CHAPTER 36

The Algebras ${}_R\mathfrak{f}$, ${}_R\mathfrak{u}$

36.1. The Algebra ${}_R\mathfrak{f}$

36.1.1. In this chapter we assume that the Cartan datum is simply laced. As in the previous chapter, we fix an integer $l \geq 1$. We preserve the assumptions of 35.1.1– 35.1.3. Note that in this case, 35.1.2(a) is automatically satisfied. Note also that in this case, we have $l_i = l$ and $\mathbf{v}_i = \mathbf{v}$ for all i.

36.1.2. We define an R-algebra ${}_R\mathfrak{f}$ as follows. If $R = \mathcal{A}'$, then ${}_R\mathfrak{f}$ is the R-subalgebra of ${}_R\mathbf{f}$ generated by the elements $\theta_i^{(n)}$ for various i, n such that $0 \leq n < l$. In the general case, we define ${}_R\mathfrak{f} = R \otimes_{\mathcal{A}'} ({}_{\mathcal{A}'}\mathfrak{f})$. We have a direct sum decomposition ${}_R\mathfrak{f} = \oplus_\nu ({}_R\mathfrak{f}_\nu)$ indexed by $\nu \in \mathbf{N}[I]$; for $R = \mathcal{A}'$, it is induced by the analogous decomposition of ${}_R\mathbf{f}$ and, in general, is obtained by extension of scalars from the special case $R = \mathcal{A}'$.

From the definitions we see that in the case where R is the quotient field of $\mathcal{A}'$, ${}_R\mathfrak{f}$ is the R-subalgebra of ${}_R\mathbf{f}$ generated by the elements $\theta_i^{(n)}$ for various i, n such that $0 \leq n < l$, or equivalently, by the elements θ_i for various i (if $l \geq 2$) and by 1. (We use the fact that $\phi([n]^!)$ is non-zero in this field for $0 \leq n < l$.) It follows that in this case, ${}_R\mathfrak{f}$ is the same as the algebra $\mathfrak{f}$ defined in 35.4.1.

We shall need the following result.

Lemma 36.1.3. *Let $i, j \in I$ be such that $\langle i, j' \rangle \neq 0$. Let $m, n \in \mathbf{N}$ be such that $m \in l\mathbf{N}$ and $n < l$.*

(a) $\theta_i^{(m)}\theta_j^{(n)} \in {}_{\mathcal{A}'}\mathbf{f}$ *is an $\mathcal{A}'$-linear combination of elements $u_s\theta_i^{(s)}$ where $s \in [0, m]$ is divisible by l and $u_s \in {}_{\mathcal{A}'}\mathfrak{f}$.*

(b) $\theta_i^{(n)}\theta_j^{(m)} \in {}_{\mathcal{A}'}\mathbf{f}$ *is an $\mathcal{A}'$-linear combination of elements $\theta_i^{(s)}u'_s$ where $s \in [0, m]$ is divisible by l and $u'_s \in {}_{\mathcal{A}'}\mathfrak{f}$.*

We have $\langle i, j' \rangle = -1$, since the Cartan datum is simply laced. We prove (a). We may assume that $m > 0$. Then $m \geq l$, hence $m \geq n + 1$ and 7.1.7 is applicable. Thus we can express $\theta_i^{(m)}\theta_j^{(n)}$ as an $\mathcal{A}'$-linear combination of terms $\theta_i^{(r)}\theta_j^{(n)}\theta_i^{(s')}$ where $r, s' \in \mathbf{N}, r + s' = m, m - n \leq s' \leq m$. For such a term, we have $r \leq n < l$, hence $\theta_i^{(r)}\theta_i^{(n)} \in {}_{\mathcal{A}'}\mathfrak{f}$ and $\theta_i^{(s')}$ is either $\theta_i^{(m)}$

or, if $s' < m$, a power of $\mathbf{v}$ times $\theta_i^{(s'-m+l)}\theta_i^{(m-l)}$ (see 34.1.2). In the last expression we have $m - l \in l\mathbf{Z}$ and $0 < s' - m + l < l$; (a) follows. Now (b) follows from (a) by using the involution σ.

