Part V

CHANGE OF RINGS

Let R be a commutative $\mathcal{A}$-algebra with 1. The main topic of Part V is the R-algebra ${}_R\dot{\mathbf{U}}$, obtained from ${}_{\mathcal{A}}\dot{\mathbf{U}}$ by tensoring with R over $\mathcal{A}$, and its modules.

Chapter 31 contains a general discussion of ${}_R\dot{\mathbf{U}}$ and its module category.

In Chapter 32, assuming that the Cartan datum is of finite type, and that a certain root of v is given in R, we show that the integrable modules of ${}_R\dot{\mathbf{U}}$ form a braided tensor category.

In Chapter 33 we consider the specialization $v = 1$ and we establish the connection with Kac-Moody Lie algebras.

Chapters 34, 35, and 36 are concerned with the case where v is a root of unity in R. In Chapter 34 we establish various properties of Gaussian binomial coefficients at roots of 1. In Chapter 35 we construct a quantum analogue of the Frobenius homomorphism (under some rather mild assumptions). This includes as a special case the classical Frobenius homomorphism over fields of positive characteristic and also the exceptional isogenies (in small characteristic) defined by Chevalley [1]. In Chapter 36 we study the Hopf algebra ${}_R\mathbf{u}$, which in some sense, is the kernel of the Frobenius homomorphism. This algebra is finite dimensional if R is a field and the Cartan datum is of finite type.

CHAPTER 31

The Algebra ${}_R\dot{\mathbf{U}}$

31.1. Definition of ${}_R\dot{\mathbf{U}}$

31.1.1. From now on, R will be a fixed commutative ring with 1, with a given invertible element $\mathbf{v}$. We shall regard R as an $\mathcal{A}$-algebra via the ring homomorphism $\phi : \mathcal{A} \to R$ such that $\phi(v^n) = \mathbf{v}^n$ for all $n \in \mathbf{Z}$.

We consider the R-algebras

$$ {}_R\mathbf{f} = R \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathbf{f}) \text{ and } {}_R\dot{\mathbf{U}} = R \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\dot{\mathbf{U}}). $$

We have a direct sum decomposition ${}_R\mathbf{f} = \oplus_\nu({}_R\mathbf{f}_\nu)$ where ν runs over $\mathbf{N}[I]$ and ${}_R\mathbf{f}_\nu = R \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathbf{f}_\nu)$. The canonical bases $\mathbf{B}$ and $\dot{\mathbf{B}}$ of ${}_{\mathcal{A}}\mathbf{f}, {}_{\mathcal{A}}\dot{\mathbf{U}}$ give rise to R-bases of ${}_R\mathbf{f}, {}_R\dot{\mathbf{U}}$ consisting of elements $1 \otimes b$ where b is in $\mathbf{B}$ or $\dot{\mathbf{B}}$; we write b instead of $1 \otimes b$. In particular the elements $1_\lambda \in {}_R\dot{\mathbf{U}}$ are well-defined for all $\lambda \in X$. They satisfy as in $\dot{\mathbf{U}}$, $1_\lambda 1_{\lambda'} = \delta_{\lambda,\lambda'} 1_\lambda$.

The structure constants $m_{ab}^c, \hat{m}_c^{ab}$ of $\dot{\mathbf{U}}$ (see 25.4.1) can be regarded as elements of R via the ring homomorphism $\phi : \mathcal{A} \to R$. The identities 25.4.1(a)–(d) are clearly satisfied in R.

The comultiplication of ${}_R\dot{\mathbf{U}}$ (a collection of maps as in 23.1.5) is defined by the same formulas as in 25.4.1.

31.1.2. The ${}_{\mathcal{A}}\mathbf{f} \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathbf{f}^{opp})$-module structure $(x \otimes x') : u \mapsto x^+ u x'^-$ on ${}_{\mathcal{A}}\dot{\mathbf{U}}$, by change of scalars, induces a ${}_R\mathbf{f} \otimes_R ({}_R\mathbf{f}^{opp})$-module structure on ${}_R\dot{\mathbf{U}}$ denoted in the same way. Similarly, the ${}_{\mathcal{A}}\mathbf{f} \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathbf{f}^{opp})$-module structure $(x \otimes x') : u \mapsto x^- u x'^+$ on ${}_{\mathcal{A}}\dot{\mathbf{U}}$, by change of scalars, induces a ${}_R\mathbf{f} \otimes_R ({}_R\mathbf{f}^{opp})$-module structure on ${}_R\dot{\mathbf{U}}$ denoted in the same way.

