CHAPTER 9

Quivers and Perverse Sheaves

9.1. The Complexes L_ν

9.1.1. By definition, a (finite) *graph* is a pair consisting of two finite sets $\mathbf{I}$ (*vertices*) and H (*edges*) and a map which to each $h \in H$ associates a two-element subset $[h]$ of $\mathbf{I}$.

We say that h is an edge joining the two vertices in $[h]$. We assume given a finite graph $(\mathbf{I}, H, h \mapsto [h])$. An *orientation* of our graph consists of two maps $H \to \mathbf{I}$ denoted $h \mapsto h'$ and $h \mapsto h''$ such that for any $h \in H$, the two elements of $[h]$ are precisely h', h''. We assume given an orientation of our graph. Thus we have an oriented graph (=quiver). Note that

(a) for any $h \in H$, we have $h' \neq h''$.

9.1.2. Let $\mathcal{V}$ be the category of finite dimensional $\mathbf{I}$-graded k-vector spaces $\mathbf{V} = \oplus_{\mathbf{i} \in \mathbf{I}} \mathbf{V}_\mathbf{i}$; the morphisms in $\mathcal{V}$ are isomorphisms of vector spaces compatible with the grading.

For each $\nu = \sum_\mathbf{i} \nu_\mathbf{i} \mathbf{i} \in \mathbf{N}[\mathbf{I}]$ we denote by $\mathcal{V}_\nu$ the full subcategory of $\mathcal{V}$ whose objects are those $\mathbf{V}$ such that $\dim \mathbf{V}_\mathbf{i} = \nu_\mathbf{i}$ for all $\mathbf{i} \in \mathbf{I}$. Then each object of $\mathcal{V}$ belongs to $\mathcal{V}_\nu$ for a unique $\nu \in \mathbf{N}[\mathbf{I}]$ and any two objects of $\mathcal{V}_\nu$ are isomorphic to each other. Moreover, $\mathcal{V}_\nu$ is non-empty for any $\nu \in \mathbf{N}[\mathbf{I}]$.

Given $\mathbf{V} \in \mathcal{V}$, we define $G_\mathbf{V} = \{g \in GL(\mathbf{V}) | g(\mathbf{V}_\mathbf{i}) = \mathbf{V}_\mathbf{i}$ for all $\mathbf{i} \in \mathbf{I}\}$ and

$$\mathbf{E}_\mathbf{V} = \oplus_{h \in H} \mathrm{Hom}(\mathbf{V}_{h'}, \mathbf{V}_{h''}).$$

Then $G_\mathbf{V}$ is an algebraic group (isomorphic to $\prod_\mathbf{i} GL(\mathbf{V}_\mathbf{i})$) acting naturally on the vector space $\mathbf{E}_\mathbf{V}$ by

$$(g, x) \mapsto gx = x' \text{ where } x'_h = g_{h''} x_h g_{h'}^{-1} \text{ for all } h \in H.$$

9.1.3. Flags. A subset $\mathbf{I}'$ of $\mathbf{I}$ is said to be *discrete* if there is no $h \in H$ such that $[h] \subset \mathbf{I}'$.

If $\nu \in \mathbf{N}[\mathbf{I}]$, we define the support of ν as $\{\mathbf{i} \in \mathbf{I} | \nu_\mathbf{i} \neq 0\}$. We say that ν is discrete if its support is a discrete subset of $\mathbf{I}$.

Let $\mathcal{X}$ be the set of all sequences $\boldsymbol{\nu} = (\nu^1, \nu^2, \ldots, \nu^m)$ in $\mathbf{N}[\mathbf{I}]$ such that ν^l is discrete for all l. Now let $\mathbf{V} \in \mathcal{V}$ and let $\boldsymbol{\nu} \in \mathcal{X}$ be such that $\dim \mathbf{V}_{\mathbf{i}} = \sum_l \nu^l_{\mathbf{i}}$ for all $\mathbf{i} \in \mathbf{I}$. A *flag* of type $\boldsymbol{\nu}$ in $\mathbf{V}$ is by definition a sequence

(a) $$f = (\mathbf{V} = \mathbf{V}^0 \supset \mathbf{V}^1 \supset \cdots \supset \mathbf{V}^m = 0)$$

of $\mathbf{I}$-graded subspaces of $\mathbf{V}$ such that, for $l = 1, 2, \ldots, m$, the graded vector space $\mathbf{V}^{l-1}/\mathbf{V}^l$ belongs to $\mathcal{V}_{\nu^l}$. If $x \in \mathbf{E}_{\mathbf{V}}$, we say that f is x-stable if $x_h(\mathbf{V}^l_{h'}) \subset \mathbf{V}^l_{h''}$ for all $l = 0, 1, \ldots, m$ and all h.

Let $\mathcal{F}_{\boldsymbol{\nu}}$ be the variety of all flags of type ν in $\mathbf{V}$. Let $\tilde{\mathcal{F}}_{\boldsymbol{\nu}}$ be the variety of all pairs (x, f) such that $x \in \mathbf{E}_{\mathbf{V}}$ and $f \in \mathcal{F}_{\boldsymbol{\nu}}$ is x-stable. Note that $G_{\mathbf{V}}$ acts (transitively) on $\mathcal{F}_{\boldsymbol{\nu}}$ by $g : f \to gf$ where f is as in (a) and $gf = (\mathbf{V} = g\mathbf{V}^0 \supset g\mathbf{V}^1 \supset \cdots \supset g\mathbf{V}^m = 0)$. Hence $G_{\mathbf{V}}$ acts on $\tilde{\mathcal{F}}_{\boldsymbol{\nu}}$ by $g : (x, f) \to (gx, gf)$.

Let $\pi_{\boldsymbol{\nu}} : \tilde{\mathcal{F}}_{\boldsymbol{\nu}} \to \mathbf{E}_{\mathbf{V}}$ be the first projection. We note the following properties which are easily checked.

(b) $\mathcal{F}_{\boldsymbol{\nu}}$ is a smooth, irreducible, projective variety of dimension

$$\sum_{\mathbf{i}; l < l'} \nu^{l'}_{\mathbf{i}} \nu^l_{\mathbf{i}};$$

the second projection $\tilde{\mathcal{F}}_{\boldsymbol{\nu}} \to \mathcal{F}_{\boldsymbol{\nu}}$ is a vector bundle of dimension

$$\sum_{h; l' < l} \nu^{l'}_{h'} \nu^l_{h''}.$$

(c) $\tilde{\mathcal{F}}_{\boldsymbol{\nu}}$ is a smooth, irreducible variety of dimension

$$f(\boldsymbol{\nu}) = \sum_{h; l' < l} \nu^{l'}_{h'} \nu^l_{h''} + \sum_{\mathbf{i}; l < l'} \nu^{l'}_{\mathbf{i}} \nu^l_{\mathbf{i}}.$$

(d) $\pi_{\boldsymbol{\nu}}$ is a proper $G_{\mathbf{V}}$-equivariant morphism.

Let $\tilde{L}_{\boldsymbol{\nu}} = (\pi_{\boldsymbol{\nu}})_! \mathbf{1} \in \mathcal{D}(\mathbf{E}_{\mathbf{V}})$. By (c),(d) and by 8.1.5, $\tilde{L}_{\boldsymbol{\nu}}$ is a semisimple complex on $\mathbf{E}_{\mathbf{V}}$. Let $L_{\boldsymbol{\nu}} = \tilde{L}_{\boldsymbol{\nu}}[f(\boldsymbol{\nu})]$. Since $D(\mathbf{1}[f(\boldsymbol{\nu})]) = \mathbf{1}[f(\boldsymbol{\nu})]$ on $\tilde{\mathcal{F}}_{\boldsymbol{\nu}}$ (see (c)) we have $D(L_{\boldsymbol{\nu}}) = L_{\boldsymbol{\nu}}$.

