CHAPTER 28

Bases for Coinvariants and Cyclic Permutations

28.1. MONOMIALS

28.1.1. In this chapter we assume that $(I, \cdot)$ is of finite type. Let $\lambda \in X^+$. For any sequence $\mathbf{i} = (i_1, i_2, \ldots, i_N)$ in I such that $s_{i_1} s_{i_2} \cdots s_{i_N}$ is a reduced expression of an element $w \in W$, we consider the element $\theta(\mathbf{i}, \lambda) = \theta_{i_1}^{(a_1)} \theta_{i_2}^{(a_2)} \cdots \theta_{i_N}^{(a_N)} \in \mathbf{f}$ where

$$a_1 = \langle s_{i_N} \cdots s_{i_2}(i_1), \lambda \rangle, \ldots, a_{N-1} = \langle s_{i_N}(i_{N-1}), \lambda \rangle, a_N = \langle i_N, \lambda \rangle;$$

note that $a_1, a_2, \ldots, a_N \in \mathbf{N}$, by 2.2.7.

Proposition 28.1.2. *The element $\theta(\mathbf{i}, \lambda)$ depends only on w and not on $\mathbf{i}$.*

Assume first that w is the longest element in the subgroup of W generated by two distinct elements i, j of I. In that case the assertion of the lemma is the quantum analogue of an identity of Verma, whose proof will be given in 39.3. We shall assume that this special case is known.

We now consider the general case. Let $\mathbf{i}' = (j_1, j_2, \ldots, j_N)$ be another sequence like $\mathbf{i}$ (for the same w). To prove that $\theta(\mathbf{i}, \lambda) = \theta(\mathbf{i}', \lambda)$, we may assume, by 2.1.2, that $\mathbf{i}'$ is obtained from $\mathbf{i}$ by replacing a subsequence $i, j, i, j, \ldots$ (m consecutive terms) of $\mathbf{i}$ by $j, i, j, i, \ldots$, (m consecutive terms), where i, j are as above and m is the order of $s_i s_j$. But this follows immediately from the special case considered above.

28.1.3. By Proposition 28.1.2, we may use the notation $\theta(w, \lambda)$ instead of $\theta(\mathbf{i}, \lambda)$ for $w, \mathbf{i}$ as above.

Proposition 28.1.4. *The element $\theta(w, \lambda)^- \eta_\lambda \in \Lambda_\lambda$ is the unique element of the canonical basis of Λ_λ which lies in the $w(\lambda)$-weight space.*

We prove this by induction on N, the length of w. If $N = 0$, there is nothing to prove. Assume now that $N \geq 1$. Let $(i_1, i_2, \ldots, i_N)$ and $(a_1, a_2, \ldots, a_N)$ be as in 28.1.1. Thus $w = s_{i_1} s_{i_2} \cdots s_{i_N}$. Let $w' =$

$s_{i_2}s_{i_3}\cdots s_{i_N}$ so that $w = s_{i_1}w'$. Let b' (resp. b) be the unique element in the canonical basis of Λ_λ which lies in the $w'(\lambda)$-weight space (resp. the $w(\lambda)$-weight space). Using the induction hypothesis, we see that it is enough to prove that $b = F_{i_1}^{(a_1)}b'$. We have $w(\lambda) = s_{i_1}w'(\lambda) = w'(\lambda) - \langle i_1, w'(\lambda)\rangle i_1' = w'(\lambda) - a_1 i_1'$ so that $F_{i_1}^{(a_1)}b'$ is a non-zero vector in the same weight space as b; thus, $F_{i_1}^{(a_1)}b' = fb$ for some $f \in \mathcal{A} - \{0\}$.

Next we note that $E_{i_1}b' = 0$, since the $(w'(\lambda) + i_1')$-weight space is zero. Otherwise, the $w'^{-1}(w'(\lambda) + i_1')$-weight space would be non-zero, hence the $(\lambda + w'^{-1}(i_1'))$-weight space would be non-zero, contradicting the fact that λ is the highest weight, since $w'^{-1}(i_1') > 0$. From the definition of $\tilde{F}_{i_1}$, it then follows that $F_{i_1}^{(a_1)}b' = \tilde{F}_{i_1}^{a_1}b'$. By the properties of the basis at ∞ of Λ_λ, the previous equality implies that $f = c \mod v^{-1}\mathbf{A}$ where c is 0 or 1. Hence we have $f = c \mod v^{-1}\mathbf{Z}[v^{-1}]$ where c is as above and $f \neq 0$.

