CHAPTER 35

The Quantum Frobenius Homomorphism

35.1. Statements of results

35.1.1. In this chapter we fix an integer $l \geq 1$. Then the integers $l_i \geq 1$, the new Cartan datum $(I, \circ)$ and the new root datum $(Y^*, X^*, \dots)$ of type $(I, \circ)$ are defined in terms of $l, (I, \cdot), (Y, X, \dots)$ as in 2.2.4, 2.2.5.

35.1.2. The assumptions (a),(b) below will be in force in this chapter.

(a) for any $i \neq j$ in I such that $l_j \geq 2$, we have $l_i \geq -\langle i, j' \rangle + 1$;

(b) $(I, \cdot)$ is without odd cycles (see 2.1.3).

Note that (a) is automatically satisfied in the simply laced case: in that case, we have $l_i = l$ for all i; in the general case, the assumption (a) can be violated only by finitely many l. Note also that (b) is automatically satisfied if $(I, \cdot)$ is of finite type.

35.1.3. Let l' be one of the integers $l, 2l$, if l is odd, and let l' be equal to $2l$, if l is even. Let $\mathcal{A}'$ be the quotient of $\mathcal{A}$ by the two-sided ideal generated by the l'-th cyclotomic polynomial $f_{l'} \in \mathcal{A}$. Thus, $(f_1, f_2, f_3, \dots) = (v - 1, v + 1, v^2 + v + 1, \dots)$.

In this chapter, we assume that the given ring homomorphism $\phi : \mathcal{A} \to R$ factors through a ring homomorphism $\mathcal{A}' \to R$, or that R is an $\mathcal{A}'$-algebra or, equivalently, that $f_{l'}(\mathbf{v}) = 0$ in R, where $\mathbf{v} = \phi(v)$.

35.1.4. When $(I, \cdot)$ is replaced by $(I, \circ)$, the element $v_i \in \mathcal{A}$, whose definition depends on the Cartan datum, becomes $v_i^* = v^{i \circ i/2} = v_i^{l_i^2}$.

For any $P \in \mathcal{A}$, we denote by P_i^* the element obtained from P by substituting v by v_i^*. For each $i \in I$, we set $\mathbf{v}_i = \phi(v_i)$ and $\mathbf{v}_i^* = \phi(v_i^*) = \mathbf{v}_i^{l_i^2}$.

Lemma 35.1.5. (a) *Let $a \in l_i\mathbf{Z}$ and $t \in l_i\mathbf{N}$. We have $\phi(\left[\begin{smallmatrix} a \\ t \end{smallmatrix}\right]_i) = \phi(\left[\begin{smallmatrix} a/l_i \\ t/l_i \end{smallmatrix}\right]_i^*)$ (equality in R).*

(b) *Let $a \in l_i\mathbf{Z}$ and let $t \in \mathbf{N}$ be non-divisible by l_i. We have $\phi(\left[\begin{smallmatrix} a \\ t \end{smallmatrix}\right]_i) = 0$ (equality in R).*

It suffices to prove this for $R = \mathcal{A}'$. This is an integral domain in which $\mathbf{v}_i^{2l_i} = 1$ and $\mathbf{v}_i^{2t} \neq 1$ for all $0 < t < l_i$. We prove (a). Applying 34.1.2(b) to v_i and l_i instead of v and l, we see that $\phi(\left[\begin{smallmatrix} a \\ t \end{smallmatrix}\right]_i) = \mathbf{v}_i^{(a+l_i)t}\binom{a/l_i}{t/l_i}$. Applying the same result to v_i^* and 1 instead of v and l, we see that $\phi(\left[\begin{smallmatrix} a/l_i \\ t/l_i \end{smallmatrix}\right]_i^*) = \mathbf{v}_i^{*(a+l_i)t/l_i^2}\binom{a/l_i}{t/l_i}$. It remains to use the equality $\mathbf{v}_i^* = \mathbf{v}_i^{l_i^2}$. The proof of (b) is entirely similar; it uses 34.1.2(a).

35.1.6. Let $\mathbf{f}^*$ (resp. $\dot{\mathbf{U}}^*$) be the $\mathbf{Q}(v)$-algebra defined like $\mathbf{f}$ (resp. like $\dot{\mathbf{U}}$), in terms of the Cartan matrix $(I, \circ)$ (resp. in terms of $(Y^*, X^*, \dots)$). Then the R-algebras ${}_R\mathbf{f}, {}_R\mathbf{f}^*, {}_R\dot{\mathbf{U}}, {}_R\dot{\mathbf{U}}^*$ are well-defined.

We state the main results of this chapter.

Theorem 35.1.7. *Recall that R is an $\mathcal{A}'$-algebra. There is a unique R-algebra homomorphism $Fr : {}_R\mathbf{f} \to {}_R\mathbf{f}^*$ such that for all $i \in I$ and $n \in \mathbf{Z}$, $Fr(\theta_i^{(n)})$ equals $\theta_i^{(n/l_i)}$ if $n \in l_i\mathbf{Z}$, and equals 0, otherwise.*

Theorem 35.1.8. *There is a unique R-algebra homomorphism Fr' : ${}_R\mathbf{f}^* \to {}_R\mathbf{f}$ such that $Fr'(\theta_i^{(n)}) = \theta_i^{(nl_i)}$ for all $i \in I$ and all $n \in \mathbf{Z}$.*

Theorem 35.1.9. *There is a unique R-algebra homomorphism $Fr : {}_R\dot{\mathbf{U}} \to {}_R\dot{\mathbf{U}}^*$ such that for all $i \in I, n \in \mathbf{Z}$ and $\zeta \in X$, we have:*

$Fr(E_i^{(n)}1_\zeta)$ equals $E_i^{(n/l_i)}1_\zeta$ if $n \in l_i\mathbf{Z}$ and $\zeta \in X^*$, and equals 0, otherwise;

$Fr(F_i^{(n)}1_\zeta)$ equals $F_i^{(n/l_i)}1_\zeta$ if $n \in l_i\mathbf{Z}$ and $\zeta \in X^*$, and equals 0, otherwise.

