CHAPTER 26

Inner Product on $\dot{\mathbf{U}}$

26.1. First Definition of the Inner Product

26.1.1. In the following theorem, $\rho : \mathbf{U} \to \mathbf{U}$ is as in 19.1.1, and $\bar{\rho} : \mathbf{U} \to \mathbf{U}$ is defined by $\bar{\rho}(u) = \overline{\rho(\bar{u})}$.

Theorem 26.1.2. *There exists a unique* $\mathbf{Q}(v)$*-bilinear pairing* $(,) : \dot{\mathbf{U}} \times \dot{\mathbf{U}} \to \mathbf{Q}(v)$ *such that* (a), (b), (c) *below hold.*

(a) $(1_{\lambda_1} x 1_{\lambda_2}, 1_{\lambda'_1} x' 1_{\lambda'_2})$ *is zero for all* $x, x' \in \dot{\mathbf{U}}$, *unless* $\lambda_1 = \lambda'_1$ *and* $\lambda_2 = \lambda'_2$;

(b) $(ux, y) = (x, \rho(u)y)$ *for all* $x, y \in \dot{\mathbf{U}}$ *and* $u \in \mathbf{U}$;

(c) $(x^- 1_\lambda, x'^- 1_\lambda) = (x, x')$ *for all* $x, x' \in \mathbf{f}$ *and all* λ *(here* (x, x') *is the inner product as in 1.2.5).*

(d) *We have* $(x, y) = (y, x)$ *for all* $x, y \in \dot{\mathbf{U}}$.

Let $\zeta \in X$. If B is a basis of $\mathbf{f}$ consisting of homogeneous elements, the elements $\rho(b^-)b'^- 1_\zeta$, with $b, b' \in B$, form a basis of $\dot{\mathbf{U}} 1_\zeta$. (We use the triangular decomposition of $\dot{\mathbf{U}}$.) Hence there is a unique $\mathbf{Q}(v)$-linear map $p : \dot{\mathbf{U}} 1_\zeta \to \mathbf{Q}(v)$ such that $p(\rho(x^-)x'^- 1_\zeta) = (x, x')$ for all $x, x' \in \mathbf{f}$; here, (x, x') is as in 1.2.5. The properties of (x, x') imply that for homogeneous x, x', we have $p(\rho(x^-)x'^- 1_\zeta) = 0$ unless x, x' have the same homogeneity, in which case $\rho(x^-)x'^- 1_\zeta = 1_\zeta \rho(x^-)x'^-$. Thus, for $\zeta' \in X$ different from ζ, we have $p(1_{\zeta'} \dot{\mathbf{U}} 1_\zeta) = 0$. It follows that, for any $\mu \in Y$, we have $p((K_\mu - v^{\langle \mu, \zeta \rangle}) \dot{\mathbf{U}} 1_\zeta) = 0$. We now define a pairing $f_\zeta : \dot{\mathbf{U}} 1_\zeta \times \dot{\mathbf{U}} 1_\zeta \to \mathbf{Q}(v)$ by $f_\zeta(u 1_\zeta, u' 1_\zeta) = p(\rho(u) u' 1_\zeta)$ where $u, u' \in \mathbf{U}$. To show independence of u, u', we must check that $p(\rho(u) u' 1_\zeta) = 0$ if either u or u' is in the left ideal of $\mathbf{U}$ generated by $(K_\mu - v^{\langle \mu, \zeta \rangle})$ for some $\mu \in Y$; in the case of u, this follows from the previous sentence, while in the case of u', this is obvious. Thus, f_ζ is well-defined. We define the bilinear pairing $(,)$ on $\dot{\mathbf{U}}$ by $(x, y) = f_\zeta(x, y)$ if $x, y \in \dot{\mathbf{U}} 1_\zeta$ and by $(x, y) = 0$ if $x \in \dot{\mathbf{U}} 1_\zeta$ and $y \in \dot{\mathbf{U}} 1_{\zeta'}$ with $\zeta \neq \zeta'$. It is clear that this pairing satisfies (a),(b),(c); the uniqueness is also clear from the proof above.

The pairing $x, y \mapsto (y, x)$ satisfies the defining properties of (x, y) (since $\rho^2 = 1$), and hence coincides with it. This proves (d).

Proposition 26.1.3. *We have* $(xu, y) = (x, y\bar{\rho}(u))$ *for all* $x, y \in \dot{\mathbf{U}}$ *and* $u \in \mathbf{U}$.