From 23.2.2 we deduce that

(a) the elements $b^+ 1_\lambda b'^-$ $(b, b' \in \mathbf{B}, \lambda \in X)$ form a basis of the R-module ${}_R\dot{\mathbf{U}}$;

(b) the elements $b^- 1_\lambda b'^+$ $(b, b' \in \mathbf{B}, \lambda \in X)$ form a basis of the R-module ${}_R\dot{\mathbf{U}}$;

(c) the R-algebra ${}_R\dot{\mathbf{U}}$ is generated by the elements $E_i^{(n)} 1_\lambda, F_i^{(n)} 1_\lambda$ for various $i \in I$, $n \geq 0$ and $\lambda \in X$.

31.1.3. We will give an alternative construction of ${}_R\dot{\mathbf{U}}$ in terms of ${}_R\mathbf{f}$.

Let ${}_RA$ be the algebra generated by the symbols $x^+1_\zeta x'^-, x^-1_\zeta x'^+$ with $x \in {}_R\mathbf{f}_\nu, x' \in {}_R\mathbf{f}_{\nu'}$, for various ν, ν', and $\zeta \in X$; these symbols are subject to the following relations:

$(\theta_i^{(a)})^+1_\zeta(\theta_j^{(b)})^- = (\theta_j^{(b)})^-1_{\zeta+ai'+bj'}(\theta_i^{(a)})^+$ if $i \neq j$;

$$(\theta_i^{(a)})^+1_{-\zeta}(\theta_i^{(b)})^- = \sum_{t\geq 0}\phi(\begin{bmatrix} a+b-\langle i,\zeta\rangle \\ t \end{bmatrix}_i)(\theta_i^{(b-t)})^-1_{-\zeta+(a+b-t)i'}(\theta_i^{(a-t)})^+;$$

$$(\theta_i^{(b)})^-1_\zeta(\theta_i^{(a)})^+ = \sum_{t\geq 0}\phi(\begin{bmatrix} a+b-\langle i,\zeta\rangle \\ t \end{bmatrix}_i)(\theta_i^{(a-t)})^+1_{\zeta-(a+b-t)i'}(\theta_i^{(b-t)})^-;$$

$x^+1_\zeta = 1_{\zeta+\nu}x^+, x^-1_\zeta = 1_{\zeta-\nu}x^-$ for $x \in \mathbf{f}_\nu$;

$(x^+1_\zeta)(1_{\zeta'}x'^-) = \delta_{\zeta,\zeta'}x^+1_\zeta x'^-$, $(x^-1_\zeta)(1_{\zeta'}x'^+) = \delta_{\zeta,\zeta'}x^-1_\zeta x'^+$;

$(x^+1_\zeta)(1_{\zeta'}x'^+) = \delta_{\zeta,\zeta'}1_{\zeta+\nu}(xx')^+$, $(x^-1_\zeta)(1_{\zeta'}x'^-) = \delta_{\zeta,\zeta'}1_{\zeta-\nu}(xx')^-$ if $x \in {}_R\mathbf{f}_\nu$;

$(rx+r'x')^+1_\zeta = rx^+1_\zeta + r'x'^+1_\zeta$, $(rx+r'x')^-1_\zeta = rx^-1_\zeta + r'x'^-1_\zeta$ if $x, x' \in {}_R\mathbf{f}_\nu$ and $r, r' \in R$.

If x or x' in $x^+1_\zeta x'^-$ or $x^-1_\zeta x'^+$ is 1, we omit writing it.

