CHAPTER 24

Canonical Bases in Certain Tensor Products

24.1. Integrality Properties of the Quasi-$\mathcal{R}$-Matrix

24.1.1. Let $M, M' \in \mathcal{C}$ be such that either ${}^{\omega}M \in \mathcal{C}^{hi}$ or $M' \in \mathcal{C}^{hi}$ (see 3.4.7). We regard $M \otimes M'$ naturally as a $\mathbf{U} \otimes \mathbf{U}$-module and we define a linear map $\Theta : M \otimes M' \to M \otimes M'$ by $\Theta(m \otimes m') = \sum_{\nu} \Theta_{\nu}(m \otimes m')$ (notation of 4.1.2.) This is well-defined since only finitely many terms of the sum are non-zero.

Lemma 24.1.2. *Let M, M' be as above. We have*

(a) $\Delta(u)\Theta(m \otimes m') = \Theta(\bar{\Delta}(u)(m \otimes m'))$.

(b) *Assume that we are given $\mathbf{Q}$-linear maps ${}^{-} : M \to M$ and ${}^{-} : M' \to M'$ such that $\overline{um} = \bar{u}\bar{m}$ and $\overline{um'} = \bar{u}\bar{m}'$ for all $u \in \mathbf{U}, m \in M, m' \in M'$. Let ${}^{-} = {}^{-}\otimes{}^{-} : M \otimes M' \to M \otimes M'$. Then $\Delta(u)\Theta(m \otimes m') = \Theta(\overline{\Delta(\bar{u})(\bar{m} \otimes \bar{m}')})$ for any $m \in M, m' \in M'$ and any $u \in \mathbf{U}$.*

The set of u for which (a) holds is clearly a subalgebra of $\mathbf{U}$ containing all K_{μ}. Hence it suffices to check (a) in the special case where u is one of the algebra generators E_i, F_i. Applying both sides of the equalities 4.2.5(c),(d) (with large p) to $m \otimes m' \in M \otimes M'$, we obtain

$$(E_i \otimes 1 + \tilde{K}_i \otimes E_i)\Theta(m \otimes m') = \Theta(E_i \otimes 1 + \tilde{K}_{-i} \otimes E_i)(m \otimes m')$$

$$(1 \otimes F_i + F_i \otimes \tilde{K}_{-i})\Theta(m \otimes m') = \Theta(1 \otimes F + F_i \otimes \tilde{K}_i)(m \otimes m').$$

This proves (a). Now (b) is a consequence of (a). The lemma follows.

24.1.3. The following property is just a reformulation of the property in Lemma 24.1.2(b). Let M, M' be as in that lemma. Then for any $m \in M, m' \in M', u \in \dot{\mathbf{U}}$ we have

(a) $$u\Theta(m \otimes m') = \Theta(\overline{\bar{u}(\bar{m} \otimes \bar{m}')}).$$

Proposition 24.1.4. *Let $\lambda, \lambda' \in X$. Consider the Verma modules $M_\lambda, M_{\lambda'}$. Note that $M = {}^\omega M_\lambda \in \mathcal{C}$ and $M' = M_{\lambda'} \in \mathcal{C}^{hi}$; hence*

$$\Theta : M \otimes M' \to M \otimes M'$$

is well-defined. Then Θ leaves stable the $\mathcal{A}$-submodule ${}^\omega_{\mathcal{A}}M_\lambda \otimes_{\mathcal{A}} ({}_{\mathcal{A}}M_{\lambda'})$.

Since the ambient space of M and M' is $\mathbf{f}$, we may regard $^- : \mathbf{f} \to \mathbf{f}$ as maps $^- : M \to M, ^- : M' \to M'$. Using the definition of Verma modules, we may identify $M' = \mathbf{U}/\sum_i \mathbf{U}E_i + \sum_\mu \mathbf{U}(K_\mu - v^{\langle \mu, \lambda' \rangle}1)$ as a $\mathbf{U}$-module so that $^- : M' \to M'$ is induced by $^- : \mathbf{U} \to \mathbf{U}$. It follows that $\overline{um'} = \bar{u}\bar{m}'$ for all $u \in \mathbf{U}$ and $m' \in M'$. Similarly, we have $\overline{um} = \bar{u}\bar{m}$ for all $u \in \mathbf{U}$ and $m \in M$.

