CHAPTER 6

Complete Reducibility Theorems

6.1. The Quantum Casimir Operator

6.1.1. In this chapter we assume that the root datum is both Y-regular and X-regular.

Let B, B_ν be as in 4.1.2. Applying $m(S \otimes 1)$ to the identities at the end of 4.2.5, where $m : \mathbf{U} \otimes \mathbf{U} \to \mathbf{U}$ is multiplication, we obtain for any $p \geq 0$:

$$\sum_{\nu:\ \mathrm{tr}\ \nu \leq p} \sum_{b \in B_\nu} (-1)^{\mathrm{tr}\ \nu} v_\nu (S(E_i b^-) b^{*+} + S(\tilde{K}_i b^-) E_i b^{*+} - S(b^- E_i) b^{*+} - S(b^- \tilde{K}_{-i}) b^{*+} E_i)$$
$$= \sum_{\nu:\ \mathrm{tr}\ \nu = p} \sum_{b \in B_\nu} (-1)^p v_\nu (S(\tilde{K}_i b^-) E_i b^{*+} - S(b^- \tilde{K}_{-i}) b^{*+} E_i),$$

and

$$\sum_{\nu:\ \mathrm{tr}\ \nu \leq p} \sum_{b \in B_\nu} (-1)^{\mathrm{tr}\ \nu} v_\nu (S(b^-) F_i b^{*+} + S(F_i b^-) \tilde{K}_{-i} b^{*+} - S(b^-) b^{*+} F_i - S(b^- F_i) b^{*+} \tilde{K}_i)$$
$$= \sum_{\nu:\ \mathrm{tr}\ \nu = p} \sum_{b \in B_\nu} (-1)^p v_\nu (S(F_i b^-) \tilde{K}_{-i} b^{*+} - S(b^- F_i) b^{*+} \tilde{K}_i);$$

equivalently, setting $\Omega_{\leq p} = \sum_{\nu:\ \mathrm{tr}\ \nu \leq p} \sum_{b \in B_\nu} (-1)^{\mathrm{tr}\ \nu} v_\nu S(b^-) b^{*+}$, we have

$$\tilde{K}_{-i} E_i \Omega_{\leq p} - \tilde{K}_i \Omega_{\leq p} E_i = \sum_{\nu:\ \mathrm{tr}\ \nu = p} \sum_{b \in B_\nu} (-1)^p v_\nu (S(\tilde{K}_{-i} b^-) E_i b^{*+} - S(b^- \tilde{K}_{-i}) b^{*+} E_i),$$

and

$$-\Omega_{\leq p} F_i + F_i \tilde{K}_i \Omega_{\leq p} \tilde{K}_i = \sum_{\nu:\ \mathrm{tr}\ \nu = p} \sum_{b \in B_\nu} (-1)^p v_\nu (S(F_i b^-) \tilde{K}_{-i} b^{*+} - S(b^- F_i) b^{*+} \tilde{K}_i).$$

6.1.2. If $M \in \mathcal{C}^{hi}$, then for any $m \in M$, we have that $\Omega(m) = \Omega_{\leq p}(m) \in M$ is independent of p for large enough p. We can write $\Omega(m) = \sum_b (-1)^{\text{ tr } |b|} v_{|b|} S(b^-) b^{*+} m$ and we have

$$\text{(a)} \qquad \tilde{K}_{-i} E_i \Omega = \tilde{K}_i \Omega E_i, \quad \Omega F_i = F_i \tilde{K}_i \Omega \tilde{K}_i, \quad \Omega K_\mu = K_\mu \Omega$$

as operators on M. It follows that for $m \in M^\lambda$, we have $\Omega E_i(m) = v_i^{-2\langle i, \lambda + i' \rangle} E_i \Omega(m)$ and $\Omega F_i(m) = v_i^{2\langle i, \lambda \rangle} F_i \Omega(m)$.

