CHAPTER 7

Higher Order Quantum Serre Relations

7.1.1. In this chapter we assume that we are given $i \neq j$ in I and $e = \pm 1$. Given $n, m \in \mathbf{Z}$, we set

$$f_{i,j;n,m;e} = \sum_{r+s=m} (-1)^r v_i^{er(-\langle i,j'\rangle n-m+1)} \theta_i^{(r)} \theta_j^{(n)} \theta_i^{(s)} \in \mathbf{f}.$$

To simplify notation we shall write $f_{n,m;e}$ instead of $f_{i,j;n,m;e}$ when convenient, and we shall set $\alpha = -\langle i,j'\rangle \in \mathbf{N}$, $\alpha' = -\langle j,i'\rangle \in \mathbf{N}$.

Lemma 7.1.2. *We have (in* $\mathbf{U}$*)*

(a) $-v_i^{e(\alpha n-2m)} E_i f^+_{n,m;e} + f^+_{n,m;e} E_i = [m+1]_i f^+_{n,m+1;e}$;

(b) $-F_i f^+_{n,m;e} + f^+_{n,m;e} F_i = [\alpha n - m + 1]_i \tilde{K}_{-ei} f^+_{n,m-1;e}$.

We prove (a). The left hand side of (a) is

$$\sum_{r+s=m} (-1)^{r+1} (v_i^{er(\alpha n-m+1)+e(\alpha n-2m)} [r+1]_i E_i^{(r+1)} E_j^{(n)} E_i^{(s)}$$
$$- v_i^{er(\alpha n-m+1)} [s+1]_i E_i^{(r)} E_j^{(n)} E_i^{(s+1)})$$
$$= \sum_{r+s=m+1} (-1)^r (v_i^{er(\alpha n-m+1)-m-1} [r]_i + v_i^{er(\alpha n-m+1)} [s]_i) E_i^{(r)} E_j^{(n)} E_i^{(s)}.$$

It remains to observe that

$$v_i^{er(\alpha n-m+1)-e(m+1))} [r]_i + v_i^{er(\alpha n-m+1)} [s]_i = v_i^{er(\alpha n-m)} [m+1]_i$$

if $r + s = m + 1$.

We prove (b). Using the identity

$$F_i E_i^{(N)} = E_i^{(N)} F_i - \frac{v_i^{-N+1} \tilde{K}_i - v_i^{N-1} \tilde{K}_{-i}}{v_i - v_i^{-1}} E_i^{(N-1)},$$

we see that the left hand side of (b) is

$$\sum_{r+s=m}(-1)^{r+1}v_i^{er(\alpha n-m+1)}(E_i^{(r)}F_iE_j^{(n)}E_i^{(s)}$$
$$-\frac{v_i^{-r+1}\tilde{K}_i-v_i^{r-1}\tilde{K}_{-i}}{v_i-v_i^{-1}}E_i^{(r-1)}E_j^{(n)}E_i^{(s)})$$
$$+\sum_{r+s=m}(-1)^r v_i^{er(\alpha n-m+1)}E_i^{(r)}E_j^{(n)}E_i^{(s)}F_i$$
$$=\sum_{r+s=m}(-1)^{r+1}v_i^{er(\alpha n-m+1)}(-E_i^{(r)}E_j^{(n)}\frac{v_i^{-s+1}\tilde{K}_i-v_i^{s-1}\tilde{K}_{-i}}{v_i-v_i^{-1}}E_i^{(s-1)}$$
$$-\frac{v_i^{-r+1}\tilde{K}_i-v_i^{r-1}\tilde{K}_{-i}}{v_i-v_i^{-1}}E_i^{(r-1)}E_j^{(n)}E_i^{(s)})$$
$$=\sum_{r+s=m}(-1)^r v_i^{er(\alpha n-m+1)}\frac{v_i^{-s+1-2r+\alpha n}\tilde{K}_i-v_i^{s-1+2r-\alpha n}\tilde{K}_{-i}}{v_i-v_i^{-1}}E_i^{(r)}E_j^{(n)}E_i^{(s-1)}$$
$$+\sum_{r+s=m}(-1)^r v_i^{er(\alpha n-m+1)}\frac{v_i^{-r+1}\tilde{K}_i-v_i^{r-1}\tilde{K}_{-i}}{v_i-v_i^{-1}}E_i^{(r-1)}E_j^{(n)}E_i^{(s)}$$
$$=\sum_{r+s=m-1}(-1)^r v_i^{er(\alpha n-m+1)}\frac{v_i^{-s-2r+\alpha n}\tilde{K}_i-v_i^{s+2r-\alpha n}\tilde{K}_{-i}}{v_i-v_i^{-1}}E_i^{(r)}E_j^{(n)}E_i^{(s)}$$
$$+\sum_{r+s=m-1}(-1)^{r-1}v_i^{e(r+1)(\alpha n-m+1)}\frac{v_i^{-r}\tilde{K}_i-v_i^{r}\tilde{K}_{-i}}{v_i-v_i^{-1}}E_i^{(r)}E_j^{(n)}E_i^{(s)}$$

