CHAPTER 25

The Canonical Basis $\dot{\mathbf{B}}$ of $\dot{\mathbf{U}}$

25.1. Stability Properties

25.1.1. In this section, the root datum is assumed to be Y-regular.

Proposition 25.1.2. *Let λ, λ' be dominant elements of X. We write $\eta = \eta_\lambda, \eta' = \eta_{\lambda'}, \eta'' = \eta_{\lambda+\lambda'}$.*

(a) *There is a unique homomorphism of $\mathbf{U}$-modules $\chi : \Lambda_{\lambda+\lambda'} \to \Lambda_\lambda \otimes \Lambda_{\lambda'}$ such that $\chi(\eta'') = \eta \otimes \eta'$.*

(b) *Let $b \in \mathbf{B}(\lambda + \lambda')$. We have $\chi(b^- \eta'') = \sum_{b_1, b_2} f(b; b_1, b_2) b_1^- \eta \otimes b_2^- \eta'$, sum over $b_1 \in \mathbf{B}(\lambda), b_2 \in \mathbf{B}(\lambda')$, with $f(b; b_1, b_2) \in \mathbf{Z}[v^{-1}]$.*

(c) *If $b^- \eta' \neq 0$, then $f(b; 1, b) = 1$ and $f(b; 1, b_2) = 0$ for any $b_2 \neq b$. If $b^- \eta' = 0$, then $f(b; 1, b_2) = 0$ for any b_2.*

The vector $\eta \otimes \eta' \in \Lambda_\lambda \otimes \Lambda_{\lambda'}$ satisfies $E_i(\eta \otimes \eta') = 0$, $K_\mu(\eta \otimes \eta') = v^{\langle \mu, \lambda \rangle + \langle \mu, \lambda' \rangle}$. This implies (a) (by 3.5.8). By the definition of comultiplication in $\mathbf{U}$, we can write $\chi(b^- \eta'') = \sum_{b_1, b_2} f(b; b_1, b_2) b_1^- \eta \otimes b_2^- \eta'$, sum over $b_1 \in \mathbf{B}(\lambda), b_2 \in \mathbf{B}(\lambda')$, with $f(b; b_1, b_2) \in \mathbf{Q}(v)$ satisfying $f(b; 1, b_2) = 1$ if $b = b_2$ and $f(b; 1, b_2) = 0$ if $b \neq b_2$. This proves (c).

By 23.2.3, we have $f(b; b_1, b_2) \in \mathcal{A}$ for all b_1, b_2. Hence to prove (b), it suffices to show that $f(b; b_1, b_2) \in \mathbf{A}$ for all b_1, b_2. We have a commutative diagram

$$\begin{array}{ccc} \mathbf{f} & \xrightarrow{\Xi} & \mathbf{f} \otimes \Lambda_{\lambda'} \\ \downarrow & & \downarrow \\ \Lambda_{\lambda+\lambda'} & \xrightarrow{\chi} & \Lambda_\lambda \otimes \Lambda_{\lambda'} \end{array}$$

where Ξ is as in 18.1.4, the left vertical map is given by $x \mapsto x^- \eta''$ and the right vertical map is given by $x \otimes y \to (x^- \eta') \otimes y$. The commutativity of the diagram follows from the definitions. Now our assertion on $f(b; b_1, b_2)$ follows from 22.1.2.

Proposition 25.1.3. *Let λ, λ' be dominant elements of X. We write $\xi = \xi_{-\lambda}, \xi' = \xi_{-\lambda'}, \xi'' = \xi_{-\lambda-\lambda'}$.*

(a) *There is a unique homomorphism of $\mathbf{U}$-modules $\chi' : {}^\omega\Lambda_{\lambda+\lambda'} \to {}^\omega\Lambda_{\lambda'} \otimes {}^\omega\Lambda_\lambda$ such that $\chi'(\xi'') = \xi' \otimes \xi$.*

(b) *Let $b \in \mathbf{B}(\lambda+\lambda')$. We have $\chi'(b^+\xi'') = \sum_{b_1,b_2} f(b; b_1, b_2) b_2^+\xi' \otimes b_1^+\xi$, sum over $b_1 \in \mathbf{B}(\lambda), b_2 \in \mathbf{B}(\lambda')$, with $f(b; b_1, b_2) \in \mathbf{Z}[v^{-1}]$. If $b^+\xi' \neq 0$, then $f(b; 1, b) = 1$ and $f(b; 1, b_2) = 0$ for any $b_2 \neq b$. If $b^+\xi' = 0$, then $f(b; 1, b_2) = 0$ for any b_2.*

We have a commutative diagram

$$\begin{array}{ccc} \mathbf{U} & \xrightarrow{\Delta} & \mathbf{U} \otimes \mathbf{U} \\ \omega \downarrow & & \downarrow \omega\otimes\omega \\ \mathbf{U} & \xrightarrow{{}^t\Delta} & \mathbf{U} \otimes \mathbf{U} \end{array}$$

(${}^t\Delta$ as in 3.3.4.) Indeed, both compositions in the diagram are algebra homomorphisms; to check that they are equal, it suffices to check that they agree on the generators E_i, F_i, K_μ and that is immediate. Using this commutative diagram, we see immediately that the proposition follows from the previous proposition.

