CHAPTER 33

Relation with Kac-Moody Lie Algebras

33.1. The Specialization $v = 1$

33.1.1. Let ${}_R{}'\mathbf{f}$ be the free associative algebra over R with generators $\theta_i \quad (i \in I)$. As for $'\mathbf{f}$, which corresponds to the case $R = \mathbf{Q}(v)$, we have a natural direct sum decomposition ${}_R{}'\mathbf{f} = \oplus_\nu ({}_R{}'\mathbf{f}_\nu)$ where ν runs over $\mathbf{N}[I]$; each $'\mathbf{f}_\nu$ is a free R-module of finite rank.

Let ${}_R\tilde{\mathbf{f}}$ be the quotient of the algebra ${}_R{}'\mathbf{f}$ by the two-sided ideal of ${}_R{}'\mathbf{f}$ generated by the elements

$$\Phi_{i,j} = \sum_{p+p'=1-\langle i,j'\rangle} (-1)^{p'} \phi(\begin{bmatrix} p+p' \\ p \end{bmatrix}_i) \theta_i^p \theta_j \theta_i^{p'}$$

for various $i \neq j$ in I. Recall that $\phi : \mathcal{A} \to R$ is given.

Let ${}_R\tilde{\mathbf{f}}_\nu$ be the image of ${}_R{}'\mathbf{f}_\nu$ under the natural map ${}_R{}'\mathbf{f} \to {}_R\tilde{\mathbf{f}}$. It is clear that we have a direct sum decomposition ${}_R\tilde{\mathbf{f}} = \oplus_\nu ({}_R\tilde{\mathbf{f}}_\nu)$. From the definition, we have, for any ν, an exact sequence of R-modules

$$\oplus_{\nu',\nu'';i\neq j} ({}_R{}'\mathbf{f}_{\nu'}) \otimes_R ({}_R{}'\mathbf{f}_{\nu''}) \to {}_R{}'\mathbf{f}_\nu \to {}_R\tilde{\mathbf{f}}_\nu \to 0$$

where the indices satisfy $\nu', \nu'' \in \mathbf{N}[I]$ and $\nu' + \nu'' + (1 - \langle i, j'\rangle)i + j = \nu$; the first map has components $x, x' \mapsto x\Phi_{i,j}x'$. If we take this exact sequence for $R = \mathbf{Q}[v, v^{-1}]$ and we tensor it over $\mathbf{Q}[v, v^{-1}]$ with $\mathbf{Q}(v)$ or with R_0 (a field of characteristic zero, regarded as an $\mathcal{A}$-algebra or $\mathbf{Q}[v, v^{-1}]$-algebra via $v \mapsto 1$), we obtain again exact sequences, by the right exactness of tensor product. We deduce that

$${}_{R_0}\tilde{\mathbf{f}}_\nu = R_0 \otimes ({}_{\mathbf{Q}[v,v^{-1}]}\tilde{\mathbf{f}}_\nu)$$

and

$${}_{\mathbf{Q}(v)}\tilde{\mathbf{f}}_\nu = \mathbf{Q}(v) \otimes ({}_{\mathbf{Q}[v,v^{-1}]}\tilde{\mathbf{f}}_\nu).$$

Since ${}_{\mathbf{Q}[v,v^{-1}]}\tilde{\mathbf{f}}_\nu$ is a finitely generated $\mathbf{Q}[v, v^{-1}]$-module and $\mathbf{Q}(v)$ is the quotient field of $\mathbf{Q}[v, v^{-1}]$, we deduce that

(a) $\dim_{\mathbf{Q}(v)}({}_{\mathbf{Q}(v)}\tilde{\mathbf{f}}_\nu) \leq \dim_{R_0}({}_{R_0}\tilde{\mathbf{f}}_\nu)$.