We denote by $\mathcal{P}_{\mathbf{V}}$ the full subcategory of $\mathcal{M}(\mathbf{E}_{\mathbf{V}})$ consisting of perverse sheaves which are direct sums of simple perverse sheaves L that have the following property: $L[d]$ appears as a direct summand of $L_{\boldsymbol{\nu}}$ for some $d \in \mathbf{Z}$ and some $\boldsymbol{\nu} \in \mathcal{X}$ such that $\dim \mathbf{V}_{\mathbf{i}} = \sum_l \nu^l_{\mathbf{i}}$ for all $\mathbf{i} \in \mathbf{I}$.

We denote by $\mathcal{Q}_{\mathbf{V}}$ the full subcategory of $\mathcal{D}(\mathbf{E}_{\mathbf{V}})$ whose objects are the complexes that are isomorphic to finite direct sums of complexes of the form $L[d']$ for various simple perverse sheaves $L \in \mathcal{P}_{\mathbf{V}}$ and various $d' \in \mathbf{Z}$. Any complex in $\mathcal{Q}_{\mathbf{V}}$ is semisimple and $G_{\mathbf{V}}$-equivariant. From 8.1.4, we see that $\mathcal{P}_{\mathbf{V}}$ and $\mathcal{Q}_{\mathbf{V}}$ are stable under Verdier duality.

9.1.4. Let $\boldsymbol{\nu} = (\nu^1, \nu^2, \ldots, \nu^m) \in \mathcal{X}$. Assume that for some j we write $\nu^j = \nu_1^j + \nu_2^j$ where $\nu_1^j, \nu_2^j \in \mathbf{N}[\mathbf{I}]$ have disjoint support. Let $\boldsymbol{\nu}' = (\nu^1, \nu^2, \ldots, \nu^{j-1}, \nu_1^j, \nu_2^j, \nu^{j+1}, \ldots, \nu^m) \in \mathcal{X}$. It is clear that $\tilde{L}_{\boldsymbol{\nu}} = \tilde{L}_{\boldsymbol{\nu}'}$ and $f(\boldsymbol{\nu}) = f(\boldsymbol{\nu}')$. Hence $L_{\boldsymbol{\nu}} = L_{\boldsymbol{\nu}'}$. Thus, in the definition of $\mathcal{P}_{\mathbf{V}}$, we may restrict ourselves to sequences $\boldsymbol{\nu} = (\nu^1, \nu^2, \ldots, \nu^m) \in \mathcal{X}$ such that each ν^j is of the form $n\mathbf{i}$ for some $\mathbf{i} \in \mathbf{I}$ and some $n > 0$. Since there are only finitely many such $\boldsymbol{\nu}$ (subject to $\dim \mathbf{V}_\mathbf{i} = \sum_l \nu_\mathbf{i}^l$ for all $\mathbf{i} \in \mathbf{I}$) we see that $\mathcal{P}_{\mathbf{V}}$ has only finitely many simple objects, up to isomorphism.

9.1.5. In the special case where $\mathbf{V}$ is such that $\sum_\mathbf{i} \dim \mathbf{V}_\mathbf{i}\mathbf{i}$ is discrete, we have $\mathbf{E}_\mathbf{V} = 0$ and $\mathcal{P}_{\mathbf{V}}$ has exactly one simple object up to isomorphism, namely $\mathbf{1}$.

9.1.6. Let $K, K' \in \mathcal{Q}_{\mathbf{V}}$. The following two conditions are equivalent:

(a) $K \cong K'$;

(b) $\dim \mathbf{D}_j(\mathbf{E}_\mathbf{V}, G_\mathbf{V}; K, DB) = \dim \mathbf{D}_j(\mathbf{E}_\mathbf{V}, G_\mathbf{V}; K', DB)$ for all simple objects $B \in \mathcal{P}_{\mathbf{V}}$ and all $j \in \mathbf{Z}$.

It is clear that (a) implies (b). Assume now that K, K' are not isomorphic. Now K is a direct sum of complexes $L[n]$ where L runs over the isomorphism classes of simple objects L of $\mathcal{P}_{\mathbf{V}}$ and $n \in \mathbf{Z}$; let $m(L, n) \in \mathbf{N}$ be the number of times that $L[n]$ appears in this direct sum. We define similarly $m'(L, n)$ by replacing K by K'. Since K, K' are not isomorphic, we can find L_0, n_0 such that $m(L_0, n_0) \neq m'(L_0, n_0)$ and such that $m(L, n) = m'(L, n)$ for all L and all $n < n_0$. By (b), we have

$$\sum_{L,n} m(L, n) \dim \mathbf{D}_{j+n}(\mathbf{E}_\mathbf{V}, G_\mathbf{V}; L, DB)$$
$$= \sum_{L,n} m'(L, n) \dim \mathbf{D}_{j+n}(\mathbf{E}_\mathbf{V}, G_\mathbf{V}; L, DB)$$

for all simple objects $B \in \mathcal{P}_{\mathbf{V}}$ and all $j \in \mathbf{Z}$.

Using 8.1.10(d), we rewrite this as follows:

$$m(B, -j) + \sum_{L} \sum_{n; n<-j} m(L, n) \dim \mathbf{D}_{j+n}(\mathbf{E}_\mathbf{V}, G_\mathbf{V}; L, DB)$$
$$\text{(c)} \qquad = m'(B, -j) + \sum_{L} \sum_{n; n<-j} m'(L, n) \dim \mathbf{D}_{j+n}(\mathbf{E}_\mathbf{V}, G_\mathbf{V}; L, DB).$$

We apply this to $B = L_0$ and $j = -n_0$. Since by our assumption, $m(L, n) = m'(L, n)$ for $n < n_0$, we see that (c) implies $m(L_0, n_0) = m'(L_0, n_0)$. This is a contradiction. Thus the equivalence of (a),(b) is proved.

9.2. The functors Ind and Res

9.2.1. Let $\mathbf{T}, \mathbf{W}$ be two objects of $\mathcal{V}$. We can form $\mathbf{E_T}, \mathbf{E_W}$ and their product $\mathbf{E_T} \times \mathbf{E_W}$. This has an action of $G_\mathbf{T} \times G_\mathbf{W}$ (product of actions as in 9.1.2).

We define a full subcategory $\mathcal{P}_{\mathbf{T},\mathbf{W}}$ of $\mathcal{M}(\mathbf{E_T} \times \mathbf{E_W})$ and a full subcategory $\mathcal{Q}_{\mathbf{T},\mathbf{W}}$ of $\mathcal{D}(\mathbf{E_T} \times \mathbf{E_W})$, as a special case of the definitions of $\mathcal{P}_\mathbf{V}, \mathcal{Q}_\mathbf{V}$ in 9.1.3; indeed, $\mathbf{T} \times \mathbf{W}$ and $\mathbf{E_T} \times \mathbf{E_W}$ are special cases of $\mathbf{V}$ and $\mathbf{E_V}$ where the oriented graph in 9.1.1 has been replaced by the disjoint union of two copies of that oriented graph.

From the definitions it is clear that any simple object $B \in \mathcal{P}_{\mathbf{T},\mathbf{W}}$ is the external tensor product $B' \otimes B''$ of two simple objects $B' \in \mathcal{P}_\mathbf{T}$ and $B'' \in \mathcal{P}_\mathbf{W}$ (and conversely). Note that any complex in $\mathcal{Q}_{\mathbf{T},\mathbf{W}}$ is semisimple and $G_\mathbf{T} \times G_\mathbf{W}$-equivariant.