We may assume that u is one of the standard generators of $\mathbf{U}$. Thus, we must verify that

$$(xE_i, y) = (x, v_i y F_i \tilde{K}_{-i}), (xF_i, y) = (x, v_i y E_i \tilde{K}_i), (xK_{-\mu}, y) = (x, yK_{-\mu})$$

for all $x, y \in \dot{\mathbf{U}}$ and $i \in I, \mu \in Y$.

We may assume that $x = u' 1_\zeta$ where $u' \in \mathbf{U}$ and $\zeta \in X$. Using 26.1.2(b), and setting $\rho(u')y = y'$, we see that the previous equalities are consequences of

$$(1_\zeta E_i, y') = (1_\zeta, v_i y' F_i \tilde{K}_{-i}), (1_\zeta F_i, y') = (1_\zeta, v_i y' E_i \tilde{K}_i)$$

and

$$(1_\zeta K_{-\mu}, y') = (1_\zeta, y' K_{-\mu})$$

for all $y' \in \dot{\mathbf{U}}$ and $i \in I, \mu \in Y$.

We can assume that $y' = \rho(y_1^-) y_2^- 1_{\zeta'}$ where $y_1, y_2 \in \mathbf{f}$ are homogeneous. Then the equalities to be proved can be rewritten as follows:

(a) $(y_1^- E_i 1_{\zeta - i'}, y_2^- 1_{\zeta'}) = v_i^{1-\langle i, \zeta' + i' \rangle} (y_1^- 1_\zeta, y_2^- F_i 1_{\zeta' + i'})$

(b) $(y_1^- F_i 1_{\zeta + i'}, y_2^- 1_{\zeta'}) = v_i^{1+\langle i, \zeta' - i' \rangle} (y_1^- 1_\zeta, y_2^- E_i 1_{\zeta' - i'})$

(c) $(y_1^- 1_\zeta K_{-\mu}, y_2^- 1_{\zeta'}) = (y_1^- 1_\zeta, y_2^- 1_{\zeta'} K_{-\mu})$.

Now (c) is obvious and (b) follows from (a), using 26.1.2(d). It remains to prove (a). We may assume that $\zeta' = \zeta - i'$. We substitute

$$y_1^- E_i 1_{\zeta'} = E_i y_1^- 1_{\zeta'} + \frac{r_i(y_1)^- \tilde{K}_{-i} - \tilde{K}_i ({}_i r(y_1)^-)}{v_i - v_i^{-1}} 1_{\zeta'}$$

and note that

$$\begin{aligned}(E_i y_1^- 1_{\zeta'}, y_2^- 1_{\zeta'}) &= (y_1^- 1_{\zeta'}, v_i \tilde{K}_i F_i y_2^- 1_{\zeta'}) \\ &= (y_1^- 1_{\zeta'}, v_i^{-1+\langle i, \zeta' - |y_2| \rangle} F_i y_2^- 1_{\zeta'}) \\ &= v_i^{-1+\langle i, \zeta' - |y_2| \rangle} (y_1, \theta_i y_2),\end{aligned}$$

$$\Big(\frac{r_i(y_1)^- \tilde{K}_{-i} - \tilde{K}_i ({}_i r(y_1)^-)}{v_i - v_i^{-1}} 1_{\zeta'}, y_2^- 1_{\zeta'}\Big) =$$

$$\frac{v_i^{-\langle i, \zeta' \rangle} (r_i(y_1), y_2) - v_i^{\langle i, \zeta' - |y_1| + i' \rangle} ({}_i r(y_1), y_2)}{v_i - v_i^{-1}},$$

$$v_i^{1-\langle i, \zeta' + i' \rangle} (y_1^- 1_\zeta, y_2^- F_i 1_{\zeta' + i'}) = v_i^{-1-\langle i, \zeta' \rangle} (y_1, y_2 \theta_i),$$

(see 26.1.2(b), (c)). We see that (a) is equivalent to:

$$v_i^{-1+\langle i,\zeta'-|y_2|\rangle}(y_1,\theta_i y_2)+\frac{v_i^{-\langle i,\zeta'\rangle}(r_i(y_1),y_2)-v_i^{\langle i,\zeta'-|y_1|+i'\rangle}({}_ir(y_1),y_2)}{v_i-v_i^{-1}}$$
$$=v_i^{-1-\langle i,\zeta'\rangle}(y_1,y_2\theta_i).$$

But this follows from the known equalities

$$(y_1,y_2\theta_i)=\frac{(r_i(y_1),y_2)}{1-v_i^{-2}},(y_1,\theta_i y_2)=\frac{({}_ir(y_1),y_2)}{1-v_i^{-2}}$$

(see 1.2.13(a)). The proposition is proved.

