CHAPTER 41

Integrality Properties of the Symmetries

41.1 Braid Group Action on $\dot{\mathbf{U}}$

41.1.1. Let $e = \pm 1$ and let $i \in I$. The symmetry $T'_{i,e} : \mathbf{U} \to \mathbf{U}$ (resp. $T''_{i,e} : \mathbf{U} \to \mathbf{U}$) induces for each λ', λ'' a linear isomorphism ${}_{\lambda'}\mathbf{U}_{\lambda''} \to {}_{s_i(\lambda')}\mathbf{U}_{s_i(\lambda'')}$ (notation of 23.1.1; $s_i : X \to X$ is as in 2.2.6). Taking direct sums, we obtain an algebra automorphism $T'_{i,e} : \dot{\mathbf{U}} \to \dot{\mathbf{U}}$ (resp. $T''_{i,e} : \dot{\mathbf{U}} \to \dot{\mathbf{U}}$) such that $T'_{i,e}(1_\lambda) = 1_{s_i(\lambda)}$ (resp. $T''_{i,e}(1_\lambda) = 1_{s_i(\lambda)}$) for all λ and $T'_{i,e}(uxx'u') = T'_{i,e}(u)T'_{i,e}(x)T'_{i,e}(x')T'_{i,e}(u')$ (resp. $T''_{i,e}(uxx'u') = T''_{i,e}(u)T''_{i,e}(x)T''_{i,e}(x')T''_{i,e}(u')$) for all $u, u' \in \mathbf{U}$ and $x, x' \in \dot{\mathbf{U}}$. Then $T'_{i,e}$ is an automorphism of the algebra $\dot{\mathbf{U}}$ with inverse $T''_{i,-e}$. These automorphisms satisfy braid group relations just like those of $\mathbf{U}$.

41.1.2. From the formulas in 37.1.3, we deduce that

$$T'_{i,e}(E_i^{(n)}1_\lambda) = (-1)^n v_i^{-en(n+\langle i,\lambda\rangle+1)} F_i^{(n)} 1_{s_i(\lambda)};$$
$$T'_{i,e}(F_i^{(n)}1_\lambda) = (-1)^n v_i^{-en(n-\langle i,\lambda\rangle-1)} E_i^{(n)} 1_{s_i(\lambda)};$$
$$T'_{i,e}(E_j^{(n)}1_\lambda) = \sum_{r+s=-\langle i,j'\rangle n} (-1)^r v_i^{er} E_i^{(r)} E_j^{(n)} E_i^{(s)} 1_{s_i(\lambda)} \text{ for } j \neq i;$$
$$T'_{i,e}(F_j^{(n)}1_\lambda) = \sum_{r+s=-\langle i,j'\rangle n} (-1)^r v_i^{-er} F_i^{(s)} F_j^{(n)} F_i^{(r)} 1_{s_i(\lambda)} \text{ for } j \neq i;$$
$$T''_{i,-e}(E_i^{(n)}1_\lambda) = (-1)^n v_i^{en(n+\langle i,\lambda\rangle-1)} F_i^{(n)} 1_{s_i(\lambda)};$$
$$T''_{i,-e}(F_i^{(n)}1_\lambda) = (-1)^n v_i^{en(n-\langle i,\lambda\rangle+1)} E_i^{(n)} 1_{s_i(\lambda)};$$
$$T''_{i,-e}(E_j^{(n)}1_\lambda) = \sum_{r+s=-\langle i,j'\rangle n} (-1)^r v_i^{er} E_i^{(s)} E_j^{(n)} E_i^{(r)} 1_{s_i(\lambda)} \text{ for } j \neq i;$$
$$T''_{i,-e}(F_j^{(n)}1_\lambda) = \sum_{r+s=-\langle i,j'\rangle n} (-1)^r v_i^{-er} F_i^{(r)} F_j^{(n)} F_i^{(s)} 1_{s_i(\lambda)} \text{ for } j \neq i.$$

It follows that $T'_{i,e}, T''_{i,e}$ restrict to $\mathcal{A}$-algebra automorphisms ${}_{\mathcal{A}}\dot{\mathbf{U}} \to {}_{\mathcal{A}}\dot{\mathbf{U}}$. They take 1_λ to $1_{s_i(\lambda)}$ for any $\lambda \in X$.

The following result is an integral version of 40.1.3.

Proposition 41.1.3. *Let $\mathbf{h} = (i_1, i_2, \dots, i_n)$ be a sequence in I such that $s_{i_1} s_{i_2} \cdots s_{i_n}$ is a reduced expression for some $w \in W$; let $t \in \mathbf{Z}$. Then*

(a) $T''_{i_1,e} T''_{i_2,e} \cdots T''_{i_{n-1},e}(E_{i_n}^{(t)}) \in {}_{\mathcal{A}}\mathbf{U}^+$;

(b) $T'_{i_1,e} T'_{i_2,e} \cdots T'_{i_{n-1},e}(E_{i_n}^{(t)}) \in {}_{\mathcal{A}}\mathbf{U}^+$.