9.2.2. We assume that we are given $\mathbf{V}, \mathbf{T}, \mathbf{W}$ in $\mathcal{V}$, that $\mathbf{W}$ is a subspace of $\mathbf{V}$ and that $\mathbf{T} = \mathbf{V}/\mathbf{W}$. We also assume that the obvious maps $\mathbf{W} \to \mathbf{V}$ and $\mathbf{V} \to \mathbf{T}$ preserve the $\mathbf{I}$-grading. Let Q be the stabilizer of $\mathbf{W}$ in $G_\mathbf{V}$ (a parabolic subgroup of $G_\mathbf{V}$). We denote by U the unipotent radical of Q. We have canonically $Q/U = G_\mathbf{T} \times G_\mathbf{W}$.

Let F be the closed subvariety of $\mathbf{E_V}$ consisting of all $x \in \mathbf{E_V}$ such that $x_h(\mathbf{W}_{h'}) \subset \mathbf{W}_{h''}$ for all $h \in H$. We denote by $\iota : F \to \mathbf{E_V}$ the inclusion. Note that Q acts on F (restriction of the $G_\mathbf{V}$-action on $\mathbf{E_V}$).

If $x \in F$, then x induces elements $x' \in \mathbf{E_T}$ and $x'' \in \mathbf{E_W}$; the map $x \mapsto (x', x'')$ is a vector bundle $\kappa : F \to \mathbf{E_T} \times \mathbf{E_W}$. Now Q acts on $\mathbf{E_T} \times \mathbf{E_W}$ through its quotient $Q/U = G_\mathbf{T} \times G_\mathbf{W}$. The map κ is compatible with the Q-actions.

We set $G_\mathbf{V} = G, Q/U = \bar{G}, \mathbf{E_V} = E, \mathbf{E_T} \times \mathbf{E_W} = \bar{E}$. We have a diagram

$$\bar{E} \xleftarrow{\kappa} F \xrightarrow{\iota} E.$$

Let $E'' = G \times_P F, E' = G \times_U F$. We have a diagram

$$\bar{E} \xleftarrow{p_1} E' \xrightarrow{p_2} E'' \xrightarrow{p_3} E$$

where $p_1(g,f) = \kappa(f)$; $p_2(g,f) = (g,f)$; $p_3(g,f) = g(\iota(f))$. Note that p_1 is smooth with connected fibres, p_2 is a $\bar{G}$-principal bundle and p_3 is proper.

Let A be a complex in $\mathcal{Q}_{\mathbf{T},\mathbf{W}}$ and let B be a complex in $\mathcal{Q}_\mathbf{V}$. We can form $\kappa_!(\iota^* B) \in \mathcal{D}(\bar{E})$. Now $p_1^* A$ is a $\bar{G}$-equivariant semisimple complex on E'; hence $(p_2)_\flat p_1^* A$ is a well-defined semisimple complex on E'' (see 8.1.7(c)). We can form $(p_3)_!(p_2)_\flat p_1^* A \in \mathcal{D}(\mathbf{V})$.

Lemma 9.2.3. $(p_3)_!(p_2)_\flat p_1^* A \in \mathcal{Q}_{\mathbf{V}}$.

The general case can be immediately reduced to the case where A is a simple perverse sheaf in $\mathcal{P}_{\mathbf{T},\mathbf{W}}$ and this is immediately reduced to the case where $A = L_{\boldsymbol{\nu}'} \otimes L_{\boldsymbol{\nu}''}$. (Note that a direct summand of a complex in $\mathcal{Q}_{\mathbf{V}}$ belongs to $\mathcal{Q}_{\mathbf{V}}$.) Thus, it suffices to prove that $(p_3)_!(p_2)_\flat p_1^*(\tilde{L}_{\boldsymbol{\nu}'} \otimes \tilde{L}_{\boldsymbol{\nu}''}) \in \mathcal{Q}_{\mathbf{V}}$, where $\boldsymbol{\nu}' = (\nu_1', \nu_2', \dots, \nu_{m'}') \in \mathcal{X}$ and $\boldsymbol{\nu}'' = (\nu_1'', \nu_2'', \dots, \nu_{m''}'') \in \mathcal{X}$ satisfy $\dim \mathbf{T}_{\mathbf{i}}' = \sum_l \nu_{\mathbf{i}}'^l$, $\dim \mathbf{W}_{\mathbf{i}} = \sum_l \nu_{\mathbf{i}}''^l$ for all $\mathbf{i} \in \mathbf{I}$.

Let $\boldsymbol{\nu}'\boldsymbol{\nu}''$ be the sequence of elements in $\mathbf{N}[\mathbf{I}]$ formed by the elements of the sequence $\boldsymbol{\nu}'$ followed by the elements of the sequence $\boldsymbol{\nu}''$. Recall that $\tilde{\mathcal{F}}_{\boldsymbol{\nu}'\boldsymbol{\nu}''}$ consists of pairs (x, f) where $x \in \mathbf{E}_{\mathbf{V}}$ and f is a flag of type $\boldsymbol{\nu}'\boldsymbol{\nu}''$ in $\mathbf{V}$ which is x-stable. Now the subspace with index m' in $f = (\mathbf{V} = \mathbf{V}^0 \supset \mathbf{V}^1 \supset \dots)$ is in the G-orbit of $\mathbf{W}$. The pairs (x, f) for which this subspace is equal to $\mathbf{W}$ form a closed subvariety $\tilde{\mathcal{F}}_{\boldsymbol{\nu}'\boldsymbol{\nu}'',0}$ of $\tilde{\mathcal{F}}_{\boldsymbol{\nu}'\boldsymbol{\nu}''}$; for such (x, f) we have $x \in F$, hence $(x, f) \to x$ defines a (proper) morphism $\tilde{\mathcal{F}}_{\boldsymbol{\nu}'\boldsymbol{\nu}'',0} \to F$. This morphism is Q-equivariant (for the natural actions of Q). Hence it induces a proper morphism $u : G \times_Q \tilde{\mathcal{F}}_{\boldsymbol{\nu}'\boldsymbol{\nu}'',0} \to G \times_Q F = E''$. Since $G \times_Q \tilde{\mathcal{F}}_{\boldsymbol{\nu}'\boldsymbol{\nu}'',0}$ is smooth, the complex $\tilde{L} = u_! \mathbf{1} \in \mathcal{D}(E'')$ is semisimple. (See 8.1.5.) It is clear from the definitions that $p_2^* \tilde{L} = p_1^*(\tilde{L}_{\boldsymbol{\nu}'} \otimes \tilde{L}_{\boldsymbol{\nu}''})$ and $(p_2)_\flat p_1^*(\tilde{L}_{\boldsymbol{\nu}'} \otimes \tilde{L}_{\boldsymbol{\nu}''}) \cong \tilde{L}$.

It remains to show that $(p_3)_! \tilde{L} \in \mathcal{Q}_{\mathbf{V}}$, or equivalently, that $(p_3 u)_! \mathbf{1} \in \mathcal{Q}_{\mathbf{V}}$. We may identify in a natural way $G \times_Q \tilde{\mathcal{F}}_{\boldsymbol{\nu}'\boldsymbol{\nu}'',0} = \tilde{\mathcal{F}}_{\boldsymbol{\nu}'\boldsymbol{\nu}''}$; then $p_3 u = \pi_{\boldsymbol{\nu}'\boldsymbol{\nu}''}$. It follows that $(p_3 u)_! \mathbf{1} = \tilde{L}_{\boldsymbol{\nu}'\boldsymbol{\nu}''}$ which is in $\mathcal{Q}_{\mathbf{V}}$ by definition. The lemma is proved.

Lemma 9.2.4. $\kappa_!(\iota^* B) \in \mathcal{Q}_{\mathbf{T},\mathbf{W}}$.

We may assume that B is a simple perverse sheaf in $\mathcal{P}_{\mathbf{V}}$. Since a direct summand of a complex in $\mathcal{Q}_{\mathbf{T},\mathbf{W}}$ belongs to $\mathcal{Q}_{\mathbf{T},\mathbf{W}}$ we see that it suffices to prove that $\kappa_!(\iota^* \tilde{L}_{\boldsymbol{\nu}}) \in \mathcal{Q}_{\mathbf{T},\mathbf{W}}$, where $\boldsymbol{\nu} \in \mathcal{X}$ satisfies $\dim \mathbf{V}_{\mathbf{i}} = \sum_l \nu_{\mathbf{i}}^l$ for all $\mathbf{i} \in \mathbf{I}$.

