Part II

GEOMETRIC REALIZATION OF f

The algebra $\mathbf{f}$ has a canonical basis $\mathbf{B}$ with very remarkable properties. This gives an extremely rigid structure for $\mathbf{f}$ and also (in the Y-regular case) for each Λ_λ. Part II will introduce the canonical basis of $\mathbf{f}$. At the same time, $\mathbf{f}$ will be constructed in a purely geometric way, in terms of perverse sheaves on the moduli space of representations of a quiver.

Chapter 8 contains a review of the theory of perverse sheaves over an algebraic variety in positive characteristic. As far as definitions are concerned, it would have been possible to stay in characteristic zero and use $\mathcal{D}$-modules instead of perverse sheaves. This would certainly have been more elementary, but would have deprived us of the possibility of using the Weil conjecture and its consequences which are available in the framework of perverse sheaves on varieties in positive characteristic.

In Chapter 9 we introduce a class of perverse sheaves attached to a quiver and the operations of induction and restriction for perverse sheaves in this class. In Chapter 10, we study the Fourier-Deligne transform of perverse sheaves in our class. This is necessary for understanding the effect of changing the orientation of the quiver. In Chapter 11 we study linear categories with a given periodic functor (a functor which has some power equal to identity). These are needed to handle the case where the Cartan datum is not symmetric. (The geometry associated to a non-symmetric Cartan datum is very closely related to that associated to a symmetric Cartan datum, together with an action of a finite cyclic group.)

In Chapter 12 we study quivers with a cyclic group action. The geometric construction of $\mathbf{f}$ and of its canonical basis (up to signs) is given in Chapter 13.

In Chapter 14, we discuss various properties of the canonical basis. For example, the property expressed in Theorem 14.3.2 is responsible for the existence of a canonical basis in the simple integrable $\mathbf{U}$-modules (see Theorem 14.4.11). Perhaps the deepest property of the canonical basis is expressed by the positivity theorem 14.4.13, which states (for symmetric Cartan data) that the structure constants of **f** are given by polynomials with positive integer coefficients.

Theorem 14.4.9 gives a natural bijection between the canonical basis of **f** for a non-symmetric Cartan datum and the fixed point set of a cyclic group action on the canonical basis of the analogous algebra corresponding to a symmetric Cartan datum.

CHAPTER 8

Review of the Theory of Perverse Sheaves

8.1.1. Let p be a fixed prime number. All algebraic varieties will be over an algebraic closure k of the finite field F_p with p elements.

Let X be an algebraic variety. We denote by $\mathcal{D}(X) = \mathcal{D}_c^b(X)$ the bounded derived category of $\bar{\mathbf{Q}}_l$-(constructible) sheaves on X (see [1, 2.2.18]); here, l denotes a fixed prime number distinct from p and $\bar{\mathbf{Q}}_l$ is an algebraic closure of the field of l-adic numbers. Objects of $\mathcal{D}(X)$ are referred to as *complexes.* For a complex $K \in \mathcal{D}(X)$, we denote by $\mathcal{H}^n K$ the n-th cohomology sheaf of K (a $\bar{\mathbf{Q}}_l$-sheaf on X). We denote by $D(K) \in \mathcal{D}(X)$ the Verdier dual of K. The constant sheaf $\bar{\mathbf{Q}}_l$ on X will be denoted by $\mathbf{1}$.

For any integer j, let $K \mapsto K[j]$ be the shift functor $\mathcal{D}(X) \to \mathcal{D}(X)$; it satisfies $\mathcal{H}^n(K[j]) = \mathcal{H}^{n+j}K$. Let $f : X \to Y$ be a morphism of algebraic varieties. There are induced functors $f^* : \mathcal{D}(Y) \to \mathcal{D}(X)$, $f_* : \mathcal{D}(X) \to \mathcal{D}(Y)$, $f_! : \mathcal{D}(X) \to \mathcal{D}(Y)$. If f is proper, we have $f_* = f_!$ and $f_!(DK) = D(f_!K)$ for $K \in \mathcal{D}(X)$.

8.1.2. Let $\mathcal{M}(X)$ be the full subcategory of $\mathcal{D}(X)$ whose objects are those K in $\mathcal{D}(X)$ such that, for any integer n, the supports of both $\mathcal{H}^n K$ and $\mathcal{H}^n D(K)$ have dimension $\leq -n$. In particular, $\mathcal{H}^n K$ and $\mathcal{H}^n D(K)$ are zero for $n > 0$. The objects of $\mathcal{M}(X)$ are called *perverse sheaves* on X.