We give a proof of the last theorem, assuming that theorem 35.1.7 is known. Using the presentation of the algebra ${}_R\dot{\mathbf{U}}$ in terms of ${}_R\mathbf{f}$ given in 31.1.3, and the analogous presentation of the algebra ${}_R\dot{\mathbf{U}}^*$ in terms of ${}_R\mathbf{f}^*$, we see that it is enough to prove that the assignment

$$x^+1_\zeta x^- \mapsto Fr(x^+)1_\zeta Fr(x^-), x^-1_\zeta x^+ \mapsto Fr(x^-)1_\zeta Fr(x^+)$$

for $x, x' \in {}_R\mathbf{f},\ \zeta \in X^*$,

$$x^+1_\zeta x^- \mapsto 0, x^-1_\zeta x^+ \mapsto 0$$

for $x, x' \in {}_R\mathbf{f},\ \zeta \in X - X^*$ respects the relations described in 31.1.3. (Here, Fr is the homomorphism given by Theorem 35.1.7.) This is immediate, using Lemma 35.1.5.

35.1.10. Remark. The homomorphism Fr constructed in Theorem 35.1.9 is called the *quantum Frobenius homomorphism.* It is compatible with the comultiplication on ${}_R\dot{\mathbf{U}}$ and ${}_R\dot{\mathbf{U}}^*$ (proof by verification on the generators $E_i^{(n)}1_\zeta$ and $F_i^{(n)}1_\zeta$).

35.1.11. The uniqueness part of Theorems 35.1.7 and 35.1.8 is clear. To prove the existence part of these theorems, we note that the general case follows by change of scalars from the case where $R = \mathcal{A}'$. Since $\mathcal{A}'$ is an integral domain, it is contained in its quotient field K and the algebras ${}_{\mathcal{A}'}\mathbf{f}, {}_{\mathcal{A}'}\mathbf{f}^*$ are naturally imbedded in the corresponding algebras over K. Thus, if the theorems are known over K then, by restriction, we see that they hold over $\mathcal{A}'$. We are thus reduced to proving the theorems assuming that R is the quotient field of $\mathcal{A}'$. The proof in this case will be given in 35.2, 35.5.

35.2 Proof of Theorem 35.1.8

35.2.1. In the rest of this chapter (except in 35.5.2, 35.5.3), we assume that R is the quotient field of $\mathcal{A}'$. Note that R is a field of characteristic zero and that the order of $\mathbf{v}^2 = \phi(v^2)$ in the multiplicative group of R is l. Thus, we have $\mathbf{v}^{2l} = 1$ and $\mathbf{v}^{2t} \neq 1$ for all $0 < t < l$. By the definition of l_i, we have $\mathbf{v}_i^{2l_i} = 1$ and $\mathbf{v}_i^{2t} \neq 1$ for all $0 < t < l_i$. In particular, $\phi([n]_i^!)$ is invertible in R, if $0 \le n < l_i$. For any i we have $\mathbf{v}_i^* = \pm 1$ since $\mathbf{v}_i^{2l_i^2} = 1$; hence, when dealing with the algebras ${}_R\mathbf{f}^*, {}_R\dot{\mathbf{U}}^*$, we are in the *quasi-classical case* (see 33.2).

Lemma 35.2.2. *The R-algebra ${}_R\mathbf{f}$ is generated by the elements $\theta_i^{(l_i)}$ $(i \in I)$ and by the elements θ_i for $i \in I$ such that $l_i \ge 2$.*

Recall that the R-algebra ${}_R\mathbf{f}$ is generated by the elements $\theta_i^{(n)}$ for various i and $n \ge 0$. Writing $n = a + l_i b$ with $0 \le a < l_i$ and $b \in \mathbf{N}$, we have $\theta_i^{(n)} = \mathbf{v}_i^{abl_i}\theta_i^{(a)}\theta_i^{(l_i b)}$ (using 34.1.2). On the other hand,

(a) $\theta_i^{(a)} = \phi([a]_i^!)^{-1}\theta_i^a$ (see 35.2.1) and

(b) $\theta_i^{(l_i b)} = (b!)^{-1}\mathbf{v}_i^{l_i^2 b(b-1)/2}(\theta_i^{(l_i)})^b$.

The equality (b) follows from the equalities

$$(\theta_i^{(l_i)})^b = [l_i b]_i^!/([l_i]_i^!)^b\theta_i^{(l_i b)} \qquad \text{in} \quad \mathbf{f}$$

and

$$\phi([l_i b]_i^!/([l_i]_i^!)^b) = b!\phi(v_i^{l_i^2 b(b-1)/2})$$

(see 34.1.3). The lemma is proved.

35.2.3. We now give the proof of Theorem 35.1.8. As observed in 35.1.11, it is enough to prove the existence statement in 35.1.8, assuming that R is as in 35.2.1.