Let $\tilde{F} \subset \tilde{\mathcal{F}}_{\boldsymbol{\nu}}$ be the inverse image of $F \subset E$ under $\pi_{\boldsymbol{\nu}}$. Let $\tilde{\pi} : \tilde{F} \to F$ be the restriction of $\pi_{\boldsymbol{\nu}}$. We have $\iota^* \tilde{L}_{\boldsymbol{\nu}} = \tilde{\pi}_! \mathbf{1}$; hence

$$\kappa_!(\iota^* \tilde{L}_{\boldsymbol{\nu}}) = \kappa_! \tilde{\pi}_! \mathbf{1} = (\kappa \tilde{\pi})_! \mathbf{1}.$$

Let $\boldsymbol{\nu} = (\nu^1, \nu^2, \dots, \nu^m)$. For any $\boldsymbol{\tau}, \boldsymbol{\omega} \in \mathcal{X}$ of the form

$$\boldsymbol{\tau} = (\tau^1, \tau^2, \dots, \tau^m), \boldsymbol{\omega} = (\omega^1, \omega^2, \dots, \omega^m)$$

such that $\tau^l + \omega^l = \nu^l$ for all l, we define a subvariety $\tilde{F}(\boldsymbol{\tau}, \boldsymbol{\omega})$ of $\tilde{F}$ as the set of all pairs (x, f) where $x \in F$ and $f = (\mathbf{V} = \mathbf{V}^0 \supset \mathbf{V}^1 \supset \cdots \supset \mathbf{V}^m = 0) \in$

$\tilde{\mathcal{F}}_{\boldsymbol{\nu}}$ is x-stable and is such that the graded vector space $(\mathbf{V}^{l-1}\cap\mathbf{W})/(\mathbf{V}^{l}\cap\mathbf{W})$ belongs to $\mathcal{V}_{\omega^l}$ for $l = 1, 2, \dots, m$.

If (x, f) is as above, then there are induced elements $(x', f') \in \tilde{\mathcal{F}}_{\boldsymbol{\tau}}$ and $(x'', f'') \in \tilde{\mathcal{F}}_{\boldsymbol{\omega}}$; here x'' is deduced from x by restriction to $\mathbf{W}$ and x' is deduced from x by passage to quotient; f' is given by the images of the subspaces in f under the projection $\mathbf{V} \to \mathbf{T}$ and f'' is given by the intersections of the subspaces in f with $\mathbf{W}$. Thus we have a morphism $\alpha : \tilde{F}(\boldsymbol{\tau}, \boldsymbol{\omega}) \to \tilde{\mathcal{F}}_{\boldsymbol{\tau}} \times \tilde{\mathcal{F}}_{\boldsymbol{\omega}}$. We have a commutative diagram

$$\begin{array}{ccc} \tilde{F}(\boldsymbol{\tau}, \boldsymbol{\omega}) & \longrightarrow & \tilde{F} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\kappa\tilde{\pi}} \\ \tilde{\mathcal{F}}_{\boldsymbol{\tau}} \times \tilde{\mathcal{F}}_{\boldsymbol{\omega}} & \longrightarrow & \bar{E}\,. \end{array}$$

where the upper horizontal arrow is the obvious inclusion and the lower horizontal arrow is $\pi_{\tau} \times \pi_{\omega}$.

It is not difficult to verify (as in [9, 4.4]) that α is a (locally trivial) vector bundle of dimension $M(\boldsymbol{\tau}, \boldsymbol{\omega}) = \sum_{h;l'<l} \tau^{l'}_{h'}\omega^{l}_{h''} + \sum_{\mathbf{i};l<l'} \tau^{l'}_{\mathbf{i}}\omega^{l}_{\mathbf{i}}$.

It is clear that the locally closed subvarieties $\tilde{F}(\boldsymbol{\tau}, \boldsymbol{\omega})$ form a partition of $\tilde{F}$. Let $\tilde{F}_j$ be the union of all subvarieties $\tilde{F}(\boldsymbol{\tau}, \boldsymbol{\omega})$ of fixed dimension j. Let Z_j be the disjoint union of the varieties $\tilde{\mathcal{F}}_{\boldsymbol{\tau}} \times \tilde{\mathcal{F}}_{\boldsymbol{\omega}}$ (union over those $(\boldsymbol{\tau}, \boldsymbol{\omega})$ such that $\tilde{F}(\boldsymbol{\tau}, \boldsymbol{\omega}) \subset \tilde{F}_j$). The maps α above can be assembled together to form a vector bundle $\tilde{F}_j \to Z_j$. The maps $\pi_{\boldsymbol{\tau}} \times \pi_{\boldsymbol{\omega}}$ can be assembled together to form a (proper) morphism $Z_j \to \bar{E}$. We have a commutative diagram

$$\begin{array}{ccc} \tilde{F}_j & \longrightarrow & \tilde{F} \\ \downarrow & & \downarrow{\scriptstyle\kappa\tilde{\pi}} \\ Z_j & \longrightarrow & \bar{E} \end{array}$$

We may therefore use 8.1.6 to conclude that $(\kappa\tilde{\pi})_!\mathbf{1}$ is a semisimple complex and that, for any i and j, there is a canonical exact sequence (in $\mathcal{M}(\bar{E})$):

(a) $$0 \to H^n(f_j)_!\mathbf{1} \to H^n(f_{\le j})_!\mathbf{1} \to H^n(f_{\le j-1})_!\mathbf{1} \to 0$$

where $f_j : \tilde{F}_j \to \bar{E}$ and $f_{\le j} : \cup_{j';j'\le j}\tilde{F}_{j'} \to \bar{E}$ are the restrictions of $\kappa\tilde{\pi}$.

The earlier arguments show that

(b) $$(f_j)_!\mathbf{1} = \oplus(\tilde{L}_{\boldsymbol{\tau}} \otimes \tilde{L}_{\boldsymbol{\omega}})[-2M(\boldsymbol{\tau}, \boldsymbol{\omega})]$$

where the direct sum is taken over all $(\boldsymbol{\tau}, \boldsymbol{\omega})$ such that $\tilde{F}(\boldsymbol{\tau}, \boldsymbol{\omega}) \subset \tilde{F}_j$.

From (a),(b) we see by induction on j that all composition factors of $H^n(f_{\leq j})_!\mathbf{1}$ are in $\mathcal{P}_{\mathbf{V}}$. Taking j large enough we see that all composition factors of $H^n(\kappa\tilde{\pi})_!\mathbf{1}$ are in $\mathcal{P}_{\mathbf{V}}$. Since $(\kappa\tilde{\pi})_!\mathbf{1}$ is semisimple, it follows that $(\kappa\tilde{\pi})_!\mathbf{1} \in \mathcal{Q}_{\mathbf{T},\mathbf{W}}$. The lemma is proved.