We have an obvious surjective R-algebra homomorphism ${}_RA \to {}_R\dot{\mathbf{U}}$. Using the relations of ${}_RA$, we easily see that the symbols $x^+1_\zeta x'^-$ generate ${}_RA$ as an R-module. In other words, the elements $b^+1_\zeta b'^-$, with $b, b' \in \mathbf{B}$ and $\zeta \in X$, generate ${}_RA$ as an R-module. Since they form an R-basis of ${}_R\dot{\mathbf{U}}$, they must also form an R-basis of ${}_RA$ and we deduce that:

(a) *the natural algebra homomorphism* ${}_RA \to {}_R\dot{\mathbf{U}}$ *is an isomorphism.*

31.1.4. There is a natural R-linear involution $\sigma : {}_R\mathbf{f} \to {}_R\mathbf{f}$; it is given by a change of rings from the analogous involution for $R = \mathcal{A}$, which is the restriction of $\sigma : \mathbf{f} \to \mathbf{f}$.

The automorphism $\omega : \dot{\mathbf{U}} \to \dot{\mathbf{U}}$ restricts to an automorphism $\omega : {}_\mathcal{A}\dot{\mathbf{U}} \to {}_\mathcal{A}\dot{\mathbf{U}}$; tensoring with R, we obtain an R-algebra automorphism $\omega : {}_R\dot{\mathbf{U}} \to {}_R\dot{\mathbf{U}}$.

31.1.5. As in 23.1.4, we say that a ${}_R\dot{\mathbf{U}}$-module M is *unital* if

(a) for any $m \in M$ we have $1_\lambda m = 0$ for all but finitely many $\lambda \in X$;

(b) for any $m \in M$ we have $\sum_{\lambda\in X} 1_\lambda m = m$.

We then have a direct sum decomposition (as an abelian group) $M = \oplus_{\lambda \in X} M^\lambda$ where $M^\lambda = 1_\lambda M$; we can regard M as an R-module by $rm = \sum_\lambda (r1_\lambda)(m)$ for $r \in R, m \in M$. Then the decomposition above is as an R-module. The unital ${}_R\dot{\mathbf{U}}$-modules are the objects of an abelian category ${}_R\mathcal{C}$ with the morphisms being homomorphisms of ${}_R\dot{\mathbf{U}}$-modules.

31.1.6. Let $M \in {}_R\mathcal{C}$, let $i \in I$ and let $n \in \mathbf{Z}$. We define R-linear maps $E_i^{(n)} : M \to M, F_i^{(n)} : M \to M$ by $E_i^{(n)} m = \sum_\lambda (E_i^{(n)} 1_\lambda) m$ and $F_i^{(n)} m = \sum_\lambda (F_i^{(n)} 1_\lambda) m$ for all $m \in M$. (Recall that $E_i^{(n)} 1_\lambda$ and $F_i^{(n)} 1_\lambda$ are elements of $\dot{\mathbf{B}} \subset \dot{\mathbf{U}}$, hence are well-defined in ${}_R\dot{\mathbf{U}}$.) It follows immediately from the definitions that $\theta_i^{(n)} \mapsto (E_i^{(n)} : M \to M)$ and $\theta_i^{(n)} \mapsto (F_i^{(n)} : M \to M)$ define two ${}_R\mathbf{f}$-module structures on M, denoted by $x, m \mapsto x^+ m$ and $x, m \mapsto x^- m$ respectively. We have

(a) $E_i^{(n)} M^\lambda \subset M^{\lambda + ni'}$, $\quad F_i^{(n)} M^\lambda \subset M^{\lambda - ni'}$ for any $i \in I, n \in \mathbf{Z}$ and $\lambda \in X$.

Moreover, for any $\zeta \in X$ and any $m \in M^\zeta$, we have

(b) $E_i^{(a)} F_j^{(b)} m = F_j^{(b)} E_i^{(a)} m$ if $i \neq j$;

(c) $E_i^{(a)} F_i^{(b)} m = \sum_{t \geq 0} \phi(\left[\begin{smallmatrix} a-b+\langle i,\zeta\rangle \\ t \end{smallmatrix}\right]_i) F_i^{(b-t)} E_i^{(a-t)} m$;

(d) $F_i^{(b)} E_i^{(a)} m = \sum_{t \geq 0} \phi(\left[\begin{smallmatrix} -a+b-\langle i,\zeta\rangle \\ t \end{smallmatrix}\right]_i) E_i^{(a-t)} F_i^{(b-t)} m$.