and (b) follows.

From Lemma 7.1.2 we deduce by induction on $p\geq 0$ the following result.

Lemma 7.1.3. *We have*
(a) $E_i^{(p)}f^+_{n,m;e}=\sum_{p'=0}^{p}(-1)^{p'}v_i^{e(2pm-\alpha pn+pp'-p')}\begin{bmatrix}m+p'\\p'\end{bmatrix}_i f^+_{n,m+p';e}E_i^{(p-p')}$;
(b) $F_i^{(p)}f^+_{n,m;e}=\sum_{p'=0}^{p}(-1)^{p'}v_i^{-e(pp'-p')}\begin{bmatrix}\alpha n-m+p'\\p'\end{bmatrix}_i\tilde{K}_{-ep'i}f^+_{n,m-p';e}F_i^{(p-p')}$.

Lemma 7.1.4. *We have*

$$F_jf^+_{n,m;e}-f^+_{n,m;e}F_j=\tilde{K}_{-ej}\frac{v_j^{e(n-1)}}{v_j^e-v_j^{-e}}f^+_{n-1,m;-e}-\tilde{K}_{ej}\frac{v_j^{-e(n-1)}}{v_j^{-e}-v_j^{e}}f^+_{n-1,m;e}.$$

We have

$$F_j f^+_{n,m;e} - f^+_{n,m;e} F_j$$

$$= - \sum_{r+s=m} (-1)^r v_i^{er(\alpha n - m+1)} E_i^{(r)} \frac{v_j^{-n+1}\tilde{K}_j - v_j^{n-1}\tilde{K}_{-j}}{v_j - v_j^{-1}} E_j^{(n-1)} E_i^{(s)}$$

$$= - \sum_{r+s=m} (-1)^r v_i^{er(\alpha n - m+1)} \frac{v_j^{-n+1+\alpha' r}\tilde{K}_j - v_j^{n-1-\alpha' r}\tilde{K}_{-j}}{v_j - v_j^{-1}} E_i^{(r)} E_j^{(n-1)} E_i^{(s)}.$$

We now use the identity $v_j^{\alpha'} = v_i^{\alpha}$; the lemma follows.

Proposition 7.1.5. (a) *If $n < 0$ or $m < 0$, then $f_{n,m;e} = 0$.*
(b) *If $m > \alpha n$, then $f_{n,m;e} = 0$.*

(a) is obvious. In particular (b) holds for $n < 0$. Hence, to prove (b), we may assume that $n \geq 0$ and that (b) holds with n replaced by $n-1$. For such fixed n, we see from 7.1.2(b) that $f^+_{n,\alpha n+1;e}$ commutes with F_i and from 7.1.4 and the induction hypothesis, that $f^+_{n,\alpha n+1;e}$ commutes with F_j. (We have $\alpha n + 1 > \alpha(n-1)$ hence the induction hypothesis is applicable to $f_{n-1,\alpha n+1;\pm 1}$.) It is trivial that $f^+_{n,\alpha n+1;e}$ commutes with F_h for any $h \neq i,j$. Thus, $f^+_{n,\alpha n+1;e}$ commutes with F_h for any $h \in I$. Using 3.2.7(a), it follows that $f_{n,\alpha n+1;e}$ is a scalar multiple of 1. On the other hand, it belongs to $\mathbf{f}_{(\alpha n+1)i+nj}$ and $(\alpha n+1)i + nj \neq 0$. It follows that $f_{n,\alpha n+1;e} = 0$.

