UNIPOTENT ELEMENTS IN SMALL CHARACTERISTIC, IV

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Introduction

Let $\mathbf{k}$ be an algebraically closed field of characteristic exponent $p \ge 1$. Let G be a connected reductive algebraic group over $\mathbf{k}$ and let $\mathfrak{g}$ be the Lie algebra of G. Note that G acts on G and on $\mathfrak{g}$ by the adjoint action and on $\mathfrak{g}^*$ by the coadjoint action. (For any $\mathbf{k}$-vector space V we denote by V^* the dual vector space.) Let $G_{\mathbf{C}}$ be the reductive group over $\mathbf{C}$ of the same type as G. Let $\mathcal{U}_G$ be the variety of unipotent elements of G. Let $\mathcal{N}_{\mathfrak{g}}$ be the variety of nilpotent elements of $\mathfrak{g}$. Let $\mathcal{N}_{\mathfrak{g}^*}$ be the variety of nilpotent elements of $\mathfrak{g}^*$ (following [KW] we say that a linear form $\xi : \mathfrak{g} \to \mathbf{k}$ is nilpotent if its kernel contains some Borel subalgebra of $\mathfrak{g}$). In [L1, L2, L3] we have proposed a definition of a partition of $\mathcal{U}_G$ and of $\mathcal{N}_{\mathfrak{g}}$ into smooth locally closed G-stable pieces which are indexed by the unipotent classes in $G_{\mathbf{C}}$ and which in many ways depend very smoothly on p. In this paper we propose a definition of an analogous partition of $\mathcal{N}_{\mathfrak{g}^*}$ into pieces which are indexed by the unipotent classes in $G_{\mathbf{C}}$. (This definition is only of interest for $p > 1$, small; for $p = 1$ or p large we can identify $\mathcal{N}_{\mathfrak{g}}$ with $\mathcal{N}_{\mathfrak{g}^*}$ and the partition of $\mathcal{N}_{\mathfrak{g}^*}$ is deduced from the partition of $\mathcal{N}_{\mathfrak{g}}$.) We will illustrate this in the case where G is of type A, C or D and p is arbitrary. (We do not treat the case where G is of type B which seems to be more complicated.)

Notation. If f is a permutation of a set X we denote $X^f = \{x \in X; f(x) = x\}$. The cardinal of a finite set X is denoted by $|X|$. For any subspace U of $\mathfrak{g}$ let $\mathrm{Ann}(U) = \{\xi \in \mathfrak{g}^*; \xi|_U = 0\}$.

1. Preliminaries

1.1. Let V be a $\mathbf{k}$-vector space of finite dimension. Let $G = GL(V)$. We have $\mathfrak{g} = \mathrm{End}(V)$. We have an isomorphism

(a) $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$

given by $X \mapsto [T \mapsto \mathrm{tr}(TX, V)]$.

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1.2. Let V be a $\mathbf{k}$-vector space of finite even dimension with a fixed nondegenerate quadratic form $Q : V \longrightarrow \mathbf{k}$. Let $(,)$ be the (nondegenerate) symmetric bilinear form $V \times V \longrightarrow \mathbf{k}$ given by $(x, y) = Q(x+y) - Q(x) - Q(y)$ for $x, y \in V$. Let $G = SO(V)$ be the special orthogonal group of Q. We have $\mathfrak{g} = \{T \in \text{End}(V); (Tx, x) = 0 \quad \forall x \in V\}$. Let $\mathfrak{S}(V)$ be the vector space consisting of all symplectic forms $V \times V \longrightarrow \mathbf{k}$. The following result is easily verified.

(a) *We have a vector space isomorphism* $\mathfrak{g} \xrightarrow{\sim} \mathfrak{S}(V)$, $T \mapsto [x, y \mapsto (Tx, y)]$. *This is compatible with the* $SO(V)$*-actions where* $SO(V)$ *acts on* $\mathfrak{S}(V)$ *by the restriction of the obvious* $GL(V)$*-action.*

By taking transpose we obtain an isomorphism $\mathfrak{S}(V)^* \xrightarrow{\sim} \mathfrak{g}^*$ compatible with the natural $SO(V)$-actions. We can find an isomorphism $\mathfrak{S}(V^*) \xrightarrow{\sim} \mathfrak{S}(V)^*$ compatible with the natural $GL(V)$-actions and, by restriction, with the natural $SO(V)$-actions. Now $(,)$ defines an isomorphism $V \longrightarrow V^*$ hence an isomorphism $\mathfrak{S}(V) \xrightarrow{\sim} \mathfrak{S}(V^*)$ compatible with the natural $SO(V)$-actions. The composition $\mathfrak{g} \xrightarrow{\sim} \mathfrak{S}(V) \xrightarrow{\sim} \mathfrak{S}(V^*) \xrightarrow{\sim} \mathfrak{S}(V)^* \xrightarrow{\sim} \mathfrak{g}^*$ is an isomorphism

(b) $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$

compatible with the $SO(V)$-actions.

1.3. Let V be a $\mathbf{k}$-vector space of finite (even) dimension with a fixed nondegenerate symplectic form $(,) : V \times V \longrightarrow \mathbf{k}$. Let $G = Sp(V)$ be the symplectic group of $(,)$. We have $\mathfrak{g} = \{T \in \text{End}(V); (Tx, y) + (x, Ty) = 0 \quad \forall x, y \in V\}$. Let $\mathfrak{Q}(V)$ be the vector space consisting of all quadratic forms $V \longrightarrow \mathbf{k}$. According to T. Xue [X]:

(a) *we have a natural vector space isomorphism* $\sigma_V : \mathfrak{g}^* \xrightarrow{\sim} \mathfrak{Q}(V)$.

Indeed, let $Z = \{X \in \text{End}(V); (Xa, a) = 0 \forall a \in V\}$. We have a diagram $\mathfrak{Q}(V) \xleftarrow{\alpha} \text{End}(V)/Z \xrightarrow{\beta} \mathfrak{g}^*$ (α is induced by $X \mapsto [a \mapsto (Xa, a)]$; β is induced by $X \mapsto [T \mapsto \text{tr}(TX, V)]$). Now α, β are isomorphisms and we set $\sigma_V = \alpha\beta^{-1}$.

We define a linear map $\mathfrak{Q}(V) \longrightarrow \mathfrak{g}, Q \mapsto A_Q$ by

$Q(x + y) - Q(x) - Q(y) = (A_Q x, y)$ for all $x, y \in V$.

If $p = 2$, then for $Q \in \mathfrak{Q}(V)$ we have $A_Q \in \mathfrak{Q}'(V)$ where $\mathfrak{Q}'(V) = \{A \in \text{End}(V); (Ax, x) = 0 \quad \forall x \in V\}$. Let $\mathfrak{Q}(V)_{nil} = \{Q \in \mathfrak{Q}(V); A_Q \text{ is nilpotent}\}$.

1.4. In this subsection we assume that $\mathbf{k}$ is an algebraic closure of a finite field $\mathbf{F}_q$ and that we are given an $\mathbf{F}_q$-rational structure on G. Then $\mathfrak{g}, \mathfrak{g}^*, \mathcal{N}_\mathfrak{g}, \mathcal{N}_{\mathfrak{g}^*}$ have induced $\mathbf{F}_q$-structures. For any $\mathbf{k}$-variety X with an $\mathbf{F}_q$-structure we denote by $F : X \longrightarrow X$ the corresponding Frobenius map. Let N be the number of roots of G. According to [S] we have

(a) $|\mathcal{N}_\mathfrak{g}^F| = q^N$.

We now state the following result.

(b) *If the adjoint group of* G *is simple of type* A, C *or* D, *then* $|\mathcal{N}_{\mathfrak{g}^*}^F| = q^N$.

We can assume that G is a general linear group, a symplectic group or an even special orthogonal group. The proof in these cases will be given in the remainder of this section. We will show elsewhere that (b) holds without any assumption on the type of G.

1.5. We preserve the setup of 1.4. Assume that there exists a G-equivariant vector space isomorphism $\iota : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ compatible with the $\mathbf{F}_q$-structures. It is easy to see that ι restricts to a bijection $\mathcal{N}_\mathfrak{g} \xrightarrow{\sim} \mathcal{N}_{\mathfrak{g}^*}$. Hence $|\mathcal{N}_\mathfrak{g}^F| = |\mathcal{N}_{\mathfrak{g}^*}^F|$ and 1.4(b) follows from 1.4(a). In particular, if G is a general linear group or an even special orthogonal group or a symplectic group (with $p \neq 2$) then ι as above exists (see 1.1(a), 1.2(b)) and 1.4(b) holds in these cases.

1.6. In this subsection we assume that $V, (,), G, \mathfrak{g}$ are as in 1.3. We set $2r = \dim V$.

Assume first that $p = 2$. Let $\xi \in \mathfrak{g}^*$, let $Q \in \mathfrak{Q}(V)$ be the element corresponding to ξ under 1.3(a) and let $A = A_Q$. The following result is due to T. Xue [X].