31.1.7. Conversely, let M be an R-module with a given direct sum decomposition $M = \oplus_{\zeta \in X} M^\zeta$ and given R-linear maps $E_i^{(n)}, F_i^{(n)} : M \to M$ (for $i \in I, n \in \mathbf{Z}$) satisfying 31.1.6(a)–(d) and such that $E_i^{(n)} = F_i^{(n)} = 0$ for $n < 0$. Assume that $\theta_i^{(n)} \mapsto (E_i^{(n)} : M \to M)$ and $\theta_i^{(n)} \mapsto (F_i^{(n)} : M \to M)$ define two ${}_R\mathbf{f}$-module structures on M, denoted by $x, m \mapsto x^+ m$ and $x, m \mapsto x^- m$, respectively. Then this structure comes from a well-defined structure of unital ${}_R\dot{\mathbf{U}}$-module on M. Indeed, it is clear that this structure gives an ${}_RA$-module structure on M hence a ${}_R\dot{\mathbf{U}}$-module structure (see 31.1.3).

31.1.8. Let $M, M' \in {}_R\mathcal{C}$. The tensor product $M \otimes_R M'$ (as R-modules) will be regarded as a ${}_R\dot{\mathbf{U}}$-module by the rule $c(x \otimes x') = \sum_{a,b} \phi(\hat{m}_c^{ab}) ax \otimes bx'$. (All but finitely many terms in the last sum are zero.) The fact that the rule above defines an ${}_R\dot{\mathbf{U}}$-module structure follows from the identity 25.4.1(c). This ${}_R\dot{\mathbf{U}}$-module is unital, by the identity 25.4.1(d). Thus $M \otimes_R M'$ is naturally an object of ${}_R\mathcal{C}$.

Now let M, M', M'' be three objects of ${}_R\mathcal{C}$. By the previous construction, the R-module $M \otimes_R M' \otimes_R M''$ can be regarded as an object of ${}_R\mathcal{C}$ in two ways, $(M \otimes_R M') \otimes_R M''$ and $M \otimes_R (M' \otimes_R M'')$. In fact these two ways coincide; this follows from the identity 25.4.1(b).

31.1.9. From the definitions it is clear that, in the case where $R = \mathbf{Q}(v), \mathrm{v} = v$, we have ${}_R\mathcal{C} = \mathcal{C}$ and the tensor product just defined coincides with the one introduced earlier for $\mathcal{C}$.

31.1.10. To any object M of ${}_R\mathcal{C}$, we associate (as in 3.4.4) a new object ${}^{\omega}M$ of ${}_R\mathcal{C}$ as follows. ${}^{\omega}M$ has the same underlying R-module as M. By definition, for any $u \in {}_R\dot{\mathbf{U}}$, the operator u on ${}^{\omega}M$ coincides with the operator $\omega(u)$ on M.

31.1.11. If $R' \to R$ is a homomorphism of commutative $\mathcal{A}$-algebras with 1, we have ${}_R\dot{\mathbf{U}} = R \otimes_{R'} ({}_{R'}\dot{\mathbf{U}})$ and for any object $M \in {}_{R'}\mathcal{C}$, we may regard $R \otimes_{R'} M$ naturally as an object in ${}_R\mathcal{C}$ with the induced ${}_R\dot{\mathbf{U}}$-module structure. This gives a functor ${}_{R'}\mathcal{C} \to {}_R\mathcal{C}$ called *change of rings*, or *change of scalars*. It commutes with tensor products (as in 31.1.8) and with the operation ω in 31.1.10.