We will show that there exists an algebra homomorphism ${}_R\mathbf{f}^* \to {}_R\mathbf{f}$ such that $\theta_i \mapsto \theta_i^{(l_i)}$ for all i. Since the assumption 35.1.2(b) is in force, the algebra ${}_R\mathbf{f}^*$ has a presentation given by the generators θ_i and the Serre-type relations; this follows from 33.2.2(c) . Since $\phi(([n]^!)_i^*) = \mathbf{v}_i^{l_i^2 n(n-1)/2} n!$ (see the proof of Lemma 35.2.2), we see that it suffices to prove that, for any $i \neq j$, we have the following identity in ${}_R\mathbf{f}$:

(a) $\sum_{p+p'=1-\langle i,j'\rangle l_j/l_i} (-1)^{p'} v_i^{l_i^2 p(p-1)/2} v_i^{l_i^2 p'(p'-1)/2} \frac{(\theta_i^{(l_i)})^p}{p!} \theta_j^{(l_j)} \frac{(\theta_i^{(l_i)})^{p'}}{p'!} = 0.$

Using $(\theta_i^{(l_i)})^b = b! \mathbf{v}_i^{l_i^2 b(b-1)/2} \theta_i^{(l_i b)}$ (equality in ${}_R\mathbf{f}$, see 34.1.3), we can rewrite (a) in the following equivalent form:

(b) $\sum_{p+p'=1-\langle i,j'\rangle l_j/l_i} (-1)^{p'} \theta_i^{(l_i p)} \theta_j^{(l_j)} \theta_i^{(l_i p')} = 0.$

It remains to prove (b). Let $\alpha = -\langle i,j'\rangle$. For any $q \in [0, l_i - 1]$ we set

$$g_q = \sum_{r+s=\alpha l_j + l_i - q} (-1)^r v_i^{r(l_i-1-q)} \theta_i^{(r)} \theta_j^{(l_j)} \theta_i^{(s)} \in {}_\mathcal{A}\mathbf{f}.$$

This is $f_{i,j;l_j,\alpha l_j+l_i-q;-1}$ in the notation of 7.1.1.

Let $g = \sum_{q=0}^{l_i-1} (-1)^q v_i^{\alpha l_j q + l_i q - q} g_q \theta_i^{(q)}$. By the higher order quantum Serre relations (see 7.1.5(b)), we have $g_q = 0$ for all $q \in [0, l_i - 1]$; hence $g = 0$. On the other hand, setting $s' = s + q$, we have $g = \sum_{r,s';r+s'=\alpha l_j+l_i} c_{r,s'} \theta_i^{(r)} \theta_j^{(l_j)} \theta_i^{(s')}$, where

$$c_{r,s'} = \sum_{q=0}^{l_i-1} (-1)^{r+q} v_i^{r(l_i-1-q)+\alpha l_j q + l_i q - q} \begin{bmatrix} s' \\ q \end{bmatrix}_i.$$

Taking the image of $g = 0$ under the obvious map ${}_\mathcal{A}\mathbf{f} \to {}_R\mathbf{f}$, we obtain

$$\sum_{r,s';r+s'=\alpha l_j+l_i} \phi(c_{r,s'}) \theta_i^{(r)} \theta_j^{(l_j)} \theta_i^{(s')} = 0 \qquad \text{in } {}_R\mathbf{f}.$$

For fixed s', we write $s' = a + l_i b$ where $0 \le a \le l_i - 1$. We have $\phi(\begin{bmatrix} s' \\ q \end{bmatrix}_i) = \mathbf{v}_i^{-bql_i} \phi(\begin{bmatrix} a \\ q \end{bmatrix}_i)$ (see 34.1.2); hence

$$\begin{aligned}
\phi(c_{r,s'}) &= (-1)^r \mathbf{v}_i^{r(l_i-1)} \sum_{q=0}^{l_i-1} (-1)^q \mathbf{v}_i^{q(s'-1)-bql_i} \phi\left(\begin{bmatrix} a \\ q \end{bmatrix}_i\right) \\
&= (-1)^r \mathbf{v}_i^{r(l_i-1)} \sum_{q=0}^{a} (-1)^q \mathbf{v}_i^{q(a-1)} \phi\left(\begin{bmatrix} a \\ q \end{bmatrix}_i\right) \\
&= \delta_{0,a} (-1)^{\alpha l_j + l_i - l_i b} \mathbf{v}_i^{(l_i-1)(\alpha l_j + l_i - l_i b)} \\
&= \delta_{0,a} (-1)^{1-b+\alpha l_j/l_i}.
\end{aligned}$$

We have used the identity $\mathbf{v}_i^{l_i(l_i-1)} = (-1)^{l_i+1}$, see 34.1.2(e). (b) follows. Thus, we have an algebra homomorphism $Fr' : {}_R\mathbf{f}^* \to {}_R\mathbf{f}$ such that $Fr'(\theta_i) = \theta_i^{(l_i)}$ for all i.

To complete the proof we must compute $Fr'(\theta_i^{(n)})$ for $n \geq 0$. We have

$$Fr'(\theta_i^{(n)}) = \phi(([n]^!)_i^*)^{-1} Fr'(\theta_i)^n = \mathbf{v}_i^{-l_i^2 n(n-1)/2}(n!)^{-1}(\theta_i^{(l_i)})^n = \theta_i^{(nl_i)}.$$

Theorem 35.1.8 is proved.