Proposition 25.1.4. *Let $\eta \in \Lambda_\lambda, \xi \in {}^\omega\Lambda_\lambda$ be as above.*

(a) *There is a unique homomorphism of $\mathbf{U}$-modules $\delta_\lambda : {}^\omega\Lambda_\lambda \otimes \Lambda_\lambda \to \mathbf{Q}(v)$, where $\mathbf{Q}(v)$ is a $\mathbf{U}$-module via the co-unit $\mathbf{U} \to \mathbf{Q}(v)$, such that $\delta_\lambda(\xi \otimes \eta) = 1$.*

(b) *Let $b, b' \in \mathbf{B}(\lambda)$. Then $\delta_\lambda(b^+\xi \otimes b'^-\eta)$ is equal to 1 if $b = b' = 1$ and is in $v^{-1}\mathbf{Z}[v^{-1}]$ otherwise.*

The following statement is equivalent to (a). There is a unique bilinear pairing $[,] : \Lambda_\lambda \times \Lambda_\lambda \to \mathbf{Q}(v)$ such that

$$[\eta, \eta] = 1$$

and

$$[F_i x, y] = -[\tilde{K}_{-i} x, E_i y], [x, F_i y] = -[E_i x, \tilde{K}_{-i} y], [K_{-\mu} x, K_\mu y] = [x, y]$$

for all $x, y \in \Lambda$, all $i \in I$ and all $\mu \in Y$. We then have $\delta_\lambda(x \otimes y) = [x, y]$.

This is also equivalent to the following statement. There is a unique bilinear pairing $[,] : \Lambda_\lambda \times \Lambda_\lambda \to \mathbf{Q}(v)$ such that

$$[\eta, \eta] = 1$$

and

$$[ux, y] = [x, \tilde{\rho}(u)y]$$

for all $x, y \in \Lambda$, and all $u \in \mathbf{U}$, where $\tilde{\rho} : \mathbf{U} \to \mathbf{U}^{opp}$ is the algebra isomorphism given by the composition $S\omega$ (S is the antipode).

This is proved exactly as in 19.1.2. It follows from the definition that $[x, y] = 0$ if $x \in (\Lambda)_\nu, y \in (\Lambda)_{\nu'}$ and $\nu \neq \nu'$. Let $(,)$ be as in 19.1.2. We show by induction on $\operatorname{tr} \nu \geq 0$, where $\nu \in \mathbf{N}[I]$, that

(c) $$[x, y] = (-1)^{\operatorname{tr} \nu} v_{-|\nu|}(x, y)$$

for all $x, y \in (\Lambda)_\nu$. This is obvious for $\nu = 0$. We assume that $\operatorname{tr} \nu \geq 1$. We can assume that $x = F_i x'$ for some i such that $\nu_i > 0$ and some $x' \in (\Lambda)_{\nu - i}$. Then

$$[x, y] = [F_i x', y] = -[\tilde{K}_{-i} x', E_i y] = -[x', \tilde{K}_{-i} E_i y]$$

and

$$(x, y) = (F_i x, y) = v_i(x', \tilde{K}_{-i} E_i y).$$

By the induction hypothesis, we have

$$-[x', \tilde{K}_{-i} E_i y] = (-1)^{\operatorname{tr} \nu} v_{-|\nu - i|}(x', \tilde{K}_{-i} E_i y);$$

hence $[x, y] = (-1)^{\operatorname{tr} \nu} v_{-|\nu|}(x, y)$, which completes the induction.

Now let b, b' be as in (b). We must show that $[b^- \eta, b'^- \eta]$ is in $v^{-1}\mathbf{Z}[v^{-1}]$ unless $b = b' = 1$. We may assume that there exists ν such that $b^- \eta, b'^- \eta$ both belong to $(\Lambda)_\nu$. The result then follows from (c) since, by 19.3.3, we have $(b^- \eta, b'^- \eta) \in \mathbf{Z}[v^{-1}]$.