Theorem 36.1.4. *The R-module ${}_R\mathfrak{f}_\nu$ is free for any $\nu \in \mathbf{N}[I]$.*

Lemma 36.1.5. . *Assume that the root datum is simply connected. Let $\lambda \in X^+$ be defined by $\langle i, \lambda \rangle = l_i - 1$ for all i. Let η be the canonical generator of ${}_R\Lambda_\lambda$. The map $x \mapsto x^-\eta$ is an isomorphism ${}_R\mathfrak{f} \to {}_R\Lambda_\lambda$.*

It suffices to prove this in the case where $R = \mathcal{A}'$; the general case follows by change of rings. The fact that this map is injective follows from Proposition 35.4.4 (over the quotient field). We prove surjectivity. The argument is similar to the one in the proof of 35.4.2. Note that ${}_R\mathbf{f}$ is spanned as an R-module by products $x_1x_2\cdots x_p$ where the factors are either of the form $\theta_i^{(n)}$ with $0 \le n < l$ (factors of the first kind) or of the form $\theta_i^{(m)}$ with $m \in l\mathbf{N}$ (factors of the second kind).

By Lemma 36.1.3, any product x_tx_{t+1} with x_t (resp. x_{t+1}) a factor of the second kind (resp. of the first kind) is an $\mathcal{A}'$-linear combination of products of the form $x'_1x'_2\cdots x'_{r-1}x'_r$ where $x'_1, x'_2, \ldots, x'_{r-1}$ are factors of the first kind and x'_r is a factor of the second kind. Applying this fact repeatedly, we see that $x_1x_2\cdots x_p$ is a linear combination of analogous words in which any factor of the first kind appears to the left of any factor of the second kind.

Since the R-module ${}_R\Lambda_\lambda$ is generated by elements $x^-\eta$ with $x \in {}_R\mathbf{f}$, we see from the previous argument that ${}_R\Lambda_\lambda$ is generated by elements $x'^-x_1^-x_2^-\cdots x_p^-\eta$ where $x' \in {}_R\mathfrak{f}$ and $x_1, x_2, \ldots, x_p \in {}_R\mathbf{f}$ are factors of the second kind.

Since $(\theta_i^{(m)})^-\eta = 0$, for any m such that $m > \langle i, \lambda \rangle = l_i - 1$, we have $(\theta_i^{(m)})^-\eta = 0$ for any $m \in l\mathbf{N}$ such that $m \neq 0$. It follows that the R-module ${}_R\Lambda_\lambda$ is generated by elements $x'^-\eta$ with $x' \in {}_R\mathfrak{f}$. The lemma follows.

36.1.6. Proof of Theorem 36.1.4. We may assume that the root datum is simply connected. Hence Lemma 36.1.5 is applicable. The isomorphism in that lemma is compatible with the direct sum decompositions according to ν; it remains to observe that the canonical basis of ${}_R\Lambda_\lambda$ provides a basis for the summand corresponding to ν.

The following result is an integral version of Theorem 35.4.2(b); here R is not assumed to be a field.

Theorem 36.1.7. *The R-linear map* $\chi : {}_R\mathbf{f}^* \otimes_R ({}_R\mathfrak{f}) \to {}_R\mathbf{f}$ *given by* $x \otimes y \mapsto Fr'(x)y$ *is an isomorphism of R-modules.*

It is enough to prove this in the case where $R = \mathcal{A}'$; the general case follows by change of rings. The fact that χ is surjective has already been proved in the course of proving Lemma 36.1.5 (actually the products in that proof are in the opposite order of what we need now, so we must apply σ to them). Next we note that χ is a homomorphism between two free $\mathcal{A}'$-modules (the freeness of ${}_R\mathbf{f}^*$ and of ${}_R\mathbf{f}$ is already known; the freeness of ${}_R\mathfrak{f}$ follows from 36.1.4). Hence to prove that χ is injective over $\mathcal{A}'$, it is enough to prove the corresponding statement for the quotient field of $\mathcal{A}'$. That statement is already known (see 35.4.2(b)). The theorem is proved.

36.1.8. Let ${}_R\mathfrak{f} \to {}_R\mathbf{f}$ be the R-algebra homomorphism induced by change of scalars from the analogous homomorphism for $R = \mathcal{A}'$, which is the obvious imbedding.

Corollary 36.1.9. *The natural algebra homomorphism* ${}_R\mathfrak{f} \to {}_R\mathbf{f}$ *is an imbedding; its image is the R-subalgebra of* ${}_R\mathbf{f}$ *generated by the elements* $\theta_i^{(n)}$ *for various* i, n *such that* $0 \le n < l$.

We shall identify ${}_R\mathfrak{f}$ with a subalgebra of ${}_R\mathbf{f}$, as above.

