ON C-SMALL CONJUGACY CLASSES IN A REDUCTIVE GROUP

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Introduction

0.1. Let G be a connected reductive algebraic group over an algebraically closed field $\mathbf{k}$ of characteristic p. Let $\mathbf{W}$ be the Weyl group of G. Let Ω_w be the double coset in G (with respect to a Borel subgroup B^*) corresponding to an element $w \in \mathbf{W}$ which has minimal length in its conjugacy class C in $\mathbf{W}$ and has no eigenvalue 1 in the reflection representation of $\mathbf{W}$. Let Z_G be the centre of G. From [L5, 5.2] it follows that the isotropy groups of the conjugation action of B^*/Z_G on Ω_w are finite abelian. One of the results of this paper is that (if G is semisimple) the set of orbits of this action is naturally an affine variety of dimension equal to the length of w, which looks very much like an affine space modulo the action of a finite group. Consider the intersection of Ω_w with a conjugacy class γ in G. Since $\Omega_w \cap \gamma$ is B^*/Z_G-stable, the result quoted above shows that, when $\Omega_w \cap \gamma$ is nonempty, it has dimension greater than or equal to $\dim(B^*/Z_G)$. As in [L5] we say that γ is C-small if $\Omega_w \cap \gamma \neq \emptyset$ and the previous inequality is an equality. (This condition depends only on C, not on w.)

In the remainder of this subsection we assume that

(i) p is 0 or a good prime for G

and that G is almost simple. In [L5] we have shown that for any C as above there is a unique unipotent class γ_C in G which is C-small. In this paper we investigate the existence of C-small semisimple classes in G. We show that such a class γ' exists in almost all cases. (There is exactly one exception to this property: it arises in type E_8 for a unique C.) Let ρ_{γ_C} be the Springer representation of $\mathbf{W}$ associated to γ_C and to the local system $\bar{\mathbf{Q}}_l$ on γ_C. We show that ρ_{γ_C} is surprisingly connected to γ' above (again with the unique exception above) as follows: ρ_{γ_C} is obtained by "j-induction" (see 0.3) from the sign representation of a reflection subgroup of $\mathbf{W}$, namely the Weyl group of the connected centralizer of an element of γ'. We will also show that the representation ρ_{γ_C} depends only on the Weyl group $\mathbf{W}$, not on the underlying root system.

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0.2. Here is some notation that we use in this paper. Let $\mathcal{B}$ the variety of Borel subgroups of G. Let $\underline{l} : \mathbf{W} \to \mathbf{N}$ be the standard length function. Let $S = \{s \in \mathbf{W}; \underline{l}(s) = 1\}$. For each $w \in \mathbf{W}$ let $\mathcal{O}_w$ be the corresponding G-orbit in $\mathcal{B} \times \mathcal{B}$. Let $\underline{\mathbf{W}}_{el}$ be the set of elliptic conjugacy classes in the Weyl group $\mathbf{W}$ of G (see [L5, 0.2].) For $C \in \underline{\mathbf{W}}_{el}$ let $d_C = \min_{w \in C} \underline{l}(w)$ and let $C_{min} = \{w \in C; l(w) = d_C\}$. For any conjugacy class γ in G and any $w \in \mathbf{W}$ we set $\mathfrak{B}_w = \{(g, B) \in G \times \mathcal{B}; (B, gBg^{-1}) \in \mathcal{O}_w\}$, $\mathfrak{B}_w^\gamma = \{(g, B) \in \mathfrak{B}_w; g \in \gamma\}$.

The cardinal of a finite set X is denoted by $|X|$ or by $\sharp(X)$. For any $g \in G$ let $Z(g)$ be the centralizer of g in G. Let ν_G be the number of positive roots of G. For an integer σ we define $\kappa_\sigma \in \{0, 1\}$ by $\sigma = \kappa_\sigma \mod 2$.

Let $C \in \underline{\mathbf{W}}_{el}$. In [L5, 5.5, 5.7(iii)] it is shown that if $\mathfrak{B}_w^\gamma \neq \emptyset$ for some/any $w \in C_{min}$ then $\dim \mathfrak{B}_w^\gamma \geq \dim(G/Z_G)$ and $\dim \gamma \geq \dim(G/Z_G) - d_C$. (Here the equivalence of "some/any" follows from [L5, 5.2(a)].) Following [L5, 5.5] we say that γ is *C-small* if for some/any $w \in C_{min}$ we have $\mathfrak{B}_w^\gamma \neq \emptyset$ and the equivalent conditions $\dim \mathfrak{B}_w^\gamma = \dim(G/Z_G)$, $\dim \gamma = \dim(G/Z_G) - d_C$ are satisfied (for the equivalence see [L5, 7.7(iv)]).

0.3. Let W be a Weyl group. Let sgn be the sign representation of W and let R_W be the reflection representation of W. Let $\mathrm{Irr}(W)$ be the set of (isomorphism classes of) irreducible representations of W. For $E \in \mathrm{Irr}(W)$ let b_E be the smallest integer ≥ 0 such that the multiplicity of E in the b_E-th symmetric power of R_W is ≥ 1. We write $E \in \mathrm{Irr}(W)^\dagger$ if this multiplicity is 1. Let W' be a subgroup of W generated by reflections. Let $E' \in \mathrm{Irr}(W')^\dagger$. There is a unique $E \in \mathrm{Irr}(W)$ such that E appears in $\mathrm{ind}_{W'}^W(E')$ and $b_E = b_{E'}$. (See [GP, 5.2.6].) We have $E \in \mathrm{Irr}(W)^\dagger$. We set $E = j_{W'}^W(E')$. The process $j_{W'}^W()$ is called j-induction.

0.4. For any unipotent class γ in G let $\rho_\gamma \in \mathrm{Irr}(\mathbf{W})$ be the Springer representation of $\mathbf{W}$ associated to γ and the local system $\bar{\mathbf{Q}}_l$ on γ. (We use the conventions of [L2].) For any $C \in \underline{\mathbf{W}}_{el}$ let γ_C be the unique C-small unipotent class of G, see [L5]; thus ρ_{γ_C} is well defined.

0.5. Let B^* be a Borel subgroup of G and let $\mathcal{T}$ be a maximal torus of B^*. Let $N_G(\mathcal{T})$ be the normalizer of $\mathcal{T}$ in G and let $\mathcal{W} = N_G(\mathcal{T})/\mathcal{T}$. For any $z \in \mathcal{W}$ let $\dot{z}$ be a representative of z in $N_G(\mathcal{T})$. We identify $\mathbf{W} = \mathcal{W}$ as follows: to $z \in \mathcal{W}$ corresponds the element $w \in \mathbf{W}$ such that $(B^*, \dot{z}B^*\dot{z}^{-1}) \in \mathcal{O}_w$. For any $s \in S$ let $\alpha_s : \mathcal{T} \to \mathbf{k}^*$ be the simple root defined by s. In the remainder of this subsection we assume that G is almost simple, simply connected and that 0.1(i) holds. Let $\alpha_0 : \mathcal{T} \to \mathbf{k}^*$ be the unique root such that for any $s \in S$, $\alpha_0\alpha_s^{-1} : \mathcal{T} \to \mathbf{k}^*$ is not a root. Let $\Delta = \{\alpha_s; s \in S\} \sqcup \{\alpha_0\}$. For any $K \subsetneqq \Delta$ let $\mathcal{W}_K$ be the subgroup of $\mathcal{W}$ generated by the reflections with respect to roots in Δ. Let G_K be the subgroup of G generated by $\mathcal{T}$ and by the root subgroups attached to roots such that the corresponding reflection in $\mathcal{W}$ is in $\mathcal{W}_K$. Note that G_K is a connected reductive subgroup of G (a "Borel-de Siebenthal subgroup"). Now $B^* \cap G_K$ is a Borel subgroup of G_K and $\mathcal{T}$ is a maximal torus of $B^* \cap G_K$. Note that $\mathcal{W}_K = N_{G_K}(\mathcal{T})/\mathcal{T} = \{w \in \mathcal{W}; \dot{w} \in G_K\}$ may be identified (using $B^* \cap G_K$,

$\mathcal{T}$) with the Weyl group $\mathbf{W}_K$ of G_K in the same way as $\mathcal{W}$ is identified with $\mathbf{W}$ (using $B^*, \mathcal{T}$). In particular $\mathbf{W}_K$ appears as a subgroup of $\mathbf{W}$. Let $\mathcal{S}_K$ be the set of semisimple conjugacy classes γ in G such that for some $\zeta \in \gamma \cap \mathcal{T}$ we have $G_K = Z(\zeta)$. Note that $\mathcal{S}_K \neq \emptyset$ and any semisimple class in G belongs to $\mathcal{S}_K$ for some $K \subsetneqq \Delta$. The following is our main result.

Theorem 0.6. *Assume that G is almost simple, simply connected and that 0.1(i) holds. Let $C \in \underline{\mathbf{W}}_{el}$. With the single exception when G is of type E_8 and for any $w \in C$, the characteristic polynomial of $w : R_{\mathbf{W}} \to R_{\mathbf{W}}$ is $(X+1)(X^2+1)^2(X^3+1)$, there exists $K \subsetneqq \Delta$ such that*

(i) for any $\gamma \in \mathcal{S}_K$, γ is a C-small semisimple class;

(ii) $\rho_{\gamma_C} = j_{\mathbf{W}_K}^{\mathbf{W}}(\mathrm{sgn})$.

In the case where G is of type E_8 and C is the class specified in the theorem, there is no $K \subsetneqq \Delta$ for which (i) holds and there is no $K \subsetneqq \Delta$ for which (ii) holds. On the other hand in this case we have $\rho_{\gamma_C} = j_{\mathbf{W}_K}^{\mathbf{W}}(\mathrm{sgn} \otimes r)$ where G_K is of type $D_5 + A_3$ and r is the irreducible representation of $\mathbf{W}_K$ on which the D_5-factor acts as the reflection representation and the A_3-factor acts trivially. Also, if $\gamma \in \mathcal{S}_K$, $\zeta \in \gamma \cap \mathcal{T}$, $Z(\zeta) = K$ and u is a unipotent element of G_K which is in a minimal unipotent class $\neq 1$ of the D_5-factor, then the G-conjugacy class γ' of ζu is C-small; although γ' is not semisimple, it is as close as possible to being semisimple.

In the case where G is of type A the theorem is immediate: C must be the conjugacy class of a Coxeter element and we can take $K = \emptyset$. The proof of the theorem in the case where G is of classical type other than A is given in §1, §3. When G is of exceptional type the proof of the theorem is given in 2.4, 3.5 (using a reduction to a computer calculation, see 2.2.)

0.7. Assume that G is almost simple, simply connected and that 0.1(i) holds. Let γ be any C-small conjugacy class in G. Let ζ (resp. u) be a semisimple (resp. unipotent) element of G such that $\zeta u = u\zeta \in \gamma$. Let $K \subsetneqq \Delta$ be such that the conjugacy class of ζ belongs to $\mathcal{S}_K$. We assume as we may that $\zeta \in \gamma \cap \mathcal{T}$ and $G_K = Z(\zeta)$. Let ρ_u be the Springer representation of $\mathbf{W}_K$ associated to the conjugacy class of u in G_K. We conjecture that

$$\rho_{\gamma_C} = j_{\mathbf{W}_K}^{\mathbf{W}}(\rho_u).$$

This is supported by Theorem 0.6.

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1. Classical groups

1.1. Let $t_1, t_2, \dots, t_m$ be commuting indeterminates. Let M be the $m \times m$ matrix whose j-th row is $t_1^j - t_1^{-j}, t_2^j - t_2^{-j}, \dots, t_m^j - t_m^{-j}$, $j \in [1, m]$. Let M' be the $(m+2) \times (m+2)$ matrix whose j-th row is $1, (-1)^j, t_1^j + t_1^{-j}, t_2^j + t_2^{-j}, \dots, t_m^j + t_m^{-j}$, $j \in [0, m+1]$. The proof of (a),(b) below is left to the reader.