25.1.5. Let $\lambda, \lambda', \lambda''$ be dominant elements of X. We define a homomorphism of $\mathbf{U}$-modules

$$t : {}^\omega\Lambda_{\lambda+\lambda'} \otimes \Lambda_{\lambda'+\lambda''} \to {}^\omega\Lambda_{\lambda'} \otimes \Lambda_{\lambda''}$$

as the composition of

$$\chi' \otimes \chi : {}^\omega\Lambda_{\lambda+\lambda'} \otimes \Lambda_{\lambda'+\lambda''} \to {}^\omega\Lambda_\lambda \otimes {}^\omega\Lambda_{\lambda'} \otimes \Lambda_{\lambda'} \otimes \Lambda_{\lambda''},$$

where χ', χ are as in 25.1.3, 25.1.2, with

$$1 \otimes \delta_{\lambda'} \otimes 1 : {}^\omega\Lambda_\lambda \otimes {}^\omega\Lambda_{\lambda'} \otimes \Lambda_{\lambda'} \otimes \Lambda_{\lambda''} \to {}^\omega\Lambda_\lambda \otimes \Lambda_{\lambda''}.$$

Lemma 25.1.6.

(a) *Let* $b \in \mathbf{B}(\lambda)$, $b'' \in \mathbf{B}(\lambda'')$. *We have*

$$t(b^+\xi_{-\lambda-\lambda'} \otimes b''^-\eta_{\lambda'+\lambda''}) = b^+\xi_{-\lambda} \otimes b''^-\eta_{\lambda''} \mod v^{-1}\mathcal{L}_{\lambda,\lambda''}$$

(notation of 24.3.1*).*

(b) *Let* $b \in \mathbf{B}(\lambda + \lambda')$, $b'' \in \mathbf{B}(\lambda' + \lambda'')$. *Assume that either* $b \notin \mathbf{B}(\lambda)$, *or* $b'' \notin \mathbf{B}(\lambda'')$. *We have* $t(b^+\xi_{-\lambda-\lambda'} \otimes b''^-\eta_{\lambda'+\lambda''}) = 0 \mod v^{-1}\mathcal{L}_{\lambda,\lambda''}$.

(c) t *is surjective.*

In this proof we shall use the symbol $\equiv$ to denote congruences modulo v^{-1} times a $\mathbf{Z}[v^{-1}]$-submodule spanned by the natural basis.

Using 25.1.2, 25.1.3, 25.1.4, we see that if b, b'' are as in (a), we have

$$\begin{aligned} t(b^+\xi_{-\lambda-\lambda'} \otimes b''^-\eta_{\lambda'+\lambda''}) &\equiv (1 \otimes \delta_{\lambda'} \otimes 1)(b^+\xi_{-\lambda} \otimes \xi_{-\lambda'} \otimes \eta_{\lambda'} \otimes b''^-\eta_{\lambda''}) \\ &= b^+\xi_{-\lambda} \otimes b''^-\eta_{\lambda''}. \end{aligned}$$

Using again 25.1.2, 25.1.3, 25.1.4, we see that if b, b'' are as in (b), we have

$$t(b^+\xi_{-\lambda-\lambda'} \otimes b''^-\eta_{\lambda'+\lambda''}) \equiv 0.$$

Now (c) follows from the fact that $\xi_{-\lambda} \otimes \eta_{\lambda'}$ is in the image of t and it generates the $\mathbf{U}$-module ${}^{\omega}\Lambda_{\lambda} \otimes \Lambda_{\lambda'}$ (see 23.3.6).

Using the definition 24.3.3 of the elements $(b\diamondsuit b')_{\lambda,\lambda'}$, we can reformulate the previous lemma as follows.

Lemma 25.1.7. (a) *Let* $b \in \mathbf{B}(\lambda)$, $b'' \in \mathbf{B}(\lambda'')$. *We have*

$$t(b\diamondsuit b'')_{\lambda+\lambda',\lambda'+\lambda''} = (b\diamondsuit b'')_{\lambda,\lambda''} \mod v^{-1}\mathcal{L}_{\lambda,\lambda''}.$$

(b) *Let* $b \in \mathbf{B}(\lambda + \lambda')$, $b'' \in \mathbf{B}(\lambda' + \lambda'')$. *Assume that either* $b \notin \mathbf{B}(\lambda)$, *or* $b'' \notin \mathbf{B}(\lambda'')$. *We have*

$$t(b\diamondsuit b'')_{\lambda+\lambda',\lambda'+\lambda''} = 0 \mod v^{-1}\mathcal{L}_{\lambda,\lambda''}.$$

25.1.8. In the following result we show that the maps

$$\Psi : {}^{\omega}\Lambda_{\lambda+\lambda'} \otimes \Lambda_{\lambda'+\lambda''} \to {}^{\omega}\Lambda_{\lambda+\lambda'} \otimes \Lambda_{\lambda'+\lambda''}$$

and

$$\Psi : {}^{\omega}\Lambda_{\lambda} \otimes \Lambda_{\lambda''} \to {}^{\omega}\Lambda_{\lambda} \otimes \Lambda_{\lambda''},$$

defined as in 24.3.2, are compatible with t.

