# CHAPTER 42

# The *ADE* Case

## 42.1. Combinatorial Description of the Left Colored Graph

**42.1.1.** In this chapter we assume that the Cartan datum is simply laced and of finite type.

**Lemma 42.1.2.** *Consider the* $\mathbf{Q}(v)$*-algebra with generators* $\alpha, \beta$ *and relations* $\alpha^2\beta - (v+v^{-1})\alpha\beta\alpha + \beta\alpha^2 = 0, \quad \beta^2\alpha - (v+v^{-1})\beta\alpha\beta + \alpha\beta^2 = 0.$ *Set* $\gamma = \alpha\beta - v^{-1}\beta\alpha$. *For* $x = \alpha, \beta$ *or* $\gamma$, *and* $n \geq 0$, *we set* $x^{(n)} = x^n/[n]^!$; *for* $n < 0$, *we set* $x^{(n)} = 0$. *We have*

(a) $\alpha\gamma = v\gamma\alpha, \quad v\beta\gamma = \gamma\beta$;

(b) $\alpha^{(p)}\beta^{(q)} = \sum_n v^{-(p-n)(q-n)}\beta^{(q-n)}\gamma^{(n)}\alpha^{(p-n)}$;

(c) $\gamma^{(m)} = \sum_{j'+j''=m}(-1)^{j'}v^{-j'}\beta^{(j')}\alpha^{(m)}\beta^{(j'')}$;

(d) $\alpha^{(p)}\beta^{(q)}\alpha^{(r)} = \sum_{m,n\geq 0; m+n=p+r-q} \begin{bmatrix} m+n \\ m \end{bmatrix}\beta^{(r-m)}\alpha^{(p+r)}\beta^{(p-n)}$, *if* $p+r \geq q$;

(e) $\beta^{(p)}\alpha^{(q)}\beta^{(r)} = \sum_{m,n\geq 0; m+n=p+r-q} \begin{bmatrix} m+n \\ m \end{bmatrix}\alpha^{(r-m)}\beta^{(p+r)}\alpha^{(p-n)}$, *if* $p+r \geq q$;

(f) $\alpha^{(p)}\beta^{(q)}\alpha^{(r)} = \sum_{n; n\leq p} v^{-(p-n)(q-n)} \begin{bmatrix} p-n+r \\ p-n \end{bmatrix}\beta^{(q-n)}\gamma^{(n)}\alpha^{(p-n+r)}$;

(g) $\beta^{(p)}\alpha^{(q)}\beta^{(r)} = \sum_{n; n\leq r} v^{-(q-n)(r-n)} \begin{bmatrix} r-n+p \\ r-n \end{bmatrix}\beta^{(r-n+p)}\gamma^{(n)}\alpha^{(q-n)}$;

(h) $\alpha^{(p)}\beta^{(p+r)}\alpha^{(r)} = \beta^{(r)}\alpha^{(p+r)}\beta^{(p)}$.

(a) is obvious.

Now (b) is obvious when $p \leq 0$ or $q \leq 0$. For $q = 1$, (b) states that $\alpha^{(p)}\beta = v^{-p}\beta\alpha^{(p)} + \gamma\alpha^{(p-1)}$; this is proved by induction on $p \geq 1$, using (a). Assume now that $q \geq 2$ and that (b) is known when $q$ is replaced by $q-1$. We write (b) for $(p, q-1)$ and multiply it on the right by $\beta$. Using the case $q = 1$, we substitute

$$\beta^{(q-1-n)}\gamma^{(n)}\alpha^{(p-n)}\beta = \beta^{(q-1-n)}\gamma^{(n)}(v^{-p+n}\beta\alpha^{(p-n)} + \gamma\alpha^{(p-n-1)}).$$

This can be rearranged using (a) and yields (b) for $(p, q)$. Thus (b) is proved.

To prove (c), we replace $\alpha^{(m)}\beta^{(j'')}$ in the right hand side of (c) by the expression provided by (b); we perform cancellations, and we obtain (c).

To prove (d), we replace $\alpha^{(p)}\beta^{(q)}$ on the left hand side and $\alpha^{(p+r)}\beta^{(p-n)}$ on the right hand side by the expressions provided by (b); we perform cancellations, and we obtain (d). Now (e) follows from (d) by symmetry; (f) and (g) follow immediately from (b) and (h) is a special case of either (d) or (e). Note that (h) is a special case of the quantum Verma identity 39.3.7.

**42.1.3.** Let $\mathbf{H}$ be the set of all sequences $\mathbf{h} = (i_1, i_2, \dots, i_n)$ in $I$ such that $s_{i_1} s_{i_2} \cdots s_{i_n}$ is a reduced expression for $w_0$. (Thus, $n = l(w_0)$.)

We shall regard $\mathbf{H}$ as the set of vertices of a graph in which $\mathbf{h} = (i_1, i_2, \dots, i_n)$ and $\mathbf{h}' = (j_1, j_2, \dots, j_n)$ are joined if $\mathbf{h}'$ is obtained from $\mathbf{h}$ by

(a) replacing three consecutive entries $i, j, i$ in $\mathbf{h}$ (with $i \cdot j = -1$) by $j, i, j$ or by

(b) replacing two consecutive entries $i, j$ in $\mathbf{h}$ (with $i \cdot j = 0$) by $j, i$.