The involution $^{-} : \Lambda_\lambda \to \Lambda_\lambda$ keeps b, b' fixed and we have $\overline{F_{i_1}^{(a_1)})b'} = \overline{F_{i_1}^{(a_1)}}\bar{b}' = F_{i_1}^{(a_1)}b'$; hence $fb = \overline{fb} = \bar{f}b$. It follows that $\bar{f} = f$; hence $f = c$. Since $f \neq 0$ and c is 0 or 1, it follows that $f = 1$. The proposition is proved.

Proposition 28.1.5. *We have $\sigma(\theta(w_0, \lambda)) = \theta(w_0, -w_0(\lambda))$.*

Let $\mathbf{i} = (i_1, i_2, \ldots, i_N)$ be a sequence in W such that $s_{i_1}s_{i_2}\cdots s_{i_N}$ is a reduced expression of w_0. Define $a_1, a_2, \ldots, a_N$ as in 28.1.1. Then $\mathbf{i}' = (i_N, i_{N-1}, \ldots, i_1)$ is such that $s_{i_N}s_{i_{N-1}}\cdots s_{i_1}$ is a reduced expression of w_0. Let $b_1, b_2, \ldots, b_N$ be defined by

$$b_1 = \langle s_{i_1}\cdots s_{i_{N-1}}(i_N), -w_0(\lambda)\rangle, \cdots, b_{N-1} = \langle s_{i_1}(i_2), -w_0(\lambda)\rangle,$$
$$b_N = \langle i_1, -w_0(\lambda)\rangle;$$

then $b_1 = a_N, b_2 = a_{N-1}, \ldots, b_N = a_1$. Using 28.1.2, we have

$$\begin{aligned}\sigma(\theta(w_0, \lambda)) &= \sigma(\theta(\mathbf{i}, \lambda)) = \sigma(\theta_{i_1}^{(a_1)}\theta_{i_2}^{(a_2)}\cdots\theta_{i_N}^{(a_N)})\\ &= \theta_{i_N}^{(a_N)}\theta_{i_{N-1}}^{(a_{N-1})}\cdots\theta_{i_1}^{(a_1)}\\ &= \theta_{i_N}^{(b_1)}\theta_{i_{N-1}}^{(b_2)}\cdots\theta_{i_1}^{(b_N)} = \theta(\mathbf{i}', -w_0(\lambda)) = \theta(w_0, -w_0(\lambda)).\end{aligned}$$

The proposition is proved.

28.1.6. We have

(a) $$S'(\theta(w_0, \lambda)^-) = (-1)^{\langle 2\rho, \lambda\rangle}v^{-\mathbf{n}(\lambda)+c_1}\tilde{K}_\nu\sigma(\theta(w_0, \lambda))^-$$

where 2ρ and $\mathbf{n}$ are as in 2.3.1,

$$\nu = \sum_i \langle \mu(i), \lambda \rangle i$$

with $\mu(i) \in Y$ as in 2.3.2, and

$$c_1 = \sum_{i \in I} \langle \mu(i), \lambda \rangle \langle i, \lambda \rangle i \cdot i/2.$$

This follows from 2.3.2(a) and 3.3.1(d), applied to $x = \theta(w_0, \lambda)$. Note that $x \in \mathbf{f}_\nu$, where ν is as above and $\operatorname{tr} \nu = \sum_{p=1}^{N} \langle s_{i_N} \cdots s_{i_{p+1}}(i_p), \lambda \rangle = \langle 2\rho, \lambda \rangle$.

28.2. The Isomorphism P

28.2.1. Coinvariants and antipode. Let M, M' be two objects of $\mathcal{C}$ and let $u \in \mathbf{U}$. For $x \in M, x' \in M'$, we have

(a) $ux \otimes x' = x \otimes S(u)x'$ in the coinvariants $(M \otimes M')_*$.

We may assume that $x \in M^\lambda, x' \in M'^{\lambda'}$. First note that $x \otimes x' = 0$ (in the coinvariants) unless $\lambda + \lambda' = 0$.

We show (a) for $u = E_i$. Both sides are zero unless $\lambda + \lambda' + i' = 0$, when we have

$$E_i x \otimes x' = -v_i^{\langle i, -\lambda' - i' \rangle} x \otimes E_i x' = -x \otimes \tilde{K}_{-i} E_i x' = x \otimes S(E_i)x'$$

(in the coinvariants).