Lemma 25.1.9. *We have* $t\Psi = \Psi t$.

We write ξ, η instead of $\xi_{-\lambda-\lambda'}, \eta_{\lambda'+\lambda''}$.

Since any element of ${}^{\omega}\Lambda_{\lambda+\lambda'} \otimes \Lambda_{\lambda'+\lambda''}$ is of the form $u(\xi \otimes \eta)$ for some $u \in \dot{\mathbf{U}}$ (see 23.3.6), it is enough to check that

$$t\Theta(\overline{u(\xi \otimes \eta)}) = \Theta\overline{t(u(\xi \otimes \eta))}$$

for all $u \in \dot{\mathbf{U}}$. Using the definition of Θ and its property 24.1.3(a), we have

$$t\Theta(\overline{u(\xi \otimes \eta)}) = t\bar{u}(\Theta(\xi \otimes \eta)) = t\bar{u}(\xi \otimes \eta) = \bar{u}t(\xi \otimes \eta) = \bar{u}(\xi_{-\lambda} \otimes \eta_{\lambda''})$$

and

$$\Theta\overline{t(u(\xi \otimes \eta))} = \Theta\overline{u(t(\xi \otimes \eta))} = \bar{u}\Theta(\xi_{-\lambda} \otimes \eta_{\lambda''}) = \bar{u}(\xi_{-\lambda} \otimes \eta_{\lambda''}).$$

The lemma is proved.

Proposition 25.1.10. (a) *Let* $b \in \mathbf{B}(\lambda)$, $b'' \in \mathbf{B}(\lambda'')$. *We have*

$$t(b\diamondsuit b'')_{\lambda+\lambda',\lambda'+\lambda''} = (b\diamondsuit b'')_{\lambda,\lambda''}.$$

(b) *Let* $b \in \mathbf{B}(\lambda + \lambda')$, $b'' \in \mathbf{B}(\lambda' + \lambda'')$. *Assume that either* $b \notin \mathbf{B}(\lambda)$, *or* $b'' \notin \mathbf{B}(\lambda'')$. *We have*

$$t(b\diamondsuit b'')_{\lambda+\lambda',\lambda'+\lambda''} = 0.$$

The difference of the two sides of the equality in (a) is in $v^{-1}\mathcal{L}_{\lambda,\lambda''}$ (by 25.1.7) and is fixed by $\Psi : {}^{\omega}\Lambda_{\lambda} \otimes \Lambda_{\lambda''} \to {}^{\omega}\Lambda_{\lambda} \otimes \Lambda_{\lambda''}$, using the definitions and Lemma 25.1.9; hence that difference is zero, by 24.3.3(d). Thus the equality in (a) holds. Exactly the same proof shows (b).

25.2. Definition of the Basis $\dot{\mathbf{B}}$ of $\dot{\mathbf{U}}$

Theorem 25.2.1. *Assume that the root datum is* Y*-regular. Let* $\zeta \in X$ *and let* $b, b'' \in \mathbf{B}$.

(a) *There is a unique element* $u = b\diamondsuit_{\zeta} b'' \in {}_{\mathcal{A}}\dot{\mathbf{U}}1_{\zeta}$ *such that*

$$u(\xi_{-\lambda} \otimes \eta_{\lambda''}) = (b\diamondsuit b'')_{\lambda,\lambda''}$$

for any $\lambda, \lambda'' \in X^+$ *such that* $b \in \mathbf{B}(\lambda), b'' \in \mathbf{B}(\lambda'')$ *and* $\lambda'' - \lambda = \zeta$.

(b) *If* $\lambda, \lambda'' \in X^+$ *are such that* $\lambda'' - \lambda = \zeta$ *and either* $b \notin \mathbf{B}(\lambda)$ *or* $b'' \notin \mathbf{B}(\lambda'')$, *then* $u(\xi_{-\lambda} \otimes \eta_{\lambda''}) = 0$ *(*u *as in (a)).*

(c) *The element u in (a) satisfies $\bar{u} = u$.*

(d) *The elements $b\diamondsuit_\zeta b''$, for various ζ, b, b'' as above, form a $\mathbf{Q}(v)$-basis of $\dot{\mathbf{U}}$ and an $\mathcal{A}$-basis of ${}_\mathcal{A}\dot{\mathbf{U}}$.*

Since the root datum is Y-regular, we can find $\lambda, \lambda'' \in X^+$ such that $b \in \mathbf{B}(\lambda), b'' \in \mathbf{B}(\lambda'')$ and $\lambda'' - \lambda = \zeta$.

