CHAPTER 27

Based Modules

27.1. ISOTYPICAL COMPONENTS

27.1.1. In this chapter we assume that $(I, \cdot)$ is of finite type.

Let $M \in \mathcal{C}$. We assume that M is finite dimensional over $\mathbf{Q}(v)$. For any $\lambda \in X^+$, we denote by $M[\lambda]$ the sum of simple subobjects of M that are isomorphic to Λ_λ. Then $M = \oplus_\lambda M[\lambda]$. We also define for any $\lambda \in X^+$:

$$M[\geq \lambda] = \oplus_{\lambda' \in X^+; \lambda' \geq \lambda} M[\lambda']$$

and

$$M[> \lambda] = \oplus_{\lambda' \in X^+; \lambda' > \lambda} M[\lambda'].$$

Clearly, $M[> \lambda]$ is a subobject of $M[\geq \lambda]$ and $M[\lambda] \oplus M[> \lambda] = M[\geq \lambda]$ as objects in $\mathcal{C}$.

27.1.2. A *based module* is an object M of $\mathcal{C}$, of finite dimension over $\mathbf{Q}(v)$ with a given $\mathbf{Q}(v)$-basis B such that

(a) $B \cap M^\zeta$ is a basis of M^ζ, for any $\zeta \in X$;

(b) the $\mathcal{A}$-submodule ${}_\mathcal{A}M$ generated by B is stable under ${}_\mathcal{A}\dot{\mathbf{U}}$;

(c) the $\mathbf{Q}$-linear involution $^- : M \to M$ defined by $\overline{fb} = \bar{f}b$ for all $f \in \mathbf{Q}(v)$ and all $b \in B$ is compatible with the $\mathbf{U}$-module structure in the sense that $\overline{um} = \bar{u}\bar{m}$ for all $u \in \mathbf{U}, m \in M$;

(d) the $\mathbf{A}$-submodule $L(M)$ generated by B, together with the image of B in $L(M)/v^{-1}L(M)$, forms a basis at ∞ for M (see 20.1.1).

We say that $^- : M \to M$ in (c) is the *associated involution* of (M, B). The direct sum of two based modules (M, B) and (M', B') is again a based module $(M \oplus M', B \sqcup B')$.

27.1.3. The based modules form the objects of a category $\tilde{\mathcal{C}}$; a morphism from the based module (M, B) to the based module (M', B') is by definition a morphism $f : M \to M'$ in $\mathcal{C}$ such that

(a) for any $b \in B$ we have $f(b) \in B' \cup \{0\}$ and

(b) $B \cap \ker f$ is a basis of $\ker f$.

27.1.4. Let (M, B) be a based module and let M' be a $\dot{\mathbf{U}}$-submodule of M such that M' is spanned as a $\mathbf{Q}(v)$-subspace of M by a subset B' of B. Then (M', B') is a based module; moreover, M/M' together with the image of $B - B'$ is a based module.

For any $\lambda \in X^+$, Λ_λ together with its canonical basis, is a based module. (See 19.3.4, 23.3.7, 20.1.4.)

27.1.5. Let (M, B) be a based module with associated involution $^-$ and let $m \in M$ be an element such that $\bar{m} = m, m \in {}_{\mathcal{A}}M$ and $m \in B + v^{-1}L(M)$ (resp. $m \in v^{-1}L(M)$). Then we have $m \in B$ (resp. $m = 0$). Indeed, we can write $m = \sum_{b \in B} c_b b$ with $c_b \in \mathcal{A}$. By our assumption, we have $c_b \in \mathbf{A}$ for all b. Hence $c_b \in \mathbf{Z}[v^{-1}]$ for all b. We have $\bar{c}_b = c_b$ for all b. Hence $c_b \in \mathbf{Z}$ for all b. Moreover, by our assumption, we have $c_b \in v^{-1}\mathbf{A}$ for all b, except possibly for a single b for which we have $c_b = 0$ or $1 \bmod v^{-1}\mathbf{A}$. It follows that $c_b = 0$ for all b, except possibly for a single b for which we have $c_b = 0$ or 1. Our assertion follows.

27.1.6. Let (M, B) be a based module. Assume that $M \neq 0$. Let $\lambda_1 \in X^+$ be such that $M^{\lambda_1} \neq 0$ and such that λ_1 is maximal with this property. Let $B_1 = B \cap M^{\lambda_1}$. It is a non-empty set. Let $M' = \oplus_{b \in B_1} \Lambda_{\lambda_1, b} \in \mathcal{C}$. Here $\Lambda_{\lambda_1, b}$ is a copy of Λ_{λ_1} corresponding to b; we denote its canonical generator η_{λ_1} by η_b.

