CHAPTER 20

Bases at ∞

20.1. THE BASIS AT ∞ OF Λ_λ

20.1.1. Let M be an object of $\mathcal{C}'$. We define a *basis at* ∞ of M to be a pair consisting of

(a) a free $\mathbf{A}$-submodule L of M such that $M = \mathbf{Q}(v) \otimes_{\mathbf{A}} L = M$ and

(b) a basis $\mathbf{b}$ of the $\mathbf{Q}$-vector space $L/v^{-1}L$;

it is required that the properties (c)–(f) below are satisfied.

(c) L is stable under the operators $\tilde{E}_i, \tilde{F}_i : M \to M$ for all i; thus, $\tilde{E}_i, \tilde{F}_i$ act on $L/v^{-1}L$;

(d) $\tilde{F}_i(\mathbf{b}) \subset \mathbf{b} \cup \{0\}$ and $\tilde{E}_i(\mathbf{b}) \subset \mathbf{b} \cup \{0\}$ for all i;

(e) we have $L = \oplus L^\lambda$ where $L^\lambda = L \cap M^\lambda$ and $\mathbf{b} = \sqcup \mathbf{b}^\lambda$ where $\mathbf{b}^\lambda = \mathbf{b} \cap (L^\lambda / v^{-1} L^\lambda)$;

(f) given $b, b' \in \mathbf{b}$ and $i \in I$, we have $\tilde{E}_i b = b'$ if and only if $\tilde{F}_i b' = b$.

The definition given above of bases at ∞ is due to Kashiwara who calls them *crystal bases.*

Lemma 20.1.2. *Let $x \in L \cap M^\lambda$ and let $i \in I$. Let $t = \langle i, \lambda \rangle$. Write $x = \sum_{s; s \geq 0; s+t \geq 0} F_i^{(s)} x_s$ where $x_s \in \ker(E_i : M^{\lambda + si'} \to M)$ and $x_s = 0$ for large enough s. (See 16.1.4.)*

(a) *For all $s \geq 0$ we have $x_s \in L$.*

(b) *If $x \mod v^{-1}L$ belongs to $\mathbf{b}$, then there exists s_0 such that $x_s \in v^{-1}L$ for $s \neq s_0$, $x_{s_0} \mod v^{-1}L$ belongs to $\mathbf{b}$ and $x = F_i^{(s_0)} x_{s_0} \mod v^{-1}L$.*

We prove (a) by induction on $N \geq 0$ such that $x_s = 0$ for $s > N$. For $N = 0$, the result is clear. Assume now that $N \geq 1$. We have $\tilde{E}_i x = \sum_{s; s \geq 1; s+t \geq 0} F_i^{(s-1)} x_s$ where $x_s = 0$ for $s > N$. By definition, we have $\tilde{E}_i x \in L \cap M^{\lambda + i'}$. Hence if $t' = \langle i, \lambda + i' \rangle = t + 2$, we have $\tilde{E}_i x = \sum_{s'; s' \geq 0; s'+t' \geq 1} F_i^{(s')} x_{s'+1}$ and $x_{s'+1} = 0$ for $s' \geq N$. By the induction hypothesis, we have $x_s \in L$ for all $s \geq 1$. Since L is stable under $\tilde{F}_i$ and

$F_i^{(s)} x_s = \tilde{F}_i^s x_s$, it follows that $F_i^{(s)} x_s \in L$ for all $s \geq 1$. Since $x \in L$, we deduce that $x_0 \in L$. This proves (a).

We prove (b) by induction on $N \geq 0$ as above. If $N = 0$, there is nothing to prove. Assume that $N \geq 1$. If $\tilde{E}_i x \in v^{-1}L$, then $v \sum_{s';s' \geq 0; s'+t' \geq 1} F_i^{(s')} x_{s'+1} \in L$ and by (a) we have $v x_{s'+1} \in L$ for all $s' \geq 0$. Hence $x_s \in v^{-1}L$ for all $s \geq 1$. As before we then have $F_i^{(s)} x_s \in v^{-1}L$ for $s \geq 1$ and $x = x_0 \mod v^{-1}L$. If $\tilde{E}_i x \notin v^{-1}L$, then $\tilde{E}_i x \mod v^{-1}L$ belongs to $\mathbf{b}$. By the induction hypothesis, there exists $s_0 \geq 1$ such that $x_s \in v^{-1}L$ for $s \neq s_0$ and $s \geq 1$. Therefore we have $\tilde{E}_i x = F_i^{(s_0-1)} x_{s_0} \mod v^{-1}L$. Equivalently, we have $\tilde{E}_i x = \tilde{F}_i^{s_0-1} x_{s_0} \mod v^{-1}L$. Applying $\tilde{F}_i$ to this and using 20.1.1(f), we obtain $x = \tilde{F}_i \tilde{E}_i x = \tilde{F}_i^{s_0} x_{s_0} = F_i^{(s_0)} x_{s_0} \mod v^{-1}L$. The lemma is proved.