For any integers N_1, N_2, let $P(N_1, N_2)$ be the $\mathcal{A}$-submodule of ${}_\mathcal{A}\dot{\mathbf{U}}$ spanned by the elements $b_1^+ b_2^- 1_\zeta$ where b_1, b_2 run through the set of pairs of elements of $\mathbf{B}$ such that $\text{tr } |b_1| \le N_1$, $\text{tr } |b_2| \le N_2$ and $|b_1| - |b_2| = |b| - |b''|$.

By arguments such as in 23.3.1 or 23.3.2, we see that any element of ${}^\omega M_\lambda \otimes M_{\lambda''}$, or ${}^\omega\Lambda_\lambda \otimes \Lambda_{\lambda''}$, of the form $\beta^+\xi_{-\lambda} \otimes \beta'^-\eta_{\lambda''}$, with $\beta, \beta' \in \mathbf{B}$, is equal to $u_1(\xi_{-\lambda} \otimes \eta_{\lambda''})$ for some $u_1 \in P(\text{ tr } |\beta|, \text{ tr } |\beta'|)$; moreover, u_1 can be taken to be equal to $\beta^+\beta'^- 1_\zeta$ plus an element in $P(\text{ tr } |\beta| - 1, \text{ tr } |\beta'| - 1)$. From this we deduce that

(e) $(b\diamondsuit b'')_{\lambda,\lambda''} \in {}^\omega\Lambda_\lambda \otimes \Lambda_{\lambda''}$ is of the form $u(\xi_{-\lambda} \otimes \eta_{\lambda''})$ for some $u \in P(\text{ tr } |b|, \text{ tr } |b''|)$; moreover, u can be taken to be equal to $b^+b''^- 1_\zeta$ plus an element in $P(\text{ tr } |b| - 1, \text{ tr } |b''| - 1)$.

Assume that u is such an element and that u' is another element with the same properties as u. Then $(u - u')(\xi_{-\lambda} \otimes \eta_{\lambda''}) = 0$; hence, by 23.3.8, we have

$$u - u' \in \sum_{i,n > \langle i, \lambda'' \rangle} {}_\mathcal{A}\dot{\mathbf{U}} F_i^{(n)} 1_\zeta + \sum_{i,n > \langle i, \lambda \rangle} {}_\mathcal{A}\dot{\mathbf{U}} E_i^{(n)} 1_\zeta.$$

Since $u - u' \in P(\text{ tr } |b|, \text{ tr } |b''|)$ we deduce that, if $\langle i, \lambda \rangle$ and $\langle i, \lambda'' \rangle$ are large enough (for all i), then we must have $u = u'$. Thus, for such λ, λ'' the element u above is uniquely determined. We denote it by $u_{\lambda,\lambda''}$.

Assume now that $\lambda, \lambda'' \in X^+$ satisfy $b \in \mathbf{B}(\lambda), b'' \in \mathbf{B}(\lambda'')$ and $\lambda'' - \lambda = \zeta$. Let $\lambda' \in X^+$ be such that $\langle i, \lambda' \rangle$ is large enough for all i, so that $u' = u_{\lambda+\lambda', \lambda'+\lambda''}$ is defined.

We show that $u'(\xi_{-\lambda} \otimes \eta_{\lambda''}) = (b\diamondsuit b'')_{\lambda,\lambda''}$. Indeed, if t is as in 25.1.5, we have

$$\begin{aligned} u'(\xi_{-\lambda} \otimes \eta_{\lambda''}) &= u'(t(\xi_{-\lambda-\lambda'} \otimes \eta_{\lambda'+\lambda''})) = t(u'(\xi_{-\lambda-\lambda'} \otimes \eta_{\lambda'+\lambda''})) \\ &= t((b\diamondsuit b'')_{\lambda+\lambda',\lambda'+\lambda''}) = (b\diamondsuit b'')_{\lambda,\lambda''} \end{aligned}$$

where the last equality follows from 25.1.10. It follows that $u_{\lambda,\lambda''}$ is independent of λ, λ'', provided that $\langle i, \lambda \rangle$ and $\langle i, \lambda'' \rangle$ are large enough (for all i); hence we can denote it as u, without specifying λ, λ''. It also follows that u satisfies the requirements of (a). This proves the existence part of (a). The previous proof shows also uniqueness. Thus (a) is proved.

Now let λ, λ'' be as in (b). Let $\lambda' \in X^+$ be such that $\langle i, \lambda' \rangle$ is large enough for all i, so that $u_{\lambda+\lambda',\lambda'+\lambda''}$ is defined (hence it is u). We have

$$\begin{aligned} u(\xi_{-\lambda} \otimes \eta_{\lambda''}) &= u(t(\xi_{-\lambda-\lambda'} \otimes \eta_{\lambda'+\lambda''})) \\ &= t(u(\xi_{-\lambda-\lambda'} \otimes \eta_{\lambda'+\lambda''})) \\ &= t((b\diamondsuit b'')_{\lambda+\lambda',\lambda'+\lambda''}) = 0, \end{aligned}$$

where the last equality follows from 25.1.10. This proves (b).