6.1.3. Remark. Let us define an isomorphism of $\mathbf{Q}(v)$-algebras $\bar{S} : \mathbf{U} \to \mathbf{U}^{opp}$ by $\bar{S}(\bar{u}) = \overline{S(u)}$ (S is the antipode.) For any $u \in \mathbf{U}$, we have $S(u)\Omega = \Omega \bar{S}(u) : M \to M$. Indeed, it suffices to check this for the generators E_i, F_i, K_μ where it follows from the formulas above.

6.1.4. Let C be a fixed coset of X with respect to the subgroup $\mathbf{Z}[I] \subset X$. Let $G : C \to \mathbf{Z}$ be a function such that

(a) $G(\lambda) - G(\lambda - i') = i \cdot i \langle i, \lambda \rangle$

for all $\lambda \in C$ and all $i \in I$. Clearly, such a function exists and is unique up to addition of an arbitrary constant function $C \to \mathbf{Z}$.

Lemma 6.1.5. *Let $\lambda, \lambda' \in C \cap X^+$. Assume that $\lambda \geq \lambda'$ and $G(\lambda) = G(\lambda')$. Then $\lambda = \lambda'$.*

We can write $\lambda' = \lambda - i'_1 - i'_2 - \cdots - i'_n$ for some sequence $i_1, i_2, \ldots, i_n$ in I. Using 6.1.4(a) repeatedly, we see that

$$G(\lambda) - G(\lambda - i'_1 - i'_2 - \cdots - i'_n) = \sum_{p=1}^{n} i_p \cdot i_p \langle i_p, \lambda \rangle - \sum_{1 \leq p < q \leq n} i_p \cdot i_q.$$

Using our assumption, we have therefore that

$$\text{(a)} \qquad \sum_{p=1}^{n} i_p \cdot i_p \langle i_p, \lambda \rangle = \sum_{1 \leq p < q \leq n} i_p \cdot i_q.$$

Since $\lambda \in X^+$, we have $\langle i, \lambda \rangle \in \mathbf{N}$ for all i, hence

$$\text{(b)} \qquad \sum_{p=1}^{n} i_p \cdot i_p \langle i_p, \lambda \rangle \geq 0.$$

Similarly, since $\lambda' \in X^+$, we have

$$\sum_{p=1}^{n} i_p \cdot i_p \langle i_p, \lambda' \rangle \geq 0,$$

or equivalently,

$$\text{(c)} \qquad \sum_{p=1}^{n} i_p \cdot i_p \langle i_p, \lambda - i'_1 - i'_2 - \cdots - i'_n \rangle \geq 0.$$

Adding (b),(c) term by term gives

$$2\sum_{p=1}^{n} i_p \cdot i_p \langle i_p, \lambda \rangle - 2 \sum_{1 \leq p < q \leq n} i_p \cdot i_q - 2\sum_{p=1}^{n} i_p \cdot i_p \geq 0.$$

Introducing here the equality (a), we obtain $-2\sum_{p=1}^{n} i_p \cdot i_p \geq 0$. Since $i_p \cdot i_p > 0$ for all p, it follows that $n = 0$; hence $\lambda = \lambda'$ as required.

6.1.6. Let $M \in \mathcal{C}$. For each $\mathbf{Z}[I]$-coset C in X we define $M_C = \oplus_{\lambda \in C} M^\lambda$. It is clear that $M = \oplus_C M_C$ as a vector space and that each M_C is a $\mathbf{U}$-submodule. Hence $M = \oplus_C M_C$ as an object in $\mathcal{C}$.

Proposition 6.1.7. *Let $M \in \mathcal{C}^{hi}$.*

(a) *Assume that there exists C as above such that $M = M_C$. Let $G : C \to \mathbf{Z}$ be as in 6.1.4. We define a linear map $\Xi : M \to M$ by $\Xi(m) = v^{G(\lambda)} m$ for all $\lambda \in C$ and all $m \in M^\lambda$. Then the operator $\Omega\Xi : M \to M$ is in the commutant of the $\mathbf{U}$-module M. Moreover, the $\mathbf{Q}(v)$-linear map $\Omega\Xi : M \to M$ is locally finite.*

