CHAPTER 29

A Refinement of the Peter-Weyl Theorem

29.1. The Subsets $\dot{\mathbf{B}}[\lambda]$ of $\dot{\mathbf{B}}$

29.1.1. In this chapter we assume that $(I, \cdot)$ is of finite type.

Let β be an element in the canonical basis $\dot{\mathbf{B}}$ of $\dot{\mathbf{U}}$. We associate to β an element $\lambda_1 \in X^+$ as follows. We have $\beta \in \dot{\mathbf{U}}1_\zeta$ for a unique $\zeta \in X$. Choose $\lambda, \lambda'' \in X^+$ such that $\lambda'' - \lambda = \zeta$ and such that $\langle i, \lambda \rangle$ is large enough for all i. Then $\beta(\xi_{-\lambda} \otimes \eta_{\lambda''})$ is in the canonical basis B of ${}^\omega\Lambda_\lambda \otimes \Lambda_{\lambda''}$, and by 27.2.1, it belongs to $B[\lambda_1]$ for a unique $\lambda_1 \in X^+$. We want to show that λ_1 depends only on β, and not on the choice of λ, λ''. It is enough to show that, if λ, λ'' are replaced by $\lambda + \lambda', \lambda' + \lambda''$, then the procedure above leads again to λ_1. This follows from 27.3.5. Thus we have a well-defined map $\dot{\mathbf{B}} \to X^+ \quad (\beta \mapsto \lambda_1)$. We shall write $\dot{\mathbf{B}}[\lambda_1]$ for the fibre of this map at λ_1. Thus we have a partition $\dot{\mathbf{B}} = \sqcup_{\lambda_1 \in X^+} \dot{\mathbf{B}}[\lambda_1]$.

29.1.2. For any $\lambda_1 \in X^+$, we denote by $\dot{\mathbf{U}}[\geq \lambda_1]$ (resp. $\dot{\mathbf{U}}[> \lambda_1]$) the $\mathbf{Q}(v)$-subspace of $\dot{\mathbf{U}}$ spanned by $\sqcup_{\lambda_2; \lambda_2 \geq \lambda_1} \dot{\mathbf{B}}[\lambda_2]$ (resp. by $\sqcup_{\lambda_2; \lambda_2 > \lambda_1} \dot{\mathbf{B}}[\lambda_2]$).

Lemma 29.1.3. *The following conditions for an element $u \in \dot{\mathbf{U}}$ are equivalent:*

(a) $u \in \dot{\mathbf{U}}[\geq \lambda_1]$;

(b) *for any $\lambda, \lambda'' \in X^+$ we have $u(\xi_{-\lambda} \otimes \eta_{\lambda''}) \in ({}^\omega\Lambda_\lambda \otimes \Lambda_{\lambda''})[\geq \lambda_1]$;*

(c) *for any object $M \in \mathcal{C}$ of finite dimension over $\mathbf{Q}(v)$ and any vector $m \in M$, we have $um \in M[\geq \lambda_1]$;*

(d) *if $\lambda_2 \in X^+$ and u acts on Λ_{λ_2} by a non-zero linear map, then $\lambda_2 \geq \lambda_1$.*

The equivalence of (a) and (b) is clear from the definition. The equivalence of (c) and (d) follows by expressing M in (c) as a direct sum of simple objects. Clearly, if u satisfies (c), then it satisfies (b). Conversely, assume that u satisfies (b); we show that it satisfies (c). We may assume that m is in a weight space of M. By 23.3.10, we can find $\lambda, \lambda'' \in X^+$ and a morphism $f : {}^\omega\Lambda_\lambda \otimes \Lambda_{\lambda''} \to M$ (in $\mathcal{C}$) such that $f(\xi_{-\lambda} \otimes \eta_{\lambda''}) = m$. We obviously have $f({}^\omega\Lambda_\lambda \otimes \Lambda_{\lambda''})[\geq \lambda_1] \subset M[\geq \lambda_1]$. Since (b) holds for u, it follows that $um = uf(\xi_{-\lambda} \otimes \eta_{\lambda''}) = f(u(\xi_{-\lambda} \otimes \eta_{\lambda''})) \in M[\geq \lambda_1]$. Thus the equivalence of (b),(c) is established. The lemma is proved.