For such joined $(\mathbf{h}, \mathbf{h}')$, i.e., in case (a) (resp. (b)) we define a map $R_{\mathbf{h}}^{\mathbf{h}'} : \mathbf{N}^n \cong \mathbf{N}^n$ as follows. This map takes $\mathbf{c} = (c_1, \dots, c_n) \in \mathbf{N}^n$ to $\mathbf{c}' = (c'_1, \dots, c'_n) \in \mathbf{N}^n$ which has the same coordinates as $\mathbf{c}$ except in the three (resp. two) consecutive positions at which $(\mathbf{h}, \mathbf{h}')$ differ; if $(a, b, c)$ (resp. $(a, b)$) are the coordinates of $\mathbf{c}$ at those three (resp. two) positions, the coordinates of $\mathbf{c}'$ at those positions are

$$(b + c - \min(a, c), \min(a, c), a + b - \min(a, c)) \qquad (\text{resp. } (b, a)).$$

It is easy to check that $R_{\mathbf{h}}^{\mathbf{h}'}$ is a bijection; its inverse is $R_{\mathbf{h}'}^{\mathbf{h}}$.

From 2.1.2, it follows that

(c) the graph $\mathbf{H}$ is connected.

**42.1.4.** Given $\mathbf{h} = (i_1, \dots, i_n) \in \mathbf{H}$ and $\mathbf{c} = (c_1, \dots, c_n) \in \mathbf{N}^n$, we define

(a) $E_{\mathbf{h}}^{\mathbf{c}} =$
$$E_{i_1}^{(c_1)} T'_{i_1,-1}(E_{i_2}^{(c_2)}) T'_{i_1,-1} T'_{i_2,-1}(E_{i_3}^{(c_3)}) \cdots T'_{i_1,-1} T'_{i_2,-1} \cdots T'_{i_{n-1},-1}(E_{i_n}^{(c_n)}).$$

According to 41.1.3, 40.2.2, the elements $E_{\mathbf{h}}^{\mathbf{c}}$ $(\mathbf{c} \in \mathbf{N}^n)$ are contained in ${}_{\mathcal{A}}\mathbf{U}^+$ and form a $\mathbf{Q}(v)$-basis of $\mathbf{U}^+$. We shall denote this basis by $B_{\mathbf{h}}$. Hence, given $\mathbf{h}, \mathbf{h}' \in \mathbf{H}$ and $\mathbf{c} \in \mathbf{N}^n$, we can write uniquely

$$E_{\mathbf{h}}^{\mathbf{c}} = \sum_{\mathbf{c}' \in \mathbf{N}^n} \gamma_{\mathbf{h},\mathbf{h}'}^{\mathbf{c},\mathbf{c}'} E_{\mathbf{h}'}^{\mathbf{c}'}$$

where $\gamma_{\mathbf{h},\mathbf{h}'}^{\mathbf{c},\mathbf{c}'} \in \mathbf{Q}(v)$.

**Proposition 42.1.5.** (a) *Assume that* $\mathbf{h}, \mathbf{h}'$ *are joined in the graph* $\mathbf{H}$. *For* $\mathbf{c}, \mathbf{c}' \in \mathbf{N}^n$, $\gamma_{\mathbf{h},\mathbf{h}'}^{\mathbf{c},\mathbf{c}'}$ *is in* $\mathbf{Z}[v^{-1}]$. *Its constant term is 1 if* $\mathbf{c}' = R_{\mathbf{h}}^{\mathbf{h}'}(\mathbf{c})$ *and is zero otherwise.*

(b) *For* $\mathbf{h} \in \mathbf{H}$, *let* $\mathcal{L}_{\mathbf{h}}$ *be the* $\mathbf{Z}[v^{-1}]$*-submodule of* $\mathbf{U}^+$ *generated by the basis* $B_{\mathbf{h}}$. *Then* $\mathcal{L}_{\mathbf{h}}$ *is independent of* $\mathbf{h} \in \mathbf{H}$. *We denote it by* $\mathcal{L}$.

(c) *For* $\mathbf{h} \in \mathbf{H}$, *let* $\pi : \mathcal{L} \to \mathcal{L}/v^{-1}\mathcal{L}$ *be the canonical projection. Then* $\pi(B_{\mathbf{h}})$ *is a* $\mathbf{Z}$*-basis of* $\mathcal{L}/v^{-1}\mathcal{L}$, *independent of* $\mathbf{h} \in \mathbf{H}$; *we denote it by* $B$.

Assume that the proposition is known in the special case in which $I$ consists of two elements $i, j$. Using the definitions and the fact that the $T'_{i,-1} : \mathbf{U} \to \mathbf{U}$ are algebra homomorphisms satisfying the braid relations, we see that (a) in the general case is a consequence of (a) in the special case. To prove (in the general case) that the objects defined in (b),(c) in terms of $\mathbf{h}, \mathbf{h}' \in \mathbf{H}$ coincide, we may assume, in view of the connectedness of the graph $\mathbf{H}$, that $\mathbf{h}, \mathbf{h}'$ are joined in $\mathbf{H}$, in which case the desired statements follow immediately from (a).