$\mathcal{M}(X)$ is an abelian category [1, 2.14, 1.3.6] in which all objects have finite length. The simple objects of $\mathcal{M}(X)$ are given by the Deligne-Goresky-MacPherson intersection cohomology complexes corresponding to various smooth irreducible subvarieties of X and to irreducible local systems on them.

For any $n \in \mathbf{Z}$, let $\tau_{\leq n} : \mathcal{D}(X) \to \mathcal{D}(X)$ and $H^n : \mathcal{D}(X) \to \mathcal{M}(X)$ be the functors of *truncation* and *perverse cohomology,* which in [1] are denoted by ${}^p\tau_{\leq n}$ and ${}^pH^n$.

There are natural morphisms

$$\tau_{\leq n-1}K \xrightarrow{\alpha_n} \tau_{\leq n}K \xrightarrow{\beta_n} (H^n K)[-n]$$

for any $K \in \mathcal{D}(X)$ and any n. For fixed K, we have $\tau_{\leq n}K = K$ for $n \gg 0$, $\tau_{\leq n}K = 0$ for $n \ll 0$ and $H^n K = 0$ for all but finitely many values of n.

For any $n \in \mathbf{Z}$, let $\mathcal{M}(X)[n]$ be the full subcategory of $\mathcal{D}(X)$ whose objects are of the form $K[n]$ for some $K \in \mathcal{M}(X)$.

8.1.3. A complex $K \in \mathcal{D}(X)$ is said to be *semisimple* if for each n,

(a) there exists $\gamma_n : (H^n K)[-n] \to \tau_{\leq n} K$ such that (α_n, γ_n) define an isomorphism $\tau_{\leq n-1} K \oplus (H^n K)[-n] \cong \tau_{\leq n} K$ and

(b) $H^n K$ is a semisimple object of $\mathcal{M}(X)$.

It follows that K is isomorphic to $\oplus_n (H^n K)[-n]$ in $\mathcal{D}(X)$.

8.1.4. Let $f : X \to Y$ be a smooth morphism with connected fibres of dimension d. We have $D(f^* K) = f^*(D(K))[2d]$ for $K \in \mathcal{D}(Y)$. (We will ignore the Tate twist.) If $K \in \mathcal{M}(Y)$, then $f^* K \in \mathcal{M}(X)[-d]$ (see [1, 4.2.4]) and $K \mapsto f^* K$ defines a fully faithful functor from $\mathcal{M}(Y)$ to $\mathcal{M}(X)[-d]$ (see [1, 4.2.5]).

8.1.5. Let $f : X \to Y$ be a proper morphism with Y smooth. Then $f_! \mathbf{1} \in \mathcal{D}(Y)$ is a semisimple complex. (See [1, 5.4.5, 5.3.8].)

8.1.6. More generally, let $f : X \to Y$ be a morphism. Assume that we are given a partition $X = X_0 \cup X_1 \cup \cdots \cup X_m$ such that $X_{\leq j} = X_0 \cup X_1 \cup \cdots \cup X_j$ is closed for $j = 0, 1, \ldots, m$. (We define $X_{\leq j} = \emptyset$ for $j < 0$.) Assume that, for each j, we are given morphisms $X_j \xrightarrow{f_j''} Z_j \xrightarrow{f_j'} Y$ such that Z_j is smooth, f_j'' is a vector bundle, f_j' is proper and $f_j' f_j'' = f_j$ where $f_j : X_j \to Y$ is the restriction of f''. Then $f_! \mathbf{1} \in \mathcal{D}(Y)$ is a semisimple complex. Moreover, for any n and j, there is a canonical exact sequence (in $\mathcal{M}(Y)$):

(a) $$0 \to H^n (f_j)_! \mathbf{1} \to H^n (f_{\leq j})_! \mathbf{1} \to H^n (f_{\leq j-1})_! \mathbf{1} \to 0$$

where $f_{\leq j} : X_{\leq j} \to Y$ is the restriction of f. The proof is essentially the same as that in [5, 3.7]; it is based on the theory of weights in [1].