As in Lemma 24.1.2, we set $^- = {}^- \otimes {}^- : M \otimes M' \to M \otimes M'$ and we have

$$u\Theta(m \otimes m') = \Theta(\overline{\bar{u}(\bar{m} \otimes \bar{m}')})$$

for all $u \in \dot{\mathbf{U}}, m \in M, m' \in M'$. In particular, taking $m = 1 = \bar{1}, m' = 1 = \bar{1}$, we obtain

(a) $u(1 \otimes 1) = \Theta(\overline{\bar{u}(1 \otimes 1)})$ for all $u \in \dot{\mathbf{U}}$,

since $1 = \bar{1}, 1 = \bar{1}$, and $\Theta(1 \otimes 1) = 1 \otimes 1$. Let $x \in {}^\omega_{\mathcal{A}}M_\lambda \otimes_{\mathcal{A}} ({}_{\mathcal{A}}M_{\lambda'})$. Then $x = \bar{x}'$ where $x' \in {}^\omega_{\mathcal{A}}M_\lambda \otimes_{\mathcal{A}} ({}_{\mathcal{A}}M_{\lambda'})$, since the involution $^- \otimes {}^- : \mathbf{f} \to \mathbf{f}$ leaves ${}_{\mathcal{A}}\mathbf{f} \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\mathbf{f})$ stable.

With the notation of 23.3.2(a), we have $x' = \pi'(u')$ for some $u' \in {}_{\mathcal{A}}\dot{\mathbf{U}}1_{\lambda'-\lambda}$. Since ${}_{\mathcal{A}}\dot{\mathbf{U}}1_{\lambda'-\lambda}$ is stable under the involution $^- : \dot{\mathbf{U}} \to \dot{\mathbf{U}}$, we have $u' = \bar{u}$ for some $u \in {}_{\mathcal{A}}\dot{\mathbf{U}}1_{\lambda'-\lambda}$. Hence $x = \bar{x}' = \overline{\bar{u}(1 \otimes 1)}$. Using (a), we have $\Theta(x) = \Theta(\overline{\bar{u}(1 \otimes 1)}) = u(1 \otimes 1) = \pi'(u)$. Using again 23.3.2(a), we have $\pi'(u) \in {}^\omega_{\mathcal{A}}M_\lambda \otimes_{\mathcal{A}} ({}_{\mathcal{A}}M_{\lambda'})$; hence $\Theta(x) \in {}^\omega_{\mathcal{A}}M_\lambda \otimes_{\mathcal{A}} ({}_{\mathcal{A}}M_{\lambda'})$. The proposition is proved.

Corollary 24.1.5. *Assume that the root datum is Y-regular. Let $\lambda, \lambda' \in X^+$. The map $\Theta : {}^\omega\Lambda_\lambda \otimes \Lambda_{\lambda'} \to {}^\omega\Lambda_\lambda \otimes \Lambda_{\lambda'}$ leaves stable the $\mathcal{A}$-submodule ${}^\omega_{\mathcal{A}}\Lambda_\lambda \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\Lambda_{\lambda'})$.*

This follows immediately from the previous proposition since ${}^\omega_{\mathcal{A}}\Lambda_\lambda \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\Lambda_{\lambda'})$ is the image of ${}^\omega_{\mathcal{A}}M_\lambda \otimes_{\mathcal{A}} ({}_{\mathcal{A}}M_{\lambda'})$ under the natural map ${}^\omega M_\lambda \otimes M_{\lambda'} \to {}^\omega\Lambda_\lambda \otimes \Lambda_{\lambda'}$.

Corollary 24.1.6. *Assume that $(I, \cdot)$ is of finite type. Write*

$$\Theta = \sum_\nu \sum_{b, b' \in \mathbf{B}_\nu} p_{b,b'} b^- \otimes b'^+ \quad (p_{b,b'} \in \mathbf{Q}(v)).$$

For any ν^0 and any $b^0, b'^0 \in \mathbf{B}_{\nu^0}$, we have $p_{b^0,b'^0} \in \mathcal{A}$.