31.1.12. Let $(Y', X', \dots)$ be another root datum of type $(I, \cdot)$ and let $f : Y' \to Y, g : X \to X'$ be a morphism of root data. This induces a homomorphism $\phi : \mathbf{U}' \to \mathbf{U}$ between the corresponding Drinfeld-Jimbo algebras (see 3.1.2). For each $\zeta' \in X'$ and $\zeta \in X$ such that $g(\zeta) = \zeta'$ let ${}_{\mathcal{A}}\dot{\phi} : {}_{\mathcal{A}}\dot{\mathbf{U}}'1_{\zeta'} \cong {}_{\mathcal{A}}\dot{\mathbf{U}}1_{\zeta}$ be as in 23.2.5. By tensoring with R this gives rise to ${}_R\dot{\phi} : {}_R\dot{\mathbf{U}}'1_{\zeta'} \cong {}_R\dot{\mathbf{U}}1_{\zeta}$. Let M be a unital ${}_R\dot{\mathbf{U}}$-module. We can regard M as a unital ${}_R\dot{\mathbf{U}}'$-module by the following rule: if $m \in M^{\zeta}$ and $u \in {}_R\dot{\mathbf{U}}'1_{\zeta'}$ then um is defined to be $({}_R\dot{\phi}(u))m$ if $\zeta' = g(\zeta)$, and 0, otherwise. This gives a functor from unital ${}_R\dot{\mathbf{U}}$-modules to unital ${}_R\dot{\mathbf{U}}'$-modules.

31.1.13. Let $\lambda \in X$. The $\mathcal{A}$-submodule ${}_{\mathcal{A}}M_{\lambda}$ of the Verma module M_{λ} is a unital ${}_{\mathcal{A}}\dot{\mathbf{U}}$-submodule (see 23.3.2); by change of scalars, it gives rise to an object ${}_RM_{\lambda}$ of ${}_R\mathcal{C}$, called an R-Verma module. We have an exact sequence in ${}_R\mathcal{C}$:

$$\oplus_{i,n>0}({}_R\dot{\mathbf{U}}1_{\lambda+ni'}) \to {}_R\dot{\mathbf{U}}1_{\lambda} \to M_{\lambda} \to 0,$$

where the first map has components given by right multiplication by $1_{\lambda+ni'}E_i^{(n)}$ and the second map is given by $u \mapsto u1$ (1 is the canonical generator of M_{λ}). This is deduced by tensoring with R from the analogous exact sequence over $\mathcal{A}$.

Let $M \in {}_R\mathcal{C}$ and let $m \in M^\lambda$ be such that $E_i^{(n)}m = 0$ for all $i \in I$ and all $n > 0$. From the previous exact sequence, we see that there is a unique morphism $t : M_\lambda \to M$ such that $t(1) = m$.

31.2. Integrable ${}_R\dot{\mathbf{U}}$-modules

31.2.1. In this section we assume that the root datum is Y-regular (except in 31.2.4). Let $\lambda, \lambda' \in X^+$. The $\mathcal{A}$-submodule ${}_{\mathcal{A}}\Lambda_{\lambda'}$ of $\Lambda_{\lambda'}$ is a unital ${}_{\mathcal{A}}\dot{\mathbf{U}}$-submodule (see 23.3.7); by change of scalars, it gives rise to an object ${}_R\Lambda_{\lambda'} = R \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\Lambda_{\lambda'})$ of ${}_R\mathcal{C}$.

Similarly, the $\mathcal{A}$-submodule ${}^{\omega}_{\mathcal{A}}\Lambda_\lambda \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\Lambda_{\lambda'})$ of ${}^{\omega}\Lambda_\lambda \otimes \Lambda_{\lambda'}$ is a unital ${}_{\mathcal{A}}\dot{\mathbf{U}}$-submodule (see 23.3.9), in fact a tensor product in ${}_{\mathcal{A}}\mathcal{C}$; by change of scalars, it gives rise to the object ${}^{\omega}_R\Lambda_\lambda \otimes_R ({}_R\Lambda_{\lambda'})$ of ${}_R\mathcal{C}$.