Lemma 29.1.4. *The following conditions for an element* $u \in \dot{\mathbf{U}}$ *are equivalent:*

(a) $u \in \dot{\mathbf{U}}[> \lambda_1]$;

(b) *for any* $\lambda, \lambda'' \in X^+$ *we have*

$$u(\xi_{-\lambda} \otimes \eta_{\lambda''}) \in ({}^{\omega}\Lambda_\lambda \otimes \Lambda_{\lambda''})[> \lambda_1].$$

(c) *for any object* $M \in \mathcal{C}$ *of finite dimension over* $\mathbf{Q}(v)$ *and any vector* $m \in M$, *we have* $um \in M[> \lambda_1]$;

(d) *if* $\lambda_2 \in X^+$ *and* u *acts on* Λ_{λ_2} *by a non-zero linear map, then* $\lambda_2 > \lambda_1$.

This follows from the previous lemma or can be proved in the same way.

Lemma 29.1.5. *Let* $\lambda_1 \in X^+$. *The subspaces* $\dot{\mathbf{U}}[\geq \lambda_1]$ *and* $\dot{\mathbf{U}}[> \lambda_1]$ *of* $\dot{\mathbf{U}}$ *are two-sided ideals. Hence* $\dot{\mathbf{U}}[\geq \lambda_1]/\dot{\mathbf{U}}[> \lambda_1]$ *is naturally a* $\dot{\mathbf{U}}$*-bimodule.*

This follows from the descriptions 29.1.3(c), 29.1.4(c) of $\dot{\mathbf{U}}[\geq \lambda_1]$ and $\dot{\mathbf{U}}[> \lambda_1]$.

29.1.6. The $\dot{\mathbf{U}}$-module structure on Λ_{λ_1} gives us a homomorphism of algebras $\dot{\mathbf{U}} \to \mathrm{End}(\Lambda_{\lambda_1})$. This restricts to a homomorphism of algebras (without 1)

$$\dot{\mathbf{U}}[\geq \lambda_1] \to \mathrm{End}(\Lambda_{\lambda_1})$$

whose kernel is, by Lemma 29.1.4, exactly $\dot{\mathbf{U}}[> \lambda_1]$; hence we have an induced homomorphism of algebras

(a) $\dot{\mathbf{U}}[\geq \lambda_1]/\dot{\mathbf{U}}[> \lambda_1] \to \mathrm{End}(\Lambda_{\lambda_1})$ which is injective.

In particular, we have

(b) $\dim(\dot{\mathbf{U}}[\geq \lambda_1]/\dot{\mathbf{U}}[> \lambda_1]) < \infty$,

or equivalently,

(c) $\dot{\mathbf{B}}[\lambda_1]$ is a finite set for any $\lambda_1 \in X^+$.

29.2. The Finite Dimensional Algebras $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P]$

29.2.1. Let P be a subset of X^+ with the following two properties:

(a) if $\lambda \in P$ and $\lambda' \in X^+$ satisfies $\lambda' \geq \lambda$, then $\lambda' \in P$;

(b) the complement of P in X^+ is finite.

Note that such P exist in abundance. We denote by $\dot{\mathbf{U}}[P]$ the subspace of $\dot{\mathbf{U}}$ generated by $\sqcup_{\lambda \in P} \dot{\mathbf{B}}[\lambda]$. From Lemma 29.1.5, we see that $\dot{\mathbf{U}}[P]$ is a two-sided ideal of $\dot{\mathbf{U}}$, and from 29.1.6(c), we see that the algebra $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P]$ is finite dimensional. Note that this algebra has a unit element (unlike $\dot{\mathbf{U}}$). Indeed, since $1_\zeta \in \dot{\mathbf{B}}$ for all $\zeta \in X$, we have that $1_\zeta \in \dot{\mathbf{U}}[P]$ for all but finitely many ζ. Then $\sum_{\zeta \in X} 1_\zeta$, which is not meaningful in $\dot{\mathbf{U}}$, is meaningful in $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P]$ and is the unit element there. Let $\lambda \in X^+ - P$. We show that $\dot{\mathbf{U}}[P]$ acts as zero on the $\dot{\mathbf{U}}$-module Λ_λ. Indeed, let β be an element of $\dot{\mathbf{B}} \cap P$ (these elements span P.) We have $\beta \in \dot{\mathbf{B}}[\lambda']$ for some $\lambda' \in P$. If the action of β on Λ_λ were non-zero, then from Lemma 29.1.3, it would follow that $\lambda \geq \lambda'$; using the definition of P, it would follow that $\lambda \in P$, a contradiction. We have proved that $\dot{\mathbf{U}}[P]$ acts as zero on Λ_λ, hence Λ_λ may be regarded as a $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P]$-module. This module is simple. Indeed, even as a $\dot{\mathbf{U}}$-module it has no proper submodules. It is clear that for $\lambda \neq \lambda'$ in $X^+ - P$, the $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P]$-modules $\Lambda_\lambda, \Lambda_{\lambda'}$ are not isomorphic (they are not isomorphic as $\dot{\mathbf{U}}$-modules).