Proposition 26.1.4. *We have $(\sigma(x),\sigma(y))=(x,y)$ for all $x,y\in\dot{\mathbf{U}}$.*

We must show that the pairing $x,y\mapsto(\sigma(x),\sigma(y))$ on $\dot{\mathbf{U}}$ satisfies the defining properties of $(,)$. Property 26.1.2(a) is obvious and property 26.1.2(c) follows from 1.2.8(b). Since $\sigma\rho=\bar{\rho}\sigma:\mathbf{U}\to\mathbf{U}$, we see that 26.1.2(b) for the pairing $x,y\mapsto(\sigma(x),\sigma(y))$ is equivalent to the identity in the previous proposition. The proposition follows.

Lemma 26.1.5. *$(x^+1_\lambda,x'^+1_\lambda)=(x,x')$ for all $x,x'\in\mathbf{f}$ and all λ; here (x,x') is as in 1.2.5.*

We may assume that x,x' are homogeneous; moreover, using 26.1.2(a), we may assume that they both belong to $\mathbf{f}_\nu$. Using 26.1.2(b), we have

$$(x^+1_\lambda,x'^+1_\lambda)=(1_\lambda,\rho(x^+)x'^+1_\lambda).$$

Using the previous proposition, we see that the last expression equals

$$\begin{aligned}(\sigma(1_\lambda),\sigma(\rho(x^+)x'^+1_\lambda))&=(1_{-\lambda},1_{-\lambda}\sigma(x'^+)\bar{\rho}(\sigma(x^+)))\\&=(1_{-\lambda},\sigma(x'^+)\bar{\rho}(\sigma(x^+))1_{-\lambda}).\end{aligned}$$

Using 26.1.2(b) and the fact that $\rho^2=1$, we see that the last expression equals

$$(\rho(\sigma(x'^+))1_{-\lambda},\bar{\rho}(\sigma(x^+))1_{-\lambda}).$$

Using the definitions, we see that there exists an integer N depending only on ν and λ such that for any $z\in\mathbf{f}_\nu$ we have $\rho(\sigma(z^+))1_\lambda=v^Nz^-1_\lambda$ and $\bar{\rho}(\sigma(z^+))1_\lambda=v^{-N}z^-1_\lambda$. Hence the last inner product is equal to

$$(v^Nx'^-1_{-\lambda},v^{-N}x^-1_{-\lambda})=(x'^-1_{-\lambda},x^-1_{-\lambda})=(x',x)=(x,x').$$

The lemma is proved.

Proposition 26.1.6. *We have $(\omega(x),\omega(y)) = (x,y)$ for all $x,y \in \dot{\mathbf{U}}$.*

We must show that the pairing $x, y \mapsto (\omega(x),\omega(y))$ on $\dot{\mathbf{U}}$ satisfies the defining properties of $(,)$. Property 26.1.2(a) is obvious and property 26.1.2(c) follows from the previous lemma. Since $\omega\rho = \rho\omega : \mathbf{U} \to \mathbf{U}$, we see that 26.1.2(b) for the pairing $x,y \mapsto (\omega(x),\omega(y))$ is equivalent to the corresponding identity for (x,y). The proposition follows.

26.2. Definition of the Inner Product as a Limit

26.2.1 In this section we assume that the root datum is Y-regular. Let $\zeta \in X$ and let $\lambda, \lambda' \in X^+$ be such that $\lambda' - \lambda = \zeta$. We consider the bilinear pairing $(,)_{\lambda,\lambda'}$ on ${}^\omega\Lambda_\lambda \otimes \Lambda_{\lambda'}$, defined by $(x \otimes x', y \otimes y') = (x,y)_\lambda (x',y')_{\lambda'}$. Here $(,)_{\lambda'}$ is the pairing on $\Lambda_{\lambda'}$ defined in 19.1.2, and $(,)_\lambda$ is the analogous pairing on Λ_λ which has the same ambient space as ${}^\omega\Lambda_\lambda$.