(a) $\det(M)$ *is equal to* $\pm\prod_i \prod_{i<j}(t_i - t_j)\prod_{i\le j}(t_i - t_j^{-1})$ *times a monomial in the* t_i*;*

(b) $\det(M')$ *is equal to* $\pm 2\prod_i(t_i - t_i^{-1})\prod_{i<j}(t_i - t_j)\prod_{i\le j}(t_i - t_j^{-1})$ *times a monomial in the* t_i*.*

1.2. Let V be a $\mathbf{k}$-vector space of finite dimension $\mathbf{n} \ge 3$. We set $\kappa = \kappa_{\mathbf{n}}$ so that $\mathbf{n} = 2n + \kappa$ with $n \in \mathbf{N}$. Assume that V has a fixed bilinear form $(,) : V \times V \longrightarrow \mathbf{k}$ and a fixed quadratic form $Q : V \longrightarrow \mathbf{k}$ such that (i) or (ii) below holds:

(i) $Q = 0$, $(x, x) = 0$ for all $x \in V$, $(,)$ is nondegenerate;

(ii) $Q \ne 0$, $(x, y) = Q(x + y) - Q(x) - Q(y)$ for $x, y \in V$, $p \ne 2$, $(,)$ is nondegenerate.

An element $g \in GL(V)$ is said to be an isometry if $(gx, gy) = (x, y) = 0$ for all $x, y \in V$ (hence $Q(gx) = Q(x)$ for all $x \in V$). Let $Is(V)$ be the group of all isometries of V (a closed subgroup of $GL(V)$). In this section we assume that G is the identity component of $Is(V)$. Let $\mathcal{F}$ be the set of all sequences $V_* = (0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_{\mathbf{n}} = V)$ of subspaces of V such that $\dim V_i = i$ for $i \in [0, \mathbf{n}]$, $Q|_{V_i} = 0$ and $\{x \in V; (x, V_i) = 0\} = V_{\mathbf{n}-i}$ for all $i \in [0, n]$. Now $Is(V)$ acts naturally (transitively) on $\mathcal{F}$.

1.3. Assume that $Q = 0$ so that $\mathbf{n} = 2n$. Let $p_* = (p_1 \ge p_2 \ge \cdots \ge p_\sigma)$ be a sequence of integers ≥ 1 such that $p_1 + p_2 + \cdots + p_\sigma = n$. For any $i \ge 1$ we set $\bar p_i = \sharp(t \in [1, \sigma]; p_t \ge i)$ so that $\bar p_1 \ge \bar p_2 \ge \ldots$ and $\sum_i \bar p_i = n$. Let $k = p_1$. We have $\bar p_k \ge 1$, $\bar p_{k+1} = 0$. We can find subspaces $\mathcal{V}_i, \mathcal{V}'_i$ ($i \in [1, k]$) of V such that

$V = \mathcal{V}_1 \oplus \mathcal{V}'_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}'_2 \oplus \ldots \oplus \mathcal{V}_k \oplus \mathcal{V}'_k$;

$\dim \mathcal{V}_i = \dim \mathcal{V}'_i = \bar p_i$ for $i \in [1, k]$;

$(,)$ is zero on $\mathcal{V}_i, \mathcal{V}'_i$ for $i \in [1, k]$;

$(\mathcal{V}_i \oplus \mathcal{V}'_i, \mathcal{V}_j \oplus \mathcal{V}'_j) = 0$ for all $i \ne j$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be a sequence elements of $\mathbf{k}^*$ such that

$\lambda_i - \lambda_j \ne 0$ for $i \ne j$,

$\lambda_i - \lambda_j^{-1} \ne 0$ for all i, j.

For any $t \in [1, \sigma]$ the system of linear equations

$$\sum_{i \in [1, p_t]} (\lambda_i^j - \lambda_i^{-j}) c_{t,i} = -\delta_{j, p_t}$$

($j \in [1, p_t]$) with unknowns $c_{t,i}$ ($i \in [1, p_t]$) has a unique solution $(c_{t,i})_{i \in [1, p_t]} \in \mathbf{k}^{p_t}$. (Its determinant is nonzero by 1.1(a) with $m = p_t$. Note that λ_i is defined for $i \in [1, p_t]$ since $p_t \le p_1 = k$.) For any $i \in [1, k]$ we choose a basis $(v_{t,i})_{t \in [1, \sigma]; p_t \ge i)}$ of $\mathcal{V}_i$ and we define vectors $v'_{t',i} \in \mathcal{V}'_i$ ($t' \in [1, \sigma], p_{t'} \ge i$) by $(v_{t,i}, v'_{t',i}) = \delta_{t,t'} c_{t,i}$ for all $t \in [1, \sigma], p_t \ge i$. Then for any $t \in [1, \sigma]$ and any $j \in [1, p_t]$ we have

(a) $\sum_{i \in [1, p_t]} (\lambda_i^j - \lambda_i^{-j})(v_{t,i}, v'_{t,i}) = -\delta_{j, p_t}$.

Hence for $j \in [-p_t, p_t - 1]$ we have

(b) $\sum_{i \in [1, p_t]} (\lambda_i^j - \lambda_i^{-j})(v_{t,i}, v'_{t,i}) = \delta_{j, -p_t}$.

(For $j \in [1, p_t - 1]$, (b) follows from (a); for $j \in [-p_t, -1]$, (b) follows from

(a) by replacing j by $-j$; for $j=0$, (b) is obvious.) For any $t\in[1,\sigma]$ we set $v_t=\sum_{i\in[1,k];i\le p_t}(v_{t,i}+v'_{t,i})\in V$. Define a linear map $g:V\to V$ by $gx=\lambda_i x$ for $x\in\mathcal{V}_i$, $gx=\lambda_i^{-1}x$ for $x\in\mathcal{V}'_i$ ($i\in[1,k]$). Then $g\in G$.

1.4. Assume that $Q\ne 0$ so that $p\ne 2$. Let $p_*=(p_1\ge p_2\ge\dots\ge p_\sigma)$ be a sequence of integers ≥ 1 such that $p_1+p_2+\dots+p_\sigma=n$. If $\kappa=0$ we assume also that $\kappa_\sigma=0$. For any $i\ge 1$ we set $\bar p_i=\sharp(t\in[1,\sigma];p_t\ge i)$ so that $\bar p_1\ge\bar p_2\ge\dots$ and $\sum_i\bar p_i=n$. Let $k=p_1$. We have $\bar p_1=\sigma$, $\bar p_k\ge 1$, $\bar p_{k+1}=0$.

We can choose subspaces $\mathcal{Z}',\mathcal{Z}''$ of V and subspaces $\mathcal{V}_i,\mathcal{V}'_i$ of V (for $i\in[2,k]$) such that:

$V=\mathcal{Z}'\oplus\mathcal{Z}''\oplus\oplus_{i\in[2,k]}(\mathcal{V}_i\oplus\mathcal{V}'_i)$;

$(,)$ is nondegenerate on $\mathcal{Z}'$ and on $\mathcal{Z}''$, $()$ is zero on $\mathcal{V}_i,\mathcal{V}'_i$ (for $i\in[2,k]$);

$\dim\mathcal{Z}'=\bar p_1+\kappa-\kappa_\sigma,\dim\mathcal{Z}''=\bar p_1+\kappa_\sigma$;

$\dim\mathcal{V}_i=\dim\mathcal{V}'_i=\bar p_i$ for $i\in[2,k]$;

$(\mathcal{V}_i\oplus\mathcal{V}'_i,\mathcal{V}_j\oplus\mathcal{V}'_j)=0$ for $i\ne j$ in $[2,k]$;

$(\mathcal{Z}',\mathcal{Z}'')=0$;

$(\mathcal{Z}'+\mathcal{Z}'',\mathcal{V}_i+\mathcal{V}'_i)=0$ for $i\in[2,k]$.

Let $\lambda_i(i\in[2,k])$ be elements of $\mathbf{k}^*$ such that

$\lambda_i-\lambda_j\ne 0$ for $i\ne j$ in $[2,k]$,

$\lambda_i-\lambda_j^{-1}\ne 0$ for all i,j in $[2,k]$.

For any $t\in[1,\sigma]$, the system of linear equations

$$c_{t,1}+(-1)^j c_{t,-1}+\sum_{i\in[2,p_t]}(\lambda_i^j+\lambda_i^{-j})c_{t,\lambda_i}=\delta_{j,p_t}$$

($j\in[0,p_t]$) with (p_t+1) unknowns ($c_{t,\lambda_i}\in\mathbf{k}$ ($i\in[2,p_t]$) and $c_{t,1}\in\mathbf{k},c_{t,-1}\in\mathbf{k}$) has a unique solution. (Its determinant is nonzero by 1.1(b) with $m=p_t-1$.) For any $i\in[2,k]$ we choose a basis $(v_{t,i})_{t\in[1,\sigma];p_t\ge i})$ of $\mathcal{V}_i$ and we define vectors $v'_{t',i}\in\mathcal{V}'_i$ ($t'\in[1,\sigma],p_{t'}\ge i$) by $(v_{t,i},v'_{t',i})=\delta_{t,t'}c_{t,\lambda_i}$ for all $t\in[1,\sigma],p_t\ge i$. (Note that if $t\in[1,\sigma]$ is such that $p_t\ge i$ then $i\in[2,p_t]$ hence c_{t,λ_i} is defined.)

We can find vectors $v'_t\in\mathcal{Z}'$ ($t\in[1,\sigma]$) such that $(v'_t,v'_{t'})=c_{t,1}\delta_{t,t'}$ for all t,t' in $[1,\sigma]$. We can find vectors $v''_t\in\mathcal{Z}''$ ($t\in[1,\sigma]$) such that $(v''_t,v''_{t'})=c_{t,-1}\delta_{t,t'}$ for all t,t' in $[1,\sigma]$. Then for any $t\in[1,\sigma]$ and any $j\in[0,p_t]$ we have

$$(v'_t,v'_t)+(-1)^j(v''_t+v''_t)+\sum_{i\in[2,p_t]}(\lambda_i^j+\lambda_i^{-j})(v_{t,i},v'_{t,i})=\delta_{j,p_t}.$$

It follows that

$$(v'_t,v''_t)+(-1)^j(v''_t,v''_t)+\sum_{i\in[2,p_t]}(\lambda_i^j+\lambda_i^{-j})(v_{t,i},v'_{t,i})=\delta_{-j,p_t}$$

for $j\in[-p_t,p_t-1]$. For any $t\in[1,\sigma]$ we set $v_t=v'_t+v''_t+\sum_{i\in[2,k];i\le p_t}(v_{t,i}+v'_{t,i})\in V$. Define a linear map $g:V\to V$ by $gx=x$ for $x\in\mathcal{Z}'$, $gx=-x$ for $x\in\mathcal{Z}''$, $gx=\lambda_i x$ for $x\in\mathcal{V}_i$ ($i\in[2,k]$), $gx=\lambda_i^{-1}x$ for $x\in\mathcal{V}'_i$ ($i\in[2,k]$). Then $g\in G$.