We now show, for our fixed n, that $f_{n,m;e} = 0$ whenever $m > \alpha n$. We argue by induction on m. If $m = \alpha n + 1$, this has been just proved. Hence we may assume that $m > \alpha n+1$. Using the induction hypothesis we see that the left hand side of the identity $-v_i^{e(\alpha n - 2m+2)} E_i f^+_{n,m-1;e} + f^+_{n,m-1;e} E_i = [m]_i f^+_{n,m;e}$ (see 7.1.2) is zero. Hence we have $[m]_i f^+_{n,m;e} = 0$. We have $m \neq 0$, hence $f^+_{n,m;e} = 0$. It follows that $f_{n,m;e} = 0$ and the induction is completed. The proposition is proved.

7.1.6. The identities $f_{n,m;e} = 0 \quad (m > \alpha n; n \geq 1)$ in $\mathbf{f}$ are called the *higher order quantum Serre relations.* For $n = 1$ and $m = \alpha + 1$, they reduce to the usual quantum Serre relations.

Corollary 7.1.7. *For any $n, m \geq 0$ such that $m \geq \alpha n + 1$, we have*

$$\theta_i^{(m)} \theta_j^{(n)} = \sum_{r+s'=m; m-\alpha n \leq s' \leq m} \gamma_{s'} \theta_i^{(r)} \theta_j^{(n)} \theta_i^{(s')}$$

where $\gamma_{s'} = \sum_{q=0}^{m-\alpha n-1} (-1)^{s'+1+q} v_i^{-s'(\alpha n - m+1+q)+q} \begin{bmatrix} s' \\ q \end{bmatrix}_i$ (identity in $\mathbf{f}$).

From 7.1.5 we see that $f_{n,m-q;1} = 0$ for $0 \le q \le m - \alpha n - 1$; hence

$$g = \sum_{q=0}^{m-\alpha n-1} (-1)^q v_i^{-mq+q} f_{n,m-q;1} \theta_i^{(q)}$$

is zero. On the other hand, using the definitions, we have

$$\begin{aligned} g &= \sum_{q=0}^{m-\alpha n-1} \sum_{r+s=m-q} (-1)^r (-1)^q v_i^{r(\alpha n-m+q+1)} v_i^{-mq+q} \theta_i^{(r)} \theta_j^{(n)} \theta_i^{(s)} \theta_i^{(q)} \\ &= \sum_{r+s'=m} c_{r,s'} \theta_i^{(r)} \theta_j^{(n)} \theta_i^{(s')} \end{aligned}$$

where $c_{r,s'} = \sum_{q=0}^{m-\alpha n-1} (-1)^{r+q} v_i^{r(\alpha n-m+q+1)-mq+q} \begin{bmatrix} s' \\ q \end{bmatrix}_i$.

If $0 \le s' \le m - \alpha n - 1$, we may replace the range of summation above to $0 \le q \le s'$ and the sum will not change, since for $0 \le s' < q$, the binomial coefficient $\begin{bmatrix} s' \\ q \end{bmatrix}_i$ is zero. Hence for such s' we have $c_{r,s'} = (-1)^r v_i^{r(\alpha n-m+1)} \sum_{q=0}^{s'} (-1)^q v_i^{q(1-s')} \begin{bmatrix} s' \\ q \end{bmatrix}_i$. By 1.3.4, the last sum is zero unless $s' = 0$. Thus, for $0 \le s' \le m - \alpha n - 1$, we have $c_{r,s'} = \delta_{0,s'} (-1)^m v_i^{m(\alpha n-m+1)}$. The corollary follows.