(a) *If $\xi \in \mathcal{N}_{\mathfrak{g}^*}$ then $A : V \to V$ is nilpotent.*

Let $H_r = \{i \in \mathbf{Z}; -2r+1 \le i \le 2r-1, i = \text{odd}\}$. A basis $(e_i)_{i \in H_r}$ of V is said to be good if $(e_i, e_j) = \delta_{i+j,0} = 0$ for all $i, j \in H_r$. We can find a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that $\xi|_\mathfrak{b} = 0$. We can find a good basis $(e_i)_{i \in H_r}$ of V such that $\mathfrak{b}$ consists of all $T \in \text{End}(V)$ with $Te_i = \sum_{j \in H_r; i \le j} t_{ij} e_j$ for all $i \in H_r$ and $t_{ij} + t_{-j,-i} = 0$ for all $i, j \in H_r$ ($t_{ij} \in \mathbf{k}$). We can find $X \in \text{End}(V)$ such that $\text{tr}(XT, V) = \xi(T)$ for all $T \in \mathfrak{g}$. Define $x_{ij} \in \mathbf{k}$ by $Xe_i = \sum_{j \in H_r} x_{ij} e_j$ for all $i \in H_r$. Then $\sum_{i,j \in H_r} x_{ij} t_{ji} = 0$ for any t_{ij}, ($j \le i$) such that $t_{ij} + t_{-j,-i} = 0$ for all i, j. It follows that $x_{ij} + x_{-j,-i} = 0$ for any $j \le i$ and $x_{i,-i} = 0$ for any $i \ge 0$. Define $y_{ij} \in \mathbf{k}$ by $Ae_i = \sum_{j \in H_r} y_{ij} e_j$ for all $i \in H_r$. We have $(Ae_i, e_j) = (Xe_i, e_j) + (e_i, Xe_j)$, $Q(e_i) = (Xe_i, e_i)$. Hence $y_{ij} = x_{ij} + x_{-j,-i}$, $Q(e_i) = x_{i,-i}$. For $j \le i$ we have $x_{ij} + x_{-j,-i} = 0$ hence $y_{ij} = 0$. Thus A is nilpotent and (a) is proved.

We show a converse to (a):

(b) *If $A : V \to V$ is nilpotent then $\xi \in \mathcal{N}_{\mathfrak{g}^*}$.*

It is enough to verify the following statement:

(c) *Let $Q \in \mathfrak{Q}(V)$. Let $A = A_Q$. Assume that $A : V \to V$ is nilpotent. Then there exists a good basis $(e_i)_{i \in H_r}$ of V such that $Ae_i = \sum_{j \in H_r; i < j} y_{ij} e_j$ for all $i \in H_r$ ($y_{ij} \in \mathbf{k}$) and $Q(e_i) = 0$ for $i \ge 0$.*

(Indeed if (c) holds then as in the proof of (a) we can define a Borel subalgebra $\mathfrak{b}$ in terms of (e_i) and we have $\xi|_\mathfrak{b} = 0$.)

We prove (c) by induction on r. When $r = 0$ the result is trivial. Now assume that $r \ge 1$. Since $x, y \mapsto (Av, x)$ is a symplectic form on an even dimensional vector space, its radical has even dimension. Thus $\dim \ker A$ is even. Since A is nilpotent, its kernel is $\neq 0$ hence it has dimension ≥ 2. Now a quadratic form on a $\mathbf{k}$-vector space of dimension ≥ 2 vanishes at some non-zero vector. Thus there exists $v \in V - \{0\}$ such that $Av = 0$, $Q(v) = 0$. Let $(\mathbf{k}v)^\perp = \{v' \in V; (v', v) = 0\}$. Let $V' = (\mathbf{k}v)^\perp / \mathbf{k}v$. Then V' inherits a nondegenerate symplectic form $(,)'$ from $(,)$, a quadratic form Q' from Q and a nilpotent endomorphism A' from A. Note that $A' = A_{Q'}$. By the induction hypothesis there exists a good basis $(e'_i)_{i \in H_{r-1}}$ (relative to $(,)'$) such that $A'e'_i = \sum_{j \in H_{r-1}; i<j} y'_{ij} e'_j$ for all $i \in H_{r-1}$ ($y'_{ij} \in \mathbf{k}$) and $Q'(e'_i) = 0$ for $i \ge 0$. For $i \in H_{r-1}$ we denote by e_i a representative of e'_i in $(\mathbf{k}v)^\perp$. We have $Ae_i = \sum_{j \in H_{r-1}; i<j} y'_{ij} e_j + c_i v$ for all $i \in H_{r-1}$ ($c_i \in \mathbf{k}$) and

$Q(e_i)=0$ for all $i\in H_{r-1}$, $i\ge 0$. We set $e_{2r-1}=v$. Let e_{-2r+1} be the unique vector in V such that $(e_{-2r+1},e_j)=\delta_{2r-1,j}$. We have $Ae_{-2r+1}=\sum_i y_{-2r+1,i}e_i$ with $y_{-2r+1,i}\in\mathbf{k}$. Since $\text{tr}(A,V)=0$ we have $y_{-2r+1,-2r+1}=0$. Thus $(e_i)_{i\in H_r}$ has the required properties.

From (a),(b) we see that $\sigma_V:\mathfrak{g}^*\xrightarrow{\sim}\mathfrak{Q}(V)$ (see 1.3(a)) restricts to a bijection

(d) $\mathcal{N}_{\mathfrak{g}^*}\xrightarrow{\sim}\mathfrak{Q}(V)_{nil}$.

Note that (d) holds also when $p\ne 2$ (with a simpler proof).

1.7. We preserve the setup of 1.6 with $p=2$. We assume that $\mathbf{k},\mathbf{F}_q$ are as in 1.4 and that we are given an $\mathbf{F}_q$-structure on V compatible with $(,)$. Then $G,\mathfrak{g},\mathfrak{g}^*,\mathfrak{Q}(V),\mathfrak{Q}'(V)$ have natural $\mathbf{F}_q$-structures with Frobenius maps denoted by F. Note that $\mathfrak{Q}(V)\to\mathfrak{Q}'(V), Q\mapsto A_Q$ (whose kernel is the set of squares of linear forms on V) induces a map $\mathfrak{Q}(V)^F\to\mathfrak{Q}'(V)^F$ with fibres of cardinal q^{2r}. From 1.6 we see that

$$|\mathcal{N}_{\mathfrak{g}^*}^F|=|\mathfrak{Q}(V)_{nil}^F|=q^{2r}|\{A\in\mathfrak{Q}'(V)^F;A\text{ nilpotent}\}|=q^{2r}q^{2r^2-2r}=q^{2r^2}.$$

(The third equality follows from 1.4(a) applied to an even special orthogonal group.) This proves 1.4(b) in our case.

2. The main results

2.1. Let $\delta\in\mathfrak{D}_G$ (see [L3, 1.1]). Let $\mathfrak{g}=\oplus_{i\in\mathbf{Z}}\mathfrak{g}_i^\delta$ be the corresponding grading of $\mathfrak{g}$ (see [L3, 1.2]). For $j\in\mathbf{Z}$ let $\mathfrak{g}_j^{*\delta}=\text{Ann}(\oplus_{i;i\ne -j}\mathfrak{g}_i^\delta)$. We have $\mathfrak{g}^*=\oplus_{j\in\mathbf{Z}}\mathfrak{g}_j^{*\delta}$. Let $G_{\ge 0}^\delta$ be as in [L3, 1.2].

Since $\mathfrak{g}_{\ge -j+1}^\delta$ is $G_{\ge 0}^\delta$-stable we see that $\mathfrak{g}_{\ge j}^{*\delta}$ is $G_{\ge 0}^\delta$-stable. For any $\xi\in\mathfrak{g}^*$ let G_ξ be the stabilizer of ξ in G for the coadjoint action. Let

$$\mathfrak{g}_2^{*\delta!}=\{\xi\in\mathfrak{g}_2^{*\delta};G_\xi\subset G_{\ge 0}^\delta\}.$$

Let $\Delta\in D_G$ (see [L3, 2.1]). As in [L3, 2.1] we write $G_{\ge 0}^\Delta$, $\mathfrak{g}_{\ge i}^\Delta$ ($i\in\mathbf{N}$) instead of $G_{\ge 0}^\delta,\mathfrak{g}_{\ge i}^\delta$ (see [L3, 1.2]) where $\delta\in\Delta$. For $j\in\mathbf{N}$ we set

$$\mathfrak{g}_{\ge j}^{*\Delta}=\text{Ann}(\mathfrak{g}_{\ge -j+1}^\Delta).$$

We have also $\mathfrak{g}_{\ge j}^{*\Delta}=\oplus_{j'\in\mathbf{Z};j'\ge j}\mathfrak{g}_{j'}^{*\delta}$. Since $\mathfrak{g}_{\ge -j+1}^\Delta$ is $G_{\ge 0}^\Delta$-stable, we see that $\mathfrak{g}_{\ge j}^{*\Delta}$ is $G_{\ge 0}^\Delta$-stable.