We show (a) for $u = F_i$. Both sides are zero unless $\lambda + \lambda' - i' = 0$, when we have

$$F_i x \otimes x' = -v_i^{\langle i, \lambda' \rangle} x \otimes F_i x' = -x \otimes F_i \tilde{K}_i x' = x \otimes S(F_i)x'$$

(in the coinvariants).

We show (a) for $u = K_\mu$. We have

$$K_\mu x \otimes x' = v^{\langle \mu, \lambda \rangle} x \otimes x', x \otimes S(K_\mu)x' = v^{-\langle \mu, \lambda' \rangle} x \otimes x'.$$

But we can assume that $v^{\langle \mu, \lambda \rangle} = v^{-\langle \mu, \lambda' \rangle}$.

Now if (a) holds for u, u', then it also holds for linear combinations of u, u' and for uu'. Indeed $uu'x \otimes x' = u'x \otimes S(u)x' = x \otimes S(u')S(u)x' = x \otimes S(uu')x'$ (in the coinvariants). Thus, (a) is proved.

An equivalent form of (a) is:

(b) $S'(u)x \otimes x' = x \otimes ux'$ in the coinvariants $(M \otimes M')_*$.

28.2.2. Let $\Lambda = \Lambda_\lambda$ where $\lambda \in X^+$, and let (M, B) be a based module. Let $\eta = \eta_\lambda$ and let ξ be the unique element in the canonical basis of Λ in the $w_0(\lambda)$-weight space. We define an isomorphism of vector spaces $P : \Lambda \otimes M \to M \otimes \Lambda$ by

$$P(x \otimes y) = (-1)^{\langle 2\rho, \zeta \rangle} v^{-\mathbf{n}(\zeta)} y \otimes x$$

for $x \in \Lambda^\zeta$ and $y \in M$. Here $2\rho \in Y, \mathbf{n} : X \to \mathbf{Z}$ are as in 2.3.1.

We show that P maps $E_i(\Lambda \otimes M)^{-i'}$ into $E_i(M \otimes \Lambda)$; $F_i(\Lambda \otimes M)^{i'}$ into $F_i(M \otimes \Lambda)$; and $(\Lambda \otimes M)^\zeta$ into $(M \otimes \Lambda)^\zeta$ for any $\zeta \in X$. Indeed, if $x \in \Lambda^\zeta, y \in M^{\zeta'}$, and $\zeta + \zeta' + i' = 0$, then

$$\begin{aligned} P(E_i(x \otimes y)) &= P(E_i x \otimes y + v^{\langle i, \zeta \rangle} x \otimes E_i y) \\ &= (-1)^{\langle 2\rho, \zeta \rangle} v^{-\mathbf{n}(\zeta + i')} y \otimes E_i x + (-1)^{\langle 2\rho, \zeta} v^{i \cdot i \langle i, \zeta \rangle / 2 - \mathbf{n}(\zeta)} E_i y \otimes x \\ &= (-1)^{\langle 2\rho, \zeta \rangle} v^{i \cdot i \langle i, \zeta \rangle / 2 - \mathbf{n}(\zeta)} (E_i y \otimes x + v^{i \cdot i \langle i, \zeta' \rangle / 2} y \otimes E_i x) \\ &= (-1)^{\langle 2\rho, \zeta \rangle} v^{i \cdot i \langle i, \zeta \rangle / 2 - \mathbf{n}(\zeta)} E_i (y \otimes x). \end{aligned}$$

If $x \in \Lambda^\zeta, y \in M^{\zeta'}$, and $\zeta + \zeta' - i' = 0$, then

$$\begin{aligned} P(F_i(x \otimes y)) &= P(x \otimes F_i y + v^{-i \cdot i \langle i, \zeta' \rangle / 2} F_i x \otimes y) \\ &= (-1)^{\langle 2\rho, \zeta \rangle} v^{-\mathbf{n}(\zeta)} F_i y \otimes x + (-1)^{2\rho, \zeta} v^{-i \cdot i \langle i, \zeta' \rangle / 2 - \mathbf{n}(\zeta - i')} y \otimes F_i x \\ &= (-1)^{\langle 2\rho, \zeta \rangle} v^{-i \cdot i \langle i, \zeta' \rangle / 2 - \mathbf{n}(\zeta - i')} (y \otimes F_i x + v^{-i \cdot i \langle i, \zeta \rangle / 2} F_i y \otimes x) \\ &= (-1)^{\langle 2\rho, \zeta \rangle} v^{-i \cdot i \langle i, \zeta' \rangle / 2 - \mathbf{n}(\zeta - i')} F_i (y \otimes x). \end{aligned}$$

It follows that P induces an isomorphism of vector spaces $P : (\Lambda \otimes M)_* \to (M \otimes \Lambda)_*$.

