CHAPTER 34

Gaussian Binomial Coefficients at Roots of 1

34.1.1. Let l be an integer ≥ 1. In this chapter we assume that the $\mathcal{A}$-algebra R, with $\phi : \mathcal{A} \to R$ and with $\mathbf{v} = \phi(v)$, is such that R is an integral domain and the following hold

$$\mathbf{v}^{2l} = 1 \text{ and } \mathbf{v}^{2t} \neq 1 \text{ for all } 0 < t < l.$$

All the identities proved in 1.3 for Gaussian binomial coefficients imply, after applying ϕ, corresponding identities in R. However, certain identities will be satisfied only in R. We shall now give some examples of such identities.

Lemma 34.1.2. (a) *If $t \geq 1$ is not divisible by l, and $a \in \mathbf{Z}$ is divisible by l, then $\phi(\left[\begin{smallmatrix} a \\ t \end{smallmatrix}\right]) = 0$.*

(b) *If $a_1 \in \mathbf{Z}$ and $t_1 \in \mathbf{N}$, then we have*

$$\phi\left(\begin{bmatrix} la_1 \\ lt_1 \end{bmatrix}\right) = \mathbf{v}^{l^2(a_1+1)t_1} \binom{a_1}{t_1}.$$

(c) *Let $a \in \mathbf{Z}$ and $t \in \mathbf{N}$. Write $a = a_0 + la_1$ with $a_0, a_1 \in \mathbf{Z}$ such that $0 \leq a_0 \leq l-1$ and $t = t_0 + lt_1$ with $t_0, t_1 \in \mathbf{N}$ such that $0 \leq t_0 \leq l-1$. We have*

$$\phi\left(\begin{bmatrix} a \\ t \end{bmatrix}\right) = \mathbf{v}^{(a_0t_1 - a_1t_0)l + (a_1+1)t_1l^2} \phi\left(\begin{bmatrix} a_0 \\ t_0 \end{bmatrix}\right) \binom{a_1}{t_1}.$$

Here $\binom{a_1}{t_1} \in \mathbf{Z}$ is an ordinary binomial coefficient. We prove (a) for $a \geq 0$ by induction on a. If $a = 0$, we have trivially $\left[\begin{smallmatrix} a \\ t \end{smallmatrix}\right] = 0$; if $a = l$, the equality $\phi(\left[\begin{smallmatrix} a \\ t \end{smallmatrix}\right]) = 0$ follows directly from the definitions.

Assume now that $a \geq 2l$ and that (a) holds for $a - l$ instead of a. By 1.3.1(e), we have

$$\phi\left(\begin{bmatrix} a \\ t \end{bmatrix}\right) = \sum_{t'+t''=t} \mathbf{v}^{(al-l)t''-lt'} \phi\left(\begin{bmatrix} al - l \\ t' \end{bmatrix}\right) \phi\left(\begin{bmatrix} l \\ t'' \end{bmatrix}\right).$$

For each term in the sum, we have that either t' or t'' is not divisible by l; hence the sum is zero by the induction hypothesis. This proves (a) for $a \geq 0$. We now prove (a), assuming that $a < 0$. Write $t = t_0 + lt_1$ with $0 < t_0 < l$. We have

$$\phi\left(\begin{bmatrix} a \\ t \end{bmatrix}\right) = (-1)^t \phi\left(\begin{bmatrix} -a+t-1 \\ t \end{bmatrix}\right)$$
$$= \sum_{t'+t''=t} \mathbf{v}^{(-a+lt_1)t''-(t_0-1)t'} \phi\left(\begin{bmatrix} -a+lt_1 \\ t' \end{bmatrix}\right) \phi\left(\begin{bmatrix} t_0-1 \\ t'' \end{bmatrix}\right).$$

Consider a term in the sum corresponding to (t', t''). Since $-a+lt_1 \geq 0$ is divisible by l, we see from the earlier part of the proof that $\phi(\left[\begin{smallmatrix} -a+lt_1 \\ t' \end{smallmatrix}\right]) = 0$ unless t' is divisible by l. But then t'' is congruent to t modulo l. Hence t'' is congruent to t_0 modulo l. It follows that $t'' \geq t_0$, hence $\left[\begin{smallmatrix} t_0-1 \\ t'' \end{smallmatrix}\right] = 0$. Hence our sum is zero and (a) is proved.

