CHAPTER 32

Commutativity Isomorphism

32.1. THE ISOMORPHISM ${}_f\mathcal{R}_{M,M'}$

32.1.1. In this chapter we assume that the Cartan datum is of finite type.

Proposition 32.1.2. *Let $M \in {}_R\mathcal{C}$.*

(a) *M is integrable if and only if it is a sum of subobjects which are finitely generated as R-modules.*

(b) *If M is integrable, then for any $m \in M$ there exists a number $N \geq 0$ such that $x^+m = 0$ for all $x \in {}_Rf_\nu$ with $\operatorname{tr} \nu \geq N$.*

Assume first that M is integrable. By 31.2.7, M is a sum of subobjects which are quotients of objects of the form ${}^\omega_R\Lambda_\lambda \otimes_R ({}_R\Lambda_{\lambda'})$ with $\lambda, \lambda' \in X^+$; these objects are finitely generated (free) R-modules. It remains to show that an object $M \in {}_R\mathcal{C}$ which is finitely generated as an R-module, is integrable and satisfies (b). This follows from the fact that there are only finitely many $\lambda \in X$ such that $M^\lambda \neq 0$, together with the fact that the root datum is X-regular. If $x \in {}_R\mathbf{f}_\nu$ and $m \in M^\lambda$, then $x^+m \in M^{\lambda+\nu}$ and $x^-m \in M^{\lambda-\nu}$.

32.1.3. Let $f : X \times X \to \mathbf{Q}$ be a function such that

$$
\begin{aligned}
&f(\zeta+\nu, \zeta'+\nu') - f(\zeta, \zeta') \\
\text{(a)} \qquad &= -\sum_i \nu_i \langle i, \zeta' \rangle (i \cdot i/2) - \sum_i \nu'_i \langle i, \zeta \rangle (i \cdot i/2) - \nu \cdot \nu'
\end{aligned}
$$

for all $\zeta, \zeta' \in X$ and all $\nu, \nu' \in \mathbf{Z}[I]$.

Such f exists: for example, we can choose a set of representatives H for the cosets $X/\mathbf{Z}[I]$ and an arbitrary function $c : H \times H \to \mathbf{Q}$, and set for any $h, h' \in H$ and $\nu, \nu' \in \mathbf{Z}[I]$:

$$f(h+\nu, h'+\nu') = c(h, h') - \sum_i \nu_i \langle i, h' \rangle (i \cdot i/2) - \sum_i \nu'_i \langle i, h \rangle (i \cdot i/2) - \nu \cdot \nu'.$$

This function satisfies (a) and conversely, any function satisfying (a), is of this form for a unique function c for fixed H. A function f satisfying (a) clearly satisfies the following identities:

(b) $f(\zeta,\zeta'+i')-f(\zeta,\zeta')=-\langle i,\zeta\rangle i\cdot i/2,$
$f(\zeta+i',\zeta')-f(\zeta,\zeta')=-\langle i,\zeta'\rangle i\cdot i/2,$
$f(\zeta-i',\zeta')-f(\zeta,\zeta')=\langle i,\zeta'\rangle i\cdot i/2,$
$f(\zeta,\zeta'-i')-f(\zeta,\zeta')=\langle i,\zeta\rangle i\cdot i/2,$
for all $\zeta,\zeta'\in X$ and $i\in I$.

32.1.4. We fix an integer $d\geq 1$ and a function $f:X\times X\to\mathbf{Q}$ as in 32.1.3, such that the values of f are contained in $\frac{1}{d}\mathbf{Z}$. Such f exists even with integer values: it suffices to take the function c in 32.1.3 with integer values. Assume that we are given an element $\tilde{\mathbf{v}}\in R$ such that $\tilde{\mathbf{v}}^d=\mathbf{v}$. For any rational number $q\in\frac{1}{d}\mathbf{Z}$, we will write $\mathbf{v}^q$ instead of $\tilde{\mathbf{v}}^{dq}$. This is the usual power of $\mathbf{v}$, when q is an integer.

