CHAPTER 22

Positivity of the Action of F_i, E_i in the Simply-Laced Case

22.1.1. In this chapter, the root datum is assumed to be Y-regular. We fix $\lambda \in X^+$ and we set $\Lambda = \Lambda_\lambda$. The main result of this chapter is Theorem 22.1.7, which asserts, in the simply laced case, that the matrices of the linear maps E_i and F_i of Λ into itself, with respect to the canonical basis of Λ, have as entries polynomials with integer, ≥ 0 coefficients.

Using Theorem 18.3.8, we see that Lemma 18.2.7 is true unconditionally. We restate it here as follows.

Theorem 22.1.2. *We have* $\Xi(\mathcal{L}(\mathbf{f})) \subset \mathcal{L}(\mathbf{f}) \odot L(\Lambda)$.

22.1.3. In the following corollary we shall use the notation

$$\nu \circ \lambda = \sum_{i \in I} \nu_i \langle i, \lambda \rangle (i \cdot i/2)$$

for any $\nu \in \mathbf{Z}[I], \lambda \in X$.

Corollary 22.1.4. *Let $b \in \mathbf{B}$. Write $r(b) = \sum h_{b;b_1,b_2} b_1 \otimes b_2$ and $\bar{r}(b) = \sum g_{b;b_1,b_2} b_1 \otimes b_2$ where $h_{b;b_1,b_2} \in \mathcal{A}$ and $g_{b;b_1,b_2} \in \mathcal{A}$; here b_1, b_2 run over $\mathbf{B}$. Thus, $g_{b;b_1,b_2} = \overline{h_{b;b_1,b_2}}$.*

If $b_2 \in \mathbf{B}(\lambda)$ and $b, b_1 \in \mathbf{B}$, we have

$$v^{-|b_1| \circ (\lambda - |b_2|)} g_{b;b_1,b_2} \in \mathbf{Z}[v^{-1}] \text{ and } v^{|b_1| \circ (\lambda - |b_2|)} h_{b;b_1,b_2} \in \mathbf{Z}[v].$$

It suffices to prove the statement about $g_{b;b_1,b_2}$.

By 3.1.5, we have $\Delta(b^-) = \sum g_{b;b_1,b_2} b_1^- \otimes \tilde{K}_{-|b_1|} b_2^-$. By the definition of Ξ, we have

$$\Xi(b) = \sum g_{b;b_1,b_2} b_1 \otimes \tilde{K}_{-|b_1|} b_2^- \eta = \sum v^{-|b_1| \circ (\lambda - |b_2|)} g_{b;b_1,b_2} b_1 \otimes b_2^- \eta.$$

By the previous theorem, if $b_2 \in \mathbf{B}(\lambda)$, the coefficient of $b_1 \otimes b_2^- \eta$ is in $\mathbf{A}$. This coefficient is clearly in $\mathcal{A}$; hence it is in $\mathbf{Z}[v^{-1}]$. The corollary follows.

Corollary 22.1.5. *Let $i \in I$. Let $b \in \mathbf{B}$. Let us write ${}_ir(b) = \sum_{b';n\in\mathbf{Z}} d_{b,\theta_i,b',n} v^n b'$ where b' runs over $\mathbf{B}$ and $d_{b,\theta_i,b',n}$ are integers.*

(a) *Let $b' \in \mathbf{B}(\lambda)$ and let $n \in \mathbf{Z}$ be such that $d_{b,\theta_i,b',n} \neq 0$. Then*

$$i \circ (\lambda - |b'|) + n \geq 0.$$

(b) *We have*

$$E_i(b^-\eta) = \sum_{b';n\in\mathbf{Z}} d_{b,\theta_i,b',n} \frac{v^{i\circ(\lambda-|b'|)+n} - v^{-i\circ(\lambda-|b'|)-n}}{v_i - v_i^{-1}} b'^-\eta$$

where b' runs over $\mathbf{B}(\lambda)$.