28.2.3. Let $^- : M \to M$ be the associated involution of the based module (M, B). Recall that on Λ we also have an involution $^-$ associated with its natural structure of based module. Then $\Lambda \otimes M$ and $M \otimes \Lambda$ are naturally based modules with associated involution Θ^- (see 27.3.3) and the spaces of coinvariants $(\Lambda \otimes M)_*$ and $(M \otimes \Lambda)_*$ inherit from them structures of based modules (see 27.2.5) with trivial action of $\mathbf{U}$.

Proposition 28.2.4. *$P : (\Lambda \otimes M)_* \cong (M \otimes \Lambda)_*$ is an isomorphism of based modules.*

The proof will be given in 28.2.8. It will be based on a number of lemmas.

Lemma 28.2.5. *For any $y \in M^{-w_0(\lambda)}$, we have*

$$P(\xi \otimes y) = \theta(w_0, -w_0(\lambda))^- y \otimes \eta$$

(equality in $(M \otimes \Lambda)_*$).

Using 28.1.4, 28.2.1(b), 28.1.6(a), 28.1.5, we have

$$\begin{aligned} P(\xi \otimes y) &= (-1)^{\langle 2\rho, \lambda \rangle} v^{-\mathbf{n}(w_0(\lambda))} y \otimes \xi = (-1)^{\langle 2\rho, \lambda \rangle} v^{-\mathbf{n}(w_0(\lambda))} y \otimes \theta(w_0, \lambda)^- \eta \\ &= (-1)^{\langle 2\rho, \lambda \rangle} v^{-\mathbf{n}(w_0(\lambda)} S'(\theta(w_0, \lambda)^-) y \otimes \eta \\ &= v^{-\mathbf{n}(w_0(\lambda))} v^{-\mathbf{n}(\lambda)+c_1} \tilde{K}_\nu \theta(w_0, -w_0(\lambda))^- y \otimes \eta \\ &= v^{-\mathbf{n}(w_0(\lambda))-\mathbf{n}(\lambda)+c_1+c_2} \theta(w_0, -w_0(\lambda))^- y \otimes \eta \end{aligned}$$

(equalities in coinvariants) where $c_2 = -\sum_i \nu_i i \cdot i\langle i, \lambda \rangle / 2 = -c_1$. Note also that $-\mathbf{n}(w_0(\lambda) - \mathbf{n}(\lambda) = 0$ by 2.3.1(b). The lemma is proved.

Lemma 28.2.6. *Let $b \in B$. Then*

$$\xi \otimes b = \xi \diamond b \in \Lambda \otimes M$$

and

$$b \otimes \eta = b \Diamond \eta \in M \otimes \Lambda.$$

From the definitions we see that $\xi \otimes b$ (resp. $b \otimes \eta$) is fixed by the involution Θ^- of $\Lambda \otimes M$ (resp. $M \otimes \Lambda$). Hence the result follows from the definition of $\xi \Diamond b$ and $b \Diamond \eta$.

In the following result, $B[\lambda]^{hi}, B[\lambda]^{lo}$ are defined in terms of (M, B) as in 27.2.3.

Lemma 28.2.7. *There is a unique bijection $B[\lambda]^{hi} \leftrightarrow B[\lambda]^{lo}$ such that the following two conditions for $b \in B[\lambda]^{hi}, b' \in B[\lambda]^{lo}$ are equivalent: $b \leftrightarrow b'$; $\theta(w_0, \lambda)^- b - b' \in M[> \lambda]$.*

Replacing M by $M[\geq \lambda]$, we are reduced to the case where $M = M[\geq \lambda]$ (see 27.2.4(a)). Then replacing M by $M/M[> \lambda]$, we are reduced to the case where $M = M[\lambda]$ (see 27.2.4(b)). Using 27.1.7, we are reduced to the case where (M, B) is Λ with its canonical basis. In this case, we have $B[\lambda]^{hi} = \{\eta\}$ and $B[\lambda]^{lo} = \{\xi\}$ and the result follows from 28.1.4.