By 1.4.3, there is a unique (surjective) algebra homomorphism ${}_{\mathbf{Q}(v)}\tilde{\mathbf{f}} \to \mathbf{f}$ which takes θ_i to θ_i for all i (and preserves 1). It is clear that this homomorphism maps ${}_{\mathbf{Q}(v)}\tilde{\mathbf{f}}_\nu$ onto $\mathbf{f}_\nu$, hence

(b) $\dim_{\mathbf{Q}(v)} \mathbf{f}_\nu \le \dim_{\mathbf{Q}(v)}({}_{\mathbf{Q}(v)}\tilde{\mathbf{f}}_\nu)$ for any $\nu \in \mathbf{N}[I]$.

Note that ${}_{R_0}\tilde{\mathbf{f}}$ is the R_0-algebra defined by the generators $\theta_i \quad (i \in I)$ and the Serre relations

$$\sum_{p+p'=1-\langle i,j'\rangle} (-1)^{p'}(\theta_i^p/p!)\theta_j(\theta_i^{p'}/p'!) = 0$$

for various $i \neq j$ in I. Thus it is the enveloping algebra of (the upper triangular part of) the corresponding Kac-Moody Lie algebra over R_0.

33.1.2. Assume now that the root datum is Y-regular and X-regular. Let $\lambda \in X^+$ and let $M = {}_{R_0}\Lambda_\lambda \in {}_{R_0}\mathcal{C}'$. The linear maps $E_i, F_i : M \to M$ satisfy in our case:

(a) $E_i M^\zeta \subset M^{\zeta+i'}$, $\quad F_i M^\zeta \subset M^{\zeta-i'}$ for any $i \in I$ and $\zeta \in X$;

(b) $(E_iF_j - F_jE_i)m = \delta_{i,j}\langle i,\zeta\rangle m$ for any $i,j \in I$ and $m \in M^\zeta$;

(c) $\sum_{p+p'=1-\langle i,j'\rangle}(-1)^{p'}(E_i^p/p!)E_j(E_i^{p'}/p'!) = 0 : M \to M$ for any $i \neq j$ in I;

(d) $\sum_{p+p'=1-\langle i,j'\rangle}(-1)^{p'}(F_i^p/p!)F_j(F_i^{p'}/p'!) = 0 : M \to M$ for any $i \neq j$ in I.

This shows that M is an integrable highest weight module of the Kac-Moody Lie algebra attached to the root datum. By results in [3], namely, the complete reducibility theorem of Weyl-Kac and the Gabber-Kac theorem, M is simple as a module of that Lie algebra and the R_0-linear map

$${}_{R_0}\tilde{\mathbf{f}}/\sum_i {}_{R_0}\tilde{\mathbf{f}}\theta_i^{\langle i,\lambda\rangle+1} \to M$$

given by

$$\theta_{i_1}\theta_{i_2}\cdots\theta_{i_p} \mapsto F_{i_1}F_{i_2}\cdots F_{i_p}\eta_\lambda$$

is an isomorphism. It follows that

(e) ${}_{R_0}\Lambda_\lambda$ is a simple object of ${}_{R_0}\mathcal{C}$ and

(f) for given $\nu \in \mathbf{N}[I]$, we have

$$\dim_{R_0}({}_{R_0}\tilde{\mathbf{f}}_\nu) = \dim_{R_0}({}_{R_0}\Lambda_\lambda^{\lambda-\nu})$$

provided that $\langle i,\lambda\rangle$ are large enough for all i.

From the definition of Λ_λ and ${}_{R_0}\Lambda_\lambda$, it is clear that

(g) $\dim_{R_0}({}_{R_0}\Lambda_\lambda^{\lambda-\nu}) = \dim_{\mathbf{Q}(v)} \Lambda_\lambda^{\lambda-\nu}$ for any λ, ν and

(h) for given $\nu \in \mathbf{N}[I]$, we have $\dim_{\mathbf{Q}(v)}(\mathbf{f}_\nu) = \dim_{\mathbf{Q}(v)} \Lambda_\lambda^{\lambda-\nu}$ provided that $\langle i, \lambda \rangle$ are large enough for all i.