Let u be the left hand side of (a). Let $\zeta' \in X$; define $\zeta \in X$ by $\zeta = s_{i_1} s_{i_2} \cdots s_{i_{n-1}}(\zeta')$. We have $u 1_\zeta = T''_{i_1,e} T''_{i_2,e} \cdots T''_{i_{n-1},e}(E_{i_n}^{(t)} 1_{\zeta'})$. Hence, by 41.1.2, we have $u 1_\zeta \in {}_{\mathcal{A}}\dot{\mathbf{U}}$. On the other hand, by 40.1.3, we have $u \in \mathbf{U}^+$. Thus, to prove (a), it suffices to prove the following statement: if $x \in \mathbf{f}$ and $\zeta \in X$ satisfy $x^+ 1_\zeta \in {}_{\mathcal{A}}\dot{\mathbf{U}}$, then $x \in {}_{\mathcal{A}}\mathbf{f}$. This follows immediately from 23.2.2. The proof of (b) is entirely similar.

The following result is an integral version of 40.2.1.

Proposition 41.1.4. *Let $w \in W$ and let $e = \pm 1$. Let $\mathbf{h} = (i_1, i_2, \dots, i_n)$ be a sequence in I such that $s_{i_1} s_{i_2} \cdots s_{i_n}$ is a reduced expression for w. Then*

(a) *the elements $E_{i_1}^{(c_1)} T'_{i_1,e}(E_{i_2}^{(c_2)}) \cdots T'_{i_1,e} T'_{i_2,e} \cdots T'_{i_{n-1},e}(E_{i_n}^{(c_n)})$ (for various sequences $\mathbf{c} = (c_1, c_2, \dots, c_n) \in \mathbf{N}^n$) form an $\mathcal{A}$-basis for an $\mathcal{A}$-submodule ${}_{\mathcal{A}}\mathbf{U}^+(w,e)$ of $\mathbf{U}^+(w,e)$ which does not depend on $\mathbf{h}$;*

(b) *the elements $E_{i_1}^{(c_1)} T''_{i_1,e}(E_{i_2}^{(c_2)}) \cdots T''_{i_1,e} T''_{i_2,e} \cdots T''_{i_{n-1},e}(E_{i_n}^{(c_n)})$ (for various sequences $\mathbf{c} = (c_1, c_2, \dots, c_n) \in \mathbf{N}^n$) form an $\mathcal{A}$-basis for ${}_{\mathcal{A}}\mathbf{U}^+(w,e)$ in (a).*

(c) *Let $i \in I$ be such that $l(s_i w) = l(w) - 1$ and let $t \in \mathbf{Z}$. Then $E_i^{(t)} {}_{\mathcal{A}}\mathbf{U}^+(w,e) \subset {}_{\mathcal{A}}\mathbf{U}^+(w,e)$.*

Using the method of 40.2.1, we see that it suffices to prove (a) in the case where I consists of two elements i, j and $h(i,j) < \infty$. In that case, the result follows from the analysis in [7]. (If $i \cdot i = j \cdot j$, this can also be deduced from Lemma 42.1.2.)

41.1.5. With the notations of 41.1.4, let $\theta_{\mathbf{h}}^{\mathbf{c}} \in {}_{\mathcal{A}}\mathbf{f}$ be the element corresponding to $E_{i_1}^{(c_1)} T'_{i_1,-1}(E_{i_2}^{(c_2)}) \cdots T'_{i_1,-1} T'_{i_2,-1} \cdots T'_{i_{n-1},-1}(E_{i_n}^{(c_n)})$ under the isomorphism $\mathbf{f} \to \mathbf{U}^+$ given by $x \mapsto x^+$.

Proposition 41.1.6. *Let $w, n, \mathbf{h}, \mathbf{c}$ be as in 41.1.4. Let $\pi : \mathcal{L}(\mathbf{f}) \to \mathcal{L}(\mathbf{f})/v^{-1}\mathcal{L}(\mathbf{f})$ be the canonical projection. We have $\theta_{\mathbf{h}}^{\mathbf{c}} \in \mathcal{L}(\mathbf{f})$ and there is a unique element b of the canonical basis $\mathbf{B}$ such that $\pi(\theta_{\mathbf{h}}^{\mathbf{c}}) = \pm\pi(b)$.*

From 38.2.3, we have $(\theta_{\mathbf{h}}^{\mathbf{c}}, \theta_{\mathbf{h}}^{\mathbf{c}}) \in 1 + v^{-1}\mathbf{Z}[[v^{-1}]] \cap \mathbf{Q}(v)$. This implies the proposition by 14.2.2(a).

Proposition 41.1.7. *Assume that the Cartan datum is of finite type. Then*

(a) ${}_{\mathcal{A}}\mathbf{U}^+(w_0, e) = {}_{\mathcal{A}}\mathbf{U}^+$.

This follows from 41.1.4 in the same way as 40.2.2 follows from 40.2.1.