For any $\delta\in\Delta$ we have an obvious isomorphism $\mathfrak{g}_2^{*\delta}\xrightarrow{\sim}\mathfrak{g}_{\ge 2}^{*\Delta}/\mathfrak{g}_{\ge 3}^{*\Delta}$. Via this isomorphism the subset $\mathfrak{g}_2^{*\delta!}$ of $\mathfrak{g}_2^{*\delta}$ can be viewed as a subset $\Sigma^{*\delta}$ of $\mathfrak{g}_{\ge 2}^{*\Delta}/\mathfrak{g}_{\ge 3}^{*\Delta}$. As in [L3, 2.3] we see that $\Sigma^{*\delta}$ is independent of the choice of δ in Δ; we will denote it by $\Sigma^{*\Delta}$. Note that $\Sigma^{*\Delta}$ is a subset of $\mathfrak{g}_{\ge 2}^{*\Delta}/\mathfrak{g}_{\ge 3}^{*\Delta}$ stable under the action of $G_{\ge 0}^\Delta$.

Let $\sigma^{*\Delta}\subset\mathfrak{g}_{\ge 2}^{*\Delta}$ be the inverse image of $\Sigma^{*\Delta}$ under the obvious map $\mathfrak{g}_{\ge 2}^{*\Delta}\to\mathfrak{g}_{\ge 2}^{*\Delta}/\mathfrak{g}_{\ge 3}^{*\Delta}$. Now $\sigma^{*\Delta}$ is stable under the coadjoint action of $G_{\ge 0}^\Delta$ on $\mathfrak{g}_{\ge 2}^{*\Delta}$ and $\xi\mapsto\xi$ is a map

$$\Psi_{\mathfrak{g}^*}:\sqcup_{\Delta\in D_G}\sigma^{*\Delta}\to\mathcal{N}_{\mathfrak{g}^*}.$$

Theorem 2.2. *Assume that the adjoint group of G is a product of simple groups of type A, C, D. Then $\Psi_{\mathfrak{g}^*}$ is a bijection.*

The general case reduces easily to the case where G is almost simple of type A, C or D. Moreover we can assume that G is a general linear group, a symplectic group or an even special orthogonal group. The proof in these cases will be given in 2.4, 2.13. We expect that the theorem holds without restriction on G.

2.3. In this subsection we assume that there exists a G-equivariant vector space isomorphism $\iota : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$. Let $\delta \in \mathfrak{D}_G$. For any $i \in \mathbf{Z}$ we have

(a) $\iota(\mathfrak{g}_i^\delta) = \mathfrak{g}_i^{*\delta}$.

Recall that $\mathfrak{g}_i^\delta = \{x \in \mathfrak{g}; \mathrm{Ad}(\delta(a))x = a^i x \quad \forall a \in \mathbf{k}^*\}$. Hence

$$\iota(\mathfrak{g}_i^\delta) = \{\xi \in \mathfrak{g}^*; \mathrm{Ad}(\delta(a))\xi = a^i \xi \quad \forall a \in \mathbf{k}^*\}.$$

If $j \in \mathbf{Z}$, $j \neq -i$ and $\xi \in \iota(\mathfrak{g}_i^\delta), x \in \mathfrak{g}_j^\delta$ then for $a \in \mathbf{k}^*$ we have

$$\xi(x) = a^{-i}(\mathrm{Ad}(\delta(a))\xi)(x) = a^{-i}\xi(\mathrm{Ad}(\delta(a)^{-1})x) = a^{-i}a^{-j}\xi(x) = a^{-i-j}\xi(x)$$

hence $\xi(x) = 0$. Thus $\iota(\mathfrak{g}_i^\delta) \subset \mathfrak{g}_i^{*\delta}$. Since $(\iota(\mathfrak{g}_i^\delta))$, $(\mathfrak{g}_i^{*\delta})$ form direct sum decompositions of $\mathfrak{g}^*$ it follows that (a) holds.

Let $\mathfrak{g}_2^{\delta !}$ be as in [L3, 1.2]. From the definitions we see that ι induces a bijection

(b) $\mathfrak{g}_2^{\delta !} \xrightarrow{\sim} \mathfrak{g}_2^{*\delta !}$.

Let $\Delta \in D_G$. Using (a) we see that for any $j \in \mathbf{N}$ we have

(c) $\iota(\mathfrak{g}_{\geq j}^\Delta) = \mathfrak{g}_{\geq j}^{*\Delta}$.

Now ι induces an isomorphism $\mathfrak{g}_{\geq 2}^\Delta / \mathfrak{g}_{\geq 3}^\Delta \xrightarrow{\sim} \mathfrak{g}_{\geq 2}^{*\Delta} / \mathfrak{g}_{\geq 3}^{*\Delta}$. This induces (using (b) and the definitions) a bijection $\Sigma^\Delta \xrightarrow{\sim} \Sigma^{*\Delta}$ (with Σ^Δ as in [L3, 2.3]) and a bijection $\sigma^\Delta \xrightarrow{\sim} \sigma^{*\Delta}$ (with σ^Δ as in [L3, A.1]).

We can find $\delta_0 \in \mathfrak{D}_G, \Delta_0 \in D_G$ such that $\delta_0 \in \Delta_0$ and $G_{\geq 0}^{\Delta_0}$ is a Borel subgroup of G. An element $\xi \in \mathfrak{g}^*$ is nilpotent if and only if for some $g \in G$ we have $\mathrm{Ad}(g)\xi \in \mathrm{Ann}(\mathfrak{g}_{\geq 0}^{\Delta_0})$ (which equals $\mathfrak{g}_{\geq 1}^{*\Delta_0} = \iota(\mathfrak{g}_{\geq 1}^{\Delta_0})$). We see that ι restricts to a bijection $\mathcal{N}_{\mathfrak{g}} \xrightarrow{\sim} \mathcal{N}_{\mathfrak{g}^*}$. Thus, if $\Psi_{\mathfrak{g}}$ (see [L3, A.1]) is a bijection, then $\Psi_{\mathfrak{g}^*}$ is a bijection.

2.4. In this subsection we assume that $G, \mathfrak{g}$ are as in 1.1 or as in 1.2. In both cases we can find an isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ compatible with the G-actions (see 1.1(a), 1.2(b)). Since $\Psi_{\mathfrak{g}}$ is a bijection (see [L3, A.2]) we see from 2.3 that Theorem 2.2 holds for G.

2.5. Let $V, (,), G, \mathfrak{g}, \mathfrak{Q}(V)$ be as in 1.3. Let $\mathfrak{Q}(V) \to \mathfrak{g}$, $Q \mapsto A_Q$ be as in 1.3.

We now fix a $\mathbf{Z}$-grading $V = \oplus_{i \in \mathbf{Z}} V_i$ which is s-good (as in [L3, 1.4]) that is, $\dim V_i = \dim V_{-i} \geq \dim V_{-i-2}$ for any $i \geq 0$, $\dim V_i$ is even for any even i and $(V_i, V_j) = 0$ whenever $i + j \neq 0$. For any i we set $V_{\geq i} = \oplus_{i'; i' \geq i} V_{i'}$.

Let $\mathfrak{Q}(V)_2$ be the vector space consisting of all $Q \in \mathfrak{Q}(V)$ such that $A_Q(V_i) \subset V_{i+2}$ for any i and $Q|_{V_i} = 0$ for any $i \neq -1$. Let $\mathfrak{Q}(V)_2^0$ be the set of all $Q \in \mathfrak{Q}(V)_2$ such that

(i) for any even $n \ge 0$, $A_Q^n : V_{-n} \longrightarrow V_n$ is an isomorphism;

(ii) for any odd $n \ge 1$, $A_Q^{(n-1)/2} : V_{-n} \longrightarrow V_{-1}$ is injective and the restriction of Q to $A_Q^{(n-1)/2}(V_{-n})$ is a nondegenerate quadratic form.
Let $\mathfrak{Q}(V)_{\ge 2}$ be the set of all $Q \in \mathfrak{Q}(V)$ such that $A_Q(V_{\ge i}) \subset V_{\ge i+2}$ for any i and $Q|_{V_{\ge 0}} = 0$. Note that $\mathfrak{Q}(V)_2 \subset \mathfrak{Q}(V)_{\ge 2} \subset \mathfrak{Q}(V)_{nil}$.