We prove (c), assuming (b). In the setup of (c) we have

$$\phi\left(\begin{bmatrix} a \\ t \end{bmatrix}\right) = \sum_{t'+t''=t} \mathbf{v}^{a_0 t'' - la_1 t'} \phi\left(\begin{bmatrix} a_0 \\ t' \end{bmatrix}\right) \phi\left(\begin{bmatrix} la_1 \\ t'' \end{bmatrix}\right).$$

By (a), the sum may be restricted to indices such that $t'' = lt''_1$ for some $t''_1 \in \mathbf{N}$ and such that $t' \leq a_0$. Then t' is congruent to t modulo l, hence t' is congruent to t_0 modulo l. Since both t', t_0 are in $[0, l-1]$, we must have $t' = t_0$ and therefore $t'' = lt_1$. Thus,

(d) $$\phi\left(\begin{bmatrix} a \\ t \end{bmatrix}\right) = \mathbf{v}^{la_0 t_1 - la_1 t_0} \phi\left(\begin{bmatrix} a_0 \\ t_0 \end{bmatrix}\right) \phi\left(\begin{bmatrix} la_1 \\ lt_1 \end{bmatrix}\right).$$

This shows that (c) is a consequence of (b).

We prove (b) for $a_1 \geq 0$ by induction on a_1. The case where $a_1 = 0$ or 1 is trivial. Assume now that $a_1 > 0$. By 1.3.1(e), we have

$$\phi\left(\begin{bmatrix} la_1 \\ lt_1 \end{bmatrix}\right) = \sum_{t'+t''=lt_1} \mathbf{v}^{l(a_1-1)t''-lt'} \phi\left(\begin{bmatrix} la_1 - l \\ t' \end{bmatrix}\right) \phi\left(\begin{bmatrix} l \\ t'' \end{bmatrix}\right).$$

By (a), we may assume that the sum is restricted to indices divisible by l, namely $t' = lt'_1, t'' = lt''_1$ with $t'_1 + t'_2 = t_1$. Using the induction hypothesis, we get

$$\phi\left(\begin{bmatrix} la_1 \\ lt_1 \end{bmatrix}\right) = \sum_{t'_1+t''_1=t_1} \mathbf{v}^{l^2(a_1 t_1 - t'_1 + t''_1)} \binom{a_1 - 1}{t'_1} \binom{1}{t''_1} = \mathbf{v}^{l^2(a_1+1)t_1} \binom{a_1}{t_1}.$$

We have used the identity 1.3.1(e), specialized for $v = 1$. This proves (b) for $a_1 \geq 0$.

We now prove (b) assuming that $a_1 < 0$. We have

$$\begin{aligned}\phi\left(\begin{bmatrix} la_1 \\ lt_1 \end{bmatrix}\right) &= (-1)^{lt_1}\phi\left(\begin{bmatrix} -la_1 + lt_1 - 1 \\ lt_1 \end{bmatrix}\right) \\ &= (-1)^{lt_1}\phi\left(\begin{bmatrix} (l-1) + l(-a_1 + t_1 - 1) \\ lt_1 \end{bmatrix}\right) \\ &= (-1)^{lt_1}\mathbf{v}^{l(l-1)t_1}\phi\left(\begin{bmatrix} l(-a_1 + t_1 - 1) \\ lt_1 \end{bmatrix}\right).\end{aligned}$$

The last equality follows from (d). By the part of (b) that is already proved, we have

$$\begin{aligned}\phi\left(\begin{bmatrix} l(-a_1 + t_1 - 1) \\ lt_1 \end{bmatrix}\right) &= \mathbf{v}^{l^2(-a_1+t_1)t_1}\begin{pmatrix} -a_1 + t_1 - 1 \\ t_1 \end{pmatrix} \\ &= \mathbf{v}^{l^2(-a_1+t_1)t_1}(-1)^{t_1}\begin{pmatrix} a_1 \\ t_1 \end{pmatrix}.\end{aligned}$$

It follows that

$$\begin{aligned}\phi\left(\begin{bmatrix} la_1 \\ lt_1 \end{bmatrix}\right) &= (-1)^{lt_1+t_1}\mathbf{v}^{l(l-1)t_1+l^2(-a_1+t_1)t_1}\begin{pmatrix} a_1 \\ t_1 \end{pmatrix} \\ &= (-1)^{(l+1)t_1}\mathbf{v}^{lt_1+l^2t_1}\mathbf{v}^{l^2(a_1+1)t_1}\begin{pmatrix} a_1 \\ t_1 \end{pmatrix}.\end{aligned}$$

It remains to observe that

(e) $\mathbf{v}^{l^2+l} = (-1)^{l+1}$.