Given two objects M,M' in ${}_R\mathcal{C}'$, we define an (invertible) linear operator $\Pi_f:M\otimes M'\to M\otimes M'$ by $\Pi_f(m\otimes m')=\mathbf{v}^{f(\lambda,\lambda')}m\otimes m'$ for $m\in M^\lambda, m'\in M'^{\lambda'}$. Let $\mathrm{s}:M'\otimes M\to M\otimes M'$ be the isomorphism of R-modules given by $\mathrm{s}(m'\otimes m)=m\otimes m'$. We define the R-linear map $\Theta:M\otimes_R M'\to M\otimes_R M'$ by

$$\Theta(m\otimes m')=\sum_\nu\sum_{b,b'\in\mathbf{B}_\nu}\phi(p_{b,b'})b^-m\otimes b'^+m'$$

where $\Theta=\sum_\nu\sum_{b,b'\in\mathbf{B}_\nu}p_{b,b'}b^-\otimes b'^+$ is as in 4.1.2, 24.1.6 (with $p_{b,b'}\in\mathcal{A}$) and $\phi:\mathcal{A}\to R$ is as in 31.1.1. By 32.1.2 applied to M', only finitely many terms in the sum are non-zero for any given m,m'.

Similarly, the R-linear map $\bar{\Theta}:M\otimes_R M'\to M\otimes_R M'$ given by

$$\bar{\Theta}(m\otimes m')=\sum_\nu\sum_{b,b'\in\mathbf{B}_\nu}\phi(\overline{p_{b,b'}})b^-m\otimes b'^+m'$$

for any $m\in M, m'\in M'$, is well-defined. From 4.1.3, we see that $\Theta,\bar{\Theta}:M\otimes_R M'\to M\otimes_R M'$ are inverse to each other.

Theorem 32.1.5. *Let ${}_f\mathcal{R}_{M,M'}=\Theta\Pi_f\mathrm{s}:M'\otimes M\to M\otimes M'$. Then ${}_f\mathcal{R}_{M,M'}$ is an isomorphism in ${}_R\mathcal{C}$.*

Let (f',d') be another pair like (f,d), but with $d'=1$; thus f' has values in $\mathbf{Z}$. Assume that the theorem holds for f replaced by f'; we show that it holds for f. Since $f(\lambda,\lambda')-f'(\lambda,\lambda')$ is constant when λ,λ' run through fixed cosets of $\mathbf{Z}[I]$ in X, the operator $\mathrm{s}\Pi_{f'}^{-1}\Pi_f\mathrm{s}:M'\otimes M\to M'\otimes M$ is an isomorphism in ${}_R\mathcal{C}$. Since ${}_f\mathcal{R}_{M,M'}={}_{f'}\mathcal{R}_{M,M'}\mathrm{s}\Pi_{f'}^{-1}\Pi_f\mathrm{s}$, our claim follows. Thus, in the rest of the proof we shall assume that $d=1$ so that f takes values in $\mathbf{Z}$; we then have $\tilde{\mathbf{v}}=\mathbf{v}$.

From the remark preceding the theorem, we see that ${}_f\mathcal{R}_{M,M'}$ is an isomorphism of R-modules; its inverse is $\mathrm{s}^{-1}\Pi_f^{-1}\bar{\Theta} : M \otimes M' \to M' \otimes M$. We must show that $u\Theta\Pi_f(m \otimes m') = \Theta\Pi_f u(m' \otimes m)$ for all homogeneous m, m' and all $u \in {}_R\dot{\mathbf{U}}$. Using the characterization 31.2.7 of integrable objects, we are reduced to the case where both M, M' are of the form ${}^{\omega}_{R}\Lambda^{\lambda} \otimes_R ({}_R\Lambda_{\lambda'})$; since such objects are obtained by change of rings from the analogous objects over $\mathcal{A}$, we may assume that $R = \mathcal{A}$. This can obviously be reduced to the case where $R = \mathbf{Q}(v)$. We may assume therefore that $R = \mathbf{Q}(v)$. It suffices to show that