Thus, we may assume that we are in the special case above. In the case where $i \cdot j = 0$, the result is trivial. Hence we may assume that $i \cdot j = j \cdot i = -1$. Now $\mathbf{H}$ consists of two elements: $\mathbf{h} = (i, j, i)$, $\mathbf{h}' = (j, i, j)$. Besides $\mathcal{L}_{\mathbf{h}}, \mathcal{L}_{\mathbf{h}'}$, we introduce the $\mathbf{Z}[v^{-1}]$-submodule $\mathcal{L}$ of $\mathbf{U}^+$ generated by the set

$$B' = \{E_i^{(p)} E_j^{(q)} E_i^{(r)} | q \ge p + r\} \cup \{E_j^{(p)} E_i^{(q)} E_j^{(r)} | q \ge p + r\}$$

in which we identify $E_i^{(p)} E_j^{(q)} E_i^{(r)} = E_j^{(r)} E_i^{(q)} E_j^{(r)}$ for $q = p + r$.

By definition, we have $T'_{i,-1}(E_j) = E_j E_i - v^{-1} E_i E_j = T''_{j,1}(E_i)$ and $T'_{j,-1}(E_i) = E_i E_j - v^{-1} E_j E_i = T''_{i,1}(E_j)$. It follows that $T'_{i,-1} T'_{j,-1} E_i = E_j$ and $T'_{j,-1} T'_{i,-1} E_j = E_i$. Hence, if $\mathbf{c} = (c_1, c_2, c_3) \in \mathbf{N}^3$ and $\mathbf{c}' = (c'_1, c'_2, c'_3) \in \mathbf{N}^3$, we have

$$E_{\mathbf{h}}^{\mathbf{c}} = E_i^{(c_1)} (E_j E_i - v^{-1} E_i E_j)^{(c_2)} E_j^{(c_3)}$$

and

$$E_{\mathbf{h}'}^{\mathbf{c}'} = E_j^{(c'_1)} (E_i E_j - v^{-1} E_j E_i)^{(c'_2)} E_i^{(c'_3)},$$

where the notation $x^{(c)}$ is as in Lemma 42.1.2. Let $(p, q, r) \in \mathbf{N}^3$ be such that $q \ge p + r$. From 42.1.2(f), we have

$$E_i^{(p)} E_j^{(q)} E_i^{(r)} = \sum_{n=0}^{p} v^{-(p-n)(q-n)} \begin{bmatrix} p - n + r \\ p - n \end{bmatrix} E_{\mathbf{h}'}^{q-n, n, p-n+r}$$

where

$$v^{-(p-n)(q-n)}\begin{bmatrix} p-n+r \\ p-n \end{bmatrix} \in v^{-(p-n)(q-n-r)}(1+v^{-1}\mathbf{Z}[v^{-1}])$$

is in $v^{-1}\mathbf{Z}[v^{-1}]$, if $n<p$ and it equals 1 if $n=p$. Similarly,

$$E_j^{(p)}E_i^{(q)}E_j^{(r)} = \sum_{n=0}^{r} v^{-(q-n)(r-n)}\begin{bmatrix} r-n+p \\ r-n \end{bmatrix} E_{\mathbf{h}'}^{r-n+p,n,q-n}$$

where

$$v^{-(q-n)(r-n)}\begin{bmatrix} r-n+p \\ r-n \end{bmatrix} \in v^{-(r-n)(q-p-n)}(1+v^{-1}\mathbf{Z}[v^{-1}])$$

is in $v^{-1}\mathbf{Z}[v^{-1}]$, if $n<r$ and it equals 1 if $n=r$. These formulas show that $\mathcal{L}_{\mathbf{h}'} = \mathcal{L}$ and, if $\pi:\mathcal{L}\to\mathcal{L}/v^{-1}\mathcal{L}$ is the canonical map, we have $\pi(B')=\pi(B_{\mathbf{h}'})$; moreover, $\pi$ maps $B'$ onto $\pi(B')$ bijectively.

By the symmetry between $i$ and $j$, there is an analogous statement for $\mathbf{h}$ (note that $\mathcal{L}, B'$ are symmetric in $i,j$). Thus, we have $\mathcal{L}_{\mathbf{h}}=\mathcal{L}$ and $\pi(B')=\pi(B_{\mathbf{h}})$. It follows that (b),(c) hold. The formulas above show also that (a) holds. The proposition is proved.

**Corollary 42.1.6.** *The $\mathcal{A}$-subalgebra ${}_{\mathcal{A}}\mathbf{U}^+$ of $\mathbf{U}^+$ coincides with the $\mathcal{A}$-submodule ${}_{\mathcal{A}}\mathcal{L}$ of $\mathbf{U}^+$ generated by $\mathcal{L}$.*

The fact that ${}_{\mathcal{A}}\mathcal{L}\subset{}_{\mathcal{A}}\mathbf{U}^+$ has been noted in 42.1.4. To prove the reverse inclusion, it suffices to show that for any $i\in I$ and any $s\in\mathbf{N}$, ${}_{\mathcal{A}}\mathcal{L}$ is stable under x multiplication by $E_i^{(s)}$. Now ${}_{\mathcal{A}}\mathcal{L}$ has an $\mathcal{A}$-basis formed by the elements $E_{\mathbf{h}}^{\mathbf{c}}$ where $\mathbf{h}$ is a fixed element of $\mathbf{H}$ which starts with $i$ and $\mathbf{c}$ runs through $\mathbf{N}^n$. Multiplication by $E_i^{(s)}$ takes each element of this basis to an $\mathcal{A}$-multiple of another element of this basis. The corollary follows.