From (f),(g),(h) we deduce that

$$\dim_{\mathbf{Q}(v)}(\mathbf{f}_\nu) = \dim_{R_0}({}_{R_0}\tilde{\mathbf{f}}_\nu)$$

for all ν. Combining this with the inequalities 33.1.1(a),(b), we see that those inequalities are in fact equalities. In particular, the natural surjective homomorphism ${}_{\mathbf{Q}(v)}\tilde{\mathbf{f}} \to \mathbf{f}$ must be an isomorphism. Similarly, the natural surjective homomorphism ${}_{R_0}\tilde{\mathbf{f}} \to {}_{R_0}\mathbf{f}$ is an isomorphism since

$$\dim_{R_0}({}_{R_0}\tilde{\mathbf{f}}_\nu) \geq \dim_{R_0}(\mathbf{f}_\nu) = \dim_{\mathbf{Q}(v)}(\mathbf{f}_\nu) = \dim_{R_0}({}_{R_0}\tilde{\mathbf{f}}_\nu)$$

for all ν.

Thus we have the following result.

Theorem 33.1.3. (a) *The natural algebra homomorphism* ${}_{\mathbf{Q}(v)}\tilde{\mathbf{f}} \to \mathbf{f}$ *is an isomorphism.*

(b) *We have* $\dim_{\mathbf{Q}(v)} \mathbf{f}_\nu = \dim_{R_0}({}_{R_0}\tilde{\mathbf{f}}_\nu)$ *for any* ν.

(c) *The natural algebra homomorphism* ${}_{R_0}\tilde{\mathbf{f}} \to {}_{R_0}\mathbf{f}$ *is an isomorphism.*

(d) *If* $\lambda \in X^+$, *then the dimension of the weight spaces of* Λ_λ *are the same as those of the simple integrable highest weight representation of the corresponding Kac-Moody Lie algebra.*

33.1.4. Remark. Parts (a), (b) and (c) of the theorem hold for arbitrary root data, since only the Cartan datum is used in their statement.

Corollary 33.1.5. *The algebra* $\mathbf{U}$ *can be defined by the generators* E_i $(i \in I)$, F_i $(i \in I)$, K_μ $(\mu \in Y)$ *and the relations 3.1.1(a)–(d), together with the quantum Serre relations for the* E_i*'s and for the* F_i*'s.*

33.2. The Quasi-Classical Case

33.2.1. In this section we assume that the $\mathcal{A}$-algebra R is a field of characteristic zero and $\phi(v_i) = \pm 1$ in R for all $i \in I$. We then say that we are in the *quasi-classical* case; this is justified by the results in this section. We also assume that $(I, \cdot)$ is without odd cycles (see 2.1.3). Then, by 2.1.3, we can find a function

(a) $i \mapsto a_i$ from I to $\{0,1\}$ such that $a_i + a_j = 1$ whenever $\langle i, j' \rangle < 0$.

Let R_0 be the ring R with a new $\mathcal{A}$-algebra structure in which $v \in \mathcal{A}$ is mapped to $1 \in R$. We want to relate the algebras ${}_R\tilde{\mathbf{f}}$ and ${}_{R_0}\tilde{\mathbf{f}}$. We cannot do this directly, but must enlarge them first as follows. Let A be the R-algebra defined by the generators $\theta_i, \tilde{K}_i \quad (i \in I)$ subject to the following relations: the θ_i satisfy the relations of ${}_R\tilde{\mathbf{f}}$, the $\tilde{K}_i$ commute among themselves and $\tilde{K}_i\theta_j = \phi(v_i)^{\langle i,j' \rangle}\theta_j\tilde{K}_i$, for all $i, j \in I$.

Let A_0 be the R-algebra defined by the generators $\theta_i, \tilde{K}_i \quad (i \in I)$ subject to the following relations: the θ_i satisfy the relations of ${}_{R_0}\tilde{\mathbf{f}}$, the $\tilde{K}_i$ commute among themselves and $\tilde{K}_i\theta_j = \phi(v_i)^{\langle i,j' \rangle}\theta_j\tilde{K}_i$, for all $i, j \in I$.