We can find $\lambda, \lambda' \in X^+$ such that $b^0 \in \mathbf{B}(-w_0(\lambda)), b'^0 \in \mathbf{B}(-w_0(\lambda'))$. By 24.1.5, $\Theta : {}^\omega\Lambda_\lambda \otimes \Lambda_{\lambda'} \to {}^\omega\Lambda_\lambda \otimes \Lambda_{\lambda'}$ maps the $\mathcal{A}$-submodule ${}^\omega_{\mathcal{A}}\Lambda_\lambda \otimes_{\mathcal{A}} ({}_{\mathcal{A}}\Lambda_{\lambda'})$ into itself. By 21.1.2, we may canonically identify as $\mathbf{U}$-modules ${}^\omega\Lambda_\lambda$ with $\Lambda_{-w_0(\lambda)}$ and $\Lambda_{\lambda'}$ with ${}^\omega\Lambda_{-w_0(\lambda')}$ respecting the canonical bases. It follows that $\Theta : \Lambda_{-w_0(\lambda)} \otimes {}^\omega\Lambda_{-w_0(\lambda')} \to \Lambda_{-w_0(\lambda)} \otimes {}^\omega\Lambda_{-w_0(\lambda')}$ maps the $\mathcal{A}$-submodule ${}_{\mathcal{A}}\Lambda_{-w_0(\lambda)} \otimes_{\mathcal{A}} ({}^\omega_{\mathcal{A}}\Lambda_{-w_0(\lambda')})$ into itself. In particular, this submodule contains the vector $\Theta(\eta \otimes \xi) = \sum_\nu \sum_{b,b'} p_{b,b'} b^- \eta \otimes b'^+ \xi$. Here, $\eta = \eta_{-w_0(\lambda)}$ and $\xi = \xi_{w_0(\lambda')}$; b runs over $\mathbf{B}_\nu \cap \mathbf{B}(-w_0(\lambda))$ and b' runs over $\mathbf{B}_\nu \cap \mathbf{B}(-w_0(\lambda'))$. Since the elements $b^- \eta \otimes b'^+ \xi$ (for all indices (ν, b, b') as in the sum) are a part of an $\mathcal{A}$-basis of ${}_{\mathcal{A}}\Lambda_{-w_0(\lambda)} \otimes_{\mathcal{A}} ({}^\omega_{\mathcal{A}}\Lambda_{-w_0(\lambda')})$, and (ν^0, b^0, b'^0) is an index in the sum, it follows that $p_{b^0,b'^0} \in \mathcal{A}$. The corollary is proved.

24.2. A Lemma on Systems of Semi-Linear Equations

Lemma 24.2.1. *Let H be a set with a partial order $\le$ such that for any $h \le h'$ in H, the set $\{h'' | h \le h'' \le h'\}$ is finite. Assume that for each $h \le h'$ in H we are given an element $r_{h,h'} \in \mathcal{A}$ such that*

(a) *$r_{h,h} = 1$ for all h;*

(b) *$\sum_{h'';h \le h'' \le h'} \bar{r}_{h,h''} r_{h'',h'} = \delta_{h,h'}$ for all $h \le h'$ in H.*

Then there is a unique family of elements $p_{h,h'} \in \mathbf{Z}[v^{-1}]$ defined for all $h \le h'$ in H such that

(c) *$p_{h,h} = 1$ for all $h \in H$;*

(d) *$p_{h,h'} \in v^{-1}\mathbf{Z}[v^{-1}]$ for all $h < h'$ in H;*

(e) *$p_{h,h'} = \sum_{h'';h \le h'' \le h'} \bar{p}_{h,h''} r_{h'',h'}$ for all $h \le h'$ in H.*