We prove (c). Let u, λ, λ'' be as in (a). We have

$$\begin{aligned} \bar{u}(\xi_{-\lambda} \otimes \eta_{\lambda''}) &= \bar{u}\Theta(\xi_{-\lambda} \otimes \eta_{\lambda''}) \\ &= \Theta\overline{u(\xi_{-\lambda} \otimes \eta_{\lambda''})} \\ &= \Theta\overline{(b\diamondsuit b'')_{\lambda,\lambda''}} \\ &= (b\diamondsuit b'')_{\lambda,\lambda''}. \end{aligned}$$

Thus $\bar{u}$ satisfies the defining property of u. By uniqueness, we have $\bar{u} = u$. This proves (c).

We prove (d). From (e) we see that, for fixed ζ, we have

$$b\diamondsuit_\zeta b'' = b^+ b''^- 1_\zeta \mod P(\text{ tr } |b| - 1, \text{ tr } |b''| - 1).$$

Since the elements $b^+ b''^- 1_\zeta$ form an $\mathcal{A}$-basis of ${}_\mathcal{A}\dot{\mathbf{U}}$, we see that (d) follows. The theorem is proved.

25.2.2. We now drop the assumption that the root datum $(Y, X, \dots)$ is Y-regular. Assume that we are given $\zeta \in X^+$. Let $(Y', X', \dots)$ be the simply connected root datum of type $(I, \cdot)$ and let $f : Y' \to Y, g : X \to X'$ be the unique morphism of root data. Let $\dot{\mathbf{U}}'$ be the algebra, defined like $\dot{\mathbf{U}}$, in terms of $(Y', X', \dots)$. Let $\zeta' = g(\zeta)$. By 23.2.5, we have a natural isomorphism

(a) $\dot{\mathbf{U}}' 1_{\zeta'} \cong \dot{\mathbf{U}} 1_\zeta$,

defined by $u^+ u'^- 1_{\zeta'} \mapsto u^+ u'^- 1_\zeta$ for all $u, u' \in \mathbf{f}$. For each $b, b'' \in \mathbf{B}$, we denote by $b\diamondsuit_\zeta b''$ the element of $\dot{\mathbf{U}} 1_\zeta$ corresponding under (a) to $b\diamondsuit_{\zeta'} b'' \in \dot{\mathbf{U}}' 1_{\zeta'}$ (which is defined by the previous theorem.)

Corollary 25.2.3. *The elements $b\diamondsuit_\zeta b''$ for various $b, b'' \in \mathbf{B}$ and various $\zeta \in X$ form an $\mathcal{A}$-basis of ${}_\mathcal{A}\dot{\mathbf{U}}$ and a $\mathbf{Q}(v)$-basis of $\dot{\mathbf{U}}$. They are all fixed by the involution* $^- : \dot{\mathbf{U}} \to \dot{\mathbf{U}}$.

25.2.4. Remark. The basis of $\dot{\mathbf{U}}$ just defined is called the *canonical basis* of $\dot{\mathbf{U}}$. We denote it by $\dot{\mathbf{B}}$. In the case where the root datum $(Y, X, \dots)$ is Y-regular, this canonical basis coincides with the one defined in 25.2.1. This follows immediately from definitions, using 23.2.5 and 23.3.4.

From the proof we see that any element of $\dot{\mathbf{B}}$ is contained in one of the summands in the direct sum decomposition 23.1.2 of $\dot{\mathbf{U}}$.

Theorem 25.2.5. *Let $\zeta \in X$ and let a, a' be as in 23.3.3. Let $P(\zeta, a, a')$, ${}_{\mathcal{A}}P(\zeta, a, a')$ be as in 23.3.3. Then $\dot{\mathbf{B}} \cap P(\zeta, a, a')$ is an $\mathcal{A}$-basis of ${}_{\mathcal{A}}P(\zeta, a, a')$ and a $\mathbf{Q}(v)$-basis of $P(\zeta, a, a')$.*

Using the definitions and 23.3.4, we are reduced to the case where the root datum is simply connected. In that case, the result follows immediately from Theorem 25.2.1.

We now show that $\dot{\mathbf{B}}$ is a generalization of $\mathbf{B}$.