(b) *Assume that M is a quotient of the Verma module $M_{\lambda'}$. Then $\Omega\Xi : M \to M$ is equal to $v^{G(\lambda')}$ times identity.*

(c) *Let M be as in* (a). *Then the eigenvalues of $\Omega\Xi : M \to M$ are of the form v^c for various integers c.*

We prove (a). We have for λ, m as above:

$$\begin{aligned}\Omega\Xi E_i(m) &= v^{G(\lambda+i')}\Omega E_i(m) = v^{G(\lambda+i')-i\cdot i\langle i,\lambda+i'\rangle} E_i\Omega(m) \\ &= v^{G(\lambda+i')-G(\lambda)-i\cdot i\langle i,\lambda+i'\rangle} E_i\Omega\Xi(m) = E_i\Omega\Xi(m)\end{aligned}$$

and

$$\begin{aligned}\Omega\Xi F_i(m) &= v^{G(\lambda-i')}\Omega F_i(m) = v^{G(\lambda-i')+i\cdot i\langle i,\lambda\rangle} F_i\Omega(m) \\ &= v^{G(\lambda-i')-G(\lambda)+i\cdot i\langle i,\lambda\rangle} F_i\Omega\Xi(m) = F_i\Omega\Xi(m).\end{aligned}$$

Moreover, $\Omega\Xi$ maps each weight space of M into itself. This proves the first assertion of (a). To prove the second assertion, it suffices to show that the restriction of $\Omega\Xi$ to any weight space is locally finite. Let $m \in M^\lambda$. Let M' be the $\mathbf{U}^+$-submodule of M generated by m. Let M'' be the $\mathbf{U}$-submodule of M generated by m. We have $M'' = \mathbf{U}^- M'$. We have $\dim M' < \infty$ since $M \in \mathcal{C}^{hi}$. It follows easily that all weight spaces of M'' are finite dimensional. In particular, the λ-weight space of M'' is finite dimensional. This weight space is stable under $\Omega\Xi$ and it contains m. Thus, $\Omega\Xi : M \to M$ is locally finite.

We prove (b). From the definition, $\Omega\Xi$ acts on the λ'-weight space of M as multiplication by $v^{G(\lambda')}$ times identity. Since this weight space generates M as a $\mathbf{U}$-module, we see that (b) follows from (a). (Note that (a) is applicable to M.)

We prove (c). Let $\tilde{M}$ be the sum of the generalized eigenspaces of $\Omega\Xi : M \to M$ corresponding to eigenvalues of form v^c for various integers c. We must show that $\tilde{M} = M$. By the argument in the proof of (a), we may assume that, for any $\lambda \in \mathcal{C}$, we have $d_M(\lambda) = \sum_{\lambda' \geq \lambda} \dim M^{\lambda'} < \infty$. We will prove that, for any $\lambda \in \mathcal{C}$, we have $M^\lambda \subset \tilde{M}$, by induction on $d = d_M(\lambda)$. If $d = 0$, there is nothing to prove. Assume now that $d \geq 1$. Let $\lambda_1 \in \mathcal{C}$ be maximal such that $\lambda_1 \geq \lambda$ and $M^{\lambda_1} \neq 0$. Let m_1 be a non-zero vector in M^{λ_1}. Let M_1 be the $\mathbf{U}$-submodule of M generated by m_1. Clearly, $d_{M/M_1}(\lambda) < d_M(\lambda)$. Hence, by the induction hypothesis, we have $(M/M_1)^\lambda \subset (M/M_1)^\sim$. On the other hand, by (b), we have $M_1 \subset \tilde{M}$. It follows that $M^\lambda \subset \tilde{M}$. The proposition is proved.

The operator $\Omega\Xi : M \to M$ is called the *quantum Casimir operator.*

6.2. Complete Reducibility in $\mathcal{C}^{hi} \cap \mathcal{C}'$

Lemma 6.2.1. *Let $M \in \mathcal{C}$. Assume that M is a non-zero quotient of the Verma module M_λ and that M is integrable. Then*

(a) *$\lambda \in X^+$ and*

(b) *M is simple.*

(a) follows from 3.5.8 applied to a non-zero vector $m \in M^\lambda$.