1.5. Assume that we are in the setup of 1.3 or 1.4. For t,t' in $[1,\sigma]$ and $j\in[-p_t,p_t-1]$ we have

(a) $$(g^jv_t,v_{t'})=\delta_{t,t'}\delta_{-j,p_t}.$$

Indeed, in the setup of 1.3, the left hand side of (a) is equal to

$$\sum_{i\in[1,k];i\le p_t,i\le p_{t'}}(\lambda_i^jv_{t,i}+\lambda_i^{-j}v'_{t,i},v_{t',i}+v'_{t',i})$$
$$=\delta_{t,t'}\sum_{i\in[1,p_t]}(\lambda_i^j-\lambda_i^{-j})(v_{t,i},v'_{t,i})=\delta_{t,t'}\delta_{j,-p_t};$$

in the setup of 1.4, the left hand side of (a) is equal to

$$(v_{t,1},v_{t',1})+(-1)^j(v'_{t,1},v'_{t',1})+\sum_{i\in[2,k];i\le p_t,i\le p_{t'}}(\lambda_i^jv_{t,i}+\lambda_i^{-j}v'_{t,i},v_{t',i}+v'_{t',i})$$
$$=\delta_{t,t'}((v_{t,1},v_{t,1})+(-1)^j(v'_{t,1},v'_{t,1})$$
$$+\sum_{i\in[2,k];i\le p_t,i\le p_t}(\lambda_i^j+\lambda_i^{-j})(v_{t,i},v'_{t,i}))=\delta_{t,t'}\delta_{j,-p_t}.$$

As in [L5, 3.3(vi)], from (a) we deduce that the vectors $(g^jv_t)_{t\in[1,\sigma],j\in[-p_t,p_t-1]}$ span a $(\mathbf{n}-\kappa)$-dimensional subspace of V on which $(,)$ is nondegenerate. (If $\kappa=0$ this subspace is V.) For any $h\in[1,n]$ we can write $h=p_1+p_2+\dots+p_{r-1}+i$ where $r\in[1,\sigma]$ and $i\in[1,p_r]$ are uniquely determined; we define V_h to be the subspace of V spanned by the vectors $g^jv_t(t\in[1,r-1],j\in[0,p_t-1])$ and $g^jv_r(j\in[0,i-1])$. Let V'_h be the subspace of V spanned by the vectors $g^jv_t(t\in[1,r-1],j\in[1,p_t])$ and $g^jv_r(j\in[1,i])$. We have $(V_h,V_h)=0$ (see (a)), $(V'_h,V'_h)=0$, $gV_h=V'_h$. There are unique sequences V_*,V'_* in $\mathcal{F}$ such that V_h,V'_h are as above for any $h\in[1,n]$. We have $V'_*=gV_*$. For any $r\in[1,\sigma]$ and $i\in[1,p_r-1]$ we have

$$\dim(V'_{p_1+p_2+\dots+p_{r-1}+i}\cap V_{p_1+p_2+\dots+p_{r-1}+i})=p_1+p_2+\dots+p_{r-1}+i-r,$$

(the intersection is spanned by the vectors $g^jv_t(t\in[1,r-1],j\in[1,p_t-1])$ and $g^jv_r(j\in[1,i-1])$),

$$\dim(V'_{p_1+p_2+\dots+p_{r-1}+i}\cap V_{p_1+p_2+\dots+p_{r-1}+i+1})=p_1+p_2+\dots+p_{r-1}+i-r+1,$$

(the intersection is spanned by the vectors $g^jv_t(t\in[1,r-1],j\in[1,p_t-1])$ and $g^jv_r(j\in[1,i])$). For any $r\in[1,\sigma]$ we have

$$\dim(V'_{p_1+p_2+\dots+p_r}\cap V_{\mathbf{n}-p_1-p_2-\dots-p_{r-1}-1})=p_1+p_2+\dots+p_r-r,$$

(the intersection is spanned by the vectors $g^j v_t (t \in [1,r], j \in [1,p_t-1])$),

$$\dim(V'_{p_1+p_2+\dots+p_r} \cap V_{\mathbf{n}-p_1-p_2-\dots-p_{r-1}}) = p_1+p_2+\dots+p_r-r+1,$$

(the intersection is spanned by the vectors $g^j v_t (t \in [1,r], j \in [1,p_t-1])$ and $g^{p_r} v_r$). (We use again (a).) As in [L5, 3.2] we deduce that $a_{V_*,V'_*} = w_{p_*} \in \mathbf{W}$ (notation of [L5, 1.4, 1.6]). Let B, B' be the stabilizers of V_*, V'_* in G. Then B, B' are Borel subgroups of G and $(B,B') \in \mathcal{O}_{w_{p_*}}$, $gBg^{-1} = B'$. Hence if γ denotes the conjugacy class of g in G, we have

(b) $$\mathfrak{B}^{\gamma}_{w_{p_*}} \neq \emptyset.$$

Note that γ is a semisimple conjugacy class and that w_{p_*} has minimal length in its conjugacy class C in $\mathbf{W}$ (which is elliptic).

Let $\delta(g) = \dim Z(g)$. Let $d = \underline{l}(w_{p_*})$. We show that

(c) $$\delta(g) = d.$$

In the setup of 1.3, $Z(g)$ is isomorphic to $GL(\bar{p}_1) \times GL(\bar{p}_2) \times \dots \times GL(\bar{p}_k)$ hence $\delta(g) = \bar{p}_1^2 + \bar{p}_2^2 + \dots + \bar{p}_k^2$. In the setup of 1.4, the identity component of $Z(g)$ is isomorphic to $SO(\bar{p}_1 + \kappa - \kappa_\sigma) \times SO(\bar{p}_1 + \kappa_\sigma) \times GL(\bar{p}_2) \times GL(\bar{p}_3) \times \dots \times GL(\bar{p}_k)$ hence

$$\delta(g) = (\bar{p}_1 + \kappa - \kappa_s)(\bar{p}_1 + \kappa - \kappa_\sigma - 1)/2 + (\bar{p}_1 + \kappa_s)(\bar{p}_1 + \kappa_s - 1)/2$$
$$+ \bar{p}_2^2 + \dots + \bar{p}_k^2 = \bar{p}_1^2 + \bar{p}_2^2 + \dots + \bar{p}_k^2 - \sigma(1-\kappa).$$

If $(1-\kappa)Q = 0$ we have $d = 2(p_2 + 2p_3 + \dots + (\sigma-1)p_\sigma) + n$; if $(1-\kappa)Q \neq 0$ we have $d = 2(p_2 + 2p_3 + \dots + (\sigma-1)p_\sigma) + n - \sigma$. Hence to prove (c) it is enough to show that

$$\bar{p}_1^2 + \bar{p}_2^2 + \dots + \bar{p}_k^2 = 2(p_2 + 2p_3 + \dots + (\sigma-1)p_\sigma) + n.$$

This follows from the equality $X = 2Y$ in [L5, 4.4]; note that f_{2h} from *loc.cit.* is the same as $\bar{p}'$ and $\sum_h f_{2h} = n$. From (b) and (c) we see that γ is a (semisimple) C-small conjugacy class. This proves 0.6(i) for our G.

2. Exceptional groups

2.1. In this subsection we assume that $\mathbf{k}$ is an algebraic closure of a finite field $\mathbf{F}_q$ with q elements; we also assume that 0.1(i) holds. We choose an $\mathbf{F}_q$-split rational structure on G with Frobenius map $F: G \to G$ such that B^* and $\mathcal{T}$ are F-stable. Note that $F(t) = t^q$ for all $t \in \mathcal{T}$. Define a class function $\Pi_G : \mathbf{W} \to \mathbf{Z}$ by $\Pi_G(w) = \sum_i \mathrm{tr}(w, H^{2i}(\mathcal{B}, \bar{\mathbf{Q}}_l))q^i$ where we use the standard $\mathbf{W}$-module structure on $H^*(\mathcal{B}, \bar{\mathbf{Q}}_l)$. For any $z \in \mathcal{W}$ let $x_z \in G$ be such that $x_z^{-1}F(x_z) = z$ and let $\mathcal{T}_z = x_z \mathcal{T} x_z^{-1}$, an F-stable maximal torus of G. For any $w \in \mathbf{W}$ we have

$$\Pi_G(w) = (-1)^{\underline{l}(w)}|G^F|q^{-\nu_G}|\mathcal{T}_w|^{-1}.$$

2.2. We assume that $\mathbf{k}$ is as in 2.1, that G is almost simple, simply connected of exceptional type and that $K \subsetneqq \Delta$. Let $\gamma \in \mathcal{S}_K$. Let $C \in \underline{\mathbf{W}}_{el}$ and let $w \in C_{min}$. We show that the condition that $\mathfrak{B}_w^\gamma \neq \emptyset$ can be tested by performing a computer calculation. We will also see that this condition depends only on K, not on γ.

We choose an $\mathbf{F}_q$-rational structure on G as in 2.1. We can assume that $g^{q-1} = 1$ for some/any $g \in \gamma$. Then γ is F-stable and $\gamma \cap \mathcal{T} = \gamma \cap \mathcal{T}^F$ is a single $\mathcal{W}$-orbit. Let $\zeta \in \gamma \cap \mathcal{T}$ be such that $Z(\zeta) = G_K$. Note that G_K is defined and split over $\mathbf{F}_q$,

Now the class function $\Pi_{G_K} : \mathbf{W}_K \to \mathbf{Z}$ is well defined, see 2.1. For $z \in \mathcal{W}$ we have

$$\begin{aligned}
&\sharp(h \in G^F; h^{-1}\zeta h \in \mathcal{T}_z) = \sharp(h \in G^F; x_z^{-1}h^{-1}\zeta h x_z \in \mathcal{T})\\
&= \sharp(h' \in G; F(h') = h'\dot{z}, h'^{-1}\zeta h' \in \mathcal{T}) = \sharp(h' \in G; F(h') = h'\dot{z}, h'^{-1}\zeta h' \in \gamma \cap \mathcal{T})\\
&= |\mathcal{W}_K|^{-1} \sum_{v \in \mathcal{W}} \sharp(h' \in G; F(h') = h'\dot{z}, h'^{-1}\zeta h' = \dot{v}\zeta\dot{v}^{-1})\\
&= |\mathcal{W}_K|^{-1} \sum_{v \in \mathcal{W}} \sharp(h'' \in G; F(h'') = h''\dot{v}^{-1}\dot{z}F(\dot{v}), h''^{-1}\zeta h'' = \zeta)\\
&= |\mathcal{W}_K|^{-1} \sum_{v \in \mathcal{W}} \sharp(h'' \in G_K; F(h'') = h''\dot{v}^{-1}\dot{z}F(\dot{v}))\\
&= |\mathcal{W}_K|^{-1}\sharp(v \in \mathcal{W}; \dot{v}^{-1}\dot{z}F(\dot{v}) \in G_K)|G_K^F| = |\mathcal{W}_K|^{-1}\sharp(v \in \mathcal{W}; v^{-1}zv \in \mathcal{W}_K)|G_K^F|.
\end{aligned}$$

(We set $h' = hx_z$; then we set $h'^{-1}\zeta h' = \dot{v}\zeta\dot{v}^{-1}$ with $v \in \mathcal{W}$; then we set $h'' = h'\dot{v}$ and we use Lang's theorem in G_K.)

