CHAPTER 30

The Canonical Topological Basis of $(\mathbf{U}^- \otimes \mathbf{U}^+)\hat{}$

30.1. The Definition of the Canonical Topological Basis

30.1.1. In this chapter we assume that $(I, \cdot)$ is of finite type.

We denote by $(\mathbf{U}^- \otimes \mathbf{U}^+)\hat{}$ the closure of $\mathbf{U}^- \otimes \mathbf{U}^+$ in $(\mathbf{U} \otimes \mathbf{U})\hat{}$ (see 4.1.1). The elements of $(\mathbf{U}^- \otimes \mathbf{U}^+)\hat{}$ are possibly infinite sums of the form $\sum_{b,b' \in \mathbf{B}} c_{b,b'} b^- \otimes b'^+$ with $c_{b,b'} \in \mathbf{Q}(v)$. In this chapter we shall construct a canonical topological basis of $(\mathbf{U}^- \otimes \mathbf{U}^+)\hat{}$ which gives rise simultaneously to the canonical bases of all tensor products of type $\Lambda_\lambda \otimes {}^\omega\Lambda_{\lambda'}$.

Let $^- : (\mathbf{U}^- \otimes \mathbf{U}^+)\hat{} \to (\mathbf{U}^- \otimes \mathbf{U}^+)\hat{}$ be the ring involution defined as the unique continuous extension of $^- \otimes {}^- : \mathbf{U}^- \otimes \mathbf{U}^+ \to \mathbf{U}^- \otimes \mathbf{U}^+$. Note that we have $\Theta \in (\mathbf{U}^- \otimes \mathbf{U}^+)\hat{}$ (see 4.1.2). By 24.1.6, we can write uniquely

$$\Theta = \sum_{b,b' \in \mathbf{B}; |b|=|b'|} a_{b,b'} b^- \otimes b'^+$$

where $a_{b,b'} \in \mathcal{A}$.

30.1.2. From 4.1.2, 4.1.3, it follows that the $\mathbf{Q}$-linear map

$$\Psi : (\mathbf{U}^- \otimes \mathbf{U}^+)\hat{} \to (\mathbf{U}^- \otimes \mathbf{U}^+)\hat{}$$

given by $\Psi(x) = \Theta \bar{x}$ (product in $(\mathbf{U}^- \otimes \mathbf{U}^+)\hat{}$) satisfies $\Psi^2 = 1$. We clearly have $\Psi(fx) = \bar{f}\Psi(x)$ for all $f \in \mathcal{A}$ and all x. Hence if we set

$$\Psi(b_1^- \otimes b_1'^+) = \sum_{b,b' \in \mathbf{B}; |b|=|b'|} a_{b,b'} b^- b_1^- \otimes b'^+ b_1'^+ = \sum_{b_2, b_2' \in \mathbf{B}} r_{b_1,b_1';b_2,b_2'} b_2^- \otimes b_2'^+$$

for all $b_1, b_1' \in \mathbf{B}$, then we have

$r_{b_1,b_1';b_2,b_2'} \in \mathcal{A}$;

$r_{b_1,b_1';b_2,b_2'} = 0$ unless $(b_1, b_1') \le (b_2, b_2')$ ($\le$ as in 24.3.1);

$r_{b_1,b_1';b_1,b_1'} = 1$;

$\sum_{b_2,b_2' \in \mathbf{B}} \bar{r}_{b_1,b_1';b_2,b_2'} r_{b_2,b_2';b_3,b_3'} = \delta_{b_1,b_3}\delta_{b_1',b_3'}$ for any $b_1, b_1', b_3, b_3' \in \mathbf{B}$.