We prove (b). Assume that M' is a subobject of M distinct from M and 0. Then clearly, $M'^\lambda = 0$. We can find $\lambda' \in X$ maximal with the property that $M'^{\lambda'} \neq 0$. Then $\lambda' < \lambda$. Let m' be a non-zero vector in $M'^{\lambda'}$. By the maximality of λ', we have $E_i m' = 0$ for all i. By 3.4.6, there exists a morphism of $\mathbf{U}$-modules from the Verma module $M_{\lambda'}$ into M' whose image contains m'. Let M'' be the image of this homomorphism. Clearly M'' is integrable (since M is integrable). Applying (a) to M'' we see that $\lambda' \in X^+$.

Applying 6.1.7(b) to M and to M'' and to the $\mathbf{Z}[I]$-coset of λ (or λ') in X, we see that $\Omega\Xi(m) = v^{G(\lambda)}m$ for all $m \in M$ and $\Omega\Xi(m) = v^{G(\lambda')}m$ for all $m \in M''$. (G as in 6.1.4.) It follows that $G(\lambda) = G(\lambda')$. This contradicts 6.1.5 since $\lambda' < \lambda$. The lemma is proved.

Theorem 6.2.2. *Let M be an integrable $\mathbf{U}$-module in $\mathcal{C}^{hi}$. Then M is a sum of simple $\mathbf{U}$-submodules.*

We may assume that $M \neq 0$. By 6.1.6, we may also assume that $M = M_C$ for some $\mathbf{Z}[I]$-coset C in X. We choose a function $G : C \to \mathbf{Z}$ as in 6.1.4 and we define $\Omega\Xi : M \to M$ (in the commutant of M) as in 6.1.7.

By writing M as a direct sum of the generalized eigenspaces of $\Omega\Xi : M \to M$ (see 6.1.7), we may further assume that there exists $c \in \mathbf{Z}$ such that $(\Omega\Xi - v^c) : M \to M$ is locally nilpotent.

Let $P = \{m \in M | E_i m = 0 \quad \forall i\}$. We have $P = \oplus_{\lambda \in C} P^\lambda$ where $P^\lambda = P \cap M^\lambda$. For any non-zero element $m \in P^\lambda$, the $\mathbf{U}$-submodule of M generated by m is of the type considered in 6.2.1; hence it is a simple subobject of M. Thus the $\mathbf{U}$-submodule M' of M generated by P is a sum of simple $\mathbf{U}$-submodules. Let $M'' = M/M'$.

Assume that $M'' \neq 0$. Then we can find $\lambda_1 \in C$ maximal such that $M''^{\lambda_1} \neq 0$. Let m_1 be a non-zero vector of M''^{λ_1}. We have $E_i m_1 = 0$ for all i. Applying 6.2.1 and 6.1.7 to the $\mathbf{U}$-submodule of M'' generated by m_1, we see that $\lambda_1 \in X^+$ and $\Omega\Xi(m_1) = v^{G(\lambda_1)}m_1$. Since $(\Omega\Xi - v^c) : M \to M$ is locally nilpotent we see that $(\Omega\Xi - v^c) : M'' \to M''$ is locally nilpotent. Hence we must have $c = G(\lambda_1)$.