As in [L5, 1.2(a)] the number of fixed points of $F : \mathfrak{B}_w^\gamma \to \mathfrak{B}_w^\gamma$, $(g, B) \mapsto (F(g), F(B))$, is given by

(a)
$$|(\mathfrak{B}_w^\gamma)^F| = |\mathbf{W}|^{-1} \sum_{E,E' \in \mathrm{Irr}\mathbf{W}, z \in \mathbf{W}, g \in \gamma^F} \mathrm{tr}(T_w, E_q)(\rho_E : R_{E'})\mathrm{tr}(z, E')\mathrm{tr}(\zeta, R^1(z)).$$

(Notation of *loc.cit.*.) Using [DL, 7.2] we see that

$$\begin{aligned}
\mathrm{tr}(\zeta, R^1(z)) &= \sharp(h \in G^F; h^{-1}\zeta h \in \mathcal{T}_z)|\mathcal{T}_z|^{-1}q^{-\nu_{G_K}}(-1)^{\underline{l}(z)}\\
&= |\mathcal{W}_K|^{-1}\sharp(v \in \mathcal{W}; v^{-1}zv \in \mathcal{W}_K)|G_K^F||\mathcal{T}_z|^{-1}q^{-\nu_{G_K}}(-1)^{\underline{l}(z)}\\
&= |\mathcal{W}_K|^{-1}\sharp(v \in \mathcal{W}; v^{-1}zv \in \mathcal{W}_K)\Pi_{G_K}(v^{-1}zv).
\end{aligned}$$

(We have used that the restriction to $\mathcal{W}_K = \mathbf{W}_K$ of the function $z \mapsto (-1)^{\underline{l}(z)}$ on $\mathcal{W} = \mathbf{W}$ is the analogous function defined in terms of G_K.) Substituting this into (a) we obtain

$$\begin{aligned}
&|(\mathfrak{B}_w^\gamma)^F| = |\gamma^F||\mathbf{W}|^{-1} \sum_{E,E' \in \mathrm{Irr}\mathbf{W}, z \in \mathbf{W}} \mathrm{tr}(T_w, E_q)(\rho_E : R_{E'})\mathrm{tr}(z, E')\\
&\times |\mathcal{W}_K|^{-1}\sharp(v \in \mathcal{W}; v^{-1}zv \in \mathcal{W}_K)\Pi_{G_K}(v^{-1}zv)\\
&= |\gamma^F||\mathbf{W}|^{-1} \sum_{E,E' \in \mathrm{Irr}\mathbf{W}, z \in \mathbf{W}} \mathrm{tr}(T_w, E_q)(\rho_E : R_{E'})\mathrm{tr}(z, E')\mathrm{tr}(z, \mathrm{ind}_{\mathcal{W}_K}^{\mathcal{W}}(\Pi_{G_K})).
\end{aligned}$$

Hence

(b) $$|(\mathfrak{B}_w^\gamma)^F| = |G^F|/|G_K^F| \sum_{E,E' \in \mathrm{Irr}\mathbf{W}} \mathrm{tr}(T_w, E_q)(\rho_E : R_{E'})(E' : \Pi_{G_K})_{\mathbf{W}_K}.$$

Here $(E' : \Pi_{G_K})_{\mathbf{W}_K}$ is the inner product of Π_{G_K} (viewed as a representation of $\mathbf{W}_K$) with the restriction of E' to $\mathbf{W}_K$. We can also write (b) as follows:

$$|(\mathfrak{B}_w^\gamma)^F| = |G^F|/|G_K^F| \sum A_{E,C}\phi_{E,E'} m_{E',E''} t_{E'',K}$$

where the sum is taken over all E, E' in $\mathrm{Irr}\mathbf{W}$, $E'' \in \mathrm{Irr}\mathbf{W}_K$ and the notation is as follows. For $C' \in \underline{\mathbf{W}}, E \in \mathrm{Irr}\mathbf{W}$ we set $A_{E,C'} = \mathrm{tr}(T_z, E_q)$ where $z \in C'_{min}$. (Note that $A_{E,C'}$ is well defined by [GP, 8.2.6(b)].) For $E, E' \in \mathrm{Irr}\mathbf{W}$ let $\phi_{E,E'} = (\rho_E : R_{E'})$. (Notation of [L5, 1.2].) For $E' \in \mathrm{Irr}\mathbf{W}$, $E'' \in \mathrm{Irr}\mathbf{W}_K$ let $m_{E',E''}$ be the multiplicity of E'' in $E'|_{\mathbf{W}_K}$. For $E'' \in \mathrm{Irr}\mathbf{W}_K$ let $t_{E'',K}$ be the multiplicity of E'' in Π_{G_K}. Thus $|(\mathfrak{B}_w^\gamma)^F|$ is $|G^F|/|G_K^F|$ times the the C-entry of the vector

$$^t(A_{E,C'})(\phi_{E,E'})(m_{E',E''})(t_{E'',K}).$$

Here the matrix $(A_{E,C'})$ is known from the works of Geck and Geck-Michel (see [GP, 11.5.11]) and is available through the CHEVIE package [C]. The matrix $\phi_{E,E'}$ has as entries the coefficients of the "nonabelian Fourier transform" in [L1, 4.15]. The matrix $(m_{E',E''})$ ("Induction table") and the vector $(t_{E'',K})$ ("Fake degree") are available through the CHEVIE package. Thus $|(\mathfrak{B}_w^\gamma)^F|$ can be obtained by calculating the product of several explicitly known matrices. The calculation was done using the CHEVIE package. It turns out that $|(\mathfrak{B}_w^\gamma)^F|$ is a polynomial in q with integer coefficients denoted by P_C^K (it depends only on K, C not on γ, w). Note that $\mathfrak{B}_w^\gamma \neq \emptyset$ if and only if $P_C^K \neq 0$ as a polynomial in q. Thus the condition that $\mathfrak{B}_w^\gamma \neq \emptyset$ can be tested. Moreover for each K such that $P_C^K \neq 0$ and for $\gamma \in \mathcal{S}_K$, the condition that γ is C-small is equivalent to the condition that $\dim(G_K) = d_C$; in this case we have $P_C^K = m_C^K |G(\mathbf{F}_q)|$ as polynomials in q where m_C^K is an integer ≥ 1 independent of q. (For any K such that $P_C^K \neq 0$ we have $\deg(P_C^K) \geq \dim(G)$ (by [L5, 5.2]). Note that m_C^K is equal to the number of connected components of $\mathfrak{B}_w^\gamma$ for $\gamma \in \mathcal{S}_K$, $w \in C_{min}$. This number can be > 1; in one example in type E_8 it is 10.

2.3. In this subsection we give (in the setup of 2.2) tables which describe for each exceptional type and each $C \in \underline{\mathbf{W}}_{el}$ (with one exception) some proper subsets K of Δ such that $P_C^K \neq 0$ and $\dim(G_K) = d_C$. The elements of $\Delta - \{\alpha_0\}$ are denoted by numbers $1, 2, 3, \dots$ as in [GP, p.20]. We write 0 instead of α_0. We specify K by marking each element of $\Delta - K$ by $\bullet$. An element $C \in \underline{\mathbf{W}}_{el}$ is specified by indicating the characteristic polynomial of an element of C acting on $R_{\mathbf{W}}$, a product of cyclotomic polynomials Φ_d (an exception is type F_4 when there are two choices for C with characteristic polynomial $\Phi_2^2\Phi_6$ in which case we use the notation $(\Phi_2^2\Phi_6)'$, $(\Phi_2^2\Phi_6)''$ for what in [GP, p.407] is denoted by D_4, C_3+A_1). The notation $d; C; \chi; (K_1)_{m_1}; (K_2)_{m_2}; \dots$ means that $C \in \underline{\mathbf{W}}_{el}$, $d = d_C$, $\chi = \rho_{\gamma_C}$ (γ_C as in 0.4), and $K_1, K_2, \dots$ are proper subsets of Δ such that $P_C^{K_i} \neq 0$ and $\dim(G_{K_i}) = d_C$; we have $m_i = m_C^{K_i}$. (We omit m_i whenever $m_i = 1$.)

The notation for irreducible representations of $\mathbf{W}$ (of type E_6, E_7, E_8) is as in [Sp]; for type F_4 it is as in [L1]; for type G_2, 1_0 is the unit representation, 2_1 is the reflection representation and 2_2 is the other two dimensional irreducible representation of $\mathbf{W}$.

Type G_2; Δ is (012)
$2; \Phi_6; 1_0;\quad (\bullet\bullet\bullet)$
$4; \Phi_3; 2_1;\quad (\bullet 1 \bullet);\quad (\bullet\bullet 2)$
$6; \Phi_2^2; 2_2;\quad (0 \bullet 2)$

Type F_4; Δ is (01234)
$4; \Phi_{12}; 1_1;\quad (\bullet\bullet\bullet\bullet\bullet)$
$6; \Phi_8; 4_2;\quad (\bullet 1 \bullet \bullet\bullet)$
$8; \Phi_6^2; 9_1;\quad (0 \bullet 2 \bullet \bullet);\quad (\bullet 1 \bullet 3 \bullet)_2$
$10; (\Phi_2^2\Phi_6)'; 8_1;\quad (\bullet\bullet\bullet 34)$
$10; (\Phi_2^2\Phi_6)''; 8_3;\quad (\bullet 12 \bullet\bullet);\quad (0 \bullet 2 \bullet 4)$
$12; \Phi_4^2; 12_1;\quad (\bullet\bullet 23 \bullet)_3;\quad (\bullet 12 \bullet 4);\quad (0 \bullet\bullet 34)$
$14; \Phi_2^2\Phi_4; 16_1;\quad (0 \bullet 23 \bullet)$
$16; \Phi_3^2; 6_1;\quad (01 \bullet 34)$
$24; \Phi_2^4; 9_4;\quad (0 \bullet 234)$

Type E_6; Δ is $\left(\begin{smallmatrix}1&3&4&5&6\\ &&2&&\\ &&0&&\end{smallmatrix}\right)$

$6; \Phi_3\Phi_{12}; 1_0;\quad \left(\begin{smallmatrix}\bullet&\bullet&\bullet&\bullet&\bullet\\ &&\bullet&&\\ &&\bullet&&\end{smallmatrix}\right)$

$8; \Phi_9; 6_1;\quad \left(\begin{smallmatrix}1&\bullet&\bullet&\bullet&\bullet\\ &&\bullet&&\\ &&\bullet&&\end{smallmatrix}\right)$

$12; \Phi_3\Phi_6^2; 30_3;\quad \left(\begin{smallmatrix}1&3&\bullet&\bullet&\bullet\\ &&\bullet&&\\ &&\bullet&&\end{smallmatrix}\right);\quad \left(\begin{smallmatrix}1&\bullet&\bullet&5&\bullet\\ &&2&&\\ &&\bullet&&\end{smallmatrix}\right)$

$14; \Phi_2^2\Phi_3\Phi_6; 15_4;\quad \left(\begin{smallmatrix}1&\bullet&4&\bullet&6\\ &&\bullet&&\\ &&0&&\end{smallmatrix}\right)$

$24; \Phi_3^3; 10_9;\quad \left(\begin{smallmatrix}1&3&\bullet&5&6\\ &&2&&\\ &&0&&\end{smallmatrix}\right)$

Type E_7; Δ is $\left(\begin{smallmatrix}0&1&3&4&5&6&7\\ &&&2&&&\end{smallmatrix}\right)$
$7; \Phi_2\Phi_{18}; 1_0;\quad \left(\begin{smallmatrix}\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet\\ &&&\bullet&&&\end{smallmatrix}\right)$
$9; \Phi_2\Phi_{14}; 7_1;\quad \left(\begin{smallmatrix}\bullet&1&\bullet&\bullet&\bullet&\bullet&\bullet\\ &&&\bullet&&&\end{smallmatrix}\right)$
$11; \Phi_2\Phi_6\Phi_{12}; 27_2;\quad \left(\begin{smallmatrix}\bullet&1&\bullet&\bullet&\bullet&\bullet&\bullet\\ &&&2&&&\end{smallmatrix}\right)$

$13; \Phi_2\Phi_6\Phi_{10}; 56_3;$ $\begin{pmatrix} \bullet & 1 & 3 & \bullet & \bullet & \bullet & \bullet \\ & & & \bullet & & & \end{pmatrix}_2;$ $\begin{pmatrix} \bullet & 1 & \bullet & \bullet & 5 & \bullet & \bullet \\ & & & 2 & & & \end{pmatrix}$

$15; \Phi_2^3\Phi_{10}; 35_4;$ $\begin{pmatrix} 0 & \bullet & 3 & \bullet & 5 & \bullet & 7 \\ & & & \bullet & & & \end{pmatrix}$

$17; \Phi_2\Phi_4\Phi_8; 189_5;$ $\begin{pmatrix} \bullet & 1 & 3 & \bullet & 5 & \bullet & \bullet \\ & & & 2 & & & \end{pmatrix};$ $\begin{pmatrix} 0 & \bullet & 3 & \bullet & 5 & \bullet & 7 \\ & & & 2 & & & \end{pmatrix}$

$21; \Phi_2\Phi_6^3; 315_7;$ $\begin{pmatrix} \bullet & 1 & \bullet & 4 & 5 & \bullet & \bullet \\ & & & 2 & & & \end{pmatrix}_3;$ $\begin{pmatrix} \bullet & 1 & 3 & \bullet & 5 & 6 & \bullet \\ & & & 2 & & & \end{pmatrix}$

