CHAPTER 21

Cartan Data of Finite Type

21.1.1. In this chapter we assume that the Cartan datum is of finite type; then the root datum is automatically Y-regular and X-regular.

Let $\lambda' = -w_0(\lambda)$. Then $\lambda' \in X^+$ and we may consider the $\mathbf{U}$-module ${}^\omega\Lambda_{\lambda'}$ as in 3.5.7. Since ${}^\omega\Lambda_{\lambda'} = \Lambda_{\lambda'}$ as a vector space, the canonical basis $\mathbf{B}(\Lambda_{\lambda'})$ of $\Lambda_{\lambda'}$ may be regarded as a basis of ${}^\omega\Lambda_{\lambda'}$.

Proposition 21.1.2. *There is a unique isomorphism of* $\mathbf{U}$*-modules* $\chi : \Lambda_\lambda \to {}^\omega\Lambda_{\lambda'}$ *such that* χ *maps* $\mathbf{B}(\Lambda_\lambda)$ *onto* $\mathbf{B}(\Lambda_{\lambda'})$.

By 6.3.4, $\Lambda_{\lambda'}$ is a finite dimensional simple object of $\mathcal{C}'$. Hence ${}^\omega\Lambda_{\lambda'}$ is a finite dimensional simple object of $\mathcal{C}'$. By definition, its $(-\lambda')$-weight space is one dimensional and the $(-\lambda' - i')$-weight space is zero for any i. By Weyl group invariance (5.2.7), it follows that the λ-weight space is one dimensional and the $(\lambda + i')$-weight space is zero for any i (we have $\lambda = w_0(-\lambda')$).

Let x be the unique element of $\mathbf{B}(\Lambda_{\lambda'})$ in this λ-weight space. Then $E_i x = 0$ for all i. By Lemma 3.5.8, there is a unique morphism (in $\mathcal{C}'$) $\chi : \Lambda_\lambda \to {}^\omega\Lambda_{\lambda'}$ which carries η to x. Since χ is a non-zero morphism between simple objects, it is an isomorphism. We can regard χ as an isomorphism of vector spaces $\Lambda_\lambda \cong \Lambda_{\lambda'}$ such that $\chi(uy) = \omega(u)\chi(y)$ for all $u \in \mathbf{U}$ and $y \in \Lambda_\lambda$ and such that $\chi(\eta) \in \mathbf{B}(\Lambda_{\lambda'})$.

We have

(a) $\chi({}_{\mathcal{A}}\Lambda_\lambda) \subset {}_{\mathcal{A}}\Lambda_{\lambda'}$.

Indeed, let $y \in {}_{\mathcal{A}}\Lambda_\lambda$. Then $y = g^-\eta$ for some $g \in {}_{\mathcal{A}}\mathbf{f}$; hence $\chi(y) = g^+\chi(\eta)$. It remains to use the fact that ${}_{\mathcal{A}}\Lambda_{\lambda'}$ is stable under g^+ (see 19.3.2).

We have

(b) $\overline{\chi(x)} = \chi(\bar{x})$ for all $x \in \Lambda_\lambda$.

Indeed, we can write $x = u\eta$ with $u \in \mathbf{U}$. We have

$$\overline{\chi(u\eta)} = \overline{\omega(u)\chi(\eta)} = \overline{\omega(u)\chi(\eta)} = \omega(\bar{u})\chi(\eta) = \chi(\bar{u}\eta) = \chi(\bar{x}),$$

as required.

We have

(c) $(x,x') = (\chi(\eta),\chi(\eta))^{-1}(\chi(x),\chi(x'))$ for $x,x' \in \Lambda_\lambda$.

Indeed, if we set $((x,x')) = (\chi(\eta),\chi(\eta))^{-1}(\chi(x),\chi(x'))$ we obtain a form satisfying the defining properties of (x,x'), hence equal to it.

Let $b \in \mathbf{B}(\Lambda_\lambda)$. Let $b' = \chi(b)$. Using (a),(b), we see that $b' \in {}_{\mathcal{A}}\Lambda_{\lambda'}$ and $\bar{b}' = b'$. Using (c) and the fact that (b,b) and $(\chi(\eta),\chi(\eta))$ are in $1+v^{-1}\mathbf{Z}[v^{-1}]$, we see that $(b',b') \in 1+v^{-1}\mathbf{A}$. Since $b' \in {}_{\mathcal{A}}\Lambda_{\lambda'}$, we have also $(b',b') \in \mathcal{A}$; hence $(b',b') \in 1+v^{-1}\mathbf{Z}[v^{-1}]$. Using Theorem 19.3.5, it follows that $\pm b' \in \mathbf{B}(\Lambda_{\lambda'})$. This argument also shows that, if $(L,\mathbf{b}),(L',\mathbf{b}')$ are the bases at ∞ of $\Lambda_\lambda,\Lambda_{\lambda'}$ defined in 20.1.4, then $\chi(L) = L'$.

We can find a sequence $i_1,i_2,\dots,i_t$ in I such that $b = \tilde{F}_{i_1}\tilde{F}_{i_2}\cdots\tilde{F}_{i_t}\eta \mod v^{-1}L$. From the definitions we have that $\chi\tilde{F}_i = \tilde{E}_i\chi$ for all i. It follows that $b' = \tilde{E}_{i_1}\tilde{E}_{i_2}\cdots\tilde{E}_{i_t}\chi(\eta) \mod v^{-1}L'$, so that $b' \mod v^{-1}L'$ belongs to $\mathbf{b}'$. Since $\pm b' \in \mathbf{B}(\Lambda_{\lambda'})$, it follows that $b' \in \mathbf{B}(\Lambda_{\lambda'})$. The proposition is proved.

21.1.3. We shall identify the $\mathbf{U}$-modules Λ_λ and ${}^{\omega}\Lambda_{\lambda'}$ via χ. In particular, the generator $\xi = \xi_{-\lambda'}$ of ${}^{\omega}\Lambda_{\lambda'}$ (see 3.5.7) is now regarded as a vector in the $w_0(\lambda)$-weight space of Λ_λ, which belongs to the canonical basis of Λ_λ.