For any $b \in B_1$, we have $E_i b = 0$ for all $i \in I$ by the maximality of λ_1. Hence there is a unique homomorphism $\phi : M' \to M$ of objects in $\mathcal{C}$ whose restriction to any summand $\Lambda_{\lambda_1, b}$ carries η_b to b. Let B' be the basis of M' given by the union of the canonical bases of the various summands $\Lambda_{\lambda_1, b}$.

Proposition 27.1.7. *In the setup above, $B \cap M[\lambda_1]$ is a basis of $M[\lambda_1]$ and ϕ defines an isomorphism $M' \cong M[\lambda_1]$ carrying B' onto $B \cap M[\lambda_1]$. Thus ϕ is an isomorphism of based modules $(M', B') \cong (M[\lambda_1], B \cap M[\lambda_1])$.*

Let $^- : M' \to M'$ be the $\mathbf{Q}$-linear involution whose restriction to each summand $\Lambda_{\lambda_1, b}$ is the canonical involution $^- : \Lambda_{\lambda_1, b} \to \Lambda_{\lambda_1, b}$. The involution $^- : M' \to M'$ is compatible under ϕ with that of M. Indeed, both involutions are the identity on B_1. (We regard B_1 as a subset of M' by $b \mapsto \eta_b$.)

Let $b' \in B' \cap \Lambda_{\lambda_1, b}$. We have $\bar{b}' = b'$; hence $\overline{\phi(b')} = \phi(\bar{b}') = \phi(b')$. Thus $\phi(b')$ is fixed by $^- : M \to M$.

We know from 19.3.5 that there exists a sequence $i_1, i_2, \ldots, i_p$ in I such that b' is equal to $\tilde{F}_{i_1} \tilde{F}_{i_2} \cdots \tilde{F}_{i_p} \eta_b$ plus a $v^{-1}\mathbf{A}$-linear combination of elements of the same kind. Now the action of $\tilde{F}_i$ on M' is compatible with the action of $\tilde{F}_i$ on M. Hence $\phi(b')$ is equal to $\tilde{F}_{i_1}\tilde{F}_{i_2}\dots\tilde{F}_{i_p}b$ plus a linear combination with coefficients in $v^{-1}\mathbf{A}$ of elements of the same kind. By property 27.1.2(d) of B, we see that either $\phi(b') \in B + v^{-1}L(M)$ or $\phi(b') \in v^{-1}L(M)$.

On the other hand, by the definition of the canonical basis of M', we have that b' belongs to the ${}_{\mathcal{A}}\dot{\mathbf{U}}$-submodule of M' generated by η_b; hence $\phi(b')$ belongs to the ${}_{\mathcal{A}}\dot{\mathbf{U}}$-submodule of M generated by b; by the property 27.1.2(b), we then have $\phi(b') \in {}_{\mathcal{A}}M$. These properties of $\phi(b')$ imply that $\phi(b') \in B$ or $\phi(b') = 0$ (see 27.1.5). The second alternative does not occur: indeed, the restriction of ϕ to the summand $\Lambda_{\lambda_1,b}$ is injective since $\Lambda_{\lambda_1,b}$ is simple. Thus we have $\phi(b') \in B$. We see that ϕ defines a bijection of the canonical basis of $\Lambda_{\lambda_1,b}$ with a subset $B(b)$ of B.

Next we consider an element $\tilde{b} \in B_1$ distinct from b. We show that $B(\tilde{b})$ is disjoint from $B(b)$. Indeed, assume that $b_1 \in B$ belongs to $B(b) \cap B(\tilde{b})$. Then we have

$$b_1 = \tilde{F}_{i_1}\tilde{F}_{i_2}\cdots\tilde{F}_{i_p}b \mod v^{-1}L(M)$$

and

$$b_1 = \tilde{F}_{j_1}\tilde{F}_{j_2}\cdots\tilde{F}_{j_q}\tilde{b} \mod v^{-1}L(M)$$

for some sequences $i_1, i_2, \dots, i_p$ and $j_1, j_2, \dots, j_q$ in I. By property 27.1.2(d), we then have

$$\tilde{b} = \tilde{E}_{j_q}\tilde{E}_{j_{q-1}}\cdots\tilde{E}_{j_1}\tilde{F}_{i_1}\tilde{F}_{i_2}\cdots\tilde{F}_{i_p}b \mod v^{-1}L(M).$$

Hence $\tilde{b}$ is equal to some element in $B(b)$ plus an element of $v^{-1}L(M)$. It follows that $\tilde{b} \in B(b)$.