$23; \Phi_2^3\Phi_6^2; 280_8;$ $\begin{pmatrix} 0 & 1 & 3 & \bullet & 5 & \bullet & 7 \\ & & & \bullet & & & \end{pmatrix}$

$25; \Phi_2\Phi_3^2\Phi_6; 70_9;$ $\begin{pmatrix} 0 & 1 & \bullet & 4 & \bullet & 6 & 7 \\ & & & 2 & & & \end{pmatrix}$

$31; \Phi_2^5\Phi_6; 84_{12};$ $\begin{pmatrix} 0 & 1 & 3 & \bullet & 5 & 6 & 7 \\ & & & \bullet & & & \end{pmatrix}$

$33; \Phi_2^3\Phi_4^2; 210_{13};$ $\begin{pmatrix} \bullet & 1 & 3 & 4 & \bullet & 6 & 7 \\ & & & 2 & & & \end{pmatrix}$

$63; \Phi_2^7; 15_{28};$ $\begin{pmatrix} 0 & 1 & 3 & 4 & 5 & 6 & 7 \\ & & & \bullet & & & \end{pmatrix}$

Type E_8; Δ is $\begin{pmatrix} 1 & 3 & 4 & 5 & 6 & 7 & 8 & 0 \\ & & 2 & & & & & \end{pmatrix}$

$8; \Phi_{30}; 1_0;$ $\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & & & & & \end{pmatrix}$

$10; \Phi_{24}; 8_1;$ $\begin{pmatrix} 1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & & & & & \end{pmatrix}$

$12; \Phi_{20}; 35_2;$ $\begin{pmatrix} 1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & 2 & & & & & \end{pmatrix}$

$14; \Phi_6\Phi_{18}; 112_3;$ $\begin{pmatrix} 1 & 3 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & & & & & \end{pmatrix}_2;$ $\begin{pmatrix} 1 & \bullet & \bullet & 5 & \bullet & \bullet & \bullet & \bullet \\ & & 2 & & & & & \end{pmatrix}$

$16; \Phi_{15}; 210_4;$ $\begin{pmatrix} 1 & \bullet & 4 & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & 2 & & & & & \end{pmatrix}_2;$ $\begin{pmatrix} 1 & \bullet & \bullet & 5 & \bullet & 7 & \bullet & \bullet \\ & & 2 & & & & & \end{pmatrix}$

$16; \Phi_2^2\Phi_{18}; 84_4;$ $\begin{pmatrix} \bullet & \bullet & \bullet & 5 & \bullet & 7 & \bullet & 0 \\ & & 2 & & & & & \end{pmatrix}$

$18; \Phi_2^2\Phi_{14}; 560_5;$ $\begin{pmatrix} 1 & 3 & \bullet & 5 & \bullet & \bullet & \bullet & \bullet \\ & & 2 & & & & & \end{pmatrix};$ $\begin{pmatrix} 1 & \bullet & \bullet & 5 & \bullet & 7 & \bullet & 0 \\ & & 2 & & & & & \end{pmatrix}$

$20; \Phi_{12}^2; 700_6;$ $\begin{pmatrix} 1 & 3 & \bullet & 5 & 6 & \bullet & \bullet & \bullet \\ & & \bullet & & & & & \end{pmatrix}_2;$ $\begin{pmatrix} 1 & 3 & \bullet & 5 & \bullet & 7 & \bullet & \bullet \\ & & 2 & & & & & \end{pmatrix}$

$22; \Phi_4^2\Phi_{12}; 400_7;$ $\begin{pmatrix} 1 & 3 & \bullet & 5 & \bullet & 7 & \bullet & 0 \\ & & 2 & & & & & \end{pmatrix}$

$22; \Phi_6^2\Phi_{12}; 1400_7;$ $\begin{pmatrix} 1 & \bullet & 4 & 5 & \bullet & \bullet & \bullet & \bullet \\ & & 2 & & & & & \end{pmatrix}_3;$ $\begin{pmatrix} 1 & 3 & \bullet & 5 & 6 & \bullet & \bullet & \bullet \\ & & 2 & & & & & \end{pmatrix}$

$24; \Phi_{10}^2; 1400_8;$ $\begin{pmatrix} 1 & \bullet & 4 & 5 & \bullet & 7 & \bullet & \bullet \\ & & 2 & & & & & \end{pmatrix}_3;$ $\begin{pmatrix} 1 & 3 & \bullet & 5 & 6 & \bullet & 8 & \bullet \\ & & 2 & & & & & \end{pmatrix}$

$24; \Phi_2^2\Phi_6\Phi_{12}; 1344_8;$ $\begin{pmatrix} \bullet & \bullet & 4 & 5 & \bullet & 7 & \bullet & 0 \\ & & 2 & & & & & \end{pmatrix}$

$26; \Phi_3^2\Phi_{12}; 448_9;$ $\begin{pmatrix} 1 & 3 & \bullet & 5 & 6 & \bullet & 8 & 0 \\ & & \bullet & & & & & \end{pmatrix}$

$26; \Phi_2^2\Phi_6\Phi_{10}; 3240_9;$ $\begin{pmatrix} 1 & 3 & 4 & \bullet & 6 & 7 & \bullet & \bullet \\ & & \bullet & & & & & \end{pmatrix};$ $\begin{pmatrix} 1 & \bullet & 4 & 5 & \bullet & 7 & \bullet & 0 \\ & & 2 & & & & & \end{pmatrix}$

$28; \Phi_3\Phi_9; 2240_{10};$ $\begin{pmatrix} 1 & 3 & \bullet & 5 & 6 & 7 & \bullet & \bullet \\ & & 2 & & & & & \end{pmatrix};$ $\begin{pmatrix} 1 & 3 & \bullet & 5 & 6 & \bullet & 8 & 0 \\ & & 2 & & & & & \end{pmatrix}$

$30; \Phi_8^2; 1400_{11};$ $\begin{pmatrix} 1 & 3 & \bullet & 5 & 6 & 7 & 8 & \bullet \\ & & 2 & & & & & \end{pmatrix}$

$32; \Phi_2^4\Phi_{10}; 972_{12};$ $\begin{pmatrix} \bullet & \bullet & 4 & 5 & \bullet & 7 & 8 & 0 \\ & & 2 & & & & & \end{pmatrix}$

$34; \Phi_2^2\Phi_4\Phi_8; 4536_{13};$ $\begin{pmatrix} 1\ 3\ 4\ \bullet\ 6\ 7\ \bullet\ \bullet \\ 2 \end{pmatrix}$; $\begin{pmatrix} 1\ \bullet\ 4\ 5\ \bullet\ 7\ 8\ 0 \\ 2 \end{pmatrix}$

$40; \Phi_6^4; 4480_{16};$ $\begin{pmatrix} 1\ \bullet\ 4\ 5\ 6\ 7\ \bullet\ \bullet \\ 2 \end{pmatrix}_{10}$; $\begin{pmatrix} 1\ 3\ 4\ 5\ \bullet\ 7\ 8\ 0 \\ \bullet \end{pmatrix}$

$42; \Phi_2^2\Phi_6^3; 7168_{17};$ $\begin{pmatrix} 1\ \bullet\ 4\ 5\ 6\ 7\ \bullet\ 0 \\ 2 \end{pmatrix}$

$44; \Phi_2^4\Phi_6^2; 4200_{18};$ $\begin{pmatrix} \bullet\ 3\ 4\ 5\ \bullet\ 7\ 8\ 0 \\ 2 \end{pmatrix}$

$44; \Phi_3^2\Phi_6^2; 3150_{18};$ $\begin{pmatrix} 1\ 3\ 4\ 5\ 6\ \bullet\ 8\ 0 \\ \bullet \end{pmatrix}$

$46; \Phi_2^2\Phi_3^2\Phi_6; 2016_{19};$ $\begin{pmatrix} 1\ 3\ \bullet\ 5\ 6\ 7\ 8\ 0 \\ 2 \end{pmatrix}$

$46; \Phi_2^2\Phi_4^2\Phi_6; 1344_{19};$

$48; \Phi_5^2; 420_{20};$ $\begin{pmatrix} 1\ 3\ 4\ \bullet\ 6\ 7\ 8\ 0 \\ 2 \end{pmatrix}$

$60; \Phi_4^4; 840_{26};$ $\begin{pmatrix} 1\ 3\ 4\ 5\ \bullet\ 7\ 8\ 0 \\ 2 \end{pmatrix}$

$64; \Phi_2^6\Phi_6; 700_{28};$ $\begin{pmatrix} \bullet\ \bullet\ 4\ 5\ 6\ 7\ 8\ 0 \\ 2 \end{pmatrix}$

$66; \Phi_2^4\Phi_4^2; 1400_{29};$ $\begin{pmatrix} 1\ \bullet\ 4\ 5\ 6\ 7\ 8\ 0 \\ 2 \end{pmatrix}$

$80; \Phi_3^4; 175_{36};$ $\begin{pmatrix} 1\ 3\ 4\ 5\ 6\ 7\ 8\ 0 \\ \bullet \end{pmatrix}$

$120; \Phi_2^8; 50_{56};$ $\begin{pmatrix} \bullet\ 3\ 4\ 5\ 6\ 7\ 8\ 0 \\ 2 \end{pmatrix}$

2.4. We prove 0.6(i) in the case where G is almost simple, simply connected of exceptional type. Let $C \in \underline{\mathbf{W}}_{el}$. Let K be a proper subset of Δ associated to C in the tables in 2.3. We can find $\gamma \in \mathcal{S}_K$ such that

(i) any element of γ has finite order.

We show that γ is C-small. By a standard argument we are reduced to the case where $\mathbf{k}$ is as in 2.1. In this case the calculations outlined in 2.2 show that γ is C-small, as claimed.

Next we consider for $w \in C_{min}$, the map $\pi : \mathfrak{B}_w \to G$, $(g, B) \mapsto G$. Let $\mathfrak{K} = \pi_! \bar{\mathbf{Q}}_l$. Using [L4, 14.2(a)], we see that the cohomology sheaves of $\mathfrak{K}$ behave smoothly when restricted to $\cup_{\gamma \in \mathcal{S}_K} \gamma$ (which is one of the pieces $Y_{L,\S}$ in [L3, 13.11]). Since we know that when $\gamma \in \mathcal{S}_K$ satisfies (i), some cohomology sheaf of $\mathfrak{K}$ is non-zero on γ, it follows that for any $\gamma \in \mathcal{S}_K$, some cohomology sheaf of $\mathfrak{K}$ is non-zero on γ; in particular, $\mathfrak{B}_w^\gamma \neq \emptyset$. It follows that any $\gamma \in \mathcal{S}_K$ is C-good. This completes the proof of Theorem 0.6(i).

3. Springer representations

3.1. In the setup of 1.2 we assume that G is the identity component of $Is(V)$. If $(1-\kappa)Q = 0$ we identify, as in [L5, 1.5], $\mathbf{W}$ with W, the group of all permutations of $[1, \mathbf{n}]$ that commute with the involution $i \mapsto \mathbf{n} + 1 - i$. Let W_n be the group of all permutations of $[1, 2n]$ that commute with the involution $i \mapsto 2n + 1 - i$. If $\mathbf{n} = 2n$ we have $W = W_n$; if $\mathbf{n} = 2n + 1$ we identify W with W_n (hence $\mathbf{W}$ with W_n) by $w \mapsto w'$ where $h(w'(i)) = w(h(i))$ for $i \in [1, 2n]$ and $h(i) = i$ if $i \in [1, n]$, $h(i) = i + 1$ if $i \in [n + 1, 2n]$. As in [L1, 4.5] we write the irreducible representations of W_n in the form $[(\lambda_1 > \lambda_2 > \cdots > \lambda_{m+1}), (\mu_1 > \mu_2 > \cdots > \mu_m)]$ where $\lambda_i, \mu_i \in \mathbf{N}$, $\sum_i \lambda_i + \sum_i \mu_i = m^2 + n$ and m is sufficiently large. For example $[(0 < 1 < 2 < \cdots < n), (1 < 2 < \cdots < n)]$ is the sign representation of W_n.