By the standard theory of finite dimensional algebras, it follows that

$$\dim(\dot{\mathbf{U}}/\dot{\mathbf{U}}[P]) \geq \sum_\lambda (\dim \Lambda_\lambda)^2$$

(sum over all $\lambda \in X^+ - P$). On the other hand, by 29.1.6(a), we have

$$\dim(\dot{\mathbf{U}}/\dot{\mathbf{U}}[P]) = \sum_\lambda \dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda] \leq \sum_\lambda (\dim \Lambda_\lambda)^2$$

(both sums over all $\lambda \in X^+ - P$).

Comparing with the previous inequality, we see that

$$\dim(\dot{\mathbf{U}}/\dot{\mathbf{U}}[P]) = \sum_\lambda (\dim \Lambda_\lambda)^2$$

and

$$\dim \dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda] = (\dim \Lambda_\lambda)^2$$

for any $\lambda \in X^+ - P$. This implies the following result.

Proposition 29.2.2. (a) *The algebra (with 1) $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P]$ is semisimple and a complete set of simple modules for it is given by Λ_λ with $\lambda \in X^+ - P$.*

(b) *For any $\lambda \in X^+$, the homomorphism $\dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda] \to End(\Lambda_\lambda)$ (see 29.1.6(a)) is an isomorphism.*

Actually, we get (b) for $\lambda \in X^+ - P$; but for any $\lambda \in X^+$ we can find P as above, not containing λ.

29.2.3. From the definition, we see that the finite dimensional semisimple algebra $\dot{\mathbf{U}}/\dot{\mathbf{U}}[P]$ inherits from $\dot{\mathbf{U}}$ a canonical basis, formed by the non-zero elements in the image of $\dot{\mathbf{B}}$.

29.3. The Refined Peter-Weyl Theorem

Lemma 29.3.1. *Let $\lambda \in X^+$.*

(a) *The anti-automorphisms $S, S', \sigma : \dot{\mathbf{U}} \to \dot{\mathbf{U}}$ carry $\dot{\mathbf{U}}[\geq \lambda]$ onto $\dot{\mathbf{U}}[\geq -w_0(\lambda)]$.*

(b) *The automorphism $\omega : \dot{\mathbf{U}} \to \dot{\mathbf{U}}$ carries $\dot{\mathbf{U}}[\geq \lambda]$ onto $\dot{\mathbf{U}}[\geq -w_0(\lambda)]$.*

Let $u \in \dot{\mathbf{U}}[\geq \lambda]$. Assume that $\lambda' \in X^+$ and $m \in \Lambda_{\lambda'}$ are such that $S(u)m \neq 0$. The map $\delta_{\lambda'} : {}^\omega\Lambda_{\lambda'} \otimes \Lambda_{\lambda'} \to \mathbf{Q}(v)$ (see 25.1.4) may be considered as a non-degenerate pairing; hence there exists $m' \in {}^\omega\Lambda_{\lambda'}$ such that $\delta_{\lambda'}(m' \otimes S(u)m) \neq 0$. Using 28.2.1, we see that $\delta_{\lambda'}(m' \otimes S(u)m) = \delta_{\lambda'}(um' \otimes m)$; hence $um' \neq 0$. Since ${}^\omega\Lambda_{\lambda'} \cong \Lambda_{-w_0(\lambda')}$, we see using 29.1.3, that $-w_0(\lambda') \geq \lambda$, or equivalently, that $\lambda' \geq -w_0(\lambda)$. Using again 29.1.3, we deduce that $S(u) \in \dot{\mathbf{U}}[\geq -w_0(\lambda)]$. An entirely similar proof shows that $S'(u) \in \dot{\mathbf{U}}[\geq -w_0(\lambda)]$. Thus $S(\dot{\mathbf{U}}[\geq \lambda]) \subset \dot{\mathbf{U}}[\geq -w_0(\lambda)]$ for all λ and $S'(\dot{\mathbf{U}}[\geq -w_0(\lambda)]) \subset \dot{\mathbf{U}}[\geq \lambda]$ for all λ. Since $SS' = S'S = 1$, the assertions about S and S' in (a) are proved. The assertion about σ follows from the assertion for S, using 23.1.7 and the fact that $\dot{\mathbf{U}}[\geq \lambda]$ is generated by elements in $\dot{\mathbf{B}}$ which are contained in the summands of the decomposition 23.1.2 of $\dot{\mathbf{U}}$.