It is clear that, as an R-vector space, A (resp. A_0) is the tensor product of ${}_R\tilde{\mathbf{f}}$ (resp. ${}_{R_0}\tilde{\mathbf{f}}$) with the group algebra of $\mathbf{Z}^I$ over R, with basis given by the monomials in $\tilde{K}_i$.

Proposition 33.2.2. (a) *The assignment* $E_i \mapsto E'_i = E_i\tilde{K}_i^{a_i}$ *and* $\tilde{K}_i \mapsto \tilde{K}_i$ *for all* i, *defines an isomorphism of* R*-algebras* $A_0 \to A$.

(b) $\dim_R({}_R\tilde{\mathbf{f}}_\nu) = \dim_R({}_{R_0}\tilde{\mathbf{f}}_\nu)$ *for all* ν.

(c) *The natural algebra homomorphism* ${}_R\tilde{\mathbf{f}} \to {}_R\mathbf{f}$ *is an isomorphism.*

Let $i, j \in I$ be distinct. A simple computation shows that we have in A:

$$E_i'^{p}E'_jE_i'^{p'} = \phi(v_i)^{a_i(p+p')(p+p'-1)}\phi(v_i)^{\langle i,j' \rangle(pa_i+p'a_j)}E_i^pE_jE_i^{p'}\tilde{K}_i^{(p+p')a_i}\tilde{K}_j^{a_j}.$$

The factor $\phi(v_i)^{a_i(p+p')(p+p'-1)}$ is 1 since $\phi(v_i) = \pm 1$ and the exponent is even. By the definition of a_i, we have

$$\phi(v_i)^{\langle i,j' \rangle(pa_i+p'a_j)} = \phi(v_i)^{\langle i,j' \rangle(p+p')a_j}\phi(v_i)^{\langle i,j' \rangle p}$$

and this equals $\phi(v_i)^{\langle i,j' \rangle p}$, if $p + p' = 1 - \langle i, j' \rangle$. Note also that $\phi([n]_i^!) = \phi(v_i)^{n(n-1)/2}n!$, since $\phi(v_i) = \pm 1$. Hence, if $p + p' = 1 - \langle i, j' \rangle$, then

$$\begin{aligned}\phi([p]_i^!)\phi([p']_i^!) &= \phi(v_i)^{(p(p-1)+p'(p'-1))/2}p!p'! \\ &= \phi(v_i)^{(p+p')(p+p'-1)/2}\phi(v_i)^{-pp'}\phi(v_i)^{p(1-p)}p!p'! \\ &= \phi(v_i)^{(p+p')(p+p'-1)/2}\phi(v_i)^{\langle i,j' \rangle p}p!p'!.\end{aligned}$$

It follows that

$$\sum_{p+p'=1-\langle i,j' \rangle}(-1)^{p'}(E_i'^{p}/p!)E'_j(E_i'^{p'}/p'!) = \phi(v_i)^{\langle i,j' \rangle(1-\langle i,j' \rangle)/2}$$
$$\sum_{p+p'=1-\langle i,j' \rangle}(-1)^{p'}(E_i^{p}/\phi([p]_i^!))E_j(E_i^{p'}/\phi([p']_i^!))\tilde{K}_i^{(1-\langle i,j' \rangle)a_i}\tilde{K}_j^{a_j}.$$

This shows that the assignment in (a) preserves relations, hence defines an algebra homomorphism $A_0 \to A$. The same proof shows that the assignment $E_i \mapsto E_i \tilde{K}_i^{-a_i}$ and $\tilde{K}_i \mapsto \tilde{K}_i$ defines an algebra homomorphism $A \to A_0$; it is clear that this is the inverse of the previous homomorphism. This proves (a).

For any ν, the isomorphism in (a) maps the subspace ${}_{R_0}\tilde{\mathbf{f}}_\nu$ isomorphically onto the subspace $({}_R\tilde{\mathbf{f}}_\nu)\tilde{K}$ where $\tilde{K}$ is a monomial in the $\tilde{K}_i$ depending only on ν. This proves (b).