**42.1.7.** From the definitions it is clear that $\mathcal{L}=\oplus_\nu\mathcal{L}_\nu$ where $\mathcal{L}_\nu=\mathcal{L}\cap\mathbf{U}_\nu^+$ for any $\nu\in\mathbf{N}[I]$. This induces a direct sum decomposition $\mathcal{L}/v^{-1}\mathcal{L} = \oplus_\nu\mathcal{L}_\nu/v^{-1}\mathcal{L}_\nu$. It is clear that $B$ is compatible with this decomposition; in other words, we have $B=\sqcup_\nu B(\nu)$ where $B(\nu)$ is the intersection of $B$ with the summand $\mathcal{L}_\nu/v^{-1}\mathcal{L}_\nu$ of $\mathcal{L}/v^{-1}\mathcal{L}$.

**42.1.8.** We consider the equivalence relation on $\mathbf{H}\times\mathbf{N}^n$ generated by $(\mathbf{h},\mathbf{c})\sim(\mathbf{h}',\mathbf{c}')$ whenever $\mathbf{h},\mathbf{h}'$ are joined in $\mathbf{H}$ and $R_{\mathbf{h}}^{\mathbf{h}'}(\mathbf{c})=\mathbf{c}'$. Let $\hat{\mathbf{H}}$ be the set of equivalence classes.

**Lemma 42.1.9.** *For any* $\mathbf{h} \in \mathbf{H}$*, the map* $f : \mathbf{N}^n \to \hat{\mathbf{H}}$*, which takes any* $\mathbf{c}$ *to the equivalence class of* $(\mathbf{h}, \mathbf{c})$*, is a bijection.*

From 42.1.5(a), we see that the (surjective) map $\mathbf{H} \times \mathbf{N}^n \to B$ given by $(\mathbf{h}, \mathbf{c}) \mapsto \pi(E_{\mathbf{h}}^{\mathbf{c}})$ is constant on equivalence classes; hence it factors through a (surjective) map $\hat{\mathbf{H}} \to B$. On the other hand, for any $\mathbf{h} \in \mathbf{H}$, the composition $\mathbf{N}^n \xrightarrow{f} \hat{\mathbf{H}} \to B$ is a bijection (again by 42.1.5). The lemma follows.

We have the following result.

**Theorem 42.1.10.** (a) *For any* $b \in B$ *there is a unique element* $\tilde{b} \in \mathcal{L}$ *such that* $\pi(\tilde{b}) = b$ *and* $\bar{\tilde{b}} = \tilde{b}$*.*

(b) *The set* $\{\tilde{b} | b \in B\}$ *is a* $\mathbf{Z}[v^{-1}]$*-basis of* $\mathcal{L}$ *and a* $\mathbf{Q}(v)$*-basis of* $\mathbf{U}^+$*.*

We shall regard the pairing $(,)$ on $\mathbf{f}$ as a pairing on $\mathbf{U}^+$ via the isomorphism $\mathbf{f} \to \mathbf{U}^+$ given by $x \mapsto x^+$. Let $\mathbf{h} \in \mathbf{H}$. By 38.2.3, the basis $E_{\mathbf{h}}^{\mathbf{c}}$ of $\mathbf{U}^+$, where $\mathbf{c}$ is running through $\mathbf{N}^n$, is almost orthonormal for $(,)$. Applying 14.2.2(b) to this basis, we see that any element $\beta \in \mathbf{B}$ satisfies $\beta^+ \in \mathcal{L}$ and $\pi(\beta^+) \in \pm B$. In particular, we have $\mathcal{L}(\mathbf{f}) \subset \mathcal{L}$. Applying 14.2.2(b) to the canonical basis $\mathbf{B}$ of $\mathbf{f} = \mathbf{U}^+$ and to $x = E_{\mathbf{h}}^{\mathbf{c}}$, which satisfies $(x, x) \in 1 + v^{-1}\mathbf{Z}[[v^{-1}]] \cap \mathbf{Q}(v)$ by 38.2.3, we see that $x \in \mathcal{L}(\mathbf{f})$; hence, by the previous sentence, $\mathcal{L} = \mathcal{L}(\mathbf{f})$ and $\pi(x) = \pm\pi(\beta^+)$ for some $\beta \in \mathbf{B}$. Since $\beta^+ \subset \mathcal{L}(\mathbf{f}) = \mathcal{L}$, we see that the existence statement in (a) holds. The uniqueness in (a), as well as statement (b) now follow from the known properties of $\mathbf{B}$. The theorem is proved.

**42.1.11.** We keep the notation from the proof of Theorem 42.1.10. We fix $i \in I$. Assume that $\mathbf{h} \in \mathbf{H}$ starts with $i$. Let $b \in B$ be such that $b = \pi(E_{\mathbf{h}}^{\mathbf{c}})$ where the first coordinate of $\mathbf{c}$ is 0. Let $\mathbf{c}' \in \mathbf{N}^n$ be such that $\mathbf{c}'$ has the same coordinates as $\mathbf{c}$ except for the first coordinate, which is $s \in \mathbf{N}$. Let $b' = \pi(E_{\mathbf{h}}^{\mathbf{c}'}) \in B$. We shall use the following notation. For $b \in B$, we define $\beta_b \in \mathcal{B}$ by $\beta_b^+ = \tilde{b}$ (see the proof of 42.1.10).