36.2. The Algebras ${}_R\dot{\mathfrak{u}}$, ${}_R\mathfrak{u}$

36.2.1. Let ${}_R\dot{\mathfrak{u}}$ be the R-subalgebra of ${}_R\dot{\mathbf{U}}$ generated by the elements $E_i^{(n)}1_\zeta, F_i^{(n)}1_\zeta$ for various i, n such that $0 \le n < l$ and various $\zeta \in X$. Note that ${}_R\dot{\mathfrak{u}}$ is the free ${}_R\mathfrak{f} \otimes_R ({}_R\mathfrak{f}^{opp})$-submodule of ${}_R\dot{\mathbf{U}}$ with basis (1_ζ) (the module structure being $(x \otimes x') : u \mapsto x^+ u x'^-$); the same statement holds for the module structure $(x \otimes x') : u \mapsto x^- u x'^+$.

Lemma 36.2.2. ${}_R\dot{\mathfrak{u}}$ *is closed under comultiplication.*

This is easily proved by checking on the algebra generators of ${}_R\dot{\mathfrak{u}}$.

36.2.3. In the rest of this chapter we assume that $l = l'$ is odd. Then $\mathbf{v}^l = 1$. We introduce a certain completion ${}_R\hat{\mathfrak{u}}$ of ${}_R\dot{\mathfrak{u}}$ as follows. Note that any element ${}_R\dot{\mathfrak{u}}$ can be written uniquely as a sum

(a) $\sum_{\zeta,\zeta' \in X} x_{\zeta,\zeta'}$

where $x_{\zeta,\zeta'} \in 1_\zeta({}_R\dot{\mathfrak{u}})1_{\zeta'}$ are zero except for finitely many pairs (ζ, ζ').

We now relax the last condition and we consider infinite formal sums (a) in which the only requirement is that there exists a finite subset $F \subset X$

such that $x_{\zeta,\zeta'} \in 1_\zeta({}_R\dot{\mathbf{u}})1_{\zeta'}$ are zero unless $\zeta - \zeta' \in F$. The set of all such formal sums is denoted by ${}_R\hat{\mathbf{u}}$. (Note that the set F varies from element to element of ${}_R\hat{\mathbf{u}}$.) The R-algebra structure of ${}_R\dot{\mathbf{u}}$ extends in an obvious way to an R-algebra structure on ${}_R\hat{\mathbf{u}}$; this algebra has a unit element $\sum_\zeta 1_\zeta$. Note that that the two ${}_R\mathfrak{f} \otimes_R ({}_R\mathfrak{f}^{opp})$-module structures on ${}_R\dot{\mathbf{u}}$ extend in an obvious way to two ${}_R\mathfrak{f} \otimes_R ({}_R\mathfrak{f}^{opp})$-module structures on ${}_R\hat{\mathbf{u}}$.

For any X^*-coset c in X, we define $1_c = \sum_{\zeta \in c} 1_c \in {}_R\hat{\mathbf{u}}$. Let J (resp. J') be the R-submodule of ${}_R\hat{\mathbf{u}}$ generated by the elements $x^+ 1_c x'^-$ (resp. $x^- 1_c x'^+$) for various $c \in X/X^*$ and $x, x' \in {}_R\mathfrak{f}$.

Lemma 36.2.4. (a) *$F_i^{(b)} u \subset J$ for any $u \in J$ and any i, b such that $0 \le b < l$.*

(b) *J is an R-subalgebra of ${}_R\hat{\mathbf{u}}$ and $J = J'$.*

To prove (a), we may assume that $u = E_{i_1}^{(a_1)} \cdots E_{i_p}^{(a_p)} 1_c x'^-$ where $a_1, \ldots, a_p \in [0, l-1], c \in X/X^*$ and $x' \in {}_R\mathfrak{f}$. We argue by induction on p. If $p = 0$, the result is trivial. Assume that $p \ge 1$. Let $x_1 = \theta_{i_2}^{(a_2)} \cdots \theta_{i_p}^{(a_p)}$. We have

$$u = 1_{c'} E_{i_1}^{(a_1)} x_1^+ x'^-$$

for some $c' \in X/X^*$. If $i \ne i_1$, the desired result follows immediately. Assume that $i = i_1$. We have

$$F_i^{(b)} u =$$

$$\sum_{\zeta \in c'} \sum_{t \ge 0; t \le a_1; t \le b} \phi\left(\begin{bmatrix} a_1 + b - \langle i, \zeta \rangle \\ t \end{bmatrix}\right) E_i^{(a_1 - t)} 1_{\zeta - (a_1 + b - t) i'} F_i^{(b-t)} x_1^+ x'^-.$$

For each t, ζ in the sum we have $0 \le t < l$ and $a_1 + b - \langle i, \zeta \rangle = a_1 + b - \langle i, \zeta_0 \rangle$ mod $l\mathbf{Z}$, for some fixed element ζ_0 of c'. We have

$$\phi\left(\begin{bmatrix} a_1 + b - \langle i, \zeta \rangle \\ t \end{bmatrix}\right) = \phi\left(\begin{bmatrix} a_1 + b - \langle i, \zeta_0 \rangle \\ t \end{bmatrix}\right)$$