For $h \le h'$ in H, we denote by $d(h, h')$ the maximum length of a chain $h = h_0 < h_1 < h_2 < \cdots < h_p = h'$ in H. Note that $d(h, h') < \infty$ by our assumption. For any $n \ge 0$, we consider the statement P_n which is the assertion of the lemma restricted to elements $h \le h'$ such that $d(h, h') \le n$. Note that property (e) is meaningful for this statement. We prove P_n by induction on n. The case $n = 0$ is trivial. Assume now that $n \ge 1$. Let $h \le h'$. If $d(h, h') < n$, then $p_{h,h'}$ is defined by P_{n-1}. If $d(h, h') = n$, we note that $q = \sum_{h'';h \le h'' < h'} \bar{p}_{h,h''} r_{h'',h'}$ is defined. We show that $\bar{q} + q = 0$.

Indeed, using P_{n-1} and (a),(b), we have

$$\begin{aligned}\bar{q}+q &= \sum_{h'';h\le h''<h'} p_{h,h''}\bar{r}_{h'',h'} + \sum_{h_1;h\le h_1<h'} \bar{p}_{h,h_1} r_{h_1,h'} \\ &= \sum_{h'',h_1;h\le h_1\le h''<h'} \bar{p}_{h,h_1} r_{h_1,h''}\bar{r}_{h'',h'} + \sum_{h'',h_1;h\le h_1<h''=h'} \bar{p}_{h,h_1} r_{h_1,h'}\bar{r}_{h'',h'} \\ &= \sum_{h'',h_1;h\le h_1\le h''\le h';h_1<h'} \bar{p}_{h,h_1} r_{h_1,h''}\bar{r}_{h'',h'} = \sum_{h_1;h\le h_1<h'} \bar{p}_{h,h_1}\delta_{h_1,h'} = 0,\end{aligned}$$

as required. Since $q \in \mathcal{A}$ satisfies $\bar{q}+q=0$, there is a unique element $q' \in v^{-1}\mathbf{Z}[v^{-1}]$ such that $q'-\bar{q}'=q$. We set $p_{h,h'}=q'$. Then properties (c),(d),(e) are clearly satisfied as far as P_n is concerned. This proves the existence in P_n. The previous proof also shows uniqueness. The lemma is proved.

24.3. The Canonical Basis of ${}^\omega\Lambda_\lambda \otimes \Lambda_{\lambda'}$

24.3.1. In this section we assume that the root datum is Y-regular.

Let $\lambda, \lambda' \in X^+$. We shall consider the following partial order on the set $\mathbf{B}\times\mathbf{B}$: we say that $(b_1,b_1') \le (b_2,b_2')$ if $\operatorname{tr}|b_1| - \operatorname{tr}|b_1'| = \operatorname{tr}|b_2| - \operatorname{tr}|b_2'|$ and if we have either

$$\operatorname{tr}|b_1| < \operatorname{tr}|b_2| \text{ and } \operatorname{tr}|b_1'| < \operatorname{tr}|b_2'|,$$

or

$$b_1 = b_2 \text{ and } b_1' = b_2'.$$

This induces, for given $\lambda,\lambda' \in X^+$, a partial order on the set $\mathbf{B}(\lambda)\times\mathbf{B}(\lambda')$.

As in 19.3.4, let $^- : \Lambda_{\lambda'} \to \Lambda_{\lambda'}$ be the unique $\mathbf{Q}$-linear involution such that $\overline{u\eta_{\lambda'}} = \bar{u}\eta_{\lambda'}$ for all $u \in \mathbf{U}$; similarly, let $^- : {}^\omega\Lambda_\lambda \to {}^\omega\Lambda_\lambda$ be the unique $\mathbf{Q}$-linear involution such that $\overline{u\xi_{-\lambda}} = \bar{u}\xi_{-\lambda}$ for all $u \in \mathbf{U}$. Let $^- = {}^-\otimes{}^- : {}^\omega\Lambda_\lambda\otimes\Lambda_{\lambda'} \to {}^\omega\Lambda_\lambda\otimes\Lambda_{\lambda'}$. The elements $b^+\xi_{-\lambda}\otimes b'^-\eta_{\lambda'}$ with $b \in \mathbf{B}(\lambda)$ and $b' \in \mathbf{B}(\lambda')$ form a $\mathbf{Q}(v)$-basis of ${}^\omega\Lambda_\lambda\otimes\Lambda_{\lambda'}$. They generate a $\mathbf{Z}[v^{-1}]$-submodule $\mathcal{L} = \mathcal{L}_{\lambda,\lambda'}$ and an $\mathcal{A}$-submodule ${}_{\mathcal{A}}\mathcal{L}$.