Let $\zeta = \lambda' - \lambda \in X$. Consider the following morphisms of ${}_R\dot{\mathbf{U}}$-modules

$$\begin{array}{c}(\oplus_{i,n>\langle i,\lambda'\rangle}({}_R\dot{\mathbf{U}}1_{\zeta-ni'})) \oplus (\oplus_{i,n>\langle i,\lambda\rangle}({}_R\dot{\mathbf{U}}1_{\zeta+ni'})) \\ \downarrow f \\ {}_R\dot{\mathbf{U}}1_\zeta \\ \downarrow \pi \\ {}^{\omega}_R\Lambda_\lambda \otimes_R ({}_R\Lambda_{\lambda'}) \\ \downarrow \\ 0\end{array}$$

where f has components given by right multiplication by $1_{\lambda-ni'}F_i^{(n)}$ (in the first group of summands), $1_{\lambda+ni'}E_i^{(n)}$ (in the second group of summands) and $\pi(u) = u(\xi_{-\lambda} \otimes \eta_{\lambda'})$. We write $\xi_{-\lambda}$ instead of $1 \otimes \xi_{-\lambda}$ and similarly for $\eta_{\lambda'}$.

Proposition 31.2.2. *The sequence above is exact.*

When R is $\mathcal{A}$ and $\mathbf{v} = v$, this is a restatement of 23.3.8. The general case follows from this by taking the tensor product with R, by the right exactness of tensor products.

We can state the previous proposition in the following equivalent form.

Corollary 31.2.3. *π is surjective and its kernel is the left ideal*

$$\sum_{i,n>\langle i,\lambda'\rangle} {}_R\dot{\mathbf{U}}F_i^{(n)}1_\zeta + \sum_{i,n>\langle i,\lambda\rangle} {}_R\dot{\mathbf{U}}E_i^{(n)}1_\zeta \qquad \text{of } {}_R\dot{\mathbf{U}}.$$

31.2.4. In this subsection, the root datum is arbitrary. An object $M \in {}_R\mathcal{C}$ is said to be *integrable* if for any $m \in M$ and any $i \in I$ there exists $n_0 \geq 1$ such that $E_i^{(n)}m = F_i^{(n)}m = 0$ for all $n \geq n_0$. In the case where $R = \mathbf{Q}(v), \mathrm{v} = v$, this coincides with the earlier definition of an integrable object of $\mathcal{C}$.

From the definitions we see immediately that:

(a) if $M, M' \in {}_R\mathcal{C}$ are integrable, then $M \otimes_R M' \in {}_R\mathcal{C}$ is integrable;

(b) if $R' \to R$ is as in 31.1.11, and if $M' \in {}_{R'}\mathcal{C}$ is integrable, then $R \otimes_{R'} M' \in {}_R\mathcal{C}$ is integrable.

Let ${}_R\mathcal{C}'$ be the the category of integrable unital ${}_R\dot{\mathbf{U}}$-modules, regarded as a full subcategory of ${}_R\mathcal{C}$.

31.2.5. Returning to the assumptions of 31.2.1, we note that ${}_R\Lambda_\lambda$ and ${}^{\omega}_{R}\Lambda_\lambda \otimes_R ({}_R\Lambda_{\lambda'})$ are integrable. Indeed, this is already known over $\mathbf{Q}(v)$; from this, the result over $\mathcal{A}$ follows, since our objects over $\mathcal{A}$ are imbedded in the corresponding objects over $\mathbf{Q}(v)$ and finally, this implies the general case, by 31.2.4(b) with $R' = \mathcal{A}$.

Proposition 31.2.6. *Let $M \in {}_R\mathcal{C}$; let $\lambda, \lambda' \in X^+$. Let $\tilde{M}$ be the R-submodule of $M^{\lambda'-\lambda}$ consisting of all m such that $E_i^{(n)}m = 0$ for all i and all $n > \langle i, \lambda\rangle$ and such that $F_i^{(n)}m = 0$ for all i and all $n > \langle i, \lambda'\rangle$. Then the map $Hom_{{}_R\dot{\mathbf{U}}}({}^{\omega}_{R}\Lambda_\lambda \otimes_R ({}_R\Lambda_{\lambda'}), M) \to \tilde{M}$ given by $f \mapsto f(\xi_{-\lambda} \otimes \eta_{\lambda'})$ is an isomorphism.*

This follows immediately from Corollary 31.2.3.