$$\Delta(u)\Theta\Pi_f(m \otimes m') = \Theta(\Pi_f {}^t\Delta(u)(m \otimes m'))$$

for all $u \in \mathbf{U}$, $m \in M, m' \in M'$. (Here ${}^t\Delta(u)$ is as in 3.3.4.) Let $\alpha : \mathbf{U} \otimes \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$ be the algebra automorphism given on the generators by

$$\begin{aligned}
\alpha(E_i \otimes 1) &= E_i \otimes \tilde{K}_{-i}, \\
\alpha(F_i \otimes 1) &= F_i \otimes \tilde{K}_i, \\
\alpha(1 \otimes E_i) &= \tilde{K}_{-i} \otimes E_i, \\
\alpha(1 \otimes F_i) &= \tilde{K}_i \otimes F_i, \\
\alpha(K_\mu \otimes K_{\mu'}) &= K_\mu \otimes K_{\mu'}.
\end{aligned}$$

We have the identity $\bar{\Delta}(u) = \alpha({}^t\Delta(u))$ for all $u \in \mathbf{U}$. Indeed, both sides can be regarded as algebra homomorphisms $\mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$; hence it suffices to check that they agree on the generators E_i, F_i, K_μ, which is immediate. Therefore, the identity in 24.1.2(a) can be rewritten as follows:

$$\Delta(u)\Theta(m \otimes m') = \Theta(\alpha({}^t\Delta(u))(m \otimes m')).$$

We will show that

(a) $$\alpha({}^t\Delta(u)) = \Pi_f {}^t\Delta(u)\Pi_f^{-1} : M \otimes M' \to M \otimes M',$$

for all $u \in \mathbf{U}$. Therefore we obtain

$$\Delta(u)\Theta(m \otimes m') = \Theta(\Pi_f {}^t\Delta(u)\Pi_f^{-1}(m \otimes m')),$$

for all m, m'. This implies

$$\Delta(u)\Theta\Pi_f(m \otimes m') = \Theta(\Pi_f {}^t\Delta(u)(m \otimes m')),$$

for all m, m', as required.

It remains to show (a). Clearly, if (a) holds for u, u' then it holds for uu', and for linear combinations of u, u'. Hence it suffices to verify (a) in the case where u is one of the generators E_i, F_i, K_μ. The verification for K_μ is trivial. We apply both sides of (a) (with $u = E_i$, resp. F_i) to $m \otimes m'$ where $m \in M^\zeta, m' \in M'^{\zeta'}$. The left hand side is

$$E_i m \otimes m' + \tilde{K}_{-i} m \otimes E_i m'$$

(resp. $m \otimes F_i m' + F_i m \otimes \tilde{K}_i m'$). The right hand side is

$$v^{-f(\zeta,\zeta')}(v^{f(\zeta,\zeta'+i')} m \otimes E_i m' + v^{f(\zeta+i',\zeta')} E_i m \otimes \tilde{K}_i m')$$

(resp. $v^{-f(\zeta,\zeta')}(v^{f(\zeta-i',\zeta')} F_i m \otimes m' + v^{f(\zeta,\zeta'-i')} \tilde{K}_{-i} m \otimes F_i m')$). It remains to use the identities 32.1.3(b) for f. The theorem is proved.

In the following corollary, we do not assume the existence of roots of $\mathbf{v}$.

Corollary 32.1.6. *If M, M' are in ${}_R\mathcal{C}'$, then $M \otimes M'$ and $M' \otimes M$ are isomorphic objects of ${}_R\mathcal{C}'$.*

Indeed, f can be chosen with integer values.

32.2. The Hexagon Property

32.2.1. Let M, M', M'' be three objects of ${}_R\mathcal{C}'$. We define a linear isomorphism ${}_f\Pi' : M \otimes M' \otimes M'' \to M \otimes M' \otimes M''$ by

$${}_f\Pi'(m \otimes m' \otimes m'') = \mathbf{v}^{f(\lambda'',\lambda+\lambda')-f(\lambda'',\lambda')-f(\lambda'',\lambda)} m \otimes m' \otimes m''$$

for all $m \in M^\lambda, m' \in M^{\lambda'}, m'' \in M^{\lambda''}$.