35.3. Structure of Certain Highest Weight Modules of ${}_R\dot{\mathbf{U}}$

Proposition 35.3.1. *Assume that $i \neq j$ in I satisfy $l_i \geq -\langle i, j' \rangle + 1$. The following identity holds in ${}_R\mathbf{f}$:*

$$\theta_i^{(l_i)}\theta_j = \sum_{r=0}^{-\langle i,j'\rangle} \mathbf{v}_i^{l_i(-\langle i,j'\rangle - r)} \phi\left(\begin{bmatrix} -\langle i,j'\rangle \\ r \end{bmatrix}_i\right) \theta_i^{(r)}\theta_j\theta_i^{(l_i-r)}.$$

We set $\alpha = -\langle i, j' \rangle$. Using Corollary 7.1.7 with $m = l_i, n = 1$, we see that we are reduced to checking the identity

$$\sum_{q=0}^{l_i-\alpha-1} (-1)^{l_i-r+1+q} \mathbf{v}_i^{-(l_i-r)(\alpha-l_i+1+q)+q} \phi\left(\begin{bmatrix} l_i - r \\ q \end{bmatrix}_i\right) = \mathbf{v}_i^{l_i(\alpha-r)} \phi\left(\begin{bmatrix} \alpha \\ r \end{bmatrix}_i\right)$$

for any $r \in [0, \alpha]$. This follows from Lemma 34.1.4.

Proposition 35.3.2. *Assume that the root datum is X-regular. Let $\lambda \in X$ be such that $\langle i, \lambda \rangle \in l_i\mathbf{Z}$ for all $i \in I$. Let M be a simple highest weight module with highest weight λ in ${}_R\mathcal{C}$ and let η be a generator of the R-vector space M^λ.*

(a) *If $\zeta \in X$ satisfies $M^\zeta \neq 0$, then $\zeta = \lambda - \sum_i l_i n_i i'$, where $n_i \in \mathbf{N}$. In particular, $\langle i, \zeta \rangle \in l_i\mathbf{Z}$ for all $i \in I$.*

(b) *If $i \in I$ is such that $l_i \geq 2$, then E_i, F_i act as zero on M.*

(c) *For any $r \geq 0$, let M'_r be the subspace of M spanned by the vectors $F_{i_1}^{(l_{i_1})} F_{i_2}^{(l_{i_2})} \cdots F_{i_r}^{(l_{i_r})}\eta$ for various sequences $i_1, i_2, \ldots, i_r$ in I. Let $M' = \sum_r M'_r$. Then $M' = M$.*

Clearly, M'_r is spanned by vectors in M^ζ where ζ is of the form $\zeta = \lambda - \sum_i l_i n_i i'$, with $n_i \in \mathbf{N}$. Such ζ satisfies $\langle i, \zeta \rangle \in l_i\mathbf{Z}$ for all $i \in I$. We use the fact that, for $j \in I$, the integer $l_j\langle i, j' \rangle$ is divisible by l_i.

We show by induction on $r \geq 0$, that

(d) $E_i M'_r = 0, F_i M'_r = 0$ for any $i \in I$ such that $l_i \geq 2$.

Assume first that $r = 0$. Then $E_i M'_0 = 0$ is obvious. Assume that for some $i \in I$ such that $l_i \geq 2$, we have $x = F_i \eta \neq 0$. For any $j \in I$, we have

$$E_j x = E_j F_i \eta = F_i E_j \eta + \delta_{i,j} \phi\left([\langle i, \lambda \rangle]_i\right) \eta = \delta_{i,j} \phi\left(\begin{bmatrix} \langle i, \lambda \rangle \\ 1 \end{bmatrix}_i\right) \eta.$$

Since $\langle i, \lambda \rangle \in l_i \mathbf{Z}$, and $l_i \geq 2$, we have $\phi(\left[\begin{smallmatrix} \langle i, \lambda \rangle \\ 1 \end{smallmatrix}\right]_i) = 0$; thus, $E_j x = 0$. If $n \geq 2$, then $E_j^{(n)} x = E_j^{(n)} F_i \eta$ is an R-linear combination of $F_i E_j^{(n)} \eta$ and of $E_j^{(n-1)} \eta$, hence is again zero. Thus, $E_j^{(n)} x = 0$ for all $j \in I$ and all $n > 0$; since $x \in M^{\lambda - i'}$, there exists a unique morphism in ${}_R\mathcal{C}$ from the Verma module $M_{\lambda - i'}$ into M which takes the canonical generator to x; its image is a subobject of M containing x but not η. Since M is simple, we must have $x = 0$. Thus (d) holds for $r = 0$.

Assume now that $r \geq 1$ and that (d) holds for $r - 1$. To show that it holds for r, it suffices to show that $E_i F_j^{(l_j)} m = 0, F_i F_j^{(l_j)} m = 0$ for any i, j in I such that $l_i \geq 2$ and any $m \in M'_{r-1}{}^{\zeta}$. If $l_j \geq 2$, then $E_i F_j^{(l_j)} m$ is an R-linear combination of $F_j^{(l_j)} E_i m$ and of $F_j^{l_j - 1} m$, hence is zero since $E_i m = 0, F_j m = 0$, by the induction hypothesis. If $l_j = 1$, then $E_i F_j m = F_j E_i m + \delta_{i,j} \phi(\left[\begin{smallmatrix} \langle i, \zeta \rangle \\ 1 \end{smallmatrix}\right]_i) m$ where $\langle i, \zeta \rangle \in l_i \mathbf{Z}$, hence $\phi(\left[\begin{smallmatrix} \langle i, \zeta \rangle \\ 1 \end{smallmatrix}\right]_i) = 0$, as above; since $E_i m = 0$, by the induction hypothesis, we have again $E_i F_j m = 0$.