**Lemma 42.1.12.** (a) *Write* $E_{\mathbf{h}}^{\mathbf{c}} = z^+$ *where* $z \in \mathbf{f}$*. Then* $z \in \mathbf{f}[i]$ *and* $E_{\mathbf{h}}^{\mathbf{c}'} = (\tilde{\phi}_i^s z)^+$*.*

(b) *We have* $\beta_{b'} = \tilde{\phi}_i^s \beta_b \mod v^{-1}\mathcal{L}(\mathbf{f})$*.*

(c) *We have* $\beta_b \in \mathcal{B}_{i;0}$*.*

(d) *We have* $\beta_{b'} = \pi_{i,s}(\beta_b)$*.*

Since $\mathbf{c}$ starts with 0, the element $E_{\mathbf{h}}^{\mathbf{c}} \in \mathbf{U}^+$ is in $T'_{i,-1}\mathbf{U}^+$, hence by 38.1.6, $z$ is in $\mathbf{f}[i]$, and ${}_ir(z) = 0$; hence $\tilde{\phi}_i^s(z) = \theta_i^{(s)}z$. It follows that $\tilde{\phi}_i^s(z)^+ = E_i^{(s)}E_{\mathbf{h}}^{\mathbf{c}} = E_{\mathbf{h}}^{\mathbf{c}'}$. This proves (a).

We prove (b). Using (a) and the definitions, we have $z = \beta_b \mod v^{-1}\mathcal{L}(\mathbf{f})$ and $\tilde{\phi}_i^s z = \beta_{b'} \mod v^{-1}\mathcal{L}(\mathbf{f})$. Since $\tilde{\phi}_i^s$ maps $v^{-1}\mathcal{L}(\mathbf{f})$ into itself it follows that $\tilde{\phi}_i^s z = \tilde{\phi}_i^s(\beta_b) \mod v^{-1}\mathcal{L}(\mathbf{f})$, hence $\tilde{\phi}_i^s(\beta_b) = \beta_{b'} \mod v^{-1}\mathcal{L}(\mathbf{f})$. This proves (b).

From the fact that the elements $E_{\mathbf{h}}^{\mathbf{c}''}$, where $\mathbf{c}''$ runs through $\mathbf{N}^n$, form a basis of $\mathbf{U}^+$, it follows immediately that

(e) for any $t \geq 0$, the elements $E_{\mathbf{h}}^{\mathbf{c}''}$ where $\mathbf{c}''$ runs through the elements of $\mathbf{N}^n$ with first coordinate $\geq t$, form a $\mathbf{Q}(v)$-basis of $E_i^t\mathbf{U}^+$.

We prove (c). Assume that $\beta_b \in \mathcal{B}_{i;t}$ with $t > 0$. Then $\beta_b^+ \in E_i^t\mathbf{U}^+$, hence it is a linear combination of elements as in (e); in particular, $E_{\mathbf{h}}^{\mathbf{c}}$ appears with coefficient 0, contradicting the definition of $\beta_b$. This proves (c). Since $\beta_b, \beta_{b'} \in \mathcal{B}$, we see from 17.3.7 that (d) follows from (b) and (c). This completes the proof.

**Corollary 42.1.13.** *We have* $\mathbf{B} = \{\tilde{b} | b \in B\}$. *(We identify* $\mathbf{f} = \mathbf{U}^+$ *as above.)*

From the proof of 42.1.10, we have that $\{\tilde{b} | b \in B\} \subset \mathcal{B}$. We show by induction on $N = \operatorname{tr} \nu$ that $\tilde{b} \in \mathbf{B}$, if $b \in B_\nu$. If $N = 0$, this is clear. Assume that $N \geq 1$. By 14.3.3, we can find $i \in I$ and $s > 0$ such that $\tilde{b} \in \mathcal{B}_{i;s}$. We then have $\tilde{b} = \pi_{i,s}\beta$ where $\beta \in \mathcal{B}_{i;0}$ (see 14.3.2). We have $\beta = \pm\tilde{b}_1$ where $b_1 \in B$. By the argument in the previous lemma we have that the sign is $+$, hence $\tilde{b} = \pi_{i,s}\tilde{b}_1$. By the induction hypothesis, we have $\tilde{b}_1 \in \mathbf{B}$; the previous equality then implies that $\tilde{b} \in \mathbf{B}$.

**42.1.14.** The basis $\{\tilde{b} | b \in B\} = \{\beta^+ | \beta \in \mathbf{B}\}$ of $\mathbf{U}^+$ is in a natural bijection with the set $B$, which in turn is in a natural bijection with the set $\hat{\mathbf{H}}$ (see the proof of 42.1.9). We thus have a purely combinatorial parametrization of the canonical basis $\mathbf{B}$.

The structure of left colored graph on $\mathbf{B}$ (see 14.4.7) corresponds to a structure of colored graph on $\hat{\mathbf{H}}$, which we will now describe in a purely combinatorial way.

For any $i \in I$, we define a function $g_i : \hat{\mathbf{H}} \to \mathbf{N}$ as follows. Let $c \in \hat{\mathbf{H}}$; we choose $\mathbf{h} \in \mathbf{H}$ such that the sequence $\mathbf{h}$ starts with $i$. By 42.1.9, $c$ is the class of $(\mathbf{h}, \mathbf{c})$ for a unique $\mathbf{c} \in \mathbf{N}^n$. We set $g_i(c) = c_1$ where $c_1$ is the first coordinate of $\mathbf{c}$. To show that this is well-defined, we consider $\mathbf{h}' \in \mathbf{H}$ such that the sequence $\mathbf{h}$ starts with $i$. Let $\mathbf{c}' \in \mathbf{N}^n$ be such that $c$ is the class of $(\mathbf{h}', \mathbf{c}')$ and let $c_1'$ be the first coordinate of $\mathbf{c}'$. We must show that $c_1 = c_1'$. Now the set $\mathbf{H}_i$ of all sequences in $\mathbf{H}$ which start with $i$ can be naturally identified with the set of reduced expressions for $s_i w_0$; applying 2.1.2, we see that $\mathbf{H}_i$, regarded as a full subgraph of $\mathbf{H}$ is connected. Hence to prove that $c_1 = c_1'$, we may assume that $\mathbf{h}, \mathbf{h}'$ are joined in the graph. Then $\mathbf{c}$ and $\mathbf{c}'$ are related by an elementary move as in 42.1.3(a) or (b). This elementary move operates on coordinates other than the first, since $\mathbf{h}, \mathbf{h}'$ start with the same element $i$. Thus, we have $c_1 = c_1'$, as desired.