In particular, we have $\tilde{b} = \phi(\tilde{b}')$ for some $\tilde{b}' \in \Lambda_{\lambda_1,b}$. Since $\tilde{b} \neq b$, we have $\tilde{b}' \neq \eta_b$; hence $\tilde{b}' \in \Lambda^{\lambda'}_{\lambda_1,b}$ with $\lambda' < \lambda_1$. It follows that $\tilde{b} \in M^{\lambda'}$ with $\lambda' < \lambda_1$. This contradicts the assumption that $\tilde{b} \in B_1$. We have proved therefore that $B(\tilde{b})$ is disjoint from $B(b)$.

Since B' is the disjoint union of the canonical bases of the various $\Lambda_{\lambda_1,b}$ and these subsets are carried by ϕ injectively onto disjoint subsets of B, it follows that ϕ restricts to an injective map $B' \to B$. Since B' is a basis of M', it follows that $\phi : M' \to M$ is injective. Thus we may identify M' with a $\dot{\mathbf{U}}$-submodule of M (via ϕ) in such a way that B' becomes a subset of B. This submodule is clearly equal to $M[\lambda_1]$. The proposition follows.

Proposition 27.1.8. *Let (M, B) be a based module and let $\lambda \in X^+$. Then*

(a) *$B \cap M[\geq \lambda]$ is a basis of the vector space $M[\geq \lambda]$ and*

(b) *$B \cap M[> \lambda]$ is a basis of the vector space $M[> \lambda]$.*

First note that (b) follows from (a). Indeed, the vector space $M[> \lambda]$ is a sum of subspaces of form $M[\geq \lambda']$ for various $\lambda' > \lambda$. To prove (a), we argue by induction on $\dim M$. If $\dim M = 0$, there is nothing to prove. Therefore we may assume that $\dim M \geq 1$.

For fixed M, we argue by descending induction on λ. To begin the induction we note that if $\sum_i \langle i, \lambda \rangle$ is sufficiently large, then $M[\geq \lambda] = 0$ and there is nothing to prove. Assume that λ is given. If $M[\lambda] = 0$, then $M[\geq \lambda]$ is a sum of subspaces $M[\geq \lambda']$ with $\lambda' > \lambda$; hence the desired result holds by the induction hypothesis (on λ). Thus we may assume that $M[\lambda] \neq 0$. Then clearly $M^\lambda \neq 0$. We can find $\lambda_1 \in X^+$ such that $\lambda_1 \geq \lambda$, $M^{\lambda_1} \neq 0$ and λ_1 is maximal with these properties.

Let $M' = M[\lambda_1]$ and let $B' = B \cap M'$. Then $(M', B') \in \tilde{\mathcal{C}}$ by 27.1.7. Hence, by 27.1.4, $M'' = M/M'$, together with the image B'' of $B - B'$, is an object of $\tilde{\mathcal{C}}$. Since $M' \neq 0$, we have $\dim M'' < \dim M$; hence the induction hypothesis (on M) is applicable to M''. We see that $B'' \cap M''[\geq \lambda]$ is a basis of $M''[\geq \lambda]$. Since $M' = M'[\lambda_1]$ and $\lambda_1 \geq \lambda$, we see that $M[\geq \lambda]$ is just the inverse image of $M''[\geq \lambda]$ under the canonical map $M \to M''$; moreover, a basis for this inverse image is given by the inverse image of $B'' \cap M''[\geq \lambda]$ under the canonical map $B \to B''$. The proposition is proved.

27.2. The Subsets $B[\lambda]$

27.2.1. Let (M, B) be a based module. Let $b \in B$. We can find $\lambda \in X^+$ such that $b \in M[\geq \lambda]$ and λ is maximal with this property. Actually, λ is unique. Indeed, assume that we also have $b \in M[\geq \lambda']$ and λ' is maximal with this property. We note that $M[\geq \lambda] \cap M[\geq \lambda']$ is a sum of subspaces $M[\geq \lambda'']$ for various λ'' such that $\lambda \leq \lambda''$ and $\lambda' \leq \lambda''$ and from 27.1.8 it follows that $b \in M[\geq \lambda'']$ for some such λ''.