We define a linear isomorphism ${}_f\Pi'' : M'' \otimes M \otimes M' \to M'' \otimes M \otimes M'$ by

$${}_f\Pi''(m'' \otimes m \otimes m') = \mathbf{v}^{f(\lambda+\lambda',\lambda'')-f(\lambda,\lambda'')-f(\lambda',\lambda'')} m'' \otimes m \otimes m'$$

for all $m \in M^\lambda, m' \in M^{\lambda'}, m'' \in M^{\lambda''}$.

Proposition 32.2.2. (a) *The map*

$${}_f\mathcal{R}_{M'',M\otimes M'}({}_f\Pi')^{-1} : M \otimes M' \otimes M'' \to M'' \otimes M \otimes M'$$

coincides with the composition

$$M \otimes M' \otimes M'' \xrightarrow{1_M \otimes {}_f\mathcal{R}_{M'',M'}} M \otimes M'' \otimes M' \xrightarrow{{}_f\mathcal{R}_{M'',M} \otimes 1_{M'}} M'' \otimes M \otimes M'.$$

(b) *The map*

$$ {}_f\mathcal{R}_{M \otimes M', M''} ({}_f\Pi'')^{-1} : M'' \otimes M \otimes M' \to M \otimes M' \otimes M''$$

coincides with the composition

$$M'' \otimes M \otimes M' \xrightarrow{{}_f\mathcal{R}_{M,M''} \otimes 1_{M'}} M \otimes M'' \otimes M' \xrightarrow{1_M \otimes {}_f\mathcal{R}_{M',M''}} M \otimes M' \otimes M''.$$

Let (f', d') be another pair like (f, d), but with $d' = 1$; thus f' has values in $\mathbf{Z}$. Assume that the proposition holds for f replaced by f'; as in the proof of Theorem 32.1.5, we see that it also holds for f'. Thus, in the rest of the proof, we shall assume that $d = 1$ so that f takes values in $\mathbf{Z}$; we then have $\tilde{\mathbf{v}} = \mathbf{v}$.

Using the characterization of integrable objects given in 31.2.7, we are reduced to the case where each of M, M', M'' is of the form ${}^{\omega}_{R}\Lambda^{\lambda} \otimes_R ({}_R\Lambda_{\lambda'})$; since such objects are obtained by change of rings from the analogous objects over $\mathcal{A}$, we may assume that $R = \mathcal{A}$; this case can be obviously reduced to the case where $R = \mathbf{Q}(v)$. We may assume therefore that $R = \mathbf{Q}(v)$.

Let $m \in M^{\lambda}, m' \in M'^{\lambda'}, m'' \in M''^{\lambda''}$. Using the definitions and 4.2.2(b), we have

$$ {}_f\mathcal{R}_{M'',M\otimes M'}(m \otimes m' \otimes m'') = v^{f(\lambda'',\lambda+\lambda')} \sum_{\nu} \Theta_{\nu}(m'' \otimes (m \otimes m'))$$

$$= v^{f(\lambda'',\lambda+\lambda')} \sum_{\nu',\nu''} v^{-f(\nu'',\lambda)+f(0,\lambda)} \Theta^{12}_{\nu'} \Theta^{13}_{\nu''} (m'' \otimes m \otimes m')$$

and

$$({}_f\mathcal{R}_{M'',M} \otimes 1_{M'})(1_M \otimes {}_f\mathcal{R}_{M'',M'})(m \otimes m' \otimes m'')$$

$$= v^{f(\lambda'',\lambda')} \sum_{\nu} ({}_f\mathcal{R}_{M'',M} \otimes 1_{M'}) \Theta^{23}_{\nu} (m \otimes m'' \otimes m')$$

$$= \sum_{\nu',\nu''} v^{f(\lambda'',\lambda')} v^{f(\lambda''-\nu'',\lambda)} \Theta^{12}_{\nu'} \Theta^{13}_{\nu''} (m'' \otimes m \otimes m').$$