Proposition 31.2.7. *Let $M \in {}_R\mathcal{C}$. Then M is integrable if and only if it satisfies the following condition:*

(a) M is a sum of subobjects each isomorphic to a quotient object of some ${}^{\omega}_{R}\Lambda_\lambda \otimes_R ({}_R\Lambda_{\lambda'})$ with $\lambda, \lambda' \in X^+$.

We know already that any object of the form ${}^{\omega}_{R}\Lambda_\lambda \otimes_R ({}_R\Lambda_{\lambda'})$ with $\lambda, \lambda' \in X^+$ is integrable. It follows immediately that, if M is as in (a), then M is integrable. We now prove the converse.

Assume that M is integrable and that $m \in M^\zeta$ where $\zeta \in X$. We can find integers $a_i, a_i' \in \mathbf{N}$ such that $E_i^{(a)}m = 0$ for all i and all $a > a_i$ and $F_i^{(a')}m = 0$ for all i and all $a' > a_i'$. Since the root datum is Y-regular, we can find $\lambda \in X$ such that $\langle i, \lambda\rangle \geq a_i$ and $\langle i, \lambda + \zeta\rangle \geq a_i'$ for all i. Let $\lambda' = \lambda + \zeta$. Then $\langle i, \lambda'\rangle \geq a_i'$ for all i. By the previous proposition, there exists a morphism $f : {}^{\omega}_{R}\Lambda_\lambda \otimes_R ({}_R\Lambda_{\lambda'}) \to M$ in ${}_R\mathcal{C}$ such that $f(\xi_{-\lambda} \otimes \eta_{\lambda'}) = m$. The image of f is a quotient of ${}^{\omega}_{R}\Lambda_\lambda \otimes_R ({}_R\Lambda_{\lambda'})$ and it contains m. Hence M satisfies (a).

31.3. Highest Weight Modules

31.3.1. In this section, we assume that the root datum is X-regular.

Let M be an object of ${}_R\mathcal{C}$. We say that M is a *highest weight module*, with highest weight $\lambda \in X$, if there exists a vector $m \in M^\lambda$ such that

(a) $E_i^{(n)} m = 0$ for all i and all $n > 0$;

(b) $M = \{x^- m | x \in {}_R\mathbf{f}\}$; and

(c) M^λ is a free R-module of rank one with generator m.

In this case, we have $M = \sum_{\lambda' \leq \lambda} M^{\lambda'}$.

Proposition 31.3.2. *Assume that R is a field.*

(a) *For any $\lambda \in X$, there exists a simple object (unique up to isomorphism) ${}_R L_\lambda$ of ${}_R\mathcal{C}$ which is a highest weight module with highest weight λ.*

(b) *If $\lambda \neq \lambda'$ then ${}_R L_\lambda$ is not isomorphic to ${}_R L_{\lambda'}$.*

(c) *If M is a highest weight module in ${}_R\mathcal{C}$ with highest weight λ, then M has a unique maximal subobject; the corresponding quotient object is isomorphic to ${}_R L_\lambda$.*

Let M be as in (c). A subobject M' of M is distinct from M if and only if $M' \subset \sum_{\lambda' < \lambda} M^{\lambda'}$. This shows that the sum of all subobjects of M distinct from M is a subobject distinct from M. Thus, M has a unique maximal subobject, hence a unique simple quotient object, which is clearly a highest weight module with highest weight λ. Applying this to the Verma module ${}_R M_\lambda$, which is a highest weight module with highest weight λ, we obtain a simple quotient ${}_R L_\lambda$ of this Verma module; this proves the existence part of (a). If L' is a simple object of ${}_R\mathcal{C}$ which is a highest weight module with highest weight λ, then, by 31.1.13, we can find a non-zero morphism ${}_R M_\lambda \to L'$. This is necessarily surjective. Since ${}_R M_\lambda$ has a unique simple quotient, we must have that L' is isomorphic to ${}_R L_\lambda$. Thus (a) and (c) are proved. (b) is now obvious.