If $\lambda \neq \lambda'$, then λ'' satisfies $\lambda < \lambda''$ and $\lambda' < \lambda''$, and we find a contradiction with the definition of λ. Thus the uniqueness of λ is proved.

Let $B[\lambda]$ be the set of all $b \in B$ which give rise to $\lambda \in X^+$ as above. These sets clearly form a partition of B. From 27.1.8, we see that, for any $\lambda \in X^+$, the set $\sqcup_{\lambda' \in X^+; \lambda' \geq \lambda} B[\lambda']$ is a basis of $M[\geq \lambda]$ and the set $\sqcup_{\lambda' \in X^+; \lambda' > \lambda} B[\lambda']$ is a basis of $M[> \lambda]$.

Proposition 27.2.2. *Let f be a morphism in $\tilde{\mathcal{C}}$ from the based module (M, B) to the based module (M', B') (see 27.1.3). For any $\lambda \in X^+$, we have $f(B[\lambda]) \subset B'[\lambda] \cup \{0\}$.*

From the definitions, we see that $f(M[\geq \lambda]) \subset M'[\geq \lambda]$ and $f(M[> \lambda]) \subset M'[> \lambda]$. Hence if $b \in B[\lambda]$, then either $f(b) \in B'[\lambda']$ for some $\lambda' \geq \lambda$ or $f(b) = 0$. Assume that $f(b) \notin B'[\lambda]$. Then $f(b) \in M'[> \lambda]$. Using the obvious inclusion $f(M) \cap M'[> \lambda] \subset f(M[> \lambda])$, we deduce that $b \in M[> \lambda] + \ker f$. Since both $M[> \lambda]$ and $\ker f$ are generated by their intersection with B, it follows that either $b \in M[> \lambda]$ or $b \in \ker f$. The first alternative contradicts $b \in B[\lambda]$; hence the second alternative holds and we have $f(b) = 0$. The proposition follows.

27.2.3. Let (M, B) be a based module. Let $\lambda \in X^+$. We define $B[\lambda]^{hi}$ to be the set of all $b \in B$ such that $b \in M^\lambda$ and $\tilde{E}_i b \in v^{-1}L(M)$ for all $i \in I$. We define $B[\lambda]^{lo}$ to be the set of all $b \in B$ such that $b \in M^{w_0(\lambda)}$ and $\tilde{F}_i b \in v^{-1}L(M)$ for all $i \in I$.

Proposition 27.2.4. (a) *We have $B[\lambda]^{hi} \subset B[\lambda]$ and $B[\lambda]^{lo} \subset B[\lambda]$.*

(b) *Let $p : M[\geq \lambda] \to M[\geq \lambda]/M[> \lambda] = \tilde{M}$ be the canonical map. Note that p defines a bijection of $B[\lambda]$ with a basis $\tilde{B}$ of $\tilde{M}$ and that $(\tilde{M}, \tilde{B})$ belongs to $\tilde{\mathcal{C}}$ so that $\tilde{B}[\lambda]^{hi}$ and $\tilde{B}[\lambda]^{lo}$ are defined. Then p restricts to bijections $B[\lambda]^{hi} \to \tilde{B}[\lambda]^{hi}$ and $B[\lambda]^{lo} \to \tilde{B}[\lambda]^{lo}$.*

We prove (a). Let $b \in B[\lambda]^{hi}$. There is a unique $\lambda' \in X^+$ such that $b \in B[\lambda']$. We must prove that $\lambda = \lambda'$. We have $b \in M[\geq \lambda']$. Replacing M with $M[\geq \lambda']$, we may assume that $M = M[\geq \lambda']$. Let π be the canonical map of M onto $M'' = M/M[> \lambda']$. Then $B[\lambda']$ is mapped by π bijectively onto a basis B'' of M'' and we have $\pi(b) \in B''$. Moreover, $\pi(b)$ belongs to $B''[\lambda]^{hi}$ and we are therefore reduced to the case where $M = M''$. Thus we may assume that $M = M[\lambda']$. Now 27.1.7 reduces us further to the case where (M, B) is $\Lambda_{\lambda'}$ with its canonical basis. In this case, there are two possibilities for b: either b is in the λ'-weight space or there exist i and $b' \in B$ such that $\tilde{F}_i b' - b \in v^{-1}L(M)$. In the first case we have $b \in M^{\lambda'}$; in the second case we have $\tilde{E}_i b - b' \in v^{-1}L(M)$; hence $\tilde{E}_i b \notin v^{-1}L(M)$, in contradiction with our assumption on b. Thus we have $b \in M^{\lambda'}$, hence $\lambda = \lambda'$, as required. We have proved that $B[\lambda]^{hi} \subset B[\lambda]$. The proof of the inclusion $B[\lambda]^{lo} \subset B[\lambda]$ is entirely similar.