If $i \neq j$, then from the identity in 35.3.1, we deduce by interchanging i, j and applying σ, that $F_i F_j^{(l_j)} m$ is an R-linear combination of $F_j^{(l_j - r)} F_i F_j^{(r)} m$ for various r with $0 \leq r \leq -\langle j, i' \rangle < l_j$. For such r we have $F_i F_j^{(r)} m = 0$. (Indeed, if $l_j \geq 2$, then $F_i F_j^{(r)} m = 0$, by the induction hypothesis; if $l_j = 1$, then $r = 0$ and $F_i F_j^{(r)} m = F_i m = 0$, again by the induction hypothesis.) Thus, we have $F_i F_j^{(l_j)} m = 0$. If $i = j$, then $F_i F_j^{(l_j)} m = F_j^{(l_j)} F_i m = 0$, by the induction hypothesis. This completes the inductive proof of (d).

Next we show by induction on $r \geq 0$ that

(e) $E_i^{(l_i)} M'_r \subset M'_{r-1}$ for any $i \in I$,

where, by convention, $M'_{-1} = 0$. This is clear for $r = 0$. Assume now that $r \geq 1$. We must show that $E_i^{(l_i)} F_j^{(l_j)} m' \in M'_{r-1}$ for any j and any $m \in M'_{r-1}$. Now $E_i^{(l_i)} F_j^{(l_j)} m'$ is an R-linear combination of $F_j^{(l_j)} E_i^{(l_i)} m'$ (which is in M'_{r-1} by the induction hypothesis) and of elements $F_j^{(l_j - t)} E_i^{(l_i - t)} m'$ with $t > 0$ such that $t \leq l_i, t \leq l_j$ (which are zero if $t < l_i$ or if $t = l_i, t < l_j$, by (d), and are in M'_{r-1} if $t = l_i = l_j$). Thus, (e) is proved.

From (d), (e), 35.2.2, and the obvious inclusion $F_i^{(l_i)} M'_r \subset M'_{r+1}$, we see that $\sum_r M'_r$ is an $_R\dot{\mathbf{U}}$-submodule of M. (It is certainly equal to the sum of its intersections with the weight spaces of M since it is spanned by homogeneous elements.) Since M is simple, we must have $M = \sum_r M'_r$. Thus (c) is proved. Now (b) follows from (d) and (c).

We prove (a). Let $\zeta \in X$ be such that $M^\zeta \neq 0$. By (c), we have $M'^\zeta \neq 0$. Then, as we have seen at the beginning of the proof, ζ is of the required form. The proposition is proved.

Corollary 35.3.3. *There is a unique unital $_R\dot{\mathbf{U}}^*$-module structure on M in which the ζ-weight space is the same as that in the $_R\dot{\mathbf{U}}$-module M, for any $\zeta \in X^* \subset X$, and such that $E_i, F_i \in {}_R\mathbf{f}^*$ act as $E_i^{(l_i)}, F_i^{(l_i)} \in {}_R\mathbf{f}$. Moreover, this is a simple highest weight module for $_R\dot{\mathbf{U}}^*$ with highest weight $\lambda \in X^*$.*

We define operators $e_i, f_i : M \to M$ for $i \in I$ by $e_i = E_i^{(l_i)}, f_i = F_i^{(l_i)}$. Using Theorem 35.1.8, we see that the e_i satisfy the Serre-type relations of $_R\mathbf{f}^*$ and that the f_i satisfy the Serre-type relations of $_R\mathbf{f}^*$.

If $\zeta \in X - X^*$ we have $M^\zeta = 0$, by 35.3.2(a). If $\zeta \in X^*$ and $m \in M^\zeta$, then, by 31.1.6(c), we have that $(e_i f_j - f_j e_i)(m)$ is equal to $\delta_{i,j}\phi(\left[\begin{smallmatrix}\langle i,\zeta\rangle \\ l_i\end{smallmatrix}\right]_i)m$ plus an R-linear combination of elements of the form $F_i^{l_i-t}E_i^{l_i-t}(m)$ with $0 < t < l_i$ which are zero by 35.3.2(b). Since $\langle i,\zeta\rangle \in l_i\mathbf{Z}$, we see from 35.1.5 that

$$\phi\left(\begin{bmatrix}\langle i,\zeta\rangle \\ l_i\end{bmatrix}_i\right) = \phi\left(\begin{bmatrix}\langle i,\zeta\rangle/l_i \\ 1\end{bmatrix}_i^*\right).$$

Therefore, $(e_i f_j - f_j e_i)(m) = \delta_{i,j}\phi(\left[\begin{smallmatrix}\langle i,\zeta\rangle/l_i \\ 1\end{smallmatrix}\right]_i^*)m$. It is clear that $e_i(M^\zeta) \subset M^{\zeta + l_i i'}$ and $f_i(M^\zeta) \subset M^{\zeta - l_i i'}$.