The last sum is finite by the previous statements. Applying 24.2.1 to the set $H = \mathbf{B} \times \mathbf{B}$ we see that there is a unique family of elements $p_{b_1,b'_1;b_2,b'_2} \in \mathbf{Z}[v^{-1}]$ defined for $b_1, b'_1, b_2, b'_2 \in \mathbf{B}$ such that

$p_{b_1,b'_1;b_1,b'_1} = 1;$

$p_{b_1,b'_1;b_2,b'_2} \in v^{-1}\mathbf{Z}[v^{-1}]$ if $(b_1, b'_1) \neq (b_2, b'_2)$;

$p_{b_1,b'_1;b_2,b'_2} = 0$ unless $(b_1, b'_1) \leq (b_2, b'_2)$;

$p_{b_1,b'_1;b_2,b'_2} = \sum_{b_3,b'_3} \bar{p}_{b_1,b'_1;b_3,b'_3} r_{b_3,b'_3;b_2,b'_2}$

for all $(b_1, b'_1) \leq (b_2, b'_2)$. Thus we have the following result.

Proposition 30.1.3. *For any $(b_1, b'_1) \in \mathbf{B} \times \mathbf{B}$, there is a unique element $\beta_{b_1,b'_1} \in (\mathbf{U}^- \otimes \mathbf{U}^+)\hat{}$ such that $\Theta\overline{\beta_{b_1,b'_1}} = \beta_{b_1,b'_1}$ and such that $\beta_{b_1,b'_1} - b_1^- \otimes b_1'^+$ is an (infinite) linear combination of elements $b_2^- \otimes b_2'^+$ with $(b_2, b'_2) > (b_1, b'_1)$ and with coefficients in $v^{-1}\mathbf{Z}[v^{-1}]$.*

We have $\beta_{b_1,b'_1} = \sum_{b_2,b'_2} p_{b_1,b'_1;b_2,b'_2} b_2^- \otimes b_2'^+$.

30.1.4. The elements $\beta_{b_1,b'_1} \in (\mathbf{U}^- \otimes \mathbf{U}^+)\hat{}$, for various $(b_1, b'_1) \in \mathbf{B} \times \mathbf{B}$, are said to form the *canonical topological basis* of $(\mathbf{U}^- \otimes \mathbf{U}^+)\hat{}$. This is not a basis in the strict sense.

Taking $b_1 = b'_1 = 1$, we obtain an element $\Upsilon = \sum_\nu \Upsilon_\nu = \beta_{1,1}$ where $\Upsilon_\nu \in \mathbf{U}_\nu^- \otimes \mathbf{U}_\nu^+$ for all ν and

(a) $$\Upsilon_0 = 1 \otimes 1.$$

Hence Υ is an invertible element of $(\mathbf{U}^- \otimes \mathbf{U}^+)\hat{}$.

By definition, we have $\Theta\bar{\Upsilon} = \Upsilon$; hence

(b) $$\Theta = \Upsilon\bar{\Upsilon}^{-1}.$$

Note also, that if $\nu \neq 0$, then Υ_ν is a linear combination of elements $b^- \otimes b'^+$ $(b, b' \in \mathbf{B}_\nu)$ with coefficients in $v^{-1}\mathbf{Z}[v^{-1}]$. This property, together with (a),(b), characterizes Υ.

30.1.5. Let $\lambda, \lambda' \in X^+$. By the general construction in 27.3.2, the $\mathbf{U}$-module $\Lambda_\lambda \otimes {}^\omega\Lambda_{\lambda'}$ has a canonical basis $B_\diamondsuit$. It consists of elements $(b^-\eta_\lambda)\diamondsuit(b'^+\xi_{-\lambda'})$ for various $b \in \mathbf{B}(\lambda)$ and $b' \in \mathbf{B}(\lambda')$.

Note that um is a well-defined element of $\Lambda_\lambda \otimes {}^\omega\Lambda_{\lambda'}$, for any $u \in (\mathbf{U}^- \otimes \mathbf{U}^+)\hat{}$ and any $m \in \Lambda_\lambda \otimes {}^\omega\Lambda_{\lambda'}$, by regarding the last space as a $\mathbf{U} \otimes \mathbf{U}$-module. In particular, $\beta_{b_1,b'_1}(\eta_\lambda \otimes \xi_{-\lambda'})$ is well-defined.