The homomorphism in (c) maps ${}_R\tilde{\mathbf{f}}_\nu$ onto ${}_R\mathbf{f}_\nu$ for any ν. Using (b), it follows that this restriction ${}_R\tilde{\mathbf{f}}_\nu \to {}_R\mathbf{f}_\nu$ is an isomorphism. This implies (c). The proposition is proved.

The next result compares the algebras ${}_{R_0}\dot{\mathbf{U}}$ and ${}_R\dot{\mathbf{U}}$.

Proposition 33.2.3. *There is a unique isomorphism of R-algebras* $f : {}_{R_0}\dot{\mathbf{U}} \to {}_R\dot{\mathbf{U}}$ *such that*

$$f(E_i 1_\zeta) = \phi(v_i)^{a_i\langle i,\zeta\rangle} E_i 1_\zeta, f(F_i 1_\zeta) = \phi(v_i)^{(1-a_i)\langle i,\zeta\rangle+1} F_i 1_\zeta, f(1_\zeta) = 1_\zeta$$

for all $i \in I, \zeta \in X$. *Here* a_i *is as in 33.2.1(a).*

To construct f, we take advantage of the fact that for the algebra ${}_{R_0}\dot{\mathbf{U}}$, we know a simple presentation by generators and relations, while for ${}_R\dot{\mathbf{U}}$ we do not. It will be easier to first construct a functor from ${}_R\mathcal{C}$ to ${}_{R_0}\mathcal{C}$ and then to show that it comes from an algebra homomorphism.

Let M be an object of ${}_R\mathcal{C}$. The linear maps $E_i, F_i : M \to M \quad (i \in I)$ satisfy

(a) $E_i M^\lambda \subset M^{\lambda+i'}, \quad F_i M^\lambda \subset M^{\lambda-i'}$ for any $i \in I$ and $\lambda \in X$;

(b) $(E_iF_j - F_jE_i)m = \delta_{i,j}\phi(v_i)^{\langle i,\lambda\rangle-1}\langle i,\lambda\rangle m$ for any $i,j \in I$ and $m \in M^\lambda$;

(c) $\sum_{p+p'=1-\langle i,j'\rangle}(-1)^{p'}\phi(v_i)^{p\langle i,j'\rangle}(E_i^p/p!)E_j(E_i^{p'}/p'!) = 0 : M \to M$ for any $i \neq j$ in I;

(d) $\sum_{p+p'=1-\langle i,j'\rangle}(-1)^{p'}\phi(v_i)^{p\langle i,j'\rangle}(F_i^p/p!)F_j(F_i^{p'}/p'!) = 0 : M \to M$ for any $i \neq j$ in I.

For any $a \in \mathbf{Z}$, we define linear maps $P_{i,a} : M \to M$ by $P_{i,a}(m) = \phi(v_i)^{a\langle i,\lambda\rangle}m$ for $m \in M^\lambda$. We define linear maps $E'_i, F'_i : M \to M$ in terms of the function 33.2.1(a) by $E'_i = E_iP_{i,a_i}, F'_i = \phi(v_i)F_iP_{i,1-a_i}$. As in the proof of the previous proposition, we can check that:

(a1) $E'_i M^\lambda \subset M^{\lambda+i'}, \quad F'_i M^\lambda \subset M^{\lambda-i'}$ for any $i \in I$ and $\lambda \in X$;

(b1) $(E'_iF'_j - F'_jE'_i)m = \delta_{i,j}\langle i,\lambda\rangle m$ for any $i,j \in I$ and $m \in M^\lambda$;

(c1) $\sum_{p+p'=1-\langle i,j'\rangle}(-1)^{p'}(E_i'^{p}/p!)E_j'(E_i'^{p'}/p'!) = 0 : M \to M$ for any $i \neq j$ in I;

(d1) $\sum_{p+p'=1-\langle i,j'\rangle}(-1)^{p'}(F_i'^{p}/p!)F_j'(F_i'^{p'}/p'!) = 0 : M \to M$ for any $i \neq j$ in I. Since ${}_{R_0}\mathbf{f} = {}_{R_0}\tilde{\mathbf{f}}$ (see 33.1.3) has the presentation given by the Serre relations, it follows that M with its weight space decomposition and with the linear maps E_i', F_i' is an object of ${}_{R_0}\mathcal{C}$.