Let $\tilde{m}_1 \in M^{\lambda_1}$ be a representative for m_1. As in the proof of 6.1.7, the $\mathbf{U}^+$-submodule M_1 of M generated by $\tilde{m}_1$ is a finite dimensional $\mathbf{Q}(v)$-vector space which is the sum of its intersections with the weight spaces of M. Hence we can find $\lambda_2 \in C$ maximal such that $M_1 \cap M^{\lambda_2} \neq 0$. Let m_2 be a non-zero vector in $M_1 \cap M^{\lambda_2}$. We have $E_i m_2 = 0$ for all i. Applying 6.2.1 and 6.1.7 to the $\mathbf{U}$-submodule of M generated by m_2, we see that $\lambda_2 \in X^+$ and $\Omega\Xi(m_2) = v^{G(\lambda_2)}m_2$. From the definition of c, we have that $G(\lambda_2) = c$. Hence $G(\lambda_1) = G(\lambda_2)$. Note that $\lambda_1 \in X^+, \lambda_2 \in X^+$; from the definitions, we see that $\lambda_2 \geq \lambda_1$. Using 6.1.5, we deduce that $\lambda_1 = \lambda_2$. It follows that M_1 is the one dimensional subspace spanned by $\tilde{m}_1$; hence we must have $E_i(\tilde{m}_1) = 0$ for all i, or equivalently, $\tilde{m}_1 \in P$. This implies that $m_1 = 0$, a contradiction.

We have proved that $M'' = 0$. Hence $M = M'$ and therefore M is a sum of simple $\mathbf{U}$-submodules. The theorem is proved.

Corollary 6.2.3. (a) *For any $\lambda \in X^+$, the $\mathbf{U}$-module Λ_λ is a simple object* *of* $\mathcal{C}'$.

(b) *If* $\lambda, \lambda' \in X^+$, *the* **U**-*modules* $\Lambda_\lambda, \Lambda_{\lambda'}$ *are isomorphic if and only if* $\lambda = \lambda'$.

(c) *Any integrable module in* $\mathcal{C}^{hi}$ *is a direct sum of simple modules of the form* Λ_λ *for various* $\lambda \in X^+$.

(a) follows from 6.2.1 since Λ_λ is integrable. To prove (b), it suffices to note the following property which follows from the definitions: given the **U**-module $M = \Lambda_\lambda$ where $\lambda \in X^+$, there is a unique element $\lambda_1 \in X$ such that $M^{\lambda_1} \neq 0$ and λ_1 is maximal with this property; we have $\lambda_1 = \lambda$.

We prove (c). From Theorem 6.2.2, it follows that any integrable module in $\mathcal{C}^{hi}$ is a direct sum of simple objects of $\mathcal{C}^{hi}$ which are necessarily integrable. Let M' be one of these simple summands. Let $\lambda \in X$ be maximal such that $M'^\lambda \neq 0$. Let m be a non-zero vector in M'^λ. Then $E_i m = 0$ for all i. Using 3.5.8, we can find a non-zero morphism $\Lambda_\lambda \to M'$ (in $\mathcal{C}'$). Since both Λ_λ and M' are simple, this must be an isomorphism.

6.3. Affine or Finite Cartan Data

6.3.1. In this section we assume that $(I, \cdot)$ has the following positivity property: $\sum_{i,j} i \cdot j x_i x_j \geq 0$ for all $(x_i) \in \mathbf{N}^I$. This certainly holds if $(I, \cdot)$ is of affine or finite type. We first prove an irreducibility result for certain Verma modules.

Proposition 6.3.2. *Let* $\lambda \in X$ *be such that* $\langle i, \lambda \rangle \leq -1$ *for all* i. *Then* M_λ *is simple.*

Assume that M has a non-zero **U**-submodule M' distinct from M. Let $\lambda' \in X$ be maximal with the property that $M'^{\lambda'} \neq 0$. Let m' be a non-zero vector in $M'^{\lambda'}$. Then $E_i m' = 0$ for all i. Hence there is a homomorphism of **U**-modules $M_{\lambda'} \to M'$ whose image contains m'. Using 6.1.7 for M_λ and $M_{\lambda'}$, we see that $\Omega\Xi(m') = v^{G(\lambda)} m'$ and $\Omega\Xi(m') = v^{G(\lambda')} m'$. It follows that $G(\lambda) = G(\lambda')$. We have $\lambda' < \lambda$ hence we can write $\lambda' = \lambda - i_1' - i_2' - \cdots - i_n'$ for some sequence $i_1, i_2, \ldots, i_n$ in I with $n \geq 1$. As in 6.1.5, from $G(\lambda) = G(\lambda')$, we deduce

(a) $\sum_{p=1}^n i_p \cdot i_p \langle i_p, \lambda \rangle = \sum_{p<q \in [1,n]} i_p \cdot i_q$.