24.3.2. Now $\Theta : {}^\omega\Lambda_\lambda\otimes\Lambda_{\lambda'} \to {}^\omega\Lambda_\lambda\otimes\Lambda_{\lambda'}$ is well-defined (see 24.1.1). Let $\Psi : {}^\omega\Lambda_\lambda\otimes\Lambda_{\lambda'} \to {}^\omega\Lambda_\lambda\otimes\Lambda_{\lambda'}$ be given by $\Psi(x) = \Theta(\bar{x})$. Since Θ and $^- : {}^\omega\Lambda_\lambda\otimes\Lambda_{\lambda'} \to {}^\omega\Lambda_\lambda\otimes\Lambda_{\lambda'}$ leave ${}_{\mathcal{A}}\mathcal{L}$ stable (see 24.1.5), we have $\Psi({}_{\mathcal{A}}\mathcal{L}) \subset {}_{\mathcal{A}}\mathcal{L}$. From 24.1.2 and 4.1.3, it follows that $\Psi^2 = 1$. We clearly have $\Psi(fx) = \bar{f}\Psi(x)$ for all $f \in \mathcal{A}$ and all x. From the definition we have for all $b_1 \in \mathbf{B}(\lambda), b'_1 \in \mathbf{B}(\lambda')$:

$$\Psi(b_1^+\xi_{-\lambda} \otimes b_1'^-\eta_{\lambda'}) = \sum_{b_2 \in \mathbf{B}(\lambda), b'_2 \in \mathbf{B}(\lambda')} \rho_{b_1,b'_1;b_2,b'_2} b_2^+\xi_{-\lambda} \otimes b_2'^-\eta_{\lambda'}$$

where $\rho_{b_1,b'_1;b_2,b'_2} \in \mathcal{A}$, and $\rho_{b_1,b'_1;b_2,b'_2} = 0$ unless $(b_1, b'_1) \geq (b_2, b'_2)$; hence the last sum is finite.

Note also that $\rho_{b_1,b'_1;b_1,b'_1} = 1$ and

$$\sum_{b_2 \in \mathbf{B}(\lambda), b'_2 \in \mathbf{B}(\lambda')} \bar{\rho}_{b_1,b'_1;b_2,b'_2} \rho_{b_2,b'_2;b_3,b'_3} = \delta_{b_1,b_3}\delta_{b'_1,b'_3},$$

for any $b_1, b_3 \in \mathbf{B}(\lambda), b'_1, b'_3 \in \mathbf{B}(\lambda')$; the last condition follows from $\Psi^2 = 1$. Applying Lemma 24.2.1 to the set $H = \mathbf{B}(\lambda) \times \mathbf{B}(\lambda')$, we see that there is a unique family of elements $\pi_{b_1,b'_1;b_2,b'_2} \in \mathbf{Z}[v^{-1}]$ defined for $b_1, b_2 \in \mathbf{B}(\lambda), b'_1, b'_2 \in \mathbf{B}(\lambda')$ such that

$\pi_{b_1,b'_1;b_1,b'_1} = 1$;
$\pi_{b_1,b'_1;b_2,b'_2} \in v^{-1}\mathbf{Z}[v^{-1}]$ if $(b_1, b'_1) \neq (b_2, b'_2)$;
$\pi_{b_1,b'_1;b_2,b'_2} = 0$ unless $(b_1, b'_1) \geq (b_2, b'_2)$;
$\pi_{b_1,b'_1;b_2,b'_2} = \sum_{b_3,b'_3} \bar{\pi}_{b_1,b'_1;b_3,b'_3} \rho_{b_3,b'_3;b_2,b'_2}$ for all $(b_1, b'_1) \geq (b_2, b'_2)$.