9.2.5. By Lemmas 9.2.3, 9.2.4, we have well-defined functors

$$\tilde{\operatorname{Ind}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}} : \mathcal{Q}_{\mathbf{T},\mathbf{W}} \to \mathcal{Q}_{\mathbf{V}} \quad (A \mapsto (p_3)_!(p_2)_\flat p_1^* A)$$

and

$$\tilde{\operatorname{Res}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}} : \mathcal{Q}_{\mathbf{V}} \to \mathcal{Q}_{\mathbf{T},\mathbf{W}} \quad (B \mapsto \kappa_!(\iota^* B)).$$

Since $\tilde{\operatorname{Ind}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}$ is defined using a direct image under a proper map and inverse images under smooth morphisms with connected fibres, it commutes with Verdier duality up to shift (see 8.1.1, 8.1.4); more precisely,

$$D(\tilde{\operatorname{Ind}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}(A)) = \tilde{\operatorname{Ind}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}(D(A))[2d_1 - 2d_2]$$

where d_1 is the dimension of the fibres of p_1 and d_2 is the dimension of the fibres of p_2. We have

$$d_1 - d_2 = \sum_h \dim \mathbf{T}_{h'} \dim \mathbf{W}_{h''} + \sum_{\mathbf{i}} \dim \mathbf{T}_{\mathbf{i}} \dim \mathbf{W}_{\mathbf{i}}.$$

We set

$$\operatorname{Ind}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}} = \tilde{\operatorname{Ind}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}[d_1 - d_2].$$

Then

$$D(\operatorname{Ind}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}(A)) = \operatorname{Ind}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}(D(A)).$$

The functor $\operatorname{Ind}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}$ is called *induction.*

9.2.6. From the proof of 9.2.3 and of 9.2.4, we see that

(a) $$\tilde{\operatorname{Ind}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}(\tilde{L}_{\nu'} \otimes \tilde{L}_{\nu''}) = \tilde{L}_{\nu'\nu''}$$

(b) $$\tilde{\operatorname{Res}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}} \tilde{L}_{\nu} \cong \oplus(\tilde{L}_{\boldsymbol{\tau}} \otimes \tilde{L}_{\boldsymbol{\omega}})[-2M(\boldsymbol{\tau},\boldsymbol{\omega})]$$

where the sum is taken over all $\boldsymbol{\tau} = (\tau^1, \tau^2, \ldots, \tau^m), \boldsymbol{\omega} = (\omega^1, \omega^2, \ldots, \omega^m)$ such that $\dim \mathbf{T}_{\mathbf{i}} = \sum_l \tau^l_{\mathbf{i}}, \dim \mathbf{W}_{\mathbf{i}} = \sum_l \omega^l_{\mathbf{i}}$ for all $\mathbf{i}$ and $\tau^l + \omega^l = \nu^l$ for all l.

9.2.7. We have $f(\nu'\nu'') = f(\nu) + f(\nu') + d_1 - d_2$; hence from 9.2.6(a) we deduce that

$$\mathrm{Ind}_{\mathbf{T},\mathbf{W}}^{\mathbf{V}}(L_{\nu'} \otimes L_{\nu''}) = L_{\nu'\nu''}.$$

9.2.8. Let Γ be a smooth irreducible variety with a free action of G. Let $\bar{\Gamma} = U\backslash\Gamma$. Then $\bar{\Gamma}$ is a smooth irreducible variety with a free action of $\bar{G}$ induced by that of G. Consider the diagram

$$E \xleftarrow{s} \Gamma \times E \xrightarrow{t} G\backslash(\Gamma \times E)$$

with the obvious maps s, t. As in 8.1.9, s^*A is a semisimple G-equivariant complex on $\Gamma \times E$ and, since t is a principal G-bundle, the semisimple complex $t_\flat s^*B \in \mathcal{D}(G\backslash(\Gamma \times E))$ is well-defined. In particular, we can replace B by $\tilde{\mathrm{Ind}}_{\mathbf{T},\mathbf{W}}^{\mathbf{V}}A$ and we obtain the semisimple complex

$$t_\flat s^*(\tilde{\mathrm{Ind}}_{\mathbf{T},\mathbf{W}}^{\mathbf{V}}A) \in \mathcal{D}(G\backslash(\Gamma \times E)).$$

Replacing E, Γ, G by $\bar{E}, \bar{\Gamma}, \bar{G}$, we obtain a similar diagram

$$\bar{E} \xleftarrow{\bar{s}} \bar{\Gamma} \times \bar{E} \xrightarrow{\bar{t}} \bar{G}\backslash(\bar{\Gamma} \times \bar{E})$$

and we can consider the semisimple complex $\bar{t}_\flat \bar{s}^*A \in \mathcal{D}(\bar{G}\backslash(\bar{\Gamma} \times \bar{E}))$. In particular, we can replace A by $\tilde{\mathrm{Res}}_{\mathbf{T},\mathbf{W}}^{\mathbf{V}}B$ and we obtain the semisimple complex

$$\bar{t}_\flat \bar{s}^*(\tilde{\mathrm{Res}}_{\mathbf{T},\mathbf{W}}^{\mathbf{V}}B) \in \mathcal{D}(\bar{G}\backslash(\bar{\Gamma} \times \bar{E})).$$

Let $u : G\backslash(\Gamma \times E) \to \{\text{point}\}$ and $\bar{u} : \bar{G}\backslash(\bar{\Gamma} \times \bar{E}) \to \{\text{point}\}$ be the obvious maps.

Lemma 9.2.9 (Adjunction). *We have a natural isomorphism*

$$\mathcal{H}^n u_!(t_\flat s^*(\tilde{Ind}_{\mathbf{T},\mathbf{W}}^{\mathbf{V}}A) \otimes t_\flat s^*(B)) \cong \mathcal{H}^n \bar{u}_!(\bar{t}_\flat \bar{s}^*(A) \otimes \bar{t}_\flat \bar{s}^*(\tilde{Res}_{\mathbf{T},\mathbf{W}}^{\mathbf{V}}B))$$

for $n \in \mathbf{Z}$. Hence, for any $j \in \mathbf{Z}$, we have

(a) $$\mathbf{D}_{j'}(E, G; \tilde{Ind}_{\mathbf{T},\mathbf{W}}^{\mathbf{V}}A, B) = \mathbf{D}_j(\bar{E}, \bar{G}; A, \tilde{Res}_{\mathbf{T},\mathbf{W}}^{\mathbf{V}}B)$$

where $j' = j + 2\dim G/Q$.

The proof (which uses 8.1.6) is given in [2]; we will not repeat it here. The shift from j to j' comes from the formula $\dim(G\backslash\Gamma) = \dim(\bar{G}\backslash\bar{\Gamma}) - \dim G/Q$.

9.2.10. In order to eliminate the shift from j to j' in the previous lemma, we define

$$\mathrm{Res}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}(B) = \tilde{\mathrm{Res}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}[d_1 - d_2 - 2\dim G/Q],$$

where d_1, d_2 are as in 9.2.5. We can now rewrite the conclusion of the previous lemma as follows:

(a) $$\mathbf{D}_j(E, G; \mathrm{Ind}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}A, B) = \mathbf{D}_j(\bar{E}, \bar{G}; A, \mathrm{Res}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}B).$$

Note that $\dim G/Q = \sum_{\mathbf{i}} \dim \mathbf{T_i} \dim \mathbf{W_i}$; hence

$$d_1 - d_2 - 2\dim G/Q = \sum_h \dim \mathbf{T}_{h'} \dim \mathbf{W}_{h''} - \sum_{\mathbf{i}} \dim \mathbf{T_i} \dim \mathbf{W_i}.$$

The functor $\mathrm{Res}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}$ is called *restriction.*

9.2.11. We can rewrite 9.2.6(b) as follows:

$$\mathrm{Res}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}} L_{\boldsymbol{\nu}} \cong \oplus (L_{\boldsymbol{\tau}} \otimes L_{\boldsymbol{\omega}})[M'(\boldsymbol{\tau}, \boldsymbol{\omega})]$$

where the sum is taken over all $\boldsymbol{\tau} = (\tau^1, \tau^2, \dots, \tau^m), \boldsymbol{\omega} = (\omega^1, \omega^2, \dots, \omega^m)$ such that $\dim \mathbf{T_i} = \sum_l \tau^l_{\mathbf{i}}, \dim \mathbf{W_i} = \sum_l \omega^l_{\mathbf{i}}$ for all $\mathbf{i}$ and $\tau^l + \omega^l = \nu^l$ for all l; we have

$$M'(\boldsymbol{\tau}, \boldsymbol{\omega}) = d_1 - d_2 - 2\dim G/Q + f(\boldsymbol{\nu}) - f(\boldsymbol{\tau}) - f(\boldsymbol{\omega}) - 2M(\boldsymbol{\tau}, \boldsymbol{\omega}).$$