We apply the previous corollary to $h_{b;b_1,b_2}$ with $b_1 = \theta_i$; we obtain

$$\sum_n v^{i\circ(\lambda-|b'|)+n} d_{b,\theta_i,b',n} \in \mathbf{Z}[v]$$

for any $b' \in \mathbf{B}(\lambda)$; (a) follows. We now prove (b). By 3.1.6(b), we have

$$E_i(b^-\eta) = (v_i - v_i^{-1})^{-1}(-r_i(b)^-\tilde{K}_{-i}\eta + v^{-|b|\cdot i+i\cdot i}{}_ir(b)^-\tilde{K}_i\eta)$$

since $E_i\eta = 0$. By 1.2.14, we have $r_i(b) = v^{|b|\cdot i-i\cdot i}\overline{{}_ir(b)}$, since $\bar{b} = b$. Note also that $\tilde{K}_{\pm i}\eta = v_i^{\pm\langle i,\lambda\rangle}\eta$. Thus,

$$E_i(b^-\eta) = \sum_{b';n} d_{b,\theta_i,b',n} \frac{v^{-|b|\cdot i+i\cdot i+n} v_i^{\langle i,\lambda\rangle} - v^{|b|\cdot i-i\cdot i-n} v_i^{-\langle i,\lambda\rangle}}{v_i - v_i^{-1}} b'^-\eta.$$

Using now $|b| = |b'| + i$, we obtain (b).

22.1.6. Let $b \in \mathbf{B}(\Lambda)$. For any $i \in I$, we set $F_i b = \sum_{b',n} f_{b,b',i,n} v^n b'$, $E_i b = \sum_{b',n} \tilde{f}_{b,b',i,n} v^n b'$ where b' runs over $\mathbf{B}(\Lambda)$ and n runs over $\mathbf{Z}$; the coefficients $f_{b,b',i,n}, \tilde{f}_{b,b',i,n}$ are integers.

Theorem 22.1.7. *Assume that the Cartan datum is simply laced. Then $f_{b,b',i,n} \in \mathbf{N}$ and $\tilde{f}_{b,b',i,n} \in \mathbf{N}$ for any b, b', i, n.*

If $\beta, \beta' \in \mathbf{B}$ are such that $\beta\eta = b, \beta'\eta = b'$, then with the notation of Theorem 14.4.13, we have

$$f_{b,b',i,n} = c_{\theta_i,\beta,\beta',n}$$

and

$$\sum_n \tilde{f}_{b,b',i,n} v^n = \sum_n [i \circ (\lambda - |b'|) + n] d_{b,\theta_i,b',n}$$

(we have used Corollary 22.1.5(b) and the equality $v_i = v$). By Theorem 14.4.13, the integers $c_{\theta_i,\beta,\beta',n}$ are ≥ 0. Hence $f_{b,b',i,n} \in \mathbf{N}$.

Again by Theorem 14.4.13, the integers $d_{b,\theta_i,b',n}$ are ≥ 0 and, by Corollary 22.1.15(a), we have $i \circ (\lambda - |b'|) + n \geq 0$ for any n such that $d_{b,\theta_i,b',n} \neq 0$. Since $[N]$ is a sum of powers of v if $N \geq 0$, we deduce that $\tilde{f}_{b,b',i,n} \in \mathbf{N}$. The theorem is proved.

Notes on Part III

1. Most results in Part III are due to Kashiwara [2]. An exception is Theorem 22.1.7, which is new.
2. Although Theorem 22.1.2 does not appear explicitly in Kashiwara's papers, it is close to results which do appear; the same applies to the results in 17.1. The proofs in 17.2 are quite different from Kashiwara's.
3. The proof in 19.2.3 is an adaptation of arguments in [3].

References

1. M. Kashiwara, *Crystallizing the q-analogue of universal enveloping algebras*, Comm. Math. Phys. **133** (1990), 249–260.
2. ______, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465–516.
3. ______, *Crystal base and Littelman's refined Demazure character formula*, Duke Math. J. **71** (1993), 839–858.
4. G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 447–498.