Indeed, if l is even, we have $\mathbf{v}^l = -1$ and both sides of (e) are -1; if l is odd, then $\mathbf{v}^l = \pm 1$ and both sides of (e) are 1. The lemma is proved.

34.1.3. Let $p \geq 0$. We have

(a) $\phi([lp]^!/([l]!)^p) = p!\mathbf{v}^{l^2p(p-1)/2}$.

We prove (a) by induction on p. If $p = 0$ or 1, then (a) is trivial. Assume that $p \geq 2$. We have

$$\phi([lp]^!/([l]!)^p) = \phi([l(p-1)]^!/([l]!)^{p-1})\phi\left(\begin{bmatrix} lp \\ l \end{bmatrix}\right).$$

Using 34.1.2 and the induction hypothesis, we see that

$$\phi([lp]^!/([l]!)^p) = (p-1)!p\mathbf{v}^{l^2(p-1)(p-2)/2+l^2(p+1)} = p!\mathbf{v}^{l^2p(p-1)}.$$

This proves (a).

Lemma 34.1.4. *Assume that* $0 \le r \le a < l$. *We have*

(a) $\sum_{q=0}^{l-a-1}(-1)^{l-r+1+q}\mathbf{v}^{-(l-r)(a-l+1+q)+q}\phi(\left[\begin{smallmatrix} l-r \\ q \end{smallmatrix}\right]) = \mathbf{v}^{l(a-r)}\phi(\left[\begin{smallmatrix} a \\ r \end{smallmatrix}\right])$.

In the left hand side of (a) we may replace $\mathbf{v}^{l^2-l}$ by $(-1)^{l+1}$. Note also that $l - r \ge 1$; hence

$$\sum_{q=0}^{l-r}(-1)^q v^{q(1-l+r)}\begin{bmatrix} l-r \\ q \end{bmatrix} = 0$$

(see 1.3.4). Hence the left hand side of (a) equals

$$(-1)^{r+1}\mathbf{v}^{r(a+1-l)-la}\sum_{q=l-a}^{l-r}(-1)^q\mathbf{v}^{q(1-l+r)}\phi\left(\begin{bmatrix} l-r \\ q \end{bmatrix}\right)$$
$$= (-1)^{r+1}\mathbf{v}^{r(a+1-l)-la}\sum_{q'=0}^{a-r}(-1)^{l-r-q'}\mathbf{v}^{(l-r-q')(1-l+r)}\phi\left(\begin{bmatrix} l-r \\ q' \end{bmatrix}\right).$$

For any q' in the last sum, we have by 34.1.2(c), $\phi(\left[\begin{smallmatrix} l-r \\ q' \end{smallmatrix}\right]) = \mathbf{v}^{lq'}\phi(\left[\begin{smallmatrix} -r \\ q' \end{smallmatrix}\right])$. Thus the left hand side of (a) is

$$(-1)^{a+l-r+1}\mathbf{v}^{r(a+1-l)-la}\sum_{q'=0}^{a-r}(-1)^{a-r-q'}\mathbf{v}^{lq'+(l-r-q')(1-l+r)}\phi\left(\begin{bmatrix} -r \\ q' \end{bmatrix}\right)$$
$$= (-1)^{a+l-r+1}\mathbf{v}^{l-(l-r)l-la}\sum_{q'=0}^{a-r}\mathbf{v}^{-q'+r(a-r-q')}\phi(\begin{bmatrix} -1 \\ a-r-q' \end{bmatrix})\phi\left(\begin{bmatrix} -r \\ q' \end{bmatrix}\right)$$
$$= (-1)^{a+l-r+1}\mathbf{v}^{l-(l-r)l-la}\phi\left(\begin{bmatrix} -r-1 \\ a-r \end{bmatrix}\right)$$
$$= (-1)^{l+1}\mathbf{v}^{l-(l-r)l-la}\phi\left(\begin{bmatrix} a \\ a-r \end{bmatrix}\right) = \mathbf{v}^{l(a-r)}\phi\left(\begin{bmatrix} a \\ r \end{bmatrix}\right).$$

We have used $\mathbf{v}^l = \mathbf{v}^{-l}$. The lemma is proved.