We show:

(a) *if $p \ne 2$, then $\mathfrak{Q}(V)_2^0$ is equal to the set S consisting of all $Q \in \mathfrak{Q}(V)_2$ such that $A_Q^n : V_{-n} \longrightarrow V_n$ is an isomorphism for any $n \ge 0$.*
Assume first that $Q \in \mathfrak{Q}(V)_2^0$. Let $n \ge 0$ and let $x \in V_{-n}$ be such that $A_Q^n(x) = 0$. If n is even then $x = 0$ by (i). If n is odd then let $x' = A_Q^{(n-1)/2}x \in V_{-1}$. For any $x_1 \in V_{-n}$ we have $(A_Q x', A_Q^{(n-1)/2}x_1) = \pm(A_Q^n x, x_1) = 0$ so that x' is in the radical of $Q|_{A_Q^{(n-1)/2}(V_{-n})}$; this radical is 0 using (ii) and the condition $p \ne 2$. Thus $x' = 0$ that is $A_Q^{(n-1)/2}x = 0$. Using the injectivity in (ii) we see that $x = 0$. Thus $A_Q^n : V_{-n} \longrightarrow V_n$ is injective for any $n \ge 0$ hence it is an isomorphism since $\dim V_n = \dim V_{-n}$. We see that $Q \in S$. Conversely assume that $Q \in S$. Assume that $n \ge 1$ is odd. Clearly $A_Q^{(n-1)/2} : V_{-n} \longrightarrow V_{-1}$ is injective. If $x \in V_{-n}$ and $A_Q^{(n-1)/2}x$ is in the radical of $Q|_{A_Q^{(n-1)/2}(V_{-n})}$ then $0 = (A_Q A_Q^{(n-1)/2}x, A_Q^{(n-1)/2}x_1) = \pm(A_Q^n x, x_1)$ for any $x_1 \in V_{-n}$ hence $A_Q^n x = 0$ so that $x = 0$ (since $Q \in S$). Thus $Q \in \mathfrak{Q}(V)_2^0$.

2.6. As in [L3, 2.6], let $\mathfrak{F}_s(V)$ be the set of all filtrations $V_* = (V_{\ge a})_{a \in \mathbf{Z}}$ such that

(i) $\{x \in V; (x, V_{\ge a}) = 0\} = V_{\ge 1-a}$ for any a;

(ii) the obvious grading of the associated vector space $gr(V_*) = \oplus_a V_{\ge a}/V_{\ge a+1}$ is an s-good grading with respect to the symplectic form $(,)_0$ on $gr(V_*)$ induced by $(,)$.
(Here condition (ii) can be replaced by the condition that there exists an s-good grading (V_i) of V such that $V_{\ge a} = \oplus_{a; a \ge i} V_i$ for any a.)

For $V_* = (V_{\ge a}) \in \mathfrak{F}_s(V)$ let $\zeta(V_*)$ be the set of all $Q \in \mathfrak{Q}(V)$ such that $A_Q(V_{\ge a}) \subset V_{\ge a+2}$ for any $a \in \mathbf{Z}$, $Q|_{V_{\ge 0}} = 0$ and such that the element $\bar{Q} \in \mathfrak{Q}(gr(V_*))_2$ induced by Q (see below) satisfies $\bar{Q} \in \mathfrak{Q}(gr(V_*))_2^0$. The element $\bar{Q}$ is defined as follows. Let $x = \sum_a x_a$ where $x_a \in V_{\ge a}/V_{\ge a+1}$. Let $\dot{x}_a \in V_{\ge a}$ be a representative of x_a. Then $\bar{Q}(x) = Q(\dot{x}_{-1}) + \sum_{a \le -2}(A_Q \dot{x}_a, \dot{x}_{-a-2})$. Note that $\zeta(V_*) \subset \mathfrak{Q}(V)_{nil}$.

Assuming that $p \ne 2$ we note that $Q \mapsto A_Q$ defines a bijection

(a) $\zeta(V_*) \xrightarrow{\sim} \tilde{\xi}'(V_*)$

(with $\tilde{\xi}'(V_*)$ as in [L3, A.3]); the inverse map associates to $A \in \tilde{\xi}'(V_*)$ the quadratic form $Q : V \longrightarrow \mathbf{k}$ given by $Q(x) = (Ax, x)/2$.

We have the following result.

Proposition 2.7. *The map $\sqcup_{V_* \in \mathfrak{F}_s(V)} \zeta(V_*) \longrightarrow \mathfrak{Q}(V)_{nil}$, $Q \mapsto Q$ is a bijection.*

When $p \neq 2$ this follows from [L3, A.3(a)] using the bijection 2.6(a) and the bijection $Q \mapsto A_Q$ of $\mathfrak{Q}(V)_{nil}$ onto $\mathcal{N}_{\mathfrak{g}}$. The proof for $p = 2$ will be given in 2.10, 2.11.

2.8. In this subsection we assume that $V \neq 0$ and that $p = 2$. For any $Q \in \mathfrak{Q}(V)_{nil}$ let $e = e_Q$ be the smallest integer ≥ 1 such that $A_Q^e = 0$ and let $f = f_Q$ be the smallest integer ≥ 0 such that $Q(A_Q^f(x)) = 0$ for all $x \in V$. Define a subset $H_Q \subset V$ as follows.

$H_Q = \{x \in V; A_Q^{e-1}x = 0\}$ if $e \geq 2f + 1$;

$H_Q = \{x \in V; A_Q^{e-1}x = 0, Q(A_Q^{f-1}(x)) = 0\}$ if $e = 2f$;

$H_Q = \{x \in V; Q(A_Q^{f-1}(x)) = 0\}$ if $e \leq 2f - 1$.

We show:

$e \leq 2f + 1$.

It is enough to show that $A_Q^{2f+1}x = 0$ for all $x \in V$ or that $(A_Q^{2f+1}x, y) = 0$ for all $x, y \in V$ or that $(A_Q^{f+1}x, A_Q^f y) = 0$ for all x, y or that $Q(A_Q^f(x + y)) - Q(A_Q^f x) - Q(A_Q^f y) = 0$ for all x, y; this is clear.

Let $V = \oplus_i V_i$ be an s-good grading of V. Let $m \geq 0$ be the largest integer such that $V_m \neq 0$. Let $Q \in \mathfrak{Q}(V)_{\geq 2}$. We set $e = e_Q, f = f_Q$, $A = A_Q$. Let $\bar{Q} \in \mathfrak{Q}(V)_2^0$. We set $\bar{e} = e_{\bar{Q}}, \bar{f} = f_{\bar{Q}}$, $\bar{A} = A_{\bar{Q}}$. We asume that $C := A - \bar{A}$ satisfies $C(V_{\geq i}) \subset V_{\geq i+3})$ for all i and that $(Q - \bar{Q})|_{V_{\geq -1}} = 0$. We show:

(i) *If m is even then $\bar{e} = m + 1$, $2\bar{f} \leq m$ (hence $2\bar{f} < \bar{e}$). Since $\bar{e} \leq 2\bar{f} + 1$ we deduce that $\bar{e} = 2\bar{f} + 1$ hence $2\bar{f} = m$. If m is odd then $2\bar{f} = m + 1$, $\bar{e} \leq m + 1$ (hence $2\bar{f} \geq \bar{e}$). In any case, $H_{\bar{Q}} = V_{\geq -m+1}$.*

We must show:

if m is even then $\{x \in V; \bar{A}^m x = 0\} = V_{\geq -m+1}$; if m is odd then $\{x \in V; \bar{A}^m x = 0, \bar{Q}(\bar{A}^{(m-1)/2}x) = 0\} = V_{\geq -m+1}$.

Let $x = \sum_{k \geq -m} x_k \in V$ with $x_k \in V_k$. We have $\bar{A}^m x = \bar{A}^m x_{-m}$ and if m is odd then $\bar{Q}(\bar{A}^{(m-1)/2}x) = \bar{Q}(\bar{A}^{(m-1)/2}x_{-m})$. Hence it is enough to show:

if m is even then $x_{-m} = 0$ if and only if $\bar{A}^m x_{-m} = 0$ (this is clear); if m is odd then $x_{-m} = 0$ if and only if $\bar{A}^m x_{-m} = 0$ and $\bar{Q}(\bar{A}^{(m-1)/2}x_{-m}) = 0$.

Assume that m is odd. Assume that $\bar{A}^m x_{-m} = 0$ and $\bar{Q}(\bar{A}^{(m-1)/2}x_{-m}) = 0$. Let $y = \bar{A}^{(m-1)/2}x_{-m} \in V_{-1}$. For any $z_{-m} \in V_{-m}$ we have $(y, \bar{A}^{(m+1)/2}z_{-m}) = (\bar{A}^m x_{-m}, z_{-m}) = 0$. Hence y is in the radical of the symplectic form $(a, b) \mapsto (a, \bar{A}b)$ on $\bar{A}^{(m-1)/2}V_{-m}$. Since $\bar{Q}$ is nondegenerate on $\bar{A}^{(m-1)/2}V_{-m}$ (with associated symplectic form $(a, \bar{A}b)$) and $\bar{Q}y = 0$ we see that $y = 0$. Since $\bar{A}^{(m-1)/2} : V_{-m} \to V_{-1}$ is injective we deduce that $x_{-m} = 0$. This proves (i).