Thus, we have a unital $_R\dot{\mathbf{U}}^*$-module structure on M. By 35.3.2(c), this is a highest weight module of $_R\dot{\mathbf{U}}^*$ with highest weight λ. This $_R\dot{\mathbf{U}}^*$-module is simple. Indeed, assume that M'' is a non-zero $_R\dot{\mathbf{U}}^*$-submodule of M. Then M'' is the sum of its intersections with the various M^ζ (with $\zeta \in X$) and is stable under all $E_i^{(l_i)}, F_i^{(l_i)} : M \to M$ (in the $_R\dot{\mathbf{U}}$-module structure). Now M'' is automatically stable under E_i^a, F_i^a (in the $_R\dot{\mathbf{U}}$-module structure) for any i and a such that $0 < a < l_i$, since these act as zero on M. Using now lemma 35.2.2, we see that M'' is stable under $E_i^{(n)}, F_i^{(n)}$ for any i and any $n \in \mathbf{N}$. Thus, M'' is a (non-zero) $_R\dot{\mathbf{U}}$-submodule of M; hence $M'' = M$. The corollary is proved.

Corollary 35.3.4. *Assume, in addition, that the root datum is Y-regular and that $\lambda \in X^+$. Then the simple highest weight module for $_R\dot{\mathbf{U}}^*$ defined in Corollary 35.3.3 is $_R\Lambda_\lambda$.*

Indeed, the module ${}_R\Lambda_\lambda$ of ${}_R\dot{\mathbf{U}}^*$ is simple. By the assumption 35.1.2(b), we may apply 33.2.4.

35.4. A Tensor Product Decomposition of ${}_R\mathbf{f}$

35.4.1. Definition. Let $\mathfrak{f}$ be the R-subalgebra of ${}_R\mathbf{f}$ generated by the elements θ_i for various i such that $l_i \geq 2$. (Note that without the assumption 35.1.2(a), the definition of $\mathfrak{f}$ should be more complicated.) We have $\mathfrak{f} = \oplus_\nu \mathfrak{f}_\nu$ where $\mathfrak{f}_\nu = {}_R\mathbf{f}_\nu \cap \mathfrak{f}$.

Theorem 35.4.2. (a) *If $i \in I$ and $y \in \mathfrak{f}_\nu$, the difference $\theta_i^{(l_i)}y - \mathbf{v}_i^{-l_i\langle i,\nu\rangle} y\theta_i^{(l_i)}$ belongs to $\mathfrak{f}$.*

(b) *The R-linear map $\chi : {}_R\mathbf{f}^* \otimes_R \mathfrak{f} \to {}_R\mathbf{f}$ given by $x \otimes y \mapsto Fr'(x)y$ is an isomorphism of vector spaces.*

We prove (a). If (a) holds for y and y', then it also holds for yy'. Hence it suffices to prove (a) when y is one of the algebra generators of $\mathfrak{f}$. Thus, we may assume that $y = \theta_j$ where j satisfies $l_j \geq 2$. By our assumption, we then have $l_i \geq -\langle i, j'\rangle + 1$. Therefore, we may use the identity in Proposition 35.3.1, and we see that $\theta_i^{(l_i)}\theta_j - \mathbf{v}_i^{-l_i\langle i,j'\rangle}\theta_j\theta_i^{(l_i)}$ is an R-linear combination of products $\theta_i^{(r)}\theta_j\theta_i^{(l_i-r)}$ with $0 < r \leq -\langle i, j'\rangle < l_i$; these products are contained in $\mathfrak{f}$, by the definition of $\mathfrak{f}$. This proves (a).

We prove (b). We first show that χ is surjective. Using Lemma 35.2.2, we see that ${}_R\mathbf{f}$ is spanned as an R-vector space by products $x_1x_2\cdots x_p$ where the factors are either in $\mathfrak{f}_\nu$ for some ν (factors of the first kind) or of the form $\theta_i^{(l_i)}$ (factors of the second kind).

By (a), any product x_sx_{s+1} with x_s (resp. x_{s+1}) a factor of the first kind (resp. of the second kind) is equal to $\mathbf{v}_i^n x_{s+1}x_s$ plus an element of $\mathfrak{f}_{\nu'}$ for some n and some ν'. Applying this fact repeatedly, we see that $x_1x_2\cdots x_p$ is a linear combination of analogous words in which any factor of the second kind appears to the left of any factor of the first kind. It follows that χ is surjective.

It remains to show that χ is injective. Recall that the elements of $\mathbf{B}$ may be regarded as an R-basis of ${}_R\mathbf{f}^*$. Assume that for each $b \in \mathbf{B}$, we are given an element $y_b \in \mathfrak{f}$ such that $y_b = 0$ for all but finitely many b and such that we have a relation $\sum_b Fr'(b)y_b = 0$ in ${}_R\mathbf{f}$. We must prove that $y_b = 0$ for all b. We may assume that each y_b belongs to $\mathfrak{f}_\nu$ for some ν. Assume that $y_b \neq 0$ for some b. Then we may consider the largest integer N such that there exists b with $y_b \neq 0$ and $\operatorname{tr}|b| = N$.

In this proof we shall assume, as we may, that $(Y, X, \dots)$ is both Y-regular and X-regular. Let $\lambda \in X^+$ and $\lambda' \in X$; assume that $\langle i, \lambda\rangle \in l_i\mathbf{Z}$ for all i (i.e., that $\lambda \in X^*$). We consider the objects $M = {}_RL_\lambda, M' = {}_RM_{\lambda'}$ of ${}_R\mathcal{C}$ (see 31.3.2, 31.1.13); let η, η' be generators of the R-vector spaces $M^\lambda, M'^{\lambda'}$. Then $M' \otimes M \in {}_R\mathcal{C}$.