We prove (b). We assume that $M = M[\geq \lambda]$. It is clear that $p(B[\lambda]^{hi}) \subset \tilde{B}[\lambda]^{hi}$ and $p(B[\lambda]^{lo}) \subset \tilde{B}[\lambda]^{lo}$. Assume that $b \in B[\lambda]$ satisfies $b \notin B[\lambda]^{hi}$. We show that $p(b) \notin \tilde{B}[\lambda]^{hi}$. By our assumption, we have that either $b \in M^{\lambda'}$ with $\lambda' \neq \lambda$ or that $\tilde{E}_i b \notin v^{-1}L(M)$ for some i.

If $b \in M^{\lambda'}$ with $\lambda' \neq \lambda$, then $p(b) \in \tilde{M}^{\lambda'}$ with $\lambda' \neq \lambda$; hence $p(b) \notin \tilde{B}[\lambda]^{hi}$, as required. If $\tilde{E}_i b \notin v^{-1}L(M)$ for some $i \in I$, then there exists $b' \in B$ such that $\tilde{E}_i b - b' \in v^{-1}L(M)$ and therefore $\tilde{F}_i b' - b \in v^{-1}L(M)$.

We consider two cases according to whether or not $b' \in M[> \lambda]$. In the first case ($b' \in M[> \lambda]$), we have $\tilde{F}_i b' \in M[> \lambda]$ (since $M[> \lambda]$ is a subobject of M) hence $b \in M[> \lambda] + v^{-1}L(M)$; this implies that $b \in M[> \lambda]$ (using that $B \cap M[> \lambda]$ is a basis of $M[> \lambda]$). Then we have $p(b) = 0$ and, in particular, $p(b) \notin \tilde{B}[\lambda]^{hi}$, as required. In the second case ($b' \notin M[> \lambda]$), we have $b' \in B[\lambda]$; hence $\pi(b') \in \tilde{B}$.

Let $L(\tilde{M})$ be the **A**-submodule of $\tilde{M}$ generated by $\tilde{B}$. From $\tilde{E}_i b - b' \in v^{-1}L(M)$, we deduce $\tilde{E}_i(\pi(b)) - \pi(b') \in v^{-1}L(\tilde{M})$. In particular, we have $\tilde{E}_i(\pi(b)) \notin v^{-1}L(\tilde{M})$; hence $p(b) \notin \tilde{B}[\lambda]^{hi}$, as required. Thus we have proved the equality $p(B[\lambda]^{hi}) = \tilde{B}[\lambda]^{hi}$. The proof of the equality $p(B[\lambda]^{lo}) = \tilde{B}[\lambda]^{lo}$ is entirely similar.

27.2.5. Coinvariants. Let $(M, B) \in \tilde{\mathcal{C}}$. Let $M[\neq 0] = \oplus_{\lambda \neq 0} M[\lambda]$. The space of *coinvariants* of M is by definition the vector space $M_* = M/M[\neq 0]$. Clearly, $M[\neq 0]$ is equal to the sum of the subspaces $M[\geq \lambda']$ for various $\lambda' \in X^+ - \{0\}$; hence, from 27.2.8, it follows that $\cup_{\lambda' \neq 0} B[\lambda']$ is a basis of $M[\neq 0]$. We deduce that under the canonical map $\pi : M \to M_*$ the subset $B[0]$ of B is mapped bijectively onto a basis B_* of M_*.

Note that π is a morphism in $\mathcal{C}$ if we regard M_* with the **U**-module structure such that $M_* = M_*[0]$. We see that

(a) (M_*, B_*) is a based module with trivial action of **U**.

Proposition 27.2.6. *We have $B[0] = B[0]^{hi} = B[0]^{lo}$. This set is mapped bijectively by $\pi : M \to M_*$ onto B_*.*

To prove the first statement, we are reduced by 27.2.4(a),(b) to the case where $M = M[0]$, where it is obvious. The second statement has already been noted.