Thus, we have defined a functor $\Gamma : {}_R\mathcal{C} \to {}_{R_0}\mathcal{C}$. This functor has the following obvious property: it associates to M an object $\Gamma(M)$ with the same underlying R-module as M and any endomorphism of M in ${}_R\mathcal{C}$ is at the same time an endomorphism of $\Gamma(M)$ in ${}_{R_0}\mathcal{C}$. Applying the functor Γ to ${}_R\dot{\mathbf{U}}$, regarded as a left module over itself, we obtain a structure of a unital left ${}_{R_0}\dot{\mathbf{U}}$-module on ${}_R\dot{\mathbf{U}}$. Thus, we have an R-bilinear pairing ${}_{R_0}\dot{\mathbf{U}} \times {}_R\dot{\mathbf{U}} \to {}_R\dot{\mathbf{U}}$, denoted by $a, b \mapsto a * b$; this has the properties $(aa') * b = a * (a' * b)$ and $(a * b)b' = a * (bb')$. The last property holds since right multiplication by b is an endomorphism of ${}_R\dot{\mathbf{U}}$ in ${}_R\mathcal{C}$, hence also in ${}_{R_0}\mathcal{C}$. We define a map $f : {}_{R_0}\dot{\mathbf{U}} \to {}_R\dot{\mathbf{U}}$ by $f(a) = \sum_{\zeta\in X} a * 1_\zeta$; only finitely many terms in the sum are non-zero. We have

$$
\begin{aligned}
f(a)f(a') &= \sum_{\zeta,\zeta'}(a * 1_\zeta)(a' * 1_{\zeta'}) \\
&= \sum_{\zeta'} a * \sum_{\zeta} 1_\zeta(a' * 1_{\zeta'}) \\
&= \sum_{\zeta'} a * (a' * 1_{\zeta'}) = \sum_{\zeta'} aa' * 1_{\zeta'} = f(aa').
\end{aligned}
$$

It is clear that f has the specified values on the algebra generators $E_i1_\zeta, F_i1_\zeta, 1_\zeta$ of ${}_{R_0}\dot{\mathbf{U}}$.

We show that f is an isomorphism. The elements $E_i1_\zeta, F_i1_\zeta, 1_\zeta$ are algebra generators of ${}_R\dot{\mathbf{U}}$, since $\phi([n]_i^!) = \pm n!$ is invertible in R for any $i \in I$ and any $n \geq 0$. It follows that f is surjective. As in 31.1.2, ${}_R\dot{\mathbf{U}}$ (resp. ${}_{R_0}\dot{\mathbf{U}}$) is a free ${}_R\mathbf{f} \otimes {}_R\mathbf{f}^{opp}$-module (resp. ${}_{R_0}\mathbf{f} \otimes {}_{R_0}\mathbf{f}^{opp}$-module) with generators 1_ζ (under $(x \otimes x') : u \mapsto x^+ux'^-$), and f carries the subspace of ${}_{R_0}\dot{\mathbf{U}}$ spanned by $x^+1_\zeta x'^-$ with

$$x \in \oplus_{\text{tr }\nu\leq N}({}_{R_0}\mathbf{f}_\nu), x' \in \oplus_{\text{tr }\nu\leq N'}({}_{R_0}\mathbf{f}_\nu)$$

and fixed ζ onto the analogous subspace of ${}_R\dot{\mathbf{U}}$. Since these subspaces are both of the same (finite) dimension over R, the restriction of f must be an isomorphism between them. This implies that f is an isomorphism. The uniqueness of f is obvious. The proposition is proved.

Corollary 33.2.4. *Assume that the root datum is Y-regular and X-regular and that $\lambda \in X^+$. Then ${}_R\Lambda_\lambda$ is a simple object of ${}_R\mathcal{C}$.*

By the proof of 33.2.3, this is equivalent to the statement that ${}_{R_0}\Lambda_\lambda$ is a simple object of ${}_{R_0}\mathcal{C}$. (See 33.1.2(e).)