8.1.7. **G-equivariant complexes.** Let $m : G \times X \to X$ be the action of a connected algebraic group G on X; let $\pi : G \times X \to X$ be the second projection. A perverse sheaf K on X is said to be *G-equivariant* if the perverse sheaves $\pi^* K[\dim G]$ and $m^* K[\dim G]$ are isomorphic. More generally, a complex $K \in \mathcal{M}(X)[n]$ is said to be G-equivariant if the perverse sheaf $K[-n]$ is G-equivariant.

We denote by $\mathcal{M}_G(X)$ the full subcategory of $\mathcal{M}(X)$ whose objects are the G-equivariant perverse sheaves on X. More generally, we denote by

$\mathcal{M}_G(X)[n]$ the full subcategory of $\mathcal{M}(X)[n]$ whose objects are of the form $K[n]$ where $K \in \mathcal{M}_G(X)$.

Here are some properties of G-equivariant complexes.

(a) If $A \in \mathcal{M}_G(X)$, and $B \in \mathcal{M}(X)$ is a subquotient of A, then $B \in \mathcal{M}_G(X)$.

(b) Assume that G acts on two varieties X', X and that $f : X' \to X$ is a morphism compatible with the G-actions. If $K \in \mathcal{M}_G(X)$, then $H^n(f^*K) \in \mathcal{M}_G(X')$ for all n. If $K' \in \mathcal{M}_G(X')$, then $H^n(f_!K') \in \mathcal{M}_G(X)$ for all n.

(c) Assume that $f : X \to Y$ is a locally trivial principal G-bundle (in particular G acts freely on X and trivially on Y). Let $d = \dim G$. If $K \in \mathcal{M}(X)[n]$, then we have $K \in \mathcal{M}_G(X)[n]$ if and only if K is isomorphic to f^*K' for some $K' \in \mathcal{M}(Y)[n+d]$. The functor $\mathcal{M}(Y)[n+d] \to \mathcal{M}_G(X)[n]$ $(K' \to f^*K')$ and the functor $\mathcal{M}_G(X)[n] \to \mathcal{M}(Y)[n+d]$ $(K \to f_\flat K := (H^{-n-d}f_*K)[n+d])$ define an equivalence of the categories $\mathcal{M}_G(X)[n], \mathcal{M}(Y)[n+d]$.

8.1.8. A semisimple complex K on a variety X with a G-action is said to be G-equivariant if for any $n \in \mathbf{Z}$, H^nK is a G-equivariant perverse sheaf on X.

Let $f : X \to Y$ be as in 8.1.7(c). If K' is a semisimple complex on Y, then f^*K' is a G-equivariant semisimple complex on X. Conversely, if K is a semisimple G-equivariant complex on X, then K is isomorphic to f^*K' for some semisimple complex K' on Y, which is unique up to isomorphism. In fact, we have $K' \cong f_\flat K$ where, by definition, $f_\flat K = \oplus_n f_\flat((H^nK)[-n])$ and $f_\flat((H^nK)[-n]) \in \mathcal{M}(Y)[-n+d]$ is as in 8.1.7(c).

8.1.9. Let A, B be two G-equivariant semisimple complexes on a variety X with G-action; let j be an integer. We choose a smooth irreducible algebraic variety Γ with a free action of G such that the $\bar{\mathbf{Q}}_l$-cohomology of Γ is zero in degrees $1, 2, \ldots, m$ where m is a large integer (compared to $|j|$).

Let us consider the diagram

$$X \xleftarrow{s} \Gamma \times X \xrightarrow{t} G\backslash(\Gamma \times X)$$

where the maps s, t are the obvious ones. Then s^*A, s^*B are semisimple G-equivariant complexes; since t is a principal G-bundle, the semisimple complexes $t_\flat s^*A, t_\flat s^*B$ on $G\backslash(\Gamma \times X)$ are well-defined. Let $u : G\backslash(\Gamma\times X) \to \{\text{point}\}$ be the obvious map. Consider the $\bar{\mathbf{Q}}_l$-vector space

$$\mathcal{H}^{j+2\dim(G\backslash\Gamma)}(u_!(t_\flat s^*A \otimes t_\flat s^*B)).$$

By a standard argument (see [6, 1.1, 1.2]), we can show that this vector space is canonically attached to A, B, j: it is independent of the choice of m and Γ provided that m is sufficiently large. We denote this vector space by $\mathbf{D}_j(X, G; A, B)$.

8.1.10. We give some properties of $\mathbf{D}_j(X, G; A, B)$.

(a) $\mathbf{D}_j(X, G; A, B) = \mathbf{D}_j(X, G; B, A)$.