We prove (b). Let $u \in \dot{\mathbf{U}}[\geq \lambda]$. Assume that $\lambda' \in X^+$ and $m \in \Lambda_{\lambda'}$ are such that $\omega(u)m \neq 0$. Then $um \neq 0$ in ${}^\omega\Lambda_{\lambda'}$ which is isomorphic to $\Lambda_{-w_0(\lambda')}$; hence, by 29.1.3, we have $-w_0(\lambda') \geq \lambda$ or equivalently, $\lambda' \geq -w_0(\lambda)$. Using again 29.1.3, we deduce that $\omega(u) \in \dot{\mathbf{U}}[\geq -w_0\lambda]$. Thus, $\omega(\dot{\mathbf{U}}[\geq \lambda]) \subset \dot{\mathbf{U}}[\geq -w_0\lambda]$. Similarly, $\omega(\dot{\mathbf{U}}[\geq -w_0(\lambda)]) \subset \dot{\mathbf{U}}[\geq \lambda]$. The lemma follows.

29.3.2. We shall use the following terminology: an element $\beta \in \dot{\mathbf{B}}$ is said to be *involutive* if $\sigma\omega(\beta) = \pm\beta$. (Recall that $\sigma\omega = \omega\sigma$ maps $\dot{\mathbf{B}}$ to $\pm\dot{\mathbf{B}}$.)

The following theorem is, in part, a summary of the results above.

Theorem 29.3.3. *Given $\lambda \in X^+$, we define $\dot{\mathbf{U}}[\geq \lambda]$ (resp. $\dot{\mathbf{U}}[> \lambda]$) as the set of all $u \in \dot{\mathbf{U}}$ with the following property: if $\lambda' \in X^+$ and u acts on $\Lambda_{\lambda'}$ by a non-zero linear map, then $\lambda' \geq \lambda$ (resp. $\lambda' > \lambda$).*

(a) *$\dot{\mathbf{U}}[\geq \lambda]$ and $\dot{\mathbf{U}}[> \lambda]$ are two-sided ideals of $\dot{\mathbf{U}}$, which are generated as vector spaces by their intersections with $\dot{\mathbf{B}}$. The quotient algebra*

$\dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda]$ is isomorphic (via the action of $\dot{\mathbf{U}}$ on Λ_λ) to the algebra End(Λ_λ); in particular, it is finite dimensional and has a unit element, denoted by $\mathbf{1}_\lambda$. Let $\pi : \dot{\mathbf{U}}[\geq \lambda] \to \dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda]$ be the natural projection.

(b) *There is a unique direct sum decomposition of $\dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda]$ into a direct sum of simple left $\dot{\mathbf{U}}$-modules such that each summand is generated by its intersection with the basis $\pi(\dot{\mathbf{B}}[\lambda])$ of $\dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda]$.*

(c) *There is a unique direct sum decomposition of $\dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda]$ into a direct sum of simple right $\dot{\mathbf{U}}$-modules such that each summand is generated by its intersection with the basis $\pi(\dot{\mathbf{B}}[\lambda])$ of $\dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda]$.*

(d) *Any summand in the decomposition (b) and any summand in the decomposition (c) have an intersection equal to a line consisting of all multiples of some element in the basis $\pi(\dot{\mathbf{B}}[\lambda])$. This gives a map from the set of all pairs consisting of a summand in the decomposition (b) and one in the decomposition (c), to the set $\pi(\dot{\mathbf{B}}[\lambda])$. This map is a bijection.*