Proposition 30.1.6. *Let $b, b' \in \mathbf{B}$.*

(a) *If $b \in \mathbf{B}(\lambda)$ and $b' \in \mathbf{B}(\lambda')$, then $\beta_{b,b'}(\eta_\lambda \otimes \xi_{-\lambda'}) = (b^- \eta_\lambda) \diamondsuit (b'^+ \xi_{-\lambda'})$.*

(b) *If either $b \notin \mathbf{B}(\lambda)$ or $b' \notin \mathbf{B}(\lambda')$, then $\beta_{b,b'}(\eta_\lambda \otimes \xi_{-\lambda'}) = 0$.*

This follows immediately from the definitions and from 27.3.2.

30.1.7. Example.. Assume $I = \{i\}$ and $X = Y = \mathbf{Z}$ with $i = 1 \in Y, i' = 2 \in X$. The canonical topological basis of $(\mathbf{U}^- \otimes \mathbf{U}^+)\hat{}$ consists of the elements

$$x_{c,d} = \sum_{s \geq 0} v_i^{-s(s+c)} \begin{bmatrix} s+d \\ s \end{bmatrix}_i F_i^{(s+c)} \otimes E_i^{(s+d)} \quad (c \geq d \geq 0)$$

and

$$y_{c,d} = \sum_{s \geq 0} v_i^{-s(s+d)} \begin{bmatrix} s+c \\ s \end{bmatrix}_i F_i^{(s+c)} \otimes E_i^{(s+d)} \quad (d \geq c \geq 0)$$

with the identification $x_{c,d} = y_{c,d}$ for $c = d$.

30.2. On the Coefficients $p_{b_1,b'_1;b_2,b'_2}$

30.2.1. The canonical topological basis in the previous section is completely determined by the set of coefficients $p_{b_1,b'_1;b_2,b'_2} \in \mathbf{Z}[v^{-1}]$ defined for all b_1, b'_1, b_2, b'_2 in $\mathbf{B}$. In this section we make a proposal for a possible topological interpretation of these coefficients, assuming that the Cartan datum is simply laced (of finite type). We shall assume that $(b_1, b'_1) \leq (b_2, b'_2)$; otherwise the coefficient is zero.

30.2.2. Let $(\mathbf{I}, H, \dots)$ be the graph of $(I, \cdot)$ (see 14.1.3); note that $\mathbf{I} = I$. Assume that we have chosen an orientation for this graph. According to 14.5.1, to give b_1, b'_1, b_2, b'_2 in $\mathbf{B}$ is the same as to give four objects $\mathbf{V}_1, \mathbf{V}'_1, \mathbf{V}_2, \mathbf{V}'_2$ of $\mathcal{V}$ and orbits O_1, O'_1, O_2, O'_2 of $G_{\mathbf{V}_1}, G_{\mathbf{V}'_1}, G_{\mathbf{V}_2}, G_{\mathbf{V}'_2}$ on $\mathbf{E}_{\mathbf{V}_1}, \mathbf{E}_{\mathbf{V}'_1}, \mathbf{E}_{\mathbf{V}_2}, \mathbf{E}_{\mathbf{V}'_2}$, respectively (notation of 9.1.2). Hence we may write $p_{O_1,O'_1;O_2,O'_2}$ instead of $p_{b_1,b'_1;b_2,b'_2}$.