41.1.8. Braid group action on ${}_R\dot{\mathbf{U}}$. Let R be as in 31.1.1. Let $e = \pm 1$. For any $i \in I$, the $\mathcal{A}$-algebra automorphism $T'_{i,e} : {}_{\mathcal{A}}\dot{\mathbf{U}} \to {}_{\mathcal{A}}\dot{\mathbf{U}}$ (resp. $T''_{i,e} : {}_{\mathcal{A}}\dot{\mathbf{U}} \to {}_{\mathcal{A}}\dot{\mathbf{U}}$) induces, upon tensoring with R, an R-algebra automorphism $T'_{i,e} : {}_R\dot{\mathbf{U}} \to {}_R\dot{\mathbf{U}}$ (resp. $T''_{i,e} : {}_R\dot{\mathbf{U}} \to {}_R\dot{\mathbf{U}}$). These automorphisms satisfy the braid group relations on ${}_R\dot{\mathbf{U}}$ just like they did over $\mathbf{Q}(v)$ (this holds over for $\mathcal{A}$ by reduction to $\mathbf{Q}(v)$, since ${}_{\mathcal{A}}\dot{\mathbf{U}}$ is imbedded in $\dot{\mathbf{U}}$, and then it holds in general by change of rings from $\mathcal{A}$ to R). Similarly, we have $T'^{-1}_{i,e} = T''_{i,-e}$ as automorphisms of ${}_R\dot{\mathbf{U}}$.

41.1.9. Braid group action and the quantum Frobenius homomorphism. Let R be as in 35.1.3. In the setup of 35.1.9, the homomorphism $Fr : {}_R\dot{\mathbf{U}} \to {}_R\dot{\mathbf{U}}^*$ is compatible with the braid group actions on ${}_R\dot{\mathbf{U}}$ and ${}_R\dot{\mathbf{U}}^*$. The proof is by checking on generators.

41.2. Braid Group Action on Integrable ${}_R\dot{\mathbf{U}}$-Modules

41.2.1. In the following proposition we assume that the root datum is Y-regular and we consider $\lambda, \lambda' \in X^+$.

Proposition 41.2.2. *The symmetries $T'_{i,e}, T''_{i,e}$ of the $\dot{\mathbf{U}}$-module ${}^{\omega}\Lambda_\lambda \otimes \Lambda_{\lambda'}$ map the ${}_{\mathcal{A}}\dot{\mathbf{U}}$-submodule ${}^{\omega}_{\mathcal{A}}\Lambda_\lambda \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\Lambda_{\lambda'})$ into itself.*

Let $m \in {}^{\omega}_{\mathcal{A}}\Lambda_\lambda \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\Lambda_{\lambda'})$. By definition (see 5.2.1), the vector $T'_{i,e}(m)$ is given by a sum of infinitely many terms such that all but a finite number of terms (depending on m) are zero. The finitely many terms that can be non-zero are of the form um where $u \in {}_{\mathcal{A}}\dot{\mathbf{U}}$. They belong to ${}^{\omega}_{\mathcal{A}}\Lambda_\lambda \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\Lambda_{\lambda'})$ since this is an ${}_{\mathcal{A}}\dot{\mathbf{U}}$-submodule. Thus this submodule is stable under $T'_{i,e}$. The same argument shows that it is stable under $T''_{i,e}$. The proposition is proved.

41.2.3. Let R be as in 31.1.1. Let M be an integrable object in ${}_R\mathcal{C}$. We define R-linear maps $T'_{i,e} : M \to M$ and $T''_{i,e} : M \to M$ by the formulas in 5.2.1, in which v is regarded as an element of R, by the $\mathcal{A}$-algebra structure on R. It is clear that these operators are well-defined.

Proposition 41.2.4. (a) *The operators* $T'_{i,e} : M \to M$ *satisfy the braid group relations. The same holds for the operators* $T''_{i,e} : M \to M$.

(b) *We have* ${T'_{i,e}}^{-1} = T''_{i,-e}$ *as operators* $M \to M$.

(c) *For any* $u \in {}_R\dot{\mathbf{U}}$ *and any* $m \in M$, *we have* $T'_{i,e}(um) = T'_{i,e}(u)T'_{i,e}(m)$ *and* $T''_{i,e}(um) = T''_{i,e}(u)T''_{i,e}(m)$.

Using the functor in 31.1.12 in the case where $(Y', X', \dots)$ is the simply connected root datum of type $(I, \cdot)$, we can reduce the general case to the case where the root datum is simply connected, hence Y-regular. In that case, using the characterization of integrable objects given in 31.2.7, we are reduced to the special case where $M = {}^{\omega}_{R}\Lambda_\lambda \otimes_R ({}_R\Lambda_{\lambda'})$ with $\lambda, \lambda' \in X^+$. Indeed, suppose that M is a sum of ${}_R\dot{\mathbf{U}}$-submodules M_α and that the proposition holds for each M_α. Then it clearly holds for M. Suppose that M is a quotient of an integrable object M' such that the proposition holds for M'; then it clearly holds for M.

In this special case, the proposition follows by change of scalars from the case $R = \mathcal{A}$ which in turn follows from the already known case where $R = \mathbf{Q}(v)$. The proposition is proved.