On the other hand, using the definitions and 4.2.2(a), we have

$$ {}_f\mathcal{R}_{M\otimes M',M''}(m'' \otimes m \otimes m') = v^{f(\lambda+\lambda',\lambda'')} \sum_{\nu} \Theta_{\nu}((m \otimes m') \otimes m'')$$

$$= v^{f(\lambda+\lambda',\lambda'')} \sum_{\nu',\nu''} v^{f(\lambda',\nu'')-f(\lambda',0)} \Theta^{23}_{\nu'} \Theta^{13}_{\nu''} (m \otimes m' \otimes m'')$$

and

$$\begin{aligned}&(1_M \otimes {}_f\mathcal{R}_{M',M''})({}_f\mathcal{R}_{M,M''} \otimes 1_{M'})(m'' \otimes m \otimes m')\\&= v^{f(\lambda,\lambda'')}(1_M \otimes {}_f\mathcal{R}_{M',M''}) \sum_{\nu} \Theta_{\nu}^{12}(m \otimes m'' \otimes m')\\&= v^{f(\lambda,\lambda'')} v^{f(\lambda',\lambda''+\nu'')} \sum_{\nu',\nu''} \Theta_{\nu'}^{23}\Theta_{\nu''}^{13}(m \otimes m' \otimes m'').\end{aligned}$$

The proposition follows since $f(\lambda''-\nu'',\lambda) - f(\lambda'',\lambda) = f(0,\lambda) - f(\nu'',\lambda)$ and $f(\lambda',\lambda''+\nu'') - f(\lambda',\lambda'') = f(\lambda',\nu'') - f(\lambda',0)$ for $\nu'' \in \mathbf{Z}[I]$.

Lemma 32.2.3. *Let $d \geq 1$ be the order of the torsion subgroup of $X/\mathbf{Z}[I]$. There exists a symmetric $\mathbf{Z}$-bilinear pairing $f : X \times X \to \frac{1}{d}\mathbf{Z}$ which satisfies 32.1.3(a).*

We can find a direct sum decomposition $X = X_1 \oplus X_2$ such that $\mathbf{Z}[I]$ is contained in X_1 as a subgroup of index d. There is a unique symmetric bilinear pairing $f_1 : X_1 \times X_1 \to \frac{1}{d}\mathbf{Z}$ such that $f_1(i',j') = -i \cdot j$ for all $i,j \in I$. For $x_1, x_1' \in X_1$ and $x_2, x_2' \in X_2$, we set $f(x_1+x_2, x_1'+x_2') = f_1(x_1,x_1')$. This has the required properties.

Proposition 32.2.4 (Hexagon property). *Let $M, M', M'' \in {}_R\mathcal{C}$. Let (f,d) be as in the previous lemma. Assume that we are given an element $\tilde{\mathbf{v}} \in R$ such that $\tilde{\mathbf{v}}^d = \mathbf{v}$. Then the map ${}_f\mathcal{R}_{M'',M\otimes M'} : M \otimes M' \otimes M'' \to M'' \otimes M \otimes M'$ coincides with the composition*

$$M \otimes M' \otimes M'' \xrightarrow{1_M \otimes {}_f\mathcal{R}_{M'',M'}} M \otimes M'' \otimes M' \xrightarrow{{}_f\mathcal{R}_{M'',M} \otimes 1_{M'}} M'' \otimes M \otimes M'$$

and the map

$${}_f\mathcal{R}_{M\otimes M',M''} : M'' \otimes M \otimes M' \to M \otimes M' \otimes M''$$

coincides with the composition

$$M'' \otimes M \otimes M' \xrightarrow{{}_f\mathcal{R}_{M,M''} \otimes 1_{M'}} M \otimes M'' \otimes M' \xrightarrow{1_M \otimes {}_f\mathcal{R}_{M',M''}} M \otimes M' \otimes M''.$$

This follows from Proposition 32.2.2, since in our case, ${}_f\Pi', {}_f\Pi''$ are the identity maps.