(e) *Each summand in the decomposition (b) and each summand in the decomposition (c) contains a unique element of the form $\pi(\beta)$ where $\beta \in \dot{\mathbf{B}}[\lambda]$ is involutive.*

(f) *Let $b, b' \in \dot{\mathbf{B}}[\lambda]$. There exists $b'' \in \dot{\mathbf{B}}[\lambda]$ and $c_{b,b',b''} \in \mathcal{A}$ such that $bb' = c_{b,b',b''} b'' \mod \dot{\mathbf{U}}[> \lambda]$.*

(a) has already been proved. (b) follows from the definitions, using 27.1.7, 27.1.8 with $M = ({}^\omega\Lambda_{\lambda'} \otimes \Lambda_{\lambda''})[\geq \lambda]/({}^\omega\Lambda_{\lambda'} \otimes \Lambda_{\lambda''})[> \lambda]$ and with $\lambda_1 = \lambda$ (for various $\lambda', \lambda'' \in X^+$).

(c) follows from (b) using the anti-automorphism $\sigma\omega = \omega\sigma$ of $\dot{\mathbf{U}}$ which maps $\dot{\mathbf{B}}$ into itself, up to signs, (see 26.3.2) and maps $\dot{\mathbf{U}}[\geq \lambda]$ and $\dot{\mathbf{U}}[> \lambda]$ into themselves (see 29.3.1).

We prove (d). The two subspaces considered in the first sentence of (d) are a minimal left ideal and a minimal right ideal in the algebra $\dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda]$ which has 1, and is finite dimensional and simple (by (a)). Their intersection is therefore a line. Since both these subspaces are spanned by a subset of the basis $\pi(\dot{\mathbf{B}}[\lambda])$, the same is true about their intersection, and the first assertion of (d) follows. The map in the second sentence of (d) is obviously surjective. It is a map between two finite sets of the same cardinality $(\dim \Lambda_\lambda)^2$ (see 29.2.2); hence it is a bijection.

We prove (e). Let G be a summand in the decomposition (b). The map $\sigma\omega : \dot{\mathbf{U}} \to \dot{\mathbf{U}}$ induces an involution ι of the vector space $\dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda]$. The image of G under ι is a summand in the decomposition (c), which by (d) intersects G in a line spanned by a vector in $\pi(\dot{\mathbf{B}}[\lambda])$. This line is necessarily stable under ι (since ι is an involution); hence our vector in this line is preserved up to a sign by ι. This proves (e) as far as G is concerned. The same proof applies to summands in the decomposition (c).

We prove (f). The product $\pi(b)\pi(b')$ is in the intersection of the left ideal of $\dot{\mathbf{U}}[\geq \lambda]/\dot{\mathbf{U}}[> \lambda]$ generated by $\pi(b')$ with the right ideal generated by $\pi(b)$, hence, by (d), is of the form $c_{b,b',b''}\pi(b'')$ for some $b'' \in \dot{\mathbf{B}}[\lambda]$ and some $c_{b,b',b''} \in \mathbf{Q}(v)$, which is necessarily in $\mathcal{A}$ since the structure constants of the algebra $\dot{\mathbf{U}}$ with respect to $\dot{\mathbf{B}}$ are in $\mathcal{A}$.

The theorem is proved.

29.4. Cells

29.4.1. The subsets $\dot{\mathbf{B}}[\lambda]$ (for various $\lambda \in X^+$) are called *two-sided cells*; they form a partition of $\dot{\mathbf{B}}$. For each λ, the two-sided cell $\dot{\mathbf{B}}[\lambda]$ is further partitioned into subsets corresponding to the bases of the various summands in the decomposition 29.3.3(b) (these are called *left cells*) and it is also partitioned into subsets corresponding to the bases of the various summands in the decomposition 29.3.3(c) (these are called *right cells*). Then 29.3.3(d) asserts that any left cell in $\dot{\mathbf{B}}[\lambda]$ and any right cell in $\dot{\mathbf{B}}[\lambda]$ have exactly one element in common; 29.3.3(e) asserts that any left cell and any right cell contain exactly one involutive element. Since the number of left cells (or right cells) in $\dot{\mathbf{B}}[\lambda]$ is $\dim \Lambda_\lambda$, it follows that the number of involutive elements in $\dot{\mathbf{B}}[\lambda]$ is also $\dim \Lambda_\lambda$.