Hence $(\sum_{p=1}^n i_p) \cdot (\sum_{q=1}^n i_q) = \sum_{p=1}^n i_p \cdot i_p (2\langle i_p, \lambda \rangle + 1)$. By our assumption, the left hand side is ≥ 0 and the right hand side is < 0. This contradiction proves the proposition.

6.3.3. *In the remainder of this section, we assume that* $(I, \cdot)$ *is of finite type.* In this case the root datum is automatically Y-regular and X-regular.

Proposition 6.3.4. (a) *For any* $\lambda \in X^+$, *we have* $\dim \Lambda_\lambda < \infty$.

(b) *Let* $M \in \mathcal{C}$ *be such that* $\dim M < \infty$. *Then* M *is integrable and* $M \in \mathcal{C}^{hi}$, *hence (by 6.2.2), it is a direct sum of simple* $\mathbf{U}$*-modules isomorphic to* Λ_λ *for various* $\lambda \in X^+$.

Let $\lambda' \in X$ be such that $\Lambda_\lambda^{\lambda'} \neq 0$. Using 5.2.7 several times, we see that we also have $\Lambda_\lambda^{w(\lambda')} \neq 0$ for any $w \in W$. In particular, we have $\Lambda_\lambda^{w_0(\lambda')} \neq 0$. It follows that $\lambda' \leq \lambda$ and $w_0(\lambda') \leq \lambda$. The last inequality implies, in view of 2.2.8, that $w_0(\lambda) \leq \lambda'$. Thus, we have $w_0(\lambda) \leq \lambda' \leq \lambda$. These inequalities are satisfied by only finitely many λ'. Since each weight space of Λ_λ is finite dimensional, we see that (a) holds. Now (b) is immediate since the root datum is X-regular. The proposition follows.

6.3.5. The following result is a variant of the complete reducibility theorem 6.2.2: we assume (see 6.3.3) that the Cartan matrix is of finite type but we do not need the condition that our module is in $\mathcal{C}^{hi}$.

Proposition 6.3.6. *Let* M *be an integrable* $\mathbf{U}$*-module. Then* M *is a sum of simple* $\mathbf{U}$*-modules of form* Λ_λ *for various* $\lambda \in X^+$.

Let $m \in M^\zeta$ and let M' be the $\mathbf{U}^+$-submodule of M generated by m. Since M is integrable, there exist $a_i \in \mathbf{N}$ such that $E_i^{(a_i+1)}m = 0$ for $i \in I$. Hence there exists $\lambda' \in X^+$ such that $E_i^{(\langle i,\lambda'\rangle+1)}m = 0$ for all i. It follows that $u \to um$ gives a surjective linear map $\mathbf{U}^+/(\sum_i \mathbf{U}^+ E_i^{(\langle i,\lambda'\rangle+1)}) \to M'$. Using 3.5.6, we see that $\mathbf{U}^+/(\sum_i \mathbf{U}^+ E_i^{(\langle i,\lambda'\rangle+1)})$ is isomorphic as a vector space to $\Lambda_{\lambda'}$, hence it is finite dimensional, by 6.3.4. Thus, $\dim M' < \infty$. Let M'' be the $\mathbf{U}$-submodule generated by M'. Since M' is stable under $\mathbf{U}^+$ and $\mathbf{U}^0$, M'' is equal to the $\mathbf{U}^-$-submodule generated by M'. By the argument above, the $\mathbf{U}^-$-submodule generated by a vector in M is finite dimensional. Since M' is finite dimensional, the $\mathbf{U}^-$-submodule generated by M' is finite dimensional. Thus, $\dim M'' < \infty$. We have shown that m is contained in a finite dimensional $\mathbf{U}$-submodule of M. Thus, M is a sum of finite dimensional $\mathbf{U}$-submodules. By 6.3.4(b), each of these is a sum of simple $\mathbf{U}$-modules of form Λ_λ for various $\lambda \in X^+$. The proposition follows.