28.2.8. Proof of Proposition 28.2.4. Let $B'_\diamondsuit$ (resp. $B''_\diamondsuit$) be the basis of $\Lambda \otimes M$ (resp. $M \otimes \Lambda$)) defined as in 27.3.3, in terms of the based modules (Λ, B_1) and (M, B). Here B_1 is the canonical basis of Λ. Let $\pi : \Lambda \otimes M \to (\Lambda \otimes M)_*$ and $\pi' : M \otimes \Lambda \to (M \otimes \Lambda)_*$ be the canonical maps. We know that π defines a bijection of $B'_\diamondsuit[0]$ onto a basis $(B'_\diamondsuit)_*$ of $(\Lambda \otimes M)_*$ and π' defines a bijection of $B''_\diamondsuit[0]$ onto a basis $(B''_\diamondsuit)_*$ of $(M \otimes \Lambda)_*$.

Let $b \in (B'_\diamondsuit)_*$. Let $\tilde{b}$ be the unique element of $B'_\diamondsuit[0]$ such that $\pi(\tilde{b}) = b$. By 27.3.8, there exists $\lambda' \in X^+$ and elements $b_1 \in B_1[-w_0(\lambda')]^{lo}, b_2 \in B[\lambda']^{hi}$ such that $\tilde{b} = b_1 \diamondsuit b_2$. In Λ, we have that $B_1[-w_0(\lambda')]$ is empty unless $-w_0(\lambda') = \lambda$ and $B_1[\lambda]^{lo} = \{\xi\}$. Thus we have $\tilde{b} = \xi \diamondsuit b_2$ where $b_2 \in B[-w_0(\lambda)]^{hi}$ and by Lemma 28.2.6, we have $\tilde{b} = \xi \otimes b_2$. By Lemma 28.2.5, we then have $P(\tilde{b}) = \theta(w_0, -w_0(\lambda))^- b_2 \otimes \eta$ modulo the kernel of π'.

By Lemma 28.2.7, we can find an element $b'_2 \in B[-w_0(\lambda)]^{lo}$ such that

$$\theta(w_0, -w_0(\lambda))^- b_2 - b'_2 \in M[> -w_0(\lambda)].$$

Then we have $P(\tilde{b}) = b'_2 \otimes \eta$ modulo the kernel of π'. Note that

$$M[> -w_0(\lambda)] \otimes \Lambda$$

is contained in the kernel of π'.

By Lemma 28.2.6, we have $b'_2 \otimes \eta = b'_2 \diamondsuit \eta$. By 27.3.8, we have that $b'_2 \diamondsuit \eta \in B''_\diamondsuit[0]$. It follows that $\pi'(P(\tilde{b}))$ belongs to $\pi(B''_\diamondsuit[0]) = (B''_\diamondsuit)_*$. We have therefore proved that P maps $(B'_\diamondsuit)_*$ into $(B''_\diamondsuit)_*$. The proposition is proved.

28.2.9. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be a sequence of elements of X^+. As in 27.3.9, the space of coinvariants $(\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2} \cdots \otimes \Lambda_{\lambda_n})_*$ has a natural based module structure (hence has a distinguished basis).

This last based module has the following property of invariance by a cyclic permutation: there is a natural isomorphism

$$(\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2} \cdots \otimes \Lambda_{\lambda_n})_* \cong (\Lambda_{\lambda_2} \otimes \Lambda_{\lambda_3} \cdots \otimes \Lambda_{\lambda_n} \otimes \Lambda_{\lambda_1})_*$$

induced by the map

$$x_1 \otimes x_2 \cdots \otimes x_n \mapsto (-1)^{\langle 2\rho, \zeta_1 \rangle} v^{-\mathrm{n}(\zeta_1)} x_2 \otimes x_3 \cdots \otimes x_n \otimes x_1$$

where $x_p \in \Lambda_{\lambda_p}^{\zeta_p}$. This isomorphism maps the distinguished basis onto the distinguished basis (see 28.2.4). If we compose the n iterates of this isomorphism, we get the identity map of $(\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2} \cdots \otimes \Lambda_{\lambda_n})_*$, since we may assume that $\zeta_1 + \zeta_2 + \cdots + \zeta_n = 0$.