Lemma 26.2.2. *If $x_1, x_2 \in {}^\omega\Lambda_\lambda \otimes \Lambda_{\lambda'}$ and $u \in \mathbf{U}$, we have*

$$(ux_1, x_2)_{\lambda,\lambda'} = (x_1, \rho(u)x_2)_{\lambda,\lambda'}.$$

It is enough to check this in the case where u is one of the algebra generators E_i, F_i, K_μ of $\mathbf{U}$. The case where $u = K_\mu$ is trivial. We now fix i and regard the $\mathbf{U}$-module ${}^\omega\Lambda_\lambda \otimes \Lambda_{\lambda'}$ as an object of $\mathcal{C}_i'$ (by restriction). It is enough to show that the form $(,)$ on this object is admissible in the sense of 16.2.2. Using 17.1.3(b), we see that this would follow if we knew that the forms $(,)_\lambda$ and $(,)_{\lambda'}$ on ${}^\omega\Lambda_\lambda$ and $\Lambda_{\lambda'}$ (regarded as objects of $\mathcal{C}_i'$) are admissible. For $(,)_{\lambda'}$ this follows from the definition. The same holds for $(,)_\lambda$ on Λ_λ. One checks easily that applying ω to an object of $\mathcal{C}_i'$ with an admissible form gives a new object of $\mathcal{C}_i'$ for which the same form is admissible. In particular, $(,)_{\lambda'}$ is admissible for ${}^\omega\Lambda_\lambda$. The lemma is proved.

Proposition 26.2.3. *Let $x, y \in \dot{\mathbf{U}}1_\zeta$. When the pair λ, λ' tends to ∞ (in the sense that $\langle i, \lambda\rangle$ tends to ∞ for all i, or equivalently, $\langle i, \lambda'\rangle$ tends to ∞ for all i, the difference $\lambda' - \lambda$ being fixed and equal to ζ), the inner product $(x(\xi_{-\lambda} \otimes \eta_{\lambda'}), y(\xi_{-\lambda} \otimes \eta_{\lambda'}))_{\lambda,\lambda'} \in \mathbf{Q}(v)$ converges in $\mathbf{Q}((v^{-1}))$ to (x,y).*

Assume first that $x = x_1^- 1_\zeta, y = y_1^- 1_\zeta$ where $x_1, y_1 \in \mathbf{f}$. In this case we have

$$x(\xi_{-\lambda} \otimes \eta_{\lambda'}) = \xi_{-\lambda} \otimes x_1^- \eta_{\lambda'}$$

and

$$y(\xi_{-\lambda} \otimes \eta_{\lambda'}) = \xi_{-\lambda} \otimes y_1^- \eta_{\lambda'};$$

hence

$$(x(\xi_{-\lambda}\otimes\eta_{\lambda'}),y(\xi_{-\lambda}\otimes\eta_{\lambda'}))_{\lambda,\lambda'}=(x_1^-\eta_{\lambda'},y_1^-\eta_{\lambda'})_{\lambda'}$$

and, by 19.3.7, this converges to (x_1,y_1) when $\lambda\to\infty$. Since $(x_1,y_1)=(x_1^-1_\zeta,y_1^-1_\zeta)$, the proposition holds in this case.

Next, we prove the proposition in the case where $x=1_\zeta$ and y is arbitrary. We may assume that $y=\rho(x_1^-)y_1^-1_\zeta$ where $x_1,y_1\in\mathbf{f}$. Using the previous lemma, we have

$$(1_\zeta(\xi_{-\lambda}\otimes\eta_{\lambda'}),y(\xi_{-\lambda}\otimes\eta_{\lambda'}))_{\lambda,\lambda'}=(x_1^-1_\zeta(\xi_{-\lambda}\otimes\eta_{\lambda'}),y_1^-1_\zeta(\xi_{-\lambda}\otimes\eta_{\lambda'}))_{\lambda,\lambda'}$$

and by the earlier part of the proof, this converges to $(x_1^-1_\zeta,y_1^-1_\zeta)$ when $\lambda\to\infty$. Since $(x_1^-1_\zeta,y_1^-1_\zeta)=(1_\zeta,y)$, the proposition holds in this case.