Let $\mathbf{V} = \mathbf{V}_2 \oplus \mathbf{V}'_2 \in \mathcal{V}$ and let $x \in \mathbf{E}_\mathbf{V}$ be an element such that $\mathbf{V}_2$ and $\mathbf{V}'_2$ are x-stable and the restriction of x to $\mathbf{V}_2$ (resp. $\mathbf{V}'_2$) is in O_2 (resp. in O'_2). Let J be the stabilizer of x in $G_\mathbf{V}$ and let Z be the J-orbit of $\mathbf{V}_2$ in the variety of all I-graded subspaces of $\mathbf{V}$. Note that $\mathbf{V}_2$ is a point of Z and that any $\mathbf{W} \in Z$ is x-stable. Let Z' be the subvariety of Z consisting of all subspaces $\mathbf{W} \in Z$ such that

(a) $\mathbf{W} \cap \mathbf{V}_2 \cong \mathbf{V}_1$ (in $\mathcal{V}$);

(b) $\mathbf{V}/(\mathbf{W} + \mathbf{V}_2) \cong \mathbf{V}'_1$ (in $\mathcal{V}$);

(c) the element of $\mathbf{E}_{\mathbf{W}\cap\mathbf{V}_2}$ defined by x corresponds under an isomorphism as in (a) to an element of $O_1 \subset \mathbf{E}_{\mathbf{V}_1}$;

(d) the element of $\mathbf{E}_{\mathbf{V}/(\mathbf{W}+\mathbf{V}_2)}$ defined by x corresponds under an isomorphism as in (b) to an element of $O'_1 \subset \mathbf{E}_{\mathbf{V}'_1}$.

One can hope that the coefficients of the various powers of v in $p_{O_1,O'_1;O_2,O'_2}$ are equal to the dimensions of the stalks of the intersection cohomology complex of the closure of Z' in Z at the point $\mathbf{V}_2 \in Z$, and that they are zero if $\mathbf{V}_2$ is not in that closure.

Notes on Part IV

1. The algebra $\dot{\mathbf{U}}$ has appeared in [1], in a geometric setting (in type A_n), but its definition in the general case is the same as that in type A_n. One of the main results of [1] was a topological definition of a canonical basis of $\dot{\mathbf{U}}$ (in type A_n), generalizing the author's definition of the canonical basis of **f**. The method of [1] works in almost the same way for affine type A_n, but the extension to other types remains to be done.
2. In [2], Kashiwara conjectured the existence of a canonical basis of $\dot{\mathbf{U}}$, for Cartan data of finite type, and constructed a basis of the quantum coordinate algebra $\mathbf{O}$ in which the structure constants were in $\mathbf{Q}[v,v^{-1}]$; this is presumably the same as the basis in 29.5.1, in which the structure constants are in $\mathbf{Z}[v,v^{-1}]$.
3. The definition of the canonical basis $\dot{\mathbf{B}}$ of $\dot{\mathbf{U}}$, in the general case was given in [6]. Most results in Chapters 24 and 25 appeared in [6].
4. Something close to Lemma 24.2.1 has been used in [3] to attach a polynomial to two elements of a Coxeter group.
5. Expressions like those in 25.3.1 appeared in [2], and are implicit in [1].
6. I do not know what is the relation, if any, between the form (,) on $\dot{\mathbf{U}}$, in 26.1.2, and the form on $\mathbf{U}$ defined in [7].
7. Propositions 27.1.7, 27.1.8 (and also results similar to 27.2.4) appear in [2].
8. Theorem 27.3.2 is similar to results in [6]; the analogous result for more than two factors (see 27.3.6) is new.
9. The existence of a canonical basis on the space of coinvariants $(\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2} \cdots \otimes \Lambda_{\lambda_n})_*$ (see 27.3.9) is new; it was known earlier for $n = 3$, see [5]. It implies that the corresponding space of coinvariants over $\mathcal{A}$ is a free $\mathcal{A}$-module; this answers a question of D. Kazhdan. (There is a somewhat analogous result about the space of "coinvariants in the tensor product" (see [4]) of several Weyl modules with the same negative central charge over an affine Lie algebra: this space has dimension independent of the central charge. There are other analogies between the two theories, for example the invariance property under cyclic permutations (28.2.9) has a counterpart in the theory over affine Lie algebras.)
10. The results in Chapters 28, 29 and 30 are new.
11. The positivity conjecture in 25.4.2 is made plausible by the results in [1].

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