We show:

(ii) *If $2n \geq m$ then $QA^n = 0, Q\bar{A}^n = 0, \bar{Q}\bar{A}^n = 0$. Hence if m is even then $f \leq m/2$; if m is odd then $f \leq (m + 1)/2$.*

To show that $QA^n = 0$ it is enough to show that $Q(A^n x) = 0$ whenever $x \in V_i$, $i \geq -m$ and $(A^{n+1}x, A^n x') = 0$ whenever $x \in V_i$, $x' \in V_j$, $i, j \geq -m$, $i \neq j$. This follows from $A^nV \in V_{\ge 0}$ and $Q_{V_{\ge 0}} = 0$, $(V_{\ge 1}, V_{\ge 0}) = 0$. The remaining equalities are proved in the same way.

The following statement is immediate.

(iii) *Let $P \in \text{End}(V)$ be a sum of products of n factors of which at least one is C and remaining ones are $\bar{A}$. Then $P(V_i) \subset V_{\ge i+2n+1}$ for all i. Hence if $n \ge m$ then $P = 0$.*

We show:

(iv) *If $n \ge m$ then $A^n = \bar{A}^n$.*

Indeed, $A^n = \bar{A}^n + P$ where P is as in (iii). Hence the result follows from (iii).

We show:

(v) *If $\bar{e} = 2\bar{f} + 1$ then $A^{\bar{e}-1} = \bar{A}^{\bar{e}-1} \ne 0$, $A^{\bar{e}} = \bar{A}^{\bar{e}} = 0$; hence $e = \bar{e}$ and $H_Q = H_{\bar{Q}}$. If $\bar{e} = 2\bar{f}$ then $A^{\bar{e}-1} = \bar{A}^{\bar{e}-1} \ne 0$, $A^{\bar{e}} = \bar{A}^{\bar{e}} = 0$; hence $e = \bar{e}$.*

In the first case we have $\bar{e} = m + 1$ hence $\bar{e} - 1 \ge m, \bar{e} \ge m$ and we use (iv); moreover, we have $e = \bar{e} = 2\bar{f} + 1 = m + 1 \ge 2f + 1$ (see (ii)), hence

$$H_Q = \{x \in V; A^{e-1}x = 0\} = \{x \in V; A^{\bar{e}-1}x = 0\} = \{x \in V; \bar{A}^{\bar{e}-1}x = 0\}$$
$$= H_{\bar{Q}}.$$

In the second case we have $2\bar{f} = m+1$ hence $\bar{e} = m+1$ and we continue as in the first case.

We show:

(vi) *If $n \ge 0$, $2n + 1 \ge m$, then $QA^n = \bar{Q}\bar{A}^n$.*

If $2n \ge m$ then both sides are 0, see (ii). Thus we may assume that $m = 2n+1$. We have $A^n = \bar{A}^n + P$, with P as in (iii). By (iii) we have $P(V) \subset V_{\ge -m+2n+1} = V_{\ge 0}$, hence $QP(V) = 0$. We have $\bar{A}^nV \subset V_{\ge -m+2n} = V_{\ge -1}$, $A\bar{A}^nV \subset AV_{\ge -1} \subset V_{\ge 1}$, $(V_{\ge 0}, V_{\ge 1}) = 0$, hence $(A\bar{A}^nV, PV) = 0$. For $x \in V$ we have

$$QA^n = Q(\bar{A}^nx + Px) = Q(\bar{A}^nx) + Q(Px) + (A\bar{A}^nx, Px)$$
$$= Q(\bar{A}^nx) + (A\bar{A}^nx, Px) = Q(\bar{A}^nx).$$

Since $\bar{A}^nx \in V_{\ge -1}$ we have $Q\bar{A}^n(x) = \bar{Q}\bar{A}^n(x)$ as required.

We show:

(vii) *If $\bar{e} \le 2\bar{f}$ (hence $\bar{f} > 0$) then $QA^{\bar{f}-1} = \bar{Q}\bar{A}^{\bar{f}-1} \ne 0$, $QA^{\bar{f}} = \bar{Q}\bar{A}^{\bar{f}} = 0$. Hence $f = \bar{f}$.*

In this case we have $2\bar{f} = m + 1$, $2(\bar{f} - 1) = m - 1$. Hence the result follows from (vi).

We show:

(viii) *If $\bar{e} < 2\bar{f}$ then $e < 2f$.*

In this case we have $f = \bar{f} = (m + 1)/2$. We must show that $e < m + 1$. By (iv) we have $A^m = \bar{A}^m$. We have $\bar{e} < 2\bar{f} = m + 1$ hence $\bar{e} \le m$ and $\bar{A}^m = 0$. Hence $A^m = 0$ and $e \le m$ as required.

Collecting together the results above we deduce:

(ix) *If $\bar{e} = 2\bar{f} + 1$ then $e = \bar{e} = m + 1 \ge 2f + 1$ (hence $e \ge 2f + 1$). If $\bar{e} = 2\bar{f}$ then $e = \bar{e} = m + 1$, $\bar{f} = f = (m + 1)/2$ hence $e = 2f$. If $\bar{e} < 2\bar{f}$*

then $f=\bar f=(m+1)/2$ *and* $e<2f$*. In each case we have* $H_Q=H_{\bar Q}$ *hence* $H_Q=V_{\ge -m+1}$ *and* $m=\max(e-1,2f-1)$.

2.9. We preserve the setup of 2.8. We assume in addition that $\bar A=0$. From the definitions we see that $V_i=0$ if $i\notin\{-1,0,1\}$ and $\dim V_{-1}=\dim V_1\le 1$. It follows that $A=0$. We have $m=0$ or $m=1$. If $m=0$ then $Q=\bar Q=0$. If $m=1$ then $Q\ne 0,\bar Q\ne 0$.

2.10. We prove the injectivity of the map in 2.7 assuming that $p=2$. We argue by induction on $\dim V$. If $\dim V=0$, the result is trivial. Assume now that $\dim V\ge 1$. Let $Q\in\mathfrak{Q}(V)$ and let $V_*=(V_{\ge a})$, $\tilde V_*=(\tilde V_{\ge a})$ be two filtrations in $\mathfrak{F}_s(V)$ such that $Q\in\zeta(V_*)$ and $Q\in\zeta(\tilde V_*)$. We must show that $V_*=\tilde V_*$. Let $\bar Q\in\mathfrak{Q}(gr(V_*))_2$, $\bar Q_1\in\mathfrak{Q}(gr(\tilde V_*))_2$ be the quadratic forms induced by Q. Let $m\ge 0$ be the largest integer such that $gr_m(V_*)\ne 0$. Let $\tilde m\ge 0$ be the largest integer such that $gr_{\tilde m}(\tilde V_*)\ne 0$. If $\bar Q=0$ then $Q=0$ hence $\bar Q_1=0$; also, $V_{\ge 1}=0$, $V_{\ge 0}=V$, $\tilde V_{\ge 1}=0$, $\tilde V_{\ge 0}=V$; hence $V_*=\tilde V_*$ as desired. Thus we can assume that $\bar Q\ne 0$, $\bar Q_1\ne 0$. Hence $m\ge 1$, $\tilde m\ge 1$. Using 2.8(ix) we see that $V_{\ge -m+1}=H_Q=\tilde V_{\ge -\tilde m+1}$, $m=\max(e_Q-1,2f_Q-1)=\tilde m$. Thus, $m=\tilde m$ and $V_{\ge -m+1}=\tilde V_{\ge -m+1}$. We have $V_{\ge m}=\{x\in V;(x,V_{\ge -m+1})=0\}$, $\tilde V_{\ge m}=\{x\in V;(x,\tilde V_{\ge -m+1})=0\}$, hence $V_{\ge m}=\tilde V_{\ge m}$. Let $V'=V_{\ge -m+1}/V_{\ge m}=\tilde V_{\ge -m+1}/\tilde V_{\ge m}$. Note that V' has a natural nondegenerate symplectic form induced by $(,)$. We set $V'_{\ge a}=$ image of $V_{\ge a}$ under $V_{\ge -m+1}\to V'$ (if $a\ge -m+1$), $V'_{\ge a}=0$ (if $a<-m+1$). We set $\tilde V'_{\ge a}=$ image of $\tilde V_{\ge a}$ under $\tilde V_{\ge -m+1}\to V'$ (if $a\ge -m+1$), $\tilde V'_{\ge a}=0$ (if $a<-m+1$). Then $V'_*=(V'_{\ge a})$, $\tilde V'_*=(\tilde V'_{\ge a})$ are filtrations in $\mathfrak{F}_s(V')$. Also Q induces an element $Q'\in\mathfrak{Q}(V')$ and we have $Q'\in\zeta(V'_*)$, $Q'\in\zeta(\tilde V'_*)$. Note also that $\dim V'<\dim V$. By the induction hypothesis we have $V'_*=\tilde V'_*$. It follows that $V_{\ge a}=\tilde V_{\ge a}$ for any $a\ge -m+1$. If $a<-m+1$ we have $V_{\ge a}=\tilde V_{\ge a}=V$. Hence $V_*=\tilde V_*$, as desired. Thus the map in 2.7 is injective.