We now consider the general case. We can write $x=u1_\zeta$ where $u\in\mathbf{U}$. Using the previous lemma, we have

$$(u1_\zeta(\xi_{-\lambda}\otimes\eta_{\lambda'}),y(\xi_{-\lambda}\otimes\eta_{\lambda'}))_{\lambda,\lambda'}=(1_\zeta(\xi_{-\lambda}\otimes\eta_{\lambda'}),\rho(u)y(\xi_{-\lambda}\otimes\eta_{\lambda'}))_{\lambda,\lambda'}$$

and by the case previously considered, this converges to $(1_\zeta,\rho(u)y)$ when $\lambda\to\infty$. Since $(1_\zeta,\rho(u)y)=(u1_\zeta,y)$, the proposition is proved.

26.3. A Characterization of $\dot{\mathbf{B}}\sqcup(-\dot{\mathbf{B}})$

In the following result there is no assumption on the root datum.

Theorem 26.3.1. (a) *Let $b,b',b_1,b_1'\in\mathbf{B}$ and let $\zeta,\zeta_1\in X$. We have*

$$(b\diamondsuit_\zeta b',b_1\diamondsuit_{\zeta_1}b_1')=\delta_{b,b_1}\delta_{b',b_1'}\delta_{\zeta,\zeta_1}\quad\text{mod }v^{-1}\mathbf{A}.$$

In particular, the canonical basis $\dot{\mathbf{B}}$ of $\dot{\mathbf{U}}$ is almost orthonormal for $(,)$.

(b) *Let $\beta\in\dot{\mathbf{U}}$. We have $\beta\in\dot{\mathbf{B}}\sqcup(-\dot{\mathbf{B}})$ if and only if β satisfies the following three conditions: $\beta\in{}_{\mathcal{A}}\dot{\mathbf{U}}$, $\bar\beta=\beta$ and $(\beta,\beta)\in 1+v^{-1}\mathbf{A}$.*

Note that (a) is trivial when $\zeta\neq\zeta_1$. Hence we may assume that $\zeta=\zeta_1$. In this case, using the definitions we are immediately reduced to the case where the root datum is simply connected. Then 26.2.3 is applicable. Hence, using the definition, we see that it is enough to prove that

(c) $((b\diamondsuit b')_{\lambda,\lambda'},(b_1\diamondsuit b_1')_{\lambda,\lambda'})_{\lambda,\lambda'}=\delta_{b,b_1}\delta_{b',b_1'}\ \text{mod }v^{-1}\mathbf{A}$

for any $\lambda,\lambda'\in X^+$ such that $\lambda'-\lambda=\zeta$ and such that $b\in\mathbf{B}(\lambda),b'\in\mathbf{B}(\lambda')$.

Since

$$(b^+\xi_{-\lambda}\otimes b'^-\eta_\lambda,b_1^+\xi_{-\lambda}\otimes b_1'^-\eta_\lambda)_{\lambda,\lambda'}=(b^-\eta_\lambda,b_1^-\eta_\lambda)_\lambda(b'^-\eta_{\lambda'},b_1'^-\eta_{\lambda'})_{\lambda'}$$
$$\in(\delta_{b,b_1}+v^{-1}\mathbf{A})(\delta_{b',b_1'}+v^{-1}\mathbf{A})=\delta_{b,b_1}\delta_{b',b_1'}+v^{-1}\mathbf{A},$$

we see that (c) follows from Theorem 24.3.3(b).

We prove (b). If $\beta \in \pm\dot{\mathbf{B}}$ then, by (a) and 25.2.1, it satisfies the three conditions listed. Conversely, if $\beta \in \dot{\mathbf{U}}$ satisfies the three conditions in (b) then, using (a) and Lemma 14.2.2(b), we see that there exists $\beta' \in \dot{\mathbf{B}}$ such that $\beta - (\pm\beta')$ is a linear combination of elements in $\dot{\mathbf{B}}$ with coefficients in $v^{-1}\mathbf{Z}[v^{-1}]$. These coefficients are necessarily 0, since $\beta - (\pm\beta')$ is fixed by $^- : \dot{\mathbf{U}} \to \dot{\mathbf{U}}$. The theorem is proved.

Corollary 26.3.2. *If $\beta \in \dot{\mathbf{B}}$, then $\sigma(\beta) \in \pm\dot{\mathbf{B}}$ and $\omega(\beta) \in \pm\dot{\mathbf{B}}$.*

σ and ω commute with $^- : \dot{\mathbf{U}} \to \dot{\mathbf{U}}$, preserve the lattice ${}_{\mathcal{A}}\dot{\mathbf{U}}$ and preserve the inner product $(,)$ (see 26.1.4, 26.1.6). Hence the corollary follows from Theorem 26.3.1(b).