(see 34.1.2); here we use the hypothesis that l is odd. Hence we have

$$F_i^{(b)} u =$$

$$\sum_{t \ge 0; t \le a_1; t \le b} \phi\left(\begin{bmatrix} a_1 + b - \langle i, \zeta_0 \rangle \\ t \end{bmatrix}\right) E_i^{(a_1 - t)} F_i^{(b-t)} \left(\sum_{\zeta \in c'} 1_{\zeta - a_1 i'}\right) x_1^+ x'^-.$$

Note that $\sum_{\zeta \in c'} 1_{\zeta - a_1 i'} = 1_{c''}$ for some $c'' \in X/X^*$. Using now the induction hypothesis, we see that $F_i^{(b)} u \in J$; (a) is proved. Using repeatedly (a) and the identities $1_c 1_{c'} = \delta_{c,c'} 1_c$, for $c, c' \in X/X^*$, we see that J is a subalgebra of ${}_R\hat{\mathbf{u}}$. Again, using (a) repeatedly, starting with $1_c x'^+ \in J$ for $c \in X/X^*, x' \in {}_R\mathbf{f}$, we see that $J' \subset J$. By symmetry, we have $J \subset J'$, hence $J = J'$. The lemma is proved.

36.2.5. Definition. ${}_R\mathbf{u}$ is the R-subalgebra $J = J'$ of ${}_R\hat{\mathbf{u}}$.

Note that the algebra ${}_R\mathbf{u}$ has a unit element $\sum_c 1_c$; here c runs through the set X/X^*. The set X/X^* is finite, since by the definition of X^*, the map $\zeta \to \langle i, \zeta \rangle \mod l$ defines an injective map $X/X^* \to (\mathbf{Z}/l\mathbf{Z})^I$.

From the definition, we see that ${}_R\mathbf{u}$ is the free ${}_R\mathfrak{f} \otimes_R ({}_R\mathfrak{f}^{opp})$-submodule of ${}_R\hat{\mathbf{u}}$ with basis $\{1_c | c \in X/X^*\}$ (the module structure being $(x \otimes x') : u \mapsto x^+ u x'^-$); the same statement holds for the module structure $(x \otimes x') : u \mapsto x^- u x'^+$.

Notes on Part V

1. The results in Chapter 32 are due to to Drinfeld, for $R = \mathbf{Q}(v)$. The extension to the case where R is a field and v is a root of 1 in R is new; it answers a question that Drinfeld asked me in January 1990.
2. The fact that the simple integrable modules of a Kac-Moody Lie algebra admit a quantum deformation (Chapter 33) was proved in [4]; for Cartan data of finite type this was also stated in [8], but the proof there has a serious gap. (It appears [2] that, for Cartan data of finite type, this result was known to Drinfeld.) The results in 33.2 are new.
3. The results in Chapter 34 have appeared (for l odd) in [5].
4. The quantum Frobenius homomorphism, for Cartan data of finite type and with some restrictions on l , was implicit in [5] and explicit in [7]; its generalization given in Chapter 35 is new.
5. In the case where R is a field of characteristic zero, v is a root of 1 in R, and the Cartan datum is of finite type, ${}_R\mathfrak{u}$ is the finite dimensional Hopf algebra defined in [6], [7] (with some restrictions on the order of v). The extension to infinite types is new.

References

1. C. Chevalley, *Séminaire sur la classification des groupes de Lie algébriques*, Ecole Norm. Sup. Paris, 1956–58.
2. V. G. Drinfeld, *On almost cocommutative Hopf algebras*, (Russian), Algebra i Analiz **1** (1989), 30–46.
3. V. G. Kac, *Infinite dimensional Lie algebras*, Birkhäuser, Boston, 1983.
4. G. Lusztig, *Quantum deformations of certain simple modules over enveloping algebras*, Adv. Math. **70** (1988), 237–249.
5. ———, *Modular representations and quantum groups*, Contemp. Math. **82** (1989), 59–77, Amer. Math. Soc., Providence, R. I..
6. ———, *Finite dimensional Hopf algebras arising from quantized universal enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 257–296.
7. ———, *Quantum groups at roots of 1*, Geom. Dedicata **35** (1990), 89–114.
8. M. Rosso, *Finite dimensional representations of the quantum analog of a complex semisimple Lie algebra*, Comm. Math. Phys. **117** (1988), 581–593.