20.1.3. In the next theorem we assume that the root datum is Y-regular. Let $\lambda \in X^+$. Let L be the $\mathbf{A}$-submodule of Λ_λ generated by the canonical basis $\mathbf{B}(\Lambda_\lambda)$ and let $\mathbf{b}$ be the image of the canonical basis in $L/v^{-1}L$.

Theorem 20.1.4. *$(L, \mathbf{b})$ is a basis at ∞ of Λ_λ.*

Property 20.1.1(c) follows from Theorem 18.3.8. We prove that property 20.1.1(d) is satisfied. Let $b \in \mathbf{b}$. There exists $\beta \in \mathbf{B}$ such that b is $\beta^- \eta_\lambda \mod v^{-1}L$. From Theorem 18.3.8 we see that $\tilde{F}_i b$ is $\tilde{\phi}_i(\beta)^- \eta_\lambda \mod v^{-1}L$ and $\tilde{E}_i b$ is $\tilde{\epsilon}_i(\beta)^- \eta_\lambda \mod v^{-1}L$ or zero. By 17.3.7, we have $\tilde{\phi}_i(\beta) = \beta' \mod v^{-1}\mathcal{L}(\mathbf{f})$ for some $\beta' \in \mathcal{B}$ and this is necessarily in $\mathbf{B}$. Then $\tilde{\phi}_i(\beta)^- \eta_\lambda = \beta'^- \eta_\lambda \mod v^{-1}L$ so that $\tilde{F}_i b$ is $\beta'^- \eta_\lambda \mod v^{-1}L$ and $\beta'^- \eta_\lambda \mod v^{-1}L$ is in $\mathbf{b} \cup \{0\}$.

By 17.3.7, we have either $\tilde{\epsilon}_i(\beta) = \beta'' \mod v^{-1}\mathcal{L}(\mathbf{f})$ for some $\beta'' \in \mathcal{B}$ (which is necessarily in $\mathbf{B}$) or $\tilde{\epsilon}_i(\beta) = 0 \mod v^{-1}\mathcal{L}(\mathbf{f})$. Then $\tilde{\epsilon}_i(\beta)^- \eta_\lambda = \beta''^- \eta_\lambda \mod v^{-1}L$ or $\tilde{\epsilon}_i(\beta)^- \eta_\lambda = 0 \mod v^{-1}L$ so that $\tilde{E}_i b$ is $\beta''^- \eta_\lambda \mod v^{-1}L$ or 0. Now $\beta''^- \eta_\lambda \mod v^{-1}L$ is in $\mathbf{b} \cup \{0\}$. This proves property 20.1.1(d).

Property 20.1.1(e) is clearly satisfied. We prove that property 20.1.1(f) is satisfied. Let $b, b' \in \mathbf{b}$. We have $b = \beta^- \eta_\lambda \mod v^{-1}L$ and $b' = \beta'^- \eta_\lambda \mod v^{-1}L$ where $\beta, \beta' \in \mathbf{B}$.

By 18.3.8, we have $\tilde{E}_i b = b'$ if and only if $(\tilde{\epsilon}_i \beta)^- \eta_\lambda = \beta'^- \eta_\lambda \mod v^{-1}L$. This is equivalent to the condition that

(a) $\tilde{\epsilon}_i \beta = \beta' \mod v^{-1}\mathcal{L}(\mathbf{f})$.

Similarly, the condition that $\tilde{F}_i b' = b$ is equivalent to the condition that

(b) $\tilde{\phi}_i \beta' = \beta \mod v^{-1}\mathcal{L}(\mathbf{f})$.

Now conditions (a) and (b) are equivalent by 17.3.7. The theorem is proved.

20.2. Basis at ∞ in a Tensor Product

20.2.1. Let $M, M' \in \mathcal{C}'$. Assume that M and M' have finite dimensional weight spaces. Assume that $(L, \mathbf{b})$ (resp. $(L', \mathbf{b}')$) is a given basis at ∞ of M (resp. M'). Consider the tensor product $M \otimes M' \in \mathcal{C}'$.

Theorem 20.2.2. *The free* $\mathbf{A}$*-submodule* $L \otimes_{\mathbf{A}} L'$ *of* $M \otimes M'$ *and the* $\mathbf{Q}$*-basis* $\mathbf{b} \otimes \mathbf{b}'$ *of* $(L \otimes_{\mathbf{A}} L')/v^{-1}(L \otimes_{\mathbf{A}} L') = (L/v^{-1}L) \otimes_{\mathbf{Q}} (L'/v^{-1}L')$ *define a basis at* ∞ *of* $M \otimes M'$.