**42.1.15.** For any $i \in I$, we define a partition $\hat{\mathbf{H}} = \sqcup_{t \geq 0} \hat{\mathbf{H}}_{i,t}$ by setting $\hat{\mathbf{H}}_{i,t} = g_i^{-1}(t)$. We define a bijection $\pi_{i,t} : \hat{\mathbf{H}}_{i,0} \cong \hat{\mathbf{H}}_{i,t}$ as follows. Let $(\mathbf{h}, \mathbf{c})$ be a representative for an element of $\hat{\mathbf{H}}_{i,0}$. Then $\mathbf{c}$ starts with 0; let $\mathbf{c}'$ be the element of $\mathbf{N}^n$ which starts with $t$ and has the same subsequent coordinates as those of $\mathbf{c}$. By definition, $\pi_{i,t}(\mathbf{h}, \mathbf{c}) = (\mathbf{h}, \mathbf{c}')$. One checks that this map is well-defined. From our earlier discussion, it is clear that the partitions of $\hat{\mathbf{H}}$ just described, together with the bijections $\pi_{i,t}$, correspond to the analogous objects for $\mathbf{B}$ which are the ingredients in the definition of the left colored graph.

**42.1.16.** We can also describe in purely combinatorial terms the left colored graph for not necessarily simply laced Cartan data, by reduction to the simply laced case, using 14.4.9 and 14.1.6.

## 42.2. Remarks on the Piecewise Linear Bijections $R_{\mathbf{h}}^{\mathbf{h}'} : \mathbf{N}^n \cong \mathbf{N}^n$

**42.2.1.** Let $\mathbf{h}, \mathbf{h}' \in \mathbf{H}$. We define a bijection $R_{\mathbf{h}}^{\mathbf{h}'} : \mathbf{N}^n \cong \mathbf{N}^n$ as a composition

(a) $R_{\mathbf{h}}^{\mathbf{h}'} = R_{\mathbf{h}(0)}^{\mathbf{h}(1)} R_{\mathbf{h}(1)}^{\mathbf{h}(2)} \cdots R_{\mathbf{h}(t-1)}^{\mathbf{h}(t)}$

where $\mathbf{h}(0), \mathbf{h}(1), \dots, \mathbf{h}(t)$ is a sequence of vertices of the graph $\mathbf{H}$ such that $\mathbf{h}(0) = \mathbf{h}, \mathbf{h}(t) = \mathbf{h}'$ and such that $\mathbf{h}(s), \mathbf{h}(s+1)$ is an edge of the graph $\mathbf{H}$ for $s = 0, 1, \dots, t-1$; the factors on the right hand side of (a) are the bijections defined in 42.1.3. (A sequence as above can always be found, by 2.1.2.) From 42.1.5, it follows that the definition of $R_{\mathbf{h}}^{\mathbf{h}'}$ is correct, that is, it does not depend on the choices made. Indeed, 42.1.5 gives us an intrinsic definition of this bijection: with the notation in 42.1.5, we have $R_{\mathbf{h}}^{\mathbf{h}'}(\mathbf{c}) = \mathbf{c}'$ if and only if $\pi(E_{\mathbf{h}}^{\mathbf{c}}) = \pi(E_{\mathbf{h}'}^{\mathbf{c}'})$. The bijections $R_{\mathbf{h}}^{\mathbf{h}'}$ are piecewise linear, since they are products of factors which are piecewise linear.

**42.2.2.** In this section we will show that the bijections $R_{\mathbf{h}}^{\mathbf{h}'}$ also appear in a completely different context.

Let $K$ be a field with a given subset $K_0 \subset K - \{0\}$ containing 1, and such that the following holds:

(a) if $f, f' \in K_0$, then $f + f' \in K_0$, $ff' \in K_0$, $\frac{f}{f+f'} \in K_0$.

For example, we could take

(b) $K = \mathbf{R}, K_0 = \mathbf{R}_{>0}$ or

(c) $K = \mathbf{R}((\epsilon))$ where $\epsilon$ is an indeterminate and $K_0$ is the subset of $K$ consisting of power series of the form $f = a_s\epsilon^s + a_{s+1}\epsilon^{s+1} + \cdots$ such that $s \geq 0$ and $a_s > 0$; we then set $|f| = s$.