If $(1-\kappa)Q \neq 0$ we identify as in [L5, 1.5] $\mathbf{W}$ with W'_n, the group of even permutations in W_n. As in [L1, 4.6] we write the irreducible representations of W'_n as unordered pairs $[(\lambda_1 > \lambda_2 > \cdots > \lambda_m), (\mu_1 > \mu_2 > \cdots > \mu_m)]$ where $\lambda_i, \mu_i \in \mathbf{N}$, $\sum_i \lambda_i + \sum_i \mu_i = m^2 - m + n$ and m is sufficiently large. (There are two irreducible representations corresponding to $[(\lambda_1 > \lambda_2 > \cdots > \lambda_m), (\mu_1 > \mu_2 > \cdots > \mu_m)]$ with $\lambda_i = \mu_i$ for all i.) For example $[(1 < 2 < \cdots < n), (0 < 1 < 2 < \cdots < n-1)]$ is the sign representation of W'_n.

3.2. Let S_n be the symmetric group in n letters. Using [L6, 5.3, 4.4(a)] we see that

(a) if $n = 2c \in 2\mathbf{N}$ then $W_c \times W'_c, S_n$ are naturally reflection subgroups of W_n and we have

$$j_{W_c \times W'_c}^{W_n}(\mathrm{sgn}) = j_{S_n}^{W_n}(\mathrm{sgn})$$
$$= [(0 < 2 < 3 < 4 < \cdots < c+1), (1 < 2 < 3 < \cdots < c)];$$

(b) if $n = 2c+1 \in 2\mathbf{N}+1$ then $W_c \times W'_{c+1}, S_n$ are naturally reflection subgroups of W_n and we have

$$j_{W_c \times W'_{c+1}}^{W_n}(\mathrm{sgn}) = j_{S_n}^{W_n}(\mathrm{sgn})$$
$$= [(1 < 2 < 3 < 4 < \cdots < c+1), (1 < 2 < 3 < \cdots < c)].$$

Using [L6, 6.3] we see that

(c) if $n = 2c \in 2\mathbf{N}$ then $W'_c \times W'_c$ is naturally a reflection subgroup of W'_n and we have

$$j_{W'_c \times W'_c}^{W'_n}(\mathrm{sgn}) = [(2 < 3 < 4 < \cdots < c+1), (0 < 1 < 2 < 3 < \cdots < c-1)].$$

Let $p_1 \geq p_2 \geq \cdots \geq p_\sigma$ be integers ≥ 1 such that $p_1 + \cdots + p_\sigma = n$. Define $\bar{p}_1 \geq \bar{p}_2 \geq \cdots \geq \bar{p}_k$ as in 1.3. Note that $\bar{p}_1 = \sigma, k = p_1$. Define $\tilde{p}_1 \geq \tilde{p}_2 \geq \cdots \geq \tilde{p}_\sigma$ by $\tilde{p}_i = p_i - 1$ if $i \in [1, \bar{p}_k]$, $\tilde{p}_i = p_i$ if $i \in [\bar{p}_k + 1, \sigma]$.

Assuming that $k > 1$ we have $\tilde{p}_\sigma = p_\sigma \geq 1$ and using [L6, 4.4(a)] we see that

(d) if $\sigma = 2\tau + 1$ we have

$$[(p_\sigma < p_{\sigma-2} + 1 < \cdots < p_3 + \tau - 1 < p_1 + \tau), (p_{\sigma-1} < p_{\sigma-3} + 1 < \cdots < p_2 + \tau - 1)]$$
$$= j_{S_{\bar{p}_k} \times W_{n-\bar{p}_k}}^{W_n}(\mathrm{sgn} \boxtimes [(\tilde{p}_\sigma < \tilde{p}_{\sigma-2} + 1 < \cdots < \tilde{p}_3 + \tau - 1 < \tilde{p}_1 + \tau),$$
$$(\tilde{p}_{\sigma-1} < \tilde{p}_{\sigma-3} + 1 < \cdots < \tilde{p}_2 + \tau - 1)]);$$

(e) if $\sigma = 2\tau$ we have

$$[(0 < p_{\sigma-1} + 1 < p_{\sigma-3} + 2 < \cdots < p_3 + \tau - 1 < p_1 + \tau),$$
$$(p_\sigma < p_{\sigma-2} + 1 < \cdots < p_2 + \tau - 1)]$$
$$= j_{S_{\bar{p}_k} \times W_{n-\bar{p}_k}}^{W_n}(\mathrm{sgn} \boxtimes [(0 < \tilde{p}_{\sigma-1} + 1 < \tilde{p}_{\sigma-3} + 2 < \cdots < \tilde{p}_3 + \tau - 1 < \tilde{p}_1 + \tau),$$
$$(\tilde{p}_\sigma < \tilde{p}_{\sigma-2} + 1 < \cdots < \tilde{p}_2 + \tau - 1)]).$$

Assuming that $k>1$, $\sigma=2\tau$ we have $\tilde p_\sigma=p_\sigma\ge 1$, $n-\bar p_k\ge 2$ and using [L6, 6.2(a)] we see that

$$\begin{aligned}&[(p_{\sigma-1}+1<p_{\sigma-3}+2<\dots<p_3+\tau-1<p_1+\tau),\\&(p_\sigma-1<p_{\sigma-2}<\dots<p_4+\tau-3<p_2+\tau-2)]\\&=j_{S_{\bar p_k}\times W'_{n-\bar p_k}}^{W'_n}(\mathrm{sgn}\boxtimes[(\tilde p_{\sigma-1}+1<\tilde p_{\sigma-3}+2<\dots<\tilde p_1+\tau),\\ \text{(f)}\quad&(\tilde p_\sigma-1<\tilde p_{\sigma-2}<\dots<\tilde p_4+\tau-3<\tilde p_2+\tau-2)]).\end{aligned}$$

We show that

(g) $j_{S_{\bar p_k}\times\dots\times S_{\bar p_2}\times S_{\bar p_1}}^{W_n}(\mathrm{sgn})$ is equal to

$$[(p_\sigma<p_{\sigma-2}+1<\dots<p_3+\tau-1<p_1+\tau),(p_{\sigma-1}<p_{\sigma-3}+1<\dots<p_2+\tau-1)]$$

if $\sigma=2\tau+1$ and to

$$\begin{aligned}&[(0<p_{\sigma-1}+1<p_{\sigma-3}+2<\dots<p_3+\tau-1<p_1+\tau),\\&(p_\sigma<p_{\sigma-2}+1<\dots<p_2+\tau-1)]\end{aligned}$$

if $\sigma=2\tau$. We argue by induction on k. If $k=1$ we have $\sigma=n$ and the result follows from (a),(b). If $k>1$ then

$$j_{S_{\bar p_k}\times\dots\times S_{\bar p_2}\times S_{\bar p_1}}^{W_n}(\mathrm{sgn})=j_{S_{\bar p_k}\times W_{n-\bar p_k}}^{W_n}(\mathrm{sgn}\boxtimes j_{S_{\bar p_{k-1}}\times\dots\times S_{\bar p_2}\times S_{\bar p_1}}^{W_{n-\bar p_k}}(\mathrm{sgn}))$$

and the result follows from (d),(e) using the induction hypothesis and the transitivity of the j-induction.

We write $\bar p_1=a+b$ where $b-a\in\{0,1\}$. We show that

(h) $j_{S_{p'_k}\times\dots\times S_{p'_2}\times W_a\times W'_b}^{W_n}(\mathrm{sgn})$ is equal to

$[(p_\sigma<p_{\sigma-2}+1<\dots<p_3+\tau-1<p_1+\tau),(p_{\sigma-1}<p_{\sigma-3}+1<\dots<p_2+\tau-1)]$

if $\sigma=2\tau+1$ and to

$$\begin{aligned}&[(0<p_{\sigma-1}+1<p_{\sigma-3}+2<\dots<p_3+\tau-1<p_1+\tau),\\&(p_\sigma<p_{\sigma-2}+1<\dots<p_2+\tau-1)]\end{aligned}$$

if $\sigma=2\tau$.

Using (a),(b) and the transitivity of j-induction we see that

$$\begin{aligned}&j_{S_{\bar p_k}\times\dots\times S_{\bar p_2}\times W_a\times W'_b}^{W_n}(\mathrm{sgn})=j_{S_{\bar p_k}\times\dots\times\dots\times S_{\bar p_2}\times W_{\bar p_1}}^{W_n}(j_{S_{\bar p_k}\times\dots\times S_{\bar p_2}\times W_a\times W'_b}^{S_{\bar p_k}\times\dots\times S_{\bar p_2}\times W_{\bar p_1}}(\mathrm{sgn}))\\&=j_{S_{\bar p_k}\times\dots\times S_{\bar p_2}\times W_{\bar p_1}}^{W_n}(j_{S_{\bar p_k}\times\dots\times S_{\bar p_2}\times S_{\bar p_1}}^{S_{\bar p_k}\times\dots\times S_{\bar p_2}\times W_{\bar p_1}}(\mathrm{sgn}))=j_{S_{\bar p_k}\times\dots\times S_{\bar p_2}\times S_{\bar p_1}}^{W_n}(\mathrm{sgn}))\end{aligned}$$

and it remains to use (g).

Assuming that $\sigma=2\tau$ we show that

$$j_{S_{\bar p_k}\times\dots\times S_{\bar p_2}\times W'_{\bar p_1/2}\times W'_{\bar p_1/2}}^{W'_n}(\mathrm{sgn})=[(p_{\sigma-1}+1<p_{\sigma-3}+2<\dots<p_1+\tau),$$
$$(p_\sigma-1<p_{\sigma-2}<\dots<p_4+\tau-3<p_2+\tau-2)]. \quad \text{(i)}$$

We argue by induction on k. If $k=1$ we have $\sigma=n$ and the result follows from (c). If $k>1$ then the left hand side of (i) is equal to

$$j_{S_{\bar p_k}\times W'_{n-\bar p_k}}^{W'_n}(\mathrm{sgn}\boxtimes j_{S_{\bar p_{k-1}}\times\dots\times S_{\bar p_2}\times W'_{\bar p_1/2}\times W'_{\bar p_1/2}}^{W'_{n-\bar p_k}}(\mathrm{sgn}))$$

and the result follows from (f) using the induction hypothesis and the transitivity of the j-induction.

3.3. Assume that we are in the setup of 1.3. Let $p_*=(p_1\ge p_2\ge\dots\ge p_\sigma)$ be as in 1.3. We consider a unipotent class γ in G such that any $u\in\gamma$ has Jordan blocks of sizes

(i) $2p_1,2p_2,\dots,2p_\sigma$.

We set $\sigma=2\tau+\kappa_\sigma$. We show:

$$\rho_\gamma=[(0<p_{\sigma-1}+1<p_{\sigma-3}+2<\dots<p_3+\tau-1<p_1+\tau),$$
$$(p_\sigma<p_{\sigma-2}+1<\dots<p_4+\tau-2<p_2+\tau-1)] \text{ if } \kappa_\sigma=0,$$
$$\rho_\gamma=[(p_\sigma<p_{s-2}+1<\dots<p_3+\tau-1<p_1+\tau),$$
$$(p_{\sigma-1}<p_{\sigma-3}+1<\dots<p_4+\tau-2<p_2+\tau-1)] \text{ if } \kappa_\sigma=1. \quad \text{(a)}$$

To the partition (i) we will apply the procedure of [L2, 11.6]. Let $M=\sigma+\kappa_\sigma$. Let $z_M\ge\dots\ge z_2\ge z_1$ be the sequence (i) if $\kappa_\sigma=0$ and $2p_1,2p_2,\dots,2p_\sigma,0$ if $\kappa_\sigma=1$. The sequence $z'_M>\dots>z'_2>z'_1$ in *loc.cit.* is

$2p_1+\sigma-1,2p_2+\sigma-2,\dots,2p_\sigma$ (if $\kappa_\sigma=0$),

$2p_1+\sigma,2p_2+\sigma-1,\dots,2p_\sigma+1,0$ (if $\kappa_\sigma=1$).