In $M' \otimes M$ we have $\sum_b Fr'(b)^- y_b^-(\eta' \otimes \eta) = 0$. We have $y_b^-(\eta' \otimes \eta) = v^{n(b)} y_b^-(\eta') \otimes \eta$, for some integer $n(b)$, since any element of $\mathfrak{f}_\nu$ with $\nu \neq 0$ annihilates η (see 35.3.2(b)). Hence we have

(c) $\sum_b v^{n(b)} Fr'(b)^-(y_b^-(\eta') \otimes \eta) = 0$ in $M' \otimes M$.

Let $M_1 = \oplus M^{\lambda_1} \subset M$ where the sum is taken over all $\lambda_1 \in X$ of the form $\lambda - \sum_i l_i p_i i'$ with $\sum_i p_i = N$. Let $\pi : M \to M_1$ be the obvious projection. We apply $1 \otimes \pi : M' \otimes M \to M' \otimes M_1$ to the equality (c). We obtain

(d) $\sum_b v^{n(b)} y_b^-(\eta') \otimes Fr'(b)^-(\eta) = 0$

where the sum is taken over b subject to $\operatorname{tr}|b| = N$.

By 35.3.3, we may regard M as a ${}_R\dot{\mathbf{U}}^*$-module; this is a simple highest weight module of ${}_R\dot{\mathbf{U}}^*$ which is just ${}_R\Lambda_\lambda$ (see 35.3.4). Note also that $Fr'(b)^-\eta$ in the ${}_R\dot{\mathbf{U}}$-module structure is the same as $b^-\eta$ in the ${}_R\dot{\mathbf{U}}^*$-module structure.

We shall assume, as we may, that $\langle i, \lambda\rangle$ are not only divisible by l_i, but are also large for all i, so that the vectors $b^-\eta \in M$ are linearly independent when b is subject to $\operatorname{tr}|b| = N$ (a finite set of b's). Here we use that $M = {}_R\Lambda_\lambda$ as a ${}_R\dot{\mathbf{U}}^*$-module. Then from (d) we deduce that $y_b^-(\eta') = 0$, hence $y_b = 0$ for all b such that $\operatorname{tr}|b| = N$. (We use the fact that M' is a Verma module.) This is a contradiction. The theorem is proved.

35.4.3. We assume that the root datum is simply connected. Then there is a unique $\lambda \in X^+$ such that $\langle i, \lambda\rangle = l_i - 1$ for all i. Let η be the canonical generator of ${}_R\Lambda_\lambda$.

Proposition 35.4.4. *The map $x \mapsto x^-\eta$ is an R-linear isomorphism $\mathfrak{f} \to {}_R\Lambda_\lambda$.*

Let $J = \sum_{i, n \geq l_i} ({}_R\mathbf{f}\theta_i^{(n)})$. It suffices to show that $J \oplus \mathfrak{f} = {}_R\mathbf{f}$. An equivalent statement is that $\sigma(J) \oplus \mathfrak{f} = {}_R\mathbf{f}$ since $\mathfrak{f}$ is σ-stable. We have $\sigma(J) = \sum_{i, n \geq l_i} \theta_i^{(n)}\, {}_R\mathbf{f}$. If i, n are such that $n \geq l_i$, then we can write $n = a + l_i b$ with $0 \leq a < l_i$ and $b \geq 1$ and we use the formulas in the proof of Lemma 35.2.2. We see that $\theta_i^{(n)} \subset \theta_i^{(l_i)}\, {}_R\mathbf{f}$. It follows that $\sigma(J) = \sum_i \theta_i^{(l_i)}\, {}_R\mathbf{f}$. The fact that $\sum_i \theta_i^{(l_i)}\, {}_R\mathbf{f}$ and $\mathfrak{f}$ are complementary subspaces of ${}_R\mathbf{f}$ follows easily from Theorem 35.4.2(b). The proposition follows.

35.5. Proof of Theorem 35.1.7

35.5.1. As we have seen in 35.1.11, we only have to prove existence in 35.1.7, assuming that R is as in 35.2.1.

By Theorem 35.4.2, there exists a unique R-linear map $P: {}_R\mathbf{f} \to {}_R\mathbf{f}^*$ such that $P(\theta_{i_1}^{(l_{i_1})} \cdots \theta_{i_n}^{(l_{i_n})}\theta_{j_1} \cdots \theta_{j_r})$ is equal to $\theta_{i_1} \cdots \theta_{i_n}$ if $r = 0$ and to 0 if $r > 0$. (Here $i_1, \ldots, i_n$ is any sequence in I and $j_1, \ldots, j_r$ is any sequence in I such that $l_{j_1} \geq 2, \ldots, l_{j_r} \geq 2$.)

We show that P is an algebra homomorphism. It suffices to show that

(a) $P(x\theta_i) = P(x)P(\theta_i)$ for any $x \in {}_R\mathbf{f}$ and any i such that $l_i \geq 2$ and

(b) $P(x\theta_i^{(l_i)}) = P(x)P(\theta_i^{(l_i)})$ for any $x \in {}_R\mathbf{f}$ and any i.