29.4.2. Let A be an associative algebra over a field K with a given basis B as a K-vector space. We do not assume that A has 1. The structure constants $c_{b,b',b''} \in K$ of A (where $b, b', b'' \in B$) are defined by $bb' = \sum_{b''} c_{b,b',b''} b''$.

Generalizing the definition of cells in Weyl groups (which goes back to A. Joseph), we will define certain preorders on B as follows. If $b, b' \in B$, we say that $b' \leq_L b$ (resp. $b' \leq_R b$) if there is a sequence $b = b_1, b_2, \dots, b_n = b'$ in B and a sequence $\beta_1, \beta_2, \dots, \beta_{n-1}$ in B such that $c_{\beta_s,b_s,b_{s+1}} \neq 0$ (resp. $c_{b_s,\beta_s,b_{s+1}} \neq 0$) for $s = 1, 2, \dots, n-1$. We say that $b' \leq_{LR} b$ if there is a sequence $b = b_1, b_2, \dots, b_n = b'$ in B and a sequence $\beta_1, \beta_2, \dots, \beta_{n-1}$ in B such that for any $s \in [1, n-1]$ we have either $c_{\beta_s,b_s,b_{s+1}} \neq 0$ or $c_{b_s,\beta_s,b_{s+1}} \neq 0$. Then $\leq_L, \leq_R, \leq_{LR}$ are preorders on B. We say that $b \sim_L b'$ if $b \leq_L b'$ and $b' \leq_L b$. This is an equivalence relation on B; the equivalence classes are called *left cells*. Similarly, $\leq_R$ (resp. $\leq_{LR}$) give rise to equivalence relations $\sim_R$ (resp. $\sim_{LR}$); the equivalence classes are called right cells (resp. two-sided cells).

In the case where $A = \dot{\mathbf{U}}$ and $B = \dot{\mathbf{B}}$, the definition of cells just given coincides with that given 29.4.1. (This can be easily checked.) The involutive elements in Theorem 29.3.3 are the analogues of the Duflo involutions from the theory of cells in Weyl groups.

29.4.3. We will describe explicitly the two-sided cells of $\dot{\mathbf{B}}$ in the simplest case where $I = \{i\}$ and $X = Y = \mathbf{Z}$ with $i = 1 \in Y, i' = 2 \in X$. (See 25.3.)

For each $n \geq 0$, we consider the subset

$$\mathfrak{S}(n) = \{E_i^{(a)} 1_{-n} F_i^{(b)}; n \geq a+b\} \cup \{F_i^{(b)} 1_n E_i^{(a)}; n \geq a+b\}$$

of the canonical basis $\dot{\mathbf{B}}$ (with the identification 25.3.1(c)). Note that $\mathfrak{S}(n)$ consists of $(n+1)^2$ elements. The product of two elements of $\mathfrak{S}(n)$ is given by the following equalities (modulo a linear combination of elements in $\mathfrak{S}(n+1) \cup \mathfrak{S}(n+2) \cup \cdots$):

$$E_i^{(a)} 1_{-n} F_i^{(b)} E_i^{(c)} 1_{-n} F_i^{(d)} = \begin{cases} \begin{bmatrix} n \\ b \end{bmatrix} E_i^{(a)} 1_{-n} F_i^{(d)} \text{ if } b = c, n \geq a+d \\ \begin{bmatrix} n \\ b \end{bmatrix} F_i^{(n-a)} 1_n E_i^{(n-d)} \text{ if } b = c, n \leq a+d \\ 0 \text{ if } b \neq c \end{cases}$$

$$E_i^{(a)} 1_{-n} F_i^{(b)} F_i^{(c)} 1_n E_i^{(d)} = \begin{cases} \begin{bmatrix} n \\ b \end{bmatrix} E_i^{(a)} 1_{-n} F_i^{(n-d)} \text{ if } b+c = n, d \geq a \\ \begin{bmatrix} n \\ b \end{bmatrix} F_i^{(n-a)} 1_n E_i^{(d)} \text{ if } b+c = n, d \leq a \\ 0 \text{ if } b+c \neq n \end{cases}$$