**42.2.3.** We consider a split semisimple algebraic group $\mathcal{G}$ over $K$, corresponding to the root datum, with a fixed maximal unipotent subgroup $\mathcal{U}^+$ and a fixed maximal torus $\mathcal{T}$ normalizing $\mathcal{U}^+$, both defined over $K$. For each $i \in I$, we denote by $\mathcal{U}_i^+$ the simple root subgroup of $\mathcal{U}^+$ corresponding to $i$; we assume that we are given an isomorphism $x_i$ of the additive group with $\mathcal{U}_i^+$, defined over $K$. Let $B^-$ be the Borel subgroup opposed to $\mathcal{U}^+$ and containing $\mathcal{T}$. We shall identify $\mathcal{G}, \mathcal{U}^+, \mathcal{T}, \mathcal{U}_i^+, B^-$ with their groups of $K$-rational points. We shall regard $x_i$ as an isomorphism of $K$ onto $\mathcal{U}_i^+$.

**Proposition 42.2.4.** *Let $w \in W$. Let $\mathbf{h} = (i_1, i_2, \dots, i_n)$ be a sequence in $I$ such that $s_{i_1}s_{i_2}\cdots s_{i_n}$ is a reduced expression for $w$.*

(a) *The map $K_0^n \to \mathcal{U}^+$ given by*

$$(p_1, p_2, \cdots, p_n) \mapsto x_{i_1}(p_1)x_{i_2}(p_2)\cdots x_{i_n}(p_n)$$

*is injective.*

(b) *The image of the map (a) is a subset $\mathcal{U}^+(w)$ of $\mathcal{U}^+$ which does not depend on $\mathbf{h}$.*

(c) *If $w' \in W$ is distinct from $w$, then $\mathcal{U}^+(w) \cap \mathcal{U}^+(w') = \emptyset$.*

Let $\mathcal{U}^+(\mathbf{h})$ be the image of the map in (a). To prove (b), it suffices, by 2.1.2, to check the following statement: if $\mathbf{h}'$ is obtained from $\mathbf{h}$ by replacing $h$ consecutive indices $i, j, i, \dots$ in $\mathbf{h}$ by the $h$ indices $j, i, j, \dots$ (for some $i \neq j$ with $h = h(i,j)$), then $\mathcal{U}^+(\mathbf{h}) = \mathbf{U}^+(\mathbf{h}')$.

To prove this statement, we may clearly assume that $I$ consists of two elements $i, j$. In the case where $i \cdot j = 0$, we have $x_i(p)x_j(p') = x_j(p')x_i(p)$ for any $p, p' \in K$. Assume now that $i \cdot j = -1$. We have the following identity, by a computation in $SL_3$:

$$x_i(t)x_j(s)x_i(r) = x_j(t')x_i(s')x_j(r')$$

where

(d) $t' = \frac{sr}{t+r}, \quad s' = t + r, \quad r' = \frac{st}{t+r}$

or equivalently,

(e) $t = \frac{s'r'}{t'+r'}, \quad s = t' + r', \quad r = \frac{s't'}{t'+r'}$.

By the definition of $K_0$, we have $s, t, r \in K_0$ if and only if $s', t', r' \in K_0$. This proves (b). We prove (c). Let $\dot{s}_i$ be an element of the normalizer of $\mathcal{T}$ in $\mathcal{G}$ which represents $s_i \in W$. If $p \in K - \{0\}$, we have $x_i(p) \in B\dot{s}_iB$. Hence if $p_1, p_2, \dots, p_n$ are in $K - \{0\}$, then

$$x_{i_1}(p_1)x_{i_2}(p_2)\cdots x_{i_n}(p_n) \in B\dot{s}_{i_1}B\dot{s}_{i_2}B\cdots\dot{s}_{i_n}B \subset B\dot{s}_{i_1}\dot{s}_{i_2}\cdots\dot{s}_{i_n}B$$

by properties of the Bruhat decomposition. Thus, $\mathcal{U}^+(w) \subset B\dot{s}_{i_1}\dot{s}_{i_2}\cdots\dot{s}_{i_n}B$ so that (c) follows from the Bruhat decomposition.

We prove (a). Assume that

$$x_{i_1}(p_1)x_{i_2}(p_2)\cdots x_{i_n}(p_n) = x_{i_1}(p'_1)x_{i_2}(p'_2)\cdots x_{i_n}(p'_n)$$

where $p_1, \dots, p_n$ and $p'_1, \dots, p'_n$ are in $K_0$. We prove that $p_l = p'_l$ for all $l$ by induction on $n$. This assumption implies

$$x_{i_1}(p_1 - p'_1)x_{i_2}(p_2)\cdots x_{i_n}(p_n) = x_{i_2}(p'_2)\cdots x_{i_n}(p'_n).$$

If $p_1 - p'_1 \neq 0$, the two sides of the last equality are in

$$B\dot{s}_{i_1}\dot{s}_{i_2}\cdots\dot{s}_{i_n}B, B\dot{s}_{i_2}\dot{s}_{i_3}\cdots\dot{s}_{i_n}B,$$

by the argument above. This is a contradiction. Thus, we must have $p_1 = p'_1$. Then we have

$$x_{i_2}(p_2)\cdots x_{i_n}(p_n) = x_{i_2}(p'_2)\cdots x_{i_n}(p'_n)$$

and the induction hypothesis shows that $p_2 = p'_2, \dots, p_n = p'_n$.