2.11. We prove the surjectivity of the map in 2.7 assuming that $p=2$. By a standard argument we can assume that $\mathbf{k}$ is an algebraic closure of the field $\mathbf{F}_2$ with 2 elements. We can also assume that $\dim V\ge 2$. We choose an $\mathbf{F}_2$-rational structure on V such that $(,)$ is defined over $\mathbf{F}_2$. Then the Frobenius map relative to the $\mathbf{F}_2$-structure acts naturally and compatibly on the source and target of the map in 2.7. We denote each of these actions by F. It is enough to show that for any $n\ge 1$ the map $\alpha_n:(\sqcup_{V_*\in\mathfrak{F}_s(V)}\zeta(V_*))^{F^n}\to\mathfrak{Q}(V)^{F^n}_{nil}$, $Q\mapsto Q$ is a bijection. Since α_n is injective (see 2.10) it is enough to show that $|(\sqcup_{V_*\in\mathfrak{F}_s(V)}\zeta(V_*))^{F^n}|=|\mathfrak{Q}(V)^{F^n}_{nil}|$. By 1.4(b), 1.6(d), we have $|\mathfrak{Q}(V)^{F^n}_{nil}|=2^{n\dim V^2/2}$. It is enough to show that

(a) $|(\sqcup_{V_*\in\mathfrak{F}_s(V)}\zeta(V_*))^{F^n}|=2^{n\dim V^2/2}$.

Now the left hand side of (a) makes sense when $\mathbf{k}$ is replaced by an algebraic closure of the prime field with p' elements where p' is any prime number; when $p'\ne 2$, this more general expression is equal to $p'^{n\dim V^2/2}$ since the map in 2.7

is already known to be a bijection in this case (we use also 1.4(b), 1.6(d)). Then (a) follows from this equality by specializing p'^n (viewed as an indeterminate) to 2^n provided that we can show that the left hand side of (a) (for general p') is "universal" in the sense that it is a polynomial in p'^n with rational coefficients independent of p', n.

We now compute the left hand side of (a) (for general p'). A collection $(f_a)_{a\in\mathbf{Z}}$ of integers is said to be admissible if $f_a = 0$ for all but finitely many a, f_a is even for any even a, $f_a = f_{-a}$ for all a, $f_0 \ge f_{-2} \ge f_{-4} \ge \dots$, $f_{-1} \ge f_{-3} \ge f_{-5} \ge \dots$ and $\sum_a f_a = \dim V$. For (f_a) as above let $\mathcal{Y}_{(f_a)}$ be the set of all $V_* \in \mathfrak{F}_s(V)$ such that $\dim(gr_a(V_*)) = f_a$ for all a. The left hand side of (a) is $\sum_{(f_a)} |\mathcal{Y}_{(f_a)}^{F^n}||\zeta(V_*)^{F^n}|$ where V_* is any fixed element in $\mathcal{Y}_{(f_a)}^{F^n}$. Since each $|\mathcal{Y}_{(f_a)}^{F^n}|$ is "universal" it is enough to show that if $V_* \in \mathcal{Y}_{(f_a)}^{F^n}$ then $|\zeta(V_*)^{F^n}|$ is "universal". By a standard argument similar to the one in [L2, 1.5(d)] we see that $|\zeta(V_*)^{F^n}| = p'^{nd}|(\mathfrak{Q}(V)_2^0)^{F^n}|$ where d is a quadratic expression in the f_a (with integral coefficients independent of n) and $|\mathfrak{Q}(V)_2^0|$ is defined with respect to a fixed s-good grading (V_i) of V which satisfies $\dim V_a = f_a$ and $F(V_a) = V_a$ for all a. It is enough to show that $|(\mathfrak{Q}(V)_2^0)^{F^n}|$ is "universal". Let s' be the number of pairs $(q, (U_{-1} \supset U_{-3} \supset U_{-5} \supset \dots))$ where q is a quadratic form $V_{-1} \to \mathbf{k}$ defined over $\mathbf{F}_{p'^n}$ and U_i are subspaces of V_{-1} defined over $\mathbf{F}_{p'^n}$ such that $\dim U_i = f_i$ and such that $q|_{U_i}$ is nodegenerate (for $i \le -1$ odd). Let s'' be the number of all $(U_0 \supset U_{-2} \supset U_{-4} \supset \dots)$ where U_i are subspaces of V_0 defined over $\mathbf{F}_{p'^n}$ such that $\dim U_i = f_i$ and such that $(,)|_{U_i}$ is nodegenerate (for $i \le 0$ even). From the definitions we have $|(\mathfrak{Q}(V)_2^0)^{F^n}| = s's''s_1$ where s_1 is "universal". It is easy to see that s'' is "universal". It remains to show that s' is "universal". If $f_0 = 0$ we have $s' = 1$ and there is nothing to prove. We now assume that $f_0 > 0$. If f_0 is odd then $s' = t\nu(f_{-1}, f_{-3}, f_{-5}, \dots)$ where t is the number of nondegenerate quadratic forms on V_{-1} and $\nu(f_{-1}, f_{-3}, f_{-5}, \dots)$ (as in [L2, 1.2]) is "universal" by [L2, 1.2(a)]); moreover, t is clearly "universal" hence s' is "universal". If f_0 is even ≥ 2 then $s' = t_1\nu^1(f_{-1}, f_{-3}, f_{-5}, \dots) + t_{-1}\nu^{-1}(f_{-1}, f_{-3}, f_{-5}, \dots)$ where t_1 (resp. t_{-1}) is the number of nondegenerate quadratic forms on V_{-1} which are split (resp. nonsplit) over $\mathbf{F}_{p'^n}$ and $\nu^\epsilon(f_{-1}, f_{-3}, f_{-5}, \dots)$ (as in [L2, 1.2]) is "universal" by [L2, 1.2(a)]); moreover, t_1, t_{-1} are clearly "universal" hence s' is "universal". Thus s' is "universal" in any case. This completes the proof of the surjectivity of the map in 2.7.

2.12. To give the s-good grading $V = \oplus_i V_i$ (see 2.5) is the same as to give an element δ of $\mathfrak{D}_G$. We can then identify $\mathfrak{Q}(V)_2$ with $\mathfrak{g}_2^{*\delta}$ under the restriction of the bijection 1.3(a). We have:

(a) $\mathfrak{Q}(V)_2 - \mathfrak{Q}(V)_2^0 \subset \mathfrak{g}_2^{*\delta} - \mathfrak{g}_2^{*\delta !}$;

(b) $\mathfrak{Q}(V)_2^0 \subset \mathfrak{g}_2^{*\delta !}$.

Let $Q \in \mathfrak{Q}(V)_2 - \mathfrak{Q}(V)_2^0$. To prove (a), we must show that there exists $B \in Sp(V)$ such that B fixes Q and B does not fix $V_{\ge i}$ for some i.

Generally x_k will denote an element of V_k. We set $A = A_Q$.

Assume first that $A : V_{-i} \to V_{-i+2}$ is not injective for some $i \ge 2$. Then A :

$V_{i-2} \longrightarrow V_i$ is not surjective and since $\dim V_{i-2} \geq \dim V_i$, we see that $A : V_{i-2} \longrightarrow V_i$ is not injective. We can find $e_{-i} \in V_{-i} - \{0\}$ such that $Ae_{-i} = 0$. We can find $e_{i-2} \in V_{i-2} - \{0\}$ such that $Ae_{i-2} = 0$. Define $B \in \text{End}(V)$ by

$$B(\sum_k x_k) = \sum_{k \neq -i, i-2} x_k + (x_{-i} + (e_{i-2}, x_{-i+2})e_{-i}) + (x_{i-2} + (e_{-i}, x_i)e_{i-2}).$$

By a computation exactly as that in the first two cases in [L3, 1.7] we have

$$(B(\sum_k x_k), B(\sum_k x'_k)) - (\sum_k x_k, \sum_k x'_k) = 0.$$

Thus $B \in Sp(V)$. We have

$$QB(\sum_k x_k) - Q(\sum_k x_k) = Q(x_{-1}) + \sum_{k \leq -2; k \neq -i} (Ax_k, x_{-k-2})$$
$$+ (Ax_{-i} + (e_{i-2}, x_{-i+2})Ae_{-i}, x_{i-2} + (e_{-i}, x_i)e_{i-2}) - Q(x_{-1})$$
$$- \sum_{k \leq -2} (Ax_k, x_{-k-2}) = -(Ax_{-i}, x_{i-2}) + (Ax_{-i}, x_{i-2} + (e_{-i}, x_i)e_{i-2})$$
$$= (Ax_{-i}, e_{i-2})(e_{-i}, x_i) = -(x_{-i}, Ae_{i-2})(e_{-i}, x_i) = 0.$$

Thus B^{-1} stabilizes Q.