This contains $M/2$ even numbers $2y_{M/2}>\dots>2y_2>2y_1$ given by

$\{2p_t+\sigma-t;t\in[1,\sigma],\kappa_t=0\}$ (if $\kappa_\sigma=0$),

$\{2p_t+\sigma-t+1;t\in[1,\sigma],\kappa_t=0\}\sqcup\{0\}$ (if $\kappa_\sigma=1$)

and $M/2$ odd numbers $2y'_{M/2}+1>\dots>2y'_2+1>2y'_1+1$ given by

$\{2p_t+\sigma-t+\kappa_\sigma;t\in[1,\sigma],\kappa_t=1\}$.

Thus, the sets $(\{y'_{M/2}>\dots>y'_2>y'_1\},\{y_{M/2}>\dots>y_2>y_1\})$ are given by

$(\{p_t+\tau-(t+1)/2;t\in[1,\sigma],\kappa_t=1\},\{p_t+\tau-t/2;t\in[1,\sigma],\kappa_t=0\})$ (if $\kappa_\sigma=0$)

$(\{p_t+\tau+(1-t)/2;t\in[1,\sigma],\kappa_t=1\},\{p_t+\tau-t/2+1;t\in[1,\sigma],\kappa_t=0\}\sqcup\{0\})$ (if $\kappa_\sigma=1$).

If $\kappa_\sigma=0$, the multisets

$(\{y'_\tau-(\tau-1)\ge\dots\ge y'_2-1\ge y'_1\ge 0\},\{y_\tau-(\tau-1)>\dots>y_2-1>y_1\})$

are given by

$(\{p_1 \ge p_3 \ge \dots \ge p_{\sigma-1} \ge 0\}, \{p_2 \ge p_4 \ge \dots \ge p_\sigma\})$.
If $\kappa_\sigma = 1$, the multisets
$(\{y'_{\tau+1} - \tau \ge \dots \ge y'_2 - 1 \ge y'_1 \ge 0\}, \{y_{\tau+1} - \tau \ge \dots \ge y_2 - 1 \ge y_1\})$
are given by
$(\{p_1 \ge p_3 \ge \dots \ge p_\sigma \ge 0\}, \{p_2 \ge p_4 \ge \dots \ge p_{\sigma-1} \ge 0\})$.
Now (a) follows from [L2, §12]. Using (a) and 3.2(g) we see that

(b) $$\rho_g = j_{S_{\bar p_k} \times \dots \times S_{\bar p_2} \times S_{\bar p_1}}^{W_n}(\mathrm{sgn}).$$

3.4. Assume that we are in the setup of 1.4. Let $p_* = (p_1 \ge p_2 \ge \dots \ge p_\sigma)$ be as in 1.4. Define $\psi : [1, \sigma] \to \{-1, 0, 1\}$ by $\psi(t) = 1$ if t is odd and $p_{t-1} > p_t$ (the last condition is regarded as satisfied when $t = 1$); $\psi(t) = -1$ if t is even and $p_t > p_{t+1}$ (the last condition is regarded as satisfied when $t = \sigma$); $\psi(t) = 0$ for all other t. We set $\sigma = 2\tau + \kappa_\sigma$. If $\mathbf{n} = 2n$ we assume that $\sigma = 2\tau$. We consider a unipotent class γ in G such that any $u \in \gamma$ has Jordan blocks of sizes
(i) $2p_1 + \psi(1), 2p_2 + \psi(2), \dots, 2p_\sigma + \psi(\sigma)$ if $\mathbf{n} = 2n$ (hence $\kappa_\sigma = 0$),
(ii) $2p_1 + \psi(1), 2p_2 + \psi(2), \dots, 2p_\sigma + \psi(\sigma)$ if $\mathbf{n} = 2n + 1$ and $\kappa_\sigma = 1$,
(iii) $2p_1 + \psi(1), 2p_2 + \psi(2), \dots, 2p_\sigma + \psi(\sigma), 1$ if $\mathbf{n} = 2n + 1$ and $\kappa_\sigma = 0$.
We show:

$$\rho_\gamma = [(p_\sigma - 1 < p_{\sigma-2} < \dots < p_4 + \tau - 3 < p_2 + \tau - 2),$$
$$(p_{\sigma-1} + 1 < p_{\sigma-3} + 2 < \dots < p_3 + \tau - 1 < p_1 + \tau)], \text{ in case (i)},$$
$$\rho_\gamma = [(p_\sigma < p_{\sigma-2} + 1 < \dots < p_3 + \tau - 1 < p_1 + \tau),$$
$$(p_{\sigma-1} < p_{\sigma-3} + 1 < \dots < p_4 + \tau - 2 < p_2 + \tau - 1)] \text{ in case (ii)},$$
$$\rho_\gamma = [0 < p_{\sigma-1} + 1 < p_{\sigma-3} + 2 < \dots < p_3 + \tau - 1 < p_1 + \tau),$$
(a) $$(p_\sigma < p_{\sigma-2} + 1 < \dots < p_4 + \tau - 2 < p_2 + \tau - 1)] \text{ in case (iii)}.$$

To the partition (i),(ii) or (iii) we will apply the procedure of [L2, 11.7]. Let $z_M \ge \dots \ge z_2 \ge z_1$ be the sequence (i),(ii) or (iii) (where $M = \sigma$ in cases (i),(ii) and $M = \sigma + 1$ in case (iii)). The sequence $z'_M > \dots > z'_2 > z'_1$ in *loc.cit.* is
$2p_1 + \psi(1) + \sigma - 1, 2p_2 + \psi(2) + \sigma - 2, \dots, 2p_\sigma + \psi(\sigma)$ (in cases (i),(ii)),
$2p_1 + \psi(1) + \sigma, 2p_2 + \psi(2) + \sigma - 1, \dots, 2p_\sigma + \psi(\sigma) + 1, 1$ (in case (iii)).
This contains $[M/2]$ even numbers $2y_{[M/2]} > \dots > 2y_2 > 2y_1$ given by
$\{2p_t + \psi(t) + \sigma - t; t \in [1, \sigma], \kappa_t = \kappa_{\psi(t)}\}$ in case (i),
$\{2p_t + \psi(t) + \sigma - t; t \in [1, \sigma], \kappa_t \ne \kappa_{\psi(t)}\}$ in case (ii),
$\{2p_t + \psi(t) + \sigma - t + 1; t \in [1, \sigma], \kappa_t \ne k_{\psi(t)}\}$ in case (iii),
and $[(M+1)/2]$ odd numbers $2y'_{[(M+1)/2]} + 1 > \dots > 2y'_2 + 1 > 2y'_1 + 1$ given by
$\{2p_t + \psi(t) + \sigma - t; t \in [1, \sigma], \kappa_t \ne \kappa_{\psi(t)}\}$ in case (i),
$\{2p_t + \psi(t) + \sigma - t; t \in [1, \sigma], \kappa_t = \kappa_{\psi(t)}\}$ in case (ii),
$\{2p_t + \psi(t) + \sigma - t + 1; t \in [1, \sigma], \kappa_t = \kappa_{\psi(t)}\} \sqcup \{1\}$ in case (iii).
Thus, the sets $(\{y'_{[(M+1)/2]} > \dots > y'_2 > y'_1\}, \{y_{[M/2]} > \dots > y_2 > y_1\})$ are given

by

$$
\begin{aligned}
&(\{p_t+\tau+(\psi(t)-t-1)/2; t\in[1,\sigma], \kappa_t\ne\kappa_{\psi(t)}\},\\
&\{p_t+\tau+(\psi(t)-t)/2; t\in[1,\sigma], \kappa_t=\kappa_{\psi(t)}\})\\
&=(\{p_t+\tau+(-t-2)/2; t\in[1,\sigma], \psi(t)=-1, \kappa_t=0\}\\
&\sqcup\{p_t+\tau+(-t-1)/2; t\in[1,\sigma], \psi(t)=0, \kappa_t=1\},\\
&\{p_t+\tau+(1-t)/2; t\in[1,\sigma], \psi(t)=1, \kappa_t=1\}\\
&\sqcup\{p_t+\tau+(-t)/2; t\in[1,\sigma], \psi(t)=0, \kappa_t=0\})\\
&=(\{p_t+\tau+(-t-2)/2; t\in[1,\sigma], \psi(t)=-1, \kappa_t=0\}\\
&\sqcup\{p_{t'}+\tau+(-t'-2)/2; t'\in[1,\sigma], \psi(t')=0, \kappa_{t'}=0\},\\
&\{p_t+\tau+(1-t)/2; t\in[1,\sigma], \psi(t)=1, \kappa_t=1\}\\
&\sqcup\{p_{t'}+\tau+(1-t')/2; t'\in[1,\sigma], \psi(t')=0, \kappa_{t'}=1\})\\
&=(\{p_t+\tau+(-t-2)/2; t\in[1,\sigma], \kappa_t=0\}\\
&\sqcup\{p_t+\tau+(1-t)/2; t\in[1,\sigma], \kappa_t=1\})
\end{aligned}
$$

in case (i),

$$
\begin{aligned}
&(\{p_t+\tau+(\psi(t)-t)/2; t\in[1,\sigma], \kappa_t=\kappa_{\psi(t)}\},\\
&\{p_t+\tau+(\psi(t)+1-t)/2; t\in[1,\sigma], \kappa_t\ne\kappa_{\psi(t)}\})\\
&=(\{p_t+\tau+(1-t)/2; t\in[1,\sigma], \psi(t)=1, \kappa_t=1\}\\
&\sqcup\{p_t+\tau-t/2; t\in[1,\sigma], \psi(t)=0, \kappa_t=0\},\\
&\{p_t+\tau-t/2; t\in[1,\sigma], \psi(t)=-1, \kappa_t=0\}\\
&\sqcup\{p_t+\tau+(1-t)/2; t\in[1,\sigma], \psi(t)=0, \kappa_t=1\})\\
&=(\{p_t+\tau+(1-t)/2; t\in[1,\sigma], \psi(t)=1, \kappa_t=1\}\\
&\sqcup\{p_{t'}+\tau+(1-t')/2; t'\in[1,\sigma], \psi(t')=0, \kappa_{t'}=1\},\\
&\{p_t+\tau-t/2; t\in[1,\sigma], \psi(t)=-1, \kappa_t=0\}\\
&\sqcup\{p_{t'}+\tau-t'/2; t'\in[1,\sigma], \psi(t')=0, \kappa_{t'}=0\})\\
&=(\{p_t+\tau+(1-t)/2; t\in[1,\sigma], \kappa_t=1\},\\
&\{p_t+\tau-t/2; t\in[1,\sigma], \kappa_t=0\})
\end{aligned}
$$

in case (ii),

$$\begin{aligned}
&(\{p_t+\tau+(\psi(t)-t)/2; t\in[1,\sigma], \kappa_t=\kappa_{\psi(t)}\}\sqcup\{0\},\\
&\{p_t+\tau+(\psi(t)+1-t)/2; t\in[1,\sigma], \kappa_t\ne\kappa_{\psi(t)}\})=\\
&(\{p_t+\tau+(1-t)/2; t\in[1,\sigma], \psi(t)=1, \kappa_t=1\}\sqcup\{0\}\\
&\sqcup\{p_t+\tau-t/2; t\in[1,\sigma], \psi(t)=0, \kappa_t=0\}\sqcup\{0\},\\
&\{p_t+\tau-t/2; t\in[1,\sigma], \psi(t)=-1, \kappa_t=0\}\\
&\sqcup\{p_t+\tau+(1-t)/2; t\in[1,\sigma], \psi(t)=0, \kappa_t=1\})\\
&=(\{p_t+\tau+(1-t)/2; t\in[1,\sigma], \psi(t)=1, \kappa_t=1\}\sqcup\{0\}\\
&\sqcup\{p_{t'}+\tau+(1-t')/2; t'\in[1,\sigma], \psi(t')=0, \kappa_{t'}=1\}\sqcup\{0\},\\
&\{p_t+\tau-t/2; t\in[1,\sigma], \psi(t)=-1, \kappa_t=0\}\\
&\sqcup\{p_{t'}+\tau-t'/2; t\in[1,\sigma], \psi(t')=0, \kappa_{t'}=0\})\\
&=(\{p_t+\tau+(1-t)/2; t\in[1,\sigma], \kappa_t=1\}\sqcup\{0\},\\
&\{p_t+\tau-t/2; t\in[1,\sigma], \kappa_t=0\}\})
\end{aligned}$$

in case (iii). (We have used that, if $\kappa_t=0$, $t<\sigma$ and $\psi(t)=0$, then $ps(t+1)=\psi(t), p_{t+1}=p_t$; if $\kappa_t=1$ and $\psi(t)=0$ then $\psi(t-1)=\psi(t), p_{t-1}=p_t$.)

