ s.t. for any $N \geq 1$ we can find $n, m$ with $N \leq n < m,$

$c_m - c_n \geq \epsilon.$ Let $N_1 = 1.$ Can find $n_1, m_1$ with $N_1 \leq n_1 < m_1,$

$c_{m_1} - c_{n_1} \geq \epsilon.$ Take $N_2 > m_1.$ Can find $n_2, m_2$ with

$N_2 \leq n_2 < m_2, c_{m_2} - c_{n_2} \geq \epsilon.$ Take $N_3 > m_2.$
Can find $n_3, m_3$ with $N_3 \leq n_3 < m_3$, $c_{m_3} - c_{n_3} \geq \epsilon$. Continue.

We have $n_1 < m_1 < n_2 < m_2 < \ldots$, $c_{m_j} - c_{n_j} \geq \epsilon$ for $j = 1, 2, \ldots$

$$(c_{m_1} - c_{n_1}) + (c_{m_2} - c_{n_2}) + \ldots + (c_{m_k} - c_{n_k}) \geq k\epsilon.$$

The lhs is

$$-c_{n_1} - (c_{n_2} - c_{m_1}) - (c_{n_3} - c_{m_2}) - \ldots - (c_{n_k} - c_{m_{k-1}}) + c_{m_k} \leq c_{m_k} - c_{n_1} \leq c - c_{n_1}$$

hence $c - c_{n_1} \geq k\epsilon$ for $k = 1, 2, \ldots$. This is absurd.
We show: (2) follows from (1) if in addition \( a_i \geq 0 \) for all \( i \).

Let \( b = \lim(b_1 + b_2 + ... + b_k) \). We have

\[
a_1 \leq a_1 + a_2 \leq a_1 + a_2 + a_3 \leq ....
\]

Also \( a_1 + a_2 + ... + a_k \leq b_1 + b_2 + ... + b_k \) for all \( k \) hence

\[
a_1 + a_2 + ... + a_k \leq b.
\]

Thus (1) is applicable and \( a_1, a_1 + a_2, a_1 + a_2 + a_3, ... \) converges.
We show: (2) follows from (3). From the previous page,

\[ |a_1| + |a_2| + \ldots + |a_k| \] is convergent hence Cauchy. Thus for any \( \epsilon > 0 \) there exists \( N \geq 1 \) s.t. for any \( n, m \) with \( N \leq n < m \) we have

\[ |a_{n+1}| + |a_{n+2}| + \ldots + |a_m| < \epsilon. \]

Now \( |a_{n+1} + a_{n+2} + \ldots + a_m| \leq |a_{n+1}| + |a_{n+2}| + \ldots + |a_m| \) hence

\[ |a_{n+1} + a_{n+2} + \ldots + a_m| < \epsilon. \] Thus \( a_1, a_1 + a_2, a_1 + a_2 + a_3 \) is Cauchy (and by (3)) it converges so that (2) holds.
We return to Tietze’s theorem. For \( x \in X \) and \( k \geq 1 \) we set

\[
s_k(x) = g_1(x) + g_2(x) + \ldots + g_k(x).
\]

Then \( s_1(x), s_2(x), s_3(x), \ldots \) converges. Since

\[
|g_n(x)| \leq 2^{n-1}/3^n,
\]

it is enough to show that

\[
1/3 + 2/3^2 + \ldots + 2^{k-1}/3^k = 1/3(1 + 2/3 + 2^2/3^2 + \ldots + 2^{k-1}/3^{k-1})
\]

is convergent. This is well known.
We have \( \lim |s_k(x)| \leq 1/3(1 + 2/3 + 2^2/3^2 + \ldots) = 1/3 \frac{1}{1-(2/3)} = 1. \)

Let \( g(x) = \lim_{k \to \infty} s_k(x) = g_1(x) + g_2(x) + \ldots. \) We have

\[ |g(x)| = \lim |s_k(x)| \leq 1. \text{ Thus } g \text{ is a function } X \to [-1, 1]. \]

We show that the convergence of \( s_k(x) \) to \( g(x) \) is uniform that is: for any \( \epsilon > 0 \) there exists \( N \) s.t. for \( n \geq N \)

and any \( x \in X \) we have \( |g(x) - s_n(x)| < \epsilon. \)
For $k > n$ we have

$$|s_k(x) - s_n(x)| = |g_{n+1}(x) + \ldots + g_k(x)| \leq |g_{n+1}(x)| + \ldots + |g_k(x)|$$

$$\leq (1/3)((2/3)^n + \ldots + (2/3)^{k-1}) \leq (1/3)((2/3)^n + (2/3)^{n+1} + \ldots)$$

$$= (1/3)(2/3)^n(1 + (2/3) + (2/3)^2 + \ldots) = (2/3)^n.$$

From $|s_k(x) - s_n(x)| \leq (2/3)^n$ we deduce, for $k \to \infty$,

$$|g(x) - s_n(x)| \leq (2/3)^n.$$
If $\epsilon > 0$ is given we can find $N$ such that $(2/3)^N < \epsilon$. If $n \geq N$ then $(2/3)^n \leq (2/3)^N < \epsilon$ and $|g(x) - s_n(x)| < \epsilon$. Thus the convergence is uniform so that $g$ is a continuous function.

Assume that $x \in A$. Recall that

$$|f(x) - g_1(x) - g_2(x) - \ldots - g_n(x)| \leq \frac{2^n}{3^n}$$

that is $|f(x) - s_n(x)| \leq \frac{2^n}{3^n}$. Taking the limit of both sides we find $|f(x) - g(x)| \leq \lim(\frac{2^n}{3^n}) = 0$. 
Thus \( |f(x) - g(x)| = 0 \) hence \( f(x) = g(x) \). This proves Tietze’s theorem.

**A variant of Tietze extension theorem**

Let \( X \) be a normal space, let \( A \subset X \) be closed

and let \( f : A \rightarrow \mathbb{R} \) be a continuous function. There exists

\[
g : X \rightarrow \mathbb{R} \text{ cont. such that } g(x) = f(x) \text{ for all } x \in A.
\]
Proof. We can replace $\mathbb{R}$ by $(-1, 1)$. Thus we have

$$f : A \rightarrow (-1, 1).$$

We can view $f$ as a cont. function $A \rightarrow [-1, 1]$.

By the original Tietze theorem we can find $g : X \rightarrow [-1, 1]$ cont.

such that $g(x) = f(x)$ for $x \in A$.

Unfortunately $g(x)$ can be 1 or $-1$ which we don’t want
to happen. Let $D = g^{-1}(-1) \cup g^{-1}(1)$. 
Then $D$ is closed in $X$. Also $A$ is disjoint from $D$

(since $f(A) \cap \{-1, 1\} = \emptyset$). We apply Urysohn lemma to $D, A$.

We find $\psi : X \to [0, 1]$ cont. such that $\psi(x) = 0$ for $x \in D$, $\psi(x) = 1$ for $x \in A$. Let $h(x) = \Psi(x)g(x)$. This is a cont.fn. $X \to [-1, 1]$. For $x \in A$ we have $h(x) = g(x) = f(x)$. We show:

$h(x) \notin \{-1, 1\}$. If $x \notin D$ then $|g(x)| < 1$

so $|h(x)| \leq |g(x)| < 1$. If $x \in D$ then $h(x) = 0$, QED.