27.3. Tensor Product of Based Modules

27.3.1. Let $(M, B), (M', B')$ be two based modules with associated involutions $^- : M \to M, ^- : M' \to M'$. We will show that *the* **U**-*module* $M \otimes M'$ *is in a natural way a based module.*

The obvious basis $B \otimes B'$ does not make $M \otimes M'$ into a based module, since the involution $^- : M \otimes M' \to M \otimes M'$ given by $\overline{m \otimes m'} = \bar{m} \otimes \bar{m}'$ is not, in general, compatible with the **U**-module structure.

We will define a new involution $\Psi : M \otimes M' \to M \otimes M'$ by $\Psi(x) = \Theta(\bar{x})$ for all $x \in M \otimes M'$; here $\Theta : M \otimes M' \to M \otimes M'$ is as in 24.1.1. Eventually, Ψ will be the associated involution of our based module.

Let $\mathcal{L}$ (resp. ${}_{\mathcal{A}}\mathcal{L}$) be the $\mathbf{Z}[v^{-1}]$-submodule (resp. $\mathcal{A}$-submodule) of $M \otimes M'$ generated by the basis $B \otimes B'$. From 24.1.6, we see that Θ leaves ${}_{\mathcal{A}}\mathcal{L}$ stable and clearly $^{-} : M \otimes M' \to M \otimes M'$ leaves ${}_{\mathcal{A}}\mathcal{L}$ stable; it follows that we have $\Psi({}_{\mathcal{A}}\mathcal{L}) \subset {}_{\mathcal{A}}\mathcal{L}$. From 24.1.2 and 4.1.3, it follows that $\Psi^2 = 1$ and $\Psi(ux) = \bar{u}\Psi(x)$ for all $u \in \mathbf{U}$ and all $x \in M \otimes M'$. We shall regard $B \times B'$ as a partially ordered set with $(b_1, b_1') \geq (b_2, b_2')$ if and only if $b_1 \in M^{\lambda_1}, b_1' \in M'^{\lambda_1'}, b_2 \in M^{\lambda_2}, b_2' \in M'^{\lambda_2'}$ where $\lambda_1 \geq \lambda_2, \lambda_1' \leq \lambda_2', \lambda_1 + \lambda_1' = \lambda_2 + \lambda_2'$.

From the definition we have, for all $b_1 \in B, b_1' \in B'$,

$$\Psi(b_1 \otimes b_1') = \sum_{b_2 \in B, b_2' \in B'} \rho_{b_1,b_1';b_2,b_2'} b_2 \otimes b_2'$$

where $\rho_{b_1,b_1';b_2,b_2'} \in \mathcal{A}$ and $\rho_{b_1,b_1';b_2,b_2'} = 0$ unless $(b_1, b_1') \geq (b_2, b_2')$. Note also that

$$\rho_{b_1,b_1';b_1,b_1'} = 1$$

and

$$\sum_{b_2 \in B, b_2' \in B'} \bar{\rho}_{b_1,b_1';b_2,b_2'} \rho_{b_2,b_2';b_3,b_3'} = \delta_{b_1,b_3} \delta_{b_1',b_3'}$$

for any $b_1, b_3 \in B$ and $b_1', b_3' \in B'$; the last condition follows from $\Psi^2 = 1$. Applying 24.2.1 to the partially ordered set $H = B \times B'$, we see that there is a unique family of elements $\pi_{b_1,b_1';b_2,b_2'} \in \mathbf{Z}[v^{-1}]$ defined for $b_1, b_2 \in B$ and $b_1', b_2' \in B'$, such that

$\pi_{b_1,b_1';b_1,b_1'} = 1$;

$\pi_{b_1,b_1';b_2,b_2'} \in v^{-1}\mathbf{Z}[v^{-1}]$ if $(b_1, b_1') \neq (b_2, b_2')$;

$\pi_{b_1,b_1';b_2,b_2'} = 0$ unless $(b_1, b_1') \geq (b_2, b_2')$;

$\pi_{b_1,b_1';b_2,b_2'} = \sum_{b_3,b_3'} \bar{\pi}_{b_1,b_1';b_3,b_3'} \rho_{b_3,b_3';b_2,b_2'}$

for all $(b_1, b_1') \geq (b_2, b_2')$.