In case (i) the multisets

$$(\{y'_\tau-\tau+1\ge\dots\ge y'_2-1\ge y'_1\},\{y_\tau-\tau+1\ge\dots>y_2-1\ge y_1\})$$

are given by

$$(\{p_2-1\ge p_4-1\ge\dots\ge p_\sigma-1\},\{p_1+1\ge p_3+1\ge\dots\ge p_{\sigma-1}+1\}).$$

In case (ii) the multisets

$$(\{y'_{\tau+1}-\tau\ge\dots\ge y'_2-1\ge y'_1\},\{y_\tau-\tau+1\ge\dots\ge y_2-1\ge y_1\})$$

are given by

$$(\{p_1\ge p_3\ge\dots\ge p_\sigma\},\{p_2\ge p_4\ge\dots\ge p_{\sigma-1}\}).$$

In case (iii) the multisets

$$(\{y'_{\tau+1}-\tau\ge\dots\ge y'_2-1\ge y'_1\},\{y_\tau-\tau+1\ge\dots\ge y_2-1\ge y_1\})$$

are given by

$$(\{p_1\ge p_3\ge\dots\ge p_{\sigma-1}\ge 0\},\{p_2\ge p_4\ge\dots\ge p_\sigma\}).$$

Now (a) follows from [L2, §13]. Using (a) and 3.2(i),(h), we see that

$$\rho_\gamma=j_{S_{\bar p_k}\times\dots\times S_{\bar p_2}\times W'_{\bar p_1/2}\times W'_{\bar p_1/2}}^{W'_n}(\text{sgn}) \text{ in case (i);}$$

$$\rho_\gamma=j_{S_{\bar p_k}\times\dots\times S_{\bar p_2}\times W_{(\bar p_1-\kappa_\sigma)/2}\times W'_{(\bar p_1+\kappa_\sigma)/2}}^{W_n}(\text{sgn})$$

(b) in case (ii),(iii).

3.5. We prove Theorem 0.6(ii). If G is of type A the result is immediate. If G is of classical type other than A, the result follows from 3.3(b) and 3.4(b). If G is of exceptional type, the result follows from the tables in 2.3 using the data on j-induction available in the CHEVIE package [C]. This completes the proof of Theorem 0.6.

3.6. Let w_{p_*} be the conjugacy class in W_n (as in 3.1) associated to $p_* = (p_1 \ge p_2 \ge \dots \ge p_\sigma)$ (where $p_1 + p_2 + \dots + p_\sigma = n$) as in [L5, 1,6]. We can view w_{p_*} as an element of $\mathbf{W}$ in the cases where G is as in 1.2 with either $Q = 0, \kappa = 0$ or with $Q \ne 0, \kappa = 1$. In both cases w_{p_*} has minimal length in its conjugacy class C which is in $\underline{\mathbf{W}}_{el}$. Let γ_C, γ'_C be the corresponding C-small unipotent classes (one is in a symplectic group, one is in an odd orthogonal group). From 3.3(b), 3.4(b), we see, using 3.2(g),(h), that

(a) *the Springer representations ρ_{γ_C}, $\rho_{\gamma'_C}$ are the same.*

We see that the map $C \mapsto \rho_{\gamma_C}$ from $\underline{\mathbf{W}}_{el}$ to $\mathrm{Irr}(\mathbf{W})$ depends only on the Weyl group $\mathbf{W}$, not on the underlying root system.

4. The variety of G-orbits on $\mathfrak{B}_w$

4.1. Let $C \in \underline{\mathbf{W}}_{el}$ and let $w \in C_{min}$. Let $U^* = U_{B^*}$. Let $U^*_w = U^* \cap \dot{w} U^* \dot{w}^{-1}$, $\mathcal{T}^w = \{t \in \mathcal{T}; \dot{w}t = t\dot{w}\}$. Let

$$\tilde{\mathfrak{B}}_w = \{(g, xU^*_w) \in G \times (G/U^*_w); x^{-1}gx \in \dot{w}U^*\},$$

a closed subvariety of $G \times (G/U^*_w)$. Now G acts on $\tilde{\mathfrak{B}}_w$ by $g_1 : (g, xU^*_w) \mapsto (g_1 g g_1^{-1}, g_1 x U^*_w)$ and $\mathcal{T}^w$ acts (freely) on $\tilde{\mathfrak{B}}_w$ by $t : (g, xU^*_w) \mapsto (g, xt^{-1}U^*_w)$; these two actions commute. Define $\pi_w : \tilde{\mathfrak{B}}_w \to \mathfrak{B}_w$ by $(g, xU^*_w) \mapsto (g, xB^*x^{-1})$. It is easy to see that, if G is semisimple, then $\mathcal{T}^w$ is a finite group and π_w is a finite principal covering with group $\mathcal{T}^w$. (In this case, the homomorphism $\mathcal{T} \to \mathcal{T}$, $t \mapsto t^{-1}\dot{w}t\dot{w}^{-1}$ is surjective, since it has finite kernel $\mathcal{T}^w$.) Note that G acts on $\mathfrak{B}_w$ by $g_1 : (g, B) \mapsto (g_1 g g_1^{-1}, g_1 B g_1^{-1})$ and that π_w is compatible with the G-actions. Note that U^*_w acts on U^* by $u_1 : u \mapsto \dot{w}^{-1}u_1\dot{w}uu_1^{-1}$.

Let $G\backslash\mathfrak{B}_w$, $G\backslash\tilde{\mathfrak{B}}_w$, $U^*_w\backslash\backslash U^*$ be the set of orbits of the G-actions on $\mathfrak{B}_w, \tilde{\mathfrak{B}}_w$ or of the U^*_w-action on U^*. Now the $\mathcal{T}^w$-action on $\tilde{\mathfrak{B}}_w$ induces a $\mathcal{T}^w$-action on $G\backslash\tilde{\mathfrak{B}}_w$; let $\mathcal{T}^w\backslash(G\backslash\tilde{\mathfrak{B}}_w)$ be the set of orbits of this action. Also, $\mathcal{T}^w$ acts on U^* by conjugation and this induces an action of $\mathcal{T}^w$ on $U^*_w\backslash\backslash U^*$; let $\mathcal{T}^w\backslash(U^*_w\backslash\backslash U^*)$ be the set of orbits of this action. Note that $u \mapsto (\dot{w}u, U^*_w)\}$ induces a bijection

(a) $U^*_w\backslash\backslash U^* \xrightarrow{\sim} G\backslash\tilde{\mathfrak{B}}_w$.

This induces a bijection $\mathcal{T}^w\backslash(U^*_w\backslash\backslash U^*) \xrightarrow{\sim} \mathcal{T}^w\backslash(G\backslash\tilde{\mathfrak{B}}_w)$. If G is semisimple then π induces a bijection $\mathcal{T}^w\backslash(G\backslash\tilde{\mathfrak{B}}_w) \xrightarrow{\sim} G\backslash\mathfrak{B}_w$; combining with the previous bijection we obtain in this case a bijection

(b) $\mathcal{T}^w\backslash(U^*_w\backslash\backslash U^*) \xrightarrow{\sim} G\backslash\mathfrak{B}_w$.

We have the following result.

Proposition 4.2. *(i) The isotropy groups of the G-action 4.1 on $\tilde{\mathfrak{B}}_w$ are $\{1\}$.*

(ii) The isotropy groups of the U^*_w-action 4.1 on U^* are $\{1\}$.

(iii) The variety $\mathfrak{B}_w$ is affine.

(iv) If G is semisimple, the variety $\tilde{\mathfrak{B}}_w$ is affine.

We prove (i). Let g_1 be an element of G such that $(g, xU^*_w) = (g_1gg_1^{-1}, g_1xU^*_w)$ for some $(g, xU^*_w) \in \tilde{\mathfrak{B}}_w$. Then $g_1gg_1^{-1} = g$, $x^{-1}g_1x \in U^*_w$. We have
$(g_1gg_1^{-1}, g_1xB_0x^{-1}g^{-1}) = (g, xB_0x^{-1}) \in \mathfrak{B}_w$
hence from [L5, 5.2] we see that the image of g_1 in G/Z_G is of finite order invertible in $\mathbf{k}$. Hence g_1 is semisimple. Since $x^{-1}g_1x \in U^*_w$ we see that g_1 is also unipotent. Hence $g_1 = 1$ and (i) is proved.

We prove (ii). Let $u_1 \in U^*_w, u \in U^*$ be such that $\dot{w}^{-1}u_1\dot{w}uu_1^{-1} = u$. We must show that $u_1 = 1$. Note that $(\dot{w}u, U^*_w) \in \tilde{\mathfrak{B}}_w$ and $(u_1\dot{w}uu_1^{-1}, u_1U^*_w) = (\dot{w}u, U^*_w)$. Thus u_1 is in the isotropy group of $(\dot{w}u, U^*_w)$ for the G-action on $\tilde{\mathfrak{B}}_w$. Using (i) we have $u_1 = 1$ and (ii) is proved.

We prove (iii) by a method inspired by [BR]. As in the proof of [L5, 5.2] we can assume that w is good in the sense of Geck and Michel. Let Y be the set of all sequences $(B_0, B_1, \dots, B_d) \in \mathcal{B}^{d+1}$ such that $(B_{i-1}, B_i) \in \mathcal{O}_w$ for $i \in [1, d]$. By [BR, Proposition 3], Y is an affine variety. Hence $G \times Y$ is an affine subvariety of $G \times \mathcal{B}^{d+1}$. Let Y' be the set of all $(g, B_0, B_1, \dots, B_d) \in G \times \mathcal{B}^{d+1}$ such that $B_i = g^iB_0g^{-i}$ for $i \in [1, d]$; this is a closed subvariety of $G \times \mathcal{B}^{d+1}$. Hence $(G \times Y) \cap Y'$ is a closed subvariety of $G \times Y$ so that it is affine. The map $\mathfrak{B}_w \to Y'$ given by $(g, B) \mapsto (g, B, gBg^{-1}, g^2Bg^{-2}, \dots, g^dBg^{-d})$ is an isomorphism of $\mathfrak{B}_w$ onto $(G \times Y) \cap Y'$. Hence $\mathfrak{B}_w$ is affine, as required.

(iv) Since $\tilde{\mathfrak{B}}_w$ is a principal bundle over $\mathfrak{B}_w$ with group $\mathcal{T}^w$ (a finite group) and $\mathfrak{B}_w$ is affine (see (iii)) we see that $\tilde{\mathfrak{B}}_w$ is affine.

4.3. Assume that G is semisimple. From 4.2(i),(iv) we see that all G-orbits on the affine variety $\tilde{\mathfrak{B}}_w$ are closed hence the set of G-orbits on $\tilde{\mathfrak{B}}_w$ has a natural structure of an affine variety. Using 4.1(a) we may identify this affine variety with $U^*_w\backslash\backslash U^*$. Using 4.2(ii) we see that this affine variety is something like an affine space of dimension $\underline{l}(w)$. Using 4.1(b) we see also that the set of G-orbits on $\mathfrak{B}_w$ is an affine variety of dimension $\underline{l}(w)$ which is something like the quotient of an affine space by the action of the finite group $\mathcal{T}^w$.

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