We show that the last expression is independent of the orientation of our graph. From the definitions we have

$$M'(\boldsymbol{\tau}, \boldsymbol{\omega}) = \sum_h \dim \mathbf{T}_{h'} \dim \mathbf{W}_{h''} - \sum_{\mathbf{i}} \dim \mathbf{T_i} \dim \mathbf{W_i} + \sum_{h;l'<l} \tau^{l'}_{h'} \omega^l_{h''}$$
$$+ \sum_{h;l'<l} \omega^{l'}_{h'} \tau^l_{h''} + \sum_{\mathbf{i};l<l'} \tau^{l'}_{\mathbf{i}} \omega^l_{\mathbf{i}} + \sum_{\mathbf{i};l<l'} \omega^{l'}_{\mathbf{i}} \tau^l_{\mathbf{i}} - 2\sum_{h;l'<l} \tau^{l'}_{h'} \omega^l_{h''} - 2\sum_{\mathbf{i};l<l'} \tau^{l'}_{\mathbf{i}} \omega^l_{\mathbf{i}}.$$

It follows that

$$\begin{aligned} M'(\boldsymbol{\tau}, \boldsymbol{\omega}) = &- \sum_{h;l'<l} (\tau^{l'}_{h'} \omega^l_{h''} + \tau^{l'}_{h''} \omega^l_{h'}) \\ &+ \sum_h (\dim \mathbf{T}_{h'} \dim \mathbf{W}_{h''} + \dim \mathbf{T}_{h''} \dim \mathbf{W}_{h'}) \\ &- \sum_{\mathbf{i};l<l'} \tau^{l'}_{\mathbf{i}} \omega^l_{\mathbf{i}} + \sum_{\mathbf{i};l>l'} \tau^{l'}_{\mathbf{i}} \omega^l_{\mathbf{i}} - \sum_{\mathbf{i}} \dim \mathbf{T_i} \dim \mathbf{W_i}, \end{aligned}$$

which is clearly independent of orientation.

9.3. The Categories $\mathcal{P}_{\mathbf{V};\mathbf{I}';\geq\gamma}$ and $\mathcal{P}_{\mathbf{V};\mathbf{I}';\gamma}$

9.3.1. Let $\mathbf{I}'$ be a discrete subset of $\mathbf{I}$ (see 9.1.3). Let $\gamma = \sum_{\mathbf{i}} \gamma_{\mathbf{i}}\mathbf{i} \in \mathbf{N}[\mathbf{I}]$ be such that $\gamma_{\mathbf{i}} = 0$ for all $\mathbf{i} \in \mathbf{I} - \mathbf{I}'$. Let $\mathcal{P}_{\mathbf{V};\mathbf{I}';\geq\gamma}$ be the full subcategory of $\mathcal{P}_{\mathbf{V}}$ consisting of perverse sheaves which are direct sums of simple perverse sheaves L that have the following property: there exists a graded subspace $\mathbf{W} \subset \mathbf{V}$ and an object $A \in \mathcal{Q}_{\mathbf{T},\mathbf{W}}$ such that $\mathbf{T} = \mathbf{V}/\mathbf{W}$ satisfies $\dim \mathbf{T}_{\mathbf{i}} \geq \gamma_{\mathbf{i}}$ if $\mathbf{i} \in \mathbf{I}'$, and $\mathbf{T}_{\mathbf{i}'} = 0$ for $\mathbf{i}' \notin \mathbf{I}'$; moreover, L is a direct summand of $\tilde{\mathrm{Ind}}_{\mathbf{T},\mathbf{W}}^{\mathbf{V}} A$.

Clearly,

(a) $\mathcal{P}_{\mathbf{V};\mathbf{I}';\geq\gamma} \supset \mathcal{P}_{\mathbf{V};\mathbf{I}';\geq\gamma'}$ if $\gamma' \in \mathbf{N}[\mathbf{I}]$ has support contained in $\mathbf{I}'$ and $\gamma_{\mathbf{i}} \leq \gamma'_{\mathbf{i}}$ for all $\mathbf{i} \in \mathbf{I}'$. Any object of $\mathcal{P}_{\mathbf{V}}$ is in $\mathcal{P}_{\mathbf{V};\mathbf{I}';\geq 0}$. Moreover, $\mathcal{P}_{\mathbf{V};\mathbf{I}',\geq\gamma}$ is empty if $\gamma_{\mathbf{i}} > \dim \mathbf{V}_{\mathbf{i}}$ for some $\mathbf{i} \in \mathbf{I}'$.

Let $\mathcal{P}_{\mathbf{V};\mathbf{I}';>\gamma}$ be the full subcategory of $\mathcal{P}_{\mathbf{V}}$ consisting of the objects which are in $\mathcal{P}_{\mathbf{V};\mathbf{i};\geq\gamma'}$ for some $\gamma' \in \mathbf{N}[\mathbf{I}]$ with support contained in $\mathbf{I}'$ such that $\gamma'(\mathbf{i}) \geq \gamma(\mathbf{i})$ for all $\mathbf{i} \in \mathbf{I}'$ and $\gamma'(\mathbf{i}) > \gamma(\mathbf{i})$ for some $\mathbf{i} \in \mathbf{I}'$.

Let $\mathcal{P}_{\mathbf{V};\mathbf{I}';\gamma}$ be the full subcategory of $\mathcal{P}_{\mathbf{V};\mathbf{i};\geq\gamma}$ consisting of those objects of $\mathcal{P}_{\mathbf{V};\mathbf{i};\geq\gamma}$ which are not in $\mathcal{P}_{\mathbf{V};\mathbf{i};>\gamma}$. If K is a simple object of $\mathcal{P}_{\mathbf{V}}$ and $\mathbf{V} \neq 0$, then K is a direct summand of some shift of $L_{\boldsymbol{\nu}}$ where $\boldsymbol{\nu}$ starts with $\nu^1 = \gamma$ which may be assumed to be of form $n\mathbf{i}$ for some $\mathbf{i} \in \mathbf{I}$ and some $n > 0$ (see 9.1.4); we see then that $K \in \mathcal{P}_{\mathbf{V};\{\mathbf{i}\};\geq n\mathbf{i}}$. Thus:

(b) if K is a simple object of $\mathcal{P}_{\mathbf{V}}$ and $\mathbf{V} \neq 0$, then there exists $\mathbf{i} \in \mathbf{I}$ such that $K \in \mathcal{P}_{\mathbf{V};\{\mathbf{i}\};\geq\mathbf{i}}$.

9.3.2. We now assume that $\mathbf{W} \subset \mathbf{V}$ and $\mathbf{T} = \mathbf{V}/\mathbf{W}$ are such that for any $h \in H$ we have $\mathbf{T}_{h'} = 0$ (hence $\mathbf{W}_{h'} = \mathbf{V}_{h'}$). It follows that $\mathbf{E_T} = 0$. Moreover, we have a natural imbedding $\iota : \mathbf{E_W} \to \mathbf{E_V}$; if $x = (x_h) \in \mathbf{E_W}$, then the h-component of $\iota(x) = x'$ is the composition $\mathbf{V}_{h'} = \mathbf{W}_{h'} \xrightarrow{x_h} \mathbf{W}_{h''} \subset \mathbf{V}_{h''}$. (In our case we have $\kappa : F \cong \mathbf{E_W}$ and the imbedding ι above may be identified with the imbedding $F \to \mathbf{E_V}$, see 9.2.2.) From our assumption it follows that the set $\{\mathbf{i} \in \mathbf{I} | \mathbf{T}_{\mathbf{i}} \neq 0\}$ is discrete; let $\mathbf{I}'$ be a discrete subset of $\mathbf{I}$ containing $\{\mathbf{i} \in \mathbf{I} | \mathbf{T}_{\mathbf{i}} \neq 0\}$.