We have the following result.

Theorem 24.3.3. (a) *For any $(b_1, b'_1) \in \mathbf{B}(\lambda) \times \mathbf{B}(\lambda')$, there is a unique element $(b_1 \diamondsuit b'_1)_{\lambda,\lambda'} \in \mathcal{L}$ such that*

$\Psi((b_1 \diamondsuit b'_1)_{\lambda,\lambda'}) = (b_1 \diamondsuit b'_1)_{\lambda,\lambda'}$ *and* $(b_1 \diamondsuit b'_1)_{\lambda,\lambda'} - b_1^+\xi_{-\lambda} \otimes b_1'^-\eta_{\lambda'} \in v^{-1}\mathcal{L}$.

(b) *The element $(b_1 \diamondsuit b'_1)_{\lambda,\lambda'}$ in (a) is equal to $b_1^+\xi_{-\lambda} \otimes b_1'^-\eta_{\lambda'}$ plus a linear combination of elements $b_2^+\xi_{-\lambda} \otimes b_2'^-\eta_{\lambda'}$ with $(b_2, b'_2) \in \mathbf{B}(\lambda) \times \mathbf{B}(\lambda')$, $(b_2, b'_2) < (b_1, b'_1)$ and with coefficients in $v^{-1}\mathbf{Z}[v^{-1}]$.*

(c) *The elements $(b_1 \diamondsuit b'_1)_{\lambda,\lambda'}$ with b_1, b'_1 as above form a $\mathbf{Q}(v)$-basis of ${}^{\omega}\Lambda_\lambda \otimes \Lambda_{\lambda'}$, an $\mathcal{A}$-basis of ${}_{\mathcal{A}}\mathcal{L}$ and a $\mathbf{Z}[v^{-1}]$-basis of $\mathcal{L}$.*

(d) *The natural homomorphism $\mathcal{L} \cap \Psi(\mathcal{L}) \to \mathcal{L}/v^{-1}\mathcal{L}$ is an isomorphism.*

The element $(b_1 \diamondsuit b'_1)_{\lambda,\lambda'} = \sum_{b_2,b'_2} \pi_{b_1,b'_1;b_2,b'_2} b_2^+\xi_{-\lambda} \otimes b_2'^-\eta_{\lambda'}$ (see 24.3.2) satisfies the requirements of (a). This shows existence in (a). It is also clear that the elements $(b_1 \diamondsuit b'_1)_{\lambda,\lambda'}$ just defined satisfy the requirements of (b),(c),(d). It remains to show the uniqueness in (a). It is enough to show that an element $x \in v^{-1}\mathcal{L}$, such that $\bar{x} = x$, is necessarily 0. But this follows from (d).

24.3.4. The basis $(b_1 \diamondsuit b_1')_{\lambda,\lambda'}$ in 24.3.3(c) is called the *canonical basis* of ${}^{\omega}\Lambda_{\lambda} \otimes \Lambda_{\lambda'}$.

24.3.5. Let $\lambda, \tilde{\lambda} \in X^+$. Let $(Y', X', \dots)$ be the simply connected root datum and let $f : Y' \to Y, g : X \to X'$ be the unique morphism of root data. Let $\dot{\mathbf{U}}'$ be the algebra defined like $\dot{\mathbf{U}}$, in terms of $(Y', X', \dots)$. Let $\lambda', \tilde{\lambda}' \in X'^+$ be defined by $\lambda' = g(\lambda), \tilde{\lambda}' = g(\tilde{\lambda})$. Then ${}^{\omega}\Lambda_{\lambda} \otimes \Lambda_{\tilde{\lambda}}$, defined in terms of $\dot{\mathbf{U}}$, has the same ambient space as ${}^{\omega}\Lambda_{\lambda'} \otimes \Lambda_{\tilde{\lambda}'}$, defined in terms of $\dot{\mathbf{U}}'$. We have a priori two definitions of the canonical basis of this space, one in terms of $\dot{\mathbf{U}}$, one in terms of $\dot{\mathbf{U}}'$. From the definitions, we easily see that these two bases coincide.