We now assume that $A : V_{-i} \longrightarrow V_{-i+2}$ is injective for all $i \geq 2$ and that for some even $n \geq 0$, $A^n : V_{-n} \longrightarrow V_n$ is not an isomorphism. Note that $n \geq 2$. As in [L3, 1.7] we can find e_{-n}, f_{-n} linearly independent in V_{-n} such that $A^n e_{-n} = 0$, $A^n f_{-n} = 0$. For $j \geq 0$ we set $e_{2j-n} = A^j e_{-n}$, $f_{2j-n} = A^j f_{-n}$. We have $e_n = 0, f_n = 0$. Also e_m, f_m are linearly independent in V_m if $m \leq 0$ is even. As in *loc.cit.*, for $j \in [0, n]$ we have $(e_{2j-n}, e_{n-2j}) = 0$, $(f_{2j-n}, f_{n-2j}) = 0$, $(e_{2j-n}, f_{n-2j}) = 0$, $(f_{2j-n}, e_{n-2j}) = 0$. Define $B \in \text{End}(V)$ by

$$B(\sum_k x_k) = \sum_{k \notin \{2h-n; h \in [0,n-1]\}} x_k$$
$$+ \sum_{j \in [0,n-1]} (x_{2j-n} + (-1)^j (f_{n-2j-2}, x_{2j-n+2})e_{2j-n}$$
$$- (-1)^j (e_{n-2j-2}, x_{2j-n+2})f_{2j-n}).$$

By a computation exactly as that in the third case in [L3, 1.7] we have

$$(B(\sum_k x_k), B(\sum_k x'_k)) - (\sum_k x_k, \sum_k x'_k) = 0.$$

Thus $B\in Sp(V)$. We have

$$QB(\sum_k x_k)-Q(\sum_k x_k)=Q(x_{-1})+\sum_{k\le -2;k\notin -n,-n+2,\dots,-2}(Ax_k,x_{-k-2})$$

$$+\sum_{j=0}^{(n-2)/2}(Ax_{-n+2j}+(-1)^j(f_{n-2j-2},x_{-n+2j+2})e_{-n+2j+2}$$

$$+(-1)^{j+1}(e_{n-2j-2},x_{-n+2j+2})f_{-n+2j+2},$$

$$x_{n-2j-2}+(-1)^{n-j-1}(f_{-n+2j},x_{n-2j})e_{n-2j-2}+(-1)^{n-j}(e_{-n+2j},x_{n-2j})f_{n-2j-2})$$

$$-Q(x_{-1})-\sum_{k\le -2}(Ax_k,x_{-k-2})$$

$$=\sum_{j=0}^{(n-2)/2}(Ax_{-n+2j},(-1)^{n-j-1}(f_{-n+2j},x_{n-2j})e_{n-2j-2}$$

$$+(-1)^{n-j}(e_{-n+2j},x_{n-2j})f_{n-2j-2})+$$

$$\sum_{j=0}^{(n-2)/2}((-1)^j(f_{n-2j-2},x_{-n+2j+2})e_{-n+2j+2}$$

$$+(-1)^{j+1}(e_{n-2j-2},x_{-n+2j+2})f_{-n+2j+2},x_{n-2j-2})$$

$$=-\sum_{j=0}^{(n-2)/2}(x_{-n+2j},(-1)^{n-j-1}(f_{-n+2j},x_{n-2j})e_{n-2j}+(-1)^{n-j}(e_{-n+2j},x_{n-2j})f_{n-2j})$$

$$+\sum_{j=0}^{(n-2)/2}((-1)^j(f_{n-2j-2},x_{-n+2j+2})(e_{-n+2j+2},x_{n-2j-2})$$

$$+\sum_{j=0}^{(n-2)/2}(-1)^{j+1}(e_{n-2j-2},x_{-n+2j+2})(f_{-n+2j+2},x_{n-2j-2})$$

$$=\sum_{j=0}^{(n-2)/2}(-1)^j(f_{-n+2j},x_{n-2j})(x_{-n+2j},e_{n-2j})$$

$$+\sum_{j=0}^{(n-2)/2}(-1)^{j+1}(e_{-n+2j},x_{n-2j})(x_{-n+2j},f_{n-2j})$$

$$+\sum_{j=1}^{n/2}(-1)^{j-1}(e_{n-2j},x_{-n+2j})(f_{-n+2j},x_{n-2j})$$

$$+\sum_{j=1}^{n/2}((-1)^j(f_{n-2j},x_{-n+2j})(e_{-n+2j},x_{n-2j})=(f_{-n},x_n)(x_{-n},e_n)-(e_{-n},x_n)(x_{-n},f_n)$$

$$+(-1)^{(n/2)-1}(e_0,x_0)(f_0,x_0)+(-1)^{(n/2)}(f_0,x_0)(e_0,x_0)=0$$

∎

since $e_n = f_n = 0$. Thus B^{-1} stabilizes Q.

Now assume that $A : V_{-i} \to V_{-i+2}$ is injective for all $i \ge 2$ and that for some odd $n \ge 1$ the restriction of Q to $A^{(n-1)/2}(V_{-n})$ is a degenerate quadratic form. Then we can find $\xi \in A^n(V_{-2n-1}) - \{0\}$ such that $(A\xi, A^n(V_{-2n-1})) = 0$, $Q(x) = 0$. We can write $\xi = A^n e_{-2n-1}$ for a unique $e_{-2n-1} \in V_{-2n-1} - \{0\}$. For any $j \ge 0$ we set $e_{-2n-1+2j} = A^j e_{-2n-1} \in V_{-2n-1+2j}$. Thus $e_{-1} = \xi$ and $(Ae_{-1}, A^n(V_{-2n-1})) = 0$, $Q(e_{-1}) = 0$. We show that $e_{2n+1} = 0$. Indeed,

$$(V_{-2n-1}, e_{2n+1}) = (V_{-2n-1}, A^{n+1}e_{-1}) = \pm(A^n V_{-2n-1}, Ae_{-1}) = 0.$$

For $j \in [0, 2n+1]$ we have

$$(e_{-2n-1+2j}, e_{2n+1-2j}) = 0.$$

It is enough to note that

$$\begin{aligned}&(A^j e_{-2n-1}, A^{2n+1-j} e_{-2n-1}) = \pm(A^{2n+1} e_{-2n-1}, e_{-2n-1})\\ &= (e_{2n+1}, e_{-2n-1}) = 0.\end{aligned}$$

Define $B \in \mathrm{End}(V)$ by

$$\begin{aligned}B(\sum_k x_k) &= \sum_{k \ne -2n-1, -2n+1, \dots, 2n-1} x_k\\ &+ \sum_{j=0}^{2n} (x_{-2n-1+2j} + (-1)^j (e_{2n-1-2j}, x_{-2n+2j+1}) e_{-2n-1+2j})\\ &= \sum_{k \ne -2n-1, -2n+1, \dots, 2n+1} x_k\\ &+ \sum_{j=0}^{2n+1} (x_{-2n-1+2j} + (-1)^j (e_{2n-1-2j}, x_{-2n+2j+1}) e_{-2n-1+2j}).\end{aligned}$$

(In the last line, $e_{2n-1-2j}$ is not defined when $j = 2n+1$; we define it to be 0.) We have

$$\begin{aligned}&(B(\sum_k x_k), B(\sum_k x'_k)) - (\sum_k x_k, \sum_k x'_k) = \sum_k (x_k, x'_{-k})\\ &+ \sum_{j=0}^{2n+1} (-1)^j (e_{2n-1-2j}, x_{-2n+2j+1})(e_{-2n-1+2j}, x'_{2n+1-2j})\\ &+ \sum_{j=0}^{2n+1} (-1)^j (e_{2n-1-2j}, x'_{-2n+2j+1})(x_{2n+1-2j}), e_{-2n-1+2j})\\ &- \sum_k (x_k, x'_{-k}) = \sum_{j=0}^{2n} (-1)^j (e_{2n-1-2j}, x_{-2n+2j+1})(e_{-2n-1+2j}, x'_{2n+1-2j})\\ &+ \sum_{j=0}^{2n} (-1)^j (e_{-2n-1+2j}, x'_{2n-2j+1})(x_{-2n+1+2j}), e_{2n-1-2j}) = 0.\end{aligned}$$