(a) is obvious. We prove (b) for $x = \theta_{i_1}^{(l_{i_1})} \cdots \theta_{i_n}^{(l_{i_n})}\theta_{j_1} \cdots \theta_{j_r}$ by induction on $r \geq 0$. The case where $r = 0$ is trivial. Assume that $r \geq 1$. We have $x = x'\theta_j$ where $j = j_r$ and $x' = \theta_{i_1}^{(l_{i_1})} \cdots \theta_{i_n}^{(l_{i_n})}\theta_{j_1} \cdots \theta_{j_{r-1}}$.

If $i \neq j$ then, using the identity in 35.3.1 (after applying σ to it), we see that $x'\theta_j\theta_i^{(l_i)}$ is equal to a multiple of $x'\theta_i^{(l_i)}\theta_j$ plus a linear combination of terms of the form $x'\theta_i^{(r)}\theta_j\theta_i^{(l_i-r)}$ where $0 < r < l_i$.

By (a) and the induction hypothesis, we have

$$P(x'\theta_i^{(l_i)}\theta_j) = P(x'\theta_i^{(l_i)})P(\theta_j) = 0$$

and

$$P(x'\theta_i^{(r)}\theta_j\theta_i^{(l_i-r)}) = P(x'\theta_i^{(r)}\theta_j)P(\theta_i^{(l_i-r)}) = 0.$$

It follows that $P(x'\theta_j\theta_i^{(l_i)}) = 0$. If $i = j$, then $x'\theta_j\theta_i^{(l_i)} = x'\theta_i^{(l_i)}\theta_j$ and the same proof as the one above shows that $P(x'\theta_j\theta_i^{(l_i)}) = 0$. On the other hand, we have from the definition $P(x) = P(x'\theta_j) = 0$. Hence (b) holds in this case; both sides are zero.

To complete the proof we must compute $P(\theta_i^{(n)})$ for $n \geq 0$. Writing $n = a + l_i b$ with $0 \leq a < l_i$ and $b \in \mathbf{N}$, we have $\theta_i^{(n)} = \mathbf{v}_i^{abl_i}\theta_i^{(a)}\theta_i^{(l_ib)}$ as in Lemma 35.2.2, hence $P(\theta_i^{(n)}) = \mathbf{v}_i^{abl_i}P(\theta_i^{(a)})P(\theta_i^{(l_ib)})$. This is zero if $a > 0$, i.e., if n is not divisible by l_i. If $a = 0$, then

$$\begin{aligned} P(\theta_i^{(n)}) &= P(\theta_i^{(l_ib)}) = (b!)^{-1}\mathbf{v}_i^{l_i^2 b(b-1)/2}P(\theta_i^{(l_i)})^b \\ &= (b!)^{-1}(\mathbf{v}_i^*)^{b(b-1)/2}\theta_i^b = \theta_i^{(b)}. \end{aligned}$$

The theorem is proved.

35.5.2. We now discuss to what extent the assumptions 35.1.2(a),(b) are necessary. Theorem 35.1.8 depends only on the assumption 35.1.2(b). This assumption can be replaced by the assumption that l is odd; then essentially the same proof will work (using the results in 33.1, instead of those in 33.2). If the Cartan datum is of finite type, then as pointed out in 35.1.2, the assumption 35.1.2(b) is automatically satisfied; hence 35.1.8 holds in this case.

We now discuss Theorem 35.1.7. Here we may again substitute the assumption 35.1.2(b) by the assumption that l is odd. If the Cartan datum is irreducible, of finite type, 35.1.2(b) is automatically satisfied, but 35.1.2(a) can fail; more precisely, if 35.1.2(a) is not satisfied, then we may assume that

(a) we have $\langle i, j' \rangle = -2$ for some $i, j \in I$ and $l = 2$ or

(b) we have $I = \{i, j\}$ with $\langle i, j' \rangle = -3$, $\langle j, i' \rangle = -1$ and $l = 2$ or 3.

In case (a), the algebra ${}_R\mathbf{f}$ is known in terms of explicit generators and relations (see [6]) and the statement of 35.1.7 can be verified by checking that these relations are satisfied in ${}_R\mathbf{f}^*$.

In case (b), the explicit presentation of the algebra ${}_R\mathbf{f}$ is not known for general l; however, for small l (for example $l = 2$ or 3), it is possible to again write generators and relations, using the formulas in [6], and with their help to verify 35.1.7. We omit further details.

We see that 35.1.8, 35.1.7 (hence also 35.1.9) hold unconditionally in the case where the Cartan datum is of finite type. It is likely that they hold without any restriction whatsoever.

35.5.3. Frobenius homomorphism in the classical case. We now assume that l is a prime number and that the $\mathcal{A}$-algebra R is such that $v = 1$ and $l = 0$ in R. (For example, R could be the finite field with l elements.) Let $l' = l$ if l is odd and let $l' = 4$ if $l = 2$. Then the value of the l'-th cyclotomic polynomial at $v = 1$ is divisible by l; hence if we define $\mathcal{A}'$ as in 35.1.3 (with the present choice of l'), we have that R is an $\mathcal{A}'$-algebra. Hence Theorems 35.1.7, 35.1.8, 35.1.9 hold in this case. (For Cartan data of finite type, the assumptions in 35.1.2 are not needed; for infinite types, 35.1.2(a) is needed, and 35.1.2(b) is needed only if $l = 2$.) For Cartan data of finite type, we thus obtain the transpose of the classical Frobenius map or of an exceptional isogeny in the sense of Chevalley.