We have the following result.

Theorem 27.3.2. (a) *For any $(b_1, b_1') \in B \times B'$, there is a unique element $b_1 \diamondsuit b_1' \in \mathcal{L}$ such that $\Psi(b_1 \diamondsuit b_1') = b_1 \diamondsuit b_1'$ and $(b_1 \diamondsuit b_1') - b_1 \otimes b_1' \in v^{-1}\mathcal{L}$.*

(b) *The element $b_1 \diamondsuit b_1'$ in (a) is equal to $b_1 \otimes b_1'$ plus a linear combination of elements $b_2 \otimes b_2'$ with $(b_2, b_2') \in B \times B'$, $(b_2, b_2') < (b_1, b_1')$ and with coefficients in $v^{-1}\mathbf{Z}[v^{-1}]$.*

(c) *The elements $b_1 \diamondsuit b_1'$ with b_1, b_1' as above, form a $\mathbf{Q}(v)$-basis $B_\diamondsuit$ of $M \otimes M'$, an $\mathcal{A}$-basis of ${}_{\mathcal{A}}\mathcal{L}$ and a $\mathbf{Z}[v^{-1}]$-basis of $\mathcal{L}$.*

$b_1 \diamondsuit b_1'$ just defined satisfy the requirements of (b),(c) and that (d) holds. It remains to show the uniqueness in (a). It is enough to show that an element $x \in v^{-1}\mathcal{L}$ such that $\bar{x} = x$ is necessarily 0. But this follows from (d).

27.3.3. The previous result, together with the known behaviour of bases at ∞ under tensor product, (see 20.2.2) shows that $(M \otimes M', B_{\diamondsuit})$ is a based module with associated involution Ψ. This is by definition the tensor product of the objects $(M, B), (M', B')$.

27.3.4. Let $\lambda, \lambda' \in X^+$. Applying the previous construction to $M = {}^{\omega}\Lambda_{\lambda}$ and $M' = \Lambda_{\lambda'}$ regarded as based modules (with respect to the canonical bases), we obtain a basis of ${}^{\omega}\Lambda_{\lambda} \otimes \Lambda_{\lambda'}$, which clearly is the same as that constructed in 24.3.3. Thus, ${}^{\omega}\Lambda_{\lambda} \otimes \Lambda_{\lambda'}$, together with its canonical basis in 24.3.3, is a based module.

Proposition 27.3.5. *Let $\lambda, \lambda', \lambda'' \in X^+$.*

(a) *The* **U***-modules $M = {}^{\omega}\Lambda_{\lambda+\lambda'} \otimes \Lambda_{\lambda'+\lambda''}$ and $M' = {}^{\omega}\Lambda_{\lambda} \otimes \Lambda_{\lambda'}$ with their canonical bases B, B' constructed in 24.3.3, are in $\tilde{\mathcal{C}}$; moreover, $t : M \to M'$ (see 25.1.5) is a morphism in $\tilde{\mathcal{C}}$.*

(b) *For any $\lambda_1 \in X^+$, we have $t(B[\lambda_1]) \subset B'[\lambda_1] \cup \{0\}$.*

The fact that $(M, B), (M', B')$ are objects of $\tilde{\mathcal{C}}$ has been pointed out in 27.3.4. The second assertion of (a) follows from Proposition 25.1.10. Now (b) follows from (a) and 27.2.2.

27.3.6. Associativity of tensor product. Let $(M, B), (M', B')$, and (M'', B'') be three based modules. On the **U**-module $M \otimes M' \otimes M''$, we can introduce two structures of based module: one by applying the construction in 27.3.2 first to $M \otimes M'$ and then to $(M \otimes M') \otimes M''$; the second one by applying the construction in 27.3.2 first to $M' \otimes M''$ and then to $M \otimes (M' \otimes M'')$. Let B_1, B_2 be the bases of $M \otimes M' \otimes M''$ obtained by these two constructions.

We show that $B_1 = B_2$. By definition, the associated involutions to these two structures are given by

$$\sum_{\nu', \nu''} (\Delta \otimes 1)(\Theta_{\nu'})\Theta^{12}_{\nu''}(^- \otimes {}^- \otimes {}^-)$$

and

$$\sum_{\nu', \nu''} (1 \otimes \Delta)(\Theta_{\nu'})\Theta^{23}_{\nu''}(^- \otimes {}^- \otimes {}^-)$$

respectively. These coincide by 4.2.4.