(b) $\mathbf{D}_j(X, G; A[n], B[m]) = \mathbf{D}_{j+n+m}(X, G; A, B)$ for all $n, m \in \mathbf{Z}$.

(c) $\mathbf{D}_j(X, G; A \oplus A_1, B) = \mathbf{D}_j(X, G; A, B) \oplus \mathbf{D}_j(X, G; A_1, B)$.

(d) If A, B are perverse sheaves, then $\mathbf{D}_j(X, G; A, B) = 0$ for $j > 0$; if, in addition, A, B are simple, then $\mathbf{D}_0(X, G; A, B) = \bar{\mathbf{Q}}_l$, if $B \cong DA$ and $\mathbf{D}_0(X, G; A, B) = 0$, otherwise.

(e) There exists $j_0 \in \mathbf{Z}$ such that $\mathbf{D}_j(X, G; A, B) = 0$ for $j \geq j_0$.

(f) If A', B' (resp. A'', B'') are G'-equivariant (resp. G''-equivariant) semisimple complexes on a variety X' (resp. X'') with a G'-action (resp. G''-action) where G', G'' are connected algebraic groups, then $A' \otimes A''$ and $B' \otimes B''$ are $G' \times G''$-equivariant semisimple complexes on $X' \times X''$ and we have a canonical isomorphism

$$\mathbf{D}_j(X' \times X'', G' \times G''; A' \otimes A'', B' \otimes B'') = \sum_{j'+j''=j} \mathbf{D}_{j'}(X', G'; A', B') \otimes \mathbf{D}_{j''}(X'', G''; A'', B'').$$

The sum is finite by (e).

Properties (a),(b),(c) are obvious; (d) follows from [5, 7.4]; (e) follows from (d); (f) follows from the Künneth formula.

8.1.11. Fourier-Deligne transform. We fix a non-trivial character $F_p \to \bar{\mathbf{Q}}_l^*$. The Artin-Schreier covering $k \to k$ given by $x \to x^p - x$ has F_p as a group of covering transformations. Hence our character $F_p \to \bar{\mathbf{Q}}_l^*$ gives rise to a $\bar{\mathbf{Q}}_l$-local system of rank 1 on k; its inverse image under any morphism $T : X' \to k$ of algebraic varieties is a local system $\mathcal{L}_T$ of rank 1 on X'.

Let $E \to X$ and $E' \to X$ be two vector bundles of constant fibre dimension d over the variety X. Assume that we are given a bilinear map $T : E \times_X E' \to k$ which defines a duality between the two vector bundles. We have a diagram $E \xleftarrow{s} E \times_X E' \xrightarrow{t} E'$ where s, t are the obvious projections.

The Fourier-Deligne transform is the functor $\Phi : \mathcal{D}(E) \to \mathcal{D}(E')$ defined by $\Phi(K) = t_!(s^*(K) \otimes \mathcal{L}_T)[d]$. Interchanging the roles of E, E' (and keeping the same T) we have a Fourier-Deligne transform $\Phi : \mathcal{D}(E') \to \mathcal{D}(E)$; it is known that the Fourier inversion formula $\Phi(\Phi(K)) = j^*K$ holds for $K \in \mathcal{D}(E)$, where $j : E \to E$ is multiplication by -1 on each fibre of E.

Φ restricts to an equivalence of categories $\mathcal{M}(E) \to \mathcal{M}(E')$; hence it defines a bijection between the set of isomorphism classes of simple objects in $\mathcal{M}(E)$ and the analogous set for $\mathcal{M}(E')$. It also commutes with the functors $K \mapsto H^n K$.

8.1.12. Let A (resp. A') be an object of $\mathcal{D}(E)$ (resp. $\mathcal{D}(E')$). Let $u, u', \dot{u}$ be the obvious maps of $E, E', E \times_X E'$ to the point. We have

$$u_!(A \otimes \Phi(A')) = u'_!(\Phi(A) \otimes A').$$

Indeed, from the definitions, we see that both sides may be identified with $\dot{u}_!(s^*A \otimes t^*A' \otimes \mathcal{L}_T[d])$.

8.1.13. Let $T : k^n \to k$ be a non-constant affine-linear function. Let $u : k^n \to \{\text{point}\}$ be the obvious map. We have $u_!(\mathcal{L}_T) = 0$. The proof is left to the reader.