Thus $B\in Sp(V)$. We have

$$
\begin{aligned}
&QB(\sum_k x_k)-Q(\sum_k x_k)\\
&Q(x_{-1}+(-1)^n(e_{-1},x_1)e_{-1})+\sum_{k\le -2;k\ne -2n-1,-2n+1,\dots,2n-1}(Ax_k,x_{-k-2})\\
&+\sum_{j=0}^{n-1}(Ax_{-2n-1+2j}+(-1)^j(e_{2n-1-2j},x_{-2n+2j+1})e_{-2n+1+2j},\\
&x_{2n-1-2j}+(-1)^j(e_{-2n-1+2j},x_{2n-2j+1})e_{2n-1-2j})\\
&-Q(x_{-1})-\sum_{k\le -2}(Ax_k,x_{-k-2})\\
&=(Ax_{-1},(-1)^n(e_{-1},x_1)e_{-1})\\
&+\sum_{j=0}^{n-1}(-1)^j(Ax_{-2n-1+2j},e_{2n-1-2j})(e_{-2n-1+2j},x_{2n-2j+1})\\
&+\sum_{j=0}^{n-1}(-1)^j(e_{2n-1-2j},x_{-2n+2j+1})(e_{-2n+1+2j},x_{2n-1-2j})\\
&=-(-1)^n(e_{-1},x_1)(x_{-1},e_1)\\
&+\sum_{j=0}^{n-1}(-1)^{j+1}(x_{-2n-1+2j},e_{2n+1-2j})(e_{-2n-1+2j},x_{2n-2j+1})\\
&+\sum_{j=1}^{n}(-1)^{j-1}(e_{2n+1-2j},x_{-2n+2j-1})(e_{-2n-1+2j},x_{2n+1-2j})\\
&=-(-1)^n(e_{-1},x_1)(x_{-1},e_1)-(x_{-2n-1},e_{2n+1})(e_{-2n-1},x_{2n+1})\\
&+(-1)^{n-1}(e_1,x_{-1})(e_{-1},x_1)=0
\end{aligned}
$$

since $e_{2n+1}=0$. Thus B^{-1} stabilizes Q.

This completes the proof of (a).

Now let $Q\in\mathfrak{Q}(V)^0_2$ and let $B\in Sp(V)$ be such that $Q(B(x))=Q(x)$ for all $x\in V$. To prove (b) it is enough to show that

$B\in G^\delta_{\ge 0}$.

The proof is similar to that in the last paragraph of [L3, 1.8]. We argue by induction on $\dim V$. Recall that $V\ne 0$. Let $A=A_Q$. We have $AB=BA$. Let m be the largest integer ≥ 0 such that $V_m\ne 0$. If $m=0$ we have $G^\delta_{\ge 0}=G$ and the result is clear. Assume now that $m\ge 1$. If m is even we have $A^mV=V_m$, $\ker(A^m:V\to V)=V_{\ge -M+1}$. Since $BA=AB$, the image and kernel of A^m are B-stable. Hence $B(V_m)=V_m$ and $B(V_{\ge -M+1})=V_{\ge -M+1}$. Hence B induces an automorphism $B'\in Sp(V')$ where $V'=V_{\ge -m+1}/V_m$, a vector space with a nondegenerate symplectic form induced by $(,)$. We have canonically $V' = V_{-m+1} \oplus V_{-m+2} \oplus \dots \oplus V_{m+1}$ and $\mathfrak{Q}(V')_2$, $\mathfrak{Q}(V')_2^0$ are defined in terms of this (s-good) grading. Now Q induces an element $Q' \in \mathfrak{Q}(V')_2^0$ and we have $Q'(B'(x')) = Q'(x')$ for all $x' \in V'$. If $V' = 0$ then clearly $B \in G^\delta_{\ge 0}$ and the result is clear. Hence we can assume that $V' \ne 0$ and that the result holds for V'. We see that for any $i \in [-m+1, m-1]$, the subspace $V_i + V_{i+1} + \dots + V_{m-1}$ of V' is B'-stable. Hence the subspace $V_{\ge i}$ of V is B-stable. We see that $B \in G^\delta_{\ge 0}$, as required. Next we assume that m is odd. We have $V_{\ge -m+1} = \{x \in V; A^m x = 0, Q(A^{(m-1)/2}x) = 0\}$ (we use that $Q \in \mathfrak{Q}(V)_2^0$). Since B commutes with A and preserves Q we see that B preserves the subspace $\{x \in V; A^m x = 0, Q(A^{(m-1)/2}x) = 0\}$ hence $BV_{\ge -m+1} = V_{\ge -m+1}$. We have $V_m = \{x \in V; (x, V_{-m+1}) = 0\}$. Since B preserves the subspace $V_{\ge -m+1}$ and B preserves Q and $(,)$ we see that $BV_m = V_m$. Hence B induces an automorphism $B' \in Sp(V')$ where $V' = V_{\ge -m+1}/V_m$, a vector space with a nondegenerate symplectic form induced by $(,)$. We have canonically $V' = V_{-m+1} \oplus V_{-m+2} \oplus \dots \oplus V_{m+1}$ and $\mathfrak{Q}(V')_2$, $\mathfrak{Q}(V')_2^0$ are defined in terms of this (s-good) grading. Now Q induces an element $Q' \in \mathfrak{Q}(V')_2^0$ and we have $Q'(B'(x')) = Q'(x')$ for all $x' \in V'$. If $V' = 0$ then clearly $B \in G^\delta_{\ge 0}$ and the result is clear. Hence we can assume that $V' \ne 0$ and that the result holds for V'. We see that for any $i \in [-m+1, m-1]$, the subspace $V_i + V_{i+1} + \dots + V_{m-1}$ of V' is B'-stable. Hence the subspace $V_{\ge i}$ of V is B-stable. We see that $B \in G^\delta_{\ge 0}$. This completes the proof of (b).

From (a),(b) we deduce

(c) $\mathfrak{Q}(V)_2^0 = \mathfrak{g}_2^{*\delta!}$.

2.13. Assume that $G, \mathfrak{g}$ are as in 1.3. In this case Theorem 2.2 follows from 2.7, in view of 2.12(c). (We identify $\mathfrak{F}_s(V)$ and D_G as follows. Let $[\delta] \in D_G$ be the equivalence class of $\delta \in \mathfrak{D}_G$ and let $V = \oplus_i V_i$ be the s-good grading corresponding to δ (see 2.12). Let $V_{\ge a} = \oplus_{i \ge a} V_i$. Then $(V_{\ge a})$ is the element of $\mathfrak{F}_s(V)$ corresponding to $[\delta]$.)

2.14. Assume that $G, \mathfrak{g}$ are as in 2.2. As in [L3] let $\mathfrak{U}_G$ be the set of G-orbits on D_G. From [L3, 2.1(b)] we see that $\mathfrak{U}_G = \mathfrak{U}_{G_{\mathbf{C}}}$. In particular, $\mathfrak{U}_G$ is a finite set which depends only on the type of G, not on $\mathbf{k}$. For any $\mathcal{O} \in \mathfrak{U}_G$ we set

$$\mathcal{N}^{\mathcal{O}}_{\mathfrak{g}^*} = \Psi_{\mathfrak{g}^*}(\sqcup_{\Delta \in \mathcal{O}} \sigma^{*\Delta}).$$

The subsets $\mathcal{N}^{\mathcal{O}}_{\mathfrak{g}^*}$ are called the *pieces* of $\mathcal{N}_{\mathfrak{g}^*}$. They form a partition of $\mathcal{N}_{\mathfrak{g}^*}$ into smooth locally closed subvarieties (which are unions of G-orbits) indexed by $\mathfrak{U}_G = \mathfrak{U}_{G_{\mathbf{C}}}$. Now assume that $\mathbf{k}$ is an algebraic closure of the finite prime field $\mathbf{F}_p$ and that we are given a split $\mathbf{F}_p$-rational structure on G. Then $\mathfrak{g}, \mathfrak{g}^*, \mathcal{N}_{\mathfrak{g}^*}$ have induced $\mathbf{F}_p$-structures and each $\mathcal{O}$ as above is defined over $\mathbf{F}_p$. Also, each piece $\mathcal{N}^{\mathcal{O}}_{\mathfrak{g}^*}$ is defined over $\mathbf{F}_p$ (with Frobenius map F). Let $n \ge 1$. We have the following result.

(a) $|(\mathcal{N}_{\mathfrak{g}^*}^{\mathcal{O}})^{F^n}|$ *is a polynomial in* p^n *with integer coefficients independent of* p, n. For type A, D this follows from [L2, 1.8]. For type C this follows from the proof in 2.11.

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