Notes on Part I

1. The Hopf algebra **U** has been defined in the simplest case (quantum analogue of SL_2) by Kulish and Reshetikhin [10] and Sklyanin [14] and, in the general case, by Drinfeld [2] and Jimbo [5], [6]. The definition given here is different from the original one; the two definitions will be reconciled in Part V.
2. The bilinear form (,) in 1.2.3 turns out eventually to be the same as that of Drinfeld [3].
3. The idea of defining the $\mathcal{A}$-form ${}_{\mathcal{A}}\mathbf{f}$ and ${}_{\mathcal{A}}\mathbf{U}$ of **f** and **U** (see 1.4.7, 3.1.13) in terms of v-analogues of divided powers appeared in [12]. (In the classical case, the **Z**-forms of enveloping algebras were defined in terms of divided powers with ordinary factorials by Chevalley and Kostant [9], for finite types, and by Tits, for infinite types.)
4. The theorem in 2.1.2 is due to Iwahori, for finite types, and to Matsumoto and Tits [1], in the general case. The statement in 2.2.7 can be deduced from a theorem of Tits on Coxeter groups, see [1], ch. 4, p.93, statement P_n.
5. The notion of Cartan datum (resp. root datum), see 1.1.1 (resp. 2.2.1), is closely related to (but not the same as) that of a generalized Cartan matrix (resp. a realization of it) in [7]. In fact, an irreducible generalized Cartan matrix is the same as an irreducible Cartan datum up to proportionality (see 1.1.1).
6. The commutation formulas in 3.1.7, 3.1.8, are closely connected with Drinfeld's description [3] of **U** (in the formal setting) as a quantum double. Their consequence, Corollary 3.1.9, is the quantum analogue of an identity of Kostant [9] (it was shown to me by V. Kac).
7. The definition 3.5.1 of integrable **U**-modules is the quantum analogue of Kac's definition [7] of integrable modules of a Kac-Moody Lie algebra.
8. The definition of universal $\mathcal{R}$-matrices is due to Drinfeld [3]. The characterization of a modified form of the $\mathcal{R}$-matrix given in 4.1.2, as well as in 4.1.3, appeared in [13]. Propositions 4.2.2 and 4.2.4 are due to Drinfeld [3].
9. The formulas for the operators $T'_{i,e}, T''_{i,e}$ (in 5.2.1) are new (they are classical for $v=1$). An identity like 5.3.4(a) (with a different definition of $T''_{i,1}$) is stated in [8] and [11].
10. The definition of the quantum Casimir operator (see 6.1) is due to Drinfeld [4]. The proof of the complete reducibility theorem 6.2.2 is inspired by the proof of the analogous result in the non-quantum case (Kac [7]).
11. A number of statements of Drinfeld in [3] were given without proof; some of the proofs were supplied by Tanisaki [15].

REFERENCES

1. N. Bourbaki, *Groupes et algèbres de Lie, Ch. 4-6*, Hermann, 1968.
2. V. G. Drinfeld, *Hopf algebras and the quantum Yang-Baxter equation*, Soviet Math. Dokl. **32** (1985), 254–258.
3. ———, *Quantum groups*, Proc. Int. Congr. Math. Berkeley 1986, vol. 1, Amer. Math. Soc., 1988, pp. 798–820.
4. ———, *On almost cocommutative Hopf algebras*, Algebra and analysis **1** (1989), 30–46.
5. M. Jimbo, *A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985), 63–69.
6. ———, *A q-analog of $U(\mathfrak{gl}(N+1))$, Hecke algebras and the Yang-Baxter equation*, Lett. Math. Phys. **11** (1986), 247–252.
7. V. G. Kac, *Infinite dimensional Lie algebras*, Birkhäuser, Boston, 1983.
8. A. N. Kirillov and N. Yu. Reshetikhin, *q-Weyl group and a multiplicative formula for universal R-matrices*, Commun. Math. Phys. **134** (1990), 421–431.
9. B. Kostant, *Groups over* **Z**, Proc. Symp. Pure Math. **9** (1966), 90–98, Amer. Math. Soc., Providence, R. I..
10. P. P. Kulish and N. Yu. Reshetikhin, *The quantum linear problem for the sine-Gordon equation and higher representations*, (Russian), Zap. Nauchn. Sem. LOMI **101** (1981), 101–110.
11. S. Z. Levendorskii and I. S. Soibelman, *Some applications of quantum Weyl groups*, J. Geom. and Phys. **7** (1990), 241–254.
12. G. Lusztig, *Quantum deformations of certain simple modules over enveloping algebras*, Adv. Math. **70** (1988), 237–249.
13. ———, *Canonical bases in tensor products*, Proc. Nat. Acad. Sci. **89** (1992), 8177–8179.
14. E. K. Sklyanin, *On an algebra generated by quadratic relations*, Uspekhi Mat. Nauk **40** (1985), 214.
15. T. Tanisaki, *Killing forms, Harish-Chandra isomorphisms and universal R-matrices for quantum algebras*, Infinite Analysis, World Scientific, 1992, pp. 941-961.