We consider the locally closed subset Θ of $\mathbf{E_V}$ consisting of all $x \in \mathbf{E_V}$ such that $\dim \mathbf{V}_{\mathbf{i}}/(\sum_{h\in H:h''=\mathbf{i}} x_h(\mathbf{V}_{h'})) = \dim \mathbf{T}_{\mathbf{i}}$ for all $\mathbf{i} \in \mathbf{I}'$, and the open subset Ξ of $\mathbf{E_W}$ consisting of all $x \in \mathbf{E_W}$ such that $\sum_{h\in H:h''=\mathbf{i}} x_h(\mathbf{W}_{h'}) = \mathbf{W}_{\mathbf{i}}$ for all $\mathbf{i} \in \mathbf{I}'$.

Let $p : G \times_Q \mathbf{E_W} \to \mathbf{E_V}$ be the unique G-equivariant map such that $(1, x) \mapsto i(x)$ for all $x \in \mathbf{E_W}$; let $p_0 : G \times_Q \Xi \to \Theta$ be the restriction of p. Note that p_0 is an isomorphism. The inverse map can be described as follows. Let $x \in \Theta$. The **I**-graded subspace

$$\oplus_{\mathbf{i}\in\mathbf{I}'}\Big(\sum_{h\in H:h''=\mathbf{i}} x_h(\mathbf{V}_{h'})\Big) \oplus (\oplus_{\mathbf{i}\in\mathbf{I}-\mathbf{I}'}\mathbf{V}_{\mathbf{i}})$$

of $\mathbf{V}$ has the same dimension in each degree as $\mathbf{W}$; hence it is equal to $g(\mathbf{W})$ for some $g \in G$. The element gx is equal to $\iota(x')$ for a well-defined $x' \in \mathbf{E_W}$. Then $p_0^{-1}(x) = (g, x')$. In particular, $\Theta, G \times_Q \Xi, G \times_Q \mathbf{E_W}$ are irreducible of the same dimension.

Since p is a proper map, its image $p(G \times_Q \mathbf{E_W})$ is a closed subset of $\mathbf{E_V}$ containing Θ. Hence $\dim G \times_Q \mathbf{E_W} \geq \dim p(G \times_Q \mathbf{E_W}) \geq \dim \Theta$. It follows that these inequalities are equalities; hence Θ is open dense in $p(G \times_Q \mathbf{E_W})$.

We have a commutative diagram

$$\begin{array}{ccccc} G \times_Q \Xi & \xrightarrow{p_0} & \Theta & \xleftarrow{\iota_0} & \Xi \\ {\scriptstyle j_0}\downarrow & & {\scriptstyle j}\downarrow & & {\scriptstyle m}\downarrow \\ G \times_Q \mathbf{E_W} & \xrightarrow{p} & \mathbf{E_V} & \xleftarrow{\iota} & \mathbf{E_W} \end{array}$$

where ι_0, j, j_0, m denote the inclusions. Both squares in the diagram are cartesian.

Let $\mathcal{P}_{\mathbf{W}}^0$ be the full subcategory of $\mathcal{P}_{\mathbf{W}}$ whose objects are those perverse sheaves A such that any simple constituent of A has support which meets Ξ. Let $\mathcal{P}_{\mathbf{W}}^1$ be the full subcategory of $\mathcal{P}_{\mathbf{W}}$ whose objects are those perverse sheaves A such that the support of A is contained in $\mathbf{E_W} - \Xi$.

Let $\mathcal{P}_{\mathbf{V}}^0$ be the full subcategory of $\mathcal{P}_{\mathbf{V}}$ whose objects are those perverse sheaves B such that any simple constituent of B has support which meets Θ and is contained in the closure of Θ. Let $\mathcal{P}_{\mathbf{V}}^1$ be the full subcategory of $\mathcal{P}_{\mathbf{V}}$ whose objects are those perverse sheaves B such that the support of B is disjoint from Θ and is contained in the closure of Θ. Clearly, any object $A \in \mathcal{P}_{\mathbf{W}}$ has a canonical decomposition $A = A^0 \oplus A^1$ where $A^0 \in \mathcal{P}_{\mathbf{W}}^0$ and $A^1 \in \mathcal{P}_{\mathbf{W}}^1$. Moreover, any object $B \in \mathcal{P}_{\mathbf{V}}$ with support contained in the closure of Θ has a canonical decomposition $B = B^0 \oplus B^1$ where $B^0 \in \mathcal{P}_{\mathbf{V}}^0$ and $B^1 \in \mathcal{P}_{\mathbf{V}}^1$.

Proposition 9.3.3. (a) *Let $A \in \mathcal{P}_{\mathbf{W}}^0$. If $n \neq 0$, we have $H^n Ind_{\mathbf{T},\mathbf{W}}^{\mathbf{V}} A \in \mathcal{P}_{\mathbf{V}}^1$. If $n = 0$, then $H^n Ind_{\mathbf{T},\mathbf{W}}^{\mathbf{V}} A \in \mathcal{P}_{\mathbf{V}}$ has support contained in the closure of Θ; hence $\xi(A) = (H^{\dim G/Q} Ind_{\mathbf{T},\mathbf{W}}^{\mathbf{V}} A)^0 \in \mathcal{P}_{\mathbf{V}}^0$ is defined.*

(b) *Let $B \in \mathcal{P}_{\mathbf{V}}^0$. If $n \neq 0$, we have $H^n Res_{\mathbf{T},\mathbf{W}}^{\mathbf{V}} B \in \mathcal{P}_{\mathbf{W}}^1$. If $n = 0$, then $H^n Res_{\mathbf{T},\mathbf{W}}^{\mathbf{V}} B \in \mathcal{P}_{\mathbf{W}}$ hence $\rho(B) = (H^{-\dim G/Q} Res_{\mathbf{T},\mathbf{W}}^{\mathbf{V}} B)^0 \in \mathcal{P}_{\mathbf{W}}^0$ is defined.*

(c) *The functors* $\xi : \mathcal{P}^0_{\mathbf{W}} \to \mathcal{P}^0_{\mathbf{V}}$ *and* $\rho : \mathcal{P}^0_{\mathbf{V}} \to \mathcal{P}^0_{\mathbf{W}}$ *establish equivalences of categories inverse to each other.*

Note that j^*B is a perverse sheaf on Θ, since the support of B is contained in the closure of Θ and Θ is open in its closure. Moreover, j^*B is a G-equivariant perverse sheaf. Since $\Theta = G \times_Q \Xi$, it follows that $\iota_0^*(j^*B)[-\dim G/Q]$ is a perverse sheaf on Ξ. But $m^*\iota^*B = \iota_0^*j^*B$, hence $m^*\iota^*B[-\dim G/Q]$ is a perverse sheaf on Ξ. Since m is an open imbedding, we have $m^*(H^n\iota^*B) = H^n(m^*\iota^*B)$ for any n, and this is zero if $n \neq -\dim G/Q$. Hence if $n \neq -\dim G/Q$, the support of $H^n\iota^*B$ is disjoint from Ξ. We have $\tilde{\mathrm{Res}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}B = \mathrm{Res}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}B[\dim G/Q] = \iota^*B$ and (b) is proved.