Proposition 25.2.6. *Let $b \in \mathbf{B}$ and let $\zeta \in X$. Then $b^- 1_\zeta \in \dot{\mathbf{B}}$ and $b^+ 1_\zeta \in \dot{\mathbf{B}}$.*

We can assume that the root datum is simply connected. Choose $\lambda, \lambda' \in X^+$ such that $\lambda' - \lambda = \zeta$ and such that $b \in \mathbf{B}(\lambda')$. We have $b^- 1_\zeta(\xi_{-\lambda} \otimes \eta_{\lambda'}) = \xi_{-\lambda} \otimes b^- \eta_{\lambda'}$. Using the definitions, we see that $\xi_{-\lambda} \otimes b^- \eta_{\lambda'}$ satisfies the defining properties of $(1 \diamondsuit b)_{\lambda, \lambda'}$; hence it is equal to $(1 \diamondsuit b)_{\lambda, \lambda'}$. It follows that $b^- 1_\zeta = 1 \diamondsuit_\zeta b$. A similar argument shows that $b^+ 1_\zeta = b \diamondsuit_\zeta 1$. The proposition is proved.

25.3. Example: Rank 1

25.3.1. In this section we assume that $I = \{i\}$ and $X = Y = \mathbf{Z}$ with $i = 1 \in Y, i' = 2 \in X$. Consider the following elements of $\dot{\mathbf{U}}$:

(a) $E_i^{(a)} 1_{-n} F_i^{(b)} \quad (a, b, n \in \mathbf{N}, n \geq a + b)$

(b) $F_i^{(b)} 1_n E_i^{(a)} \quad (a, b, n \in \mathbf{N}, n \geq a + b)$.

Note that

(c) $E_i^{(a)} 1_{-n} F_i^{(b)} = F_i^{(b)} 1_n E_i^{(a)}$ for $n = a + b$.

Proposition 25.3.2. *The canonical basis of $\dot{\mathbf{U}}$ consists of the elements 25.3.1(a) and (b), with the identification 25.3.1(c). More precisely, if $n \geq a + b$, we have*

$$E_i^{(a)} 1_{-n} F_i^{(b)} = \theta_i^{(a)} \diamondsuit_{-n+2b} \theta_i^{(b)}$$

and

$$F_i^{(b)} 1_n E_i^{(a)} = \theta_i^{(a)} \diamondsuit_{n-2a} \theta_i^{(b)}.$$

We compute the image of the elements 25.3.1(a),(b) under the map $\dot{\mathbf{U}} \to {}^\omega\Lambda_p \otimes \Lambda_q$, with $p, q \geq 0$, given by $u \mapsto u(\xi_{-p} \otimes \eta_q)$. The image of the element 25.3.1(a) is zero unless $-n + 2b = q - p$, in which case it is

$$\begin{aligned}
&E_i^{(a)} F_i^{(b)} (\xi_{-p} \otimes \eta_q) = E_i^{(a)} (\xi_{-p} \otimes F_i^{(b)} \eta_q) \\
&= \sum_{a'+a''=a} v_i^{a'a''-a''p} E_i^{(a')} \xi_{-p} \otimes E_i^{(a'')} F_i^{(b)} \eta_q \\
&= \sum_{a'+a''=a} \sum_{t \geq 0} v_i^{a'a''-a''p} E_i^{(a')} \xi_{-p} \otimes \begin{bmatrix} a'' - b + q \\ t \end{bmatrix}_i F_i^{(b-t)} E_i^{(a''-t)} \eta_q \\
&= \sum_{a'+a''=a} v_i^{a'a''-a''p} \begin{bmatrix} a'' - b + q \\ a'' \end{bmatrix}_i E_i^{(a')} \xi_{-p} \otimes F_i^{(b-a'')} \eta_q \\
&= \sum_{s \geq 0; s \leq a, s \leq b} v_i^{s(a-s-p)} \begin{bmatrix} s - b + q \\ s \end{bmatrix}_i E_i^{(a-s)} \xi_{-p} \otimes F_i^{(b-s)} \eta_q.
\end{aligned}$$

This element is fixed by the involution Ψ of ${}^\omega\Lambda_p \otimes \Lambda_q$, since the element 25.3.1(a) is fixed by $^- : \dot{\mathbf{U}} \to \dot{\mathbf{U}}$. Using the definitions, we see that this element is $(\theta_i^{(a)} \diamondsuit \theta_i^{(b)})_{p,q}$. Hence the element 25.3.1(a) is $\theta_i^{(a)} \diamondsuit_{-n+2b} \theta_i^{(b)}$.