Only properties (c),(d),(f) in the definition 20.1.1 need to be verified. In verifying these properties, we shall fix $i \in I$ and write L^t for the sum $\oplus L^\lambda$ over all λ such that $\langle i, \lambda \rangle = t$. The notation L'^t has a similar meaning.

Let G^t be the set of all $z \in L^t$ such that $z \mod v^{-1}L$ belongs to $\mathbf{b}$ and such that $E_i z = 0$. Let G'^t be the set of all $z' \in L'^t$ such that $z' \mod v^{-1}L'$ belongs to $\mathbf{b}'$ and such that $E_i z' = 0$. From the definitions, all elements of the form $F_i^{(s)} z \quad (z \in G^t, s \in [0, t])$ belong to $\mathbf{b}$ modulo $v^{-1}L$ and according to 20.1.2, all elements of $\mathbf{b}$ are obtained in this way.

Similarly, all elements of the form $F_i^{(s')} z' \quad (z' \in G'^{t'}, s' \in [0, t'])$ belong to $\mathbf{b}'$ modulo $v^{-1}L'$ and all elements of $\mathbf{b}'$ are obtained in this way.

Using Nakayama's lemma, which is applicable since the weight spaces are assumed to be finite dimensional, we deduce that the elements $F_i^{(s)} z \quad (z \in G^t, s \in [0, t])$ generate the $\mathbf{A}$-module L; similarly, the elements $F_i^{(s')} z' \ (z' \in G'^{t'}, s' \in [0, t'])$ generate the $\mathbf{A}$-module L'.

Let $z \in G^t, z' \in G'^{t'}, s \in [0, t], s' \in [0, t']$. According to 17.2.4, we have

$$\tilde{F}_i(F_i^{(s)} z \otimes F_i^{(s')} z') = F_i^{(s)} z \otimes F_i^{(s'+1)} z' \mod v^{-1}(L \otimes_{\mathbf{A}} L')$$

if $s + s' < t'$;

$$\tilde{F}_i(F_i^{(s)} z \otimes F_i^{(s')} z') = F_i^{(s+1)} z \otimes F_i^{(s')} z' \mod v^{-1}(L \otimes_{\mathbf{A}} L')$$

if $s + s' \geq t'$;

$$\tilde{E}_i(F_i^{(s)} z \otimes F_i^{(s')} z') = F_i^{(s)} z \otimes F_i^{(s'-1)} z' \mod v^{-1}(L \otimes_{\mathbf{A}} L')$$

if $s + s' \leq t'$;

$$\tilde{E}_i(F_i^{(s)} z \otimes F_i^{(s')} z') = F_i^{(s-1)} z \otimes F_i^{(s')} z' \mod v^{-1}(L \otimes_{\mathbf{A}} L')$$

if $s + s' > t'$.

It follows that $\tilde{E}_i, \tilde{F}_i$ map a set of generators of the $\mathbf{A}$-module $L \otimes_{\mathbf{A}} L'$ into $L \otimes_{\mathbf{A}} L'$; hence they map $L \otimes_{\mathbf{A}} L'$ into $L \otimes_{\mathbf{A}} L'$. This verifies property 20.1.1(c) of a basis at ∞. Properties 20.1.1(e),(f) of a basis at ∞ are also clear from the previous formulas. The theorem is proved.

20.2.3. Assume that $z \in G^t, z' \in G'^{t'}$ and that $s \in [0,t], s' \in [0,t']$ are such that $t + t' = 2(s + s')$. By the formulas in 20.2.2, the condition that $\tilde{F}_i(F_i^{(s)} z \otimes F_i^{(s')} z') \in v^{-1}(L \otimes_{\mathbf{A}} L')$ is that either $s' = t'$ and $s + s' < t'$, or $s = t$ and $s + s' \geq t'$. The first case cannot occur since $s \geq 0$. Hence the condition is that $s = t$ and $s + s' \geq t'$. But if $s = t$ then $t' = s + 2s'$ hence $s + s' \geq s + 2s'$ so that $s' = 0$. Thus the condition is $s = t = t', s' = 0$. We can reformulate this as follows.

Proposition 20.2.4. *Let $(M, L, \mathbf{b}), (M', L', \mathbf{b}')$ be as above. Let $b \in \mathbf{b}, b' \in \mathbf{b}'$. Assume that $b \in \mathbf{b}^\lambda$ and $b' \in \mathbf{b}'^{\lambda'}$ and $\langle i, \lambda \rangle + \langle i, \lambda' \rangle = 0$. Then the following two conditions are equivalent:*

(a) $\tilde{F}_i(b \otimes b') = 0$ *in* $(L \otimes_{\mathbf{A}} L')/v^{-1}(L \otimes_{\mathbf{A}} L')$;

(b) $\tilde{F}_i(b) = 0$ *in* $L/v^{-1}L$ *and* $\tilde{E}_i(b') = 0$ *in* $L'/v^{-1}L'$.