Next from the definitions, we see that the $\mathbf{Z}[v^{-1}]$-submodules of $M \otimes M' \otimes M''$ generated by B_1 or B_2 coincide; they both coincide with the $\mathbf{Z}[v^{-1}]$-submodule $\mathcal{L}$ of $M \otimes M' \otimes M''$ generated by $B \otimes B' \otimes B''$; moreover, if $\pi : \mathcal{L} \to \mathcal{L}/v^{-1}\mathcal{L}$ is the canonical projection, then $\pi(B_1) = \pi(B_2) = \pi(B \otimes B' \otimes B'')$.

To show that $B_1 = B_2$, it suffices to show that $(b\diamondsuit b')\diamondsuit b'' = b\diamondsuit(b'\diamondsuit b'')$ for any $b \in B, b' \in B', b'' \in B''$. Let $b_1 = (b\diamondsuit b')\diamondsuit b'' \in B_1$ and $b_2 = b\diamondsuit(b'\diamondsuit b'') \in B_2$. From the definitions, we have that $\pi(b_1) = \pi(b \otimes b' \otimes b'')$ and $\pi(b_2) = \pi(b \otimes b' \otimes b'')$. Hence $\pi(b_1) = \pi(b_2)$. Then $b_1 - b_2 \in v^{-1}\mathcal{L}$ and $b_1 - b_2$ is fixed by the associated involution. This forces $b_1 = b_2$, as required. Thus we may omit brackets and write $b\diamondsuit b'\diamondsuit b''$ instead of $(b\diamondsuit b')\diamondsuit b''$ or $b\diamondsuit(b'\diamondsuit b'')$. This implies automatically that the analogous associativity result is also true for more than three factors.

27.3.7. Coinvariants in a tensor product. Let $(M, B), (M', B')$ be two based modules. We form their tensor product $(M \otimes M', B_\diamondsuit)$. The following result describes the subset $B_\diamondsuit[0]$ of $B_\diamondsuit$.

Proposition 27.3.8. *Let $b \in B, b' \in B'$. We have*

$$B_\diamondsuit[0] = \cup_{\lambda' \in X^+}\{b\diamondsuit b' | b \in B[-w_0(\lambda')]^{lo}, b' \in B'[\lambda']^{hi}\}.$$

Let $b \in B, b' \in B'$ be two elements such that $b \in M^\lambda, b' \in M'^{\lambda'}$. According to 27.2.6, the condition that $b\diamondsuit b'$ belongs to $B_\diamondsuit[0]$ is that $\lambda + \lambda' = 0$ and $\tilde{F}_i(b\diamondsuit b') \in v^{-1}L(M \otimes M')$ for all i; the last condition is clearly equivalent to the condition that $\tilde{F}_i(b \otimes b') \in v^{-1}L(M \otimes M')$. By 20.2.4, our condition is equivalent to the following one: $\lambda + \lambda' = 0$, $\tilde{F}_i(b) \in v^{-1}L(M)$ and $\tilde{E}_i(b') \in v^{-1}L(M')$ for all $i \in I$. The proposition follows.

27.3.9. We consider a sequence $\lambda_1, \lambda_2, \ldots, \lambda_n$ of elements of X^+. According to 27.3.6, the tensor product $\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2} \cdots \otimes \Lambda_{\lambda_n}$ is in a natural way a based module (hence has a distinguished basis) and according to 27.2.5, the space of coinvariants $(\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2} \cdots \otimes \Lambda_{\lambda_n})_*$ inherits a natural based module structure (hence has a distinguished basis).

27.3.10. Let us assume, for example, that the root datum is simply connected of type D_m, that $n = 2n'$ and that $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$ is such that Λ_λ is the standard $(2m)$-dimensional module. Then we may identify the space of coinvariants $(\Lambda_{\lambda_1} \otimes \Lambda_{\lambda_2} \cdots \otimes \Lambda_{\lambda_n})_*$ naturally with the dual space of $\mathrm{End}_{\mathbf{U}}(\Lambda_\lambda^{\otimes n'})$. Hence, from 27.3.9, we obtain a distinguished basis for the algebra $\mathrm{End}_{\mathbf{U}}(\Lambda_\lambda^{\otimes n'})$, the quantum analogue of the *Brauer centralizer algebra*. This basis is of the same nature as the basis of the Hecke algebra of type A defined in [3].