**Corollary 42.2.5.** *The subset $\cup_{w\in W}\mathcal{U}^+(w)$ of $\mathcal{U}^+$ is closed under multiplication. It coincides with the submonoid (with 1) of $\mathcal{U}^+$ generated by the elements $x_i(p)$, for various $i \in I$ and $p \in K_0$.*

Let $i \in I$ and $p \in K_0$. Let $\mathbf{h} = (i_1, i_2, \dots, i_n)$ be as in 42.2.4. If $s_is_{i_1}s_{i_2}\cdots s_{i_n}$ is a reduced expression in $W$, then $x_i(p)\mathcal{U}^+(\mathbf{h}) \subset \mathcal{U}^+(\mathbf{h}')$ where $\mathbf{h}' = (i, i_1, i_2, \dots, i_n)$. If $s_is_{i_1}s_{i_2}\cdots s_{i_n}$ is not a reduced expression in $W$, then we have $s_{i_1}s_{i_2}\cdots s_{i_n} = s_is_{j_1}s_{j_2}\cdots s_{j_{n-1}}$. for some $j_1, j_2, \dots, j_{n-1}$. Set $\mathbf{h}' = (i, j_1, j_2, \dots, j_{n-1})$. Clearly, $x_i(p)\mathcal{U}^+(\mathbf{h}') \subset \mathcal{U}^+(\mathbf{h}')$ and, by 42.2.4(b), we have $\mathcal{U}^+(\mathbf{h}') = \mathcal{U}^+(\mathbf{h})$. It follows that $x_i(p)\mathcal{U}^+(\mathbf{h}) \subset \mathcal{U}^+(\mathbf{h}')$.

We have thus proved that the set $\cup_{w\in W}\mathcal{U}^+(w)$ is stable under left multiplication by elements of the form $x_i(p)$ as above. The corollary follows.

**42.2.6.** From now on we assume that $K, K_0$ are as in 42.2.2(c). Recall that we then have a well-defined map $f \mapsto |f|$ from $K_0$ to $\mathbf{N}$.

For any $\mathbf{h} = (i_1, i_2, \ldots, i_n)$ in $\mathbf{H}$ and any $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbf{N}^n$, we define a subset $\mathcal{U}^+(\mathbf{h}, \mathbf{c})$ of $\mathcal{U}^+$ as follows. By definition, $\mathcal{U}^+(\mathbf{h}, \mathbf{c})$ consists of all elements of $\mathcal{U}^+$ which are of the form $x_{i_1}(p_1)x_{i_2}(p_2)\cdots x_{i_n}(p_n)$ where $p_1, p_2, \ldots, p_n$ are elements of $K_0$ such that $|p_1| = c_1, |p_2| = c_2, \ldots, |p_n| = c_n$. From 42.2.4, we see that we have a partition

(a) $\mathcal{U}^+(w_0) = \sqcup_{\mathbf{c}} \mathcal{U}^+(\mathbf{h}, \mathbf{c})$.

**Proposition 42.2.7.** *Let $\mathbf{h}, \mathbf{h}'$ be elements of $\mathbf{H}$ and let $\mathbf{c}, \mathbf{c}'$ be elements of $\mathbf{N}^n$ such that $R_{\mathbf{h}}^{\mathbf{h}'}(\mathbf{c}) = \mathbf{c}'$. We have $\mathcal{U}^+(\mathbf{h}, \mathbf{c}) = \mathcal{U}^+(\mathbf{h}', \mathbf{c}')$. In particular, the partition 42.2.6(a) of $\mathcal{U}^+(w_0)$ is independent of $\mathbf{h}$.*

We may clearly assume that $\mathbf{h}, \mathbf{h}'$ are joined in the graph $\mathbf{H}$. That case reduces immediately to the case where $I$ consists of two elements $i, j$. The case where $i \cdot j = 0$ is trivial.

Assume now that $i \cdot j = -1$. Using the identities (d),(e) in the proof of 42.2.4, we see that it is enough to verify the following statement. Let $t, s, r, t', s', r' \in K_0$ be such that $t' = \frac{sr}{t+r}, s' = t + r, r' = \frac{st}{t+r}$. Then

$$|t'| = |s|+|r|-\min(|t|, |r|), \quad |s'| = \min(|t|, |r|), \quad |r'| = |t|+|s|-\min(|t|, |r|).$$

This is immediate. The proposition is proved.

**42.2.8.** We now see that the set of subsets in the partition 42.2.6(a) of $\mathcal{U}^+(w_0)$ (which is intrinsic, by 42.2.7) is in natural $1-1$ correspondence with the set $\hat{\mathbf{H}}$, hence also with the canonical basis $\mathbf{B}$. At the same time we have obtained a new interpretation of the piecewise linear bijections $R_{\mathbf{h}}^{\mathbf{h}'}: \mathbf{N}^n \cong \mathbf{N}^n$ in terms of the geometry of the group $\mathcal{G}$.

# Notes on Part VI

1. The braid group action on **U** has been introduced (with a different normalization) in [5], in the simply laced case, and in [6], for arbitrary Cartan data of finite type. Another approach (for Cartan data of finite type) to the braid group action has been found by Soibelman [8]. The general case has not been treated before in the literature. The fact that the braid group acts naturally on integrable modules over arbitrary ground rings (see 41.2) is also new.
2. The paper[3] of Levendorskii and Soibelman contains several results relating braid group actions (for finite type) with comultiplication and with the inner product. In particular, an identity like 37.3.2(a) appears (for finite type) in [3]. Our lemma 38.1.8 is closely related to [3, 2.4.2]; however, neither of these two results implies the other. Propositions 38.2.3 and 40.2.4 are generalizations of [3, 3.2].
3. Corollary 40.2.2 and Proposition 41.1.7 appeared in [6] and [2].
4. Most results in 42.1 appeared in [7]. The results in 42.2 are new; in obtaining them, I have been stimulated by a question of B. Kostant.

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