Let $\tilde{A}$ be the perverse sheaf on $G \times_Q \mathbf{E}_{\mathbf{W}}$ such that $\tilde{A} = r'_\flat r^*A[\dim G/Q]$ in the diagram $\mathbf{E}_{\mathbf{W}} \xleftarrow{r} G/U \times \mathbf{E}_{\mathbf{W}} \xrightarrow{r'} G \times_Q \mathbf{E}_{\mathbf{W}}$. By definition,

$$\tilde{\mathrm{Ind}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}A = p_! r'_\flat r^* A = p_! \tilde{A}[-\dim G/Q].$$

This shows that $\tilde{\mathrm{Ind}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}A$ has support contained in the image of p, hence in the closure of Θ. We have

$$j^*(\tilde{\mathrm{Ind}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}A) = j^*p_!\tilde{A}[-\dim G/Q] = (p_0)_! j_0^*\tilde{A}[-\dim G/Q].$$

Since j_0 is an open imbedding, we see that $j_0^*\tilde{A}$ is a perverse sheaf on $G \times_Q \Xi$. Since $p_0 : G \times_Q \Xi \to \Theta$ is an isomorphism, it follows that $(p_0)_! j_0^*\tilde{A}$ is a perverse sheaf on Θ. Thus $j^*(\tilde{\mathrm{Ind}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}A)[\dim G/Q]$ is a perverse sheaf on Θ.

Since $\tilde{\mathrm{Ind}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}A$ has support contained in the closure of Θ, it follows that $H^n\tilde{\mathrm{Ind}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}A$ has support contained in the closure of Θ and $j^*(H^n\tilde{\mathrm{Ind}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}A) = H^n(j^*\tilde{\mathrm{Ind}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}A)$. This is zero if $n \neq \dim G/Q$. Hence for $n \neq \dim G/Q$, the support of $H^n\tilde{\mathrm{Ind}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}A$ is disjoint from Θ. This proves (a), since $\mathrm{Ind}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}A = \tilde{\mathrm{Ind}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}A[\dim G/Q]$.

From the proof of (b), we have

$$\begin{aligned} m^*(\rho(B)) &= m^*(H^{-\dim G/Q}\iota^*B) \\ &= m^*(\iota^*B[-\dim G/Q]) \\ &= \iota_0^*j^*B[-\dim G/Q]. \end{aligned}$$

This implies that $j_0^*\tilde{\rho}(B) = p_0^*j^*B$. From the proof of (a), we have

$$\begin{aligned} j^*(\xi(A)) &= j^*(H^{\dim G/Q}\tilde{\mathrm{Ind}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}A) \\ &= j^*(\tilde{\mathrm{Ind}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}}A[\dim G/Q]) = (p_0)_! j_0^*\tilde{A}. \end{aligned}$$

Hence

$$m^*(\rho(\xi(A)) = \iota_0^* j^*(\xi(A))[-\dim G/Q] = \iota_0^*(p_0)_! j_0^* \tilde{A}[-\dim G/Q] = m^* A$$

and

$$j^*(\xi(\rho(B)) = (p_0)_! j_0^* \tilde{\rho}(B) = (p_0)_! p_0^* j^* B = j^* B.$$

Since $A \in \mathcal{P}^0_{\mathbf{W}}$ and $B \in \mathcal{P}^0_{\mathbf{V}}$, it follows that $\rho(\xi(A)) = A$ and $\xi(\rho(B)) = B$. The proposition is proved.

9.3.4. Assume that $\mathbf{I}'$ is a subset of $\mathbf{I}$ such that $h' \notin \mathbf{I}'$ for any $h \in H$, that is, $\mathbf{i}$ is a *sink* of our quiver, for any $\mathbf{i} \in \mathbf{I}'$. Let $\mathbf{V} \in \mathcal{V}$. For any $\gamma = \sum_{\mathbf{i}} \gamma_{\mathbf{i}} \mathbf{i} \in \mathbf{N}[\mathbf{I}]$ with support contained in $\mathbf{I}'$, let $\mathbf{E}_{\mathbf{V};\gamma}$ be the set of all $x \in \mathbf{E}_{\mathbf{V}}$ such that

$$\dim \mathbf{V}_{\mathbf{i}} / (\sum_{h \in H : h'' = \mathbf{i}} x_h(\mathbf{V}_{h'})) = \gamma_{\mathbf{i}}$$

for any $\mathbf{i} \in \mathbf{I}'$. The sets $\mathbf{E}_{\mathbf{V};\gamma}$ form a partition of $\mathbf{E}_{\mathbf{V}}$ with the following property: for any γ as above, the union $\mathbf{E}_{\mathbf{V};\geq\gamma} = \cup_{\gamma'} \mathbf{E}_{\mathbf{V};\gamma'}$ (with γ' running over the elements of $\mathbf{N}[\mathbf{I}]$ with support contained in $\mathbf{I}'$ such that $\gamma'_{\mathbf{i}} \geq \gamma_{\mathbf{i}}$ for all $\mathbf{i} \in \mathbf{I}$) is a closed subset of $\mathbf{E}_{\mathbf{V}}$. Hence for any simple object B of $\mathcal{P}_{\mathbf{V}}$, there is a unique element $\gamma = \gamma^B \in \mathbf{N}[\mathbf{I}]$ with support contained in $\mathbf{I}'$ such that the support of B is contained in $\mathbf{E}_{\mathbf{V};\geq\gamma}$ and meets $\mathbf{E}_{\mathbf{V};\gamma}$. We have $\gamma_{\mathbf{i}} \leq \dim \mathbf{V}_{\mathbf{i}}$ for all $\mathbf{i} \in \mathbf{I}'$.

Lemma 9.3.5. *Assume that $B \in \mathcal{P}_{\mathbf{V};\mathbf{I}';\gamma'}$ where $\gamma' \in \mathbf{N}[\mathbf{I}]$ has support contained in $\mathbf{I}'$. Then $\gamma^B = \gamma'$.*

We write γ instead of γ^B. Let $\mathbf{W}$ be a graded subspace of $\mathbf{V}$ such that $\mathbf{T} = \mathbf{V}/\mathbf{W}$ satisfies $\dim \mathbf{T}_{\mathbf{i}} = \gamma_{\mathbf{i}}$ for all $\mathbf{i} \in \mathbf{I}'$ and $\mathbf{T}_{\mathbf{i}'} = 0$ for all $\mathbf{i}' \in \mathbf{I} - \mathbf{I}'$. We may apply Proposition 9.3.3 to $\mathbf{I}'$ and B. With notations there, let $A = \rho(B) \in \mathcal{P}_{\mathbf{W}}$; we have that some shift of B is a direct summand of $\widetilde{\mathrm{Ind}}^{\mathbf{V}}_{\mathbf{T},\mathbf{W}} A$. Hence $B \in \mathcal{P}_{\mathbf{V};\mathbf{I}';\geq\gamma}$.

From the definition of induction we see that any perverse sheaf in $\mathcal{P}_{\mathbf{V};\mathbf{I}';\geq\gamma'}$ has support contained in $\mathbf{E}_{\mathbf{V};\geq\gamma'}$. In particular, the support of B is contained in $\mathbf{E}_{\mathbf{V};\geq\gamma'}$. By definition, the support of B meets $\mathbf{E}_{\mathbf{V};\gamma}$; hence $\mathbf{E}_{\mathbf{V};\gamma}$ meets $\mathbf{E}_{\mathbf{V};\geq\gamma'}$, so that $\gamma_{\mathbf{i}} \geq \gamma'_{\mathbf{i}}$ for all $\mathbf{i} \in \mathbf{I}'$.

Assume that $\gamma_{\mathbf{i}} > \gamma'_{\mathbf{i}}$ for some $\mathbf{i} \in \mathbf{I}'$. Since $B \in \mathcal{P}_{\mathbf{V};\mathbf{I}';\geq\gamma}$, it follows that $B \in \mathcal{P}_{\mathbf{V};\mathbf{I}';>\gamma'}$, which contradicts our assumption that $B \in \mathcal{P}_{\mathbf{V};\mathbf{I}';\gamma'}$. Thus, we must have $\gamma_{\mathbf{i}} = \gamma'_{\mathbf{i}}$ for all $\mathbf{i} \in \mathbf{I}'$. The lemma is proved.