$$F_i^{(d)} 1_n E_i^{(c)} F_i^{(b)} 1_n E_i^{(a)} = \begin{cases} \begin{bmatrix} n \\ b \end{bmatrix} F_i^{(d)} 1_n E_i^{(a)} \text{ if } b = c, n \geq a+d \\ \begin{bmatrix} n \\ b \end{bmatrix} E_i^{(n-d)} 1_{-n} F_i^{(n-a)} \text{ if } b = c, n \leq a+d \\ 0 \text{ if } b \neq c \end{cases}$$

$$F_i^{(a)} 1_n E_i^{(b)} E_i^{(c)} 1_{-n} F_i^{(d)} = \begin{cases} \begin{bmatrix} n \\ b \end{bmatrix} F_i^{(a)} 1_n E_i^{(n-d)} \text{ if } b+c = n, d \geq a \\ \begin{bmatrix} n \\ b \end{bmatrix} E_i^{(n-a)} 1_{-n} F_i^{(d)} \text{ if } b+c = n, d \leq a \\ 0 \text{ if } b+c \neq n \end{cases}.$$

Hence $\mathfrak{S}(n)$ are the *two-sided cells.* The involutive elements in $\mathfrak{S}(n)$ are $E_i^{(a)} 1_{-n} F_i^{(a)}$ with $a \geq 0, 2a \leq n$ and $F_i^{(a)} 1_n E_i^{(a)}$ with $a \geq 0, 2a \leq n$, with the identification $E_i^{(a)} 1_{-n} F_i^{(a)} = F_i^{(a)} 1_n E_i^{(a)}$ if $2a = n$.

29.5. THE QUANTUM COORDINATE ALGEBRA

29.5.1. Let $\mathbf{O}$ be the vector space of all $\mathbf{Q}(v)$-linear forms $f : \dot{\mathbf{U}} \to \mathbf{Q}(v)$ with the following property: f vanishes on $\dot{\mathbf{U}}[\geq \lambda]$ for some $\lambda \in X^+$. If $a \in \dot{\mathbf{B}}$, then the linear form $\tilde{a} : \dot{\mathbf{U}} \to \mathbf{Q}(v)$ given by $\tilde{a}(a') = \delta_{a,a'}$, for all $a \in \dot{\mathbf{B}}$, belongs to $\mathbf{O}$ and $\{\tilde{a} | a \in \dot{\mathbf{B}}\}$ is a basis of $\mathbf{O}$. This follows from 29.3.3. We define an algebra structure on $\mathbf{O}$ by the rule $\tilde{a}\tilde{b} = \sum_c \hat{m}_c^{ab} \tilde{c}$, where c runs over $\dot{\mathbf{B}}$ and $\hat{m}_c^{ab}$ are as in 25.4.1. The previous sum is well-defined: all but finitely many terms are zero. This product is associative by 25.4.1(b).

The linear map $\Delta : \mathbf{O} \to \mathbf{O} \otimes \mathbf{O}$ given by $\Delta(\tilde{c}) = \sum_{a,b} m_{ab}^c \tilde{a} \otimes \tilde{b}$ (with m_{ab}^c as in 25.4.1) is well-defined. All but finitely many terms in the sum are zero. This map is called comultiplication. It is coassociative by 25.4.1(a) and it is an algebra homomorphism by 25.4.1(c). The element $\tilde{1}_0$ is a unit element for this algebra. Consider the linear function $\mathbf{O} \to \mathbf{Q}(v)$ which takes $\tilde{a}$ to 1 if $a = 1_\lambda$ for some $\lambda \in X$, and otherwise, to zero. This is an algebra homomorphism. Thus $\mathbf{O}$ becomes a Hopf algebra called *the quantum coordinate algebra*. It is easy to see that this definition is the same as the usual one.

We call $\{\tilde{a} | a \in \dot{\mathbf{B}}\}$ the *canonical basis* of $\mathbf{O}$.

29.5.2. Let ${}_{\mathcal{A}}\mathbf{O}$ be the $\mathcal{A}$-submodule of $\mathbf{O}$ generated by the basis $(\tilde{a})$. Since the structure constants $m_{ab}^c, \hat{m}_c^{ab}$ are in $\mathcal{A}$, it follows that ${}_{\mathcal{A}}\mathbf{O}$ inherits from $\mathbf{O}$ a structure of Hopf algebra over $\mathcal{A}$. If now R is any commutative $\mathcal{A}$-algebra with 1, we can define a Hopf algebra over R by ${}_R\mathbf{O} = R \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathbf{O})$.