The image of the element 25.3.1(b) is zero unless $n - 2a = q - p$, in which case it is

$$\begin{aligned}
&F_i^{(b)} E_i^{(a)} (\xi_{-p} \otimes \eta_q) = F_i^{(b)} (E_i^{(a)} \xi_{-p} \otimes \eta_q) \\
&= \sum_{b'+b''=b} v_i^{b'b''-b'q} F_i^{(b')} E_i^{(a)} \xi_{-p} \otimes F_i^{(b'')} \eta_q \\
&= \sum_{b'+b''=b} \sum_{t \geq 0} v_i^{b'b''-b'q} \begin{bmatrix} -a + b' + p \\ t \end{bmatrix}_i E_i^{(a-t)} F_i^{(b'-t)} \xi_{-p} \otimes F_i^{(b'')} \eta_q \\
&= \sum_{s \geq 0; s \leq a, s \leq b} v_i^{s(b-s-q)} \begin{bmatrix} s - a + p \\ s \end{bmatrix}_i E_i^{(a-s)} \xi_{-p} \otimes F_i^{(b-s)} \eta_q.
\end{aligned}$$

As before, we see that this is equal to $(\theta_i^{(a)} \diamondsuit \theta_i^{(b)})_{p,q}$. The proposition follows.

25.4. Structure Constants

25.4.1. For any triplet $a, b, c \in \dot{\mathbf{B}}$, we define elements $m_{ab}^c \in \mathcal{A}$ by $ab = \sum_c m_{ab}^c c$ (ab is the product in $\dot{\mathbf{U}}$). We also define elements $\hat{m}_c^{ab} \in \mathcal{A}$ by the following requirement: for any $\lambda'_1, \lambda''_1, \lambda'_2, \lambda''_2 \in X$ and any $c \in \dot{\mathbf{B}} \cap ({}_{\lambda'_1+\lambda'_2}\mathbf{U}_{\lambda''_1+\lambda''_2})$ we have

$$\Delta_{\lambda'_1, \lambda''_1, \lambda'_2, \lambda''_2}(c) = \sum_{a,b} \hat{m}_c^{ab} a \otimes b;$$

in the last formula, $\Delta_{\lambda'_1,\lambda''_1,\lambda'_2,\lambda''_2}$ is as in 23.1.5; a runs over $\dot{\mathbf{B}} \cap ({}_{\lambda'_1}\mathbf{U}_{\lambda''_1})$; b runs over $\dot{\mathbf{B}} \cap ({}_{\lambda'_2}\mathbf{U}_{\lambda''_2})$. If $a \in \dot{\mathbf{B}} \cap ({}_{\lambda'_1}\mathbf{U}_{\lambda''_1}), b \in \dot{\mathbf{B}} \cap ({}_{\lambda'_2}\mathbf{U}_{\lambda''_2})$ and $c \in \dot{\mathbf{B}} \cap ({}_{\lambda'_3}\mathbf{U}_{\lambda''_3})$, and either $\lambda'_3 \neq \lambda'_1 + \lambda'_2$ or $\lambda''_3 \neq \lambda''_1 + \lambda''_2$, then $\hat{m}_c^{ab}$ is defined to be 0.

The elements $m_{ab}^c, \hat{m}_{ab}^c$ are called the *structure constants* of $\dot{\mathbf{U}}$. They satisfy the following identities, for all $a, b, d, e \in \dot{\mathbf{B}}$ and $\lambda \in X$:

(a) $\sum_c m_{ab}^c m_{cd}^e = \sum_c m_{ac}^e m_{bd}^c$;

(b) $\sum_c \hat{m}_c^{ab} \hat{m}_e^{cd} = \sum_c \hat{m}_e^{ac} \hat{m}_c^{bd}$;

(c) $\sum_c m_{ab}^c \hat{m}_c^{ed} = \sum_{a',b',c',d'} \hat{m}_a^{a'b'} \hat{m}_b^{c'd'} m_{a'c'}^e m_{b'd'}^d$.

(d) $\hat{m}_{1_\lambda}^{ab} = 1$ if $a = 1_{\lambda'}, b = 1_{\lambda''}, \lambda' + \lambda'' = \lambda$ and $\hat{m}_{1_\lambda}^{ab} = 0$ otherwise.

In each sum, all but finitely many terms are zero. The identity (a) expresses the associativity of multiplication in $\dot{\mathbf{U}}$; (b) is a consequence of the coassociativity of comultiplication in $\mathbf{U}$; (c),(d) are consequences of the fact that the comultiplication $\Delta : \mathbf{U} \to \mathbf{U} \otimes \mathbf{U}$ is an algebra homomorphism preserving 1.

Conjecture 25.4.2. *If the Cartan datum is symmetric, then the structure constants* $m_{ab}^c, \hat{m}_c^{ab}$ *are in* $\mathbf{N}[v, v^{-1}]$.

This would generalize the positivity theorem 14.4.13. For the proof an interpretation of $(\dot{\mathbf{U}}, \dot{\mathbf{B}})$ in terms of perverse sheaves, generalizing that of $(\mathbf{f}, \mathbf{B})$ will be required.